Essays in Microeconomic Theory

by

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Abstract

This dissertation consists of two independent essays. In the first essay, Coordination in Complex Environments, I introduce a framework to study coordination in highly uncertain environments. Coordination is an important aspect of innovative contexts, where: the more innovative a course of action, the more uncertain its outcome. To explore the interplay of coordination and informational complexity, I embed a beauty-contest game into a complex environment. I uncover a new conformity phenomenon. The new effect may push towards exploration of unknown alternatives, or constitute a status quo bias, depending on the network structure of the connections among players.

In the second essay, The Extensive Margin of Bayesian Persuasion, I study the persuasion of a receiver who accesses information only if she exerts attention effort. The sender uses the information to incentivize the receiver to pay attention. I show that persuasion mechanisms are equivalent to signals. In a model of media capture, the sender finds it optimal to censor high states.

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This work is dedicated to my family.

Contents

Part I

Essay 1: Coordination in Complex Environments

1 Introduction

Coordination poses challenges in highly uncertain environments. Consider retailers that share the same manufacturer and choose marketing strategies.[1] Innovative advertisement comes with uncertainty about the brand image of the manufacturer. Moreover, retailers need to coordinate their advertisements and succeed in different markets. Does uncertainty lead to a unified brand image, and do the marketing campaigns align with the interests of the manufacturer? Coordination is also an important aspect of technological innovation. Developers of messaging apps benefit from interoperability, as it addresses the uncertainty surrounding which apps will be popular. Similarly, tech companies often converge on standards for universal connectors. Do coordination motives lead to more exploration? This paper studies coordination problems in the face of "incremental" uncertainty, referred to as *complexity*, such that: the more innovative a decision is, the more uncertain its outcome becomes.

I introduce a model of coordination within a complex environment. In the model, every player wants the outcome of her action to be close to a target. The target of a player combines her fixed favorite outcome with the individual outcomes of the opponents, leading to a coordination-adaptation tradeoff. A given network of players determines how much each target weighs each individual outcome. Analogous coordination motives arise in several

[1] This type of marketing for the manufacturer's product is known as co-op advertising with multiple retailers (Jørgensen and Zaccour, 2014).

settings, including financial markets, oligopolistic competition, organizations, and labor markets (Keynes, 1936; Topkis, 1998; Marschak and Radner, 1972; Diamond, 1982).

Complexity is modeled by the uncertainty about how actions translate into outcomes, to capture that more innovative actions lead to more volatile outcomes. This informational complexity involves a status quo and a *covariance structure*. The status quo is an action that implies relatively low uncertainty. The covariance structure describes the likelihood that two actions yield similar outcomes. For example, this complexity is relevant when deciding about a financial investment, the adoption of novel pricing strategies, and how boldly to innovate in new technologies. In the model, players simultaneously choose policies, and there is an outcome for every policy, given by an *outcome function*. Players know that the outcome function is the realized path of a Brownian motion. The initial point of the Brownian motion represents the status quo: a known outcome corresponds to the initial policy. Instead, different policies than the status-quo (initial) policy lead to outcomes known only up to a noise. The more an outcome differs in expectation from the status-quo outcome, the higher its variance; this approach to modeling complex environments is introduced by Callander (2011a).

I show that the interplay of coordination and complexity leads to a novel conformity phenomenon. In particular, expected outcomes are closer across players than in an environment without complexity, in all equilibria when the network is complete. This conformity occurs in addition to the status-quo bias identified by Callander (2011a) and the conformity due merely to the presence of coordination motives. To separate the new conformity from previously studied phenomena, I decompose equilibrium expected outcomes in terms of more primitive objects: the equilibrium outcomes in a non-complex environment, the status-quo bias absent strategic interactions, and a new strategic-uncertainty element due to the interplay of complexity and coordination.

The new element in the equilibrium characterization arises from an endogenous leader-follower relationship among players introduced by the covariance structure. In the model, the *follower* in a pair of players is the one with the closest policy to the status quo. Consider the two ways in which the policy of a player influences the incentives of her opponents. First, policies enter into the expected targets of players, due to standard coordination motives. Second, the policy of a player determines the correlation between her outcome and her opponents' outcomes. Given a pair of players with different policies, the only player whose policy directly affects the covariance is the follower, not the leader.[2] As a result, the follower has an extra incentive to explore by choosing a policy in the direction of the leader. The new incentive of the follower is the source of conformity. In general, the leader-follower relationship induces an asymmetry among players that interacts with the exogenous structure of connections.

Conformity has a delicate interaction with the network of players. A player may exert substantial influence on a follower player through the network. This influence can be so strong that it steers the follower away from a third player. In this case, "counter-formity" emerges, leading to expected outcomes that are more distant between certain players than in the no-complexity case. In general, the leader-follower relationship is determined in equilibrium. The equilibrium decomposition serves to verify that a certain leader-follower structure can be sustained.

To illustrate the conformity phenomenon, I study applications of the model. In oligopolistic competition, coordination motives arise from strategic complementarities whenever the incentives to raise prices increase with the prices of competitors. Moreover, a pricing algorithm may rely on data not available when algorithmic pricing is adopted (Brown and

[2] This property is due to independent increments, a reasonable assumption in innovative contexts owing to a maximum-uncertainty principle (Jovanovic and Rob, 1990). However, the covariance structures implied by other Gaussian processes have features that reminisce about a leader-follower relationship; for instance, the Ornstein-Uhlenbeck covariance between two "outcomes" is increasing only in one "policy" (Bardhi, 2023).

MacKay, 2023). Hence, complexity arises when innovative pricing rules are associated with high uncertainty. In this case, conformity takes the form of concentrated expected prices across firms. The presence of conformity suggests a downward bias when firm heterogeneity is estimated from price data and the analyst does not control for complexity.[3] The equilibrium decomposition provides a tool for isolating the new conformity effect.

I show that conformity increases in the complexity of the environment, whenever two players exist who are the leader and the follower for each of their opponents. This order of players occurs in applications, such as in an oligopoly with two firms with extreme marginal costs. The measure of complexity is the additional uncertainty implied by a change in expected outcome away from the status quo.[4] The intuition for this comparative statics follows from the "first-order" effect of an increase in complexity. In particular, matching the outcome of a leader becomes more "cost-effective" for a follower, relative to targeting a favorite outcome. The reason is that the two outcomes are the same when players choose the same policy, regardless of the level of complexity. This comparative statics is consistent with findings in social psychology. Since Asch (1951), psychologists observe that conformity "is far greater on difficult items than on easy ones." The "difficulty" is typically obtained by asking experimental subjects about their "certainty of judgement" (Krech et al., 1962).[5]

New coordination problems arise in complex environments. The source of equilibrium multiplicity is the presence of endogenous "kinks" in payoffs. Intuitively, at the margin there is a premium to choosing the same policy as another player, because two individual outcomes are the same whenever policies are the same. Hence, coordination problems

[3] Since Bresnahan (1987), a common empirical exercise is to infer the cost parameters from data, under certain hypotheses about market structure and equilibrium behavior.

[4] Letting μ and ω be the drift and variance parameters of the Brownian motion, the measure of complexity is $\omega/(2|\mu|)$.

[5] I also show that conformity increases in the strength of coordination motives and the number of players, matching the observation that "yielding to the group pressures" is easier for higher "group cohesion" and "group size" Krech et al. (1962).

are intimately linked to the leader-follower relationship: by choosing the same policy of an opponent, a player neutralizes the asymmetry. The location of kinks is determined in equilibrium: a player's payoff has a kink at every policy of an opponent. To make predictions for coordination in complex environments, I study an equilibrium-selection rule. The coordination game admits a "potential" with a unique maximizer, which acts as an equilibrium selection (Monderer and Shapley, 1996).[6] I characterize the potential-maximizer equilibrium, and I leverage the characterization in applications, as a means to study welfare, select among multiple equilibria, and for comparison with the no-complexity case (without complexity, the unique equilibrium maximizes the potential.)

I study the interaction between the conformity motive and the network of players' connections. In a two-type network, a decrease in inter-group heterogeneity below a tipping point triggers coordination problems: every player faces an interval of policies sustainable in equilibrium. This result is important for policy interventions that change favorite outcomes of players (Galeotti et al., 2020): certain interventions may bring about coordination problems. For sufficiently high complexity, extreme conformity prevails: all players choose the same policy. The equilibrium selection allows to retrieve the heterogeneity between groups given such homogeneous behavior. In particular, extreme conformity is observationally equivalent to the optimal choice of a representative player. The equilibrium selection pins down the weighted average of favorite outcomes that constitutes the "representative" favorite outcome.

Complexity has implications for management of organizations with decentralized authority, which includes practices such as co-op advertising and multi-branding. In multi-division organizations, a division manager trades off coordination with other managers and adap-

[6] The uniqueness of a potential-maximizer equilibrium obtains jointly with the multiplicity of equilibria because the potential is not smooth. Two papers study specific nondifferentiable potentials as counterexamples to the results for smooth potentials (Radner, 1962; Neyman, 1997).

tation to idiosyncratic needs. Moreover, communication frictions induce noise over the implementation of managerial instructions. The noise is typically minimal if the instruction is about maintaining the current situation. I show that an organization with decentralized authority can implement profit maximization in sufficiently complex environments. Hence, complexity is a rationale for decentralized organizations that leave the holding company with only oversight authority.[7]

To investigate robustness of my results, I consider generalizations of the model. I establish that the status-quo bias and the leader-follower intuition survive status-quo heterogeneity. In particular, I study a general model that incorporates incomplete information about a heterogeneous status quo across players. In the model, a vector of status-quo policies is common knowledge and players have private information about their own status-quo outcomes. The set of equilibria has a similar structure as in the homogeneous-status-quo case: there exists a greatest and a least equilibrium, and they are in nondecreasing strategies. In equilibrium, every player expects to be a leader for every opponent with a certain probability.

I separately identify the role played by variance and covariance of the environment in a general model in which players have "correlated" outcome functions. In particular, the interplay between coordination and complexity takes the form of a linear combination of two effects — in the decomposition of equilibrium expected outcomes. First, a pure status-quo bias, which arises with uncorrelated outcomes across players. This effect pushes every player towards the status quo, and is magnified by the network of players. Second, a pure experimentation motive that arises only with correlated outcomes. This effect pulls players away from the status quo and it is introduced by the correlation component.

[7] This result complements the literature that studies informational asymmetries within organizations, see, e.g., Alonso et al. (2008); Rantakari (2008); Dessein and Santos (2006); the present model is biased towards favoring centralization because it abstracts away from division managers' private information.

Related Literature I borrow the model of complexity from the literature initiated with Callander (2011a), which studies a dynamic exploration-exploitation tradeoff using a Brownian motion. The main role of the covariance structure in the dynamic interaction is to discipline learning over time. Cetemen et al. (2023) study a similar complex environment in which discoveries are correlated over time and members of a team contribute resources for exploration. I contribute to the complexity literature by studying coordination motives and network games in a complex environment with the Brownian covariance structure. I also show that the status-quo bias survives the introduction of coordination motives and incomplete information about a heterogeneous status quo. Other work considers strategic interactions and Gaussian processes. In particular, the covariance structure has a direct role in the principal-agent settings of Bardhi and Bobkova (2023) and Bardhi (2023), in which a principal incentivizes agents to provide information about an underlying outcome function. These authors study covariance structures that are characterized by the "nearest-attribute" property, including the Brownian covariance.[8] My paper focuses on the Brownian covariance because it has two characteristics. First, the Brownian covariance preserves the strategic complementarities of the coordination game (Lemma 1); second, such covariance contains a leader-follower asymmetry that leads to conformity (Section 3). Other covariances are "asymmetric" but not supermodular (e.g., squared-exponential covariance), and vice versa (squared-polynomial). Garfagnini (2018) studies the rich welfare properties of complexity in a network game, under an environment that does not exhibit a covariance structure, because the decision-outcome mappings are drawn from player-specific

[8] Other strategic settings include: the dynamic models in Callander and Matouschek (2019), Callander and Hummel (2014), and Garfagnini and Strulovici (2016), which analyze intertemporal interactions; the communication models in Callander (2008), Callander et al. (2021), and Aybas and Callander (2023), in which a sender informs a receiver about the underlying outcome function; and the electoral competition in Callander (2011b). Gaussian processes are used in a similar way as in the complexity literature to study innovation, price rigidity, and in psychology (Jovanovic and Rob, 1990; Ilut and Valchev, 2022; Ilut et al., 2020; Anderson, 1960).

independent Brownian motions. In Section 6, I study the generalization of my model with imperfectly correlated outcome functions that includes independent Brownian motions as a special case.

The literature on coordination games with quadratic ex-post payoffs includes models of oligopolistic competition, peer effects, and network games (surveyed in Choné and Linnemer (2020) and Jackson and Zenou (2015).) I show that complexity introduces coordination problems under a common upper bound on the strength of coordination motives maintained in this paper, also for payoffs that admit a unique correlated equilibrium without complexity (Neyman, 1997). Moreover, complexity makes best responses nonlinear. The nonlinearity is due to the kinks in expected payoffs and it implies that equilibrium strategies are necessarily without constant slope in the heterogenous-status-quo game. Instead, the leading models of quadratic-payoff beauty contests with incomplete information admit a unique equilibrium, and the unique equilibrium features linear strategies in player's privately known types (Radner, 1962; Morris and Shin, 2002; Angeletos and Pavan, 2007). As an implication, the general game in this paper does not rely on results valid for incomplete-information beauty contests with linear best replies. Instead, status-quo heterogeneity is modeled as an interim Bayesian game (Van Zandt and Vives, 2007).[9]

Outline After introducing the model in Section 2, I study the conformity phenomenon in Section 3, with an application to oligopoly pricing. Section 4 analyzes an equilibrium selection and applications to network games and organizational economics. Section 5 contains the general model. Section 6 discusses further generalizations and directions for future research.

[9] The results of Van Zandt and Vives (2007) and Van Zandt (2010) cannot be applied off-the-shelf, so I leverage the additional structure of preferences to establish measurability of the greatest-best-reply mapping.

2 Model

2.1 Players and Payoffs

Every player $i \in N := \{1 \dots n\}$ has preferences over outcome profiles.

Payoffs An outcome profile is a list of individual outcomes $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbf{R}^n$. The payoff to player i from the outcome profile $\boldsymbol{x}$ is

$$\pi_i(\boldsymbol{x}) = -\left(x_i - (1-\alpha)\delta_i - \alpha \sum_{j \neq i} \gamma^{ij} x_j\right)^2,$$

in which $\alpha \in [0, 1)$ measures the strength of coordination motives, δ_i is the favorite outcome of player i, and $\gamma^{ij} \geq 0$ is the weight of the connection between player j and player i. Connections are symmetric, so $\gamma^{ij} = \gamma^{ji}$ for all players $i, j \in N$. Payoffs reflect a desire for coordination because $\alpha\gamma^{ij}$ is nonnegative. Similar payoffs are used to model organizations and peer effects (Jackson and Zenou, 2015).

Environment Every player i chooses a policy $p_i \in P = [\underline{p}, \overline{p}]$ simultaneously, for $\underline{p}, \overline{p} \in \mathbf{R}$ with $\underline{p} < \overline{p}$. The outcome corresponding to policy $p \in P$ is given by the *outcome function* $\chi \colon \mathbf{R} \to \mathbf{R}$, evaluated at p. The outcome function is the realized path of a Brownian motion with drift $\mu < 0$, variance parameter $\omega > 0$, and starting point $(p_0, \chi(p_0))$.[10] Figure 1 illustrates one such outcome function. Players know the status-quo policy $p_0 \in (\underline{p}, \overline{p})$, the corresponding status-quo outcome $\chi(p_0) \in \mathbf{R}$, and the parameters of the Brownian motion, μ and ω. The Brownian motion disciplines the beliefs of players about outcomes. Player i believes that $\chi(p)$ and $\chi(q)$ are jointly Gaussian random variables, for all pairs of policies $p, q \in P \setminus \{p_0\}$. This structure of uncertainty captures a complex environment

[10] See Definition 1.1 and 5.19 in Karatzas and Shreve (1998), Chapter 2, for the definition of a Brownian motion with these parameters.

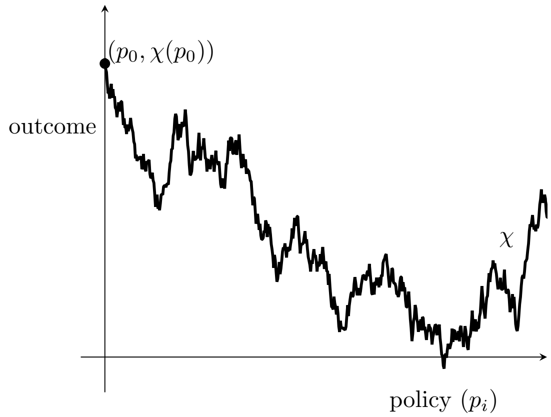


Figure 1: An outcome function, mapping individual policies to individual outcomes, given by the realized path of a Brownian motion.

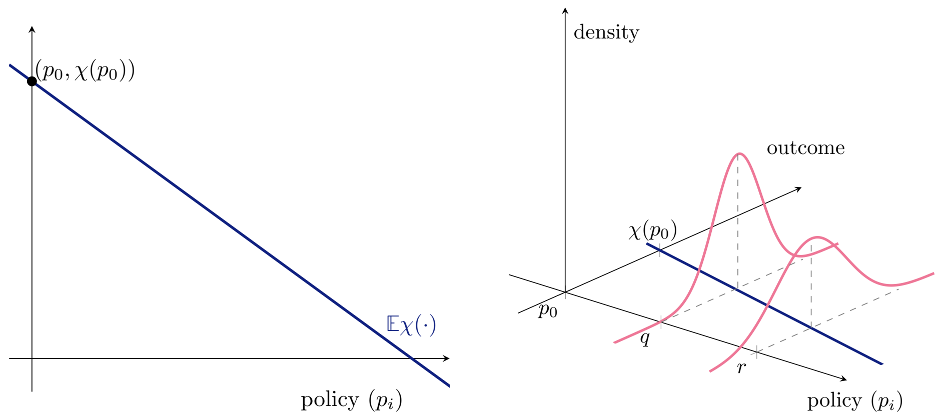
because a player is more certain about the outcome of a policy the closer the policy is to the status-quo policy (Figure 2). This way of modeling the complexity of an environment is first used by Callander (2011a). The measure of the complexity is $k := \frac{\omega}{2|\mu|}$.

Player i's payoff from the outcomes corresponding to the policy profile $\boldsymbol{p} \in P^n$ is given by $\pi_i(\chi(p_1), \ldots, \chi(p_n))$, which we denote by $\pi_i(\boldsymbol{\chi}(\boldsymbol{p}))$. Player i's expected payoff from the policy profile $\boldsymbol{p}$ given the status-quo outcome $\chi(p_0)$ is denoted by $\mathbb{E}\pi_i(\boldsymbol{\chi}(\boldsymbol{p}))$.

2.2 Strategies and Equilibrium

The main focus of the paper is the game $G(x_0)$ in which the strategy space of player i is the policy space P and player i's utility is her expected payoff given the status-quo outcome $x_0 \in \mathbf{R}$. In particular, I study the strategic-form game $\langle N, \{P, \mathbb{E}\pi_i(\boldsymbol{\chi}(\cdot))\}_{i \in N}\rangle$ given that $\chi(p_0) = x_0$. An equilibrium is a profile of policies $\boldsymbol{p}$ such that: for every player i, p_i maximizes expected payoff of player i given that her opponents choose policies according to $\boldsymbol{p}$.[11]

[11] In the equilibrium definition, "$\ldots, \chi(p_{i-1}), \chi(q_i), \chi(p_{i+1}), \ldots$" denotes the outcome profile corresponding to $(\chi(q_i), (\chi(p_j))_{j \in N \setminus \{i\}})$. Due to strict concavity of $p_i \mapsto \mathbb{E}\pi_i(\boldsymbol{\chi}(\boldsymbol{p}))$, player i's best response is unique (Appendix, Lemma 23); hence, focusing on pure strategies is without loss. The operator $\mathbb{E}$ denotes the



(a) Expected value of outcomes.

(b) Expected value and variance of outcomes.

Figure 2: Player i believes that outcomes are given by normal random variables. The expectations of these random variables are determined by the drift line of the Brownian motion (panel (a)). The closer the policy r is to the status-quo policy, the lower the variance of outcome $\chi(r)$, as inpanel (b).

Definition 1. *The policy profile $\boldsymbol{p} \in P^n$ is an equilibrium if:*

$$\mathbb{E}\pi_i(\boldsymbol{\chi}(\boldsymbol{p})) \geq \mathbb{E}\pi_i(\ldots, \chi(p_{i-1}), \chi(q_i), \chi(p_{i+1}), \ldots), \text{ for all } q_i \in P \text{ and } i \in N.$$

In the specific case of no complexity, which is the limit game when $\omega = 0$, the policy-outcome mapping is given by $\psi \colon p_i \mapsto \chi(p_0) + \mu(p_i - p_0)$, as argued in the next section, and the profile of outcomes corresponding to the policy profile $\boldsymbol{p}$ is $\boldsymbol{\psi}(\boldsymbol{p})$. An *equilibrium without complexity* is a a Nash equilibrium of the strategic-form game $\langle N, \{P, \pi_i(\boldsymbol{\psi}(\cdot))\}_{i \in N}\rangle$.

2.3 Discussion and Interpretation

This section interprets the connections between players as arising from a network studies certain implications of the Brownian-motion structure of uncertainty. The reader who is interested in results and applications may skip the present section.

expectation given $\chi(p_0) = x_0$.

Network of Players The matrix of connections is $\mathbf{\Gamma} := [\gamma^{ij} : i, j \in N]$, which is interpreted as the adjacency matrix of a network of players, letting $\gamma^{ii} = 0$ for all $i \in N$. I use $\boldsymbol{\delta}$ for the column vector of favorite outcomes, $\boldsymbol{I}$ for the identity matrix and $\boldsymbol{B}(\boldsymbol{M}) := (\boldsymbol{I} - \boldsymbol{M})^{-1}$ for the Leontief inverse of the n-by-n matrix $\boldsymbol{M}$, when $\boldsymbol{I} - \boldsymbol{M}$ is nonsingular. The Katz-Bonacich centrality of players in the network is useful in the study of equilibria.

Definition 2. *The centrality of player i is the ith entry of the column vector $\boldsymbol{\beta}$ given by:*

$$\boldsymbol{\beta} = (1 - \alpha)\boldsymbol{B}(\alpha\mathbf{\Gamma})\boldsymbol{\delta}.$$

The graph of the network $\langle N, \mathbf{\Gamma} \rangle$ offers an interpretation for centrality.[12] The ij entry of the Leontief inverse of $\alpha\mathbf{\Gamma}$ counts the walks of every length from node i to node j and discounts walks of length ℓ by α^ℓ, given that $\boldsymbol{B}(\alpha\mathbf{\Gamma}) = \sum_{\ell=0}^{\infty} \alpha^\ell \mathbf{\Gamma}^\ell$. The centrality of player i counts all "α-discounted" walks starting from i and weighs every walk to player j by $(1 - \alpha)\delta_j$.

Complexity The following formulas are useful to analyze the implications of the Brownian-motion structure of uncertainty, derived in the Appendix (Section 11.2). The parameters of the distribution of $(\chi(p), \chi(q))$, given the status-quo outcome $\chi(p_0)$ are denoted by $\mathbb{E}\chi(p)$,

[12] The matrix $\boldsymbol{I} - \alpha\mathbf{\Gamma}$ is positive definite due to Assumption 1 (next section) so centralities are well-defined and $\boldsymbol{B}(\alpha\mathbf{\Gamma}) = \sum_{\ell=0}^{\infty} \alpha^\ell \mathbf{\Gamma}^\ell$. Other definitions of Katz-Bonacich centrality do not adjust by $(1 - \alpha)$ or use the term "weighted" if $\delta_i \neq 1, i \in N$.

$\mathbb{V}\chi(p)$ and $\mathbb{C}(\chi(p), \chi(q))$. For all policies $p, q \in P$, we have

$$\mathbb{E}\chi(p) = \chi(p_0) + \mu(p - p_0),$$

$$\mathbb{V}\chi(p) = |p - p_0|\omega,$$

$$\mathbb{C}(\chi(p), \chi(q)) = \begin{cases} \min\{\mathbb{V}\chi(p), \mathbb{V}\chi(q)\} & \text{if } p, q \geq p_0 \text{ or } p_0 \leq p, q, \\ 0 & \text{if } p > p_0 > q \text{ or } q > p_0 > p. \end{cases}$$

Larger changes in individual expected outcomes are associated with high variance of the corresponding outcomes. The measure of complexity, k, is the additional variance implied by a marginal change of expected outcome, away from the status quo, scaled by 1/2. The covariance expression is due to the independent-increments property of the Brownian motion, and is determined by the closest policy to the status quo.

No-Coordination Benchmark When $\alpha = 0$ there isn't any strategic interaction. The game reduces to a collection of decision problems and corresponds to the static version of Callander (2011a). In that case, player i's optimal policy p_i^* trades off closeness of the expected outcome to δ_i with the variance induced by the distance of p_i^* from the status-quo policy p_0. Hence, player i does not optimally choose the policy p_i° such that $\mathbb{E}\chi(p_i^\circ) = \delta_i$; except possibly in the knife-edge case in which $\chi(p_0) = \delta_i$. Player i's optimal policy reflects a *status-quo bias*, because it's closer to the status quo than the policy p_i° is. To find the optimal policy, player i does not consider the correlation between outcomes of distinct policies because only her own outcome is payoff-relevant. In particular, player i's expected

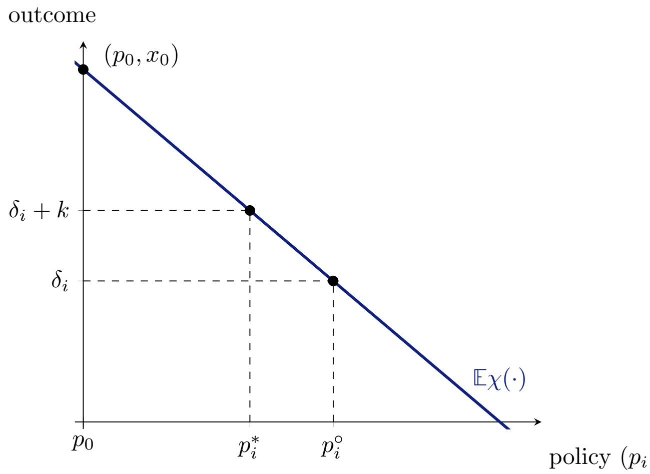


Figure 3: If $\alpha = 0$, player i has a unique optimal policy p_i^*. The policy p_i^* trades off closeness of the expected outcome to δ_i with the variance induced by the distance from the status-quo policy p_0. (For this figure: $\delta_i = 1, \mu = -1/2, \omega = 1/2, \alpha = 0, p_0 = 0 = \underline{p}, \chi(0) = 2.5$, and $\overline{p} \geq 3$.)

payoff is

$$\begin{aligned}\mathbb{E}\pi_i(\boldsymbol{\chi}(\boldsymbol{p})) &= -\mathbb{E}(\chi(p_i) - \delta_i)^2 \\ &= -\underbrace{(\mathbb{E}\chi(p_i) - \delta_i)^2}_{\text{quadratic}} - \underbrace{\mathbb{V}\chi(p_i)}_{\text{piecewise-linear}}.\end{aligned}$$

The first equality follows from the definition of π_i and the second from mean-variance decomposition. The variance term is a continuous and piecewise-linear function of player i's policy with a kink at the status-quo policy.[13] The presence of this kink leads to a second form of the status-quo bias: for an interval of status-quo outcomes, the optimal policy is the status-quo policy (Callander (2011a) and Corollary 1.)

Coordination and Complexity Players take into account the correlation between outcomes of different policies, because the outcomes of opponents are payoff-relevant. In

[13] I adopt the convention of calling a function linear when it is affine.

particular, the same distance in expected outcome from the status quo is "less expensive" — in terms of uncertainty — if it implies a high covariance with the outcomes of other players. The interplay of strategic interaction ($\alpha > 0$) and complexity of the environment ($k > 0$) gives rise to *endogenous* kinks in expected payoffs. Player i's expected payoff in the two-player case with $\delta_i = 0$ and $\gamma^{ij} = 1, j \neq i$, is as follows,

$$\begin{aligned}\mathbb{E}\pi_i(\boldsymbol{\chi}(\boldsymbol{p})) &= -\mathbb{E}(\chi(p_i) - \alpha\chi(p_j))^2 \\ &= -\underbrace{(\mathbb{E}\chi(p_i) - \alpha\mathbb{E}\chi(p_j))^2}_{\text{quadratic}} - \underbrace{\mathbb{V}\chi(p_i)}_{\text{piecewise-linear}} + \underbrace{2\alpha\mathbb{C}(\chi(p_i), \chi(p_j))}_{\text{piecewise-linear}} - \alpha^2\,\mathbb{V}\,\chi(p_j).\end{aligned}$$

If $k > 0$ and $\alpha > 0$, the mean-variance decomposition is "kinked" due to the presence of covariance terms. The location of kinks is endogenous: the expected payoff of player i has a kink at the policy of player j. A second type of kink is located at the status-quo policy and it leads to a status-quo bias (as in Callander (2011a) and similarly to Ilut et al. (2020).)

No-Complexity Benchmark The special case of the model without complexity is essentially equivalent to the linear-best-response game $S := \langle N, \{\mathbf{R}, \pi_i\}_{i \in N}\rangle$, studied in the literature on games played over networks (Ballester et al., 2006). There exists a unique Nash equilibrium in S, under a commonly used upper bound on the magnitude of coordination motives: the strategy profile $(\beta_1, \ldots, \beta_n)$ (Corollary 2). The result holds because the best-reply mapping of the game S is affine and contractive. Furthermore, Neyman (1997) establishes uniqueness of the correlated equilibrium. With complexity, best responses are not as smooth because of endogenous kinks, and they admit a multiplicity of equilibria under the same upper bound on coordination motives.

Notation The set of strategy profiles, P^n, and the set of profiles of opponents' strategies, P^{n-1}, are endowed with the product order. $\leq$ denotes all partial orders and $<$ the asymmetric part of $\leq$. For posets S and T, the function $g\colon S \times T \to \mathbf{R}$ exhibits strictly increasing differences if $t \mapsto g(s', t) - g(s, t)$ is increasing for all $s', s \in S$ with $s < s'$. $-i$ denotes $N \setminus \{i\}$. The column vector corresponding to the list of real numbers $(x_1, \ldots, x_\ell)$ is denoted by $\boldsymbol{x}$, and the column vector of ones by $\mathbf{1}$. The Hadamard (element-by-element) product of matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ is denoted by $\boldsymbol{A} \odot \boldsymbol{B}$. Proofs are in the Appendix.

2.4 Analysis

Coordination motives often lead to multiple equilibria. The following requirement ensures existence and uniqueness of an equilibrium absent complexity, and is common in the literature on games played over networks (Jackson and Zenou, 2015).

Assumption 1. *Let $\lambda(\mathbf{\Gamma})$ denote the largest eigenvalue of $\mathbf{\Gamma}$, then:*

$$\alpha\lambda(\mathbf{\Gamma}) < 1.$$

This requirement upper bounds the magnitude of overall coordination motives and isolates coordination problems induced by the introduction of complexity.[14]

The game $G(x_0)$ is of strategic complementarities.

Lemma 1 (Strategic Complementarities)**.** *For every player i, the expected payoff $\mathbb{E}\pi_i(\boldsymbol{\chi}(\boldsymbol{p}))$ exhibits strictly increasing differences in $(p_i, \boldsymbol{p}_{-i})$.*

Intuitively, the returns to choosing higher policies are increasing in the opponents' policies. The key observation in the proof leverages the covariance structure given by the Brownian motion discussed in Section 2.3. When opponents increase their policies, a higher

[14] The square matrix $\mathbf{\Gamma}$ is nonnegative, so $\lambda(\mathbf{\Gamma})$ is equal to the spectral radius of $\mathbf{\Gamma}$ (Theorem 8.3.1 in Horn and Johnson (2013)). To see why the assumption imposes an upper bound on the magnitude of coordination motives, note that $\lambda(\mathbf{\Gamma})$ is nonnegative and nondecreasing in γ^{ij}, so the upper bound on α is more stringent when players are more interconnected.

own policy implies (i) a closer expected outcome to the opponents' expected outcomes, (ii) a different volatility of own outcome, and (iii) a change in the covariance between the outcomes of players. The willingness to incur volatility stems from variance and covariance elements, and it varies with opponent's policies. By the results discussed in Section 2.3, the covariance between two outcomes is supermodular in the associated policies. The reason is that only the player with the least-volatile outcome is "controlling" the covariance directly, in every pair of players. Thus, if player i is a *follower* of player j — player i incurs less volatility than player j —, then she has an incentive to adjust her policy towards player j's policy. Moreover, the incentives of the *leader* player — player i — are not affected by player j's policy, except via the target.

Due to strategic complementarities, the set of equilibria is nonempty and admits an order structure.

Proposition 1 (Structure of the Equilibrium Set)**.** *There exist a greatest and least equilibrium.*

Strategy spaces are compact intervals and the expected payoff function of player i is strictly supermodular in $(p_i, \boldsymbol{p}_{-i})$ by Lemma 1. A known argument based on Tarski's fixed-point theorem establishes existence (Milgrom and Roberts, 1990; Vives, 1990).

The following result offers a characterization of equilibria in the form of a decomposition of equilibrium expected outcomes.

Proposition 2 (Equilibrium Decomposition)**.** *The profile of policies $\boldsymbol{p} \in (\underline{p}, \overline{p})^n$ is an equilibrium if, and only if:*

$$\mathbb{E}\boldsymbol{\chi}(\boldsymbol{p}) = \boldsymbol{\beta} + \boldsymbol{b}k + \alpha(\boldsymbol{I} - \alpha\boldsymbol{\Gamma})^{-1}(\boldsymbol{\Gamma} \odot \boldsymbol{A})\mathbf{1}k,$$

for a matrix $\boldsymbol{A} = [a_{ij} : i, j \in N]$ *and a vector* $\boldsymbol{b}$ *such that* $a_{ij}, b_i \in [-1, 1]$ *and*

$$b_i = \begin{cases} 1 & \text{if } p_i > p_0, \\ -1 & \text{if } p_i < p_0, \end{cases} \quad \text{and } a_{ij} = \begin{cases} 1 & \text{if } p_i > p_j, \\ -1 & \text{if } p_i < p_j. \end{cases}$$

The decomposition is stated for equilibria in which all players choose interior policies.[15] The decomposition provides a tool to verify whether a policy profile is an equilibrium via the induced expected outcomes. If the expected outcomes satisfy the decomposition for a matrix $\boldsymbol{A}$, which is constrained by the induced location of players in the policy interval, then the policy profile is an equilibrium.

The three summands that constitute equilibrium expected outcomes are labeled in order to study the interplay between coordination and complexity:

$$\mathbb{E}\boldsymbol{\chi}(p) = \underbrace{\boldsymbol{\beta}}_{\substack{\text{equilibrium outcomes} \\ \text{without complexity}}} + \underbrace{\boldsymbol{b}k}_{\substack{\text{status-quo} \\ \text{bias}}} + \underbrace{\alpha(\boldsymbol{I} - \alpha\boldsymbol{\Gamma})^{-1}(\boldsymbol{\Gamma} \odot \boldsymbol{A})\mathbf{1}k}_{\substack{\text{additional strategic-uncertainty} \\ \text{effect}}} .$$

If $k = 0$, the decomposition characterizes the unique equilibrium without complexity, which is determined by the centrality vector (see Lemma 2 below.) If $\alpha = 0$, the decomposition characterizes the unique equilibrium without coordination motives, which is determined by the vector of favorite outcomes and a status-quo-bias term (Corollary 1 below.) The interplay of coordination motives and complexity generates an additional term: the *endogenous* matrix $\boldsymbol{A}$, which keeps track of leader-follower asymmetries in every pair of players.

The decomposition leaves room for multiple equilibria and coordination problems: possibly for multiple policy profiles there exists a matrix $\boldsymbol{A}$ satisfying the decomposition. Figure 4 shows that a two-player game admits an interval of policies that can be sustained

[15] The complete characterization accounts for the boundary cases of players' equilibrium best responses, and it is stated in Appendix 13.

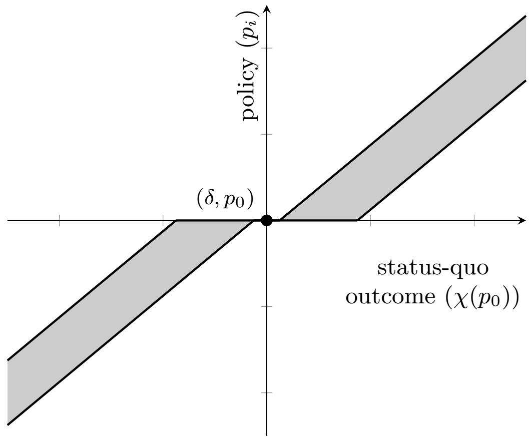


Figure 4: The grey area — including black lines and the point (δ, p_0) — illustrates the equilibrium set, represented by player i's policy, for every status-quo outcome. In particular, if $n = 2$ and $\delta_1 = \delta_2 =: \delta$, then every equilibrium $\boldsymbol{p}$ is symmetric, i.e., $p_1 = p_2$. (For this figure: $n = 2, \delta_1 = \delta_2 = 0, \mu = -1/2, \omega = 1/2, \alpha = 1/3$.)

in equilibrium.

In order to attribute the multiplicity to the interplay between coordination motives and complexity, the following results focus on the particular cases of no complexity and no coordination motives. In both benchmark cases there exists a unique equilibrium.

Corollary 1 (No Coordination)**.** *Let $\alpha = 0$. There exists a unique equilibrium of $G(x_0)$. Moreover, the profile of policies $\boldsymbol{p} \in (\underline{p}, \overline{p})^n$ is an equilibrium of $G(x_0)$ if, and only if:*

$$\mathbb{E}\boldsymbol{\chi}(\boldsymbol{p}) = \boldsymbol{\beta} + \boldsymbol{b}k,$$

for a vector $\boldsymbol{b}$ such that $b_i \in [-1, 1]$ and

$$b_i = \begin{cases} 1 & \text{if } p_i > p_0, \\ -1 & \text{if } p_i < p_0. \end{cases}$$

Corollary 2 (No Complexity)**.** *There exists a unique equilibrium of the game $G(x_0)$ without complexity. Moreover, the profile of policies $\boldsymbol{p} \in (\underline{p}, \overline{p})^n$ is an equilibrium of $G(x_0)$ without complexity if, and only if:*

$$\boldsymbol{\psi}(\boldsymbol{p}) = \boldsymbol{\beta}.$$

The following remark focuses on identical players. In contrast to the single-player case (Callander (2011a), Corollary 1), there exist multiple equilibria. Moreover, coordination problems increase in α: the equilibrium set grows in the inclusion sense as α increases (Appendix, Corollary 3).

Remark 1 (n Identical Players)*.* Let $\gamma^{ij} = \gamma$ and $\delta_i = 0$ for all players $i, j \in N$ with $i \neq j$. In every equilibrium $\boldsymbol{p}$, $p_i = p_j$ for all players $i, j \in N$ (proofs for this remark as in Appendix, Section 14.) Moreover, let $\underline{q}(a)$ and $\overline{q}(a)$ be, respectively, the policies in the least and greatest equilibrium when the degree of coordination motives is $\alpha = a$. If $\alpha_1 < \alpha_2$ and $\underline{q}(\alpha_1), \overline{q}(\alpha_1), \underline{q}(\alpha_2), \overline{q}(\alpha_2) \in (p_0, \overline{p})$, then $\underline{q}(\alpha_2) < \underline{q}(\alpha_1)$ and $\overline{q}(\alpha_2) > \overline{q}(\alpha_1)$. For intuition, suppose the policy space is $[p_0, \overline{p}]$. Then, the least equilibrium decreases in α and the greatest equilibrium increases in α for a complete network. As shown in the Appendix, the equilibrium set for a complete network with $\boldsymbol{\delta} = \boldsymbol{0}$ gets larger in set inclusion as α increases. As a result, the greatest equilibrium (i.e., the equilibrium with the least volatile outcomes) gets closer to the status quo, and the least equilibrium (i.e., the equilibrium with the most uncertain outcomes) involves more exploration, as α increases.

3 Conformity

This section uncovers a new conformity phenomenon. Conformity is due to the interplay between coordination and complexity that is present in the decomposition of equilibrium expected outcomes (Proposition 2), via the endogenous matrix $\boldsymbol{A}$ that keeps track of leader-follower relationships.

3.1 Example

To develop the intuition for how the conformity phenomenon arises, I start with a two-player example, i.e., $n = 2$. Furthermore, assume that the favorite outcomes are sufficiently distinct, $\delta_1 - \delta_2 > 2k\alpha/(1-\alpha)$. This ensures that the centralities are strictly ordered, $\beta_1 > \beta_2$, there exists a unique equilibrium $p^\star$, and player 1 is the follower ($\overline{p} > p_2^\star > p_1^\star > p_0$, for sufficiently large $\chi(p_0)$ and $\overline{p}$.)[16] Recall that each policy choice implies a unique expected outcome, hence, in what follows, I use the expected outcomes instead of the policies as the players' choice variable.

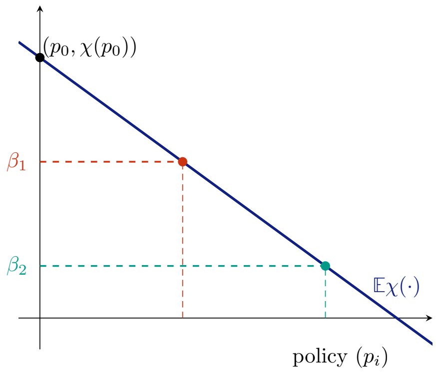
The best response of player i in the game without complexity is the expected outcome

$$(1-\alpha)\delta_i + \alpha \mathbb{E}\chi(p_j), \tag{1}$$

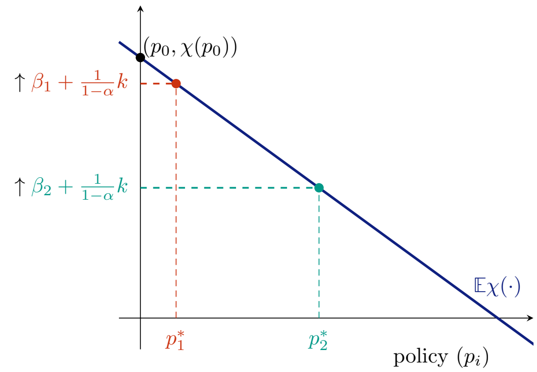
which is a function of the expected outcome of player j. There exists a unique pair of expected outcomes that induces an equilibrium: $\big(\beta_1, \beta_2\big)$ (Corollary 2 and Panel (a) in Figure 5.)[17] The distance between equilibrium expected outcomes is given by centralities: $\beta_1 - \beta_2$.

Complexity introduces two elements to the best-response analysis, a status-quo bias and a leader-follower asymmetry, reflecting variance and covariance features of the environment. First, consider a model with noisy and *independent* outcomes (which is illustrated in panel (b) of Figure 5, see also Section 6.) In this case, the best response of player i is the expected

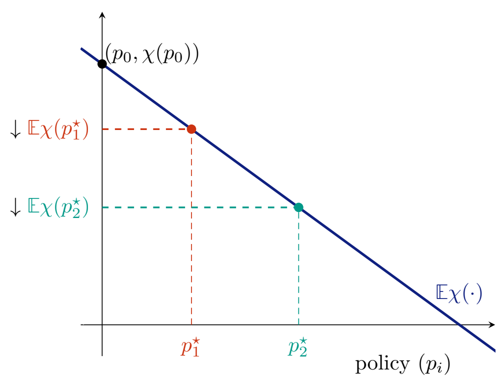
[16] The remaining cases are considered in Appendix, Section 15.
[17] To make the discussion simpler, best responses are restricted on $\big(\mathbb{E}\chi(\overline{p}), \chi(p_0)\big)$.



(a) The equilibrium in the game without complexity. The expected outcomes are given by the centrality of players, (β_1, β_2).



(b) Noisy and independent outcomes. The equilibrium expected outcomes are given by centrality of players and the adjusted status-quo bias, $(\beta_1 + mk, \beta_2 + mk)$. The arrows indicate the equilibrium status-quo bias: expected outcomes are higher than in the game without complexity (panel (a)).



(c) Equilibrium in $G(x_0)$. The expected outcomes are given by the decomposition in Proposition 2, which includes the leader-follower asymmetry, $(\beta_1 + k - k\frac{\alpha}{1+\alpha}, \beta_2 + k + k\frac{\alpha}{1+\alpha})$. The arrows indicate the extra exploration induced by the covariance structure: expected outcomes are lower than in the game without correlation (panel (b)).

Figure 5: Panel (a) illustrates the equilibrium in the game without complexity. Panel (b) illustrates the equilibrium when outcomes are noisy but independent across policies, given $\mathbb{V}\chi(p) = 0.5p$ and $\mathbb{C}(\chi(p), \chi(q)) = 0$, for $p, q > p_0$. Panel (c) illustrates the equilibrium in the game $G(x_0)$ when $\omega = 1/2$. (For the figures: $\delta_1 = 2, \delta_2 = 0, \mu = -1/2, \omega = 1/2, \alpha = 1/3, p_0 = 0 = \underline{p}, \chi(0) = 2.5, \overline{p} > 2.75$.)

outcome

$$(1-\alpha)\delta_i + \alpha\mathbb{E}\chi(p_j) + k. \tag{2}$$

The best response shifts upwards, with respect to the case of no complexity, i.e., expression (1), by the same amount as in the single-player game (Callander, 2011a). An incentive to stay close to the status quo emerges and there is not any leader-follower asymmetry. There exists a unique pair of equilibrium expected outcomes: $(\beta_1 + mk, \beta_2 + mk)$, in which $m = 1/(1-\alpha)$ is the social multiplier, studied in network games (Jackson and Zenou, 2015). The multiplier magnifies the status-quo bias identified by Callander: when player i moves towards the status quo, player j has an incentive to do the same (due to the presence of $\alpha\mathbb{E}\chi(p_i)$ in the best response of player j.) Player 1 is a "follower" only in the sense that she incurs less uncertainty than player 2. In equilibrium, the distance between expected outcomes is pinned down by centralities, $\beta_1 - \beta_2$, because best responses shift by the same amount. Hence, an increase in uncertainty alone does not lead to further conformity.

Consider the complex environment in the game $G(x_0)$, i.e., with noisy and correlated outcomes according to the Brownian motion. The best response of player 1 is:

$$(1-\alpha)\delta_1 + \alpha\mathbb{E}\chi(p_2) + k - 2\alpha k, \tag{3}$$

while the best response of player 2 is the same as with uncorrelated outcomes, i.e., expression (2). The introduction of correlation makes player 1 willing to explore more. Hence, the follower has an incentive to catch up with the leader, which clashes with the push towards the status quo. This exploration motive is reflected by a downward shift of the best response of player 1 — relative to the uncorrelated-outcomes case of expression (2). There is a unique equilibrium $\boldsymbol{p}^\star$ for the given leader-follower relationship, described by the pair of expected outcomes $\left(\beta_1 + k - k\frac{\alpha}{1+\alpha}, \beta_2 + k + k\frac{\alpha}{1+\alpha}\right)$. In general, the equilibrium exhibits three features, studied in the rest of this section.

(1) Conformity. Additional conformity arises due to complexity. In particular,

$$\mathbb{E}\chi(p_1^\star) - \mathbb{E}\chi(p_2^\star) - (\beta_1 - \beta_2) < 0.$$

(2) The new conformity increases (locally) in complexity. The difference in expected outcomes, netting out $\beta_1 - \beta_2$, is:

$$\mathbb{E}\chi(p_1^\star) - \mathbb{E}\chi(p_2^\star) - (\beta_1 - \beta_2) = -2\frac{\alpha}{1+\alpha}k.$$

Strict monotonicity is local. If complexity exceeds the cutoff implied by our requirement — i.e., $\delta_1 - \delta_2 > 2k\alpha/(1-\alpha)$) —, then players have the same equilibrium expected outcome.

(3) The leader "pulls" the follower away from the status quo. With the introduction of complexity, the follower is facing two new incentives. First, she is pushed towards the status quo, via the status-quo bias that is present also without correlation in outcomes. Second, she is pulled away from the status quo, via the conformity that is introduced by the covariance structure. The interplay between the covariance of the environment and coordination motives leads to an extra exploration incentive, when "controlling" for the variance effect that is isolated in the uncorrelated-outcomes case (Figure 5).

In general, conformity is "scaled" by the correlation between outcomes. In particular, suppose two Brownian motions, with same initial points, drift and variance, that are correlated with parameter ρ (see Section 6.) While the best response of the leader is identical to the no-correlation case (expression 2), the best response of the follower is

$$(1-\alpha)\delta_1 + \alpha\mathbb{E}\chi(p_2) + k - 2\alpha\rho k,$$

in which the follower's exploration motive is scaled by ρ. Hence, the higher the correlation, the stronger the conformity effect. In particular, $\mathbb{E}\chi(\tilde{p}_1) - \mathbb{E}\chi(\tilde{p}_2) - (\beta_1 - \beta_2) = \rho(-2\frac{\alpha}{1+\alpha}k)$, in an equilibrium $\tilde{p}$. The presence of a nontrivial covariance structure induces players to explore more without sacrificing coordination.

3.2 Pairwise Conformity

Under a complete network, complexity unambiguously leads to a strong form of conformity, that holds for all pairs of players and equilibria of $G(x_0)$.

Lemma 2. *Let $\gamma^{ij} = \gamma$ for all players $i, j \in N$ with $i \neq j$, and $\boldsymbol{p} \in (\underline{p}, \overline{p})^n$ be an equilibrium. If $p_i < p_j$, then:*

$$\mathbb{E}\chi(p_i) - \mathbb{E}\chi(p_j) < \beta_i - \beta_j.$$

The above result compares the expected outcomes of every pair of players in equilibrium to the no-complexity case, across all equilibria. The introduction of complexity makes players choose closer policies.

An equilibrium $\boldsymbol{p}$ is *ordered* if it satisfies $p_0 < p_1 < p_2 < \cdots < p_n < \overline{p}$.[18] For ordered equilibria, conformity (locally) increases with the complexity of the environment.

Lemma 3. *Let $\gamma^{ij} = \gamma$ for all players $i, j \in N$ with $i \neq j$, and $\boldsymbol{p} \in (p_0, \overline{p})^n$ be an ordered equilibrium. Then, for all $i \in \{1, \ldots, n-1\}$,*

$$\mathbb{E}\chi(p_i) - \mathbb{E}\chi(p_{i+1}) = \beta_i - \beta_{i+1} - 2\frac{\alpha\gamma}{1+\alpha\gamma}k.$$

[18] Equilibrium actions are naturally ordered by the primitives of certain economic environments. In oligopolistic competition, for instance, demand intercepts and marginal costs order equilibrium prices (Section 3.5).

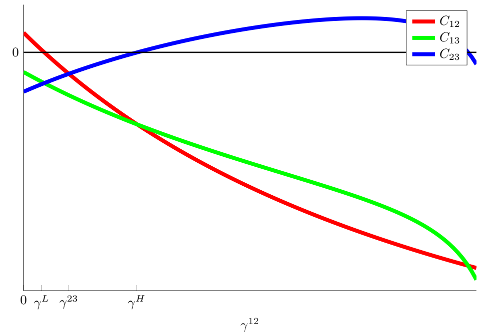


Figure 6: The additional conformity is defined by $C_{ij} = \mathbb{E}\chi(p_i^\star) - \mathbb{E}\chi(p_j^\star) - \beta_i + \beta_j$, for players $i, j \in N$ in an equilibrium $p^\star$. Suppose that there exists a "middle" player, player 2. In particular, player 2 is the follower to player 3 and the leader to player 1. When the connection between player 1 and 2 is sufficiently weak ($\gamma^{12} < \gamma^L$), the middle player values the pull of the "global leader" more than the push towards the status quo of the global follower. As a result, counterformity arises between player 1 and 2. A similar phenomenon occurs between player 2 and 3 when γ^{12} is sufficiently large. (For this figure: $n = 3, \gamma^{23} = 0.2, \gamma^{13} = 0, \delta_1 = 1, \delta_2 = 0, \delta_3 = -1, k = 2, \alpha = 0.45, p_0 = 0 = \underline{p}$ and sufficiently large $\chi(0), \overline{p}$.)

The comparative statics holds locally. If the conformity motive is sufficiently strong, the difference in favorite outcomes does not sustain the leader-follower asymmetry. This is the case, for instance, if complexity exceeds the cutoffs implied Lemma 3. In this case, extreme conformity arises: the relevant players choose the same policy. A second instance of complete conformity is when players are identical (Remark 1).

3.3 Counterformity

Conformity interacts with the network of players. A player may exert substantial network influence on a follower player. If this influence is strong enough, it drives the follower away from a third player. "Counter-Formity" emerges when equilibrium expected outcomes in a pair of players are more distant than in a non-complex environment. This situation is illustrated in Figure 6, with a three-player example.

In general, conformity has a delicate interaction with the network of players. Consider an ordered equilibrium. Player n is a leader for every other player, while player 1 is a follower for every opponent. The first term of the infinite sum induced by $\alpha(\boldsymbol{I}-\alpha\boldsymbol{\Gamma})^{-1}(\boldsymbol{\Gamma}\odot\boldsymbol{A})\mathbf{1}k$, i.e., $\alpha k(\boldsymbol{\Gamma}\odot\boldsymbol{A})\mathbf{1}k$, represents a "first-order" conformity effect. While player n's opponents are choosing policies closer to the status quo than her, player 1's opponents are incurring more uncertainty than him.[19] Hence, player n has an extra incentive than player 1 to choose a policy close to the status quo. This incentive is an *endogenous* status-quo bias for player n relative to player 1 because it is determined in equilibrium. I tentatively define the "extra status-quo bias" for player n that takes into account the connections among players by averaging the entries in the nth row of $\boldsymbol{A}$, each weighted according to the connection of player n with the corresponding opponent; this average yields

$$\sum_j a_{nj}\gamma^{nj} > 0.$$

The same intuition leads to an "extra exploration motive" for player 1,

$$\sum_j a_{1j}\gamma^{1j} < 0.$$

The vector $\alpha k(\boldsymbol{\Gamma}\odot\boldsymbol{A})\mathbf{1}$ collects these first-order incentives of all players, each scaled by αk. The complete intuition takes into account how the extra status-quo biases and exploration motives feed into the network of players. The resulting *equilibrium* strategic-uncertainty effect is

$$(\boldsymbol{\Gamma}\odot\boldsymbol{A})\mathbf{1}\alpha k + \alpha\boldsymbol{\Gamma}(\boldsymbol{\Gamma}\odot\boldsymbol{A})\mathbf{1}\alpha k + (\alpha\boldsymbol{\Gamma})^2(\boldsymbol{\Gamma}\odot\boldsymbol{A})\mathbf{1}\alpha k + \dots,$$

which yields the vector $\boldsymbol{B}(\alpha\boldsymbol{\Gamma})(\alpha\boldsymbol{\Gamma}\odot\boldsymbol{A})\mathbf{1}k$, present in the decomposition of equilibrium

[19] This configuration of players implies that $a_{nj}=1$, $j\neq n$, and $a_{1k}=-1$, $k\neq 1$ (Proposition 2).

expected outcomes. Thus, player i's strategic-uncertainty effect counts all the discounted walks starting from i and weighs each walk to player j by the endogenous status-quo bias $\alpha k \sum_\ell a_{j\ell}\gamma^{j\ell}$.

As the next result suggests, heterogeneity in network connections is related to counterformity. We say that $\mathbf{\Gamma}$ is a *line* if: (i) $\gamma^{ii+1} = 1$ for all $i \in \{1, \ldots, n-1\}$, (ii) $\gamma^{ii-1} = 1$ for all $i \in \{2, \ldots, n\}$, and (iii) $\gamma^{ij} = 0$ otherwise. In a line network, conformity emerges pairwise, and it increases in complexity.

Lemma 4. *Let $\mathbf{\Gamma}$ be a line, $\alpha \leq 1/2$, and $\boldsymbol{p} \in (p_0, \overline{p})^n$ be an ordered equilibrium. Then, for all $i \in \{1, \ldots, n-1\}$,*

$$\mathbb{E}\chi(p_i) - \mathbb{E}\chi(p_{i+1}) = \beta_i - \beta_{i+1} - c_i k,$$

for some $c_i > 0$.

In Figure 6, $\mathbf{\Gamma}$ is a line only when $\gamma^{12} = \gamma^{23}$, in which case there is "only" conformity.

Remark 2 (Interventions). The design of network interventions studies changes in favorite outcomes that induce certain equilibrium behavior of players (Galeotti et al., 2020). Suppose an ordered equilibrium in a complete network or in a line. Moderate changes in favorite outcomes do not affect conformity. Hence, if a policymaker adopts a "small" intervention, the presence of complexity does not lead to unintended consequences; the results about optimal interventions under a "small budget" are robust to a low level of complexity. Substantial interventions, on the other hand, change the leader-follower relationships, and, so, the pattern of conformity.

3.4 Discussion

The conformity effect is not specific to the abstract coordination game $G(x_0)$. Incremental uncertainty and coordination motives are present in many economic environments.

- In oligopolistic competition, firms that rely on algorithmic pricing face uncertainty over their own listed prices. This uncertainty arises because an algorithm conditions prices on data not available when the algorithm is selected (Brown and MacKay, 2023). Price competition exhibits strategic complementarities in many models of oligopoly. In Section 3.5, we model firms that choose pricing policies knowing the resulting listed prices up to some noise — which may reflect market uncertainty or the recent introduction of algorithmic pricing. As the environment becomes more complex, firms choose more similar pricing policies. This result suggests that, without considering the complexity of the relevant industry, estimates of firm parameters from price data show reduced heterogeneity across firms.

- In social psychology, it is documented that conformity increases in the difficulty of the task and in the "cohesion" of the group (Krech et al., 1962). By the comparative statics results, conformity increases in complexity, the strength of coordination motives, and the number of players.[20]

- Peer recognition is important in scientific research (Partha and David, 1994). In general, coordination motives are present in certain interactions in which exploration of unknown alternatives is important. If society values exploration, conformity may limit learning about the underlying outcome function.[21] The presence of conformity

[20] The main comparative statics is in Lemma 3, a simple corollary is that "overall" conformity increases in the number of players: $\mathbb{E}\chi(p_1) - \mathbb{E}\chi(p_n) = \beta_i - \beta_{i+1} - 2(n-1)\frac{\alpha}{1+\alpha}k$. Similar results follow from Lemma 4.

[21] In Brownian-motion models, however, learning occurs in two ways: radical and incremental experimentation, given, respectively, by the extreme ($\max\{p_1, \ldots, p_n\}$ and $\min\{p_1, \ldots, p_n\}$) and non-extreme policies that are chosen (similarly to Garfagnini and Strulovici (2016).) If conformity increases, less is known about radical experimentation, but, possibly, more about incremental experimentation.

is important for the design of incentives for research and innovation.

- The management of every subsidiary owned by the same holding company coordinates with other subsidiaries and adapts to idiosyncratic circumstances. Communication frictions are a source of noise in the implementation of production processes. This noise may be particularly relevant for the adoption of innovative technologies. In Section 4.3, I show that an organization with decentralized decision-making — e.g., a holding company with only oversight capacities — can implement profit maximization in sufficiently complex environments. This result suggests that centralized decision-making may be less desirable in the presence of coordination problems. The analysis also points to a responsibility of the holding company's management: leveraging the coordination problems induced by the environment and making maximization of the holding's profits a focal point for the management of subsidiaries.

- In primary elections, career concerns determine the choice of platforms of politicians, because the winner has authority over the campaign in a future general election. Often, the consequences of extreme policies are unknown. In separate work, I study elections under complexity, in which each competitor represents a combination of (i) a constituency of voters and (ii) a career-concerned politician. I find that complexity lessens the polarization of platforms. This result suggests that better information of political parties about the policy-outcome mapping — from, e.g., lobbies and interest groups — may increase political polarization.

In order to study different games in which a similar equilibrium analysis holds, I define an auxiliary utility function of player i over outcomes, $v_i(\boldsymbol{x}) = 2(1-\alpha)\delta_i x_i - x_i^2 + 2\alpha \sum_{j \in N} \gamma^{ij} x_i x_j$. The next result studies the strategic-form game $F(x_0)$, in which players and strategy spaces are the same as in $G(x_0)$ and utility functions are $\mathbb{E}v_1(\boldsymbol{\chi}(\cdot)), \ldots, \mathbb{E}v_n(\boldsymbol{\chi}(\cdot))$.

Lemma 5 (Equivalence)**.** *For every player $i \in N$, there exists a function $g_i \colon P^{n-1} \times \mathbf{R} \to \mathbf{R}$ such that:*

$$\mathbb{E}\pi_i(\boldsymbol{\chi}(\boldsymbol{p})) = \mathbb{E}v_i(\boldsymbol{\chi}(\boldsymbol{p})) + g_i(\boldsymbol{p}_{-i}, x_0) \textit{ for all } \boldsymbol{p} \in P^n, x_0 \in \mathbf{R}.$$

The game $F(x_0)$ has the same set of equilibria as $G(x_0)$ because the games are Von-Neumann-Morgenstern equivalent (Morris and Ui, 2004). The applications in this paper leverage the above result to apply the analysis in the preceding section.

3.5 Application 1: Oligopoly Pricing

I study the implications of conformity for oligopoly pricing. I model competition among firms who set pricing policies, or algorithms, knowing the resulting price only in expectation. Conformity takes the form of closer pricing policies across firms in more complex environments.

Model A representative consumer has quasi-linear preferences over bundles of $n+1$ goods, which are represented by the quadratic utility function U such that

$$U(q_1, \ldots, q_n, m) = \sum_i a_i q_i - \frac{1}{2} b \sum_i q_i^2 - \frac{1}{2} c \sum_{i,j : j \neq i} q_i q_j + m,$$

in which m denotes the numéraire good, and $b > c \geq 0$. The last condition is to study substitute goods and a well-defined demand system leading to strategic complementarities in the resulting price-setting firm interaction. The coefficients of the Marshallian demand of the representative consumer are normalized so that the own-price coefficient is -1 in the demand for every good $i \in \{1, \dots, n\}$.[22]

Each price is set through the decision of one of n firms. Firm i has constant marginal costs — parametrized by c_i — and no fixed costs. We define a strategic-complementarity coefficient $\zeta := \frac{1-(b-c)}{b-c} \in \left[0, \frac{2}{n-1}\right)$ and the net demand intercept for product i, $\widehat{a}_i := a_i - c_i - \zeta \sum_{j \neq i}(a_j - c_j)$.[23] Given a profile of prices net of marginal costs, $\boldsymbol{x}$, the profits of firm i are

$$\pi_i^B(\boldsymbol{x}) = \left(\widehat{a}_i - x_i + \zeta \sum_{j \in -i} x_j\right) x_i.$$

Each firm chooses a pricing policy p_i. The function χ specifies the markup that is eventually realized from every pricing policy.[24] Firm i's profits from the policy profile $\boldsymbol{p}$ are given by $\pi_i^B(\boldsymbol{\chi}(\boldsymbol{p}))$. Firms choose pricing policies simultaneously in the *pricing game*, $\left\langle N, \{\mathbb{E}\pi_i^B(\boldsymbol{\chi}(\cdot)), [p_0, \overline{p}]\}_{i \in N}\right\rangle$. There exists a unique vector of equilibrium markups in the pricing game without complexity, which we denote by $\boldsymbol{\beta}^B$ (Lemma 2 and 5.)

Results Complexity leads to less dispersed expected prices across products, by leveraging a natural ordering property of equilibrium policies. If the net demand intercepts are sufficiently heterogeneous, then every equilibrium is ordered; which may arise in practice if firms are sufficiently different in their production efficiency.

Proposition 3. *Let $\boldsymbol{p} \in (p_0, \overline{p})^n$ be an equilibrium of the pricing game. If $p_1 < p_2 \leq \cdots \leq$*

[22] The Marshallian demand is well-defined because the Hessian of the quadratic form $(q_1, \dots, q_n) \mapsto U(q_1, \dots, q_n, m)$ is negative definite whenever $b > c \geq 0$ Amir et al. (2017). The matrix of demand coefficients arising from the representative consumer $[D_{ij} : i, j \in N]$ is normalized via $D_{ii} = -1$; see Appendix 14.

[23] The inequality $\zeta < \frac{2}{n-1}$ is the content of Assumption 1 in the pricing game under the normalization on demand coefficients. The inequality $\zeta \geq 0$ is assumed following the normalization of demand coefficients. These two constraints are not needed without the normalization, and the normalization is used only to ease the connection between the game $F(x_0)$ and the pricing game; for a formal discussion, see Appendix 14.

[24] The same structure can be applied to a model in which the outcome of policy p_i is a price, and not a markup, and the findings are qualitatively unchanged. The present section works with markups as outcomes to ease the connection between the pricing game and the game F_0.

$p_{n-1} < p_n$, then:

$$\mathbb{E}\chi(p_1) - \mathbb{E}\chi(p_n) - (\beta_1^B - \beta_n^B) = -(n-1)\frac{\zeta}{2+\zeta}k.$$

Moreover, if $\widehat{a}_i - \widehat{a}_{i+1} > 2\zeta k$ for all $i \in \{1, \ldots, n-1\}$, then: every equilibrium $\boldsymbol{p} \in (p_0, \overline{p})^n$ is ordered such that $p_1 < \cdots < p_n$, and there exists at most one interior equilibrium.

The impact of complexity on conformity of markup policies is increasing in the level of complexity and in the strategic-complementarity coefficient in ordered equilibria. The more substitutable products, the greater the impact of complexity on price conformity, measured by $\mathbb{E}\chi(p_i) - \mathbb{E}\chi(p_j) - (\beta_i^B - \beta_j^B)$. The reason is that the strength of strategic complementarities (ζ) increases in product substitutability c.

Discussion The pricing game models quasi-Bertrand competition with differentiated products in which negative quantities and prices are theoretically available, and the consumer's income is sufficiently large.[25] A reason for the presence of correlated noise in the mapping from pricing policies to listed prices — or, equivalently, to markups — is that firms buy pricing services from the same provider.

4 Equilibrium Selection

4.1 Potential Maximizer

I propose an equilibrium selection based on the observation that the game $G(x_0)$ is a potential game (Monderer and Shapley, 1996).

[25] The probability of negative prices is made arbitrarily small, for sufficiently large status-quo price. The terminology is inspired by Monderer and Shapley (1996), who refer to quantity competition as quasi-Cournot competition when negative quantities are possible.

A game is a common-interest game if all players have the same payoff function. A game is a potential game if it is "best-response equivalent" to an auxiliary game that is a common-interest game (definitions are in the Appendix.) For a potential game, the common payoff function in the auxiliary game is called the potential function, which maps strategy profiles into real numbers.

The *potential* is the function $V\colon P^n \to \mathbf{R}$ given by

$$V(\boldsymbol{p}) = \mathbb{E}\Big[2(1-\alpha)\boldsymbol{\delta}^\top \boldsymbol{\chi}(\boldsymbol{p}) - \boldsymbol{\chi}(\boldsymbol{p})^\top(\boldsymbol{I} - \alpha\boldsymbol{\Gamma})\boldsymbol{\chi}(\boldsymbol{p})|\chi(p_0) = x_0\Big].$$

I study the maximizers of the potential V. A *potential maximizer* is a policy profile $\boldsymbol{p}^\star$ that maximizes the potential, so

$$\boldsymbol{p}^\star \in \arg\max_{\boldsymbol{p}\in P^n} V(\boldsymbol{p}).$$

Proposition 4. *The following properties of the potential maximizer hold.*

(1) If the policy profile $\boldsymbol{p} \in P^n$ is a potential maximizer, then $\boldsymbol{p}$ is an equilibrium.

(2) If $P = [p_0, \overline{p}]$, there exists a unique potential maximizer.

For part (1), I establish von-Neumann-Morgenstern equivalence (Morris and Ui, 2004) between the two strategic-form games played in the outcome space with utility functions $\{\pi_1, \pi_2, \dots, \pi_n\}$ and $\{v, v, \dots, v\}$, in which $v(\boldsymbol{x}) = 2(1-\alpha)\boldsymbol{\delta}^\top\boldsymbol{x} - \boldsymbol{x}^\top(\boldsymbol{I} - \alpha\boldsymbol{\Gamma})\boldsymbol{x}$. This result extends to the induced games played in the policy space, and so it establishes that $G(x_0)$ is a potential game, a fortiori.[26] Since a strategy profile that maximizes the potential is

[26] In particular, for every player $i \in N$ there exists a function $g_i\colon P^{n-1} \times \mathbf{R} \to \mathbf{R}$ such that: $\mathbb{E}\pi_i(\boldsymbol{\chi}(\boldsymbol{p})) = \mathbb{E}v(\boldsymbol{\chi}(\boldsymbol{p})) + g_i(\boldsymbol{p}_{-i}, x_0)$ for all $\boldsymbol{p} \in P^n$ and $x_0 \in \mathbf{R}$. The last step of the proof verifies that von-Neumann-Morgenstern equivalence is consistent with the definition of a potential game.

necessarily an equilibrium of the potential game (Radner, 1962), part (1) follows. Moreover, the potential for $G(x_0)$ is uniquely defined up to a constant term.[27] These two observations imply that the potential maximizer provides a valid equilibrium selection for $G(x_0)$.

The potential V is not differentiable whenever $p_i = p_j$ for a pair of players, due to the covariance structure (see Section 2.3 and 11.3.) However, strict concavity on $[p_0, \overline{p}]$ leads to existence and uniqueness of the potential maximizer. I study the (well-defined) superdifferential of V to characterize the potential maximizer.

Proposition 5 (Potential Maximizer)**.** *Let $P = [p_0, \overline{p}]$. The policy profile $\boldsymbol{p} \in (p_0, \overline{p})^n$ is a potential maximizer if, and only if:*

$$\mathbb{E}\boldsymbol{\chi}(\boldsymbol{p}) = \boldsymbol{\beta} + \mathbf{1}k + \alpha(\boldsymbol{I} - \alpha\boldsymbol{\Gamma})^{-1}(\boldsymbol{\Gamma} \odot \boldsymbol{A})\mathbf{1}k,$$

for a skew-symmetric matrix $\boldsymbol{A} = [a_{ij} : i, j \in N]$ such that $a_{ij} \in [-1, 1]$ and $a_{ij} = 1$, if $p_i > p_j$.

The decomposition for the potential maximizer has a similar structure as the equilibrium decomposition. The main difference is the skew-symmetry property of the endogenous matrix $\boldsymbol{A}$ that implies the uniqueness result.

The uniqueness and characterization of the potential maximizer allow to make predictions about strategic interactions in complex environments using the potential maximizer as equilibrium selection. With quadratic ex-post payoffs, the selection is useful precisely due to complexity. If $k > 0$, the strictly concave potential is not smooth and there are multiple equilibria. If $k = 0$, the strictly concave potential is differentiable everywhere and

[27] In Appendix 13, I establish that V is an exact potential; Monderer and Shapley (1996) introduce the notion of exact potential, a particular case of the weighted potential; Morris and Ui (2004) study the equivalence between weighted potential games and potential games in connection with von-Neumann-Morgenstern equivalence.

there exists a unique equilibrium: the potential maximizer.[28] It follows that studying the potential maximizer is useful to compare $G(x_0)$ with the case in which $k = 0$.

I study the welfare in the game $F(x_0)$ using the tools developed for the maximization of the potential of $G(x_0)$. The *utilitarian welfare in* $F(x_0)$ is given by the function $W : \boldsymbol{p} \mapsto \sum_i \mathbb{E} v_i(\boldsymbol{\chi}(\boldsymbol{p}))$. A *welfare maximizer* is a policy profile $\boldsymbol{p}^W$ that maximizes utilitarian welfare in $F(x_0)$, so

$$\boldsymbol{p}^W \in \arg\max_{\boldsymbol{p} \in P^n} W(p).$$

The following result characterizes the welfare maximizer.

Proposition 6 (Welfare Maximizer)**.** *Let* $P = [p_0, \overline{p}]$ *and* $2\alpha\lambda(\boldsymbol{\Gamma}) < 1$. *There exists a unique welfare maximizer. Moreover, the policy profile* $\boldsymbol{p} \in (p_0, \overline{p})^n$ *is a welfare maximizer if, and only if:*

$$\mathbb{E}\boldsymbol{\chi}(\boldsymbol{p}) = (1-\alpha)\boldsymbol{B}(2\alpha\boldsymbol{\Gamma})\boldsymbol{\delta} + \mathbf{1}k + 2\alpha\boldsymbol{B}(2\alpha\boldsymbol{\Gamma})(\boldsymbol{\Gamma} \odot \boldsymbol{A})\mathbf{1}k,$$

for a matrix $\boldsymbol{A} = [a_{ij} : i, j \in N]$ *such that* $a_{ij} \in [-1, 1]$, $a_{ij} = -a_{ji}$, *and* $a_{ij} = 1$ *if* $p_i > p_j$.

In the proof, I leverage the known observation that utilitarian welfare maximization in $F(x_0)$ is equivalent to maximization of the potential of an auxiliary game in which the magnitude of cost externalities is doubled. The reason is that the game $F(x_0)$ is a coordination game, by the results established for $G(x_0)$ (Lemma 1 and 5), in which players do not internalize all the externality of their policy. This intuition resonates with the results for games played over networks (Jackson and Zenou, 2015), and allows to use the characterization of the potential-maximizer equilibrium in Proposition 5.

[28] To establish this observation, it suffices that: if $\boldsymbol{p}^\circ \in P^n$ satisfies $\boldsymbol{\psi}(\boldsymbol{p}^\circ) = \boldsymbol{\beta}$, then it maximizes $\boldsymbol{p} \mapsto v(\boldsymbol{\psi}(\boldsymbol{p}))$ on P^n. This claim is established by showing that $\boldsymbol{p} \mapsto v(\boldsymbol{\psi}(\boldsymbol{p}))$ is a potential for the game $G(x_0)$ without complexity (Appendix).

4.2 Application 2: Network of Players

This section presents a characterization of the potential-maximizer equilibrium for a class of network games. For sufficiently high complexity, extreme conformity prevails: all players choose the same policy. The equilibrium behavior is observationally equivalent to the optimal choice of a single player with a favorite outcome that is characterized under the potential-maximizer equilibrium selection.

I study the game in which every player is part of only one of two groups, A and B, and players in the same group have the same favorite outcomes and connections. γ denotes the connection between a player in group A and a player in B, by $\delta_g, \gamma^{gg}, \beta_g$ and n_g, respectively, the favorite outcome, the weight of an intra-group connection, the centrality of a player and the number of players for group $g \in \{A, B\}$.

The *two-type network game* is the game $G(x_0)$ with the restriction described in the above paragraph. In every equilibrium of a two-type game, player i chooses the same policy as player j if they are in the same group.[29] Hence, an equilibrium is represented by a pair (p_A, p_B), such that $i \in A$ plays p_A, and $j \in B$ plays p_B. I use $\alpha_A := \frac{\alpha\gamma n_B}{1-\alpha\gamma^{AA}(n_A-1)}$ and $\alpha_B := \frac{\alpha\gamma n_A}{1-\alpha\gamma^{BB}(n_B-1)}$. By Assumption 1, $\alpha_A, \alpha_B \in [0,1]$ and $\frac{\alpha_A+\alpha_B-2\alpha_A\alpha_B}{1-\alpha_A\alpha_B} \in [0,1]$.[30]

Lemma 6 (Two-Type Network)**.** *Let $\beta_A \geq \beta_B$ and $(p_A, p_B) \in (p_0, \overline{p})^2$ be the unique potential maximizer of the two-type network game.*

(1) If $\beta_A - \beta_B \geq \frac{\alpha_A+\alpha_B-2\alpha_A\alpha_B}{1-\alpha_A\alpha_B}k$, then $p_A < p_B$ and

$$\mathbb{E}\chi(p_A) - \mathbb{E}\chi(p_B) = \beta_A - \beta_B - \frac{\alpha_A + \alpha_B - 2\alpha_A\alpha_B}{1 - \alpha_A\alpha_B}k.$$

[29] The proof of this result uses the fact that the game $G(x_0)$ is a potential game, and that, for given policies chosen in group g', the "reduced potential" that includes only members of g is "symmetric"; see, e.g., Vives (1999), Chapter 2, Footnote 23.

[30] These results are established in the Appendix, Section 15.

(2) If $\beta_A - \beta_B \le \frac{\alpha_A+\alpha_B-2\alpha_A\alpha_B}{1-\alpha_A\alpha_B}k$*, then* $p_A = p_B$ *and*

$$\mathbb{E}\chi(p_A) = \frac{\alpha_B(1-\alpha_A)\beta_A + \alpha_A(1-\alpha_B)\beta_B}{\alpha_B(1-\alpha_A) + \alpha_A(1-\alpha_B)} + k.$$

The result shows that the strategic-uncertainty effect increases in the number of players. In particular, $\mathbb{E}\chi(p_A) - \mathbb{E}\chi(p_B) - (\beta_A - \beta_B)$ is decreasing in n_A and n_B. Moreover, for sufficiently high complexity, conformity is extreme: all players choose the same policy. In this case, the expected outcome is the same as if a representative player were choosing an optimal policy, in isolation and with a favorite outcome equal to $\frac{\alpha_B(1-\alpha_A)\beta_A+\alpha_A(1-\alpha_B)\beta_B}{\alpha_B(1-\alpha_A)+\alpha_A(1-\alpha_B)}$, which is a weighted average of centralities in the two groups.

4.3 Application 3: Centralization in Organizations

This section considers a stylized model of an organization in which division managers choose production processes knowing the produced quantity of alternative choices only up to some noise.

Model A firm is made of two divisions, each producing a different good. When quantity produced by division i is x_i, the cost of division i is

$$c_i x_i - g x_1 x_2,$$

in which the parameter $g > 0$ measures the degree of cost externalities and $c_i > 0$. An increase in the quantity produced by one division reduces the marginal costs of the other division, as in Alonso et al. (2015). The inverse demand function for product i is given by

$$a_i - \frac{1}{b} x_i,$$

where $b > 0$ measures the price elasticity of demand. The profits of division i given the profile of quantities $\boldsymbol{x}$ are

$$\pi_i^O(\boldsymbol{x}) := \left(a_i - \frac{1}{b}x_i - c_i + gx_j\right)x_i.$$

The CEO's objective is the maximization of total profits $\pi_1^O + \pi_2^O$. I impose an upper bound on the strength of cost externalities for the CEO's profit maximization to be well-behaved: $bg < 1$.[31]

Each division manager chooses a production policy $p_i \in [p_0, \overline{p}]$. The function χ specifies the quantity produced by a division for every production policy. Division i's profits given the pair of policies $\boldsymbol{p}$ are given by $\pi_i^O(\boldsymbol{\chi}(\boldsymbol{p}))$. The division managers set production policies simultaneously and independently in the *production game*, $\left\langle \{1,2\}, \{\mathbb{E}\pi_i^O(\boldsymbol{\chi}(\cdot)), [p_0, \overline{p}]\}_{i\in\{1,2\}} \right\rangle$.

Results I investigate whether managerial incentives are compatible with total-profit maximization. The rest of the analysis assumes that $a_1 - c_1 = a_2 - c_2 =: \widehat{a}$, which implies that managers choose the same policy in equilibrium and for total-profit maximization; $\widehat{a}$ is the net demand intercept for the two goods.[32]

Proposition 7. *There exists a unique policy profile $\boldsymbol{p}^O$ that maximizes expected total profits. Moreover, $\boldsymbol{p}^O$ is an equilibrium of the production game if and only if:*

$$\widehat{a}\frac{b}{1 - bg} \leq 2k.$$

The result gives conditions under which $\boldsymbol{p}^O$ is in the equilibrium set. First, I show that the CEO's objectives are well-defined by studying the maximization of total profits, which is equivalent to the maximization of utilitarian welfare in the coordination game between the

[31] The Hessian of total profits $\pi_i^O + \pi_j^O$ is negative definite iff: $bg < 1$.

[32] More general results are given in the Appendix, Section 15.

division managers. The maximization of expected total profits is solved using the welfare analysis in Proposition 6 and the equilibrium set is characterized using Proposition 2 and Lemma 5.

The result associates multi-division firms with weaker cost externalities and operating in more complex environments with an equilibrium that implements the CEO's optimal production policy. A necessary and sufficient condition to for maximization of total profits to be implemented in equilibrium is that complexity exceeds the threshold $\widehat{a}\frac{b}{2(1-bg)}$. The threshold increases in the net demand intercept $\widehat{a}$ and price sensitivity of demand, reflecting that the interests of division managers move farther apart from the CEO's interests for favorable individual market conditions. The threshold also increases in g, because the "non-internalized" externalities increase in g.

Discussion A reason for the presence of noise in the mapping from production processes to quantities is frictions in the command chain. Suppose that each division manager only instructs lower-end division managers about production decisions, who in turn interact with store managers, and so forth. The division manager is unsure about how her instructions are communicated along the chain of command and finally implemented. Complexity captures the noise perceived by the division manager; e.g., the longer the chain, the less predictable the outcome of the original instruction. To capture that bold decisions are unpredictable, in the model division managers do not know the shape of the mapping from production policies to quantities, and there is a status-quo policy leading to a certain quantity. In particular, if a division manager opts for the status-quo policy, the quantity produced by that division is known to both managers and CEO. The status quo is common between divisions, which may arise in practice if the two divisions are just starting to operate separately and have operated under a centralized authority until now.

5 Heterogeneous Status Quo

This section considers an incomplete-information extension of the game $G(x_0)$ introduced in Section 2.

5.1 General Model

Ex-Post payoffs are the same as in Section 2.1. The following description of interim beliefs defines a Bayesian game parametrized by a profile of status-quo policies, $\mathcal{G}(\boldsymbol{p}_0)$, which is defined explicitly in the Appendix (Section 13).

Player i believes that the outcome function χ is the realized path of a Brownian motion with drift $\mu < 0$, variance parameter $\omega > 0$ and starting point $(p_0^i, \chi(p_0^i))$. Every player knows the profile of status-quo policies $\boldsymbol{p}_0 = (p_0^1, \ldots, p_0^n) \in \mathbf{R}^n$. The status-quo outcome of player i is known to player i and not known to her opponents: $\chi(p_0^i)$ is player i's *type*. Beliefs are consistent with the limit of a common prior over a Brownian motion.[33] I denote by $\mathbb{P}^i$ the probability of an event and by $\mathbb{E}^i$ the expectation operator induce by player i's beliefs at a given type $\chi(p_0^i)$ (see the Appendix, Section 11.2, for more details.)

Every player simultaneously chooses a policy. In this section, $P_i = [\underline{p}_i, \overline{p}_i]$ is the policy space of player i, for $\underline{p}_i, \overline{p}_i \in \mathbf{R}$ with $\underline{p}_i \leq p_0^i \leq \overline{p}_i$, and $P = \times_i P_i$ to ease readability, with a slight inconsistency of notation with respect to the previous sections. A strategy for player i is a measurable function $\sigma_i \colon \mathbf{R} \to P_i$. The set of strategies for player i is denoted by Σ_i, the set of strategy profiles by $\Sigma := \times_{i \in N} \Sigma_i$, and the set of profiles of strategies for players other than i by $\Sigma_{-i} = \times_{i \in -i} \Sigma_j$; Σ_i is endowed with the pointwise order to be a lattice, Σ_{-i} and Σ are endowed with the product order. The following notation is used, given a profile

[33] Given a Brownian motion with starting point $(0, z)$ and realized path denoted by ξ, suppose that each player observes the point $(p_0^i, \xi(p_0^i))$ and a signal about z with Gaussian noise that is i.i.d. across players. As the noise grows, player i's belief about $\xi(q)$ given $\xi(p_0^i) = x_0^i$ converges to her belief in $\mathcal{G}(\boldsymbol{p}_0)$ about $\chi(q)$ when her type is $\chi(p_0^i) = x_0^i$.

of strategies of player i's opponents σ_{-i}:

$$(\chi(p_i), \chi(\sigma_{-i})) = (\ldots, \chi(\sigma_{i-1}(\chi(p_0^{i-1}))), \chi(p_i), \chi(\sigma_{i+1}(\chi(p_0^{i+1}))), \ldots),$$

The expected payoff of player i, given σ_{-i}, is

$$\Pi_i(p_i, x_0^i; \sigma_{-i}) := \mathbb{E}^i[\pi_i(\chi(p_i), \chi(\sigma_{-i}))]$$

An equilibrium of $\mathcal{G}(\boldsymbol{p}_0)$ is an interim Bayesian Nash equilibrium; the definition uses $\varphi_i(x_0^i; \sigma_{-i}) := \arg\max_{p_i \in P_i} \Pi_i(p_i, x_0^i; \sigma_{-i})$.

Definition 3. *The strategy profile $\sigma \in \Sigma$ is an equilibrium of $\mathcal{G}(\boldsymbol{p}_0)$ if, and only if:*

$$\sigma_i(x_0^i) \in \varphi_i(x_0^i; \sigma_{-i}), \quad \textit{for all } x_0^i \in \mathbf{R}, i \in N.$$

Remark 3. Consider the game $\mathcal{G}((p_0, \ldots, p_0))$, in which players have the same status-quo policy p_0. This game is effectively the collection of strategic-form games $\{G(x_0)\}_{x_0 \in \mathbf{R}}$, because the profile of status-quo outcomes is common knowledge. Hence, the game $G(x_0)$ is the subgame of $\mathcal{G}((p_0, \ldots, p_0))$ starting at $\chi(p_0) = x_0$.

5.2 Results

The assumption that status-quo policies are different across players is maintained in this section.

Assumption 2 (Incomplete Information)**.** *Status-Quo policies are different across players: $p_0^i \neq p_0^j$ for all $i, j \in N$ with $j \neq i$.*

Player i's belief about $\chi(q)$ is nondecreasing in $\chi(p_0^i)$ in the sense of first-order stochastic dominance (FOSD) and satisfies a translation-invariance property studied in Mathevet

(2010).[34]

Lemma 7 (FOSD Monotonicity and Translation Invariance of Beliefs)**.** *Player i's belief about the outcome of policy q is nondecreasing in $\chi(p_0^i)$ according to first-order stochastic dominance. Moreover, player i's belief satisfies the following translation invariance property:*

$$\mathbb{P}^i\{\chi(q) < x | \chi(p_0^i) = x_0^i\} = \mathbb{P}^i\{\chi(q) < x + \Delta | \chi(p_0^i) = x_0^i + \Delta\}, \text{ for all } \Delta \in \mathbf{R}.$$

FOSD monotonicity is used to establish the single-crossing property of expected payoffs in own policy and type.

A more stringent upper bound on the strength of coordination motives than Assumption 1 is used to establish single-crossing of expected payoffs, which is used for the existence of equilibria in monotone strategies.

Assumption 3. *For every player i,*

$$\alpha \sum_{j \in N} \gamma^{ij} < 1.$$

Assumption 3 implies that $\boldsymbol{I} - \alpha \boldsymbol{\Gamma}$ has strictly dominant diagonal, which is a known sufficient condition for Assumption 1.

The incomplete-information game $\mathcal{G}(\boldsymbol{p}_0)$ exhibits strategic complementarities.

Lemma 8 (Single Crossing and Strategic Complementarities)**.** *For all $i \in N$, the expected payoff $(\boldsymbol{p}, \chi(p_0^i)) \mapsto \mathbb{E}^i \pi_i(\boldsymbol{\chi}(\boldsymbol{p}))$ exhibits strictly increasing differences in p_i, p_j, $j \in -i$, and in $(p_i, \chi(p_0^i))$.*

The upper bound on coordination motives is key for increasing differences in own policy and type. To establish this property, the right-derivative of $p_i \mapsto \mathbb{E}^i \pi_i(\boldsymbol{\chi}(\boldsymbol{p}))$ is shown to be

[34] For notational convenience, in the following result I use the symbol "—", even though the beliefs of players do not necessarily arise as conditional probabilities, because $\mathcal{G}(\boldsymbol{p}_0)$ is an interim Bayesian game.

an affine function of x_0^i, where the coefficient on x_0^i is $1 - \alpha \sum_j \gamma^{ij}$ (Appendix). The upper bound on coordination motives is necessary for the single-crossing property of expected payoffs in (p_i, x_0^i), which associates higher policies to higher types.

The following result establishes existence of Bayesian Nash equilibrium in nondecreasing strategies.

Proposition 8. *There exist a greatest and a least Bayesian Nash equilibrium, $\overline{\sigma}$ and $\underline{\sigma}$, respectively. Moreover, $\overline{\sigma}$ and $\underline{\sigma}$ are profiles of nondecreasing strategies.*

Because the type spaces are necessarily unbounded, results from the literature on incomplete-information games with strategic complementarities do not apply directly. However, I establish that the expected payoff function $p_i \mapsto \Pi_i(p_i, x_0^i; \sigma_{-i})$ is strictly concave for a profile of nondecreasing strategies σ_{-i}. Given strict concavity of Π_i, compactness of P_i and strategic complementarities, type spaces can be compactified to establish similar results as Van Zandt and Vives (2007). In particular, the greatest-best-reply mapping $x_0^i \mapsto \sup \varphi_i(x_0^i, \sigma_{-i})$ is measurable; see Lemma 26 in Appendix.)

Remark 4. Let $\alpha = 0$. From the analysis in Callander (2011a) and Corollary 1, it follows that: (i) there exists a unique Bayesian Nash equilibrium, and (ii) in the unique Bayesian Nash equilibrium, the strategy of each player is nondecreasing in her type.

The following result shows a status-quo effect.

Lemma 9 (Status-Quo Bias)**.** *For every Bayesian Nash equilibrium in nondecreasing strategies σ and player i, the following holds:*

There exist cutoffs $c_1^i, c_2^i \in \mathbf{R}$ with $c_1^i < c_2^i$ such that: $\sigma_i(x) = p_0^i$ for all $x \in [c_1^i, c_2^i]$, and $\sigma_i(x) \neq p_0^i$ for all $x \in \mathbf{R} \setminus [c_1^i, c_2^i]$.

There are two takeaways. First, the reason why the slope of equilibrium strategies is not constant is the presence of a status quo: if the status-quo outcome of player i is in an interval

$[x_1^i, x_2^i]$, player i prefers to stick to the status-quo policy, than to incur the uncertainty implied by a change of expected outcome. This equilibrium behavior is consistent with the optimal strategy in the game without coordination motives (Corollary 1).

Secondly, equilibrium strategies do not have a constant slope, differently from general models of beauty contest under incomplete information. Strategies with constant slope are either the focus or constitute the unique possibility in equilibrium in standard beauty-contest models of incomplete information. In Lambert et al. (2018) — where the environment is "informationally complex" because of the arbitrarily large, though finite, dimensionality of the state and type profile —, the authors establish the existence of an equilibrium in strategies with constant slope.

The following result offers a partial characterization of equilibria in nondecreasing strategies, using χ_j for $\chi(\sigma_j(p_0^j))$, given $\sigma_j \in \Sigma_j$ and $j \in N$.

Lemma 10. *Let $P_i = [p_0^i, \infty)$ for all $i \in N$. The profile of nondecreasing strategies σ is an equilibrium if, and only if, the following condition holds. For all $i \in N$ and $x_0^i \in \mathbf{R}$ such that $\sigma_i(x_0^i) > p_0^i$, there exists a vector $[a_{ij} : j \in N]$, such that:*

$$\mathbb{E}^i \chi_i - \alpha \sum_{j \in N} \gamma^{ij} \mathbb{E}^i \chi_j = \beta_i - \alpha \sum_{j \in N} \gamma^{ij} \beta_j + k + \alpha k \sum_{j \in N} \gamma^{ij} a_{ij},$$

and $a_{ij} \in \Big[2\mathbb{P}^i\{\sigma_j(\chi(p_0^j)) < \sigma_i(x_0^i) | \chi(p_0^i) = x_0^i\} - 1, 2\mathbb{P}^i\{\sigma_j(\chi(p_0^j)) \leq \sigma_i(x_0^i) | \chi(p_0^i) = x_0^i\} - 1\Big].$

The next result studies the multiplicity of equilibria, letting d denote the sup-norm distance between two strategies for player i.[35]

[35] The sup-norm of a strategy for a player is well-defined because policy spaces are bounded. Moreover, in the Appendix I establish that (i) equilibrium strategies are continuous and (ii) type spaces can be compactified, so that the sup can be replaced by the max in d by Weierstrass' Theorem (Lemmata 24 and 25).

Proposition 9. *The following holds:*

$$\max_{i\in N} d(\overline{\sigma}_i, \underline{\sigma}_i) \leq 2k \max_{i\in N} \frac{\alpha\sum_j \gamma^{ij}}{1-\alpha\sum_j \gamma^{ij}} \frac{1}{|\mu|}.$$

By Proposition 8, all equilibria lie between two extreme strategy profiles, $\overline{\sigma}$ and $\underline{\sigma}$. Therefore, the distance between player i's strategies in any two equilibria is at most the distance between the extremal equilibria, i.e. $d(\overline{\sigma}_i, \underline{\sigma}_i)$, which is upper bounded by the Proposition.

In the Appendix, I study the game with finite policy spaces. With two players and finite policy spaces, there exists a unique equilibrium in nondecreasing strategies. The key step of the proof is the observation that increasing differences — which yield strategic complementarities in $G(x_0)$ and single-crossing in $\mathcal{G}(\boldsymbol{p}_0)$ — are constant in own type. This "constant-type" monotonicity, and the translation invariance and FOSD monotonicity properties of beliefs suffice establish uniqueness by using the results in Mathevet (2010); the author shows that under "translation-monotone" and FOSD-nondecreasing beliefs, a class of coordination games admits a unique equilibrium because the best-response mapping to nondecreasing strategies is a contraction.

6 Imperfectly Correlated Outcome Functions

In many strategic interactions, players face distinct decision-outcome mappings. Firms buy pricing services from different providers, and pricing algorithms are trained on separate datasets. Similarly, the communication noise may be only partially correlated across multiple divisions of the same organization. To capture these features in the case of 2 players, suppose that the outcome function of player 1 is $X^1 = Y^1$, while the outcome function of player 2 is $X^2 = \rho Y^1 + \sqrt{1-\rho^2}Y^2$, for $\rho \in [0, 1]$ and a 2-dimensional Brownian motion (Y^1, Y^2) with common drift μ, variance parameter ω, and independent coordinates.[36] The analysis in this paper leads to the following characterization of equilibria.[37] A policy profile $\boldsymbol{p} \in (p_0, \overline{p})^n$ is an equilibrium if, and only if:

$$\mathbb{E}\boldsymbol{\chi}(\boldsymbol{p}) = \boldsymbol{\beta} + (\boldsymbol{I} - \alpha\boldsymbol{\Gamma})^{-1}(\boldsymbol{I} + \rho\alpha\boldsymbol{\Gamma} \odot \boldsymbol{C})\mathbf{1}k,$$

for a matrix $\boldsymbol{C}$ such that $C_{ij} \in [-1, 0]$, $C_{ij} = 0$ if $p_i > p_j$ and $C_{ij} = -1$ if $p_i < p_j$.

This general model allows for a finer decomposition that separates the two elements of the complexity of the environment: variance of outcomes and covariance of pairs of outcomes. The new term in the decomposition is a linear combination of two effects. First, a *pure status-quo bias*, which arises with independent outcomes across players (i.e., the positive vector $(\boldsymbol{I} - \alpha\boldsymbol{\Gamma})^{-1}\mathbf{1}k$, discussed in Section 3.)[38] This component pushes every player towards the status quo, and is magnified by the network of players. Second, a *pure experimentation motive*, that arises only with correlated outcomes (i.e., the nonpositive vector $(\boldsymbol{I} - \alpha\boldsymbol{\Gamma})^{-1}(\rho\alpha\boldsymbol{\Gamma} \odot \boldsymbol{C})\mathbf{1}k$.) This component pulls players away from the status quo.

[36] See Definition 5.19 in Karatzas and Shreve (1998), Chapter 2.

[37] The model considered in this paragraph is constructed as in Section 2, except that $\boldsymbol{p} \mapsto \pi_i(X^i(p_i), X^j(p_j))$ replaces $\boldsymbol{p} \mapsto \pi_i(\chi(p_i), \chi(p_j))$. This construction generalizes for$n$ players via a suitable linear combination of the coordinates of an n-dimensional standard Brownian motion (Definition 5.1 in Karatzas and Shreve (1998), Chapter 2); see Exercise 4.16 in Shreve (2004).

[38] The vector $(\boldsymbol{I} - \alpha\boldsymbol{\Gamma})^{-1}\mathbf{1}k$ scales the "unweighted" centralities by the degree of complexity.

Part II

Essay 2: The Extensive Margin of Persuasion

7 Introduction

In the "information age," consumers of information decide whether an information source deserves attention because information acquisition is costly (Floridi, 2014; Simon, 1996). The information-design literature studies a *sender* who supplies information to a *receiver*, to persuade the receiver to take a certain action (Bergemann and Morris, 2019; Kamenica, 2019). When attention is costly, the sender faces the dual problem of (i) persuading the receiver to take a certain action and (ii) inducing her to pay attention. In this paper, I study the persuasion of a receiver who is privately informed about her cost and benefit of information, in which the sender uses information to reward the receiver for her effort.

The intensive margin of persuasion captures intensity of the sender's persuasion on the receiver's *action* decision, while the extensive margin of persuasion refers to whether or not the receiver pays *attention* to the sender's information. The study of the extensive margin of persuasion is important to determine which consumers have access to information. In a persuasion game, the sender effectively allocates information to a heterogeneous audience. For instance, today's central banks use "layered communication" to reach the general public, characterized by heterogeneous and limited information-processing ability. In my model, I investigate the following questions: Who accesses information? Does the receiver benefit from the limit to her information-processing ability?

In order to study the *extensive* and *intensive* margin of persuasion, I model the persuasion of an inattentive receiver who takes a binary action, 1 or 0. There is a state θ unknown to two players: Sender (he) and Receiver (she). Receiver chooses 1 only if she expects the state θ to exceed her outside option. Sender wants Receiver to choose 1 regardless of the state. In the baseline model, Sender designs a random variable S correlated with the state θ, called *signal*. Knowing the signal S, but not its realization, Receiver chooses her attention effort e: high effort is costly and increases the probability of observing the signal realization. The choice of effort captures the choice of acquiring information about the state, and the cost of effort may be monetary or psychological. The timing is as follows.

(1) Sender chooses signal S, without knowing the Receiver's *type*, which includes her effort cost and outside option.

(2) Receiver chooses her effort e;

(3) Receiver observes the realization of S with probability e, and observes an uninformative signal with the remaining probability. She chooses action 1 or 0 given her posterior belief.

For instance, let's suppose that a university (Sender) wants its graduates to find employment at a renowned firm (Receiver), regardless of their skills (state), while the firm finds it profitable only to hire high-skill graduates. The university decides how to best advertise its graduates to maximize the probability that the graduates are hired by the renowned firm. The university's marketing policy includes: grading policy, social-media presence, advertisement of graduates' achievements, and so on. There are two main forces that determine the optimal marketing: the university wants the firm to (i) pay attention to the marketing campaign, and (ii) hire the graduates. Paying attention refers to the extensive margin of persuasion: is the firm reached by the marketing efforts? The hiring decision refers to the intensive margin of persuasion: does the firm hire the graduates, given the information acquired from the marketing campaign? The private information of Receiver, in this example, captures the fact that the university is not fully informed about (i) the extent to which the firm is hiring, and the platforms where firms seek job candidates (cost of effort); and (ii) the firm's hiring process, including intervew questions and tests (outside option).

The extensive margin of persuasion arises because Receiver is privately informed about her type. In particular, the sender takes into account that increasing the correlation between the state and the signal has two effects: on Receiver's attention effort e — the extensive margin of persuasion —, and on Receiver's action if she observes the realization of S — the intensive margin of persuasion.

I show the equivalence between persuasion mechanisms and signals. Let's suppose that Sender commits to a persuasion mechanism, which is a menu of signals $S_\bullet$, as opposed to a single signal. Under a persuasion mechanism, Receiver reports a type and chooses an effort level. In particular, Receiver chooses the probability with which to observe the signal from the menu that corresponds to her reported type. A mechanism is incentive-compatible if Receiver finds it optimal to report her type truthfully. For every incentive-compatible persuasion mechanism $S_\bullet$, I construct a signal S that induces the same action and effort distributions over Receiver types (Theorem 1). The key is to establish a supermodularity property of type-t Receiver's expected utility: the return from effort is increasing in a t-specific informativeness order, which agrees with Blackwell's order whenever possible. I construct a single signal S that attaches to each Receiver's type the same t informativeness as the incentive-compatible mechanism $S_\bullet$. This result shows that Sender does not need to offer a fine collection of information structures, and allows the study of persuasion to focus on single signals.

I characterize the optimal information structure in commonly-studied applications,

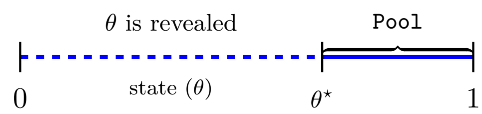


Figure 7: An upper censorship is a signal that reveals states below a cutoff state $\theta^\star$, and sends a single realization, `Pool`, if the state is above the cutoff.

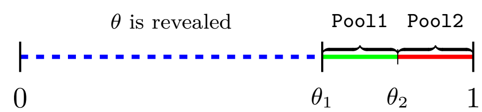


Figure 8: A bi-upper censorship is a signal that reveals low states and separates high from very high states.

which censors high states. An upper censorship is a signal that reveals low states, and censors high states, as in Figure 7. Upper censorships are optimal signals if the Receiver's outside option admits a single-peaked distribution (Theorem 2). Given the equivalence between persuasion mechanisms and signals, we can focus on upper censorships to study the extensive margin of the Sender's persuasion in applications. I apply my results to the problem of media censorship. If Sender knows Receiver's attention cost and has preferences over the extensive margin, inspired by models of media capture à la Gehlbach and Sonin (2014), bi-upper censorships are optimal signals (see Figure 8). I study the effect of changes in Receiver's attention cost on the information provided by the Sender, measured à la Blackwell, through the optimal upper censorship. I do so by isolating the effect of each of the two dimensions of Receiver's private information. Sender provides more information as Receiver's attention cost stochastically increases, if he knows Receiver's outside option (Proposition 11). Moreover, Sender provides more information as the Receiver's attention cost increases, if he knows the attention cost and that cost is sufficiently small (Proposition 12).

Related Literature If the receiver's attention is costless, prior work determines the extent of a sender's intensive margin of persuasion (Kamenica and Gentzkow, 2011; Kolotilin et al., 2017). To study the extensive margin of persuasion, the model of this paper features either the receiver's attention cost, and the receiver's private information. The attention costs lead Receiver to decide whether to become informed, and the private information captures the heterogeneity of attention choice in the audience of a sender. Persuasion of an inattentive receiver has been studied in three models, which do not include Receiver's private information. Wei (2021) studies a receiver who incurs a cost to reduce her uncertainty about the state. **?** study a receiver who acquires costly information about the state from a third party. Differently from these papers, I consider a receiver whose attention cost is not within the rational-inattention paradigm. In the main model of Bloedel and Segal (2021), the receiver bears a cost proportional to the mutual information between the sender's signal and the receiver's signal about the sender's one. In a separate model, the authors study the same cost structure as in my paper.[39] Differently from these models, I include Receiver's private information to study a rich extensive margin of persuasion. The connection with these papers is further discussed in Section 8.

If attention effort is costless, optimality properties of upper-censorship signals are known (Gentzkow and Kamenica, 2016; Kolotilin, 2018; Dworczak and Martini, 2019; Kleiner et al., 2021; Kolotilin et al., 2022; **?**), and the equivalence between persuasion mechanisms and signals is shown by Kolotilin et al. (2017) (see also Guo and Shmaya (2019)). I generalize these results to the case of receiver's costly and privately known attention effort.

The literature on incomplete-information beauty contests studies the supply of Gaussian

[39] Either in Wei (2021) and in the special case of Bloedel and Segal (2021), the analysis assumes that every signal has at most two realizations with positive probability, which is without loss of generality, although for different reasons in the two models. This assumption would imply a loss of generality in my model because Receiver has private information.

signals to inattentive receivers.[40] The restriction to Gaussian signals renders many questions about optimal information structures moot. The literature on media capture considers the provision of information to receivers who are privately informed, either about the opportunity cost of supporting an incumbent politician, or about their attention cost (respectively, Kolotilin et al. (2022) and Gehlbach and Sonin (2014)).[41]

Outline I present the model in the next section. In Section 9.1, I describe the equivalence between persuasion mechanisms and signals. In Section 9.2, I characterize the extensive margin of persuasion. In Section 10.1, I study optimal signals and welfare implications of changes in Receiver's attention cost. In Section 10.2, I discuss implications for the theory of media capture.

8 Model

A Sender (he) and a Receiver (she) play the following persuasion game. Before the *state* $\theta \in \Theta := [0, 1]$ is realized, players have a common prior $\mu_0 \in \Delta\Theta$, which admits an absolutely continuous CDF F_0.[42] Receiver's type $t = (\zeta_t, \lambda_t) \in T$, where $T = [0, 1]^2$, is distributed independently of the state θ, according to a CDF H. ζ_t is Receiver's threshold type, or outside option, λ_t is Receiver's attention type, or attention cost. Receiver's *material payoff* from taking action $a \in \{0, 1\}$ is $u_R(a, \theta, \zeta_t) = a(\theta - \zeta_t)$, when her threshold type is ζ_t and the state is θ. Receiver's effort cost, if her attention *effort* is $e \in [0, 1]$, is given by $\lambda_t k(e)$, where k is a continuous function. The Receiver's utility is given by the difference between

[40] Several models characterize the optimal supply of Gaussian signals to inattentive receivers, see Cornand and Heinemann (2008); Chahrour (2014); Myatt and Wallace (2014); Galperti and Trevino (2020); see also Nimark and Pitschner (2019), and references therein, for related models.

[41] See Prat (2015) for a survey of the literature on media censorship.

[42] $\Delta\mathcal{X}$ denotes the set of Borel probability measures over the set $\mathcal{X}$.

her material payoff and her effort cost:

$$U_R(a,\theta,e;t) := u_R(a,\theta,\zeta_t) - \lambda_t k(e).$$

Sender always wants Receiver to take action 1, and his utility when Receiver chooses action a is $U_S(a) = a$.

The timing of the game is as follows.

- Sender publicly commits to a signal, which is a measurable function $\sigma : \Theta \to \Delta M$, where M is an exogenous rich space of signal realizations.[43]
- Nature draws Receiver's type t according to H.
- Receiver chooses an effort $e \in [0,1]$, knowing her type t.
- Nature draws the state θ according to μ_0, and a message $m \in M \cup \{\phi\}$. m is drawn from $\sigma(\theta)$ with probability e, and m is equal to ϕ, where $\phi \notin M$, with the remaining probability.
- Receiver observes the message m, and then updates her belief about θ, using Bayes' rule and knowledge of σ and the prior μ_0. Given her *posterior belief*, she chooses an action $a \in \{0,1\}$.

We analyze Sender-optimal Perfect Bayesian Equilibria of this game, in line with the literature on Bayesian persuasion. We denote by $\overline{F}$ the CDF corresponding to full mass at x_0 which is the prior mean of θ (see the Appendix for the defision of equilibrium.)

Receiver's optimal action and effort Let's describe type-t Receiver's optimal action, given her posterior belief $\mu \in \Delta\Theta$. Letting $t = (\zeta, \lambda)$, the optimal action is 1 if the expected

[43] It is sufficient that $M = [0,1]$.

state according to μ, x, exceeds her threshold type ζ, and the optimal action is 0 if the expected state according to μ is such that $x < \zeta$.[44] Thus, Receiver's optimal action depends on belief μ only through its mean $x_\mu := \int_\Theta \theta \, \mathrm{d}\mu(\theta)$. The Receiver's *material payoff at belief* μ is her expected material payoff when her belief is μ:

$$v_t(\mu) := \int_0^1 u_R([x_\mu \geq \zeta], \zeta, \theta) \, \mathrm{d}\mu(\theta),$$

where $[P]$ is the Iverson bracket of the statement P: $[P] = 1$ if the statement P is true, and $[P] = 0$ otherwise. We note that $v_t(\mu)$ depends on the belief μ only through its induced mean x_μ.

Sender's maximization problem After Sender chooses a signal that induces the distribution over posterior beliefs $p \in \Delta\Delta\Theta$, type-t Receiver chooses her effort to maximize her expected utility. In particular, she faces the maximization problem given by

$$\max_{e\in[0,1]} e \int_{\Delta\Theta} v_t(\mu) \, \mathrm{d}p(\mu) + (1-e)v_t(\mu_0) - \lambda_t k(e). \tag{4}$$

If k is smooth, the optimal effort is obtained by a simple marginal-cost-marginal-benefit analysis. Type-t Receiver compares the marginal benefit of committing to observing the signal with probability e to the marginal cost of such a commitment. The marginal benefit is the difference between the expected material payoff when Receiver updates her beliefs according to p and the material payoff at the prior belief: $\int_{\Delta\Theta} v_t(\mu) \, \mathrm{d}p(\mu) - v_t(\mu_0)$. We refer to this difference as the marginal benefit of effort at the random posterior p.[45] If k is differentiable, the marginal cost of effort e is given by $\lambda_t \frac{\partial k}{\partial e}(e)$. Since $v_t(\mu)$ depends

[44] We break the Receiver's indifference in favor of Sender. This assumption is without loss of generality given our assumption that H is absolutely continuous. This assumption is necessary for Sender optimality when Sender knows Receiver's threshold type (see, e.g., Gentzkow and Kamenica (2016)).

[45] The marginal benefit of effort at a random posterior p is commonly referred to as the value of the information of the signal that induces p.

on belief μ only through its induced mean, the random posterior p influences Receiver's effort decision only through the marginal benefit of effort. In particular, if the signal's informativeness increases in the Blackwell order, the marginal benefit of effort shifts upward for every Receiver's type; while the marginal cost of effort does not change. We denote by $E(p;t)$ the nonempty set of maximizers of the above program, which we study in Section 9.2.

Let's describe the role of the extensive and the intensive margin of persuasion in the Sender's incentives. We use the formalism of random posteriors, as done in the literature on persuasion. Let $\mathcal{R}$ be the set of *feasible random posteriors*: distributions of the Receiver's belief satisfying the martingale condition.[46] We describe the Sender's choice of a feasible random posterior, which is without loss of generality.[47] We define the Sender's payoff at belief μ using Receiver's optimal action as: $V_S(\mu;t) := [x_\mu \geq \zeta_t]$. The Sender's problem is:

$$\sup_{p,e(\cdot)} \int_{\Delta\Theta} \int_T e(t)(V_S(\mu;t) - V_S(\mu_0;t))\,\mathrm{d}H(t)\,\mathrm{d}p(\mu)$$

$$\text{s.t. } p \in \mathcal{R} \text{ and } e(t) \in E(p;t) \text{ for all } t \in T.$$

We decompose the persuasion of a Receiver's type into two terms. The Receiver's optimal action depends only on the mean of the Receiver's belief, which is either the (random) posterior mean following the information policy p, or the prior mean $\int_\Theta \theta\,\mathrm{d}\mu_0(\theta) =: x_0$. The effort chosen by Receiver is the probability that the mean of the Receiver's belief is the (random) posterior mean following the information policy p. Thus, the Sender's

[46] In particular:

$$\mathcal{R} := \left\{ p \in \Delta\Delta\Theta \;:\; \int_{\Delta\Theta} \mu \mathrm{d}p(\mu) = \mu_0 \right\}.$$

[47] Every signal induces a distribution in $\mathcal{R}$, by the martingale property of Bayesian updating. Moreover, for all $p \in \mathcal{R}$, there exists a signal that induces p as the distribution of the posterior belief; see, e.g., Kamenica and Gentzkow (2011) and Appendix C.2 in **?**.

expected payoff depends on the feasible random posterior p in two ways, which can be ascribed to the intensive and the extensive margin of Bayesian persuasion. First, type t acts if the posterior mean is higher than type-t outside option ζ_t. Letting $a^\circ(t) = [x_0 \geq \zeta_t]$ and $a^\star(\mu, t) = [x_\mu \geq \zeta_t]$, and assuming maximum effort, the equilibrium expected action is larger than under an uninformative signal by the following amount:[48]

$$\int_{\Delta\Theta} V_S(\mu; t)\,\mathrm{d}p(\mu) - V_S(\mu_0; t) = \mathbb{E}\{a^\star - a^\circ \mid t,\ e(p,t) = 1\}.$$

Second, each type t has some probability of updating her belief, which is t's effort decision $e(p; t)$. Letting $e(p,t)$ be the effort chosen by type-t Receiver, the expected change in Receiver's action is:

$$\underbrace{e(p,t)}_{\text{extensive margin}} \underbrace{\left(\int_{\Delta\Theta} V_S(\mu; t)\,\mathrm{d}p(\mu) - V_S(\mu_0; t)\right)}_{\text{intensive margin}} = \underbrace{\mathbb{E}\{a^\star - a^\circ \mid t\}}_{\text{persuasion of type } t}.$$

The term $e(p,t)$ captures the extensive margin of persuasion: different posterior distributions may lead to different effort decisions of type t. The second term captures the intensive margin of persuasion: different posterior distributions may lead to different distributions of $a^\star - a^\circ$, given Receiver's type t.

Benchmark cases If infomation is costless, the model is equivalent to persuasion of a privately informed receiver as studied in prior work (e.g., Kolotilin et al. (2017); Kolotilin (2018)). The extensive-margin term in the persuasion decomposition is moot. If information is costly and Receiver's type t is known to Sender, our framework specifies to the model

[48] The following conditional expectation given t is taken with respect to the random posterior that is distributed according to p.

studied by Bloedel and Segal (2021), in which Sender solves

$$\sup_{p,e} \int_{\Delta\Theta} e(V_S(\mu;t) - V_S(\mu_0;t))\, \mathrm{d}p(\mu) \tag{5}$$

$$\text{s.t. } p \in \mathcal{R} \text{ and } e \in E(p;t). \tag{6}$$

As Bloedel and Segal (2021) observe, we can use a first-order approach when k is sufficiently smooth; moreover, there exists an optimal signal that is a binary signal by a revelation-principle argument. The problem in Equation 5 is similar to that studied in the *attention-management* literature (Lipnowski et al., 2020, 2022a; Wei, 2021). If we assume that the attention-management Sender wants Receiver to take action 1 regardless of the state, the maximization in 5 is a *constrained* version of the attention-management one.[49] In particular, in our model, Receiver effectively chooses an element from a specific set of garblings of the posterior: the mixtures of the Sender's signal and an uninformative signal. In the attention-management literature, Receiver's choice of garbling is unrestricted.

Information policies Receiver chooses her effort to maximize her expected utility (Problem 4), and the marginal benefit of effort depends on the random posterior p only through the distribution of posterior means. Thus, we identify a feasible random posterior with the induced posterior mean distribution, and here we formalize this representation (similarly to, e.g., Gentzkow and Kamenica (2016)). Let $\mathcal{D}$ be the collection of CDF's over $[0,1]$. A CDF F is feasible if it represents the posterior mean distribution of a feasible random posterior. By Blackwell's theorem, a CDF F is feasible if, and only if: F is a mean preserving

[49] Sender wants Receiver to take action 1 regardless of the state in Wei (2021), while Sender maximizes Receiver's material payoff in Lipnowski et al. (2020, 2022a). The optimal signal for a Sender who wants action 1 regardless of the state is not characterized in attention management, except in the binary-state case (Wei (2021)).

contraction of F_0. Let's define the *information policy of a CDF* $F \in \mathcal{D}$ as:

$$\begin{aligned} I_F \colon \mathbb{R}_+ &\to \mathbb{R}_+ \\ x &\mapsto \int_0^x F(y)\,\mathrm{d}y. \end{aligned}$$

The information policy of a feasible F, I_F, is upper bounded pointwise by F_0, due to Blackwell's theorem. I_F is lower bounded pointwise by $\overline{F}$, because the uninformative signal does not change the mean of the receiver's belief. Moreover, I_F is convex because F is nondecreasing. These are the only three constraints on feasible information policies, so we identify a feasible random posterior with its induced information policy (Gentzkow and Kamenica, 2016). The set of feasible information policies is:

$$\mathcal{I} := \{I \colon \mathbb{R}_+ \to \mathbb{R}_+ \mid I \text{ is convex and } I_{F_0}(x) \geq I(x) \geq I_{\overline{F}}(x) \text{ for all } x \in \mathbb{R}_+\}.$$

We analyze the Sender's problem as a choice of an information policy $I \in \mathcal{I}$. There are two reasons why this choice of formalism pays off. First, I is a measure of the Blackwell informativeness of the corresponding signal. In particular, σ is a more informative signal than τ if, and only if, $I_\sigma(x) \geq I_\tau(x)$, $x \in [0, 1]$, where I_S denotes the information policy corresponding to the posterior mean's CDF induced by signal S. Thus, the pointwise ranking of information policies correspond to Blackwell's information order. Second, information policies offer a tractable characterization of optimal effort, as the next Lemma shows.

Preliminary results Receiver chooses effort by comparing her payoff from updating her belief and her payoff from remaining uninformed. Let's develop notation to deal with this comparison. We define the operator Δ as:

$$\Delta\colon I \mapsto I - I_{\overline{F}}.$$

We denote by ΔI the composite function $\Delta(I)$. For the information policy I, ΔI is a measure of the "net" informativeness, where the Blackwell's informativeness of the uniformative signal, given by $I_{\overline{F}}$, is used as a benchmark. We characterize Receiver's marginal benefit of effort in terms of the Sender's information policy I. For an information policy I, we let $I'(x)$ denote the right derivative evaluated at x, which is the value attained by a CDF evaluated at x, and $I'(x^-)$ denote its left derivative.

Lemma 11 (Marginal Benefit of Effort)**.** *Receiver's marginal benefit of effort given the information policy I and her type (ζ, λ) is:*

$$\int_0^1 u_R([x \geq \zeta], x, \zeta)\,\mathrm{d}I'(x) - \int_0^1 u_R([x \geq \zeta], x, \zeta)\,\mathrm{d}I'_{\overline{F}}(x) = \Delta I(\zeta).$$

Proof. For an information policy $I \in \mathcal{I}$:

$$\begin{aligned}\int_0^1 u_R([x \geq \zeta], x, \zeta)\,\mathrm{d}I'(x) &= \int_\zeta^1 x - \zeta\,\mathrm{d}I'(x)\\ &= x_0 - \zeta + I(\zeta)\end{aligned}$$

The second equality follows from Riemann-Stjeltes integration by parts, using $I'(1) = 1$ and $I(1) = 1 - x_0$. ■

The marginal benefit of effort is increasing in the informativeness of I, measured à la

Blackwell. We define type-(c, λ) indirect utility V at the information policy I

$$V(\Delta I(c); \lambda) = \max_{e \in [0,1]} e\Delta I(c) - \lambda k(e) - v_t(\mu_0),$$

and we refer to $V(\Delta I(c); \lambda)$ as the value of information policy I to type (c, λ). The optimal effort of type (c, λ), given information policy I, is an element of $E(\Delta I(c); \lambda)$, where:

$$E(\Delta I(c); \lambda) = \arg\max_{e \in [0,1]} e\Delta I(c) - \lambda k(e).$$

Type-t Receiver's payoff is increasing in the Blackwell information of the Sender's signal, by Blackwell's theorem. Thus, type-t Receiver's value of information policy$I$ is increasing in the informativeness of I. This fact arises as an implication of monotone comparative statics and the envelope theorem, stated in the next Lemma.

Lemma 12. *Type-t Receiver's value of information $V(\Delta I(\zeta_t), \lambda_t)$ is a nondecreasing, absolutely continuous and convex function of $\Delta I(\zeta_t)$.*

Proof. We observe that $f\colon (e, \Delta I(\zeta_t)) \mapsto e\Delta I(\zeta_t)$ is supermodular. The result follows from the envelope theorem for supermodular optimization (Lemma 57 in the Appendix). ■

Unsurprisingly, this result states that Receiver's payoff is increasing in the Blackwell information of the Sender's information policy. However, Blackwell's order is incomplete. We leverage the Lemma to construct a natural type-specific completion of the Blackwell order over information policies. Let's construct a t-specific informativeness order over information policies: $\preceq_t$ over $\mathcal{I}$, such that

$$J \preceq_t I \quad \text{iff} \quad \Delta J(\zeta_t) \leq \Delta I(\zeta_t), \quad \text{for every } J, I \in \mathcal{I}.$$

$\preceq_t$ is a complete order that agrees with Blackwell's order whenever possible. $\preceq_t$ is a local informativeness measure that is a sufficient to caracterize a type-t Receiver's behavior. To prove Lemma 12, we leverage the fact that type-t Receiver's expected utility is supermodular in informativeness and effort, ordering informativeness by $\preceq_t$. In the next section we leverage this observation to prove a strong equivalence between persuasion mechanisms and signals, and in the following section we leverage this observation to study the extensive margin of Bayesian persuasion.

9 Main Results

9.1 Equivalence of Persuasion Mechanisms and Signals

In this section, we consider a more general setup than the previous model. We expand the Sender's strategy space, to include "menues" of signals. We ask whether Sender attains a larger payoff by committing to such a menu, so that each Recever's type self-selects into her preferred signal, than by choosing a single information policy.

A *persuasion mechanism* is a collection of information policies: $(I_r)_{r \in T}$, where $I_r \in \mathcal{I}$ for all reports $r \in T$. We refer to a persuasion mechanism as $I_\bullet$, omitting the reference to reports. A persuasion mechanism $I_\bullet$ is *incentive compatible* (IC) if:

$$V(\Delta I_t(\zeta_t), \lambda_t) \geq V(\Delta I_r(\zeta_t), \lambda_t), \quad \text{for all types } t \in T \text{ and reports } r \in T.$$

We interpret a persuasion mechanism as a rule allocating a signal to every report of Receiver's type. Thus, a persuasion mechanism is IC if it is optimal for Receiver to report her type truthfully. In particular, after her report r, Receiver chooses an effort optimally given the information policy I_r. By a revelation-principle arguments, a Sender who commits to a persuasion mechanisms can, without loss of optimality, commit to an IC persuasion mechanism. We focus on IC persuasion mechanisms in what follows.

The following two definitions characterize a notion of equivalence between an IC persuasion mechanism $I_\bullet$ and a single information policy J. An IC persuasion mechanism $I_\bullet$ and an information policy J induce the same effort and action distributions if the following two conditions hold.

(1)

$$E(\Delta I_t(\zeta_t); \lambda_t) \subseteq E(\Delta J(\zeta_t); \lambda_t), \quad \text{for all } t \in T. \tag{7}$$

(2)

$$\partial I_t(\zeta_t) \subseteq \partial I(\zeta_t) \quad \text{if } (0,1] \cap E(\Delta I_t(\zeta_t); \lambda_t) \neq \emptyset.$$

The following result allows us to study effort and action distributions of persuasion mechanisms via information policies, thus bypassing the screening problem.

Theorem 1. *For every IC persuasion mechanism $I_\bullet$ there exists an information policy J such that: $I_\bullet$ and J induce the same effort and action distributions.*

Proof. Section 17 in the Appendix. ■

There is a simple intuition for this result, which leverages the local informativeness order $\preceq_t$. $\preceq_t$ determines the choice of type-t Receiver from the collection of information policies of a persuasion mechanism $I_\bullet$, by Lemma 12. Letting c be t's threshold type, we know that type t chooses that information policy $I_\star$ from the mechanism $(I_r)_{r\in T}$ such that $\Delta I_\star(c) \geq \Delta I_r(c)$ for every $r \in T$. It is readily established that $J := \sup_{r\in T} I_r$ is a feasible information policy: J is pointwise bounded by I_{F_0} and $I_{\overline{F}}$ because I_r is an information policy for every $r \in T$, and J is convex because J the pointwise supremum of convex functions. In the proof we show that J replicates effort and action decisions of every type t given the IC mechanism $(I_r)_{r\in T}$.

In the next section we study Sender's optimization by choice of a single information policy. In light of Theorem 1, the following results are relevant to the study of persuasion mechanisms. In particular, a takeaway of Theorem 1 is that single information policies are without loss of generality for welfare analysis.

Remark 1. *Let's recall that type-t Receiver's expected utility is supermodular in informativeness, as orderd by $\preceq_t$, and effort. As a confirmation that supermodularity the key intuition for Theorem 1, in Section 17 of the Appendix, we prove the result assuming supermodular Receiver's interim payoff, as a function of informativeness and effort. This specification nests the original model where Receiver's ex-post utility is given by U_R.*

9.2 Characterization of the extensive margin

We characterize effort decisions assuming smoothness conditions on k.

Assumption 4 (Smooth Effort Cost)**.** *k is a differentiable convex function on $[0,1]$, and satisfies: $k'(1) > 1 - x_0$.*

Under Assumption 4, we denote by $k'(e)$ the derivative of k at effort e.

Lemma 13. *Let Assumption 4 hold, $I \in \mathcal{I}$, and e_t be any element of $E(\Delta I(\zeta_t), \lambda_t)$.*

$$\Delta I(\zeta_t) \leq \lambda_t k'(e_t),$$

with equality if $e_t > 0$.

Proof. Receiver maximizes a concave function over a compact set. A solution exists and, by differentiability of k, we can use standard Lagrangean arguments to show that it has the prescribed form, so long as optimal effort is in $[0,1)$. Let's see that the requirement that $k'(1) > 1 - x_0$ assumes away boundary solutions at 1. Because $I(1) = 1 - x_0$, we have $\Delta I(\zeta) \leq 1 - x_0 < k(1)$, for every $\zeta \in [0,1]$. ■

The marginal-cost-marginal-benefit analysis of Receiver's effort decision is depicted in Figure 9. Net informativeness ΔI defines a continuous function of Receiver's outside option, with a peak at the cutoff type that is equal to the prior mean x_0. The proof of this statement is in the Appendix (Lemma 54). The intuition for single-peakedness comes from the observation that the marginal benefit of effort is $\Delta I(\zeta)$, given I (Lemma 11). Type x_0 finds it valuable to observe any signal about θ, in order to make a more informed choice than if she is left at the prior. Extreme types have, instead, the least to gain from committing to observe a signal: the ex-ante probability that a signal realization modifies t's optimal action is low because only extreme realizations of the state are pivotal for their optimal action. If we intersect the marginal benefit of effort ΔI with $\lambda_t k'(0)$, we observe that, in general, intermediate types will exert a positive effort, and extreme types will not exert any effort. This result is depicted in figure 9, and implies that the set of Receiver types who become informed is defined by the two cutoff types who are just indifferent between exerting positive effort and not exerting any effort. Under Assumption 4, the cutoff types given information policy I and attention cost λ are:

$$\overline{c}^{\lambda}(\Delta I) := \min\{c \in [0,1] \mid \Delta I(c) \geq \lambda k'(0)\},$$

$$\overline{c}^{\lambda}(\Delta I) := \max\{c \in [0,1] \mid \Delta I(c) \geq \lambda k'(0)\},$$

with the requirement that if either of the two sets is empty, the relevant cutoff type is x_0. The next observation is that the interval shape of the extensive margin generalizes. By the supermodularity property of the Receiver's value function, type-t's optimal effort is

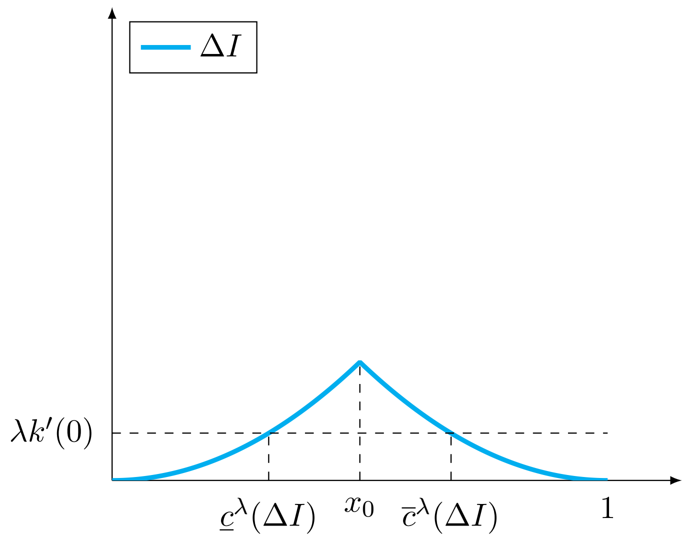


Figure 9: The set of types c who exert positive attention effort is an interval.

nondecreasing in $\Delta I(\zeta_t)$ without differentiability hypotheses.

Proposition 10. *Let I be an information policy, and $e(c, \lambda) \in E(\Delta I(c), \lambda)$ for every type (c, λ). Then, $e(\cdot, \lambda)$ is single-peaked, with a peak at x_0.*

Proof. We observe that $f : (e, \Delta I(\zeta_t)) \mapsto e\Delta I(\zeta_t)$ is strictly supermodular. Thus, by Lemma 11, we establish that every selection from the optimal effort correspondence is nondecreasing, using monotone comparative statics results (Lemma 57 in the Appendix). The result follows from single-peakedness of ΔI, with a peak at x_0, established in Lemma 54 in the Appendix. ■

By our result, every selection from the optimal effort correspondence exhibits cutoff outside-option types, given an attention cost type. In particular, extreme types — whose outside option is above or below the cutoffs — do not exert any effort.

9.3 Sender's Value of an Information Policy

We now express the Sender's problem as a maximization by choice of a feasible information policy, using the previous results on the extensive margin. First, we describe the extensive and intensive margins.

Lemma 14. *The Sender's maximization problem is given by:*

$$\sup -\int_T e(t)\Delta I'(\zeta_t^-)\,\mathrm{d}H(t) \tag{8}$$

$$s.t.\ I \in \mathcal{I} \text{ and } e(t) \in E(\Delta I(\zeta_t), \lambda_t) \text{ for all } t \in T. \tag{9}$$

Proof. Letting G be the marginal CDF of information cost consistent with H. Sender's value of I, given $e(\zeta,\lambda) \in E(\Delta I(\zeta),\lambda)$ for all $(\zeta,\lambda) \in T$, is:

$$\int_{[0,1]}\int_{[0,1]}(1 - I'(\zeta^-) - [\zeta \leq x_0])e(\zeta,\lambda)\,\mathrm{d}H(\zeta|\lambda) + (1 - H(x_0|\lambda))\,\mathrm{d}G(\lambda),$$

because type-t Receiver chooses action 1 when indifferent, by Sender-optimality. The Lemma follows after normalizing Sender's expected payoff from $I_{\overline{F}}$ to 0, and the observation that: $1 - [\zeta_t \leq x_0] = I'_{\overline{F}}(\zeta^-)$. ■

The Sender's value of the information policy I depends on I in two ways: the intensive and the extensive margin of Bayesian Persuasion. First, the probability that threshold type c chooses action 1 is the probability that the posterior mean is higher than her outside option c: $1-I'(c^-)$. We note that what matters for Sender is not the probability of action 1, but the degree to which the information policy changes the prior action decisions towards action 1. Thus, to $1-I'(c^-)$ we subtract $[c < x_0]$, and we note that: $1-I'(c^-)-[c < x_0] = -\Delta I'(\zeta^-)$. Second, a Receiver's type updates her beliefs with probability equals to her effort decision. The next result re-writes the Sender's problem in way to shows that Sender is effectively allocating information to Receiver's types, without smoothness assumptions.

Lemma 15. *Let H admit a PDF h that is decomposed as: $h(\zeta, \lambda) = h_{\zeta|\lambda}(\zeta|\lambda)h_\lambda(\lambda)$, and let the conditional PDF $h_{\zeta|\lambda}$ admit a derivative with respect to ζ, $h'_{\zeta|\lambda}$. The Sender's maximization problem is given by:*

$$\max_{I \in \mathcal{I}} - \int_{[0,1]} \int_{[0,1]} V(\Delta I(\zeta), \lambda) h'_{\zeta|\lambda}(\zeta|\lambda) h_\lambda(\lambda) \, \mathrm{d}\zeta \, \mathrm{d}\lambda. \tag{10}$$

Proof. See Section 18 in the Appendix. ■

Under the hypotheses of this Lemma, we define the Sender's value of an information policy I as $v(I)$, which is the maximand in the optimization above. And we say that an information policy I is optimal if it solves the maximization in 10. From the above result, we know that Sender prefers to allocate (Blackwell) informativeness to a Receiver's type (ζ, λ) so long as the measure induced by $h'_{\zeta|\lambda}(\zeta|\lambda)h_\lambda(\lambda)$ is positive, and he prefers to not allocate informativeness to types such that $h'_{\zeta|\lambda}(\zeta|\lambda)h_\lambda(\lambda)$ is negative. In the next section, we make use of this intuition to solve for the optimal signal in applied models.

Remark 2. *Shishkin (2023) uses a similar information-allocation intuition, in a model without the extensive margin because Receiver's attention is costless.*

10 Applications

10.1 Single-Peaked Distribution of Receiver's Outside Option

In applications, it's common to assume that the distribution of Receiver's outside option is single-peaked (Shishkin (2023); Gitmez and Molavi (2023), and also particular cases considered by Kolotilin (2018); Lipnowski et al. (2022b)).

Assumption 5 (Single-Peakedness of Outside Option Distribution)**.** *(1) The attention cost λ is independent of threshold c, and distributed according to the CDF H.*

(2) The distribution of the threshold ζ admits an absolutely continuous quasiconcave PDF f, with CDF F.

10.1.1 Optimality of Upper Censorships

Under Assumption 5, the two dimensions of Receiver's type are independently distributed and $h'_{\zeta|\lambda}(\zeta|\lambda)$ is nonpositive before a peak threshold type, and nonnegative after the peak (Lemma 15). Thus, it is optimal to reveal a lot of information through low posterior means, and not much information through high posterior means. There exists a class of signals that achieve this "information allocation," the class of upper-censorship signals. An upper-censorship signal implies full revelation conditional on the state being lower than a cutoff, and full censorship conditional on the state being above the cutoff (Figure 7). Since we work in the space of information policies, we define an upper censorship as an information policy which is induced by an upper-censorship signal.

Definition 4. *The $\theta^\star$ upper censorship is the unique information policy $I_{\theta^\star} \in \mathcal{I}$ such that:*

$$I_{\theta^\star}(x) = \begin{cases} I_{F_0}(x) & , x \in [0, \theta^\star] \\ I_{F_0}(\theta^\star) + (x - \theta^\star)F_0(\theta^\star) & , x \in (\theta^\star, x^\star] \\ I_{\overline{F}}(x) & , x \in (x^\star, \infty), \end{cases}$$

where $x^\star = \int_{\theta^\star}^{1} \theta \, \mathrm{d}\frac{F_0(\theta)}{1-F_0(\theta^\star)}$.

The next result shows that upper censorships are optimal information policies under independent cost and threshold types, whenever threshold types are single-peakedly distributed.

Theorem 2. *Let Assumption 5 hold. There exists a $\theta^\star \in \Theta$ such that the $\theta^\star$ upper censorship is an optimal information policy.*

Proof. Section 18.1 in the Appendix ■

A reading of this result is as a revelation principle result. In particular, in order to maximize Sender's payoff, Sender can focus on upper censorships under single-peakedness assumptions. A similar result can be proved using single-dipped distributions, where "lower censorships" arise as optimal signals. In light of Theorem 1, we know that the study of upper censorships informs us about properties of persuasion mechanisms. Theorem 1 nests many known results about the optimality of upper censorships in models without Receiver's private information, or effort cost. The next remark discusses uniqueness issues.

10.1.2 Welfare Analysis

Does Receiver benefit from her attention cost? In particular, Receiver's inattentiveness may act as a bargaining power: Sender is forced to increase the informativeness of his signal to induce Receiver to pay attention. This observation holds without Receiver's private information, as we establish in the Appendix (Section 18.3). In this section, we assume that k is linear: $k(e) = \kappa e$, $\kappa > 0$. In order to isolate the effect of each of the two dimensions of Receiver's private information, we ask whether Receiver is better off as her attention cost λ increases in two separate cases: (1) when Sender knows Receiver's outside option, and (2) when Sender knows Receiver's attention cost. In the first case, we study an increase of the distribution of the attention cost in a stochastic order.

The next result characterizes the optimal upper censorship with known action threshold.

Proposition 11. *Let the distribution of attention cost admit a log-concave CDF with support $[0, \bar{\kappa}]$, and a continuous PDF, with $\bar{\kappa} > 1 - x_0$, and the outside option be known to*

Sender. Moreover, let's assume that k is linear: $k(e) = \kappa e$. There exists a solution to the Sender's problem that is a θ upper censorship, where $\theta \in \{0, \theta^\star, \zeta\}$ and $\theta^\star$ solves:

$$(1 - F_0(\theta))(\zeta - \theta) = (\chi(\Delta I_\theta(\zeta)))^{-1}$$

Proof. Without loss of generality: $\kappa = 1$ and

$$K(\lambda) = \exp\Big(-\int_\lambda^{\bar\kappa} \chi(t)\,\mathrm{d}t\Big),$$

for some $\chi : (0, \bar\kappa) \to \mathbb{R}_+$ such that: $\int_\lambda^{\bar\kappa} \chi(t)\,\mathrm{d}t < \infty$ and $\lim_{\lambda\to 0} \int_\lambda^{\bar\kappa} \chi(t)\,\mathrm{d}t = \infty$. Our primitive is the nonincreasing reverse hazard rate χ. Without loss of optimality, Sender maximizes his payoff by choice of an upper censorship (see Section 18 of the Appendix). In particular, Sender's optimization is:

$$\max_{I \in \mathcal{I}^u} (1 - I'(\zeta^-))K(\Delta I(\zeta)),$$

where $\mathcal{I}^u \subseteq \mathcal{I}$ is the collection of upper censorships. Suppose there exists a solution $I \in \mathcal{I}^u$, such that $I = I_{\theta^\star}$, for some $\theta^\star \in (0, 1)$. By definition of I, at $y = I(\zeta)$ the next condition holds:

$$I_{F_0}(\theta^\star) + F_0(\theta^\star)(\zeta - \theta^\star) - y = 0.$$

By the implicit function theorem, there exists a differentiable function t:

$$t\colon (0,1) \to (0,1)$$
$$y \mapsto \theta^\star,$$

such that:

$$t'(y) = \begin{cases} \frac{1}{(\zeta - t(y))F_0'(t(y))} & , 0 < \zeta < t(y) \\ \frac{1}{F_0'(t(y))} & , 1 > \zeta \geq t(y). \end{cases}$$

Let the value of I_θ be:

$$\begin{aligned} v \colon (0,1) &\to [0,1] \\ \theta &\mapsto \big(1 - I_\theta'(\zeta^-)\big) K(\Delta I_\theta(\zeta)). \end{aligned}$$

Since $I'_{\theta^\star}(\zeta^-) = F_0(\theta^\star)$, v is differentiable in θ at $\theta^\star$. Let $\zeta > \theta^\star$. Using the chain rule, the derivative of v with respect to $I(\zeta)$ is nonnegative if:

$$(1 - F_0(\theta^\star))(\zeta - \theta^\star) \geq (\chi(\Delta I_{\theta^\star}(\zeta)))^{-1},$$

and nonpositive if:

$$(1 - F_0(\theta^\star))(\zeta - \theta^\star) \leq (\chi(\Delta I_{\theta^\star}(\zeta)))^{-1}.$$

The function $\theta \mapsto (1 - F_0(\theta))(\zeta - \theta)$ is decreasing on $(0, \zeta)$, and the function $\theta \mapsto \chi(\Delta I_\theta(\zeta))$ is decreasing on $(0, \zeta)$. As a result, finding the optimal $\theta^\star$ upper censorship is a concave program. ■

As a corollary to the above result, let full-information and no-information not be optimal information policies when the reverse hazard rate is χ_1 and when it is χ_2. The optimal upper censorship under χ_1 has a lower censorship point than the optimal upper censorship under χ_2 if $\chi_1(\lambda) \leq \chi_2(\lambda)$ for all λ. Thus, if attention cost stochastically increases (in the reverse hazard rate order), the optimal upper censorship is more Blackwell informative. This result is consistent with the symmetric-information special case of our model, where Sender knows Receiver's type.

Is it a robust feature of Bayesian persuasion that information costs increase Receiver's information? Let's consider the case of symmetric information (about Receiver's type, in particular). As pointed out by Wei (2021); Bloedel and Segal (2021) and Matysková and Montes (2023), the answer is no. In particular, under mutual-information cost the informativeness of Sender's signal is nonmonotone in the commonly known information-cost parameter. To see this, consider the following two extremes. If information is costless, Sender uses a partially informative signal, as established in the literature. If information is prohibitevely costly, only poorly informative signals induce Receiver to acquire some information. Our result uncovers a natural way to model larger cost in a stochastic sense to maintain symmetric-information comparative statics, by using the reverse-hazard rate dominance order.

Does Receiver benefit from small, known, attention cost? The next result shows that Receiver benefits from a small, known, cost when she is privately informed about her belief threshold for action.

Proposition 12. *Let Assumption 5 hold, attention cost be known to Sender, and f be strictly single-peaked. For a sufficiently small $\varepsilon > 0$:*

(1) There exists a unique optimal upper censorship when $\lambda = 0$: I°;

(2) There exists a unique optimal upper censorship when $\lambda = \varepsilon$: $I^\star$;

(3) $I^\star$ is more Blackwell informative than I°, that is: $I^\circ(x) \leq I^\star(x)$, $x \in \mathbb{R}_+$.

Proof. By Lemma 15, the derivative of the Sender's value of the $\overline{\theta}$ upper censorship with respect to the $\overline{\theta}$ is:

$$\partial F_0/\partial\theta(\overline{\theta})\int_{\overline{\theta}}^{\overline{c}^{\lambda}}(x-\overline{\theta})\partial h/\partial\zeta(x)\,\mathrm{d}x \leq \partial F_0/\partial\theta(\overline{\theta})\int_{\overline{\theta}}^{1}(x-\overline{\theta})\partial h/\partial\zeta(x)\,\mathrm{d}x,$$

where the inequality is strict if λ is sufficiently small, because h is decreasing on $(p, 1]$, $p < 1$. The right-hand side of the inequality is the derivative of the Sender's value of the $\overline{\theta}$ upper censorship with respect to the $\overline{\theta}$ when $\lambda = 0$. Because h is increasing on $[0, p)$ and on $(p, 1]$, $p < 1$, and ε is small, both sides of the above inequality are decreasing in $\overline{\theta}$. As a result, there exists a unique optimal upper censorship either when $\lambda = 0$, and when $\lambda = \varepsilon$.[50] ■

In Wei (2021), Receiver is better off with $\varepsilon > 0$ costs than with 0 costs, due to the "bargaining-power" idea described above. In **?**, Receiver is worse off with $\varepsilon > 0$ costs than with 0 costs, because Receiver becomes arbitrarily informed at almost 0 cost. So, the welfare analysis of attention cost is dependent on the information-cost model.

10.2 Media Censorship

Let's suppose a government wants people to stay home, and has control over the media. If media start to suggest to stay home, for instance by showing how bad a pandemic situation is, individuals may change their behavior only so long as they paid attention to the media. Thus the government must take into account that releasing information has two effects: information impacts behavior if individuals are attentive, a change in the intensive margin of persuasion; information determines attention decisions, a change in the extensive margin of persuasion. We introduce an advertising market á la Gehlbach and Sonin (2014) in the model of Kolotilin et al. (2022), thus providing a bridge between the two approaches to model media censorship.

[50] Uniqueness with costless information is readily established also using the tools from Kolotilin (2018); Kolotilin et al. (2022).

Assumption 6 (Media Censorship)**.** *We assume that:*

(1) Sender knows Receiver's attention-cost type λ*.*

(2) k *is linear, and Assumption 5 (Single-Peakedness of Outside Option) holds.*

(3) Sender's payoff is given by:

$$U_G(a, e; \cdot) = \psi a + \gamma e.$$

Part (1) is isomorphic to binary effort decision, which is the assumption in Gehlbach and Sonin (2014). Part (3) makes our model's Sender correspond to the government of the media censorship model of Gehlbach and Sonin (2014). Part (2) is made for tractability. ψ captures the mobilizing character of the government. A larger mobilizing character implies that Sender cares more about the support from the population of agents, where an agent is identified by a Receiver's type. γ captures the size of the media market. A larger market implies that Sender cares more about providing information, due to larger advertising revenues. If $\gamma = 0$, we know that an upper censorship is optimal, be previous results. Let's recall that an upper censorship leads to poorly informative large posterior means. Thus, because Sender with $\gamma > 0$ cares more about the extensive margin, it may be suboptimal to provide so little information as an upper censorship does. Let's define a class of information policies that nests upper censorships.

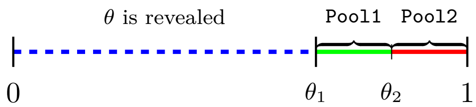


Figure 10: A bi-upper censorship is a signal that reveals low states and separates high from very high states.

Definition 5. *A bi-upper censorship is an information policy I such that, for $\theta_1, \theta_2 \in \Theta$:*

$$I_{\theta^\star}(x) = \begin{cases} I_{F_0}(x) & , x \in [0, \theta_1] \\ I_{F_0}(\theta_1) + (x - \theta_1)F_0(\theta_1) & , x \in (\theta_1, x_1] \\ I_{F_0}(x_1) + (x - x_1)F_0(x_1) & , x \in (x_1, x_2] \\ I_{\overline{F}}(x) & , x \in (x_2, \infty), \end{cases}$$

where $x_1 = \int_{\theta_1}^{\theta_2} \theta \, \mathrm{d} \frac{F_0(\theta)}{F_0(\theta_2) - F_0(\theta_1)}$, $x_2 = \int_{\theta_2}^{1} \theta \, \mathrm{d} \frac{F_0(\theta)}{1 - F_0(\theta_2)}$.

Proposition 13. *Let Assumption 6 hold, and the peak of the PDF of outside option be p, with $p \geq x_0$. There exists an optimal signal that is a bi-upper censorship.*

Proof. The definition of a bi-upper censorship and the proof are in Section 18.2. ■

We can interpret the assumption that the peak of the PDF of ζ is sufficiently large as a sufficiently high disagreement between Sender and the ex-ante Receiver (Shishkin (2023)). The current results nests the media-censorship result in Kolotilin et al. (2022), which shows that an upper censorship is an optimal signal without attention cost.

Part III

Appendix 1: Proofs for Essay 1

11 Preliminaries

In this section, we study the properties of payoffs over outcomes defined in Section 2, the outcome distribution discussed in Section 2.3, and the potential of $G(x_0)$. In Section 11.2.3, we extend the model to study a common-prior model. The analysis maintains Assumption 1.

11.1 Ex-Post Payoffs

In this section, we study the ex-post payoff functions. Player $i \in N = \{1, \ldots, n\}$ has preferences over outcome profiles $x \in \mathbf{R}^n$ that are represented by the *payoff* $u_i \colon \mathbf{R}^n \to \mathbf{R}$, which takes a quadratic-loss form:

$$\pi_i(x_i, x_{-i}) = -\left(x_i - (1-\alpha)\delta_i - \alpha \sum_{j \in N} \gamma^{ij} x_j \right)^2,$$

in which $\delta_i \in \mathbf{R}$, $\alpha \in [0, 1)$, $\gamma^{ij} \geq 0$, and $\gamma^{ii} = 0$.

We note that: $\pi_i(x_i, x_{-i}) = 2(1-\alpha)\delta_i x_i - x_i^2 + 2\alpha \sum_{j \in N} \gamma^{ij} x_i x_j + h_i(x_{-i})$, in which $h_i(x_{-i})$ is constant with respect to x_i. Player i's *effort-game payoff* is: $v_i \colon \mathbf{R}^n \to \mathbf{R}$, with

$$v_i(x_i, x_{-i}) = 2(1-\alpha)\delta_i x_i - x_i^2 + 2\alpha \sum_{j \in N} \gamma^{ij} x_i x_j.$$

We let $\boldsymbol{\delta}$ and $\boldsymbol{\Gamma}$ be, respectively, the column vector of favorite outcomes $(\delta_1, \ldots, \delta_n)^\mathsf{T}$ and the interactions matrix $[\gamma^{ij} : i, j \in N]$. We let $\boldsymbol{G} := \alpha\boldsymbol{\Gamma}$, $\boldsymbol{Q} := \boldsymbol{I} - \boldsymbol{G}$, $\boldsymbol{b} := (1-\alpha)\boldsymbol{\delta}$. We define $\boldsymbol{\beta} := \boldsymbol{Q}^{-1}\boldsymbol{b}$. $\mathbf{1}$ and $\boldsymbol{I}$ denote, respectively, a column vector of ones and the $n \times n$ identity matrix. For a matrix $\boldsymbol{A}$, we let a_{ij} be the entry in the ith row and jth column of $\boldsymbol{A}$, and $a_{i\bullet}$ be the column vector corresponding to the ith row of $\boldsymbol{A}$.

We let $\boldsymbol{x}$ be the column vector given by the outcome profile $(x_1, \ldots x_n)$. We define the *potential* $v \colon \mathbf{R}^n \to \mathbf{R}$, such that

$$v(\boldsymbol{x}) = 2(1-\alpha)\boldsymbol{\delta}^\mathsf{T}\boldsymbol{x} - \boldsymbol{x}^\mathsf{T}(\boldsymbol{I} - \alpha\boldsymbol{\Gamma})\boldsymbol{x}.$$

We note that: $v(\boldsymbol{x}) = -(\boldsymbol{x} - \boldsymbol{\beta})^\mathsf{T}\boldsymbol{Q}(\boldsymbol{x} - \boldsymbol{\beta}) + \boldsymbol{\beta}^\mathsf{T}\boldsymbol{Q}\boldsymbol{\beta}$. The *effort-game utilitarian welfare* is $\sum_{i \in N} v_i$, so that

$$\sum_{i \in N} v_i(\boldsymbol{x}) = 2(1-\alpha)\boldsymbol{\delta}^\mathsf{T}\boldsymbol{x} - \boldsymbol{x}^\mathsf{T}(\boldsymbol{I} - 2\alpha\boldsymbol{\Gamma})\boldsymbol{x}.$$

The following Lemma states that player i's payoff is best-response equivalent to the effort-game payoff and to the potential. In particular, we show that the three strategic-form games $(N, (\pi_i, \mathbf{R})_{i \in N})$, $(N, (v_i, \mathbf{R})_{i \in N})$ and $(N, (v, \mathbf{R})_{i \in N})$ are von Neumann–Morgenstern equivalent (Morris and Ui, 2004). We adopt the following notational conventions: $\boldsymbol{x}$ denotes (x_i, x_{-i}), and $-i := N \setminus \{i\}$, for all $i \in N$.

Lemma 16 (VNM Equivalence)**.** *For all $i \in N$, there exist $h_i, g_i \colon \mathbf{R}^{n-1} \to \mathbf{R}$ such that:*

$$u_i(\boldsymbol{x}) - v_i(\boldsymbol{x}) = h_i(x_{-i}) \text{ and } u_i(\boldsymbol{x}) - v(\boldsymbol{x}) = g_i(x_{-i}) \text{ for all } \boldsymbol{x} \in \mathbf{R}^n.$$

Proof. The second result is a consequence of symmetry of $\boldsymbol{\Gamma}$. In particular, we note that: $\sum_{(i,j) \in N^2} \gamma^{ij} x_i x_j -$

$2\sum_{j\in N}\gamma^{ij}x_i x_j$ is constant with respect to x_i, and:

$$\begin{aligned}
\pi_i(\boldsymbol{x}) - v_i(\boldsymbol{x}) &= -\left((1-\alpha)\delta_i + \alpha\sum_{j\in N}\gamma^{ij}x_j\right)^2,\\
v(\boldsymbol{x}) - v_i(\boldsymbol{x}) &= \sum_{j\in -i}\Big(2(1-\alpha)\delta_j x_j - x_j^2\Big) + \alpha\sum_{(i,j)\in N^2}\gamma^{ij}x_i x_j - 2\alpha\sum_{j\in N}\gamma^{ij}x_i x_j.
\end{aligned}$$

■

11.2 Interim Beliefs

In this section, we study player i's beliefs in the game $\mathcal{G}(\boldsymbol{p}_0)$, given $p_0^i \neq p_0^j$, for all $i, j \in N$ with $i \neq j$.

Every player knows the profile of status-quo policies $(p_0^1, \ldots, p_0^n) \in \mathbf{R}^n$. Player i privately knows the outcome corresponding to her own status quo policy: $\chi(p_0^i)$. Player i believes that the outcome function $\chi\colon \mathbf{R} \to \mathbf{R}$ is the realized path of a Brownian motion with drift $\mu < 0$, variance parameter $\omega > 0$ and starting point $(p_0^i, \chi(p_0^i))$. This belief structure is consistent with a common prior that is studied in section 11.2.3

11.2.1 Expectation and Covariance

We define $\mathbb{E}^i, \mathbb{V}^i, \mathbb{C}^i$ as, respectively, the conditional expectation, variance and covariance operators given knowledge of $\chi(p_0^i)$.

Lemma 17 (Interim expectation, variance, and covariance)**.** *The following formulas hold. For all policies $p, q \in \mathbf{R}$ we have:*

$$\begin{aligned}
\mathbb{E}^i\chi(p) &:= \mathbb{E}\Big[\chi(p) \mid \chi(p_0^i)\Big] = \chi(p_0^i) + \mu(p - p_0^i),\\
\mathbb{V}^i\chi(p) &:= \mathrm{Var}\Big[\chi(p) \mid \chi(p_0^i)\Big] = |p - p_0^i|\omega,\\
\mathbb{C}^i(\chi(p), \chi(q)) &:= \mathrm{Cov}\Big[\chi(p), \chi(q) \mid \chi(p_0^i)\Big]\\
&= \begin{cases} \min\{\mathbb{V}^i\chi(p), \mathbb{V}^i\chi(q)\} & \text{if } \operatorname{sgn}(p - p_0^i) = \operatorname{sgn}(q - p_0^i),\\ 0 & \text{if } p > p_0^i > q \text{ or } q > p_0^i > p. \end{cases}
\end{aligned}$$

Proof. The formulas for the expectation and the variance operators are known in the experimentation literature (Callander, 2011a). Let's show that the covariance formula is a consequence of the Markov property of beliefs. By the law of iterated expectations:

$$\begin{aligned}
\mathbb{C}^i(\chi(p), \chi(q)) =& \mathbb{E}\Big[\chi(p)\mathbb{E}\Big[\chi(q) \mid \chi(p), \chi(p_0^i)\Big] \mid \chi(p_0^i)\Big]\\
&- \mathbb{E}^i\chi(p)\mathbb{E}\Big[\mathbb{E}\Big[\chi(q) \mid \chi(p), \chi(p_0^i)\Big] \mid \chi(p_0^i)\Big].
\end{aligned}$$

Moreover, if $q \geq p \geq p_0^i$, then: $\mathbb{E}[\chi(q) \mid \chi(p), \chi(p_0^i)] = \mathbb{E}[\chi(q) \mid \chi(p)]$, by the Markov property, so the covariance expression simplifies to

$$\begin{aligned}\mathbb{C}^i(\chi(p),\chi(q)) =&\mathbb{E}\Big[\chi(p)\mathbb{E}[\chi(q)\mid\chi(p)]\mid\chi(p_0^i)\Big]-\mathbb{E}^i\chi(p)\mathbb{E}\Big[\mathbb{E}[\chi(q)\mid\chi(p)]\mid\chi(p_0^i)\Big]\\ =&\mathbb{E}\Big[\chi(p)(\chi(p)+\mu(q-p))\mid\chi(p_0^i)\Big]\\ &-\mathbb{E}^i\chi(p)\mathbb{E}\Big[\chi(p)+\mu(q-p)\mid\chi(p_0^i)\Big]\\ =&\mathbb{V}^i\chi(p),\end{aligned}$$

in which the second equality uses $\mathbb{E}[\chi(q)\mid\chi(p)]=\chi(p)+\mu(q-p)$. Instead, if $q>p_0^i>p$, then: $\mathbb{E}[\chi(q)\mid\chi(p),\chi(p_0^i)]=\mathbb{E}[\chi(q)\mid\chi(p_0^i)]$, by the Markov property, so the covariance expression simplifies to

$$\begin{aligned}\mathbb{C}^i(\chi(p),\chi(q)) &=\mathbb{E}\Big[\chi(p)\mathbb{E}\Big[\chi(q)\mid\chi(p_0^i)\Big]\mid\chi(p_0^i)\Big]-\mathbb{E}^i\chi(p)\mathbb{E}\Big[\chi(q)\mid\chi(p_0^i)\Big]\\ &=0.\end{aligned}$$

Thus, the covariance formula holds. ■

The Brownian motion structure implies that the conditional distribution of $\chi(p)$ and $\chi(q)$ given $\chi(p_0^i)$ is jointly Gaussian, for all $p,q\in\mathbf{R}\setminus\{p_0^i\}$. The CDF of $\chi(q)\mid\chi(p_0^i)$ is denoted by $F(\cdot,q|\chi(p_0^i),p_0^i)$. The next result states that beliefs are monotone in status-quo outcome and admit an invariance property.

Proof of Lemma 7

Lemma 18 (FOSD and Translation Invariance of beliefs.). *For all $y,y'\in\mathbf{R}$ such that $y\geq y'$, we have:*

$$F(\cdot,q|y,p_0^i)\leq F(\cdot,q|y',p_0^i)\quad\textit{pointwise for all } q,p_0^i\in\mathbf{R},$$

moreover: $F(x+\Delta,q|y+\Delta,p_0^i)=F(\cdot,q|y',p_0^i)$ for all $\Delta,x,y,q,p_0^i\in\mathbf{R}$.

Proof. Letting Φ be the CDF of a standard Gaussian random variable, we observe that $F(x',q|y',p_0^i)=\Phi\left(\frac{x'-y'-\mu(q-p_0^i)}{\sqrt{|q-p_0^i|\omega}}\right)$. ■

11.2.2 Derivatives of Variance and Covariance terms

We define the left and right derivatives of $\mathbb{V}^i\chi(p)$ and $\mathbb{C}^i(\chi(p),\chi(q))$ with respect to p, using Iverson brackets ($[Y]=1$ if Y is true, and $[Y]=0$ otherwise). First, let's observe that:

$$\mathbb{C}^i(\chi(p),\chi(q))=\begin{cases}(q-p_0^i)_+\omega & \text{if } q<p \text{ and } p\geq p_0^i,\\ (p-p_0^i)\omega & \text{if } p\leq q \text{ and } p\geq p_0^i,\\ (p_0^i-p)\omega & \text{if } q<p \text{ and } p<p_0^i,\\ (p_0^i-q)_+\omega & \text{if } p<q \text{ and } p<p_0^i,\end{cases}$$

from which it follows that:

$$\partial_{-}\mathbb{V}^i\chi(p) = \begin{cases} -\omega & p \leq p_0^i, \\ \omega & p > p_0^i, \end{cases} \qquad \partial_{+}\mathbb{V}^i\chi(p) = \begin{cases} -\omega & p < p_0^i, \\ \omega & p \geq p_0^i, \end{cases}$$

$$\partial_{-}\mathbb{C}^i(\chi(p),\chi(q)) = \begin{cases} [p \leq q]\omega & p > p_0^i, \\ -[p > q]\omega & p \leq p_0^i, \end{cases} \qquad \partial_{+}\mathbb{C}^i(\chi(p),\chi(q)) = \begin{cases} [p < q]\omega & p \geq p_0^i, \\ -[p \geq q]\omega & p < p_0^i. \end{cases}$$

In particular, we have that:

$$\begin{aligned} \partial\mathbb{C}^i(\chi(p),\chi(q)) &= \begin{cases} \partial_p(\min\{p,q\} - p_0^i)\omega & \text{if } p \geq p_0^i, \\ -\partial_p(\max\{p,q\} - p_0^i)\omega & \text{if } p < p_0^i. \end{cases} \\ &= \begin{cases} \left(\frac{1}{2} - \frac{1}{2}\partial_p|p-q|\right)\omega & \text{if } p \geq p_0^i, \\ \left(-\frac{1}{2} - \frac{1}{2}\partial_p|p-q|\right)\omega & \text{if } p < p_0^i. \end{cases} \\ &= \frac{1}{2}\Big(1 - 2[p < p_0^i] - \partial_p|p-q|\Big)\omega \\ \partial\mathbb{V}^i\chi(p) &= 1 - 2[p < p_0^i] \end{aligned}$$

Lemma 19 (Concavity of VCV)**.** *The function* $p_i \mapsto \sum_{(i,j)\in N^2} q^{ij} \operatorname{Cov}[\chi(p_i),\chi(p_j) \mid \chi(p_0^\star) = x_0^\star]$ *is convex on* $\mathbf{R}$ *for all* $i \in N$ *and* $p_0^\star \in \mathbf{R}$*, and*

$$\begin{aligned} g_i(p_i, p_{-i}) &:= \partial_{+p_i} \sum_{(i,j)\in N^2} q^{ij} \operatorname{Cov}[\chi(p_i),\chi(p_j) \mid \chi(p_0^\star) = x_0^\star] \\ &= \boldsymbol{q}_{i\bullet}^\top\mathbf{1} - 2\boldsymbol{q}_{i\bullet}^\top\mathbf{1}[p_i < p_0^i] + \alpha\sum_{j\in N} g_{ij}\partial_{+p_i}|p_i - p_j| \\ &= \boldsymbol{q}_{i\bullet}^\top\mathbf{1} - 2\boldsymbol{q}_{i\bullet}^\top\mathbf{1}[p_i < p_0^i] + \alpha\sum_{j\in N} g_{ij}([p_i \geq p_j] - [p_i < p_j]), \end{aligned}$$

and $g_i(p_i,\cdot)$ *is nonincreasing on* $\mathbf{R}^{n-1}$*. Moreover, the function* $\boldsymbol{p} \mapsto \sum_{(i,j)\in N^2} q^{ij} \operatorname{Cov}[\chi(p_i),\chi(p_j) \mid \chi(p_0^\star) = x_0^\star]^n$ *is convex on* $[p_0^\star, \overline{p}]^n$*.*

Proof. First, we show that the function $f\colon p_i \mapsto \sum_{(i,j)\in N^2} q^{ij} \operatorname{Cov}[\chi(p_i),\chi(p_j) \mid \chi(p_0^\star) = x_0^\star]$ is convex. By definition of $\boldsymbol{Q}$, we have that:

$$f(p_i) = \sum_{i\in N} \operatorname{Var}[\chi(p_i) \mid \chi(p_0^\star) = x_0^\star] - \sum_{(i,j)\in N^2} g_{ij} \operatorname{Cov}[\chi(p_i),\chi(p_j) \mid \chi(p_0^\star) = x_0^\star]$$

Thus, for all $i \in N$, assuming $\omega = 1$ without loss of generality, we have:

$$\begin{aligned} \partial_{+p_i} f(p_i) &= 1 - 2[p_i < p_0^i] - \alpha\sum_{j\in N} g_{ij}(-\partial_{+p_i}|p_i - p_j| - 2[p_i < p_0^i]) - \alpha\sum_{j\in N} g_{ij} \\ &= \boldsymbol{q}_{i\bullet}^\top\mathbf{1} - 2\boldsymbol{q}_{i\bullet}^\top\mathbf{1}[p_i < p_0^i] + \alpha\sum_{j\in N} g_{ij}\partial_{+p_i}|p_i - p_j|. \end{aligned}$$

Thus, $\partial_{+p_i} f$ is a nondecreasing function and so$f$ is convex on $\mathbf{R}$ (Rockafellar, 1970).

Let's show the second part of the lemma. Let's observe that:

$$\boldsymbol{p} \in [p_0^\star, \overline{p}]^n \implies f(p_i) = \sum_{(i,j)\in N^2} q^{ij} \min\{p_i - p_0^\star, p_j - p_0^\star\}\omega.$$

Joint convexity follows. ■

11.2.3 Common Prior

In this section, we define a common prior over the outcome function, parametrized by the amount of noise about the initial value of the Brownian motion. As the noise grows unboundedly large, the interim beliefs converge to the beliefs of the heterogeneous status quo game introduced in Section 5 and analyzed in Section 12.

Timeline Let's describe a timeline of a game. Every player knows the profile of status-quo policies $(p_0^1, \dots, p_0^n) \in \mathbf{R}^n$.

(1) Nature draws the initial value $X(0)$ from a normal distribution with mean 0 and variance $\sigma_0^2 \geq 0$.

(2) Nature draws the outcome function $X\colon \mathbf{R} \to \mathbf{R}$ from a Brownian motion with drift $\mu < 0$, variance parameter $\omega > 0$ and starting point $(0, X(0))$.

(3) Player i privately observes the realization of signal S_i about $X(0)$ and the outcome corresponding to her own status quo policy: $X(p_0^i)$.

After (3), players update their beliefs using Bayes' rule, and then simultaneously choose real-valued policies. i's payoff from the policy profile $\boldsymbol{p}$ is $u_i(X(p_1), \dots, X(p_n))$. We assume that $S_i = X(0) + \sigma\varepsilon_i$, for $\sigma \geq 0$ and a standard Gaussian random variable ε_i, and that for all pairs of players $i \neq k$, ε_i is independent from ε_k and from $X(0)$. To ease on notation, we assume that $\omega = 1$. In the limit as $\sigma_0 \to \infty$ and $\sigma \to \infty$, Bayes' rule for jointly Gaussian random variables gives us

$$\begin{aligned} \mathbb{E}[X(0) \mid I] &\to X(p_0^i), \\ \mathrm{Var}[X(0) \mid I] &\to p_0^i. \end{aligned}$$

To verify the second formula, let's observe that $X(p_0^i) - \mu p_0^i$ is an unbiased signal about $X(0)$, with precision $1/p_0^i$. In particular, for a Wiener process $W(\cdot)$, we have that:

$$X(p_0^i) - \mu p_0^i = X(0) + \omega(W(p_0^i) - W(0)),$$

and $W(p_0^i) - W(0)$ is Gaussian, centered at 0, with variance p_0^i. $W(p_0^i) - W(0)$ is independent of $X(0)$ and $(\varepsilon_i)_{i\in N}$ by our hypotheses.

Interim beliefs The information structure is parametrized by (σ_0, σ). In this section, we derive interim beliefs as a function of (σ_0, σ) and study the behavior as $(\sigma_0, \sigma) \to (\infty, \infty)$. Beliefs are described by Gaussian random variables, thus we study the expectation, variance and covariance terms of the outcomes $X(p), X(q)$ given the realization of $(S_i, X(p_0^i)) = I$, for $(p, q) \in \mathbf{R}^2$, with $q \leq p$. We claim that $\mathbb{E}[X(p) \mid I] \to \mathbb{E}\big[X(p) \mid X(p_0^i)\big]$ and $\mathrm{Cov}[X(p), X(q) \mid I] \to \mathrm{Cov}\big[X(p), X(q) \mid X(p_0^i)\big]$ for all $(p, q) \in \mathbf{R}^2$.

Case 1: $0 \leq p_0^i \leq q$. By the Markov property: $\mathbb{E}[X(p) \mid I] = \mathbb{E}\big[X(p) \mid X(p_0^i)\big]$, $\mathbb{E}[X(q) \mid I] = \mathbb{E}\big[X(q) \mid X(p_0^i)\big]$, and $\mathrm{Cov}[X(p), X(q) \mid I] = \mathrm{Cov}\big[X(p), X(q) \mid X(p_0^i)\big]$.

Case 2: $0 \le q \le p_0^i \le p$. By the Brownian bridge properties, $\mathbb{E}[X(q) \mid I] = \mathbb{E}\big[X(q) \mid X(p_0^i)\big]$, using $\mathbb{E}[X(0) \mid I] = X(p_0^i)$. By the Markov property: $\mathbb{E}[X(p) \mid I] = \mathbb{E}\big[X(p) \mid X(p_0^i)\big]$. By the law of iterated covariance:

$$\begin{aligned}\mathrm{Cov}[X(p), X(q) \mid I] = \mathbb{E}[&\mathrm{Cov}[X(p), X(q) \mid X(0), I]|I] \\ &+ \mathrm{Cov}[\mathbb{E}[X(p) \mid X(0), I], \mathbb{E}[X(q) \mid X(0), I] \mid I],\end{aligned}$$

By the Markov property, both terms on the right-hand side are 0.

Case 3: $0 \le q \le p \le p_0^i$. By the Brownian bridge properties, $\mathbb{E}[X(q) \mid I] \to \mathbb{E}\big[X(q) \mid X(p_0^i)\big]$, using the formula for $\mathbb{E}[X(0) \mid I]$. Similarly, we obtain that $\mathbb{E}[X(p) \mid I] \to \mathbb{E}\big[X(p) \mid X(p_0^i)\big]$. Towards using the law of iterated covariance, we observe that, by the Brownian bridge properties

$$\mathrm{Cov}\Big[X(p), X(q) \mid X(p_0^i), X(0)\Big] = \frac{(p_0^i - p)q}{p_0^i}.$$

Moreover, for a, b, c, d given by the Brownian bridge properties

$$\begin{aligned}\mathrm{Cov}\Big[\mathbb{E}\Big[X(p) \mid X(0), X(p_0^i)\Big], \mathbb{E}\Big[X(q) \mid X(0), X(p_0^i)\Big] \mid X(p_0^i), S_i\Big] = \\ \mathrm{Cov}\Big[aX(0) + bX(p_0^i), cX(0) + dX(p_0^i) \mid X(p_0^i), S_i\Big],\end{aligned}$$

from which it follows that:

$$\mathrm{Cov}\Big[\mathbb{E}\Big[X(p) \mid X(0), X(p_0^i)\Big], \mathbb{E}\Big[X(q) \mid X(0), X(p_0^i)\Big] \mid X(p_0^i), S_i\Big] = ab\,\mathrm{Var}[X(0) \mid I].$$

By the Brownian bridge properties $ab = \frac{p_0^i - p}{p_0^i}\frac{p_0^i - q}{p_0^i}$. Using the law of iterated covariance and the formula for $\mathrm{Var}[X(0) \mid I]$, we observe that

$$\begin{aligned}\mathrm{Cov}[X(p), X(q) \mid I] &\to \frac{(p_0^i - p)q}{p_0^i} + \frac{p_0^i - p}{p_0^i}(p_0^i - q) \\ &\to p_0^i - p.\end{aligned}$$

The remaining cases are dealt with similarly.

11.3 Potential

For a profile of policies $\boldsymbol{p} \in P$, we denote the corresponding column vector of outcomes as $\boldsymbol{\chi}(\boldsymbol{p})$, or $\boldsymbol{\chi}$ if the policy profile is unambiguous. In this section, we study the following function:

$$\begin{aligned}V(\cdot, x_0) \colon P &\to \mathbf{R} \\ \boldsymbol{p} &\mapsto \mathbb{E}\{v(\boldsymbol{\chi}(\boldsymbol{p}))|\chi(p_0) = x_0\},\end{aligned}$$

under the assumption that $P_i = [p_0, \overline{p}]$ for all $i \in N$, for given $p_0, x_0 \in \mathbf{R}$.

Definition 6. *Let $x_0 \in \mathbf{R}$. An element of $\arg\max_{\boldsymbol{p} \in [p_0, \overline{p}]^n} V(\boldsymbol{p}, x_0)$ is called the potential maximizer given x_0.*

It will be useful to study $f(\boldsymbol{p}, x_0) = -V(\boldsymbol{p}, x_0)$, and also to omit the dependence on x_0 when it leads to no confusion. Moreover, we let $\mathbb{E}\boldsymbol{\chi}(\boldsymbol{p}) = \mathbb{E}[\boldsymbol{\chi}(\boldsymbol{p})|\chi(p_0) = x_0]$.

Lemma 20. *$f\colon \boldsymbol{p} \to -V(\boldsymbol{p}, x_0)$ is a strictly convex function on $\mathbf{R}^n$, and*

$$f(\boldsymbol{p}) = (\mathbb{E}\boldsymbol{\chi}(\boldsymbol{p}) - \boldsymbol{\beta})^\top \boldsymbol{Q}(\mathbb{E}\boldsymbol{\chi}(\boldsymbol{p}) - \boldsymbol{\beta}) + \omega \boldsymbol{p}^\top \boldsymbol{Q}\mathbf{1} + \omega \sum_{(i,j)\in N^2} g_{ij}\frac{|p_i - p_j|}{2} - \boldsymbol{\beta}^\top \boldsymbol{Q}\boldsymbol{\beta}.$$

Proof. First, we observe that v is a quadratic function of the outcome profile. So, we have the next chain of equalities:

$$\begin{aligned}
V(\boldsymbol{p}) &= -(\mathbb{E}\boldsymbol{\chi}(\boldsymbol{p}) - \boldsymbol{\beta})^\top \boldsymbol{Q}(\mathbb{E}\boldsymbol{\chi}(\boldsymbol{p}) - \boldsymbol{\beta}) - \sum_{(i,j)\in N^2} q_{ij}(\min\{p_i, p_j\} - p_0)\omega + \boldsymbol{\beta}^\top \boldsymbol{Q}\boldsymbol{\beta} \\
&= -(\mathbb{E}\boldsymbol{\chi}(\boldsymbol{p}) - \boldsymbol{\beta})^\top \boldsymbol{Q}(\mathbb{E}\boldsymbol{\chi}(\boldsymbol{p}) - \boldsymbol{\beta}) - \sum_{(i,j)\in N^2} q_{ij}(p_i/2 + p_j/2)\omega + \\
&+ \sum_{(i,j)\in N^2} q_{ij}|p_i - p_j|\omega/2 + \boldsymbol{\beta}^\top \boldsymbol{Q}\boldsymbol{\beta} \\
&= -(\mathbb{E}\boldsymbol{\chi}(\boldsymbol{p}) - \boldsymbol{\beta})^\top \boldsymbol{Q}(\mathbb{E}\boldsymbol{\chi}(\boldsymbol{p}) - \boldsymbol{\beta}) + \\
&+ \sum_{i\in N}(1 - \boldsymbol{g}_{i\bullet}^\top \mathbf{1})p_i\omega - \sum_{(i,j)\in N^2} g_{ij}|p_i - p_j|\omega/2 + \boldsymbol{\beta}^\top \boldsymbol{Q}\boldsymbol{\beta}.
\end{aligned}$$

The second equality expresses $\min\{p_i, p_j\} = \frac{p_i + p_j - |p_i - p_j|}{2}$, and the third uses the definition of $\boldsymbol{Q}$. ■

Towards finding the potential maximizer, we find the subdifferential of f, and ∂ denotes the subdifferential operator with respect to the vector of policies $\boldsymbol{p}$. By the above Lemma, we have that:

$$\begin{aligned}
\partial f(\boldsymbol{p}) &= 2\mu \boldsymbol{Q}(\mathbb{E}\boldsymbol{\chi}(\boldsymbol{p}) - \boldsymbol{\beta}) + \boldsymbol{Q}\mathbf{1}\omega + \frac{\omega}{2}\partial \sum_{(i,j)\in N^2} g_{ij}|p_i - p_j| \\
\frac{\partial f(p)}{2\mu} &= \boldsymbol{Q}(\mathbb{E}\boldsymbol{\chi}(\boldsymbol{p}) - \boldsymbol{\beta} - \mathbf{1}k) - \frac{k}{2}\partial \sum_{(i,j)\in N^2} g_{ij}|p_i - p_j|.
\end{aligned}$$

The subdifferential of f is

$$\begin{aligned}
\partial f(\boldsymbol{p}) = \{\boldsymbol{y} \in \mathbf{R}^n : \frac{\boldsymbol{y}}{2\mu} = \boldsymbol{Q}(\mathbb{E}\boldsymbol{\chi}(\boldsymbol{p}) - \boldsymbol{\beta} - \mathbf{1}k) - (\boldsymbol{G} \odot \boldsymbol{A})\mathbf{1}k,\ &\text{for some } \boldsymbol{A} \text{ such that} \\
a_{ij} = -a_{ji}, p_i > p_j \implies a_{ij} = 1, p_i = p_j \implies &a_{ij} \in [-1, 1]\}.
\end{aligned}$$

Let $\mathbf{0}$ be a column of zeroes and $I_S\colon \mathbf{R}^n \to \mathbf{R}$ be the characteristic function of $S \subseteq \mathbf{R}^n$. By strict convexity of f and convexity of P, standard results in convex analysis (Rockafellar, 1970) imply that the potential maximizer is the unique $\boldsymbol{p} \in P$ such that:

$$\mathbf{0} \in \partial f(\boldsymbol{p}) + \partial I_P(\boldsymbol{p}).$$

Lemma 21. *There exists a unique potential maximizer given $x_0 \in \mathbf{R}$. Moreover, $\boldsymbol{p} \in (p_0, \overline{p})^n$ is the unique potential maximizer given $x_0 \in \mathbf{R}$ if, and only if:*

$$\mathbb{E}\boldsymbol{\chi}(\boldsymbol{p}) = \boldsymbol{\beta} + \mathbf{1}k + (\boldsymbol{I} - \boldsymbol{G})^{-1}(\boldsymbol{G} \odot \boldsymbol{A})\mathbf{1}k,$$

for some skew-symmetric $\boldsymbol{A} = [a_{ij} : i, j \in N]$ *such that:*

$$a_{ij} = 1 \text{ if } p_i > p_j, \text{ and } a_{ij} \in [-1, 1] \text{ if } p_i = p_j, \text{ for all } i, j \in N.$$

Proof. For interior $\boldsymbol{p}$, it is necessary and sufficient that $\mathbf{0} \in \partial f(\boldsymbol{p})$. The result follows from the preceding derivation. ■

12 Proofs for Section 5

12.1 General Model

In this section, we study the heterogeneous-status-quo game. We formulate it as a Bayesian game and study its Bayesian Nash equilibria. The definition of the Bayesian game and of Bayesian Nash equilibria are in terms of interim beliefs, and follow closely the respective definitions in Van Zandt and Vives (2007). The following definitions depend on a vector of status-quo policies $\boldsymbol{p}_0$ such that: $p_0^i \neq p_0^j$, for all players i, j with $i \neq j$. Thus, the heterogeneous status-quo game given $\boldsymbol{p}_0$ is $\mathcal{G}(\boldsymbol{p}_0)$. In this section, we maintain Assumption 3.

Components of the game

(1) The set of players is N.

(2) The type space of player i is $(\mathbf{R}, \mathcal{B})$, in which $\mathcal{B}$ is the Borel sigma-algebra; the typical type of player i is denoted by x_0^i.

(3) Player i's type-dependent beliefs are represented by an $n-1$-dimensional Gaussian random vector $(\chi(p_0^j))_{j \in -i}$ with expectation and variance-covariance that are functions of i's type. Let $j, k \in N \setminus \{i\}$, and x_0^i be i's type, then: the expectation and variance-covariance of $(\chi(p_0^j))_{j \in -i}$ are given, respectively, by $\mathbb{E}\Big[\chi(p_0^j) \mid \chi(p_0^i) = x_0^i\Big]$ and $\mathrm{Cov}\Big[\chi(p_0^j), \chi(p_0^k) \mid \chi(p_0^i) = x_0^i\Big]$, which are defined in Section 11.2. Let $f_i(\cdot|x_0^i) \colon \mathbf{R}^{n-1} \to \mathbf{R}$ be the PDF of the Gaussian random vector $(\chi(p_0^j))_{j \in -i}$ with mean and variance-covariance as above. We note that f_i is well-defined because $p_0^i \neq p_0^j$, for all players i, j with $i \neq j$. Thus, player i's type-dependent belief is such that: for every measurable $A \subseteq \mathbf{R}^{n-1}$ and type $x_0^i \in \mathbf{R}$, we have the following formula for the probability of A:

$$\mathbb{P}((\chi(p_0^j))_{j \in -i} \in A | x_0^i) = \int_A f(x_0^{-i} | x_0^i) \, \mathrm{d}x_0^{-i}.$$

In particular, let's define $p_i(x_0^i)$ as the probability measure on $\mathbf{R}^{n-1}$ induced by the set-valued mapping $A \mapsto \int_A f(x_0^{-i} | x_0^i) \, \mathrm{d}x_0^{-i}$. The function $x_0^i \mapsto p_i(x_0^i)$ gives player i's interim beliefs.

(4) The action set of player i is $P_i = [\underline{p}_i, \overline{p}_i]$, for $\underline{p}_i < \overline{p}_i$, and $\underline{p}_i, \overline{p}_i \in \mathbf{R}$; we let $P := \times_{i \in N} P_i$ and $P_{-i} := \times_{j \in -i} P_j$.

(5) The payoff of player i is $u_i \colon P \times \mathbf{R} \to \mathbf{R}$, such that:

$$u_i(\boldsymbol{p}, x_0^i) = \mathbb{E}\Big[\pi_i(\chi(p_1), \dots \chi(p_n)) \mid \chi(p_0^i) = x_0^i\Big].$$

Properties of the components of the game In this section, the superdifferential operator ∂ refers to differentiation with respect to i's policy p_i.

Lemma 22 (Best-Response Equivalence)**.** *For all $i \in N$, there exist $h_i, g_i \colon P_{-i} \times \mathbf{R} \to \mathbf{R}$ such that, letting $\boldsymbol{\chi} = (\chi(p_1), \ldots, \chi(p_n))$:*

$$\mathbb{E}\Big[\pi_i(\boldsymbol{\chi}) \mid \chi(p_0^i) = x_0^i\Big] - \mathbb{E}\Big[v_i(\boldsymbol{\chi}) \mid \chi(p_0^i) = x_0^i\Big] = h_i(p_{-i}, x_0^i)$$
$$\textit{and } \mathbb{E}\Big[\pi_i(\boldsymbol{\chi}) \mid \chi(p_0^i) = x_0^i\Big] - \mathbb{E}\Big[v(\boldsymbol{\chi}) \mid \chi(p_0^i) = x_0^i\Big] = g_i(p_{-i}, x_0^i) \textit{ for all } \boldsymbol{p} \in P, x_0^i \in \mathbf{R}.$$

Proof. Follows from VNM Equivalence established in Lemma 16. ∎

Proof of Lemma 8

Lemma 23 (Lemma 8)**.** *The function $u_i(\cdot, x_0^i)$ exhibits strictly increasing differences in (p_i, p_{-i}) for all $x_0^i \in \mathbf{R}$, and the function $u_i((\cdot, p_{-i}), \cdot)$ exhibits strictly increasing differences in (p_i, x_0^i) for all $p_{-i} \in P_{-i}$. Moreover, $u_i((\cdot, p_{-i}), x_0^i)$ is strictly concave.*

Proof. First, we establish strict concavity of $u_i((\cdot, p_{-i}), x_0^i)$. For a profile of policies of i's opponents p_{-i} and $x_0^i \in \mathbf{R}$, we study the function

$$\begin{aligned}\boldsymbol{p} \mapsto -(x_0^i\mathbf{1} + \mu(\boldsymbol{p} - p_0^i\mathbf{1}) - \boldsymbol{\beta})^\intercal \boldsymbol{Q}(x_0^i\mathbf{1} + \mu(\boldsymbol{p} - p_0^i\mathbf{1}) - \boldsymbol{\beta})\\ - \sum_{(i,j)\in N^2} q^{ij} \operatorname{Cov}[\chi(p_i), \chi(p_j) \mid \chi(p_0^i) = x_0^i]\end{aligned}$$

First, we observe that $\boldsymbol{p} \mapsto -(x_0^i\mathbf{1} + \mu(\boldsymbol{p} - p_0^i\mathbf{1}) - \boldsymbol{\beta})^\intercal \boldsymbol{Q}(x_0^i\mathbf{1} + \mu(\boldsymbol{p} - p_0^i\mathbf{1}) - \boldsymbol{\beta})$ is strictly concave on $\mathbf{R}^n$ because $\boldsymbol{Q}$ is positive definite. Strict concavity follows from previous results and Best-Response Equivalence.

Let's establish strictly increasing differences in (p_i, x_0^i). By absolute continuity of the concave function $u_i((\cdot, p_{-i}), x_0^i)$:

$$u_i((r_i, p_{-i}), x_0^i) - u_i((q_i, p_{-i}), x_0^i) = \int_{q_i}^{r_i} \partial_- u_i((p_i, p_{-i}), x_0^i)\, \mathrm{d}p_i.$$

By the formulas from Lemma 19

$$\begin{aligned}\partial_- u_i(p_i, p_{-i}, x_0^i) = -2\mu \boldsymbol{q}_{i\bullet}^\intercal(\mathbb{E}[\boldsymbol{\chi} \mid \chi(p_0^i) = x_0^i] - \boldsymbol{\beta})\\ -\boldsymbol{q}_{i\bullet}^\intercal\mathbf{1}\omega + 2\boldsymbol{q}_{i\bullet}^\intercal\mathbf{1}[p_i < p_0^i]\omega - \alpha\sum_{j\in N} g_{ij}\partial_{-p_i}|p_i - p_j|\omega.\end{aligned}$$

We observe that: (i) monotonicity of $F(\cdot, p_j; t_i, p_0^i)$ in i's own type (Lemma 18) and (ii) strict diagonal dominance of $\boldsymbol{Q}$ jointly imply that $\partial_- u_i(p_i, p_{-i}, x_0^i)$ is strictly increasing in x_0^i, thus the function $u_i((\cdot, p_{-i}), \cdot)$ has strictly increasing differences in (p_i, x_0^i) for all $p_{-i} \in P_{-i}$.

Similarly, we establish that the function $u_i(\cdot, x_0^i)$ has strictly increasing differences in (p_i, p_{-i}) for all $x_0^i \in \mathbf{R}$ by monotonicity of $\partial_- u_i(p_i, p_{-i}, x_0^i)$ with respect to p_{-i}, established in Lemma 19. ∎

Given the strategic complementarities established in Lemma 23, we draw on the toolset developed by the literature on incomplete-information games with complementarities to show that a greatest and a least equilibria exist and are in monotone strategies. Since payoffs in $\mathcal{G}(\boldsymbol{p}_0)$ are not necessarily bounded, we leverage strict concavity of expected payoffs in own action and compactness of action spaces to establish similar results to (Van Zandt and Vives, 2007).

Remark 3. *Let's observe that: "Assumption 1.", "Assumption 2.", "Assumption 3.", "Part (1) of Assumption 4.", and "Part (2) of Assumption 4." from Van Zandt and Vives (2007) hold. Assumption 1. holds because we endow the type space of player i, $\mathbf{R}$, with the usual order. Assumption 2. holds because P_i is a compact interval of the real line, and we endow P_i with the usual metric, so P_i is a lattice. Let's show that Assumption 3. holds by verifying that $x_0^i \to \int_A f(x_0^{-i}|x_0^i)\,\mathrm{d}x_0^{-i}$ is measurable. Measurability holds because f is a well-defined and a continuous real-valued function of x_0^i on $\mathbf{R}$. In particular, x_0^i enters f only through the expected value of $(\chi(p_0^j))_{j\in -i}$. $u_i(\boldsymbol{p},\cdot)$ is a real-valued continuous function on $\mathbf{R}$ for all $\boldsymbol{p}\in P$, and $u_i(\cdot,x_0^i)$ defines a real-valued continuous function on $\mathbf{R}^n$ by concavity of $u_i(\cdot,x_0^i)$; thus, parts (1) and (2) of Assumption 4. hold.*

Strategies and equilibrium A strategy for player i is a measurable function $\sigma_i\colon \mathbf{R}\to P_i$. Let Σ_i denote the set of strategies for player i. Let $\Sigma := \times_{i\in N}\Sigma_i$ denote the set of strategy profiles, and let $\Sigma_{-i} = \times_{i\in -i}\Sigma_j$ denote the set of profiles of strategies for players other than i. Σ_i is endowed with the pointwise order to be a lattice, Σ_{-i} and Σ are endowed with the product order and $\leq$ denotes every partial order

We use the following shorthand notation given a profile of strategies of i's opponents $\sigma_{-i} = (\dots,\sigma_{i-1},\sigma_{i+1},\dots)$:

$$\chi_{-i} = \chi(\sigma_{-i}) = (\dots,\chi(\sigma_{i-1}(\chi(p_0^{i-1}))),\chi(\sigma_{i+1}(\chi(p_0^{i+1}))),\dots)$$
$$(\chi_i,\chi_{-i}) = (\chi(p_i),\chi(\sigma_{-i})) = (\dots,\chi(\sigma_{i-1}(\chi(p_0^{i-1}))),\chi(p_i),\chi(\sigma_{i+1}(\chi(p_0^{i+1}))),\dots),$$

and $\boldsymbol{\chi}$ is the column vector of outcomes corresponding to (χ_i,χ_{-i}).

The expected payoff of player i, given σ_{-i}, is

$$U_i(p_i,x_0^i;\sigma_{-i}) := \mathbb{E}\{u_i(\chi(p_i),\chi(\sigma_{-i}))|\chi(p_0^i)=x_0^i\},\ x_0^i,p_i\in\mathbf{R}.$$

We use $U_i(p_i,x_0^i;\sigma_{-i},\boldsymbol{p}_0)$ when the particular status-quo policy profile is important; we note that $U_i(p_i,x_0^i;\sigma_{-i},\boldsymbol{p}_0)$ depends on p_0^j through $F(\cdot,p_0^j;x_0^i,p_0^i)$ if $j\neq i$. Let $\varphi_i(x_0^i;\sigma_{-i})$ be the set of policies that maximize $U_i(p_i,x_0^i;\sigma_{-i})$,

$$\varphi_i(x_0^i;\sigma_{-i}) := \arg\max_{p_i\in P_i} U_i(p_i,x_0^i;\sigma_{-i}).$$

Then, we have that $\sigma\in\Sigma$ is a Bayesian Nash equilibrium if, and only if:

$$\sigma_i(x_0^i)\in\varphi_i(x_0^i;\sigma_{-i}),\quad \text{for all } x_0^i\in\mathbf{R},\ i\in N.$$

Let $\beta_i\colon\Sigma_{-i}\to\Sigma_i$ denote player i's best-response correspondence

$$\beta_i(\sigma_{-i}) := \{\sigma_i\in\Sigma_i : \sigma_i(x_0^i)\in\varphi_i(x_0^i;\sigma_{-i}) \text{ for all } x_0^i\in\mathbf{R}\}.$$

Lemma 24. *The expected payoff to player i is, up to a term that is constant with respect to i's policy*

p_i:

$$
\begin{aligned}
U_i(p_i, x_0^i; \sigma_{-i}) = &- (\mathbb{E}[\boldsymbol{\chi} \mid \chi(p_0^i) = x_0^i] - \boldsymbol{\beta})^\top \boldsymbol{Q} (\mathbb{E}[\boldsymbol{\chi} \mid \chi(p_0^i) = x_0^i] - \boldsymbol{\beta}) \\
&- \mathbb{V}[\chi(p_i) \mid \chi(p_0^i) = x_0^i] \\
&- 2 \sum_{j \in -i} q^{ij} \int_{x_0^j \in \mathbf{R}} \operatorname{Cov}[\chi(p_i), \chi(s_j(x_0^j)) \mid \chi(p_0^i) = x_0^i] \, \mathrm{d}F(x_0^j, p_0^j; x_0^i, p_0^i).
\end{aligned}
$$

Moreover:

(1) $U_i(p_i, x_0^i; \sigma_{-i})$ *is strictly concave in* p_i.

(2) $U_i(p_i, x_0^i; \sigma_{-i})$ *exhibits strictly increasing differences in* (p_i, x_0^i) *if* σ_{-i} *is a profile of nondecreasing strategies.*

Proof. First, we establish strict concavity using a result in Radner (1962) ("Lemma", p. 863) and Lemma 23.

Let's establish strictly increasing differences in (p_i, x_0^i). By absolute continuity of the concave function $U_i(\cdot, x_0^i; \sigma_{-i})$, we have

$$
U_i(r_i, x_0^i; \sigma_{-i}) - U_i(q_i, x_0^i; \sigma_{-i}) = \int_{q_i}^{r_i} \partial_- U_i(p_i, x_0^i; \sigma_{-i}) \, \mathrm{d}p_i.
$$

We inspect monotonicity of $\partial_- U_i(p_i, x_0^i; \sigma_{-i})$ with respect to t_i, using the formulas in Lemma 23 and Lemma 19. Our proof is complete given: (i) monotonicity of $F(\cdot, p_0^j; x_0^i, p_0^i)$ in the sense of FOSD with respect to x_0^i (Lemma 18), and (ii) strict diagonal dominance of $\boldsymbol{Q}$. ■

Remark 4. *Item (2) in Lemma 24 implies that the Single Crossing Condition for games of incomplete information (Athey, 2001) is satisfied in $\mathcal{G}(\boldsymbol{p}_0)$. The reason is that strictly increasing differences imply the Milgrom-Shannon single-crossing property of incremental returns.*

The following result restricts the type spaces to compact sets.

Lemma 25 (Compact type spaces)**.** *For all i, there exist types $\underline{x}_0^i, \overline{x}_0^i \in \mathbf{R}$, such that:*

$$
\begin{aligned}
x_0^i > \overline{x}_0^i &\implies \varphi_i(x_0^i, \sigma_{-i}) = \{\overline{p}_i\}, \text{ for all } \sigma_{-i} \in \Sigma_{-i} \\
\text{and } x_0^i < \underline{x}_0^i &\implies \varphi_i(x_0^i, \sigma_{-i}) = \{\underline{p}_i\}, \text{ for all } \sigma_{-i} \in \Sigma_{-i}.
\end{aligned}
$$

Proof. We establish the first claim. Let $\underline{\sigma}_{-i}$ be the least element in Σ_{-i}, which is given by a profile of constant functions. Let $\overline{x}_0^i$ be such that: $\overline{p}_i \in \varphi_i(\overline{x}_0^i, \underline{\sigma}_{-i})$. $\overline{x}_0^i$ is well-defined by an application of Topkis' Theorem, because (i) $\varphi_i(\cdot, \underline{\sigma}_{-i})$ is nonempty-valued and continuous correspondence (by strict concavity of $U_i(p_i, x_0^i; \underline{\sigma}_{-i})$ as a function of p_i and Berge's Theorem, respectively), and (ii) $U_i(p_i, x_0^i; \underline{\sigma}_{-i})$ exhibits strictly increasing differences in p_i, x_0^i on $P_i \times \mathbf{R}$. $U_i(p_i, x_0^i; \underline{\sigma}_{-i})$ exhibits increasing differences in (p_i, x_0^i) (Lemma 24), thus $x_0^i > \overline{x}_0^i \implies \varphi_i(x_0^i, \underline{\sigma}_{-i}) = \{\overline{p}_i\}$. The first follows because $U_i(p_i, x_0^i; \sigma_{-i})$ exhibits increasing differences in (p_i, σ_{-i}). The second claim is established analogously. ■

Lemma 26 (Measurability of GBR)**.** *The mapping $x_0^i \to \sup \varphi_i(x_0^i; \sigma_{-i})$ is measurable.*

Proof. By strict concavity of $U_i(\cdot, x_0^i; \sigma_{-i})$, its maximizer on P_i exists and is unique, so $\sup \varphi_i(x_0^i; \sigma_{-i}) = \varphi_i(x_0^i; \sigma_{-i})$. $U_i(p_i, \cdot; \sigma_{-i})$ is continuous, so by Berge's maximum theorem the unique selection from $\varphi_i(\cdot; \sigma_{-i})$ is a real-valued continuous function on $\mathbf{R}$. The claim follows from Corollary 4.26 in Aliprantis and Border (2006). ■

Remark 5. *Lemma 26 admits a different proof that is similar to the apprach taken by Van Zandt and Vives (2007). Let's observe that $U_i(p_i, \cdot; \sigma_{-i})$ is a continuous real-valued function on $\mathbf{R}$ by Lemma 24. Let's observe that $U_i(\cdot; \sigma_{-i})$ is continuous in i's own policy, and measurable in i's own type. Thus, $U_i(\cdot; \sigma_{-i})$ is a Carathéodory function. Therefore, the Measurable Maximum Theorem (Aliprantis and Border (2006), Theorem 18.19) holds.*

If σ_i is a nondecreasing function, by Lemma 25 its generalized inverse σ_i^{-1} is well-defined:

$$\sigma_i^{-1}(p_i) = \inf\Big\{x_0^i \in \mathbf{R} \;:\; p_i \leq \sigma_i(x_0^i)\Big\}, \quad p_i \in P_i.$$

Moreover, if σ_i is nondecreasing, σ_i^{-1} is nondecreasing, left-continuous and admits a limit from the right at each point given Lemma 25. We define σ_i^- to be the generalized inverse of σ_i extended by continuity to be a correspondence:

$$\begin{aligned}\sigma_i^- &: P_i \rightrightarrows \mathbf{R}\\ &p_i \mapsto [\sigma_i^{-1}(p_i), \lim_{p_i' \to p_i^+} \sigma_i^{-1}(p_i')] =: [\sigma_{i1}^-(p_i), \sigma_{i2}^-(p_i)].\end{aligned}$$

Proof of Lemma 9 The result is a consequence of the following Lemma.

Lemma 27. *If σ is a Bayesian Nash equilibrium, the left and right derivatives of $U_i(p_i, x_0^i; \sigma_{-i})$ with respect to p_i and evaluated at $p_i = \sigma_i(x_0^i)$ are, respectively:*

$$\partial_- U_i(p_i, x_0^i; \sigma_{-i}) = \begin{cases} -2\mu \boldsymbol{q}_{i\bullet}^\top(\mathbb{E}[\boldsymbol{\chi} \mid \chi(p_0^i) = x_0^i] - \boldsymbol{\beta}) - \boldsymbol{q}_{i\bullet}^\top \mathbf{1}\omega - & \\ \sum_j g_{ij}[2F(\sigma_{j1}^{-1}(p_i), p_0^j; x_0^i, p_0^i) - 1]\omega & \text{if } p_i > p_0^i, \\ -2\mu \boldsymbol{q}_{i\bullet}^\top(\mathbb{E}[\boldsymbol{\chi} \mid \chi(p_0^i) = x_0^i] - \boldsymbol{\beta}) + \boldsymbol{q}_{i\bullet}^\top \mathbf{1}\omega - & \\ \sum_{j\in -i} g_{ij}[2F(\sigma_{j1}^{-1}(p_i), p_0^j; x_0^i, p_0^i) - 1]\omega & \text{if } p_i \leq p_0^i, \end{cases}$$

$$\partial_+ U_i(p_i, x_0^i; \sigma_{-i}) = \begin{cases} -2\mu \boldsymbol{q}_{i\bullet}^\top(\mathbb{E}[\boldsymbol{\chi} \mid \chi(p_0^i) = x_0^i] - \boldsymbol{\beta}) - \boldsymbol{q}_{i\bullet}^\top \mathbf{1}\omega - & \\ \sum_j g_{ij}[2F(\sigma_{j2}^{-1}(p_i), p_0^j; x_0^i, p_0^i) - 1]\omega & \text{if } p_i \geq p_0^i, \\ -2\mu \boldsymbol{q}_{i\bullet}^\top(\mathbb{E}[\boldsymbol{\chi} \mid \chi(p_0^i) = x_0^i] - \boldsymbol{\beta}) + \boldsymbol{q}_{i\bullet}^\top \mathbf{1}\omega - & \\ \sum_j g_{ij}[2F(\sigma_{j2}^{-1}(p_i), p_0^j; x_0^i, p_0^i) - 1]\omega & \text{if } p_i < p_0^i. \end{cases}$$

Proof. The result follows from Lemma 24 and the expression for the covariance in Section 11.2.2. ■

Lemma 28. *Let σ_{-i} be a profile of nondecreasing strategies of i's opponents. Then: $\varphi_i(\cdot; \sigma_{-i})$ is nonempty-valued, uniquely-valued, continuous and nondecreasing in the strong set order.*

Proof. $\varphi_i(\cdot; s_{-i})$ is nonempty-valued, uniquely-valued and continuous by Berge's Theorem, since: (i) P_i is nonempty and compact, and (ii) $U_i(\cdot, x_0^i; \sigma_{-i})$ is strictly concave (Lemma 24), and $U_i(p_i, x_0^i; \sigma_{-i})$ is a continuous function of x_0 (Lemma 24, noting that $U_i(p_i, x_0; s_{-i})$ is a strictly concave function of x_0).

$\varphi_i(\cdot; \sigma_{-i})$ is nondecreasing by Topkis' Theorem (Topkis (1978), Theorem 6.3), because $U_i(p_i, x_0^i; \sigma_{-i})$ exhibits strictly increasing differences in (p_i, x_0^i) (Lemma 24). ■

Lemma 29. *The strategy profile of nondecreasing strategies σ is a Bayesian Nash equilibrium if, and* *only if, the following conditions are satisfied for all $i \in N$, $x_0^i \in \mathbf{R}$.*

$$k \sum_{j \in N} g_{ij}[2F(\sigma_{j2}^{-1}(\sigma_i(x_0^i)), p_0^j; x_0^i, p_0^i) - 1] \geq \boldsymbol{q}_{i\bullet}^\mathsf{T}(\mathbb{E}[\boldsymbol{\chi} \mid \chi(p_0^i) = x_0^i] - \boldsymbol{\beta} - \mathbf{1}k)$$
$$\geq k \sum_{j \in N} g_{ij}[2F(\sigma_{j1}^{-1}(\sigma_i(x_0^i)), p_0^j; x_0^i, p_0^i) - 1] \text{ if } \sigma_i(x_0^i) > p_0^i,$$
$$k \sum_{j \in N} g_{ij}[2F(\sigma_{j2}^{-1}(\sigma_i(x_0^i)), p_0^j; x_0^i, p_0^i) - 1] \geq \boldsymbol{q}_{i\bullet}^\mathsf{T}(\mathbb{E}[\boldsymbol{\chi} \mid \chi(p_0^i) = x_0^i] - \boldsymbol{\beta} + \mathbf{1}k)$$
$$\geq k \sum_{j \in N} g_{ij}[2F(\sigma_{j1}^{-1}(\sigma_i(x_0^i)), p_0^j; x_0^i, p_0^i) - 1] \text{ if } \sigma_i(x_0^i) < p_0^i,$$
$$k\boldsymbol{q}_{i\bullet}^\mathsf{T}\mathbf{1} + k \sum_{j \in N} g_{ij}[2F(\sigma_{j2}^{-1}(\sigma_i(x_0^i)), p_0^j; x_0^i, p_0^i) - 1] \geq \boldsymbol{q}_{i\bullet}^\mathsf{T}(\mathbb{E}[\boldsymbol{\chi} \mid \chi(p_0^i) = x_0^i] - \boldsymbol{\beta})$$
$$\geq -k\boldsymbol{q}_{i\bullet}^\mathsf{T}\mathbf{1} + k \sum_{j \in N} g_{ij}[2F(\sigma_{j1}^{-1}(\sigma_i(x_0^i)), p_0^j; x_0^i, p_0^i) - 1] \text{ if } \sigma_i(x_0^i) = p_0^i.$$

Proof of Lemma 10

Proof. The result is a consequence of the above Lemma. Assuming $\underline{p}_i = p_0^i$, the strategy profile of nondecreasing strategies σ is a Bayesian Nash equilibrium if, and only if, the following condition is satisfied. For all $i \in N$ and $x_0^i \in \mathbf{R}$ such that $\sigma_i(x_0^i) > p_0^i$, there exists a matrix $\boldsymbol{A} = [a_{ij}]$, such that:

$$\mathbb{E}[\boldsymbol{\chi} \mid \chi(p_0^i) = x_0^i] = \boldsymbol{\beta} + \mathbf{1}k + \boldsymbol{Q}^{-1}\boldsymbol{G} \odot \boldsymbol{A}\mathbf{1}k,$$

and $a_{ij} \in [2F(\sigma_{j1}^{-1}(\sigma_i(x_0^i)), p_0^j; x_0^i, p_0^i) - 1, 2F(\sigma_{j2}^{-1}(\sigma_i(x_0^i)), p_0^j; x_0^i, p_0^i) - 1]$. ∎

Existence of Bayesian Nash equilibria

Lemma 30 (Properties of GBR mapping)**.** *The following hold.*

(1) $\beta_i(\sigma_{-i})$ has a greatest element, which we call $\overline{\beta}_i(\sigma_{-i})$, for all $\sigma_{-i} \in \Sigma_{-i}$.

(2) For $\sigma'_{-i}, \sigma_{-i} \in \Sigma_{-i}$ such that $\sigma'_{-i} \geq \sigma_{-i}$, we have that $\beta_i(\sigma'_{-i}) \geq \beta_i(\sigma_{-i})$.

(3) If the strategies in σ_{-i} are nondecreasing, then the unique strategy given by $\overline{\beta}_i(\sigma_{-i})$ is nondecreasing (in i's type).

Proof. $U_i(p_i, x_0^i; \sigma_{-i})$ is continuous as a function of p_i and has increasing differences in p_i, σ_{-i} because increasing differences are preserved by integration. Thus, by "Lemma 7" in Van Zandt and Vives (2007), $\varphi_i(x_0^i; \sigma_{-i})$ is a nonempty complete lattice, and (2) holds.

(3) is established in Lemma 28.

(1) is a consequence of Lemma 26. ∎

Proof of Proposition 8

Lemma 31 (Proposition 8)**.** *There exist a greatest and a least Bayesian Nash equilibrium, and they are in nondecreasing strategies.*

Proof. $u_i(\cdot, x_0^i)$ is a continuous real-valued function on the compact set P, so $u_i(\cdot, x_0^i)$ is bounded. Given Lemma 30, the proof follows from the same argument as that of "Lemma 6" in Van Zandt and Vives (2007). ∎

Proof of Proposition 9 The result is a consequence of the following result, which upper bounds the distance between two equilibrium strategies of any player, in the sense of the sup norm.

Lemma 32. *If $\overline{\sigma}_i(x_0^i) - \underline{\sigma}_i(x_0^i) > c$, for $i \in N, x_0^i \in \mathbf{R}, c > 0$, then:*

$$\omega > \mu^2 \frac{1}{\frac{\alpha \sum_j g_{ij}}{1-\alpha \sum_j g_{ij}}} c$$

Equivalently, if $\omega \leq v$, then: $\max_{i\in N} |\overline{\sigma}_i - \underline{\sigma}_i| \leq v \frac{\alpha \sum_j g_{ij}}{1-\alpha \sum_j g_{ij}} / (\mu^2)$.

Proof. Let $\overline{\sigma}, \underline{\sigma} \in \Sigma$ be, respectively, the greatest and least Bayesian Nash equilibria, and suppose that they are distinct elements of Σ. Let's take $i \in N$ be such that: $i \in \arg\max_{i' \in N} \max_{x_0^{i'} \in \mathbf{R}} \overline{\sigma}_{i'}(x_0^{i'}) - \underline{\sigma}_{i'}(x_0^{i'})$. First, we verify that i is well defined. By hypothesis, $\overline{\sigma}_{i'} \geq \underline{\sigma}_{i'}$ pointwise. Thus, $x_0^{i'} \mapsto \overline{\sigma}_{i'}(x_0^{i'}) - \underline{\sigma}_{i'}(x_0^{i'})$ is bounded below pointwise by a constant function that takes value 0, and bounded above pointwise by a constant function that takes value $\max_{j\in N} \overline{p}_j - \underline{p}_j > 0$. It follows that $\sup\{\overline{\sigma}_{i'}(x_0^{i'}) - \underline{\sigma}_{i'}(x_0^{i'}) : x_0^{i'} \in [\underline{x}_0^{i'}, \overline{x}_0^{i'}]\}$ is well defined, and $\sup\{\overline{\sigma}_{i'}(x_0^{i'}) - \underline{\sigma}_{i'}(x_0^{i'}) : x_0^{i'} \in [\underline{x}_0^{i'}, \overline{x}_0^{i'}]\} = \max\{\overline{\sigma}_{i'}(x_0^{i'}) - \underline{\sigma}_{i'}(x_0^{i'}) : x_0^{i'} \in [\underline{x}_0^{i'}, \overline{x}_0^{i'}]\}$ because $\underline{\sigma}_{i'}, \overline{\sigma}_{i'}$ are continuous by Berge's Theorem (Lemma 28). By Lemma 25 result, $\max\{\overline{\sigma}_{i'}(x_0^{i'}) - \underline{\sigma}_{i'}(x_0^{i'}) : x_0^{i'} \in [\underline{x}_0^{i'}, \overline{x}_0^{i'}]\} \geq \max\{\overline{\sigma}_{i'}(x_0^{i'}) - \underline{\sigma}_{i'}(x_0^{i'}) : x_0^{i'} \notin [\underline{x}_0^{i'}, \overline{x}_0^{i'}]\} = \{0\}$. It follows that $\arg\max_{x_0^{i'} \in \mathbf{R}} \overline{\sigma}_{i'}(x_0^{i'}) - \underline{\sigma}_{i'}(x_0^{i'}) \subseteq [\underline{x}_0^{i'}, \overline{x}_0^{i'}]$. Thus, $\max_{i'\in N} \max_{x_0^{i'} \in \mathbf{R}} \overline{\sigma}_{i'}(x_0^{i'}) - \underline{\sigma}_{i'}(x_0^{i'})$ has a solution. It follows that i is well defined.

Let $y_{i'} \in \arg\max_{x_0^{i'} \in \mathbf{R}} \overline{\sigma}_{i'}(x_0^{i'}) - \underline{\sigma}_{i'}(x_0^{i'})$, for all $i' \in N$. The problem $\max_{i'\in N} \overline{\sigma}_{i'}(y_{i'}) - \underline{\sigma}_{i'}(y_{i'})$ has a solution, which we denote by j, and we define $t := y_j$. By definition of $y_{i'}, i' \in N$, we have that $\max_{i'\in N} \overline{\sigma}_{i'}(y_{i'}) - \underline{\sigma}_{i'}(y_{i'}) \geq \max_{i'\in N} \max_{x_0^{i'} \in \mathbf{R}} \overline{\sigma}_{i'}(x_0^{i'}) - \underline{\sigma}_{i'}(x_0^{i'})$. Therefore, $i = j$ and t is the type (of player i) for which $\overline{\sigma}_j(x_0^j) - \underline{\sigma}_j(x_0^j)$ is maximized across players (j) and types (x_0^j). By the definition of Bayesian Nash equilibrium, we have

$$\partial_{+p_i} U_i(\overline{\sigma}_i(t), t; \overline{\sigma}_{-i}) \geq 0 \text{ and } \partial_{-p_i} U_i(\underline{\sigma}_i(t), t; \underline{\sigma}_{-i}) \leq 0.$$

Therefore:

$$\partial_{+p_i} U_i(\overline{\sigma}_i(t), t; \overline{\sigma}_{-i}) - \partial_{-p_i} U_i(\underline{\sigma}_i(t), t; \underline{\sigma}_{-i}) \geq 0.$$

Let's verify that:

$$A := -2\mu(\mathbb{E}\Big[\chi(\overline{\sigma}_i(t))|\chi(p_0^i) = t\Big] - \mathbb{E}\Big[\chi(\underline{\sigma}_i(t))|\chi(p_0^i) = t\Big]$$
$$- \sum_j g_{ij}\mathbb{E}\Big[\chi(\overline{\sigma}_j(\chi(p_0^j)))|\chi(p_0^i) = t\Big] - g_{ij}\mathbb{E}\Big[\chi(\underline{\sigma}_j(\chi(p_0^j)))|\chi(p_0^i) = t\Big]) < -2\mu^2 c \boldsymbol{q}_{i\bullet}^\top \mathbf{1}.$$

The claim follows from the next inequality,

$$\begin{aligned} A &= -2\mu^2(\overline{\sigma}_i(t) - \underline{\sigma}_i(t)) + 2\mu^2 \sum_j g_{ij}\mathbb{E}\Big[\chi(\overline{\sigma}_j(\chi(p_0^j)))|\chi(p_0^i) = t\Big] - \mathbb{E}\Big[\chi(\underline{\sigma}_j(\chi(p_0^j)))|\chi(p_0^i) = t\Big] \\ &\leq -2\mu^2(\overline{\sigma}_i(t) - \underline{\sigma}_i(t)) \boldsymbol{q}_{i\bullet}^\top \mathbf{1}, \end{aligned}$$

which holds by definition of i and t.

We have that:

$$\partial_{+p_i} U_i(\overline{\sigma}_i(t), t; \overline{\sigma}_{-i}) - \partial_{-p_i} U_i(\underline{\sigma}_i(t), t; \underline{\sigma}_{-i}) =$$
$$A + B - [\overline{\sigma}_i(t) > p_0^i] 2\boldsymbol{q}_{i\bullet}^\top \mathbf{1}\omega + [\underline{\sigma}_i(t) > p_0^i] 2\boldsymbol{q}_{i\bullet}^\top \mathbf{1}\omega.$$

With:

$$B := -2\omega \sum_{j \in N} g_{ij} F(\overline{\sigma}_{j2}^{-1}(\overline{\sigma}_i(t)), p_0^j; t, p_0^i) - g_{ij} F(\underline{\sigma}_{j1}^{-1}(\underline{\sigma}_i(t)), p_0^j; t, p_0^i) \in [-2\omega(1 - \boldsymbol{q}_{i\bullet}^\top \mathbf{1}), 2\omega(1 - \boldsymbol{q}_{i\bullet}^\top \mathbf{1})]$$

Then:

$$\begin{aligned} A + B - [\overline{\sigma}_i(t) > p_0^i] 2\boldsymbol{q}_{i\bullet}^\top \mathbf{1}\omega + [\underline{\sigma}_i(t) > p_0^i] 2\boldsymbol{q}_{i\bullet}^\top \mathbf{1}\omega &> 0 \\ B &> -A \\ 2\omega(1 - \boldsymbol{q}_{i\bullet}^\top \mathbf{1}) &> 2\mu^2 c \boldsymbol{q}_{i\bullet}^\top \mathbf{1} \\ \omega \frac{\alpha \sum_j g_{ij}}{1 - \alpha \sum_j g_{ij}} &> \mu^2 c \end{aligned}$$

■

12.2 Finite Policy Spaces

12.2.1 Auxiliary results

The expected payoff of player i given symmetric information, σ_{-i}, and a profile of status quo outcomes $(x_0^1, \ldots, x_0^n)^\top = \boldsymbol{x}_0 \in \mathbf{R}^n$ is

$$U_i(p_i, \boldsymbol{x}_0; \sigma_{-i}) := \mathbb{E}\{u_i(\chi(p_i), \chi(\sigma_{-i})) | \chi(p_0^1) = x_0^1, \ldots, \chi(p_0^n) = x_0^n\},$$

for all $p_i \in \mathbf{R}$. We use $U_i(p_i, \boldsymbol{x}_0; \sigma_{-i}, \boldsymbol{p}_0)$ when the status-quo policy profile is important.

We derive a second expression for the right and left derivatives of expected payoffs, based on v_i. For given policy p and nondecreasing strategy s_j:

$$\mathbb{C}^i(\chi(p), \chi(s_j)) = \begin{cases} \omega \int_{\left(-\infty, s_{j1}^-(p)\right)} s_j(x_0^j) - p_0^i \, \mathrm{d}F^i(x_0^j) + \omega \Big[1 - F^i(s_{j1}^-(p))\Big](p - p_0^i) & , p > p_0^i, \\ 0 & , p = p_0^i, \\ \omega F^i(s_{j2}^-(p))(p - p_0^i) - \omega \int_{\left(s_{j2}^-(p), \infty\right)} s_j(x_0^j) - p_0^i \, \mathrm{d}F^i(x_0^j) & , p < p_0^i. \end{cases}$$

Thus, we have

$$\partial \mathbb{C}^i(\chi(p_i), \chi(s_j)) = \begin{cases} \omega \Big[1 - F^i(s_j^-(p^i))\Big] & , p^i > p_0^i, \\ \Big[-\omega F^i(s_{j2}^-(p_0^i)), \omega - \omega F^i(s_{j1}^-(p_0^i))\Big] & , p^i = p_0^i \\ -\omega F^i(s_j^-(p^i)) & , p^i < p_0^i. \end{cases}$$

We express the left and right derivatives of the conditional expected payoff at $p_i \neq p_0^i$ as follows.

$$\partial_- U_i(p_i, x_0^i; s_{-i}) \propto \mathbb{E}^i\chi(p_i) - \delta_i - \alpha \sum_j \gamma^{ij}\mathbb{E}^i\chi(s_j) - \frac{1}{-2\mu}\frac{\partial}{\partial p_i}\mathbb{V}^i\chi(p_i) + \\ +2\alpha\frac{1}{-2\mu}\sum_j \gamma^{ij}\partial_-\mathbb{C}^i(\chi(p_i), \chi(s_j))$$

$$\partial_+ U_i(p_i, x_0^i; s_{-i}) \propto \mathbb{E}^i\chi(p_i) - \delta_i - \alpha \sum_j \gamma^{ij}\mathbb{E}^i\chi(s_j) - \frac{1}{-2\mu}\frac{\partial}{\partial p_i}\mathbb{V}^i\chi(p_i) + \\ +2\alpha\frac{1}{-2\mu}\sum_j \gamma^{ij}\partial_+\mathbb{C}^i(\chi(p_i), \chi(s_j)),$$

in which -2μ is the proportionality constant.

Lemma 33 (Continuity)**.** *Let s be a strategy profile and $x_0 := (x_0^1, \dots, x_0^n)$ be the profile of status-quo outcomes corresponding to status-quo policies $p_0 := (p_0^1, \dots, p_0^n)$. Then:*

(1) $\overline{U}_i(p_i, x_0; s_{-i}, p_0)$ is a continuous function of $(\dots, s_{i-1}(x_0^{i-1}), p_i, s_{i+1}(x_0^{i+1}), \dots)$.

(2) If $\dots < p_0^{\ell-1} < p_0^\ell < p_0^{\ell+1} < \dots$, then: $\overline{U}_i(p_i, x_0; s_{-i}, p_0)$ is a continuous function of p_0^ℓ on $(p_0^{\ell-1}, p_0^{\ell+1})$, $\ell \in N$.

Proof. We prove (1) first. We have:

$$\overline{U}_i(p_i, x_0; s_{-i}, p_0) = \int \cdots \int u_i(\dots, \chi(s_{i-1}(x_0^{i-1})), \chi(p_i), \chi(s_{i+1}(x_0^{i+1})), \dots) \\ \mathrm{d}m(\dots, \chi(s_{i-1}(x_0^{i-1})), \chi(p_i), \chi(s_{i+1}(x_0^{i+1})), \dots),$$

Where m is the distribution of a random vector that we describe in what follows. Because u_i is quadratic, the mean vector and the variance-covariance matrix of the random vector described by G determine $\overline{U}_i(p_i, x_0; s_{-i}, p_0)$. Thus, we prove (1) by means of the next two claims:

$\mathbb{E}[\chi(q)|\chi(p_0^1) = x_0^1, \dots, \chi(p_0^n) = x_0^n]$ *is a continuous function of q.* By the properties of Brownian bridges:

$$\mathbb{E}\Big[\chi(q)|\chi(p_0^1) = x_0^1, \dots, \chi(p_0^N) = x_0^n\Big] = \\ \begin{cases} \chi(p_1) + \frac{\chi(p_2)-\chi(p_1)}{p_2-p_1}(q-p_1) & p_1 \leq q \leq p_2, p_1 = \max\{p_0^i : p_0^i \leq q\}, p_2 = \min\{p_0^i : p_0^i \geq q\} \\ \chi(\max p_0) + \mu(q - \max p_0) & q \geq \max p_0 \\ \chi(\min p_0) + \mu(q - \min p_0) & q \leq \min p_0 \end{cases}$$

$\mathrm{Cov}[\chi(q), \chi(q')|\chi(p_0^1) = x_0^1, \dots, \chi(p_0^n) = x_0^n]$ *is a continuous function of q, q'.* Let $q \leq q'$:

$$\mathbb{C}\mathrm{ov}\Big[\chi(q), \chi(q')|\chi(p_0^1) = x_0^1, \dots, \chi(p_0^n) = x_0^n\Big] = \\ \begin{cases} \omega\frac{(p_2-q')(q-p_1)}{p_2-p_1} & p_1 \leq q \leq p_2, p_1 = \max\{p_0^i : p_0^i \leq q\}, p_2 = \min\{p_0^i : p_0^i \geq q\} \\ \mathbb{C}\mathrm{ov}[\chi(q'), \chi(q) \mid \chi(\max p_0)] & q' \geq q \geq \max p_0 \\ \mathbb{C}\mathrm{ov}[\chi(q'), \chi(q) \mid \chi(\min p_0)] & q \leq q' \leq \min p_0 \\ 0 & \text{else.} \end{cases}$$

Let's establish (2). Let $p_0^1 < p_0^2 < \dots$. The expressions above show that mean and covariance terms of the pair of random variables $\chi(q), \chi(q') \mid \chi(p_0^1), \dots, \chi(p_0^n)$ are locally continuous in $p_0^1, \dots, p_0^n$. ■

12.2.2 Definitions and Assumptions

We consider the same interim Bayesian game as the heterogeneous status quo game, except that the policy space of every agent is a finite nonempty set and that $n = 2$. In particular, we consider the two-player heterogeneous status quo game $\mathcal{F}$, for fixed status quo policy profile $\boldsymbol{p}_0 \in \mathbf{R}^2$ and the finite policy spaces defined in what follows, under the maintained assumption that $p_0^1 \neq p_0^2$.

Let $A_i = \{a_{i,1}, \dots, a_{i,M_i}\}$, for given $M_i \in \mathbf{N}$ and every $i \in N$. We define the following payoff differences, towards studying strategic complementarities

$$du_i(a_i, a_i', a_{-i}, x_0^i) = \int_{a_i'}^{a_i} u_i(p_i, a_{-i}, x_0^i)\,\mathrm{d}p_i$$
$$\delta_i(a_i, a_i', a_{-i}, a_{-i}', x_0^i) = du_i(a_i, a_i', a_{-i}, x_0^i) - du_i(a_i, a_i', a_{-i}', x_0^i).$$

Lemma 34 (Dominance Region)**.** *There exists $\underline{x}, \overline{x} \in \mathbf{R}$ such that: $\underline{x} < \overline{x}$ and, for all $i \in N$, $a_{-i} \in A_{-i}$ it holds that*

$$du_i(a_{i,M_i}, a_i', a_{-i}, x_0^i) > 0 \textit{ if } a_i \neq a_{i,M_i} \textit{ and } x_0^i > \overline{x},$$
$$\textit{and } du_i(a_{i,1}, a_i', a_{-i}, x_0^i) > 0 \textit{ if } a_i \neq a_{i,1} \textit{ and } x_0^i < \underline{x}.$$

Proof. The result follows from Lemma 25. In particular, in the notation of the aforementioned result, we define

$$\overline{x} := \max\{\overline{x}_1, \overline{x}_2\}$$
$$\underline{x} := \max\{\underline{x}_1, \underline{x}_2\}.$$

■

Lemma 35 (Strategic Complementarities)**.** *The function $u_i(\cdot, x_0^i)$ exhibits increasing differences in (a_i, a_{-i}), for all $i \in N$ and $x_0^i \in \mathbf{R}$.*

Proof. The result follows from Lemma 23. ■

Lemma 36 (Type Monotonicity)**.** *The function $u_i(\cdot, a_{-i}, x_0^i)$ exhibits strictly increasing differences in (a_i, x_0^i), for all $i \in N$ and $a_{-i} \in A_{-i}$.*

Proof. The result follows from Lemma 23. ■

Lemma 37 (Constant Type Monotonicity)**.** *For all $i \in N$, $a_i'', a_i' \in A_i$ with $a_i'' > a_i'$, and all $a_{-i}'', a_{-i}' \in A_{-i}$ with $a_{-i}'' > a_{-i}'$, the function $\delta_i(a_i'', a_i', a_{-i}'', a_{-i}', \cdot)$ is constant on $\mathbf{R}$.*

Proof. In the proof of Lemma 23, we show an expression for $du_i(a_i, a_i', a_{-i}, x_0^i)$, which we use to write:

$$\delta_i(a_i'', a_i', a_{-i}'', a_{-i}', x_0^i) =$$
$$\int_{a_i'}^{a_i''} -2\mu(-g_{i-i})\mu(a_{-i}'' - a_{-i}') - g_{i-i}(\partial_-|p_i - a_{-i}''| - \partial_-|p_i - a_{-i}'|)\omega\,\mathrm{d}p_i.$$

The result follows. ■

Lemma 38 (Existence of Cutoffs)**.** *For all $i \in N$, $a_i'', a_i' \in A_i$ and all $a_{-i} \in A_{-i}$, there exists $\widetilde{x} \in \mathbf{R}$ such that*

$$du_i(a_i'', a_i', a_{-i}, \widetilde{x}) = 0.$$

Proof. In the proof of Lemma 23, we show that u_i is strictly concave in i's policy. The result follows. ■

Lemma 39 (Payoff Continuity)**.** *For all $i \in N$, $a_i \in A_i$ and $a_{-i} \in A_{-i}$, the function $u_i(a_i, a_{-i}, \cdot)$ is continuous on $\mathbf{R}$.*

Proof. $u_i(a_i, a_{-i}, \cdot)$ is a strictly concave function of the column vector $(\mathbb{E}[\chi(a_j)|\chi(p_0^i) = x_0^i], \mathbb{E}[\chi(a_{-j})|\chi(p_0^i) = x_0^i])^\mathsf{T}$, for a given $j \in N$, by positive definiteness of $\boldsymbol{Q}$. The result follows since $u_i(a_i, a_{-i}, x_0^i)$ is a function of x_0^i only through $(\mathbb{E}[\chi(a_j)|\chi(p_0^i) = x_0^i], \mathbb{E}[\chi(a_{-j})|\chi(p_0^i) = x_0^i])^\mathsf{T}$, and the function $x_0^i \mapsto (\mathbb{E}[\chi(a_j)|\chi(p_0^i) = x_0^i], \mathbb{E}[\chi(a_{-j})|\chi(p_0^i) = x_0^i])^\mathsf{T}$ is affine. ■

In $\mathcal{F}$, a strategy for player i is a function $\alpha_i \colon \mathbf{R} \to A_i$. We study Bayesian Nash equilibria of $\mathcal{F}$ defined at the interim stage.

12.2.3 Existence of Bayesian Nash Equilibria

The proof is an adaptation of the one in Athey (2001). For simplicity of exposition, we prove the theorem in the where $A := A_1 = A_2$ and $M_1 - 1 =: M$, so that we may relabel policies as in $A = \{a_0, \ldots, a_M\}$. We say that strategy α_i' improves upon strategy α_i given α_{-i} if: $U_i(\alpha_i(x_0^i), x_0^i; \alpha_{-i}) \leq U_i(\alpha_i'(x_0^i), x_0^i; \alpha_{-i})$ for all x_0^i.

We define the set of i's cutoffs as

$$\widehat{\Sigma}_i := \{(x_1, \ldots, x_M) \in (\mathbf{R} \cup \{-\infty, \infty\})^M : x_1 \leq x_2 \leq \cdots \leq x_M\},$$

$\widehat{\Sigma} = \times_{i \in N} \widehat{\Sigma}_i$, and $\widehat{\Sigma} = \times_{j \in -i} \widehat{\Sigma}_j$, We say that a strategy α_i has finite cutoffs if $a_0, a_M \in \alpha_i(\mathbf{R})$.

Lemma 40 (Finite Cutoffs)**.** *Let's fix $i \in N$. If α_i does not have finite cutoffs, there exists strategy α_i' that has finite cutoffs and improves upon α_i given some nondecreasing strategy profile of i's opponents.*

Proof. Let's suppose $a_0 \in \alpha_i(\mathbf{R})$ and $a_M \notin \alpha_i(\mathbf{R})$. Let's define $b = \inf\{x_0^i \in \mathbf{R} : \alpha_i(x_0^i) = \max \alpha_i(\mathbf{R})\}$. There exists $k > 0$ such that $\partial_{-} U_i(A_M, b + k; \alpha_{-i}) > 0$, because $\partial_{-} U_i'(A_M, \cdot; \alpha_{-i})$ is increasing for nondecreasing α_{-i} Let's define the strategy α_i' for player i as follows:

$$\alpha_i' : y \mapsto \begin{cases} \alpha_i(y) & , y \leq b + k \\ a_M & , y > b + k \end{cases}$$

The other cases can be dealt with similarly. ■

Definition 7. *(i) Given a nondecreasing strategy α_i, $x \in \widehat{\Sigma}_i$ represents α_i if the following holds for all $m \in \{0, \ldots, M\}$.*

$x_m = \infty$ if $a_m > \max \alpha_i(\mathbf{R})$, $x_m = -\infty$ if $a_m < \min \alpha_i(\mathbf{R})$, and:

$$x_m = \inf\{x_0^i \in \mathbb{R} \ : \ \alpha_i(x_0^i) \geq a_m\}, \ \textit{otherwise}.$$

(ii) Given a vector $x \in \widehat{\Sigma}_i$, strategy α_i is consistent with x if:

$$\alpha_i(x_0^i) = \begin{cases} a_0 & , x_0^i \leq x_1 \\ a_1 & , x_1 < x_0^i \leq x_2 \\ \vdots & \\ a_M & , x_M < x_0^i. \end{cases}$$

For fixed cutoff profile of i's opponents, $X^{-i} = (x^j)_{j \in -i} \in \widehat{\Sigma}_{-i}$, we denote i's expected payoff from policy p as her expected payoff from $(\chi(p), \chi(\alpha_{-i}(x_0^{-i})))$, in which α_j is consistent with x^j, $j \in -i$; thus, we have

$$\widehat{U}_i(p, x_0^i; X^{-i}) := U_i(p, x_0^i; \alpha_{-i}).$$

We define the best response to X^{-i} of i as:

$$\widehat{a}_i^{BR}(x_0^i, X^{-i}) = \arg\max_{a \in \mathcal{A}_i} \widehat{U}_i(a, x_0^i; X^{-i})$$

Lemma 41 (Bounds of best-response cutoffs)**.** *There exists $\underline{t}, \overline{t}$ such that the following holds. For every $i \in N, X^{-i} \in \widehat{\Sigma}_{-i}$, nondecreasing selection ζ from $\widehat{a}_i^{BR}(x_0^i, X^{-i})$ and cutoffs $x^i \in \widehat{\Sigma}_i$ representing ζ, we have:*

$$-\infty < \underline{t} \leq x_1^i \leq \cdots \leq x_M^i \leq \overline{t} < \infty.$$

Proof. The result follows from Lemma 25. ■

Proposition 14 (Existence in Discrete Game)**.** *In the game $\mathcal{F}$, there exists an equilibrium in nondecreasing strategies.*

Proof. We apply Kakutani's theorem to the following correspondence. Let's define the set of cutoff vectors that represent best response strategies to the profile X:

$$\Gamma_i(X^{-i}) = \{y \in \widehat{\Sigma}_i : \text{there exists a strategy for } i \text{ consistent with } y \text{ that} \\ \text{is a selection from } a_i^{BR}(\cdot, X^{-i})\}.$$

We claim that there exists a fixed point of the correspondence $(\Gamma_1, \ldots, \Gamma_I) : \Sigma \to \Sigma$, where:

$$\Sigma := \times_{i \in N} \Sigma_i \quad \text{and} \quad \Sigma_i := \{x \in [\underline{t}, \overline{t}]^M \ : \ x_1 \leq x_2 \leq \cdots \leq x_M\}.$$

Σ_i is compact, convex subset of $\mathbf{R}^{nM}$. Γ is nonempty-valued because action spaces are finite and the Single Crossing Condition for games of incomplete information holds. Γ is convex-valued due to "Lemma 2" in Athey (2001), and the Single Crossing Condition for games of incomplete information. Γ has closed graph, as established in the proof of "Lemma 3" in Athey (2001). Thus, by Kakutani's theorem, there exists a fixed point of Γ.

Next, we claim that a fixed point of Γ is an equilibrium of $\mathcal{F}$. It follows from Lemma 41, because if a strategy is a best-response against X^{-i}, than it admits a representation with finite uniformly bounded cutoffs. ■

Remark 6. *We note that the proof of existence of Bayesian Nash equilibria in $\mathcal{F}$ does not rely on the assumption that $n = 2$. Thus, it also establishes existence with finite policy spaces and n players.*

Remark 7 (Existence in $\mathcal{G}(\boldsymbol{p}_0)$)**.** *Following the approach in Athey (2001), there is a second existence proof for nondecreasing strategy equilibria in $\mathcal{G}(\boldsymbol{p}_0)$, which uses a purification argument given existence of an equilibrium in nondecreasing strategies in $\mathcal{F}$.*

Lemma 42. *In $\mathcal{G}(\boldsymbol{p}_0)$, there exists an equilibrium in which every player's strategy is nondecreasing.*

Proof. For each player i, let's consider a sequence of action spaces $P_i^\bullet$, in which

$$P_i^k = \left\{ \underline{p}_i + \frac{m}{10^k}(\overline{p}_i - \underline{p}_i) \ : \ m = 0, \dots, 10^k \right\} \quad , k \in \mathbf{N}.$$

For every k, the game where finite action spaces $P_1^k,\ P_2^k, \dots$ replace $A_1,\ A_2, \dots$ has an equilibrium, by Lemma 14. Let's fix a sequence of equilibria in nondecreasing strategies, $s^\bullet$. Because action spaces $P_1^k,\ P_2^k, \dots$ are bounded by $\min \underline{p}_i$ and $\max \overline{p}_i$, $s^\bullet$ is a sequence of uniformly bounded nondecreasing functions. By Helly's selection theorem, $s^\bullet$ admits a pointwise convergent subsequence, so we define $s^\star := \lim s^\bullet$. Because s^k is an equilibrium, it holds that $U_i(s_i^k(x_0^i), x_0^i; s_{-i}^k) \geq U_i(p, x_0^i; s_{-i}^k)$, for all k and $p \in P_i^k$. $U_i(p, \boldsymbol{x}_0; s_{-i}^k)$ is a continuous function of $(\dots, s_{i-1}^k(x_0^{i-1}), s_{i+1}^k(x_0^{i+1}), \dots)$, by lemma 63. Thus, $U_i(p, x_0^i; s_{-i}^k)$, which is the expectation of $U_i(p, \boldsymbol{x}_0; s_{-i}^k)$, converges as $k \to \infty$. Therefore: it holds that $U_i(s_i^\star(t_i), x_0^i; s_{-i}^\star) \geq U_i(p, x_0^i; s_{-i}^\star)$, for all $p \in P_i$. $s^\star$ is an equilibrium of the game $\mathcal{G}(p_0^1, \dots, p_0^N)$. ■

12.2.4 Uniqueness of Bayesian Nash equilibria with 2 players

First, we establish two properties of beliefs in $\mathcal{F}$, which we leverage to establish uniqueness of nondecreasing strategy equilibrium.

Let C_i denote the space of nondecreasing strategies for player $i \in N$, in which a nondecreasing strategy is identified by its finite sequence of "real cutoffs" (Mathevet, 2010). For $k > 1$, let's compute the probability that i attaches to her opponent playing strictly less than $g = a_{-i,k} \in A_{-i}$, given that i's type is x_0^i and $-i$'s strategy is α_{-i}:

$$\Phi\left(\frac{\alpha_{-i1}^-(g) - x_0^i - \mu(p_0^{-i} - p_0^i)}{\sqrt{\omega |p_0^i - p_0^{-i}|}} \right),$$

in which $\alpha_{-i1}^-(g)$ is the real cutoff between $a_{-i,k-1}$ and $a_{-i,k}$ implied by α_{-i}. For $k = 1$, that probability is 0.

Towards a definition of the above probability as a function of real cutoffs, we make the following definitions. Given a policy $g \in A_{-i}$, we let $k_{-i}(g)$ be such that: $g = a_{-i,k_{-i}(g)}$. A real cutoff between $a_{-i,k}$ and $a_{-i,k+1}$ is denoted by $c_{-i,k}^r$, for $k \in \{1, \dots, M_i - 1\}$ (the interpretation for $c_{-i,k}^r$ is that types below $c_{-i,k}^r$ play $a_{-i,k}$ and types above $c_{-i,k}^r$ play $a_{-i,k+1}$).

Given a nondecreasing strategy $c_{-i} \in C_{-i}, g \in A_{-i}, x_0^i \in \mathbf{R}$, we define:

$$\Lambda_i(g | c_{-i}, x_0^i) = \begin{cases} \Phi\left(\frac{c_{-i,k_{-i}(g)-1}^r - x_0^i - \mu(p_0^{-i} - p_0^i)}{\sqrt{\omega |p_0^i - p_0^{-i}|}} \right) & \text{if } k_{-i}(g) > 1, \\ 0 & \text{if } k_{-i}(g) = 1. \end{cases}.$$

Lemma 43 (FOSD and Translation Invariance)**.** *For all $i \in N$, and $y_0^i, x_0^i \in \mathbf{R}$ with $y_0^i > x_0^i$, we have:*

$$\Phi\left(\frac{s - y_0^i - \mu(p_0^{-i} - p_0^i)}{\sqrt{\omega|p_0^i - p_0^{-i}|}}\right) < \Phi\left(\frac{s - x_0^i - \mu(p_0^{-i} - p_0^i)}{\sqrt{\omega|p_0^i - p_0^{-i}|}}\right).$$

Moreover, let c_{-i} be a column vector real cutoffs with M_{-i} columns corresponding to an element of C_{-i}, we have that

$$\Lambda_i(g|c_{-i} + \Delta\mathbf{1}, x_0^i + \Delta) = \Lambda_i(g|c_{-i}, x_0^i),$$

for all $i \in N$, $g \in A_{-i}$ and $\Delta \in [0, \overline{x} - \underline{x}]$.

Proof. The first part follows from Lemma 18. The second part follows from the definition of Λ_i. ■

Proposition 15. *In the game $\mathcal{F}$, there exists a unique equilibrium in nondecreasing strategies.*

Proof. Given that we established existence of an equilibrium in nondecreasing strategies, it suffices to establish that there exists at most one equilibrium in nondecreasing strategies. The proof uses the same argument as “Proposition 2” and “Theorem 1” in Mathevet (2010). In particular, Lemmata 34 through 39 imply “Assumptions 1, 2, 3, 4, 5, 6” in Mathevet (2010), and beliefs in $\mathcal{F}$ satisfy FOSD and Translation Invariance. ■

Remark 8. *This remark explains why the results for $\mathcal{G}(\boldsymbol{p}_0)$, either for existence and for the characterization of extremal equilibria, are not used in $\mathcal{F}$. This remark is informed by the approach taken in Mathevet (2010) to establish uniqueness. For notational convenience, our next definition is valid under the assumption that $A_i \subseteq P_i$ for all $i \in N$,*

$$\varphi_i^F(x_0^i, \alpha_{-i}) = \arg\max_{p_i \in A_i} U_i(p_i, x_0^i; \alpha_{-i}).$$

We note that φ_i^F differs from φ_i because the respective optimization problems have different feasible sets: A_i and P_i, respectively. If the mapping $x_0^i \to \sup \varphi_i^F(x_0^i, \alpha_{-i})$ is measurable, then there exists a unique equilibrium in $\mathcal{F}$.[51] *However, $\varphi_i^F(x_0^i, \alpha_{-i})$ is not necessarily single-valued, so the Caratheodory-function argument used in $\mathcal{G}(\boldsymbol{p}_0)$ does not hold in $\mathcal{F}$.*

13 Proofs for Section 2

Proof of Lemma 1.

Proof. By strict concavity of expected payoff in own policy (Lemma 23), it is enough to verify that, up

[51] Here is the reason. Let’s order individual strategies and strategy profiles in $\mathcal{F}$ as in the heterogeneous status quo game. To establish uniqueness, by Proposition 15, it suffices to establish that there exists a largest and a smallest equilibrium, and that they are in nondecreasing strategies. Once we establish that the “GBR” mapping is measurable — ie, the equivalent in $\mathcal{F}$ of Lemma 26 in $\mathcal{G}(\boldsymbol{p}_0)$ —, the same argument that we adopt to establish Proposition 31 in $\mathcal{G}(\boldsymbol{p}_0)$ is valid in $\mathcal{F}$.

to a positive proportionality constant of -2μ, the right derivative of expected payoff in own policy is:

$$\partial_{p_i+}\mathbb{E}\pi_i(\boldsymbol{\chi}(p)) \propto \mathbb{E}\chi(p_i) - \beta_i - \alpha\sum_j \gamma^{ij}(\mathbb{E}\chi(p_j) - \beta_j + \partial_{p_i+}|p_i - p_j|k)$$
$$-\left(1 - \alpha\sum_j \gamma^{ij}\right)\frac{1}{-2\mu}\partial_{p_i+}\mathbb{V}\chi(p_i),$$

which follows from the independent Proposition 16. The result follows because $p_{-i} \mapsto \partial_{p_i+}\mathbb{E}\pi_i(\boldsymbol{\chi}(p))$ is increasing (this step is shown explicitely in the proof of 23, and it is omitted here for the sake of brevity.) ■

Proof of Proposition 1.

Proof. In G_0, strategy spaces are compact intervals and player i's payoff function is continuous in p_i for all p_{-i} (Lemma 23) and strictly supermodular in (p_i, p_{-i}) (Lemma 8). The result follows from Tarski's fixed point theorem, and the argument is known in the literature on supermodular games (Milgrom and Roberts, 1990; Vives, 1990). ■

Proof of Proposition 2.

Proof. Without loss of generality, we set $p_0 = 0$ to ease on notation. By right and left differentiation of the strictly concave expected payoff of player i in own payoff (Lemma 23), at policy profile p, and by the best-response equivalence established in Lemma 16, the best response constraints for i are equivalent to the following pair of inequalities:

$$\mathbb{E}\chi(p_i) - \beta_i - \alpha\sum_j \gamma^{ij}(\mathbb{E}\chi(p_j) - \beta_j) \leq ([p_i \geq 0] - [p_i < 0])k$$
$$+\alpha\sum_j \gamma^{ij}([p_i \geq p_i] - [p_i < p_j])k$$
$$\text{and } ([p_i < 0] - [p_i \leq 0])k + \alpha\sum_j \gamma^{ij}([p_i < p_i] - [p_i \geq p_j])k \geq \mathbb{E}\chi(p_i) - \beta_i$$
$$-\alpha\sum_j \gamma^{ij}(\mathbb{E}\chi(p_j) - \beta_j),$$

which are found by left and right differentiation of the strictly concave potential, *separately* in each individual policy (i.e. for all p_i's). The result follows from rearranging the above inequalities in matrix notation. ■

Proof of Lemma 2.

Proof. The result follows directly from the results in Belhaj et al. (2014), and also the analysis in Ballester et al. (2006). ■

Proof of Corollary 1.

Proof. The result follows from the analysis of Callander (2011a), or the same arguments leading to Lemma 1 and Proposition 2. ■

14 Proofs for Section 3

Proofs of Section 3

Proof of Lemma 2.

Proof. The present proof uses the notation described in Section 11. By the equilibrium decomposition:

$$\boldsymbol{Q}\mathbb{E}\boldsymbol{\chi} = \boldsymbol{b} + \boldsymbol{Q}\boldsymbol{1}k + (\boldsymbol{G}\odot\boldsymbol{A})\boldsymbol{1}k$$

Thus:

$$\boldsymbol{q}_{i\bullet}^{\mathsf{T}}\mathbb{E}\boldsymbol{\chi} = b_i + \boldsymbol{q}_{i\bullet}^{\mathsf{T}}\boldsymbol{1}k + \sum_j g_{ij}a_{ij}k$$

So, by symmetry of $\boldsymbol{G}$

$$\begin{aligned}(\mathbb{E}\chi(p_i) - \mathbb{E}\chi(p_j))(1+g_{ij}) &= b_i - b_j + \boldsymbol{q}_{i\bullet}^{\mathsf{T}}\boldsymbol{1}k - \boldsymbol{q}_{j\bullet}^{\mathsf{T}}\boldsymbol{1}k + \\ + \sum_{\ell\notin\{i,j\}}(g_{i\ell}-g_{j\ell})\mathbb{E}\chi_\ell &+ \sum_{\ell\notin\{i,j\}}(g_{i\ell}a_{i\ell} - g_{j\ell}a_{j\ell})k + g_{ij}(a_{ij}-a_{ji})k\end{aligned}$$

Which simplifies to:

$$(\mathbb{E}\chi(p_i) - \mathbb{E}\chi(p_j))(1+g_{ij}) = b_i - b_j + \alpha\gamma\sum_{\ell\notin\{i,j\}}(a_{i\ell}-a_{j\ell})k - 2gk$$

From the equilibrium decomposition, it holds that: (i) $a_{i\ell} - a_{j\ell} \in [-2, 0]$ if $p_i < p_j$, and (ii) $a_{i\ell} - a_{j\ell} = 0$ only if: $p_\ell \in \{p_i, p_j\}$ or $p_\ell \in P_i \setminus [p_i, p_j]$. The result follows. ∎

Proof of Lemma 3

Proof. We use the notation developed in Section 11. We have that, for all $i, m \in N$

$$\begin{aligned}\mathbb{E}\chi(p_i) &= \beta_i + k + (\boldsymbol{I}-\boldsymbol{G})^{-1}_{ii}\sum_{\ell\in N} g_{i\ell}a_{i\ell}k + \\ + \sum_{j\in N\setminus\{i,m\}}(\boldsymbol{I}-\boldsymbol{G})^{-1}_{ij}\sum_{\ell\in N} g_{j\ell}a_{j\ell}k &+ (\boldsymbol{I}-\boldsymbol{G})^{-1}_{im}\sum_{\ell\in N} g_{m\ell}a_{m\ell}k.\end{aligned}$$

Thus:

$$\mathbb{E}\chi(p_i) - \mathbb{E}\chi(p_m) = \beta_i - \beta_m + \left[(\boldsymbol{I}-\boldsymbol{G})^{-1}_{ii} - (\boldsymbol{I}-\boldsymbol{G})^{-1}_{mi}\right]\left(\sum_{\ell\in N} g_{i\ell}a_{i\ell} - \sum_{\ell\in N} g_{m\ell}a_{m\ell}\right)k$$

Letting $g := \alpha\gamma$, by computation of $(\boldsymbol{I}-\boldsymbol{G})^{-1}$, we have that the diagonal element is $\frac{1-g(n-1)+g}{(1-g(n-1))(1+g)}$ and the off-diagonal element is: $\frac{g}{(1-g(n-1))(1+g)}$, so that:

$$(\boldsymbol{I}-\boldsymbol{G})^{-1}_{ii} - (\boldsymbol{I}-\boldsymbol{G})^{-1}_{im} = \frac{1}{1+g}.$$

Thus, by the preceding equality we have:

$$\begin{aligned}\mathbb{E}\chi(p_i) - \mathbb{E}\chi(p_m) &= \beta_i - \beta_m + \frac{g}{1+g}\left(\sum_{\ell \in N} a_{i\ell} - \sum_{\ell \in N} a_{m\ell}\right)k \\ &= \beta_i - \beta_m - \frac{g}{1+g}2k + \frac{g}{1+g}\left(\sum_{\ell \in N\setminus\{i,m\}} a_{i\ell} - a_{m\ell}\right)k.\end{aligned}$$

The result follows from the equilibrium decomposition in Proposition 2 and the hypotheses on p. ■

Proof of Lemma 5.

Proof. The result follows from Lemma 16. ■

Towards the proof of Lemma 3, we establish an auxiliary result. We say that $\mathbf{\Gamma}$ is *complete* if: $\gamma^{ij} = 1$ for all $j \in N \setminus \{i\}$ and $\gamma^{ii} = 0$ for all $i \in N$. We say that the equilibrium $\boldsymbol{p}$ is ordered if: $p_1 < p_2 < \cdots < p_n$, and a the equilibrium $\boldsymbol{p}$ is interior if: $p_i \in (p_0, \overline{p}), i \in N$.

Lemma 44. *Let $\mathbf{\Gamma}$ be complete. Then, Assumption 1 is satisfied if, and only if: $\alpha < 1/(n-1)$. Moreover, if $\boldsymbol{p} \in (p_0, \overline{p})^n$ is an ordered equilibrium and $i \in \{1, \ldots, n-1\}$, then*

$$\mathbb{E}\chi(p_i) - \mathbb{E}\chi(p_{i+\ell}) = \beta_i - \beta_{i+\ell} - 2\ell\frac{\alpha}{1+\alpha}k, \ell \in \{1, \ldots, n-i\}.$$

Furthermore, if $\delta_i - \delta_{i+1} > 2\frac{\alpha}{1-\alpha}k$, then: every interior equilibrium is ordered, and there exists at most one ordered interior equilibrium.

Proof. **Assumption 1 is satisfied if, and only if:** $\alpha < 1/(n-1)$**.** The result follows from the largest eigenvalue of $\mathbf{\Gamma}$ being $\lambda(\mathbf{\Gamma}) = n-1$.

"Moreover" part. By the Decomposition of equilibrium expected outcomes, $p_i < p_j$ implies

$$\mathbb{E}\chi(p_i) - \mathbb{E}\chi(p_j) = \beta_i - \beta_j + \frac{\alpha}{1+\alpha}\sum_{\ell \in N\setminus\{i,j\}}(a_{i\ell} - a_{j\ell})k - 2\frac{\alpha}{1+\alpha}k,$$

in which $a_{i\ell}, a_{j\ell}$ are elements of the matrix A in the decomposition, and we used the properties of the complete $\mathbf{\Gamma}$. The formula for $\mathbb{E}\chi(p_i) - \mathbb{E}\chi(p_{i+\ell})$ in the Lemma follows from the properties of A stated in the decomposition given that $\boldsymbol{p}$ is ordered.

It remains to verify that $\mathbb{E}\chi(p_i) \geq \beta_i$. We set $\widehat{\alpha} = \alpha(n-1)$ for $\widehat{\alpha} \in (0,1)$ — if $\widehat{\alpha} = 0$, then $\mathbb{E}\chi(p_i) = \beta_i + k \geq \beta_i$. After computation of the Leontieff inverse $\boldsymbol{B}$, it is established that:

$$1 + \widehat{\alpha}\sum_{j \in N} B_{ij}a_{ij} = 1 - \frac{(n-1)(1-\widehat{\alpha}) + \widehat{\alpha}}{(n-1+\widehat{\alpha})(1-\widehat{\alpha})}\widehat{\alpha} + \frac{\widehat{\alpha}}{(n-1+\widehat{\alpha})(1-\widehat{\alpha})}\widehat{\alpha}(n-1),$$

using the properties of the matrix $\boldsymbol{A}$ for an interior ordered equilibrium $\boldsymbol{p}$ (and the entries of $\boldsymbol{B}$, described in the proof of Proposition 3).

We verify that

$$1 + \widehat{\alpha}\sum_{j \in N} B_{ij}a_{ij} \leq 0 \iff (n-1)(1-\alpha) + \alpha + 2\alpha^2(n-2) \leq 0$$

Since the left-hand side of the above inequality is always positive, the result follows.

“Furthermore” part. This result is established in the proof of Proposition 3. ■

Proof of Lemma 3.

Proof. The result is an implication of Lemma 44 ■

Towards the proof of Lemma 4, we establish an auxiliary result. We say that $\mathbf{\Gamma}$ is a *line* if: (i) $\gamma^{ii+1} = 1$ for all $i \in \{1, \dots, n-1\}$, (ii) $\gamma^{ii-1} = 1$ for all $i \in \{2, \dots, n\}$, and (iii) $\gamma^{ij} = 0$ otherwise. We say that the equilibrium $\boldsymbol{p}$ is ordered if: $p_1 < p_2 < \dots < p_n$.

Lemma 45. *Let $\mathbf{\Gamma}$ be a line and $0 < \alpha < 1/2$. Then, Assumption 1 is satisfied. Moreover, if $\boldsymbol{p} \in (p_0, \overline{p})^n$ is an ordered equilibrium and $i \in \{1, \dots, n-1\}$, then*

$$\mathbb{E}\chi(p_i) - \mathbb{E}\chi(p_{i+\ell}) = \beta_i - \beta_{i+\ell} - a(i, \ell, n, \alpha)k, \ell \in \{1, \dots, n-i\},$$

for some $a(i, \ell, n, \alpha) > 0$.

Furthermore, $\mathbb{E}\chi(p_i) \geq \beta_i$.

Proof. **(1) Characterization of the inverse of $\boldsymbol{I} - \alpha\mathbf{\Gamma}$ using Toeplitz matrices.**

We have that $\boldsymbol{I} - \alpha\mathbf{\Gamma} =: \boldsymbol{S} = [S^{ij} : i, j \in N]$ in which (i) $S^{ii+1} = -\alpha$ for all $i \in \{1, \dots, n-1\}$, (ii) $S^{ii-1} = -\alpha$ for all $i \in \{2, \dots, n\}$, (iii) $S^{ij} = 1$ and (iv) $S^{ij} = 0$ otherwise. This matrix $\boldsymbol{S}$ Toeplitz becase it is constant on each diagonal. We study the following transformation $\boldsymbol{T}$ of $\boldsymbol{S}$.

$$\boldsymbol{T} = \frac{1}{\alpha}\boldsymbol{S},$$

so that $\boldsymbol{T}$ in which (i) $T^{ii+1} = -1$ for all $i \in \{1, \dots, n-1\}$, (ii) $T^{ii-1} = -1$ for all $i \in \{2, \dots, n\}$, (iii) $T^{ij} = a := 1/\alpha$ and (iv) $T^{ij} = 0$ otherwise. $\boldsymbol{T}$ is Toepliz, and the entries of its inverse can be characterized starting from the two solutions to $r^2 - ar + 1 = 0$. If $0 < \alpha < 1/2$, there exists two distinct roots, defined as:

$$\begin{aligned} r_- &:= \frac{1 - \sqrt{(1+2\alpha)(1-2\alpha)}}{2\alpha} \\ r_+ &:= \frac{1 + \sqrt{(1+2\alpha)(1-2\alpha)}}{2\alpha}. \end{aligned}$$

It is straightforward to establish that $0 < r_- < 1 < 1/\alpha < r_+ < 1/\alpha + 1$. By the characterization of inverse of Toeplitz matrices (e.g., Theorem 2.8 in Meurant (1992)), we have: $\boldsymbol{T}^{-1} = [T^{-1}_{ij} : i, j \in N]$ and

$$T^{-1}_{ij} = \frac{(r_+^i - r_-^i)(r_+^{n-j+1} - r_-^{n-j+1})}{(r_+ - r_-)(r_+^{n+1} - r_-^{n+1})}, j \geq i.$$

(2) Characterization of vector $\alpha\mathbf{\Gamma} \odot \boldsymbol{A}\mathbf{1}k$, given an ordered equilibrium. We have that:

$$\alpha\mathbf{\Gamma} \odot \boldsymbol{A}\mathbf{1}k = \alpha \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} k.$$

(3) Characterization of vector $\boldsymbol{e} := (\boldsymbol{I} - \alpha\boldsymbol{\Gamma})^{-1}\alpha\boldsymbol{\Gamma} \odot \boldsymbol{A}\mathbf{1}k$, given an ordered equilibrium. By using the definition of $\boldsymbol{T}^{-1}$, and $\boldsymbol{e} = [e_i : i \in N]$ we have that:

$$\begin{aligned} e_i &= -k\Big(- \frac{r_+^i - r_-^i}{r_+^{n+1} - r_-^{n+1}} + \frac{r_+^{n-i+1} - r_-^{n-i+1}}{r_+^{n+1} - r_-^{n+1}} \Big) \\ &= -\frac{r_+^{n-i+1} - r_+^i - r_-^{n-i+1} + r_-^i}{r_+^{n+1} - r_-^{n+1}} k. \end{aligned}$$

It follows that

$$e_i - e_{i+\ell} \propto -\big(r_+^{n-i+1} - r_+^{n-i-\ell+1} - r_+^i + r_+^{i+\ell} - r_-^{n-i+1} + r_-^{n-i-\ell+1} + r_-^i - r_-^{i+\ell}\big),$$

which is a positive number. We take:

$$a(i,\ell,n,\alpha) = \frac{r_+^{n-i+1} - r_+^{n-i-\ell+1} - r_+^i + r_+^{i+\ell} - r_-^{n-i+1} + r_-^{n-i-\ell+1} + r_-^i - r_-^{i+\ell}}{r_+^{n+1} - r_-^{n+1}}.$$

(4) Largest Eigenvalue of $\boldsymbol{\Gamma}$. The adjacency matrix $\boldsymbol{\Gamma}$ is Toeplitz. By known results (Theorem 2.2 in Kulkarni et al., 1999), the largest eigenvalue is

$$\lambda(\boldsymbol{\Gamma}) = -2\cos(\pi n/(n+1)) \in [0, 2).$$

"Furthermore" Part. We verify that $e_i \geq -k$. In particular,

$$\begin{aligned} -e_i/k > 1 &\iff r_+^{n-i+1} - r_+^i - r_-^{n-i+1} + r_-^i > r_+^{n+1} - r_-^{n+1} \\ &\iff -r_+^i + r_-^i > r_+^{n+1}(1 - r_+^{-i}) - r_-^{n+1}(1 - r_-^{-i}). \end{aligned}$$

The right-hand side of the above inequality is positive and the left-hand side is negative, by definition of r_+, r_- and $\alpha \in (0, 1/2), i \in N$. Thus, it holds that $-e_i/k \leq 1$. ■

Proof of Lemma 3.

Proof. The result is an implication of Lemma 45. ■

Proofs for section 3.5

A representative consumer has quasi-linear preferences over bundles of $n+1$ goods, which are represented by the quadratic utility function U such that

$$U(q_1, \ldots, q_n, z) = \sum_i \widehat{a}_i q_i - \frac{1}{2} b \sum_i q_i^2 - \frac{1}{2} c \sum_{i,j:j\neq i} q_i q_j + z,$$

in which r denotes the numéraire good. Let $\boldsymbol{B} = c\mathbf{1}\mathbf{1}^\mathsf{T} + (b - c)\boldsymbol{I}$ be the matrix with b on the main diagonal and c in off-diagonal entries.

Lemma 46. *Let $b > c > 0$. Then: $\boldsymbol{B}$ is a symmetric and positive definite matrix. Its inverse $\boldsymbol{B}^{-1}$ is symmetric, positive definite, its entries given by $\frac{b-c+(n-1)c}{(b-c)[(n-1)c+b]}$ on the main diagonal, and $-\frac{c}{(b-c)([(n-1)c+b])}$ in off-diagonal entries.*

Proof. $\boldsymbol{B}$ is symmetric. The eigenvalues of $\frac{1}{b}\boldsymbol{B}$ are $1-c/b$ and $1+\frac{n-1}{b}c$, so $\boldsymbol{B}$ is positive definite. Then, $\boldsymbol{B}^{-1}$ is well-defined, positive definite and has eigenvalues $(b-c)^{-1}$ and $(b+(n-1)c)^{-1}$.

We verify that $\boldsymbol{B}^{-1} = r\mathbf{1}\mathbf{1}^\intercal + \frac{1}{b-c}\boldsymbol{I}$, for $r = -\frac{c}{(b-c)((n-1)c+b)}$. Let's observe that $\mathbf{1}\mathbf{1}^\intercal\mathbf{1}\mathbf{1}^\intercal = n\mathbf{1}\mathbf{1}^\intercal$, and:

$$\begin{aligned}\boldsymbol{B}\boldsymbol{B}^{-1} = \boldsymbol{I} &\iff r\mathbf{1}\mathbf{1}^\intercal c\mathbf{1}\mathbf{1}^\intercal + \boldsymbol{I} + r(b-c)\mathbf{1}\mathbf{1}^\intercal\boldsymbol{I} + \frac{c}{b-c}\mathbf{1}\mathbf{1}^\intercal\boldsymbol{I} = \boldsymbol{I}\\ &\iff rcn\mathbf{1}\mathbf{1}^\intercal + \left[r(b-c) + \frac{c}{b-c}\right]\mathbf{1}\mathbf{1}^\intercal = \boldsymbol{I} - \boldsymbol{I}\\ &\iff r = -\frac{c}{(b-c)((n-1)c+b)}.\end{aligned}$$

■

By normalizing the main-diagonal entries of $\boldsymbol{B}^{-1}$ to 1, the off-diagonal elements are $1-\frac{1}{b-c}$. We note that $1-\frac{1}{b-c} < 0 \iff 1-(b-c) > 0$. Thus, in what follows we assume $1 > b-c$. Moreover, we assume that $\zeta := \frac{1-(b-c)}{b-c} < \frac{2}{n-1}$. Our parameter assumptions are summarized as follows

Assumption 7 (Demand System 2)**.** *We assume that*

(1) Goods are utility-substitute and U is strictly concave, which is equivalent to what is assumed in the main body of the text.

(2) Own-price coefficients of demand are all equal to -1 and that the degree of utility substitutability c is bounded above by $b-\frac{n-1}{n+1}$.

The two assumptions are jointly represented by:

$$c \geq 0 \text{ and } 1 > b-c > \frac{n-1}{n+1}.$$

$b > c \geq 0$ is equivalent to requiring that the following two conditions jointly hold: (i) goods are utility-substitute (U is submodular) and (ii) U is strictly concave. The requirement $1 > b-c$ is needed following the normalization that own-price coefficient of demand is -1, and $\frac{1-(b-c)}{b-c} < \frac{2}{n-1}$ is the content of Assumption 1 in the current setup after the normalization (we note that $\frac{1-(b-c)}{b-c} < \frac{2}{n-1} \iff b-c > \frac{n-1}{n+1}$). In the following remark, we verify that the additional assumptions can be dispensed of, which justifies that in the main text we only assume $b > c \geq 0$.

Remark 9 (Comparison of Assumption 7 with the model of oligopoly in Section 3)**.** *Under our assumptions, goods are mutually direct substitutes (Weinstein, 2022), substitutes in the sense of Hedgeworth and Marshallian demand satisfies the Law of Demand (Amir et al., 2017). Moreover, for a positive price vector $\boldsymbol{v}$ and sufficiently large income, demand for the goods excluding the numeraire is given by $\boldsymbol{B}^{-1}(\widehat{\boldsymbol{a}}-\boldsymbol{x})$.*

Let's show that under $b > c \geq 0$ the analysis goes through without the extra content in Assumption 7. First, let's observe that the concavity assumption on demand — positive definiteness of $\boldsymbol{B}$ following from $b > c \geq 0$ according to Lemma 46 — guarantees positive definiteness of $\boldsymbol{B}^{-1}$, and induces a contractive property on the best-response mapping of the game $\left\langle N, \{\pi_i^B, \mathbf{R}\}_{i\in N}\right\rangle$. Letting $\mathrm{Diag}(\boldsymbol{M})$ *return an $n\times n$ diagonal matrix whose entries are the n elements in the main diagonal of matrix $\boldsymbol{M}$, such best-response* *mapping follows form first-order conditions and is given by:*

$$\begin{aligned}\mathrm{BR}(\boldsymbol{x}) &= -2\,\mathrm{Diag}(\boldsymbol{B}^{-1})\boldsymbol{x} + \Big[\mathrm{Diag}(\boldsymbol{B}^{-1}) - \boldsymbol{B}^{-1}\Big]\boldsymbol{x} + \boldsymbol{B}^{-1}\widehat{\boldsymbol{a}} + \mathrm{Diag}(\boldsymbol{B}^{-1})\widehat{\boldsymbol{x}} \\ &= -\Big[\mathrm{Diag}(\boldsymbol{B}^{-1}) + \boldsymbol{B}^{-1}\Big]\boldsymbol{x} + \boldsymbol{B}^{-1}\widehat{\boldsymbol{a}} + \mathrm{Diag}(\boldsymbol{B}^{-1})\widehat{\boldsymbol{x}}.\end{aligned}$$

The Jacobian of $\mathrm{BR}(\boldsymbol{x})$ *is given by* $-[\mathrm{Diag}(\boldsymbol{B}^{-1}) + \boldsymbol{B}^{-1}]$, *which is negative definite iff* $\mathrm{Diag}(\boldsymbol{B}^{-1}) + \boldsymbol{B}^{-1}$ *is positive definite. The diagonal entries of* $\boldsymbol{B}^{-1}$ *are positive (Lemma 46). Thus, the best-reply mapping is a contraction.*

Secondly, to establish that the normalization on demand coefficients is innocuous, we show that the coefficients of $\boldsymbol{B}^{-1}$ *are negative, shown in Lemma 46.*

We assume that each of the prices of n goods is set by one of n firms that compete in prices. Each of n firms has constant marginal costs and no fixed costs. Let $\boldsymbol{D} := -\boldsymbol{B}^{-1} = [D_{ij} : i, j \in N]$ be the matrix of demand coefficients. Given a profile of prices $\widehat{x}$ and marginal costs $\widehat{m}$, profits of firm i are:

$$\begin{aligned}\pi_i^B(\widehat{x}) &:= (\widehat{x}_i - \widehat{m}_i)\left[\sum_{j\in N} D_{ij}(\widehat{x}_j - \widehat{a}_j)\right] \\ &= \left(\widehat{m}_i + \widehat{a}_i - \sum_{j\in -i} D_{ij}a_j\right)\widehat{x}_i - \widehat{x}_i^2 + \sum_{j\in -i} D_{ij}\widehat{x}_i\widehat{x}_j + F,\end{aligned}$$

for a term $F = -\widehat{m}_i\Big(\widehat{a}_i - \sum_{j\in -i} D_{ij}\widehat{a}_j\Big) - \widehat{m}_i \sum_{j\in -i} D_{ij}\widehat{x}_j$ that is constant with respect to $\widehat{x}_i$. We can equivalently express profits in terms of markups, $x := \widehat{x} - \widehat{m}$, letting $a := \widehat{a} - \widehat{m}$, to write

$$\pi_i^B(x) := \left(a_i - \zeta \sum_{j\in -i} a_j\right)x_i - x_i^2 + \zeta \sum_{j\in -i} x_j x_i,$$

for $\zeta = \frac{1-(b-c)}{b-c}$. In particular, we note that we may set:

$$\begin{aligned}2\alpha\gamma^{ij} &= \zeta \\ 2(1-\alpha)\delta_i &= \left(a_i - \zeta \sum_{j\in -i} a_j\right).\end{aligned}$$

So that the largest eigenvalue of $\boldsymbol{\Gamma}$ is $\frac{\zeta}{2}(n-1)$ and the content of Assumption 7 is justified in light of Assumption 1.

Proof of Proposition 3.

Proof. First, the pricing game has the same set of equilibria as the particular case of $G(x_0)$ in which: $\underline{p} - p_0$, the favorite outcome of i is $\widehat{a}_i/[2(1-\alpha)]$, coordination motives are $\zeta/2$ and Γ is the adjacency matrix of a network in which $\gamma^{ij} = 1, i \in N, j \in -i$, which we refer to as a *complete* network for the present proof. This result follows from Lemma 5. This observation implies the first part of the proposition via Lemma 3.

Second, let's establish a property of equilibria. Let $\boldsymbol{p}$ be an equilibrium. By the decomposition in

Proposition 2, if the network is complete and $p_i = p_j$, then

$$(1+\alpha)[\mathbb{E}\chi(p_i) - \mathbb{E}\chi(p_j)] = (1-\alpha)(\delta_i - \delta_j) - m\alpha k,$$

for $m \in [0, M]$, in which- $M = |\{\ell \in N : p_\ell \in [p_i, p_j]\}|$. In particular, a similar derivation is described in the proof of Lemma 2, and it is omitted in the present proof. From the above equality it follows that: $p_i = p_j$ implies that $m\alpha k \geq (1-\alpha)(\delta_i - \delta_j)$. In the pricing game, then, $p_i = p_j$ implies that

$$m\zeta k \geq \widehat{a}_i - \widehat{a}_j. \tag{11}$$

Third, we establish that: if $\min_{i\in N, j\in -i} |\widehat{a}_i - \widehat{a}_j| > 2\zeta k$, the no two players choose the same policy in equilibrium. In what follows, we fix an equilibrium $\boldsymbol{p}^\star \in (p_0, \overline{p})$, and a policy $p \in (p_1^\star, \ldots, p_n^\star)$ that is played in equilibrium by a number of players $m \in \{2, n\}$. For fixed number of players $m \in \{2, \ldots, n\}$ who play the same policy p in equilibrium $\boldsymbol{p}^\star$, there exist players i', j' who play p and with

$$\widehat{a}_{i'} - \widehat{a}_{j'} > (m-1) \min_{i\in N, j\in -i} |\widehat{a}_i - \widehat{a}_j| \tag{12}$$

In particular, this observation holds by taking i', j' to be the players choosing, respectively, $\min\{p_1^\star, \ldots, p_n^\star\}$ and $\max\{p_1^\star, \ldots, p_n^\star\}$. Let's observe that: if $\min_{i\in N, j\in -i} |\widehat{a}_i - \widehat{a}_j| > 2\zeta k$, then $\min_{i\in N, j\in -i} |\widehat{a}_i - \widehat{a}_j| > \frac{m'}{m'-1}\zeta k$ for all $m' \in \{2, \ldots, n\}$, so:

$$(m-1) \min_{i\in N, j\in -i} |\widehat{a}_i - \widehat{a}_j| > m\zeta k.$$

Hence, if $\min_{i\in N, j\in -i} |\widehat{a}_i - \widehat{a}_j| > 2\zeta k$, inequality 11 contradicts inequality 12.

Fourth, we show that the only interior equilibrium in which no two players choose the same policy is $p_1 <, \ldots, < p_n$ if $\min_{i\in N, j\in -i} |\widehat{a}_i - \widehat{a}_j| > 2\zeta k$. By the proof of Lemma 2, if the network is complete and $\boldsymbol{p} \in (p_0, \overline{p})^n$ is an equilibrium with $p_0 < p_1 < \cdots < p_n < \overline{p}$, then

$$(1+\alpha)[\mathbb{E}\chi(p_i) - \mathbb{E}\chi(p_j)] = (1-\alpha)(\delta_i - \delta_j) - 2\alpha k,$$

whenever $p_i < p_j$. We note that $\alpha < 1$ under a complete network, by Assumption 1.

Hence, if $\min_{i\in N, j\in -i} |\widehat{a}_i - \widehat{a}_j| > 2\zeta k$ and $\boldsymbol{p} \in (p_0, \overline{p})^n$ is an equilibrium of the pricing game, then $p_0 < p_1 < \cdots < p_n < \overline{p}$ up to a permutation of players. Moreover, by the decomposition in Proposition 2, if $\min_{i\in N, j\in -i} |\widehat{a}_i - \widehat{a}_j| > 2\zeta k$ there exists at most one interior equilibrium.

■

Proofs for Remark 1 We say that players have the *same unweighted centrality* if $\boldsymbol{u} := (\boldsymbol{I} - \alpha\boldsymbol{\Gamma})^{-1}\boldsymbol{1}$ is such that $u_i = u_j$ for all players $i, j \in N$. An equilibrium $\boldsymbol{p} \in P^n$ is *symmetric* if $p_i = p_j$ for all players $i, j \in N$.

Lemma 47. *Let players have the same centrality, same unweighted centrality, and $\underline{p} = p_0$. If $\chi(p_0)$ and $\overline{p}$ are sufficiently large, there exist a greatest and a least symmetric equilibrium, respectively $\boldsymbol{q}$ and $\boldsymbol{s}$. Moreover:*

$$\begin{aligned}\mathbb{E}\boldsymbol{\chi}(\boldsymbol{q}) &= \boldsymbol{\beta} + \boldsymbol{1}k \\ \mathbb{E}\boldsymbol{\chi}(\boldsymbol{s}) &= \boldsymbol{\beta} + 2\boldsymbol{1}k - \boldsymbol{u}k.\end{aligned}$$

Proof. **Application of the Decomposition of Equilibrium Expected Outcomes**
Let $([p_i < p_j],\ i, j \in N)$ and $([p_i \le p_j],\ i, j \in N)$ be two n-by-n matrices, in which $[Y]$ is the Iverson bracket of the statement Y, so $[Y] = 1$ if the statement Y is true, and $[Y] = 0$ otherwise. We define $\mathbf{\Gamma}_+(p) = \mathbf{\Gamma} \odot ([p_i < p_j],\ i, j \in N)$ and $\mathbf{\Gamma}_-(p) = \mathbf{\Gamma} \odot ([p_i \le p_j],\ i, j \in N)$. By the decomposition in Proposition 2, $p \in (p_0, \overline{p})^n$ is an interior equilibrium if, and only if:

$$k(\boldsymbol{I} - 2\alpha\mathbf{\Gamma}_-(p))\mathbf{1} \le (\boldsymbol{I} - \alpha\mathbf{\Gamma})(\mathbb{E}\boldsymbol{\chi}(p) - \boldsymbol{\beta}) \le k(\boldsymbol{I} - 2\alpha\mathbf{\Gamma}_+(p))\mathbf{1}.$$

Implications of symmetric equilibria
If $p \in (p_0, \overline{p})^n$, then:

$$\begin{aligned}\boldsymbol{\beta} + (\boldsymbol{I} - \alpha\mathbf{\Gamma})^{-1}(\boldsymbol{I} - 2\alpha\mathbf{\Gamma}_-(p))\mathbf{1}k &= \boldsymbol{\beta} + 2\mathbf{1}k - \boldsymbol{u}k \\ \boldsymbol{\beta} + (\boldsymbol{I} - \alpha\mathbf{\Gamma})^{-1}(\boldsymbol{I} - 2\alpha\mathbf{\Gamma}_+(p))\mathbf{1}k &= \boldsymbol{\beta} + \mathbf{1}k.\end{aligned}$$

(The first equality follows from the definition of $\boldsymbol{B}$.)

The result follows. ∎

Corollary 3. *Let $\delta_i = 0$ for all $i \in N$ and players have the same unweighted centrality. Then, $\boldsymbol{p} \in (p_0, \overline{p})^n$ is an equilibrium if, and only if:*

$$\mathbb{E}\boldsymbol{\chi}(\boldsymbol{p}) \in [(2\mathbf{1} - \boldsymbol{u})k, \boldsymbol{u}k].$$

Moreover: $\boldsymbol{u}k$ is increasing in α and k, $(2\mathbf{1} - \boldsymbol{u})k$ is decreasing in α, and $(2 - u_i)k$ is increasing in k iff $u_i < 2$.

Proof of Lemma 1

Proof. The first part of the proof is a consequence of an observation made in Vives (1999), Chapter 2, Footnote 23, and the potential structure of the game (Proposition 22.) The second part follows from Corollary 3, after noting that players have the same unweighted centralities under a complete network. ∎

15 Proofs for Section 4

Proofs for Section 4.1

Towards the proof of Proposition 4, introduce a definitions and several lemmata.

Definition 8 (Monderer and Shapley (1996))**.** *The game in strategic form $\langle I, \{S_i, u_i\}_{i \in I}\rangle$ is a potential game if there exists a function $U\colon \times_i S_i \to \mathbf{R}$ such that, for all $i \in I$, $s_{-i} \in \times_{j \ne i} S_j$ and $s_i, s_i' \in S_i$:*

$$u_i((s_i, s_{-i})) > u_i((s_i', s_{-i})) \text{ iff } U((s_i, s_{-i})) > U((s_i', s_{-i}));$$

the function U is called a potential for the game.

Towards the study of a selection rule for equilibria of $G(x_0)$, we introduce a function that is related to the potential of the game without complexity. The no-complexity potential is the function $v\colon \mathbf{R}^n \to \mathbf{R}$ given by

$$v(\boldsymbol{x}) = 2(1 - \alpha)\boldsymbol{\delta}^\top \boldsymbol{x} - \boldsymbol{x}^\top(\boldsymbol{I} - \alpha\mathbf{\Gamma})\boldsymbol{x}.$$

And the expected no-complexity potential $V\colon P^n \to \mathbf{R}$ is given by

$$V(p) = \mathbb{E}v(\boldsymbol{\chi}(p)), \text{ for all } p \in P^n.$$

The expected no-complexity potential, or *potential*, provides a potential for the game $G(x_0)$, as established by the next results. The function v is the potential of the game S defined in Section 2.3; this result is a corollary to Proposition 16 and is known (Jackson and Zenou, 2015).

Lemma 48. *The game $G(x_0)$ is a potential game. Moreover, for every player $i \in N$ there exists a function $g_i\colon P^{n-1} \times \mathbf{R} \to \mathbf{R}$ such that:*

$$\mathbb{E}\pi_i(\boldsymbol{\chi}(p)) = \mathbb{E}v(\boldsymbol{\chi}(p)) + g_i(p_{-i}, x_0^i) \textit{ for all } p \in P^n \textit{ and } x_0 \in \mathbf{R},$$

and a potential for $G(x_0)$ is the expected no-complexity potential $V\colon p \mapsto \mathbb{E}v(\boldsymbol{\chi}(p))$ given the status-quo outcome x_0.

Proof of Lemma 48.

Proof. We first establish von-Neumann-Morgenstern equivalence (Morris and Ui, 2004) between the two strategic-form games S and $\langle N, \{P, v\}_{i\in N}\rangle$. Thus, we show that: for all $i \in N$, there exists a function $h_i\colon \mathbf{R}^{n-1} \to \mathbf{R}$ such that

$$\pi_i(\boldsymbol{x}) - v(\boldsymbol{x}) = h_i(x_{-i}) \text{ for all } \boldsymbol{x} \in \mathbf{R}^n.$$

The claim is a consequence of $\boldsymbol{\Gamma}$ being a symmetric matrix. In particular, we note that $\sum_{(i,j)\in N^2} \gamma^{ij} x_i x_j - 2\sum_{j\in N} \gamma^{ij} x_i x_j$ is constant with respect to x_i, and:

$$v(\boldsymbol{x}) - v_i(\boldsymbol{x}) = \sum_{j\in -i} \Big(2(1-\alpha)\delta_j x_j - x_j^2\Big) + \alpha \sum_{(i,j)\in N^2} \gamma^{ij} x_i x_j - 2\alpha \sum_{j\in N} \gamma^{ij} x_i x_j.$$

The second part of the Lemma follows, by observing that $v_i(\boldsymbol{x}) - \pi_i(\boldsymbol{x})$ is constant in x_{-i}, as shown in Section 11, and taking expectations given the status-quo outcome.

It remains to establish that von-Neumann-Morgenstern equivalence between G_0 and $\langle N, \{P, \mathbb{E}v(\boldsymbol{\chi}(p))\}_{i\in N}\rangle$ implies that G_0 is a potential game according to the definition in Monderer and Shapley (1996). We prove a stronger statement: V is a w-potential for $G(x_0)$ with $w_i = 1$ for all $i \in N$, that is, $G(x_0)$ is an weighted and exact potential game, and V is a weighted and exact potential. The intuition for the observation is the same underlining Lemma 1 in Morris and Ui (2004), we include a proof solely because the authors assume finite strategy spaces.

Let $\Pi_i(q_i, p_{-i}) := \mathbb{E}\pi_i(\chi(p_1), \ldots, \chi(q_i), \chi(p_{i+1}), \ldots)$. By the definitions of Monderer and Shapley (1996), pages 127-128, V is an exact potential for $G(x_0)$ if $\Pi_i(p_i, \cdot) - \Pi_i(p_i', \cdot) = V((p_i, \cdot)) - V((p_i', \cdot))$ for all $p_i, p_i' \in P$. By our preceding results:

$$\Pi_i(p_i, p_{-i}) - V((p_i, p_{-i})) = g_i(p_{-i}, x_0) \text{ and } \Pi_i(p_i', p_{-i}) - V((p_i', p_{-i})) = g_i(p_{-i}, x_0).$$

Thus, we have

$$\Pi_i(p_i, p_{-i}) - V((p_i, p_{-i})) = \Pi_i(p_i', p_{-i}) - V((p_i', p_{-i})),$$

which we rearrange to write:

$$\Pi_i(p_i, p_{-i}) - \Pi_i(p_i', p_{-i}) = V((p_i, p_{-i})) - V((p_i', p_{-i})).$$

■

Lemma 49. *If U is a potential for the game $G(x_0)$, there exists a constant $c \in \mathbf{R}$ such that*

$$U(p) = V(p) + c, \text{ for all } p \in P^n.$$

Moreover, if p is a potential maximizer, then p is an equilibrium of $G(x_0)$.

Proof of Lemma 49.

Proof. Let $p \in P^n$ be a potential maximizer and $i \in N, q_i \in P$ such that

$$\mathbb{E}\pi_i(\boldsymbol{\chi}(p)) < \mathbb{E}\pi_i(\dots, \chi(p_{i-1}), \chi(q_i), \dots).$$

By Lemma 48, we have

$$\mathbb{E}v(\boldsymbol{\chi}(p)) < \mathbb{E}v(\dots, \chi(p_{i-1}), \chi(q_i), \dots),$$

Which contradicts the definition of p.

The second part of the Lemma follows from Lemma 2.7 in Monderer and Shapley (1996) if $G(x_0)$ is an exact potential game, using a definition in Monderer and Shapley (1996), pages 127-128. In the proof of Lemma 48, we establish that $G(x_0)$ is an exact potential game when we show that V is an exact potential for $G(x_0)$. ■

Proposition 16. *The game $G(x_0)$ is a potential game and $V\colon P^n \to \mathbf{R}$ is a potential for $G(x_0)$. Moreover,*

(1) If $U\colon P^n \to \mathbf{R}$ is a potential for $G(x_0)$, there exists a constant $c \in \mathbf{R}$ such that

$$U(p) = V(p) + c, \text{ for all } p \in P^n.$$

(2) If the policy profile $p \in P^n$ maximizes V, then p is an equilibrium of $G(x_0)$.

Proof of Proposition 16

Proof. The Proposition follows directly from Lemmata 48 and 49. ■

We establish an auxiliary Lemma towards the proof of Proposition 5. Towards a characterization of the potential maximizer, we note that the no-complexity potential can be expressed as $v(\boldsymbol{x}) = -(\boldsymbol{x} - \boldsymbol{\beta})^\mathsf{T}(\boldsymbol{I} - \alpha\boldsymbol{\Gamma})(\boldsymbol{x} - \boldsymbol{\beta}) + \boldsymbol{\beta}^\mathsf{T}(\boldsymbol{I} - \alpha\boldsymbol{\Gamma})\boldsymbol{\beta}$, which directly implies the following expression for V.

Lemma 50. *For all policy profiles $p \in P^n$, we have that*

$$V(p) = -(\mathbb{E}\boldsymbol{\chi}(p) - \boldsymbol{\beta})^\mathsf{T}(\boldsymbol{I} - \alpha\boldsymbol{\Gamma})(\mathbb{E}\boldsymbol{\chi}(p) - \boldsymbol{\beta}) - \sum_{i \in N} \mathbb{V}\chi(p_i) + \alpha \sum_{i,j \in N} \gamma^{ij}\mathbb{C}[\chi(p_i), \chi(p_j)],$$

up to a term that is constant in p.

Proof of Lemma 50.

Proof. We observe that the potential function v is a quadratic form, so $V(p) = -(\mathbb{E}\boldsymbol{\chi}(p) - \boldsymbol{\beta})^{\mathsf{T}}(\boldsymbol{I} - \alpha\boldsymbol{\Gamma})(\mathbb{E}\boldsymbol{\chi}(p) - \boldsymbol{\beta}) - \mathrm{tr}((\boldsymbol{I} - \alpha\boldsymbol{\Gamma})\boldsymbol{\Omega}) + \boldsymbol{\beta}^{\mathsf{T}}(\boldsymbol{I} - \alpha\boldsymbol{\Gamma})\boldsymbol{\beta}$, in which $\boldsymbol{\Omega}$ is the variance-covariance matrix of $\boldsymbol{\chi}(p)$ given $\chi(p_0) = x_0$, which is well-defined by joint Gaussianity of outcomes and $\omega > 0$. ■

Proposition 17 (Potential Maximizer)**.** *Let $P = [p_0, \overline{p}]$. There exists a unique potential maximizer. Moreover, the policy profile $p \in (p_0, \overline{p})^n$ is a potential maximizer if, and only if:*

$$\mathbb{E}\boldsymbol{\chi}(p) = \boldsymbol{\beta} + \mathbf{1}k + \alpha(\boldsymbol{I} - \alpha\boldsymbol{\Gamma})^{-1}(\boldsymbol{\Gamma} \odot \boldsymbol{A})\mathbf{1}k,$$

for a skew-symmetric matrix $\boldsymbol{A} = [a_{ij} : i, j \in N]$ such that $a_{ij} \in [-1, 1]$ and $a_{ij} = 1$, if $p_i > p_j$.

Proof of Proposition 17.

Proof. The first part of the result is a consequence of standard tools in convex analysis. First, we claim that there exists at most one potential maximizer. This follows from strict concavity of V, proved in Section 11.3. For existence given strict convexity of $-V$ see, e.g., Proposition 9.3.2, part (iv), in Briceño-Arias and Combettes (2013), stated in a game-theoretic environment.

The characterization of the potential maximizer is established in Lemma 21. ■

Proof of Proposition 4.

Proof. Part (1) follows from Proposition 16. Part (2) follows from Proposition 17. ■

Proof of Proposition 5.

Proof. The result follows from Proposition 17. ■

Proof of Proposition 6.

Proof. We use the notation developed in Section 11, in which we define v_i as the "effort-game ex-post payoff", defined over outcome profiles. It holds that:

$$v(\boldsymbol{x}) = \sum_i v_i(\boldsymbol{x}) - \alpha\boldsymbol{x}^{\mathsf{T}}\boldsymbol{\Gamma}\boldsymbol{x}.$$

Thus, we have that:

$$W(p) = \mathbb{E}\Big[v(\boldsymbol{\chi}(p)) + \alpha\boldsymbol{\chi}(p)^{\mathsf{T}}\boldsymbol{\Gamma}\boldsymbol{\chi}(p) | \chi(p_0) = x_0\Big].$$

Strict concavity of W on $[p_0, \overline{p}]^n$ follows from the same argument as Lemma 19. Thus, the superdifferential of W is well-defined. By standard subgradient calculus (Rockafellar, 1970), we write the following expression for ∂W, using $+$ for (Minkowski) set addition,

$$\partial W(p) = \partial V(p) + \partial\mathbb{E}\Big[\alpha\boldsymbol{\chi}(p)^{\mathsf{T}}\boldsymbol{\Gamma}\boldsymbol{\chi}(p) | \chi(p_0) = x_0\Big].$$

Using the decomposition of expectation of quadratic forms, we have:

$$\partial W(p) = \partial V(p) + 2\alpha\boldsymbol{\Gamma}\partial\mathbb{E}[\boldsymbol{\chi}(p)] + \alpha\partial \sum_{(i,j)\in N^2} \gamma^{ij}\, \mathbb{C}(\chi(p_i), \chi(p_j)),$$

for which we also apply symmetry of $\mathbf{\Gamma}$. The result follows from the characterization of $\partial V(p)$ in Lemma 21, in which we also characterize $\partial \sum_{(i,j)\in N^2} \gamma^{ij}\, \mathbb{C}(\chi(p_i),\chi(p_j))$. ■

Proofs for Section 4.2 and Section 4.3

In this section, we assume that $P=[p_0,\overline{p}]$.

Lemma 51. *Let $|a_1-c_1-a_2+c_2|\le -gk$. For sufficiently large $\chi(p_0)$, total profits are maximized by*

$$\mathbb{E}\chi(p_i)=\min\left\{b\frac{a-c_1+a-c_2}{4(1+gb)}+k,\chi(p_0)\right\}.$$

The maximization of total profits is implemented in equilibrium if, and only if: $a-c_1+a-c_2\le\frac{1+bg}{b}2k$.

Proof. By Lemma 5, we find the set of equilibria using Proposition 2. By Proposition 6 and Lemma 5, we find the maximizer of total profits by using 52 and $2g$ in place of g. ■

Dyad

We assume that $N=2$, and we use $\widehat{\alpha}:=\alpha\gamma^{12}$. We use $\chi_i:=\chi(p_i)$, $\boldsymbol{\chi}$ for the column vector of outomes $(\chi(p_1),\chi(p_2))'$, and ∂_{p_i} for the subdifferential with respect to p_i. The expectation operators are conditional on $\chi(p_0)=x_0$. Let $y_+:=\max\{\beta_1,\beta_2\}+k\Big(1-\frac{\widehat{\alpha}}{1+\widehat{\alpha}}\Big)$, $y_-:=\min\{\beta_1,\beta_2\}+k\Big(1+\frac{\widehat{\alpha}}{1+\widehat{\alpha}}\Big)$.

Lemma 52 (Dyad)**.** *Let $y_+\ge x_0$ and $\mathbb{E}\chi(\overline{p})\ge y_-$. The following hold.*

(1) If $(1-\alpha)(\delta_2-\delta_1)\ge 2\widehat{\alpha}k$, then there exists a unique equilibrium in $G|_{x_0}$. Moreover, in equilibrium:

$$\mathbb{E}\chi_1=\beta_1+k\Big(1+\frac{\widehat{\alpha}}{1+\widehat{\alpha}}\Big)$$
$$\mathbb{E}\chi_2=\beta_2+k\Big(1-\frac{\widehat{\alpha}}{1+\widehat{\alpha}}\Big),$$

which imply

$$\mathbb{E}\chi_2-\mathbb{E}\chi_1=\beta_2-\beta_1-2\frac{\widehat{\alpha}}{1+\widehat{\alpha}}k.$$

.

(2) If $(1-\alpha)(\delta_2-\delta_1)<2\widehat{\alpha}k$, then there exist multiple equilibria in $G|_{x_0}$. Moreover, in equilibrium:

$$\begin{aligned}(1-\alpha)(\delta_2-\delta_1)&=\widehat{\alpha}(d_1-d_2),\ \textit{for some } d_2,d_1\in[-1,1]\\ \mathbb{E}\chi_1=\mathbb{E}\chi_2&=\frac{\beta_1+\beta_2}{2}+k+\frac{\widehat{\alpha}}{1-\widehat{\alpha}}\frac{d_1+d_2}{2}k\\ &\in\left[\frac{\beta_1+\beta_2}{2}+k-\frac{\widehat{\alpha}}{1-\widehat{\alpha}}k,\frac{\beta_1+\beta_2}{2}+k+\frac{\widehat{\alpha}}{1-\widehat{\alpha}}k\right].\end{aligned}$$

(3) If $0\le(1-\alpha)(\delta_2-\delta_1)<2\widehat{\alpha}k$, then there exists a unique potential maximizer in $G|_{x_0}$. Moreover,

in the potential maximizer: $(1-\alpha)(\delta_2-\delta_1) = 2\widehat{\alpha}d_1k$, $d_1 \in [0,1)$, *and:*

$$\mathbb{E}\chi_1 = \beta_1 + k\left(1 + \frac{\widehat{\alpha}}{1+\widehat{\alpha}}d_1\right)$$
$$\mathbb{E}\chi_2 = \beta_2 + k\left(1 - \frac{\widehat{\alpha}}{1+\widehat{\alpha}}d_1\right),$$

which imply

$$\mathbb{E}\chi_1 = (\beta_1+\beta_2)/2 + k.$$

Proof. The expected effort-game payoff to player i is:

$$\mathbb{E}v_i(\chi_i,\chi_j) = 2(1-\alpha)\delta_i\mathbb{E}\chi_i - (\mathbb{E}\chi_i)^2 + 2\widehat{\alpha}\mathbb{E}\chi_i\mathbb{E}\chi_j - \mathbb{V}\chi_i + 2\widehat{\alpha}\mathbb{C}\chi_i\chi_j,$$

up to a term that is constant with respect to p_i. The superdifferential of $\mathbb{E}v_i(\chi_i,\chi_j)$ with respect to p_i is:

$$2\mu(1-\alpha)\delta_i - 2\mu\mathbb{E}\chi_i + 2\mu\widehat{\alpha}\mathbb{E}\chi_j - \omega + \widehat{\alpha}\omega - \widehat{\alpha}\omega\partial_{p_i}|p_i-p_j|.$$

In any interior equilibrium p:

$$0 \in \begin{pmatrix} 1 & -\widehat{\alpha} \\ -\widehat{\alpha} & 1 \end{pmatrix}\mathbb{E}\boldsymbol{\chi} - (1-\alpha)\boldsymbol{\delta} - \begin{pmatrix} 1 & -\widehat{\alpha} \\ -\widehat{\alpha} & 1 \end{pmatrix}\mathbf{1}k - \widehat{\alpha}\begin{pmatrix} \partial_{p_1}|p_1-p_2| \\ \partial_{p_2}|p_2-p_1| \end{pmatrix}k$$

Thus, we obtain the following interior equilibrium condition. $p \in (p_0, \overline{p})$ is an equilibrium if, and only if:

$$\mathbb{E}\boldsymbol{\chi} \in \frac{1-\alpha}{1-\widehat{\alpha}^2}\begin{pmatrix} 1 & \widehat{\alpha} \\ \widehat{\alpha} & 1 \end{pmatrix}\delta + k\mathbf{1} + \frac{\widehat{\alpha}}{1-\widehat{\alpha}^2}\begin{pmatrix} 1 & \widehat{\alpha} \\ \widehat{\alpha} & 1 \end{pmatrix}\begin{pmatrix} \partial_{p_1}|p_1-p_2| \\ \partial_{p_2}|p_2-p_1| \end{pmatrix}k,$$

and in an equilibrium in which $p_1 > p_2$ the last term simplifies to a singleton:

$$\frac{\widehat{\alpha}}{1-\widehat{\alpha}^2}\begin{pmatrix} 1 & \widehat{\alpha} \\ \widehat{\alpha} & 1 \end{pmatrix}\begin{pmatrix} \partial_{p_1}|p_1-p_2| \\ \partial_{p_2}|p_2-p_1| \end{pmatrix}k = \left\{\frac{\widehat{\alpha}}{1+\widehat{\alpha}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}k\right\}.$$

In an equilibrium p in which $p_1 = p_2$, the last term can be written as:

$$\frac{\widehat{\alpha}}{1-\widehat{\alpha}^2}\begin{pmatrix} 1 & \widehat{\alpha} \\ \widehat{\alpha} & 1 \end{pmatrix}\begin{pmatrix} \partial_{p_1}|p_1-p_2| \\ \partial_{p_2}|p_2-p_1| \end{pmatrix}k = \frac{\widehat{\alpha}}{1-\widehat{\alpha}^2}\begin{pmatrix} \partial_{p_1}|p_1-p_2| + \widehat{\alpha}\partial_{p_2}|p_2-p_1| \\ \partial_{p_2}|p_2-p_1| + \widehat{\alpha}\partial_{p_1}|p_1-p_2| \end{pmatrix}k.$$

In the potential maximizer p, we have that: $\partial_{p_1}|p_1-p_2| = -\partial_{p_2}|p_2-p_1|$, and so the last term simplifies to:

$$\frac{\widehat{\alpha}}{1-\widehat{\alpha}^2}\begin{pmatrix} 1 & \widehat{\alpha} \\ \widehat{\alpha} & 1 \end{pmatrix}\begin{pmatrix} \partial_{p_1}|p_1-p_2| \\ \partial_{p_2}|p_2-p_1| \end{pmatrix}k = \frac{\widehat{\alpha}}{1+\widehat{\alpha}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\partial_{p_1}|p_1-p_2|k.$$

■

Two-Type Network

We assume that there are two groups of players, A and B, such that: $N = A \cup B$, and $A \cap B = \emptyset$. We let $n_G := |G|$, $G \in \{A, B\}$, and $G(\ell), -G(\ell)$ denote, respectively, the group of player ℓ and the other group. Moreover, we assume that: $\delta_\ell = \delta_{G(\ell)}$, and

$$\gamma^{\ell k} = \gamma^{G(\ell)G(k)}, \text{ for all } \ell, k \in N.$$

We note that, by our maintained assumptions: $\gamma^{AB} = \gamma^{BA}$, and: $\gamma^{GF} = o(n)$, because $n_F\gamma^{GF} + (n_G - 1)\gamma^{GG} \leq 1$, for all $G, F \in \{A, B\}, G \neq F$.

The potential function is such that: $G(i) = G(j) \implies v(x_1, \ldots x_i, \ldots x_j, \ldots x_n) = v(x_1, \ldots x_j, \ldots x_i, \ldots x_n)$, so every equilibrium is represented by a pair (p_A, p_B), such that $i \in A$ plays p_A, and $j \in B$ plays p_B. We let $\mathbb{E}\chi_{G(i)} = \mathbb{E}\chi_i$ in the potential maximizer p. We use $\alpha_A := \frac{\alpha\gamma^{AB}n_B}{1-\alpha\gamma^{AA}(n_A-1)}$ and $\alpha_B := \frac{\alpha\gamma^{BA}n_A}{1-\alpha\gamma^{BB}(n_B-1)}$. We note that: $\alpha_A \leq \frac{\alpha\gamma^{AB}n_B}{\alpha\gamma^{AB}n_B+\alpha\gamma^{AA}(n_A-1)-\alpha\gamma^{AA}(n_A-1)} = 1$, and, similarly, $\alpha_B \leq 1$.

We note that $\frac{\alpha_A+\alpha_B-2\alpha_A\alpha_B}{1-\alpha_A\alpha_B} \in [0, 1]$, because:

$$\alpha_A + \alpha_B - 2\alpha_A\alpha_B > 0 \iff \frac{\alpha_A}{1-\alpha_A} + \frac{\alpha_B}{1-\alpha_B} > 0,$$

and

$$\frac{\alpha_A + \alpha_B - 2\alpha_A\alpha_B}{1-\alpha_A\alpha_B} = 1 - \frac{(1-\alpha_A)(1-\alpha_B)}{1-\alpha_A\alpha_B}.$$

Also, we note that $\frac{\partial}{\partial\alpha_{G(i)}}\frac{\alpha_A+\alpha_B-2\alpha_A\alpha_B}{1-\alpha_A\alpha_B} = \left(\frac{1-\alpha_{-G(i)}}{1-\alpha_A\alpha_B}\right)^2$.

Lemma 53. *Let Γ be a two-type network, such that: $\beta_A \geq \beta_B$, and let $x_0 \geq \beta_A + k - \alpha_A(1-\alpha_B)\frac{1}{1-\alpha_A\alpha_B}k$ and $\beta_B + k + \alpha_B(1-\alpha_A)\frac{1}{1-\alpha_A\alpha_B}k \geq \mathbb{E}\chi(\overline{p})$.*

(1) If $\beta_A - \beta_B \geq \frac{\alpha_A+\alpha_B-2\alpha_A\alpha_B}{1-\alpha_A\alpha_B}k$, then $p_A \leq p_B$ in the unique interior potential maximizer. Moreover:

$$\mathbb{E}\chi_A = \beta_A + k - \alpha_A(1-\alpha_B)\frac{1}{1-\alpha_A\alpha_B}k$$
$$\mathbb{E}\chi_B = \beta_B + k + \alpha_B(1-\alpha_A)\frac{1}{1-\alpha_A\alpha_B}k,$$

which imply:

$$\mathbb{E}\chi_A - \mathbb{E}\chi_B = \beta_A - \beta_B - \frac{\alpha_A + \alpha_B - 2\alpha_A\alpha_B}{1-\alpha_A\alpha_N}k.$$

(2) If $\beta_A - \beta_B < \frac{\alpha_A+\alpha_B-2\alpha_A\alpha_B}{1-\alpha_A\alpha_B}k$, then $p_A = p_B$ in the unique interior potential maximizer. Moreover:

$$\mathbb{E}\chi_A = \beta_A + k - \frac{\alpha_A(1-\alpha_B)}{1-\alpha_A\alpha_B}dk$$
$$\mathbb{E}\chi_B = \beta_B + k + \frac{\alpha_B(1-\alpha_A)}{1-\alpha_A\alpha_B}dk, \ d \in [0, 1].$$

and $\beta_A - \beta_B = \frac{\alpha_A+\alpha_B-2\alpha_A\alpha_B}{1-\alpha_A\alpha_B}dk$*, which imply:*

$$\mathbb{E}\chi_A = \frac{\alpha_B(1-\alpha_A)\beta_A + \alpha_A(1-\alpha_B)\beta_B}{\alpha_B(1-\alpha_A)+\alpha_A(1-\alpha_B)} + k.$$

Proof. The superdifferential of $\mathbb{E}v_i(\chi_1,\dots,\chi_n)$ with respect to p_i, $i \in A$, evaluated at an equilibrium, is:

$$2\mu(1-\alpha)\delta_A - 2\mu\mathbb{E}\chi_i + 2\mu\alpha\gamma^{AA}(n_A-1)\mathbb{E}\chi_A + 2\mu\alpha\gamma^{AB}n_B\mathbb{E}\chi_B +$$
$$-\omega + \alpha\gamma^{AA}(n_A-1)\omega + \alpha\gamma^{AB}(n_B)\omega - \alpha\gamma^{AA}(n_A-1)\partial_{p_i}|p_i-p_A|\omega - \alpha\gamma^{AB}n_B\partial_{p_i}|p_i-p_B|.$$

If p is the potential maximizer, then: $p_i = p_{G(i)}$, and:

$$\begin{aligned}
0 \in\ & 2\mu(1-\alpha)\delta_A - 2\mu\mathbb{E}\chi_A + 2\mu\alpha\gamma^{AA}(n_A-1)\mathbb{E}\chi_A + 2\mu\alpha\gamma^{AB}n_B\mathbb{E}\chi_B + \\
& -\omega + \alpha\gamma^{AA}(n_A-1)\omega + \alpha\gamma^{AB}(n_B)\omega - \alpha\gamma^{AB}n_B\partial_{p_A}|p_A-p_B| \\
0 \in\ & 2\mu(1-\alpha)\delta_B - 2\mu\mathbb{E}\chi_B + 2\mu\alpha\gamma^{BB}(n_B-1)\mathbb{E}\chi_B + 2\mu\alpha\gamma^{BA}n_A\mathbb{E}\chi_A + \\
& -\omega + \alpha\gamma^{BB}(n_B-1)\omega + \alpha\gamma^{BA}(n_A)\omega - \alpha\gamma^{BA}n_A\partial_{p_B}|p_B-p_A|.
\end{aligned}$$

We use $\alpha_A := \frac{\alpha\gamma^{AB}n_B}{1-\alpha\gamma^{AA}(n_A-1)}$ and $\alpha_B := \frac{\alpha\gamma^{BA}n_A}{1-\alpha\gamma^{BB}(n_B-1)}$. We note that: $\alpha_A \leq \frac{\alpha\gamma^{AB}n_B}{\alpha\gamma^{AB}n_B+\alpha\gamma^{AA}(n_A-1)-\alpha\gamma^{AA}(n_A-1)} = 1$, and, similarly, $\alpha_B \leq 1$. Thus, if p is the potential maximizer, then $p_i = p_{G(i)}$, and, for some $d \in \partial_{p_A}|p_A - p_B|$:

$$0 = 2\mu(1-\alpha)\begin{pmatrix} \frac{\delta_A}{1-\alpha\gamma^{AA}(n_A-1)} \\ \frac{\delta_B}{1-\alpha\gamma^{BB}(n_B-1)} \end{pmatrix} - 2\mu\begin{pmatrix} 1 & -\alpha_1 \\ -\alpha_2 & 1 \end{pmatrix}\begin{pmatrix} \mathbb{E}\chi_A \\ \mathbb{E}\chi_B \end{pmatrix} - \begin{pmatrix} 1 & -\alpha_1 \\ -\alpha_2 & 1 \end{pmatrix}\mathbf{1}\omega + \begin{pmatrix} \alpha_A \\ -\alpha_B \end{pmatrix}\omega d.$$

Thus, $p \in (p_0, \overline{p})^n$ is the unique potential maximizer if, and only if: $p_i = p_{G(i)}$, $i \in N$, and:

$$\begin{pmatrix} \mathbb{E}\chi_A \\ \mathbb{E}\chi_B \end{pmatrix} = \begin{pmatrix} \beta_A \\ \beta_B \end{pmatrix} + k\mathbf{1} + \begin{pmatrix} 1 & -\alpha_A \\ -\alpha_B & 1 \end{pmatrix}^{-1}\begin{pmatrix} -\alpha_A \\ \alpha_B \end{pmatrix}kd,\ d \in \partial_{p_A}|p_A - p_B|.$$

In the unique potential maximizer for $p_A < p_B$, we have:

$$\begin{pmatrix} 1 & -\alpha_A \\ -\alpha_B & 1 \end{pmatrix}^{-1}\begin{pmatrix} -\alpha_A \\ \alpha_B \end{pmatrix}kd = \frac{1}{1-\alpha_A\alpha_B}\begin{pmatrix} -(1-\alpha_B)\alpha_A \\ (1-\alpha_A)\alpha_B \end{pmatrix}k,$$

and:

$$\mathbb{E}\chi_A - \mathbb{E}\chi_B = \beta_A - \beta_B - \frac{\alpha_A + \alpha_B - 2\alpha_A\alpha_B}{1-\alpha_A\alpha_B}k.$$

∎

Part IV

Appendix 2: Proofs for Essay 2

16 Mathematical preliminaries

Let $\mathcal{D}$ be the set of distributions over $[0,1]$ identified by their cumulative distribution functions, $F_0 \in \mathcal{D}$ be continuous, and $x_0 := \int_0^1 \theta \, \mathrm{d}F_0(\theta)$. Let's define the information policy of $F \in \mathcal{D}$ as

$$\begin{aligned} I_F \colon \mathbb{R}_+ &\to \mathbb{R}_+ \\ x &\mapsto \int_0^x F(y) \, \mathrm{d}y. \end{aligned}$$

We define the set of feasible cumulative distribution functions as

$$\mathcal{F} := \{F \in \mathcal{D} : I_F(1) = I_{F_0}(1), \text{ and } I_F(x) \leq I_{F_0}(x) \text{ for all } x \in \mathbb{R}_+\}.$$

In order to use information policies to represent feasible CDF's, we use the following notation. Let $(\cdot)_+ := \max\{\cdot, 0\}$, and $\overline{F} : x \mapsto (x - x_0)_+$; we note that $\overline{F} \in \mathcal{F}$. The set of information policies is:

$$\mathcal{I} := \{I \colon \mathbb{R}_+ \to \mathbb{R}_+ : I \text{ is convex and } I_{F_0}(x) \geq I(x) \geq I_{\overline{F}}(x) \text{ for all } x \in \mathbb{R}_+\}.$$

For a function I, we denote the right and left derivatives by, respectively, $\partial_+ I$ and $\partial_- I$, and the subdifferential of I by ∂I. Let $I \in \mathcal{I}$, then: (i) $\partial_+ I(x)$ exists for all $x \in [0, \infty)$, and (ii) $\partial_- I(x)$ exists for all $x \in [0, \infty)$, once I is extended to take value 0 at $x < 0$. We define I on nonnegative reals to simplify the exposition, so, when we write $\partial_- I(x)$, we implicitely assume that I is extended to take value 0 on $(-\infty, 0)$. For two convex functions, the subdifferential of $I + J$ at x is the (Minkowski) addition of $\partial I(x)$ and $\partial J(x)$, we denote it by $\partial(I + J)(x)$. We also use $I'(x)$ and $I'(x^-)$ for, respectively, $\partial_+ I$ and $\partial_- I$.

Fact 1. *The following hold:*

(1) If $F \in \mathcal{F}$, then $I_F \in \mathcal{I}$;

(2) If $I \in \mathcal{I}$, then $I' \in \mathcal{F}$, once I is extended to take value 0 at every $x < 0$.

Proof. See Gentzkow and Kamenica (2016) and Kolotilin (2018). ■

We define the operator Δ as:

$$\Delta \colon I \mapsto I - I_{\overline{F}}.$$

And we denote by ΔI the composite function $\Delta \circ I$.[52] It holds that $0 \leq \Delta I(x) \leq 1 - x_0$, $x \in \mathbb{R}$, by

[52] Information policies are also called "left side integrals" (An, 1995; Bagnoli and Bergstrom, 2005) and "integral CDFs" (Shishkin, 2023). A belief distribution that has barycenter equal to a given prior belief (e.g., the distribution induced by the cumulative distribution function F_0,) is also called an information policy (Lipnowski and Mathevet, 2018; Lipnowski et al., 2020; Lipnowski and Ravid, 2020; Lipnowski et al., 2022a; Lipnowski and Ravid, 2023; Lipnowski et al., 2024). Ravid et al. (2022) use the notation "I_F" for, in our notation, the function $I_{F_0} - I_F$.

definition of I. The set of *information allocations* is:

$$\begin{aligned}\mathcal{A} := \{A\colon \mathbb{R}_+ \to \mathbb{R}_+ : A \text{ is convex on } [0, x_0],\ A \text{ is convex on } [x_0, 1],\\ A \text{ is continuous at } x_0,\ A(x) \leq I_{F_0}(x) - I_{\overline{F}}(x) \text{ for all } x \in \mathbb{R}_+,\\ \text{there exist } m \in [0,1) \text{ and } m' \in [m,1] \text{ such that } \partial_- A(x_0) = m \text{ and}\\ \partial_+ A(x_0) = m' - 1\}.\end{aligned}$$

Lemma 54. *The following hold:*

(1) If $A \in \mathcal{A}$, then: $A + I_{\overline{F}} \in \mathcal{I}$.

(2) If $I \in \mathcal{I}$, then $\Delta I \in \mathcal{A}$.

Proof. The only non trivial step is to show convexity of $A + I_{\overline{F}}$. We note that m, in the definition of A, is a subgradient of $A + I_{\overline{F}}$ at x_0. ■

Remark 10. *We note that any element of $\mathcal{F}$, $\mathcal{I}$ and $\mathcal{A}$ can be identified with its restriction on $[0,1]$, which without loss has codomain $[0,1]$. By the previous lemmata, there exists a bijection between any two of $\mathcal{F}$, $\mathcal{I}$, and $\mathcal{A}$; e.g., take $F \mapsto I_F$, with inverse $I \mapsto I'$. Moreover, if we endow $\mathcal{F}$ with the Blackwell order and use the component-wise order for $\mathcal{A}$ and $\mathcal{F}$, then the bijection is an order isomorphism.*

We state a result from convex analysis.

Fact 2 (Subdifferential of Convex Functions)**.** *Let $S \subseteq \mathbb{R}$, $f\colon S \to \mathbb{R}$ be convex, and $\varphi\colon \mathbb{R} \to \mathbb{R}$ be a nondecreasing convex function on the range of f.*

(1) The function $\varphi \circ f$ is convex on S.

(2) For all $y \in S$, letting $t = f(y)$, we have:

$$\{\alpha u : (\alpha, u) \in \partial\varphi(t) \times \partial f(y)\} = \partial\varphi \circ f(y).$$

Proof. See Proposition 8.21 and Corollary 16.72 in Bauschke and Combettes (2011). ■

We endow $\mathcal{I}$ with the component-wise order, we use $\leq$ to denote all partial orders and $<$ for the asymmetric part of $\leq$. The next result appears also in Curello and Sinander (2024).

Lemma 55 (Lattice Structure of $\mathcal{I}$)**.** *$\mathcal{I}$, endowed with the component-wise order, is a complete lattice.*

Proof. First, $(\mathcal{I}, \leq)$ is a poset because $\leq$ is reflexive, asymmetric and transitive on $\mathcal{I}$.

Second, we show that $H\colon x \mapsto \max\{I(x), J(x)\}$ is well-defined, it is equal to $\sup\{I, J\}$ pointwise, and $H \in \mathcal{I}$. H is well-defined by boundedness of I, J; H is convex being the upper envelope of a nonempty family of proper convex functions, and H satisfies the Blackwell bounds, i.e., $I_{\overline{F}} \leq H \leq I_{F_0}$. Thus, $H \in \mathcal{I}$. By definition, $H \geq \sup\{I, J\}$. If $H > K$, $K \in \mathcal{I}$ and $K \geq \sup\{I, J\}$, then $K(x) < \max\{I(x), J(x)\}$ for some $x \in \mathbb{R}$. Thus, $H = \sup\{I, J\}$.

Third, we show that the lower convex envelope C of $M\colon x \mapsto \min\{I(x), J(x)\}$ is well-defined, it is equal to $\inf\{I, J\}$ pointwise, and $C \in \mathcal{I}$. The definition of C is:

$$C(x) = \sup\{h(x) : h \text{ is convex and } h(y) \leq M(y) \text{ for all } y \in [0,1]\}, x \in [0,1],$$

and $C(x) = I(x)$ on $\mathbb{R} \setminus [0,1]$. C is well-defined because: (i) M is well-defined, and (ii) $I_{\overline{F}}$ lower bounds M and it is convex. C is the pointwise sup of a family of convex functions, so C is convex. $I_{\overline{F}}$ is a convex lower bound of M, so C satisfies the lower Blackwell bound, and C satisfies the upper Blackwell bound by definition: $C \leq M \leq I_{F_0}$. Thus, $C \in \mathcal{I}$. If $C < K \leq \inf\{I, J\}$, for some $K \in \mathcal{I}$, then C is not the pointwise maximum of all convex functions that lie below I, J.

It remains to show that $\mathcal{I}$ is complete. We claim that every set $C \subseteq \mathcal{I}$ has a sup and inf in $\mathcal{I}$. This claim is established following the same steps as above. ∎

Definition 9. *The θ upper censorship, for $\theta \in [0,1)$, is the unique information policy $I_\theta \in \mathcal{I}$ such that:*

$$I_\theta(x) = \begin{cases} I_{F_0}(x) & , x \in [0,\theta] \\ I_{F_0}(\theta) + (x-\theta)F_0(\theta) & , x \in (\theta, \overline{x}_\theta] \\ I_{\overline{F}}(x) & , x \in (\overline{x}_\theta, \infty), \end{cases}$$

in which $\overline{x}_\theta = \int_\theta^1 \tilde{\theta} \, \mathrm{d}\frac{F_0(\tilde{\theta})}{1-F_0(\theta)}$; the 1 upper censorship is $I_{\overline{F}}$.

The next lemma shows a property of upper censorships, a version of which appears in Lipnowski et al. (2021).

Lemma 56. *Let $I \in \mathcal{I}$, $\zeta \in [0,1]$, and F_0 be continuous. There exists $\theta \in [0,\zeta]$ such that:*

(1.) $I_\theta(\zeta) = I(\zeta)$.

(2.) $I'_\theta(\zeta^-) \leq I'(\zeta^-)$ *and:*

$$\begin{aligned} I_\theta(x) - I(x) &\geq 0, x \in [0,\zeta] \\ I_\theta(x) - I(x) &\leq 0, x \in [\zeta, \infty). \end{aligned}$$

.

Proof. Let $\zeta \in [0,1]$. Let $M := \{m \in [0, I'(\zeta^-)] \mid I(\zeta) + m(x-\zeta) \leq I_{F_0}(x)$ for all $x \in [0,\zeta]\}$, and $m := \min M$. We construct an information policy starting from the line $x \mapsto I(\zeta) + m(x-\zeta)$, via the next three claims.

(1) m is well-defined. (i) M is nonempty, because $0 \leq I'(\zeta^-) \leq 1$ (which follows from $I \in \mathcal{I}$), $I'(\zeta^-) \in \partial I(\zeta^-)$ and $I(x) \leq I_{F_0}(x)$ for all x; (ii) M is closed, becase the mapping $m \mapsto I(\zeta) + m(x-\zeta)$ is a continuous function on $[0, I'(\zeta^-)]$; (iii) M is bounded because $I'(\zeta^-) \leq 1$, since $I \in \mathcal{I}$.

(2) There exists $\theta \in [0,\zeta]$ such that $I_{F_0}(\theta) = I(\zeta) + m(\theta - \zeta)$. If $m = 0$, then $0 = I_{F_0}(0) \geq I(\zeta) \geq 0$. Hence, taking $\theta = 0$ verifies our claim. Let $m > 0$, and suppose there does not exist $\theta \in [0,\zeta]$ such that $I_{F_0}(\theta) = I(\zeta) + m(\theta-\zeta)$. There exists $\overline{\varepsilon} > 0$ such that: $I(\zeta) + (m-\varepsilon)(x-\zeta) < I_{F_0}(x)$ for all $x \in [0,\zeta]$ and $0 < \varepsilon \leq \overline{\varepsilon}$. Moreover, for a sufficiently small $\varepsilon > 0$, we have $m - \varepsilon \in M$. Thus, we have a contradiction with the definition of m.

(3) $m \in \partial I_{F_0}(\theta)$ and $I(\zeta) + m(x-\zeta) = I_{F_0}(\theta) + (x-\theta)F_0(\theta)$ for all x. First, we argue that $m \in \partial I_{F_0}(\theta)$. By convexity of I_{F_0} and definition of θ, $x \mapsto I(\zeta) + m(x-\zeta)$ is tangent to I_{F_0} at θ. Thus, m is a subgradient of I_{F_0} at θ. Now, we argue that $I(\zeta) + m(x-\zeta) = I_{F_0}(\theta) + (x-\theta)F_0(\theta)$ for all x. $m = F_0(\theta)$ because I_{F_0} is differentiable (by the fact that $F_0(x^-) = F_0(x), x \in \mathbb{R}$.) The equality follows because $x \mapsto I(\zeta) + m(x-\zeta)$ is equal to I_{F_0} at $x = \theta$.

We define the following function.

$$
\begin{aligned}
I^u \colon \mathbb{R}_+ &\to \mathbb{R}_+ \\
x &\mapsto \begin{cases} I_{F_0}(x) & , x \in [0, \theta] \\ I(\zeta) + m(x - \zeta) & , x \in (\theta, \zeta] \\ \max\{I(\zeta) + m(x - \zeta), I_{\overline{F}}(x)\} & , x \in (\zeta, \infty). \end{cases}
\end{aligned}
$$

Now, we claim that $I^u = I_\theta$. It suffices to show that: (i) for some $x_u \in [0, 1]$

$$
I^u(x) = \begin{cases} I_{F_0}(x) & , x \in [0, \theta] \\ I_{F_0}(\theta) + (x - \theta)F_0(\theta) & , x \in (\theta, x_u] \\ I_{\overline{F}}(x) & , x \in (x_u, \infty), \end{cases}
$$

and (ii) $I^u \in \mathcal{I}$. We claim that (i) holds by means of the next three claims.

There exists $x_u \in [\zeta, 1]$ *such that:*

$$
\begin{aligned}
I(\zeta) + m(x - \zeta) &\geq I_{\overline{F}}(x) \quad , x \in [0, x_u] && (13) \\
I(\zeta) + m(x - \zeta) &\leq I_{\overline{F}}(x) \quad , x \in (x_u, 1]. && (14)
\end{aligned}
$$

Let's note that: (a) $I(\zeta) \geq I_{\overline{F}}(\zeta)$; (b) by $m \in \partial I_{F_0}(\theta)$ and $I_{F_0}(1) = I_{\overline{F}}(1)$, we have that $I_{\overline{F}}(1) \geq I(\zeta)+m(1-\zeta)$, and (c) the two functions, $x \mapsto I(\zeta)+m(x-\zeta)$ and $I_{\overline{F}}$, are affine with slopes, respectively, m and 1, such that: $m \leq 1$.

We proceed to verify that (ii) holds, i.e. $I^u \in \mathcal{I}$, via the next two claims.

(1) $I_{\overline{F}}(x) \leq I^u(x) \leq I_{F_0}(x)$ *for all* $x \in \mathbb{R}_+$ *and* I^u *locally convex at all* $x \notin \{\theta, x_u\}$. If $x \in [0, \theta)$, I^u is locally convex and $I_{\overline{F}}(x) \leq I^u(x) \leq I_{F_0}(x)$. If $x \in (\theta, \zeta)$, I^u is affine, $I_{\overline{F}}(x) \leq I(x) \leq I^u(x)$ by construction of I^u and definition of I, and $I^u(x) \leq I_{F_0}(x)$ by $m \in \partial I_{F_0}(x)$. If $x \in [\zeta, \infty)$, I is locally convex (because it is the maximum of affine functions), $I_{\overline{F}}(x) \leq I^u(x)$ by construction of I^u, $I^u(x) \leq I_{F_0}(x)$ because: (i) $m \in \partial I_{F_0}(\zeta)$ and (ii) $I_{\overline{F}}(x) \leq I_{F_0}(x)$. To verify global convexity, it suffices to verify the next claim.

(2) I^u *is subdifferentiable at* $x \in \{\theta, x_u\}$. First, we argue that m is a subgradient of I^u at θ. This follows from the fact that the slope of I^u at θ is a subgradient of I_{F_0} at θ, and $I^u(\theta) = I_{F_0}(\theta)$. On $[0, \theta]$, $I^u = I_{F_0}$, and on $[\theta_u, \infty]$ I^u is above the line $x \mapsto I(\zeta) + m(x - \zeta)$. Thus, $m \in \partial I^u(\theta)$. Second, the fact that m is a subgradient of I^u at x_u follows from the claim in (13).

We have established that $I^u = I_\theta$. (1.) and (2.) hold by construction. ■

For posets S and T, the function $g \colon S \times T \to \mathbb{R}$ exhibits increasing differences if $t \mapsto g(s', t) - g(s, t)$ is nondecreasing for all $s', s \in S$ with $s < s'$; the function $g \colon S \times T \to \mathbb{R}$ exhibits strictly increasing differences if $t \mapsto g(s', t) - g(s, t)$ is increasing for all $s', s \in S$ with $s < s'$; the function $g \colon S \to \mathbb{R}$ is single-crossing from above if: (i) $g(s) \leq 0$ implies $g(s') \leq 0$ and (ii) $g(s) < 0$ implies $g(s') < 0$ for all $s', s \in S$ with $s < s'$. The function $g \colon S \to \mathbb{R}$ is strictly single-crossing from above if: (i) $g(s) \leq 0$ implies $g(s') < 0$ for all $s', s \in S$ with $s < s'$.

The following lemma states known facts from the envelope theorem and monotone comparative statics.

Lemma 57. *Let* $f \colon [0, 1] \times [0, 1] \to \mathbb{R}^2$ *exhibit increasing differences, and be such that:* $f(\cdot, a)$ *is upper semi-continuous for all* $a \in [0, 1]$, $f(e, \cdot)$ *is nondecreasing for all* $e \in [0, 1]$, *the derivative with respect to* *the variable a, $\partial f/\partial a(e,\cdot)$, exists and is bounded for all $e \in [0,1]$. The following hold:*

(1) $\arg\max_{e\in[0,1]} f(e,a) \neq \emptyset$ *for all* $a \in [0,1]$*:*

(2) $a \mapsto \max_{e\in[0,1]} f(e,a)$ *is nondecreasing and absolutely continuous.*

(3) If $a \mapsto \frac{\partial f}{\partial a}(e,a)$ *is nondecreasing for all* $e \in [0,1]$, $a \mapsto \max_{e\in[0,1]} f(e,a)$ *is convex.*

(4) If f exhibits strictly increasing differences, $a \mapsto \frac{\partial f}{\partial a}(e,a)$ *is nondecreasing,* $f(e,\cdot)$ *is increasing for all* $e \in (0,1]$, $\arg\max_{e\in[0,1]} f(e,a) \cap (0,1] \neq \emptyset$*, and* $1 \geq a' > a \geq 0$*, then:*

$$\max_{e\in[0,1]} f(e,a') > \max_{e\in[0,1]} f(e,a).$$

Proof. By upper semi-continuity of f, $\arg\max_{e\in[0,1]} f(e,a) \neq \emptyset$, so (1) holds. Then, by the increasing-differences property of f, there exists a nondecreasing selection $e^\star\colon a \mapsto \arg\max_{e\in[0,1]} f(e,a)$ on $[0,1]$ (Milgrom and Shannon, 1994). By our hypotheses, we apply the envelope theorem (Milgrom and Segal, 2002), letting $V(a) := \max_{e\in[0,1]} f(e,a)$, to establish that V is absolutely continuous and

$$V(a) = V(0) + \int_0^a \partial f/\partial a(e^\star(\tilde{a}),\tilde{a})\,\mathrm{d}\tilde{a}.$$

Since $\partial f/\partial a$ is nonnegative, V is nondecreasing. Hence, (2) holds.

Let's establish that V is convex if $a \mapsto \frac{\partial f}{\partial a}(e,a)$ is nondecreasing. By the increasing-differences property of f: (i) $e \mapsto \partial f/\partial a(e,a)$ is nondecreasing, and (ii) there exists a nondecreasing $e^\star\colon a \mapsto \arg\max_{e\in[0,1]} f(e,a)$. As a result, $a \mapsto \frac{\partial f}{\partial a}(e^\star(a),a)$ is nondecreasing. Thus, V is convex (Theorem 24.8 in Rockafellar (1970), noting that $a \mapsto \partial f/\partial a(e^\star(a),a)$ is uni-dimensional.) Hence, (3) holds.

Let $a' > a$, for $a', a \in [0,1]$, and $e' \in \arg\max_{e\in[0,1]} f(e,a) \cap (0,1]$. Then: $V(a') - V(a) = \int_a^{a'} \partial f/\partial a(e^*(\tilde{a}),\tilde{a})\,\mathrm{d}\tilde{a}$ for every selection e^* of $\arg\max_{e\in[0,1]} f(e,a) \cap (0,1]$. We have the following chain of inequalities under the additional hypotheses stated in part (4):

$$\begin{aligned} V(a') - V(a) &\geq \int_a^{a'} \partial f/\partial a(e',\tilde{a})\,\mathrm{d}\tilde{a} \\ &\geq \int_a^{a'} \partial f/\partial a(e',a)\,\mathrm{d}\tilde{a}, \end{aligned}$$

in which the first inequality follows from the strict increasing-differences property of f and the definition of e', the second inequality holds because $a \mapsto \frac{\partial f}{\partial a}(e,a)$ is nondecreasing (for the first inequality, in particular, we note that: (i) every selection e^* of $\arg\max_{e\in[0,1]} f(e,a) \cap (0,1]$ is nondecreasing, (ii) there exists a selection e^* of $\arg\max_{e\in[0,1]} f(e,a) \cap (0,1]$ such that $e^*(a) = e'$.) Since $\int_a^{a'} \partial f/\partial a(e',a)\,\mathrm{d}\tilde{a} = (a'-a)\partial f/\partial a(e',a)$, (4) holds. ■

17 Proofs for Section 9.1

In this section, we prove Theorem 1 in a more general setup than that of Section 9.1.

For this section, we fix a function $f\colon [0,1]\times[0,1] \to \mathbb{R}^2$ that satisfies strictly increasing differences, and such that: $f(\cdot,a)$ is upper semi-continuous for all $a \in [0,1]$, $f(e,\cdot)$ is nondecreasing for all $e \in [0,1]$, the derivative with respect to the variable a, $\partial f/\partial a(e,\cdot)$, exists, is nonnegative and bounded for all $e \in [0,1]$, and $f(e,\cdot)$ is increasing for all $e \in (0,1]$. We define the value of an information policy $I \in \mathcal{I}$

as $V_\lambda(e, \Delta I(c)) := f(e, \Delta I(c)) - K(e, \lambda)$, and we use the shorthand $t = (c_t, \lambda_t)$. Let's define the set of optimal efforts as

$$E_{\lambda_t}(\Delta I(\zeta_t)) := \arg\max_{e \in [0,1]} V_{\lambda_t}(e, \Delta I_t(\zeta_t))$$

Definition 10. *(1) A persuasion mechanism is a menu of information policies $(I_r)_{r\in R}$, with $I_r \in \mathcal{I}$ for all reports $r \in R$, and $R = T$.*

(2) A persuasion mechanism $(I_r)_{r\in R}$ is incentive compatible (IC) if:

$$t \in \arg\max_{r \in R} \left\{ \max_{e\in[0,1]} f(e, \Delta I_r(\zeta_t)) - K(e, \lambda_t) \right\}, \quad \textit{for all types } t \in T.$$

Definition 11. *An IC persuasion mechanism $(I_r)_{r\in R}$ and an information policy I induce the same effort and action distribution if:*

(1)

$$E_{\lambda_t}(\Delta I_t(\zeta_t)) \subseteq E_{\lambda_t}(\Delta I(\zeta_t)) \quad \textit{for all } t \in T. \tag{15}$$

(2)

$$\partial I_t(\zeta_t) \subseteq \partial I(\zeta_t) \quad \textit{if } (0,1] \cap E_{\lambda_t}(\Delta I_t(\zeta_t)) \neq \emptyset.$$

Proposition 18. *For every IC persuasion mechanism $(I_r)_{r\in R}$ there exists an information policy J such that $(I_r)_{r\in R}$ and J induce the same effort and action distributions.*

Proof. Let's fix an IC persuasion mechanism $(I_r)_{r\in R}$. First, we define an information policy J, and then we show that it induces the same effort and action distributions as $(I_r)_{r\in R}$.

(1) Definition of information policy J. Let's fix an IC persuasion mechanism $(I_r)_{r\in R}$. Let's define the function $I\colon [0,1] \to \mathbb{R}_+$ as follows:

$$I(c) := \sup_{r\in R} I_r(c), \; c \in [0,1] \tag{16}$$

$I(c)$ is well defined because $0 \leq I_r(c) \leq I_{F_0}(c) \leq 1 - x_0$, $c \in [0,1]$. I is the pointwise supremum of a family of convex functions, so I is convex. It holds that $I_{\overline{F}}(c) \leq I(c) \leq I_{F_0}(c)$, $c \in [0,1]$, because $I_r \in \mathcal{I}, r \in R$. We extend I on $(1,\infty)$, so that the resulting extended function $J\colon \mathbb{R}_+ \to \mathbb{R}_+$ is an information policy, by defining $J(c) = I_{F_0}(c)$, $c \in (1,\infty)$, and $J(c) = I(c)$, $c \in [0,1]$. We have that $J \in \mathcal{I}$.

(2) Effort distribution.

There are two cases.

1. $E_{\lambda_t}(\Delta I_t(\zeta_t)) \cap (0,1] \neq \emptyset$.
2. $E_{\lambda_t}(\Delta I_t(\zeta_t)) = \{0\}$.

First, we consider case (1.). By envelope theorem, we have:

$$V_{\lambda_t}(a) - V_{\lambda_t}(\Delta I_t(\zeta_t)) = \int_{\Delta I_t(\zeta_t)}^{a} \frac{\partial f}{\partial e}(\tilde{a}, e(\tilde{a}))\, \mathrm{d}\tilde{a},$$

for a selection e of E_{λ_t}. Because f exhibits strictly increasing differences, $e(\tilde{a}) \geq e(\Delta I_t(\zeta_t))$ if $\tilde{a} \geq \Delta I_t(\zeta_t)$. By the assumption that $\frac{\partial f}{\partial e}(\tilde{a}, \cdot) > 0$ on $(0, 1]$ for all $\tilde{a}$

$$V_{\lambda_t}(a) - V_{\lambda_t}(\Delta I_t(\zeta_t)) > 0, \text{ for all } a > \Delta I_t(\zeta_t).$$

Thus, in case (1.) IC implies that

$$\sup_{r \in R} \Delta I_r(\zeta_t) = \Delta I_t(\zeta_t).$$

Let's consider case (2.), and, towards a contradiction, let's assume $0 \notin E_{\lambda_t}(\Delta J(\zeta_t))$. By Berge's Maximum Theorem, E_{λ_t} is upper hemi continuous. Therefore, there exists $\overline{a} \in (\Delta I_t(\zeta_t), \Delta J(\zeta_t))$ and $f > 0$ such that $f \in E_{\lambda_t}(\overline{a})$. By the assumption that $\frac{\partial f}{\partial e}(\tilde{a}, \cdot) > 0$ on $(0, 1]$ for all $\tilde{a}$

$$V_{\lambda_t}(\Delta J(\zeta_t)) - V_{\lambda_t}(\overline{a}) > 0.$$

The above inequality and the envelope theorem imply that

$$V_{\lambda_t}(\Delta J(\zeta_t)) - V_{\lambda_t}(\Delta I_t(\zeta_t)) > 0.$$

Hence, IC does not hold, which is a contradiction.

Towards a contradiction, let's assume $f \in E_{\lambda_t}(\Delta J(\zeta_t))$ and $f > 0$. By the same steps as above:

$$V_{\lambda_t}(\Delta J(\zeta_t)) - V_{\lambda_t}(\Delta I_t(\zeta_t)) > 0.$$

Hence, IC does not hold, which is a contradiction.

Action distribution. Let's suppose that $d \in \partial I_s(\zeta_s)$ and $d \notin \partial J(\zeta_s)$ for some type $s \in T$. Because I_s and J are information policies, they have the same extension on $(-\infty, 0)$ and $\zeta_s > 0$. We have that d is a subgradient of I_s at ζ_s, and d is not subgradient of J at ζ_s; since $J(\zeta_s) = I_s(\zeta_s)$ — as established above —, there exists $x \in \mathbb{R}$ such that

$$I_s(x) \geq I_s(\zeta_s) + d(x - \zeta_s) > J(x),$$

which implies $I_s(x) > J(x)$. The last inequality contradicts the definition of J. ■

18 Optimal signal characterization

After a lemmata, we prove a result (Lemma 62) that implies Lemma 10.

Model Primitives Receiver's material payoff from taking action $a \in \{0, 1\}$, when the state is $\theta \in [0, 1]$, is $a(\theta - c)$, for the Receiver's outside option $c \in [0, 1]$. Receiver's cost of effort $e \in [0, 1]$ is $K(e, \lambda)$ for a lower semi continuous function $K(\cdot, \lambda)$ and the Receiver's attention cost $\lambda \in \mathbb{R}$. We define the Receiver's ex-post payoff as her material payoff net of effort cost, $u(\theta, a, e, c, \lambda) = a(\theta - c) - K(e, \lambda)$. We let the Receiver's type (c, λ) be supported on T and its distribution admit a conditional density function, so that the probability density function is $(c, \lambda) \mapsto g(c, \lambda) = g_{c|\lambda}(c|\lambda) g_\lambda(\lambda)$.

Given that $\theta \mapsto u(\theta, a, e, c, \lambda)$ is affine, the value of Recever's optimal action at a posterior belief with mean x is:

$$U(x,e,c,\lambda) := \max_{a\in\{0,1\}} u(x,a,e,c,\lambda).$$

Given $F \in \mathcal{F}$ and an effort $e \in [0,1]$, we define $e \odot F = eF + (1-e)\overline{F}$, and note that $e \odot F \in \mathcal{F}$. An equilibrium is a tuple $\langle F, e, \alpha\rangle$, in which $F \in \mathcal{F}$ is the Sender's choice of experiment, $e(\cdot,\hat{F})\colon T \to [0,1]$ is measurable for all $\hat{F} \in \mathcal{F}$, $\alpha(\cdot,x)\colon T \to [0,1]$ is measurable for all $x \in [0,1]$, and $\alpha(c,\lambda,\cdot)\colon [0,1] \to [0,1]$ is measurable for all $(c,\lambda) \in T$, such that:

(1) α satisfies a Opt:

$$\alpha(c,\lambda,x) > 0 \text{ only if } 1 \in \arg\max_{a\in\{0,1\}} u(x,a,e,c,\lambda) \text{ for all } x \in [0,1],\ (c,\lambda) \in T.$$

(2) e satisfies e Opt:

$$e(c,\lambda,\hat{F}) \in \arg\max_{e\in[0,1]} \int_{[0,1]} U(x,e,c,\lambda)\,\mathrm{d}(e(c,\lambda,F)\odot F)(x), \text{ for all } (c,\lambda)\in T, \hat{F}\in\mathcal{F}.$$

(3) F is rational for Sender, given (α, e), that is: F maximizes

$$F \mapsto \int_{\mathbb{R}}\int_{[0,1]}\int_{[0,1]} \alpha(x,c,\lambda)\,\mathrm{d}(e(c,\lambda,F)\odot F)(x)\,\mathrm{d}G_{c|\lambda}(c|\lambda)\,\mathrm{d}G_{\lambda}(\lambda), \text{ on } \mathcal{F}.$$

[53]

18.0.1 Properties of Equilibrium Objects

Lemma 58 (Value of Information)**.** *The following holds:*

$$\int_{[0,1]} U(x,e,c,\lambda) - U(x_0,e,c,\lambda)\,\mathrm{d}\hat{F}(x) = \Delta I_{\hat{F}}(c), \text{ for all } c,\lambda,e,\hat{F}.$$

Proof. By definition of U, letting $\alpha(c,x)$ be any distribution over $\{0,1\}$ such that $\alpha(c,x)\Big(\arg\max_{a\in\{0,1\}} a(x-c)\Big) = 1$, we have:

$$\begin{aligned}\int_{[0,1]} U(x,e,c,\lambda)\,\mathrm{d}F(x) + K(e,\lambda) &= \int_{[c,1]} x - c\,\mathrm{d}F(x) - (1-\alpha(c,c)(\{1\}))(F(c)-F(c^-))(c-c)\\ &= \int_{[c,1]} x - c\,\mathrm{d}F(x).\end{aligned}$$

By Riemann-Stieltjes integration by parts:

$$\int_{[c,1]} x - c\,\mathrm{d}F(x) = (1-c) - F(c)(c-c) - \int_{[c,1]} F(x)\,\mathrm{d}x.$$

[53] The last condition implicitely assumes that Sender cannot signal what she does not know. The reason is that Sender directly chooses a Bayes-plausible Receiver's belief distribution, i.e., an element of $\mathcal{F}$. One could extend the notation to allow Sender to choose a distribution supported on $\mathcal{F}$. This extension does not change our results, since Sender does not signal what she does not know, and the choice of random experiment is publicly committed to. By the definition of a Opt, α denotes the probability of action 1.

By continuity of I_F:

$$\int_{[c,1]} x - c \,\mathrm{d}F(x) = 1 - c - I_F(1) + I_F(0) + I_F(c). \tag{17}$$

Since $I_F(1) = 1 - x_0$ and $I_F(0) = 0$, we have that:

$$\int_{[0,1]} U(x, e, c, \lambda)\,\mathrm{d}\hat{F}(x) - \int_{[0,1]} U(x, e, c, \lambda)\,\mathrm{d}\overline{F}(x) = \Delta I_{\hat{F}}(c)$$

Since $\int_{[0,1]} U(x, e, c, \lambda)\,\mathrm{d}\overline{F}(x) = U(x_0, e, c, \lambda)$, the result follows. ■

Lemma 59 (Recever's Rationality)**.** *If $\langle F, e, \alpha\rangle$ is an equilibrium, then:*

(1) $1 - \int_{[0,1]} \alpha(c, \lambda, x)\,\mathrm{d}\hat{F}(x) \in \partial I_{\hat{F}}(c)$, *for all* $(c, \lambda) \in T$ *and* $\hat{F} \in \mathcal{F}$*;*

(2) $e(c, \lambda, \hat{F}) \in \arg\max_{e\in[0,1]} e\Delta I_{\hat{F}}(c) - K(e, \lambda)$, *for all* $(c, \lambda) \in T, \hat{F} \in \mathcal{F}$*.*

Proof. For part (1), we observe that, by a Opt:

$$1 - \int_{[0,1]} \alpha(c, \lambda, x)\,\mathrm{d}\hat{F}(x) \in \left[\int_{[0,c)} \mathrm{d}\hat{F}(x), \int_{[0,c]} \mathrm{d}\hat{F}(x)\right].$$

For part (2), we use Lemma 58 to express e Opt, given F, as:

$$e(c, \lambda) \in \arg\max_{e\in[0,1]} e\Delta I_F(c) + U(x_0, e, c, \lambda).$$

By the derivation in the proof of Lemma 58 (Equation 17) and $\int_{[0,1]} U(x, e, c, \lambda)\,\mathrm{d}\overline{F}(x) = U(x_0, e, c, \lambda)$, we have:

$$e\Delta I_F(c) + U(x_0, e, c, \lambda) = e\Delta I_F(c) + 1 - c - I_{\overline{F}}(1) + I_{\overline{F}}(0) + I_{\overline{F}}(c) - K(e, \lambda)$$

Using $I_{\overline{F}}(1) = 1 - x_0$ and $I_{\overline{F}}(0) = 0$:

$$e\Delta I_F(c) + U(x_0, e, c, \lambda) = e\Delta I_F(c) + x_0 - c + I_{\overline{F}}(c) - K(e, \lambda).$$

Part (2) follows because $x_0 - c + I_{\overline{F}}(c)$ is a constant with respect to e. ■

Lemma 60. *There exist: (i) a measurable selection from* $(c, \lambda) \mapsto \max_{a\in\{0,1\}} u(x, a, e, c, \lambda)$ *for all* $e, x \in [0, 1]$*, (ii) a measurable selection from* $x \mapsto \max_{a\in\{0,1\}} u(x, a, e, c, \lambda)$ *for all* $(c, \lambda) \in T, e \in [0, 1]$*, and (iii) a measurable selection from* $(c, \lambda) \mapsto \arg\max_{e\in[0,1]} e\Delta I_F(c) - K(e, \lambda)$ *for all* $F \in \mathcal{F}$*.*

Proof. The nontrivial part is to show (iii). By Lemma 59, Receiver is maximizing a real-valued function that is continuous in c, λ, and the choice variable e. Thus, the Measurable Maximum Theorem (Aliprantis and Border (2006), Theorem 18.19) holds. ■

Definition 12. *The Receiver's value of an experiment* $F \in \mathcal{F}$ *is* $V_\lambda(e, \Delta I_F(c)) := e\Delta I_F(c) - K(e, \lambda)$*.*

Lemma 61 (Interval Structure of the Extensive Margin)**.** *If $\langle \hat{F}, e, \alpha\rangle$ is an equilibrium, and* $e^\star_\lambda \colon c \mapsto e(c, \lambda, F), F \in \mathcal{F}$*, then:* $e^\star_\lambda((0, 1])$ *is an interval.*

Proof. Let $\langle \hat{F}, e, \alpha\rangle$ is an equilibrium. First, we make the preliminary observation that: $e(\cdot,\lambda,F)$, in the statement of the lemma, is equal to $e^*\circ\Delta I_F$ for some selection e^* from $\Delta I_F(c)\mapsto \arg\max_{e\in[0,1]} V_\lambda(e,\Delta I_F(c))$.

Every selection e^* from $\Delta I_F(c)\mapsto \arg\max_{e\in[0,1]} V_\lambda(e,\Delta I_F(c))$ is nondecreasing, because V_λ satisfies strictly increasing differences on $[0,1]^2$ and $\Delta I_F(c)\in[0,1]$. Since ΔI_F is nondecreasing on $[0,x_0]$ and ΔI_F is nonincreasing on $[x_0,1]$, $e^*\circ\Delta I$ is nondecreasing on $[0,x_0]$, and nonincreasing on $[x_0,1]$. We define:

$$\begin{aligned}\underline{c} &:= \inf\{c\in[0,x_0] : e^*\circ\Delta I(c)>0\}\\ \overline{c} &:= \sup\{c\in[x_0,1] : e^*\circ\Delta I(c)>0\},\end{aligned}$$

if the relevant set is nonempty; we set $\underline{c}=x_0$ and $\overline{c}=x_0$ if the relevant set is empty. The claim follows from the next two observations. First, we note that $e^*\circ\Delta I(c)>0$ only if: $c\in[\underline{c},\overline{c}]$. Second, we note that: $c\in(\underline{c},\overline{c})$ only if $e^*\circ\Delta I(c)>0$. ■

Definition 13. *(1) $\hat{F}\in\mathcal{F}$ is an equilibrium experiment if there exists an equilibrium $\langle F,e,\alpha\rangle$ with $\hat{F}(x)=F(x)$ for all $x\in\mathbb{R}$. (2) The Receiver's value from the experiment $F\in\mathcal{F}$ is: $V_\lambda(\Delta I_F(c)) := \max_{e\in[0,1]}\hat{V}_\lambda(e,\Delta I_F(c))$, in which $\hat{V}_\lambda(e,\Delta I_F(c)) = e\Delta I_F(c)-K(e,\lambda)$. (3) $F\in\mathcal{F}$ is an optimal experiment if:*

$$F\in\arg\max_{\hat{F}\in\mathcal{F}}\int_{\mathbb{R}}\int_{[0,1]} V_\lambda(\Delta I_{\hat{F}}(c))\frac{\partial g_{c|\lambda}}{\partial c}(c|\lambda)\,\mathrm{d}c\,\mathrm{d}G_\lambda(\lambda).$$

(4) We say that there are multiple Sender's payoffs if: there exist distinct equilibria $\langle F,e,\alpha\rangle$ and $\langle\hat{F},\hat{e},\hat{\alpha}\rangle$ such that:

$$\int_{[0,1]}\int_{[0,1]}\alpha(x,c,\lambda)\,\mathrm{d}(e(c,\lambda)\odot F)(x)\,\mathrm{d}G_{c|\lambda}(c|\lambda)\neq\int_{[0,1]}\int_{[0,1]}\hat{\alpha}(x,c,\lambda)\,\mathrm{d}(\hat{e}(c,\lambda)\odot\hat{F})(x)\,\mathrm{d}G_{c|\lambda}(c|\lambda).$$

Lemma 62 (Uniqueness of Sender's Payoff)**.** *$F\in\mathcal{F}$ is an equilibrium experiment if, and only if: F is an optimal experiment. Moreover: if $G_{c|\lambda}(\cdot|\lambda)$ and $g_{c|\lambda}(\cdot|\lambda)$ are absolutely continuous for all λ, then there are not multiple Sender's payoffs.*

Proof. We show that: F is optimal if, and only if: F is rational for Sender, given (α,e), α satisfies a Opt, and e satisfies e Opt. This is accomplished by establishing that the mapping $D_\lambda(\cdot,\alpha,e)$ such that

$$D_\lambda(\cdot,\alpha,e)\colon F\mapsto\int_{[0,1]}\int_{[0,1]}\alpha(x,c,\lambda)\,\mathrm{d}(e(c,\lambda,F)\odot F)(x)\,\mathrm{d}G_{c|\lambda}(c|\lambda)-\int_{[0,1]}V_\lambda(\Delta I_{\hat{F}}(c))\frac{\partial g_{c|\lambda}}{\partial c}(c|\lambda)\,\mathrm{d}c$$

is constant (in F,) for all λ. As a preliminary step, we note that $e(c,\lambda,F)=e^*_\lambda(\Delta I_F(c))$, for all $c\in[0,1]$ and a selection e^* from $\arg\max_{a\in\{0,1\}} e\Delta I(c)-K(e,\lambda)$, by e Opt, given F.

First, let's write $W(F):=\int_{[0,1]}\int_{[0,1]}\alpha(x,c,\lambda)\,\mathrm{d}(e^*_\lambda(\Delta I_F(c))\odot F)(x)\,\mathrm{d}G_{c|\lambda}(c|\lambda)$ as follows:

$$W(F)=\int_{[0,1]}\int_{[0,1]}e^*_\lambda(\Delta I_F(c))(\alpha(x,c,\lambda)-\alpha(x_0,c,\lambda))\,\mathrm{d}F(x)\,\mathrm{d}G_{c|\lambda}(c|\lambda)+\int_{[0,1]}\alpha(x_0,c,\lambda)\,\mathrm{d}G_{c|\lambda}(c|\lambda).$$

Thus, by Lemma 59, there exists a selection d_I^1 from the subdifferential of ΔI_F on $[0,x_0]$ and a selection

d_I^2 from the subdifferential of ΔI_F on $(x_0, 1]$ such that:

$$-\Big(W(F) - W(\overline{F})\Big) = \int_{[0,x_0]} e_\lambda^*(\Delta I_F(c)) d_I^1(c)\, \mathrm{d}G_{c|\lambda}(c|\lambda) + \int_{(x_0,1]} e_\lambda^*(\Delta I_F(c)) d_I^2(c)\, \mathrm{d}G_{c|\lambda}(c|\lambda)$$

By the envelope theorem (Lemma 57), e_λ^* is a selection from the subdifferential of the convex and nondecreasing function V_λ. By $\Delta I_F \in \mathcal{A}$, ΔI_F is: (i) convex on $[0, x_0]$, and (ii) convex on $(x_0, 1]$. Hence: by the rules of subdifferential calculus (Fact 2), there exists a selection d from the subdifferential of $V_\lambda \circ \Delta I_F$ such that: $d(c) = e_\lambda^*(\Delta I_F(c)) d_I^1(c)$, for all $c \in [0, x_0]$, and $d(c) = e_\lambda^*(\Delta I_F(c)) d_I^2(c)$, for all $c \in (x_0, 1]$. Hence:

$$\begin{aligned}
-\Big(W(F) - W(\overline{F})\Big) &= \int_{[0,x_0]} d(c)\, \mathrm{d}G_{c|\lambda}(c|\lambda) + \int_{(x_0,1]} d(c)\, \mathrm{d}G_{c|\lambda}(c|\lambda) \\
&= \int_{[0,x_0]} d(c)\, \mathrm{d}G_{c|\lambda}(c|\lambda) + \int_{[x_0,1]} d(c)\, \mathrm{d}G_{c|\lambda}(c|\lambda),
\end{aligned}$$

in which the second equality uses absolute continuity of $G_{c|\lambda}(\cdot|\lambda)$. By Fact 2, the composition $V_\lambda \circ \Delta I_F$ is a convex function on $[0, x_0]$, so $V_\lambda \circ \Delta I_F$ is the integral of any selection from the its subdifferential (Corollary 24.2.1 in Rockafellar (1970)) on $[0, x_0]$. Similarly, $V_\lambda \circ \Delta I_F$ is a convex function on $[x_0, 1]$. By absolute continuity of $g_{c|\lambda}(\cdot|\lambda)$, we integrate by parts to obtain:

$$-\Big(W(F) - W(\overline{F})\Big) = V_\lambda \circ \Delta I_F(1) g_{c|\lambda}(1|\lambda) - V_\lambda \circ \Delta I_F(0) g_{c|\lambda}(0|\lambda) - \int_{[0,1]} V_\lambda \circ \Delta I_F(c) \frac{\partial g_{c|\lambda}}{\partial c}(c|\lambda)\, \mathrm{d}c.$$

Since $\Delta I_F(1) = \Delta I_F(0) = 0$, we have:

$$-\Big(W(F) - W(\overline{F})\Big) = \Big(g_{c|\lambda}(1|\lambda) - g_{c|\lambda}(0|\lambda)\Big) V_\lambda(0) - \int_{[0,1]} V_\lambda \circ \Delta I_F(c) \frac{\partial g_{c|\lambda}}{\partial c}(c|\lambda)\, \mathrm{d}c.$$

Hence:

$$W(F) = \int_{[0,1]} V_\lambda \circ \Delta I_F(c) \frac{\partial g_{c|\lambda}}{\partial c}(c|\lambda)\, \mathrm{d}c + W(\overline{F}) - \Big(g_{c|\lambda}(1|\lambda) - g_{c|\lambda}(0|\lambda)\Big) V_\lambda(0).$$

So:

$$D_\lambda(F, \alpha, e) = \int_{\mathbb{R}} \Big(g_{c|\lambda}(1|\lambda) - g_{c|\lambda}(0|\lambda)\Big) V_\lambda(0) + \int_{[0,1]} \alpha(x_0, c, \lambda)\, \mathrm{d}G_{c|\lambda}(c|\lambda)\, \mathrm{d}G_\lambda(\lambda).$$

Hence, $D_\lambda(\cdot, \alpha, e)$ is constant on $\mathcal{F}$. Hence, F is optimal if, and only if: F is rational for Sender, given (α, e), α satisfies a Opt, and e satisfies e Opt. It follows that: if $\langle \hat{F}, e, \alpha \rangle$ is an equilibrium, then $\hat{F}$ is optimal.

For the second direction, let F be optimal. By Lemma 60, there exist e and α that satisfy the equilibrium measurability conditions, a Opt, and e Opt, given F. Since F is optimal, F is rational for Sender, given (α, e), by the above equivalence (i.e., because $D_\lambda(\cdot, \alpha, e)$ is constant on $\mathcal{F}$.) Thus, $\langle F, e, \alpha \rangle$ is an equilibrium.

Let's verify that there are not multiple Sender's payoff. Suppose that there are multiple Sender's payoffs, given by distinct equilibria $\langle F, e, \alpha \rangle$ and $\langle \hat{F}, \hat{e}, \hat{\alpha} \rangle$. By the above result, F and $\hat{F}$ are optimal. Hence: $\int_{\mathbb{R}} D_\lambda(F, \alpha, e)\, \mathrm{d}G_\lambda(\lambda) \neq \int_{\mathbb{R}} D_\lambda(\hat{F}, \hat{\alpha}, \hat{e})\, \mathrm{d}G_\lambda(\lambda)$. Since D_λ is constant in the experiment, it must be that $D_\lambda(F, \alpha, e)$ varies when e and α are substituted by $\hat{e}$ and $\hat{\alpha}$, for some λ. However,

$D_\lambda(F,\alpha,e) = D_\lambda(F,\alpha,\hat{e})$, and $g_{c|\lambda}(\cdot|\lambda)$ is absolutely continuous. Hence, $D_\lambda(F,\alpha,e) = D_\lambda(F,\hat{\alpha},\hat{e})$. As an implication, there are not multiple Sender's payoff. ■

Lemma 63. *The real-valued function*

$$W\colon F \mapsto \int_{\mathbb{R}}\int_{[0,1]} V_\lambda(\Delta I_F(c))\frac{\partial g_{c|\lambda}}{\partial c}(c|\lambda)\,\mathrm{d}c\,\mathrm{d}G_\lambda(\lambda)$$

is continuous on $\mathcal{F}$, endowed with the L_1 norm.

Proof. Let's fix λ, $\delta > 0$ and $F \in \mathcal{F}$, and define $p_\lambda := \int_{[0,1]}\left|\frac{\partial g_{c|\lambda}}{\partial c}(c|\lambda)\right|\mathrm{d}c \geq 0$.

Let's define $\varepsilon = \frac{\delta}{p_\lambda}$ if $p_\lambda > 0$, and let ε be an arbitrary positive number otherwise. Let $H \in \mathcal{F}$ such that:

$$\int_{[0,1]}|H(x) - F(x)|\,\mathrm{d}x < \varepsilon.$$

The proof consists of three steps.

First, we stablish the preliminary claim that: $|V_\lambda(\Delta I_H(c)) - V_\lambda(\Delta I_F(c))| < \varepsilon$. By definition of V_λ and the envelope theorem (Lemma 57), there exists a selection e from $c \mapsto \arg\max_{e\in[0,1]} e\Delta I_F(c) - K(e,\lambda)$ such that:

$$|V_\lambda(\Delta I_H(c)) - V_\lambda(\Delta I_F(c))| = \int_{[\min\{\Delta I_H(c),\Delta I_F(c)\},\max\{\Delta I_H(c),\Delta I_F(c)\}]} e(a)\,\mathrm{d}a.$$

Since the codomain of e is $[0,1]$, by the above equality:

$$|V_\lambda(\Delta I_H(c)) - V_\lambda(\Delta I_F(c))| \leq |\Delta I_H(c) - \Delta I_F(c)|.$$

We have the following chain of inequalities,

$$\begin{aligned}|V_\lambda(\Delta I_H(c)) - V_\lambda(\Delta I_F(c))| &\leq \left|\int_{[0,c]} H(x) - F(x)\,\mathrm{d}x\right|\\ &\leq \int_{[0,c]}|H(x) - F(x)|\,\mathrm{d}x\\ &\leq \varepsilon,\end{aligned}$$

which establishes the preliminary claim.

Second, we establish the continuity of the function $W_\lambda\colon F \mapsto \int_{[0,1]} V_\lambda(\Delta I_F(c))\frac{\partial g_{c|\lambda}}{\partial c}(c|\lambda)\,\mathrm{d}c$ on $\mathcal{F}$. We have the following chain of inequalities:

$$\begin{aligned}|W_\lambda(H) - W_\lambda(F)| &\leq \int_{[0,1]}\left|V_\lambda(\Delta I_H(c))\frac{\partial g_{c|\lambda}}{\partial c}(c|\lambda) - V_\lambda(\Delta I_F(c))\right|\left|\frac{\partial g_{c|\lambda}}{\partial c}(c|\lambda)\right|\mathrm{d}c\\ &\leq \varepsilon p_\lambda\\ &\leq \delta.\end{aligned}$$

Thirdly, we have the following chain of inequalities:

$$\begin{aligned} |W(H) - W(F)| &\leq \int_{\mathbb{R}} |W_\lambda(H) - W_\lambda(F)| \, \mathrm{d}G(\lambda) \\ &\leq \delta. \end{aligned}$$

■

Proposition 19. *An equilibrium exists.*

Proof. First, we observe that the set $\mathcal{F}$, when we identify functions that are equal almost everywhere, is compact in the topology induced by the L^1 norm (this result is a known consequence of Helly's Selection Theorem, see, e.g., Proposition 1 in Kleiner et al. (2021).)

The result follows from Weierstrass' Theorem and Lemma 62 via upper semi continuity of the Sender's maximand function in the definition of an optimal experiment (Lemma 63). ■

Remark 11. *The L^1 norm metrizes weak convergence (Lemma 1 in Machina (1982).)*

18.0.2 Optimality Properties of Upper Censorships

Definition 14. *(1) The distribution of Receiver's type (c, λ) satisfies single-peakedness if: (i) $g_{c|\lambda}(\cdot|\lambda)$ is absolutely continuous for all λ, and (ii) there exists $p \in [0, 1]$ such that: for all λ, $g_{c|\lambda}(\cdot|\lambda)$ is nondecreasing on $[0, p]$ and nonincreasing on $[p, 1]$. (2) The distribution of Receiver's type (c, λ) satisfies strict single-peakedness if: (i) $g_{c|\lambda}(\cdot|\lambda)$ is absolutely continuous for all λ, and (ii) there exists $p \in [0, 1]$ such that: for all λ, $g_{c|\lambda}(\cdot|\lambda)$ is increasing on $[0, p]$ and decreasing on $[p, 1]$. (3) The distribution of Receiver's type (c, λ) satisfies conditional single-peakedness if: (i) $g_{c|\lambda}(\cdot|\lambda)$ is absolutely continuous for all λ, and (ii) for all λ, there exists $p_\lambda \in [0, 1]$ such that $g_{c|\lambda}(\cdot|\lambda)$ is nondecreasing on $[0, p_\lambda]$ and nonincreasing on $[p_\lambda, 1]$.*

Remark 12. *Log-Concave probability density functions are quasi-concave (An (1995), and references therein.) Hence, if the conditional density of c given λ is absolutely continuous and log-concave for all λ, then (c, λ) satisfies single-peakedness. A second sufficient condition is that: (i) the joint distribution of (c, λ) is log-concave, and (ii) the conditional density of c given λ is absolutely continuous. Single-Peakedness implies conditional single peakedness but the converse does not necessarily hold.*

Lemma 64. *Let the distribution of Receiver's type (c, λ) satisfy single-peakedness. There exists an optimal experiment that is an upper censorship.*

Proof. By Lemma 62, the optimal experiment maximizes W defined as:

$$W(F) \colon F \mapsto \int_{\mathbb{R}} \int_{[0,p]} V_\lambda(\Delta I_{\hat{F}}(c)) \frac{\partial g_{c|\lambda}}{\partial c}(c|\lambda) \, \mathrm{d}c + \int_{[p,1]} V_\lambda(\Delta I_{\hat{F}}(c)) \frac{\partial g_{c|\lambda}}{\partial c}(c|\lambda) \, \mathrm{d}c \, \mathrm{d}G_\lambda(\lambda).$$

Suppose two experiments $F, H \in \mathcal{F}$ have information policies given by $I = I_F, J = I_H$ such that: $I(x) \geq J(x)$ for all $x \in [0, p]$ and $I(x) \leq J(x)$ for all $x \in [p, 1]$. Because (i) V_λ is nondecreasing, (ii) $\frac{\partial g_{c|\lambda}}{\partial c}(\cdot|\lambda)$ is nonnegative on $[0, p]$ and nonpositive on $[p, 1]$, it follows that $I_F \geq I_H$, it follows that $W(F) \geq W(H)$.

The result follows from Lemma 56. ■

18.1 Proof of Proposition 2

Proof. If an optimal information policy exists, the Proposition follows from Lemma 15 and Lemma 56. Existence established. ■

18.2 Proof of Proposition 13

We prove several lemmata.

Lemma 65. *Let $I \in \mathcal{I}$ such that $p \geq \underline{c}(\Delta I)$, there exists another information policy $I^\star$ such that:*

(FEAS) $I^\star$ is feasible: $I^\star \in \mathcal{I}$,

(EM) $I^\star$ produces the same extensive margin as I: $\underline{c}(\Delta I^\star) = \underline{c}(\Delta I)$, $\bar{c}(\Delta I^\star) = \bar{c}(\Delta I)$.

(IMPR)

$$\Delta I^\star(x) \geq 0, \text{ for all } x \in [\bar{c}(\Delta I), \bar{c}(\Delta I)]$$

.

(CENS) There exist $x_\ell, \theta_\ell, \theta_m, x_m$ such that $0 \leq x_\ell \leq \theta_\ell \leq \theta_m \leq x_m \leq 1$, and:

$$I^\star(x) = \begin{cases} I_{\overline{F}}(x) & , x \in [0, x_\ell] \\ I_{F_0}(\theta_\ell) + F_0(\theta_\ell)(x - \theta_\ell) & , x \in (x_\ell, \theta_\ell] \\ I_{F_0}(x) & , x \in (\theta_\ell, \theta_m] \\ I_{F_0}(\theta_m) + F_0(\theta_m)(x - \theta_m) & , x \in (\theta_m, x_m] \\ \overline{I}_{\overline{F}}(x) & , x \in (x_m, \infty]. \end{cases}$$

Proof. We use the following notation: $\underline{c}(I - \overline{I}) =: \underline{c}$, $\bar{c}(I - \overline{I}) =: \bar{c}$. In the first step, we prove the lemma in the case where there is a feasible information policy that is a straight line between the points $\overline{p} := (\bar{c}, I(\bar{c}))$ and $\underline{p} := (\underline{c}, I(\underline{c}))$. In the second step, we prove the lemma in the case where there is not a feasible information policy that is a straight line between the points $\overline{p}$ and $\underline{p}$.

First Step. Let's define the line line i such that $x \mapsto I(\bar{c}) + \lambda^*(x - \bar{c})$, with slope $\lambda^* := \frac{I(\underline{c}) - I(\bar{c})}{\underline{c} - \bar{c}}$. We claim that $i^\star(x) := \max\{i(x), \overline{I}_0(x)\}$ satisfies all properties. It is FEAS by hypothesis. It is EXT because $i(\bar{c}) = I(\bar{c})$ and $i(\underline{c}) = I(\underline{c})$. It is IMPR because I is convex and $i^\star$ is EXT. It is CENS with $\theta_\ell = \theta_m = x_m$, because: (i) EXT of $i^\star$ and convexity of I imply that $i^\star$ is affine in $[\bar{c}, \underline{c}]$, (ii) $\lambda^* \in [0, 1]$ and EXT imply, with $I \in \mathcal{I}_0$ that there are intersections $\widetilde{x}_1, \widetilde{x}_2$, with $\widetilde{x}_1 \leq \bar{c} \leq \underline{c} \leq \widetilde{x}_2$, where: $i^\star(x) = \overline{I}(x)$ if $x \in [0, \widetilde{x}_1] \cup [\widetilde{x}_2, 1]$.

Second Step. In this case, $i^\star$ is not FEAS. Since $i^\star$ satisfies FEAS at x if $x \leq \bar{c}$ and if $x \geq \underline{c}$, there is a point $x^* \in (\bar{c}, \underline{c})$ such that $i(x^*) > I_{F_0}(x^*)$.

$$L := \{\lambda \in [I'(\bar{c}), 1] \mid I(\bar{c}) + \lambda(x - \bar{c}) \leq I_{F_0}(x) \text{ for all } x \in [\bar{c}, \infty)\},$$
$$M := \{\lambda \in [0, I'(\underline{c})] \mid I(\underline{c}) + \lambda(x - \underline{c}) \leq I_{F_0}(x) \text{ for all } x \in [0, \bar{c}]\}.$$

$\ell := \max L$, $m := \min M$. We define two lines:

$$y_\ell \text{ is: } x \mapsto I(\bar{c}) + \ell(x - \bar{c})$$
$$y_m \text{ is: } x \mapsto I(\bar{c}) + m(x - \bar{c}).$$

We prove a lemmata.

Lemma 66. *ℓ, m are well-defined.*

Proof. L is nonempty because $I'(\overline{c}) \in L$, which follows from: (i) $I_{F_0}(x) \geq I(x)$ for all x and (ii) $I'(\overline{c}) \in \partial I(\overline{c})$. M is nonempty because $I'(\underline{c}) \in M$, which follows from: (i) $I_{F_0}(x) \geq I(x)$ for all x and (ii) $I'(\underline{c}) \in \partial I(\underline{c})$. L, M are closed because I_{F_0} is continuous. L, M are bounded. ∎

Lemma 67. *that there exists a unique pair of numbers $(\theta_\ell, \theta_m) \in [\overline{c}, 1] \times [0, \overline{c}]$ such that:*

$$y_\ell(\theta_\ell) = I_{F_0}(\theta_\ell)$$
$$y_m(\theta_m) = I_{F_0}(\theta_m)$$

Proof. Suppose there does not exists such a ℓ. There exists a sufficiently small $\varepsilon > 0$ such that: (i) $\ell + \varepsilon \in L$ and (ii) $I(\overline{c}) + (\ell + \varepsilon)(x - \overline{c}) < I_{F_0}(x)$ for all $x \in [\overline{c}, \infty)$; we note that $\ell = 1$ contradicts $\ell \in L$ because $I'_{F_0}(x) < 1$ if $x < 1$. Uniquenss of ℓ follows from convexity of I_{F_0}.

Suppose there does not exists such an m. There exists a sufficiently small $\varepsilon > 0$ such that: (i) $\ell - \varepsilon \in M$ and (ii) $I(\overline{c}) + (m - \varepsilon)(x - \overline{c}) < I_{F_0}(x)$ for all $x \in [0, \overline{c})$; we note that $m = 0$ contradicts $I \neq I_{\overline{F}}$. Uniquenss of m follows from convexity of I_{F_0}. ∎

Lemma 68. *$\theta_\ell \leq \theta_m$.*

Proof. Let's prove that it suffices to show that: $\ell \leq m$. Suppose $\ell \leq m$, then: since $\ell \in \partial I_{F_0}(\theta_\ell)$ and $m \in \partial I_{F_0}(\theta_m)$, and I_{F_0} is strictly convex, we have: $\theta_\ell \leq \theta_m$.

First, we show that $\ell \leq \lambda^*$. Suppose that: $\ell > \lambda^*$. Then: $I(x) + \ell(x - \overline{c}) > I(\overline{c}) + \lambda^*(x - \overline{c})$ for all $x > \overline{c}$. Therefore, since $\ell > 0$:

$$I_{F_0}(x^*) \geq I(\overline{c}) + \lambda^*(x^* - \overline{c}).$$

We reached a contradiction with the definition of x^*, so: $\ell \leq \lambda^*$.

Let's prove that $m \geq \lambda^*$. Suppose $m < \lambda^*$. Then: $I(x) + m(x - \underline{c}) > I(\underline{c}) + \lambda^*(x - \underline{c})$ for all $x < \underline{c}$. Therefore, since $m > 0$:

$$I_{F_0}(x^*) \geq I(\overline{c}) + \lambda^*(x^* - \overline{c}).$$

We reached a contradiction with the definition of x^*, so: $m \geq \lambda^*$. Therefore, we have $m \geq \lambda^* \geq \ell$, which implies $\theta_m \geq \theta_\ell$. ∎

We define a candidate $I^\star$ and we verify that it has the desired properties.

$$I^\star(x) := \begin{cases} \max\{I_{\overline{F}}(x), I(\overline{c}) + \ell(x - \overline{c})\} & , x \in [0, \theta_\ell] \\ I_{F_0}(x) & , x \in [\theta_\ell, \theta_m] \\ \max\{I_{\overline{F}}(x), I(\underline{c}) + m(x - \underline{c})\} & , x \in [\theta_m, \infty] \end{cases}$$

Let's first verify that $I^\star$ is well-defined. We know that $\ell \in \partial I_{F_0}(\theta_\ell)$ and $m \in \partial I_{F_0}(\theta_m)$. Since $I(\overline{c}) + \ell(0 - \overline{c}) < I_{F_0}(0)$ and $I(\overline{c}) \geq I_{F_0}(\overline{c})$, $\max\{I_{F_0}(x), I(\overline{c}) + \ell(x - \overline{c})\} = I_{F_0}(x)$ if $x < x_0$; and $\max\{I_{F_0}(x), I(\overline{c}) + \ell(x - \overline{c})\} = I(\overline{c}) + \ell(x - \overline{c})$ if $x > x_0$; for some $x_0 \in [0, \theta_\ell]$. In a similar way, we can show that there exists a $x_2 \in [\theta_m, 1]$ such that: $\max\{I_{F_0}(x), I(\underline{c}) + m(x - \underline{c})\} = I_{F_0}(x)$ if $x > x_2$, and $\max\{I_{F_0}(x), I(\underline{c}) + m(x - \underline{c})\} = I(\underline{c}) + m(x - \underline{c})$ if $x < x_2$.

(CENS) follows from the definition of $I^\star$ and its well-definedness, using $y_0 = x_0$, $y_1 = x_1$, $y_2 = x_3$, $y_3 = x_4$, and $\alpha_1 = \lambda_1$ and $\alpha_2 = \lambda_3$.

(IMPR) IMPR on $[\overline{c}, x_1]$ and $[x_3, \underline{c}]$ follows from convexity of I, and on $[x_1, x_3]$ follows from FEAS of I in that region.

(EM) follows from $I^\star(\overline{c}) = I(\overline{c}) + \lambda_1(x - \overline{c})$, and $I^\star(\underline{c}) = I(\underline{c}) + \lambda_3(x - \underline{c})$.

(FEAS) First, $I^\star$ is always above $\overline{I}_0$. Second $I^\star$ is always below $\underline{I}_0$, which follows from $\lambda_\ell \in \partial \underline{I}_0(x_\ell)$ for all $\ell \in \{1, 3\}$. The maximum of affine functions is convex, and $\underline{I}_0$ is convex. Global convexity then follows if $I^\star$ is subdifferentiable at x_1 and x_3. We now claim that $\lambda_\ell \in \partial I^\star(x_\ell)$ for all $\ell \in \{1, 3\}$. This claim follows from $\lambda_\ell \in \partial \underline{I}_0(x_\ell)$ for all $\ell \in \{1, 3\}$, and the fact that $\underline{I}_0(x_1) = I(\overline{c}) + \lambda_1(x_1 - \overline{c})$ and $\underline{I}_0(x_3) = I(\underline{c}) + \lambda_3(x_3 - \underline{c})$ (together with convexity of $I^\star$ in $[0, x_1]$ and $[x_3, 1]$). We established that the subdifferential of $I^\star$ at x_1 and x_3 nonempty, which finalizes the proof that $I^\star$ is globally convex.

■

Proof of Proposition 13

Proof. By the previous lemmata, to prove Proposition 13 we only need to prove the following claim.

Let $I \in \mathcal{I}$ such that $p < \underline{c}(\Delta I)$, there exists another information policy I° such that:

(FEAS) I° is feasible: $I^\circ \in \mathcal{I}$,

(EM) I° produces the same extensive margin as I: $\underline{c}(\Delta I^\circ) = \underline{c}(\Delta I)$, $\overline{c}(\Delta I^\circ) = \overline{c}(\Delta I)$.

(IMPR)

$$\Delta I^\circ(x) \geq 0, \text{ for all } x \in [\overline{c}(\Delta I), \overline{c}(\Delta I)]$$

.

(CENS) There exist $x_\ell, \theta_\ell, \theta_m, x_m, \theta_u, x_u$ such that $0 \leq x_\ell \leq \theta_\ell \leq \theta_m \leq x_m^\circ \leq x_u \leq 1$, and:

$$I^\circ(x) = \begin{cases} I_{\overline{F}}(x) & , x \in [0, x_\ell] \\ I_{F_0}(\theta_\ell) + F_0(\theta_\ell)(x - \theta_\ell) & , x \in (x_\ell, \theta_\ell] \\ I_{F_0}(x) & , x \in (\theta_\ell, \theta_m] \\ I_{F_0}(\theta_m) + F_0(\theta_m)(x - \theta_m) & , x \in (\theta_m, x_m^\circ] \\ I_{F_0}(\theta_u) + F_0(\theta_u)(x - \theta_u) & , x \in (x_m^\circ, x_u] \\ \overline{I}_{\overline{F}}(x) & , x \in (x_u, \infty). \end{cases}$$

The claim follows from taking $I^\star$ from the previous lemmata until the point x_m° where $I^\star$ intercepts the line j: $x \mapsto I(\overline{c}) + I'(\overline{c})(x - \overline{c})$, and $\max\{I_{\underline{F}}, j\}$ after x_m°. ■

18.3 Known ζ and κ

We assume, in this section only, that Sender knows both ζ and κ. If $\zeta > 1$, any information policy is optimal. If $\zeta \leq \theta_0$, $I_{\overline{F}}$ is optimal. Let $1 \geq \zeta \geq \theta_0$.

The Sender's problem is:

$$\max_{I \in \mathcal{I}} (1 - I'(\zeta_-))[\Delta I(\zeta) \geq \kappa].$$

Lemma 69. *There exists a solution to the Sender's problem $I \in \mathcal{I}$ such that: for $\theta \in [0, \zeta]$, I is the θ upper censorship and:*

$$\Delta I_\theta \leq \kappa,$$

with equality if $\theta > 0$.

Proof. Let $\mathcal{I}^u := \{I \in \mathcal{I} \ : \ I = I_\theta, \text{ for some } \theta \in [0,1] \text{ such that } \theta \leq \zeta\}$. Suppose the solution is not I_{F_0}. The Sender's problem is, without loss of optimality by lemma 56:

$$\max_{I \in \mathcal{I}^u} (1 - I'(\zeta_-))[\Delta I(\zeta) \geq \kappa].$$

Suppose there exists a solution $I \in \mathcal{I}^u$, such that $I = I_{\theta^\star}$, for some $\theta^\star \in (0,1)$. We distinguish three cases.

(1) If $\Delta I(\zeta) < \kappa$, then $I_{\overline{F}}$ achieves the same Sender payoff. (2) If $\Delta I(\zeta) = \kappa$, the lemma holds. (3) Let's suppose $\Delta I(\zeta) > \kappa$. By definition of I, at $y = I(\zeta)$ the next condition holds:

$$I_{F_0}(\theta^\star) + F_0(\theta^\star)(\zeta - \theta^\star) - y = 0.$$

By the implicit function theorem, there exists a differentiable function t:

$$\begin{aligned} t\colon (0,1) &\to (0,1) \\ y &\mapsto \theta^\star, \end{aligned}$$

such that:

$$t'(y) = \begin{cases} \frac{1}{(\zeta - t(y))F_0'(t(y))} & , 0 < \zeta < t(y) \\ \frac{1}{F_0'(t(y))} & , 1 > \zeta \geq t(y). \end{cases}$$

Let the value of I_θ be:

$$\begin{aligned} v\colon (0,1) &\to [0,1] \\ \theta &\mapsto (1 - I_\theta'(\zeta_-)) \end{aligned}$$

Because $I_{\theta^\star}'(\zeta_-) = F_0(\theta^\star)$, v is differentiable in θ at $\theta^\star$. Using the chain rule, the derivative of v with respect to $I(\zeta)$ is:

$$-F_0'(t(I(\zeta))) \frac{1}{(\zeta - t(I(\zeta)))F_0'(t(I(\zeta)))},$$

whenever $\zeta > t(I(\zeta))$, and -1 otherwise. It follows that we can consider without loss solutions $I \in \mathcal{I}^u$ that satisfy: $\Delta I_\theta(\zeta) = \kappa$ and $I = I_\theta$, or $\Delta I(\zeta) < \kappa$. ■

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