Heegaard Floer Homology and Link Detection

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Heegaard Floer homology is a family of invariants in low dimensional topology due originally to Ozsváth-Szabó. We discuss various aspects of Heegaard Floer homology and give several link detection results for versions of Heegaard Floer homology for links. In particular, we show that knot and link Floer homology detect various infinite families of cable links. We also give classification results for the Heegaard Floer theoretic invariants of a type of knot called an "almost L-space knot" and an infinite family of detection results for annular Khovanov homology.

Contents

List of Figures

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Chapter 1

Introduction

We begin by providing some general background in knot theory, as well as a brief discussion of the main results contained in this thesis.

1.1 Background and context

Let Y be a three manifold. A *knot* is an oriented embedding $S^1 \to Y$. A *link* is an oriented embedding $\bigsqcup S^1 \to Y$. We are interested in studying knots and links up to isotopy. Knots are natural objects of study in their own right and have been examined mathematically since the early 19th century. They have also proven essential for understanding low dimensional topology more generally – all three and four manifolds can be encoded using appropriately decorated links by work of Kirby [Kir78].

The central goal of knot theory is to classify all knots, a task initiated in earnest by the Scottish mathematician Tait in the 19th century. In some sense it is straightforward to verify that isotopic knots are indeed isotopic. Knots in $\mathbb{R}^3$ can be encoded via diagrams in $\mathbb{R}^2$ by projecting onto the x-y plane and any isotopy can be realised by a sequence of simple diagrammatic moves by work of Reidemeister [Rei27]. On the other hand, showing that two knots are not isotopic is in general a difficult task. To do so we build algebraic invariants. Examples of invariants include the Alexander and Jones polynomials as well as *knot Floer homology* and *Khovanov homology* which are stronger, vector space valued versions of the two preceding invariants. Knot Floer homology, which is due independently to Ozsváth-Szabó [OS04b] and J.Rasmussen [Ras03], is a member of the larger family of Heegaard Floer invariants. These invariants are defined using Heegaard diagrams, symplectic topology and analysis. Remarkably each Heegaard Floer theoretic invariant turns out to be an invariant of the underlying topological object, be that a knot or a 3 or 4-manifold.

We will be interested in the following question;

Question 1.1.1 (The Botany Question)**.** Which links have knot Floer homology of a given type?

If there is a unique link with given knot Floer homology, we say knot Floer homology "detects" that link. The first detection results given were for simple knots; Ozsváth-Szabó showed knot Floer homology detects the unknot [OS04a], Ghiggini proved that knot Floer homology detects the trefoil and figure eight knots [Ghi08]. New detection results have been given more recently by Farber-Reinoso-Wang [FRW22] and Baldwin-Sivek [BNS22]. On the other hand there are infinite families of links that are not distinguished by their knot Floer homologies, with examples given by Hedden-Watson [HW18] and Wang [Wan20].

We are also interested in the following natural question;

Question 1.1.2 (The Geography Question)**.** Which vector spaces arise as the knot Floer homology of some link?

There are a number of known results in this direction, most notably restraints on $\widehat{\mathrm{HFK}}(K)$ in terms of the genus of K by work of Ozsváth-Szabó [OS04a] and whether or not K is fibered by work of Ghiggini [Ghi08], Ni [Ni07] and Baldwin-Vela-Vick [BVV18].

1.2 Summary of results

Some of the results in this thesis have appeared in two papers, one that was joint work of the author and Martin [BM20], the other which was joint work with Dey [BD22a]. Results concerning almost L-space knots will appear in a forthcoming paper, while the remaining results may or may not eventually appear in published form outside of this thesis.

Knot Floer homology, link Floer homology and link detection

The paper "Knot Floer homology, link Floer homology and link detection" [BM20] is concerned with providing link detection results for various simple links. The work carried out in that paper which appears most directly in this thesis is the work on Annular Khovanov homology discussed in Section 4.3

Cable links, annuli and sutured Floer homology

The paper "Cable links, annuli and sutured Floer homology" [BD22a] is concerned with the study of links with the knot of link Floer homology type of certain cable knots. It is the basis for Chapter 3. The main results include the following;

Theorem *3.0.1*. Suppose K is a non-trivial L-space knot and $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(K_{2,2n})$ with $n > 2g(K) - 1$. Then L is isotopic to $K'_{2,2n}$ where K' is an L-space knot.

As well as the following corollary;

Corollary *3.0.4*. Knot Floer homology detects:

1. $T(2, 2n)$ for all $n \neq 0$
2. $T(2, 3)_{2,2n}$ for all $n > 1$
3. $T(2, 5)_{2,2n}$ for all $n > 3$

Here all of these links are oriented as the boundary of annuli.

Almost L-space knots

Chapter 5 is largely based on a forthcoming paper concerning almost L-space knots. The main result is the following;

Theorem *5.1.1.* Let K be an almost L-space knot. Then $\mathrm{CFK}^\infty(K)$ has the filtered chain homotopy type of one of the following complexes;

1. A staircase complex direct sum a box complex.
2. An almost staircase complex.

One consequence of this result – which can be proven using techniques similar to those used to prove Theorem 3.0.1 – is the following proposition;

Proposition *5.8.9.* Let K be an almost L-space knot. Suppose L is a link such that $\widehat{\mathrm{HFL}}(L) \cong \widehat{\mathrm{HFL}}(K_{m,mn})$ with $m > 2g(K) - 1$. Then L is the (m, mn)-cable of an almost L-space knot K' such that $\widehat{\mathrm{HFK}}(K') \cong \widehat{\mathrm{HFK}}(K)$.

Just as in the case of Theorem 3.0.1, this result yields a number of detection results as corollaries;

Proposition *5.8.10.* Link Floer homology detects the (m, mn)-cables of $T(2, -3)$, the figure eight knot and the mirror of 5_2 for $n > 1$.

Miscellaneous Results

Section 4 contains some material that has not appeared in papers to date and may or may not eventually appear in the future. The main two such results are the following;

Theorem *4.0.1.* Knot Floer homology detects $T(2, 2n)$ for all n.

Note that here $T(2, 2n)$ is not oriented as the boundary of an annulus, contrary to the case in Corollary 3.0.4. This allows one to prove, using results that first appeared in work of the author and Martin [BM20], the following result;

Theorem *4.3.1.* Annular Khovanov Homology detects $\widehat{\beta_{2n}}$ for all n.

Here β_n is the braid $\sigma_1\sigma_2\cdots\sigma_{n-1}$, where σ_i is the ith standard Artin generator of the braid group. We note that special cases of the above two results appeared in work of the author and Martin [BM20] as well as the author and Dey [BD22b].

1.3 Organization

This thesis is organised as follows; in Chapter 2 we survey background material on sutured manifolds and various Heegaard Floer homology theories; in Chapter 3 we discuss cables of L-space knots; in Chapter 4 we discuss involutions on know Floer homology and applications to the geography problem; in Chapter 5 we discuss almost L-space knots.

Conventions and notation

1. $Y^3_q(K)$ denotes q-surgery on a knot K in a 3-manifold Y.
2. K_n denotes the core of n surgery on a knot K.
3. $Sym^g(\Sigma)$ is the n-fold symmetric product of the space Σ

Chapter 2

A Survey of Heegaard Floer Homology

In this chapter we survey background material that will recur in subsequent chapters. In particular we will discuss sutured manifolds and variants of Heegaard Floer homology. The chapter is organised as follows: in Section 2.1 we discuss sutured manifolds, in 2.2 we discuss sutured Floer homology; in Section 2.3 we review Heegaard Floer homology for closed 3-manifolds; in Section 2.4 we review link Floer homology; and in Section 2.5 we conclude by discussing knot Floer homology.

2.1 Topological Underpinnings

Sutured manifolds are a type of 3-manifold with boundary that were first studied by Gabai, who used them to study foliations [Gab83].

Definition 2.1.1. A *balanced sutured manifold* is an oriented 3-manifold Y together with a decomposition of ∂Y into three parts; R_-, R_+ and γ such that;

1. γ is a collection of annuli,

2. $\partial R_+ \cap \partial R_- = \emptyset$,

3. each annulus shares one boundary component with R_+ and the other with R_-,

4. and $\chi(R_+) = \chi(R_-)$ for each connected compoennt of Y.

Here R_+ is given the outward normal orientation induced by ∂Y and R_- is given the opposite orientation. We note that Gabai originally gave a more general definition of a sutured manifold, allowing toroidal boundary componenets to count as sutured. However, sutured Floer homology is only defined for balanced sutured manifolds, whence we restrict our attention to balanced sutured manifolds. We duly suppress the term "balanced".

It is often helpful to consider the *core* of γ – i.e. the homologically essential simple closed curve in γ – which we denote $s(\gamma)$. Observe that ∂R_+ induces an orientation $s(\gamma)$ and conversely an orientation on $s(\gamma)$ induces orientations on γ, R_+ and R_-.

We give two examples of sutured manifolds which we will be interested in in subsequent sections and chapters.

Example 2.1.2. The exterior of a link L can be endowed with the structure of a sutured manifold be taking $s(\gamma)$ to be a collection of pairs of meridians for each link component.

Example 2.1.3. The exterior of a surface with boundary Σ in a 3-manifold Y can be endowed with the structure of a sutured manifold by taking as sutures neighborhoods of appropriate push-offs of the boundary components.

Definition 2.1.4. Let Y be a 3-manifold. The *Thurston norm* is a map; $| - | : H_2(Y, \partial Y) \to \mathbb{Z}$ given by

$$||[\alpha]|| = \min\{-\chi(\Sigma), 0 : \Sigma \text{ is a surface embedded in } Y \text{ such that } [\Sigma] = \alpha\} \qquad (2.1)$$

This definition was introduced by Thurston [Thu86] and can be thought of a generalization of the Seifert genus of a knot.

We can now give the following definition;

Definition 2.1.5. A sutured manifold (Y, γ) is *taut* if Y is irreducible and $R_\pm$ are Thurston norm-minimizing in $H_2(Y, \gamma)$.

We will in fact only be interested in taut sutured manifolds for the reason that the sutured Floer homology – an invariant we will define in Section 2.2 – of sutured manifolds which are not taut vanish [Juh08, Proposition 9.18].

We will be interested in a variety of surfaces properly embedded in sutured manifolds. In particular we will need the following definition;

Definition 2.1.6. A *decomposing surface* in a sutured manifold (Y, γ) is a compact, oriented surface with boundary $(S, \partial S) \subset (Y, \partial Y)$ such that for every component a of $\partial S \cap \gamma$ and we have that either;

1. a is a properly embedded non-separating arc in γ.

2. a is a circle which is essential in the component γ_i of γ it is contained in. In this case we require that the orientation of a agrees with that of $s(\gamma_i)$.

Decomposing surfaces are useful because they allow us to construct new sutured manifolds from old sutured manifolds. For the following let $\nu(S)$ denote a tubular neighborhood of a surface S in some ambient 3-manifold.

Definition 2.1.7. Let (Y, γ) be a sutured manifold and S be a decomposing surfaces S. Define a new sutured manifold (Y', γ') by setting $Y' = Y - \nu(S)$ topologically. Let $S_\pm$ be positive and negative push offs of S. Set $R_\pm = (R_\pm \cap \partial Y') \cup S_\pm$, and let γ be the remaining components of the boundary.

Note that technically Y' has corners. We smooth them and suppress any mention of such. This operation is called *sutured decomposition*. We us the short hand $(Y, \gamma) \overset{\Sigma}{\rightsquigarrow} (Y', \gamma')$ to indicate that (Y', γ') is obtained by decomposing (Y, γ) along the decomposing surface Σ.

One of the primary examples of a sutured manifold decomposition that we will be interested in is the following;

Example 2.1.8. If Σ is a Seifert surface for a link L then the exterior of L, viewed as a sutured manifold as in Example 2.1.2, admits a sutured decomposition along $\Sigma \cap X(L)$ to the exterior of Σ, viewed as a sutured manifold as in Example 2.1.3.

We will be especially interested in decomposing sutured manifolds along especially well behaved surfaces. We need the following definitions;

Definition 2.1.9. A curve c in a surface Σ is *boundary coherent* if either

1. $[c] \neq 0 \in H_1(\Sigma)$
2. $[c] = 0$ and c is oriented as the boundary of the component of $\Sigma - c$ that is disjoint from $\partial\Sigma$.

Definition 2.1.10. Two parallel arcs or curves, c_1, c_2 in a surface Σ are *coherently oriented* if $[c_1] = [c_2] \in H_1(\Sigma, \partial\Sigma)$.

For a given sutured manifold (Y, γ) (equipped with a Riemannian metric) there exists a non-vanishing vector field v_0 on ∂Y that points out of Y on R_+ and into Y along R_- and is the gradient of a height function $\gamma \cong s(\gamma) \times [0, 1] \to \mathbb{R}$.

Definition 2.1.11. A *nice* surface in a sutured manifold (Y, γ) is a surface Σ with non-empty boundary such that:

1. There is a v_0 as above such that v_0 is nowhere parallel to the normal vector field of Σ.
2. For each component r of $R_+ \cup R_-$ the set of closed components $\Sigma \cap r$ consists of parallel coherently oriented, boundary coherent simple closed curves.

Decomposition along nice decomposing surfaces will play nicely with sutured Floer homology, as we will see in Theorem 2.2.22.

2.2 Sutured Floer Homology

In this section we review aspects of Sutured Floer homology which will be relevant in later sections. We refer the reader to [Juh06] and [Juh08] for details. This section is organized as follows; in Subsection 2.2.1 we discuss Heegaard diagrams for sutured manifolds, in Subsection 2.2.2 we discuss the definition of sutured Floer homology, and in Subsection 2.2.3 we discuss some properties of Sutured Floer homology.

2.2.1 Sutured Heegaard Diagrams

Heegaard diagrams for closed three manifolds have been of interest to topologists for some time, given that every closed three manifold admits a Heegaard diagram and that any two Heegaard diagrams for a three manifold are related by a sequence of simple moves. In this subsection we discuss a version of Heegaard diagrams for sutured manifolds.

Definition 2.2.1. A *sutured Heegaard diagram* is a surface with boundary Σ and collections $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ $\{\beta_1, \beta_2, \cdots, \beta_n\}$ of pairwise disjoint simple closed homologically essential curves in Σ.

From a sutured Heegaard diagram one can obtain a sutured manifold.

Example 2.2.2. A sutured Heegaard diagram (Σ, α, β) encodes a sutured manifold (Y, γ) as follows;

1. As a topological manifold Y is obtained from $\Sigma \times [-1, 1]$ by attaching 3-dimensional 2-handles along each $\alpha_i \times \{-1\}$, $\beta_i \times \{1\}$ for all i.

2. γ is given by $\partial\Sigma \times [-1, 1]$ while $R_\pm$ consists of the component of $\partial Y - \gamma$ containing $(x, \pm 1)$ for all $x \notin \alpha_i, \beta_i$ for all i.

It might seem that sutured manifolds that are obtained by the above process are special but in fact they are not, by the following result of Juhasz;

Proposition 2.2.3. [Juh06, Proposition 2.13] Every sutured manifold admits a sutured Heegaard diagram.

This can be proven using techniques from Morse theory, just as the more classical case of closed manifold can be proven. In fact, a given sutured manifold can be represented by infinitely many sutured Heegaard diagrams. Fortunately any two such diagrams can be related by a sequence of relatively simple moves. We describe one of these operations now;

Definition 2.2.4. Consider $(S^1 \times S^1, a, b)$ where a, b are essential simple closed curves in $S^1 \times S^1$ which intersect transversely at a single point. Given a Heegaard diagram (Σ, α, β), the *stabilization* of H is given by $(\Sigma \# S^1 \times S^1, \alpha \cup a, \beta, \cup b)$, where the connect sum is taken away from the α, β, a, b curves.

It is straightforward to see that stabilization and destabilization do not change the underlying sutured manifold.

Theorem 2.2.5. [Juh06, Proposition 2.15] If H and H' are sutured Heegaard diagrams representing the same sutured manifold then H and H' are related by a sequence of the following moves;

1. Isotopies of the α and β curves.

2. Handleslides of α-curves over α-curves.

3. Handleslides of β-curves over β-curves.

4. Stabilizations and destabilizations.

This can again be proven using techniques from Morse theory.

Let S_n denote the symmetric group on n letters. Let $\times^n Y$ denote the n fold Cartesian product of a space Y. Note that S_n acts naturally on $\times^n Y$.

Definition 2.2.6. Given a surface Σ we can define the *symmetric product*, $\mathrm{Sym}^n(\Sigma)$, as $\times^n\Sigma/S_n$.

Given a sutured Heegaard diagram (Σ, α, β) We will be interested in two submanifolds of $\mathrm{Sym}^n(\Sigma)$. Let T_α denote the image of $\alpha_1 \times \alpha_2, \cdots \times \alpha_n$ in $\mathrm{Sym}^g(\Sigma)$ and $\mathbb{T}_\beta$ denote the image of $\beta_1 \times \beta_2 \times \cdots \times \beta_n$ in $\mathrm{Sym}^g(\Sigma)$

Proposition 2.2.7. [Per08] $\mathrm{Sym}^n(\Sigma)$ admits a symplectic structure such that $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ are Lagrangian submanifolds.

Symplectic manifolds and their Lagrangian submanifolds are objects of interest to a wide variety of mathematicians. We can then define sutured Floer homology essentially as the Lagrangian Floer homology of the two Lagrangian submanifolds $\mathbb{T}_\alpha, \mathbb{T}_\beta$ in $\mathrm{Sym}^g(\Sigma)$. Note that defining sutured Floer homology in this way is ahistorical; Perutz's result followed a number of years after the introduction of Heegaard Floer homology and sutured Floer homology. In the original version of Heegaard Floer homology $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ were not Lagrangians, but rather similar objects called *totally real tori*. We discuss sutured Floer homology more thoroughly in the next section.

2.2.2 Sutured Floer homology

Sutured Floer homology is an invariant of sutured manifolds defined by Juhasz [Juh06]. To each sutured manifold sutured Floer homology assigns a chain complex whose filtered chain homotopy type is an invariant of the sutured manifold.

We start with the underlying vector space for the chain complex. Let $\mathbb{F}$ denote the field with two elements. We note that Sutured Floer homology can be defined with coefficients in $\mathbb{Z}$ though to do so requires additional technical work, and our work in later sections will not require $\mathbb{Z}$-coefficients. In passing we note that the following question is open;

Question 2.2.8. Does there exist a sutured manifold (Y, γ) such that $\mathrm{SFH}(Y, \gamma; \mathbb{Z})$ contains torsion?

Khovanov homology, a related invariant a version of which we will discuss in Section 4.3, contains a great deal of torsion, so at a purely formal level it is odd that the above question should be open.

Definition 2.2.9. Set $\mathrm{SFC}(Y, \gamma) := \mathbb{F}\langle \mathbb{T}_\alpha \cap \mathbb{T}_\beta \rangle$.

This is a finitely generated vector space. The differential is given by the following equation;

$$\partial(x) = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\phi \in \pi_2(x,y), \mu(\phi)=1} |\mathcal{M}(\phi)| y \tag{2.2}$$

We discuss the elements of this formula.

Given $z \in \mathbb{C}$, we let $\Re(z)$ denote its real part.

Definition 2.2.10. $\pi_2(x, y)$ is the set of homotopy classes of *Whitney disks*; that is maps $f : \{z \in \mathbb{C} : |z| \leq 1\} \to \mathrm{Sym}^n(\Sigma)$ such that $f(z) \in \mathbb{T}_\alpha$ if $\Re(z) \leq 1$, $f(z) \in \mathbb{T}_\beta$ if $\Re(z) \geq 1$,

Given $\phi \in \pi_2(x, y)$, $\mu(\phi)$ is the *Maslov index* of ϕ. We do not give the definition here, as it is rather technical, and instead refer the reader to [Alt13, Section 2.3]. We note, however, that $\mu(\phi)$ can be thought of as the expected dimension of a certain moduli space we will presently discuss, and that it also admits a pleasant combinatorial formula, by work of Lipshitz [Lip06].

Definition 2.2.11. Let $\phi \in \pi_2(x, y)$. $\widetilde{\mathcal{M}}(\phi)$ is the space of non-constant pseudo-holomorphic representatives of $\pi_2(x, y)$.

Here we have endowed $\mathrm{Sym}^g(\Sigma)$ with an almost complex structure. There is an $\mathbb{R}$-action on $\widetilde{\mathcal{M}}(\phi)$ induced by the 1-parameter family of conformal automorphisms of $\{z \in \mathbb{C} : |z| \leq 1\}$ that fix $\pm i$. We let $\mathcal{M}(\phi)$ denote the quotient of $\widetilde{\mathcal{M}}(\phi)$ by this action. Both moduli spaces are dependent on the choice of almost complex structure. We suppress this in our notation for the sake of readability.

This completes our discussion of the formula for the differential. Of course, to show that the filtered chain homotopy type of $(\mathrm{SFC}(Y, \gamma), \partial)$ is a link invariant there is a great deal more work to do. Specifically we need the following results;

Theorem 2.2.12 (Juhasz)**.** $\partial^2 = 0$

This is proven by looking at the ends of the moduli space $\mathcal{M}(\phi)$ where $\phi \in \pi_2(x, y)$ satisfies $\mu(\phi) = 2$.

Theorem 2.2.13 (Juhasz)**.** The filtered chain homotopy type of $(\mathrm{SFC}(Y, \gamma), \partial)$ is independent of the sutured Heegaard diagram used to encode (Y, γ).

This result in fact require that the Heegaard diagrams satisfy a property called *weak admissibility*, which we do not discuss. This is proven using counts of objects called "pseudo-holomorphic triangles", which we also do not discuss.

Theorem 2.2.14 (Juhasz)**.** The filtered chain homotopy type of $(\mathrm{SFC}(Y, \gamma), \partial)$ is independent of the choice of almost complex structure.

This is a result one expects to hold given that it does so for other versions of Lagrangian Floer homology.

2.2.3 Properties of Sutured Floer homology

In this subsection we discuss various formal properties of sutured Floer homology.

First we show sutured Floer homology admits a grading by "Spin^c-structures". Let (Y, γ) be a sutured manifold endowed with a Riemannian metric. As in the previous subsection let v_0 be a non-vanishing vector field on ∂Y that points into Y on R_-, out of Y on R_+ and that is the gradient of the height function $\gamma \cong s(\gamma) \times [-1, 1] \to [-1, 1]$.

Definition 2.2.15. Let v and w be two non-vanishing vector fields on Y. We say v and w are *homologous* if there exists a ball B contained in the interior of Y such that v and w are homotopic through a family of non-vanishing vector fields that restrict to v_0 on the ∂Y.

This is an equivalence relation.

Definition 2.2.16. $\mathrm{Spin}^c(Y, \gamma, v_0)$ is the set of equivalence classes of vector fields that restrict to v_0 on ∂Y. We call such an equivalence class a *relative Spin^c-structure.*

Sutured Floer homology decomposes as a direct sum along relative Spin^c structures.

Proposition 2.2.17. $\mathrm{SFC}(Y, \gamma)$ splits over relative Spin^c-structures, as does ∂.

Of course it follows in turn that Sutured Floer homology splits over relative Spin^c-structures. That is we can write;

$$\mathrm{SFH}(Y, \gamma) = \bigoplus_{\mathfrak{s} \in \mathrm{Spin}^c(Y,\gamma)} \mathrm{SFH}(Y, \gamma, \mathfrak{s}) \tag{2.3}$$

Moreover, each summand $\mathrm{SFH}(Y, \gamma, \mathfrak{s})$ carries a "relative Maslov grading". To define it we require the following definitions;

Definition 2.2.18. Let $\mathfrak{s}$ be a a relative Spin^c structure on a sutured manifold (Y, γ). The *Chern class* of $\mathfrak{s}$, denoted $c_1(\mathfrak{s}) \in H^2(Y)$, is defined as the first Chern class of the two-plane field perpendicular to a vector field v with $[v] = \mathfrak{s}$.

Of course, one should check that this definition is independent of the choice of representative v.

Definition 2.2.19. For $\mathfrak{s} \in \mathrm{Spin}^c(Y, \gamma)$ the *divisibility* of $\mathfrak{s}$ is given by

$$d(\mathfrak{s}) = \gcd_{S \in H_2(Y;\mathbb{Z})} \langle c_1(\mathfrak{s}), S \rangle \tag{2.4}$$

Definition 2.2.20. Let x, y be generators of $\mathrm{SFC}(Y, \mathfrak{s})$ and $\phi \in \pi_2(x, y)$. Define a *relative Maslov grading* $m(x, y)$ by;

$$m(x, y) = \mu(\phi) \mod d(\mathfrak{s}) \tag{2.5}$$

Of course, for this to be well defined we require that the above definition is independent of the choice of ϕ. We do not show this here. We note also that under appropriate hypotheses these relative Maslov gradings can be upgraded to absolute gradings, for example in the setting of Heegaard Floer homology as well as knot and link Floer homology.

Remark 2.2.21. Gripp-Huang define a more general grading on $\mathrm{SFC}(Y, \gamma)$ using homotopy classes of 2-plane fields, which encodes the grading by Spin^c-structures and the relative Maslov grading [HR15, HR17].

Sutured Floer homology behaves nicely under sutured manifold decompositions along nice decomposing surfaces.

Theorem 2.2.22. [Juh08, Theorem 1.3] Let (Y, γ) be a balanced sutured manifold and $(Y, \gamma) \stackrel{S}{\rightsquigarrow} (Y', \gamma')$ be a sutured manifold decomposition along a nice surface. Then;

$$\mathrm{SFH}(Y', \gamma') \cong \bigoplus_{\mathfrak{s} \in O_S} \mathrm{SFH}(Y, \gamma) \tag{2.6}$$

In other words sutured decompositions yield specific summands at the level of sutured Floer homology. Of course, it remains to explain what the set O_S is.

Definition 2.2.23. Let (Y, γ) be a sutured manifold and $(\Sigma, \partial\Sigma) \subset (Y, \partial Y)$ be a decomposing surface. $\mathfrak{s} \in \mathrm{Spin}^c(Y, \gamma)$ is *outer with respect to* Σ if there is a non-vanishing vector field v with $[v] = \mathfrak{s}$ and v different from a normal vector field to Σ.

There is linear functional $f_\Sigma : \mathrm{Spin}^c \to \mathbb{R}$ defined in terms of topological quantities which determines the Spin^c-structures that are outer [Juh08]. Indeed, under appropriate hypotheses the summand $\mathrm{SFH}(Y', \gamma')$ is related by an affine isomorphism to the sutured Floer homology of (Y, γ) supported in a particular face of the convex hull of the Spin^c structures in which $\mathrm{SFH}(Y, \gamma)$ has non-trivial support [Juh10].

2.3 Heegaard Floer homology for closed 3-manifolds

Heegaard Floer homology, denoted $\widehat{\mathrm{HF}}(-)$, is an invariant of three manifolds due to Ozsváth-Szabó. As in the case of sutured Floer homology it is defined using Heegaard diagrams, symplectic topology and analysis [OS04c]. Indeed, it can be viewed as a special case of sutured Floer homology where we define the Heegaard Floer homology of a manifold to be the sutured Floer homology of the three manifold with the interior of a 3-ball removed. It can be defined with integer coefficients but we will take coefficients in $\mathbb{Z}/2$ throughout this thesis. $\widehat{\mathrm{HF}}(Y)$ splits as a direct sum over Spin^c-structures.

Let Y be a *rational homology sphere*; which is to say $H_*(Y) \cong H_*(S^3)$. There is a non-canonical bijection between the set of Spin^c-structures on Y and $H_1(Y)$. Suppose Y is given by performing n-surgery on a knot K in S^3. Then K determines a canonical bijection between $\mathrm{Spin}^c(Y)$ and $\mathbb{Z}/n \cong H_1(Y)$ as follows. Observe that the trace of n surgery, $X_n(K)$, gives a cobordism from $S^3_n(K)$ to S^3. Fix a Seifert surface Σ for K. Consider the surface $\widehat{\Sigma}$ obtained by capping off Σ in $X_n(K)$. Suppose $\mathfrak{s}$ is a Spin^c structure over $S^3_n(K)$ that admits an extension $\mathfrak{t}$ over $X_n(K)$ with the property that $\langle c_1(\mathfrak{t}), [\widehat{\Sigma}]\rangle - n \equiv 2i \mod 2n$. Then the map $\mathfrak{s} \mapsto i$ is an isomorphism by a result of Ozsváth-Szabó [OS08b, Lemma 2.2].

The relative Maslov grading on each Spin^c-structure on Heegaard Floer homology arising from sutured Floer homology can be upgraded to an absolute Maslov grading.

2.4 Link Floer Homology

Link Floer homology is a link invariant originally due to Ozsváth-Szabó [OS08a]. The simplest version of link Floer homology can be thought of as the sutured Floer homology of a link exterior with pairs of meridional sutures, as in Example 2.1.2. We call the Spin^c grading on link Floer homology the *Alexander grading.* If L is an n-component link the Alexander grading can be thought of as $\mathbb{Z}^n$ valued, or indeed as $H_1(X(L))$ valued. Link Floer homology detects the Thurston norm of a link exterior, see [OS08c] for a precise statement.

The relative Maslov grading on each Spin^c-structure on Link Floer homology arising from sutured Floer homology can be upgraded to an absolute Maslov grading.

We will be interested in stronger versions of link Floer homology than can be obtained as a special case of sutured Floer homology, however. To define these versions, we encode n component links with pointed Heegaard diagrams without boundary $(\Sigma, \alpha, \beta, \mathbf{z}, \mathbf{w})$. Here there is a single z and w baspeoint for each component L_i of L, Σ is a closed surface, and α and β are maximal collections of homologically independent simple closed curves in $\Sigma - (\mathbf{z} \cup \mathbf{w})$. We will count pseudoholomorhic disks weighted by counts of intersections with certain submanifolds of $\text{Sym}(\Sigma)$, one for each basepoint.

Some of the results referenced in this thesis are most concisely stated using Zemke's formalism. That it, we view Link Floer homology as a freely generated $\mathbb{F}[U_1^{\pm 1}, U_2^{\pm 1}, \dots U_n^{\pm 1}, V_1^{\pm 1}, V_2^{\pm 1}, \dots V_n^{\pm 1}]$-module with differential given by;

$$\partial(x) = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\phi \in \pi_2(x,y), \mu(\phi)=1} |\mathcal{M}(\phi)| \prod_{1 \leq i \leq n} U_i^{n_{z_i}(\phi)} V_i^{n_{w_i}(\phi)} y \tag{2.7}$$

To ensure that $\partial^2 = 0$ we need to set $U_i V_i = V_i U_i$ for all i.

The more traditional version of link Floer homology, CFL^∞, can be viewed as an $\mathbb{F}[U_1^{\pm 1}, U_2^{\pm 1}, \dots U_n^{\pm 1}]$-module with;

$$\partial(x) = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\phi \in \pi_2(x,y), \mu(\phi)=1} |\mathcal{M}(\phi)| \prod_{1 \leq i \leq n} U_i^{n_{z_i}(\phi)} y \tag{2.8}$$

This can be recovered from Zemke's invariant. Each U action decreases the Maslov grading by 2. The associated graded complex for $\mathrm{CFK}^\infty(K)$ carries an Alexander grading and a U grading for each component. The U action lowers the Alexander grading by 1 and the U grading by 1.

From $\mathrm{CFK}^\infty(L)$ one can recover a still weaker invariant, $\widehat{\mathrm{CFL}}$, by setting $U_{L_i} = 0$ for all i. The associated graded carries an Alexander grading for each component L_i of L.

2.5 Knot Floer homology

Knot Floer homology can simply be though of as link Floer homology in the special case that the link under consideration is a knot. We note that knot Floer homology, which is due independently to Ozsváth-Szabó [OS04c] and J. Rasmussen [Ras03], predates link Floer homology and sutured Floer homology. Different conventions for the absolute Maslov grading on knot Floer homology exist, however. In this thesis we use the convention that the absolute Maslov grading in knot Floer homology is $\dfrac{n-1}{2}$ larger than the absolute Maslov grading on link Floer homology, where n is the number of components of the link under consideration.

We again call the Spin^c grading on knot Floer homology the *Alexander grading*, and can think of it as taking values in $\mathbb{Z}$. We note that knot Floer homology can be extended to an invariant of links in S^3 by the process of *knotification* which assigns to each n-component link in S^3 a knot in $\#^{n-1} S^1 \times S^2$, see [OS04c] for details. Important properties of knot Floer homology include that it detects the Euler characteristic of a link [Ni06],[OS04c] and that it categorifies the Alexander polynomial [OS04b]. It follows that link Floer homology detects the linking number of two component links by a result of Hoste [Hos85a].

Dehn-surgery is a well studied operation on 3-manifolds. Let $q \in \mathbb{Q}$ and K be a knot in a S^3. q-surgery on K, denoted $S^3_q(K)$ is the manifold obtained by removing a tubular neighborhood of K from S^3, then regluing it with framing determined by q. An important result, that we shall use extensively in Chapter 5, is that $\mathrm{CFK}^\infty(K)$ determines $\widehat{\mathrm{HF}}(S^3_q(K))$ for all $q \in \mathbb{Q}$. This is due to Ozsváth-Szabó [OS10].

Chapter 3

$(2, 2n)$-cables of L-space knots

Let Y be a 3-manifold. Heegaard Floer homology satisfies the following rank inequality;

$$\mathrm{rank}(\widehat{\mathrm{HF}}(Y)) \geq |H_1(Y)| \tag{3.1}$$

Here $|H_1(Y)|$ is the number of elements in $H_1(Y)$. This equation results from the fact that an appropriate decatigorification of $\widehat{\mathrm{HF}}(Y)$ yields $|H_1(Y)|$. An *L-space* is a rational homology sphere for which Inequality 3.1 is tight. Understanding the geometric and algebraic properties of L-spaces is of central interest in low dimensional topology and the subject of Boyer-Gordon-Watson's "L-space conjecture" [BGW13].

An *L-space knot* is a knot K for which $S^3_n(K)$ is an L-space for for some $n \in \mathbb{Z}^{\geq 0}$. The knot Floer homology of L-space knots is well understood, by work of Ozsváth-Szabó [OS05a].

Alternatively, if we let K_n denote the core of n-surgery on a knot K, L-space knots can be defined as knots K for which $\mathrm{rank}(\widehat{\mathrm{HFK}}(K_n)) = n$ for some $n \geq 0$. Note that this is a non-standard definition of an L-space knot, but it follows quickly from the immersed curve interpretation for the surgery formula in Heegaard Floer homology that it is equivalent to the traditional definition, see [HRW18] for details.

Alternatively this follows from work of Hedden [Hed07], [Hed11, Theorem 1.4], or Eftekary [Eft11]. Positive torus knots are examples of L-space knots. In particular the unknot U is an L-space knot. Note that $\text{rank}(\widehat{\text{HFK}}(U_n)) = |n|$ for all n. If K is a non-trivial L-space knot then $\text{rank}(\widehat{\text{HFK}}(K_n)) = n$ if and only if $n > 2g(k) - 1$, again see [HRW18] for details. L-space knots have many strong properties, including that their knot Floer homologies are determined by their Alexander polynomials.

Let K be a knot. Recall that *the* (p, q)*-cable of* K, which we denote $K_{2,2n}$, is the link obtained by taking the knot $T(p, q)$ and tying it into K.

The goal of this section is to prove the following theorem;

Theorem 3.0.1. Let K be an L-space knot. Suppose L is a link such that $\widehat{\text{HFK}}(L) \cong \widehat{\text{HFK}}(K_{2,2n})$. Then L is the $(2, 2n)$-cable of an L-space knot K' such that $\widehat{\text{HFK}}(K') \cong \widehat{\text{HFK}}(K)$.

Note here that the conclusion that K' is an L-space knot is redundant, since it follows from the fact that $\widehat{\text{HFK}}(K') \cong \widehat{\text{HFK}}(K)$. This is readily seen via Hanselman-Rasmussen-Watson's interpretation of Heegaard Floer homology via immersed curves [HRW16, HRW18].

We begin by proving the knot Floer homology of $(2, 2n)$-cables of L-space knots satisfy various properties;

Proposition 3.0.2. Let K be a non-trivial L-space knot, $n > 2g(K) - 1$. Then $\widehat{\text{HFK}}(K_{2,2n})$ satisfies the following;

1. $\max\{A : \widehat{\text{HFK}}(K_{2,2n}, A) \neq 0\} = 1$
2. $\text{rank}(\widehat{\text{HFK}}(K_{2,2n}, 1)) = n$
3. The maximum Maslov grading of $\widehat{\text{HFK}}(K_{2,2n})$ is $\frac{1}{2}$.

Here we orient $(2, 2n)$ cables as the boundary of annuli, as we do throughout this chapter. These conditions can be deduced from more general work of Gorsky-

Hom [GH17], but we provide a proof using Juhasz's surface decomposition theorem [Juh08] and the skein exact triangle for knot Floer homology.

For this proof – and indeed for the rest of this thesis – we let K_n denote the core of n-surgery on K.

Proof. The maximal Euler characteristic surface bounding a $(2,2n)$-cable knot is an annulus if K is non-trivial, whence condition 1 follows. Decomposing along that annulus gives the exterior of K_n, which has sutured Floer homology of rank n since K is an L-space knot and $n > 2g(K) - 1$, proving condition two holds.

To see condition three holds true, note that $\widehat{\mathrm{HFK}}(K_{2,2n+1})$ and $\widehat{\mathrm{HFK}}(K_{2,2n-1})$ can be computed readily via immersed curves [HW19]. $\widehat{\mathrm{HFK}}(K_{2,2n})$ can then be computed via the skein exact triangle [OS04b], where here $K_{2,2n}$ is not oriented as the boundary of an annulus. From here, using link Floer homology and the fact that the linking number is n, one determines that the maximum Maslov grading is $\frac{1}{2}$. □

With the properties of $(2,2n)$-cables of L-space knots given in Proposition 3.0.2 we can now prove the following characterization of $(2,2n)$-cables of L-space knots;

Lemma 3.0.3. Suppose K is a non-trivial L-space knot and $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(K_{2,2n})$ with $n > 2g(K) - 1$. Then L is isotopic to $K'_{2,2n}$ where K' is an L-space knot.

Proof. Note that L has at most two components since the maximal Maslov grading of a generator is $\frac{1}{2}$ and $\widehat{\mathrm{HFK}}(L)$ must admit a spectral sequence to $\widehat{\mathrm{HF}}(\#^{(n-1)}S^1 \times S^2)$, which has maximal Maslov grading $\frac{n-1}{2}$. Since $\widehat{\mathrm{HFK}}(L)$ is of even rank, L must have two components.

Since the maximal Alexander grading is 1 and L has non-zero linking number L bounds an annulus. Decomposing along this annulus yields a sutured manifold with sutured Floer homology given by $\widehat{\mathrm{HFK}}(K'_n)$ for some knot K'. Note that $\mathrm{rank}(\widehat{\mathrm{HFK}}(K'_n)) = n$ so that K' is an L-space knot. □

Proof of Theorem 3.0.1. Suppose K, L are as in the statement of the theorem. Lemma 3.0.3 implies that L is given by $K'_{2,2n}$ for some L-space knot K'. Noting that the Alexander polynomial of a knot is determined by the Alexander polynomial of a $(2, 2n)$-cable implies that $\Delta_K(t) = \Delta_{K'}(t)$. This in turn implies that $\widehat{\mathrm{HFK}}(K) \cong \widehat{\mathrm{HFK}}(K')$, since for L-space knots the knot Floer homology is determined by the Alexander polynomial, as desired. □

Corollary 3.0.4. Knot Floer homology detects:

1. $T(2, 2n)$ for all $n \neq 0$
2. $T(2, 3)_{2,2n}$ for all $n > 1$
3. $T(2, 5)_{2,2n}$ for all $n > 3$

Proof. Suppose L is a link with knot Floer homology of one of the three given types. Note that the unknot, $T(2, 3)$ and $T(2, 5)$ are L-space knots. Thus Theorem 3.0.1 implies that L the $(2, 2n)$-cable of a knot with the same knot Floer homology as the unknot, $T(2, 3)$ or $T(2, 5)$ respectively. The result then follows from the fact that knot Floer homology detects each of these three knots [OS04a],[Ghi08], [FRW22]. □

We conclude with a corollary we will use in the next chapter;

Corollary 3.0.5. Link Floer homology detects $T(2, 2n)$ with the orientation induced by viewing $T(2, 2n)$ as the $(2, 2n)$-cable of an unknot.

Proof. Suppose L is a link with $\widehat{\mathrm{HFL}}(L)$. L is a two component link and a result of Hoste [Hos85b] implies that the linking number of L is n. It follows that after revering the orientation of a component of L, the link Floer homology of L agrees with that of $T(2, 2n)$ oriented as the boundary of an annulus, whence in turn after revering the orientation of a component of L the knot Floer homology of L agrees with that of $T(2, 2n)$ oriented as the boundary of an annulus. The result then follows from Corollary 3.0.4. □

Chapter 4

Involutions and Detection Results

In the previous chapter we showed – amongst other things – that knot Floer homology detects $T(2,2n)$ oriented as the boundary of an annulus. In this chapter one of our main goals is to prove that knot Floer homology detects $T(2,2n)$ with the other orientation;

Theorem 4.0.1. Knot Floer homology detects $T(2,2n)$ for all n.

The $n = 0$ case of this result follows readily from a result of Ni [Ni06]. We take $n \neq 0$ for the remainder of this chapter. This result was previously known for small $n \in \{\pm1, \pm2, \pm3, \pm4, \pm5\}$ due to work of the author and Martin [BM20] and the author and Dey [BD22b].

The proof of this result uses a symmetry of knot Floer homology that arises via a natural symmetry of knots. This extra symmetry allows us to prove a useful geography result.

This chapter is organised as follows; in Section 4.1 we discuss Sarkar's basepoint pushing map, in Section 4.2 we prove Theorem 4.0.1, and in Section 4.3 we prove an infinite family of detection results for an invariant called Annular Khovanov homology.

4.1 Sarkar's Basepoint pushing map

Given a doubly pointed knot there is a natural diffeomorphism corresponding to pushing each basepoint to the other along the orientation of the knot. This operation was studied by Sarkar [Sar15] and Zemke [Zem17].

The following is a consequence of a theorem of Zemke [Zem17, Theorem B];

Theorem 4.1.1. Suppose L is a link in S^3 and K is a component of L, with basepoints z_k and w_k. Let ζ denote the diffeomorphism resulting from a finger move around K, in the direction dictated by the orientation induced by K. The induced map ζ_* on $\mathrm{CFL}^\infty(L)$ has the filtered equivariant chain homotopy type $\zeta_* \sim 1 + \Phi_K \Psi_K$

To make sense of the statement of this theorem we need to define Φ_{L_i}, Ψ_{L_i}. This is done as follows $\Phi_{L_i} : \mathrm{CFL}^\infty(L) \to \mathrm{CFL}^\infty(L)$ can be defined up to filtered chain homotopy;

$$\Phi_{L_i} \sim U_i^{-1} \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\phi \in \pi_2(x,y), \mu(\phi)=1} n_{w_i}(\phi) |\mathcal{M}(\phi)| \prod U_i^{n_{w_i}(\phi)} V_i^{n_{z_i}(\phi)} y \tag{4.1}$$

We can think of this as the derivative of ∂ with respect to U_{L_i}

Similarly $\Psi_K : \mathrm{CFL}^\infty(L) \to \mathrm{CFL}^\infty(L)$ is given by;

$$\Psi_{L_i} \sim V_i^{-1} \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\phi \in \pi_2(x,y), \mu(\phi)=1} n_{w_i}(\phi) |\mathcal{M}(\phi)| \prod U_i^{n_{w_i}(\phi)} V_i^{n_{z_i}(\phi)} y \tag{4.2}$$

Of course, we are taking coefficients in $\mathbb{Z}/2$ so only the parity of $n_{w_i}(\phi)$ and $n_{z_i}(\phi)$ are important. We will need the following Lemma;

Proposition 4.1.2. [Sar15, Lemma 4.4] $\Phi_K \Psi_k + \Psi_K \Phi_k$, Φ_K^2 and Ψ_K^2 are chain homotopic to 0.

In the following section we will be interested in applying the above proposition to chain complexes satisfying certain algebraic restrictions, namely complexes which are "thin". We will need the following proposition;

Proposition 4.1.3. Suppose C is a finitely generated $\mathbb{Z} \oplus \mathbb{Z}$ graded vector space over $\mathbb{Z}/2$. Suppose C is supported in a single δ-grading, where here the δ-grading is given by $\delta(x, y) = x - y$. Suppose ∂_1 and ∂_2 are commuting differentials such that $\partial_1 : C(c, y) \mapsto C(x-1, y-1)$, $\partial_2 : C(x, y) : C(x+1, y+1)$ and $(\partial_1\partial_2)^2 = 0$. Suppose moreover that $H_*(C, \partial_1) \cong 0$ and $H_*(C, \partial_2) \cong 0$. Then there is a change of basis such that C splits as summands of the form $\mathbb{F}\langle x, y, z, w\rangle$ such that x is of bigrading (a, b) for some $a, b \in \mathbb{Z}$ while y, z are of bigrading $(a+1, b+1)$ and w is of bigrading $(a+2, b+2)$ and the non-trivial components of the differential are given by $\partial_1 y = x, \partial_1 w = z, \partial_2 x = z, \partial_2 y = w$.

Sarkar [Sar15, Theorem 6.1] notes that this result is essentially proven by Petkova [Pet13], though in a slightly different context. We give the proof here for completeness. We note in passing that we will be interested in applying this result in the setting in which ∂_1 and ∂_2 are Ψ_K and Φ_K respectively.

Proof. We construct a basis of the desired form. Consider an element x of minimal i grading. Let it be a generator. Observe that $\partial_2 x \neq 0$ and that there exists an element y with $\partial_1 y = x$ By adding y to any other element y' with this property, we may assume y is the unique such element. Let y be a generator. Suppose $\partial_2 x = z$. Let z be a generator. Then since ∂_1 and ∂_2 commute $\partial_1(\partial_2 y) = z$. Let $w = \partial_2(y)$ be a generator. By adding y to any other generator y' with w a component of $\partial_2(y')$, we may assume that y is the unique generator with this property. We thus have a subcomplex B of C of the desired form. We need to show that B is a summand, i.e. that there are no components of the differential from $C - B$ to B.

By construction there is no other component of the differential ∂_1 or ∂_2 to or from x. Likewise there is no other component of ∂_1 or ∂_2 from y. There can be no component of ∂_1 or ∂_2 to y as this would violate the condition that $\partial_1^2 = 0$ or $\partial_2^2 = 0$

respectively.

It remains to show that there is no component of ∂_1 to w or z. Suppose a generator w' had a component of ∂_2 to z. Then we could add w to w' to remove the unwanted component. Any component of ∂_1 to w would then violate the condition that $\partial_1^2 = 0$, a contradiction. □

4.2 Knot Floer homology detects $T(2, 2n)$

Our goal in this section is to prove Theorem 4.0.1. This result had been proven for certain small values of n by the author and Martin [BM20] as well as the author and Dey [BD22b]. The knot Floer homology of $T(2, 2n)$ with the orientation induced by viewing $T(2, 2n)$ as the $(2, 2n)$-cable of the unknot is given by;

$$\widehat{\mathrm{HFK}}(T(2,2n);i) = \begin{cases} \mathbb{F}_{1/2} & \text{if } i = n \\ \mathbb{F}^2_{i-n+1/2} & \text{for } |i| < n \\ \mathbb{F}_{1/2-2n} & \text{if } i = -n \end{cases} \tag{4.3}$$

We start by showing that in general if $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(T(2, 2n))$ then L consists of two unknotted components.

Lemma 4.2.1. Suppose $\widehat{\mathrm{HFK}}(L) \cong \widehat{HFK}(T(2, 2n))$. Then L has two components.

Proof. Suppose L is as in the statement of the theorem. Since $\mathrm{rank}(\widehat{\mathrm{HFK}}(L))$ is even, L has at least two components. To see that L has at most two components note that the maximum Maslov grading of a generator is $1/2$, and if L has n-components then it needs to admit a spectral sequence to V^n – where $V \cong \mathbb{F}_{1/2} \oplus \mathbb{F}_{-1/2}$ – which has maximum Maslov grading $\frac{n-1}{2}$. □

We now show that if $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(T(2, 2n))$ then $\widehat{\mathrm{CFL}}(L)$ is of an especially simple form. Let L be a two component link. Observe that $\widehat{\mathrm{HFL}}(L)$ can be viewed as the E_1 page of a spectral sequence from $\widehat{\mathrm{CFL}}(L)$ to $\cong \mathbb{F}_0 \oplus \mathbb{F}_{-1}$. Ozsváth-Szabó observe that if $\widehat{\mathrm{HFL}}(L)$ is thin in the sense that there is a constant δ such that for every generator of grading (A_1, A_2, m), $A_1 + A_2 - m = \delta$ then the differential on $\widehat{\mathrm{HFL}}(L)$ decomposes into a direct summand of particularly simple pieces. We define them here;

$$(B_d)_{i,j} = \begin{cases} \mathbb{F}_d & \text{if } (i,j) = (0,0) \\ \mathbb{F}_{d+1} & \text{if } (i,j) \in \{(0,1),(1,0)\} \\ \mathbb{F}_{d+2} & \text{if } (1,1) \end{cases} \tag{4.4}$$

$$(V_d^l)_{i,j} = \begin{cases} \mathbb{F}_d & \text{if } (i,j) = (-j,j) \text{ with } j = 0,\dots,l-1 \\ \mathbb{F}_{d-1} & \text{if } (i,j) = (-j-1,j) \text{ with } j = 0,\dots,l-1 \end{cases} \tag{4.5}$$

$$(H_d^l)_{i,j} = \begin{cases} \mathbb{F}_d & \text{if } (i,j) = (i,-i) \text{ with } i = 0,\dots,l-1 \\ \mathbb{F}_{d-1} & \text{if } (i,j) = (i,-i-1) \text{ with } i = 0,\dots,l-1 \end{cases} \tag{4.6}$$

$$(X_d^l)_{i,j} = \begin{cases} \mathbb{F}_d & \text{if } i+j = l \text{ and } i,j \geq 0 \\ \mathbb{F}_{d+1} & \text{if } i+j = l+1 \text{ and } i,j > 0 \end{cases} \tag{4.7}$$

$$(Y_d^l)_{i,j} = \begin{cases} \mathbb{F}_d & \text{if } i+j = l \text{ and } i,j \geq 0 \\ \mathbb{F}_{d-1} & \text{if } i+j = l-1 \text{ and } i,j > 0 \end{cases} \tag{4.8}$$

Each of these complexes are trivial aside from in the specified gradings. Moreover, the horizontal and vertical components of the differential are non-trivial. This is result is analogous to Proposition 4.1.3. We are now in a position to prove the following;

Lemma 4.2.2. Suppose $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(T(2, 2n))$. Then $(\widehat{\mathrm{CFL}}(L), \partial)$ consists of B_d summands and a single $Y_0^0[\frac{n}{2}, \frac{n}{2}] \oplus Y_{-1}^1[\frac{n-1}{2}, \frac{n-1}{2}]$ summand. In particular L has unknotted components.

Here $[x, y]$ indicates a shift in the bi-Alexander grading.

Proof of Lemma 4.2.2. Let L be as in the statement of the Lemma and m be the number of components of L. Then L has two components by 4.2.1. Let L_1 and L_2 be the two components of L. It follows that there is a unique Maslov grading 0 generator. Consider $\widehat{\mathrm{HFL}}(L)$. Since the linking number is n and the Alexander grading of the Maslov index 0 generator is n, it follows that there is a summand $Y_0^0[\frac{n}{2}, \frac{n}{2}]$ and since there are two Maslov index -1 generators there is a summand $Y_{-1}^1[\frac{n-1}{2}, \frac{n-1}{2}]$. As observed in the proof of [OS08a, Theorem 12.1], $\widehat{\mathrm{HFL}}(L)$ consists of pairs of summands $V_d^l(x, y) \oplus V_{d-1}^l(x, y-1)$, $H_d^l(x, y) \oplus H_{d-1}^l(x, y-1)$, $Y_0^l(x, y) \oplus Y_{-1}^{l-1}(x+1, y+1)$, $X_0^l(x, y) \oplus X_{-1}^{l-1}(x, y)$, as well as a collection of copies of B_d. Note that for the case at hand the rank in each Alexander grading is at most two, so we have that the rest of the complex is given by summands of the form B_d or $V_d^1[x, y] \oplus V_{d-1}^1[x, y-1]$ or $H_d^1[x, y] \oplus H_{d-1}^1[x-1, y]$. Indeed we readily see that d is even.

Suppose L_1 is not unknotted. Consider the $V_d^1[x, y] \oplus V_{d-1}^1[x, y-1]$ summands with maximal x. Note that $x > \frac{n}{2}$. Amongst these consider the summands with minimal d. It follows that there is a generator in $\widehat{\mathrm{HFK}}(L_1)$ of minimal Alexander grading with Maslov grading $d - 2(x - \frac{n}{2})$. Note that this is an even number. But this is a contradiction, since if there is a generator of even Maslov index in A_1 grading $\frac{n}{2} - x$ which persists under the spectral sequence to $\widehat{\mathrm{HFK}}(L_1) \otimes V$ then there must be a generator of odd Maslov index in A_1 grading $\frac{n}{2} - x - 1$, since the summands of $\widehat{\mathrm{HFL}}(L)$ that persist under the spectral sequence to $\widehat{\mathrm{HFK}}(L_1)\otimes V$ are of the form $V_d^1[x,y]\oplus V_{d-1}^1[x,y-1]$ with d even.

Thus there are no $V_d^1[x,y]\oplus V_{d-1}^1[x,y-1]$ summands. A similar proof shows that there are no $H_d^1[x,y]\oplus H_{d-1}^1[x-1,y]$ summands. Thus the remaining summands are all of the form B_d and each component of L is unknotted, as desired. □

It remains to determine the bigradings of all the B_d summands. This can be deduced from the following Lemma.

Lemma 4.2.3. Suppose L is a δ-thin two component link with unknotted components. Then $\widehat{\mathrm{HFL}}(L;A_i=C)$ can be decomposed into summands of the form $\mathbb{F}_{d+1}[\delta+d+1-C]\oplus\mathbb{F}_d^2[\delta+d-C]\oplus\mathbb{F}_{d-1}[\delta+d-1-C]$ for all $C\neq\pm\frac{\ell\mathrm{k}(L)}{2}$.

Here by $\widehat{\mathrm{HFL}}(L;A_i=C)$ we mean the summand of $\widehat{\mathrm{HFL}}(L)$ with A_i grading equal to C. $\mathbb{F}_d[C']$ indicates a summand of $\widehat{\mathrm{HFL}}(L;A_i=C)$ with Maslov grading d and the remaining Alexander grading equal to C'.

We proceed by an argument used by Sarkar in the proof of Theorem 6.1 in [Sar15]

Proof. This follows from Lemma 4.1.3. Specifically, suppose L is as in the statement of the Lemma. Observe that since $C\neq\pm\frac{\ell\mathrm{k}(L)}{2}$ there exists a spectral sequence which starts at $(\widehat{\mathrm{HFL}}(L;A_i=C),\Phi_{L_i})$ and converges to 0 where the differential on the ith page shift the (A_j,m) grading by $(-i,-1)$. Likewise there exists a spectral sequence which starts at $(\widehat{\mathrm{HFL}}(L;A_i=C),\Psi_{L_i})$ and converges to 0 where the differential on the ith page shift the (A_j,m) grading by $(2i-1,i)$. However, given that $\widehat{\mathrm{HFL}}(L)$ is thin these spectral sequences must collapse immediately. Whatsmore, the differentials Φ_{L_i} and Ψ_{L_i} commute, so Proposition 4.1.3 implies that $\widehat{\mathrm{HFL}}(L)$ decomposes into components of the desired form. □

We can now prove Theorem 4.0.1.

Proof of Theorem 4.0.1. Suppose L is as in the statement of the theorem. Lemma 4.2.1 implies that L has two components. Lemma 4.2.2 implies that these two components are unknotted and indeed determines $\widehat{\mathrm{HFL}}(L)$ up to the location of B_d summands.

In the case of $T(2, \pm 2)$ there are no B_d summands and the result follows directly from Corollary 3.0.5. Thus we take $|n| > 1$. The locations of these B_d summands are determined by applying Lemma 4.2.3 to Alexander gradings $A_i \neq \pm\frac{n}{2}$. It follows that $\widehat{\mathrm{HFL}}(L) \cong \widehat{\mathrm{HFL}}(T(2, 2n))$, whence the result follows from Corollary 3.0.5. □

4.3 Annular Khovanov Homology Detects $\widehat{\beta_{2n}}$

Khovanov homology is a bigraded vector space valued invariant due to Khovanov [K+00] which shares many formal properties with Knot Floer homology. It has the advantage over knot Floer homology that its definition is manifestly combinatorial. The topological content of Khovanov homology is not well understood.

Annular Khovanov homology was defined by Asaeda-Przytycki-Sikora [APS04] as a categorification of the Kauffman bracket skein module of the thickened annulus. The resulting theory is an invariant of links in the thickened annulus $A \times I$ or alternatively the complement of an unknot in the 3-sphere $S^3 \setminus U$.

In this section we apply some earlier knot Floer detection results to show that annular Khovanov homology detects certain braid closures. The proofs will rely on the spectral sequence from annular Khovanov homology of a link L to the knot Floer homology of the lift of the annular axis in $\Sigma(L)$ [GN14, Rob13].

Let β_n denote the braid $\sigma_1\sigma_2\cdots\sigma_{n-1}$. We use knot Floer detection results for $T(2, 2n)$ to show that annular Khovanov homology detects the closure of β_n for all even n.

Theorem 4.3.1. Annular Khovanov homology detects $\widehat{\beta_{2n}}$ for all n.

Here $\widehat{\alpha}$ indicates the braid closure of the braid α. We note that versions of this result appeared in work of the author and Martin [BM20], as well as the author and Dey [BD22b]. Annular Khovanov homology is also known to detect the closures of the trivial braids [BG15] as well as all two braids [GLW18].

Our approach is as follows; first we use properties of annular Khovanov homology to deduce necessary topological properties of the annular knot like braidedness or unknottedness. Then we use a knot Floer detection result to show that the lift of the annular axis is $T(2,2n)$. Finally we translate this into information about the annular link.

The spectral sequence from the annular Khovanov homology of an annular link L to the knot Floer homology of the lift of the annular axis in $\Sigma(L)$ is defined with $\mathbb{Z}/2\mathbb{Z}$ coefficients. At times, however, we will work with annular Khovanov homology over $\mathbb{C}$ because with these coefficients annular Khovanov homology has the structure of an $\mathfrak{sl}_2(\mathbb{C})$ representation [GLW18, Proposition 3].

Proposition 4.3.2 ([LM])**.** Suppose γ is an n-braid and β is a periodic n-braid. If $BH(\gamma)$ and $BH(\beta)$ are conjugate then so too are γ and β.

Remark 4.3.3. An alternative proof of this Proposition was originally communicated to the author and Gage Martin by Marissa Loving and Dan Margalit [LM].

Proof. Let β and γ be as in the statement of the proposition. Note that both conjugation and the Birman-Hilden correspondence preserve the Nielsen-Thurston classification so we know that γ is periodic as well. That is a power of γ is some power of the full twist Δ^2. Thus there are numbers N and M so that $\beta^N = \gamma^M$.

Now we consider the fractional Dehn twist coefficient of β and γ. We know that $FDTC(\beta) = k/m$ for some fixed k, m. The Birman-Hilden correspondence either preserves the fractional Dehn twist coefficient of n-braids or halves it depending on the parity of n. The fractional Dehn twist coefficient is preserved under conjugation by a combination of [IK17, Corollary 4.17] and [Ghy01, Proposition 5.3] so we know

$FDTC(BH(\beta)) = FDTC(BH(\gamma))$ so then we also know $FDTC(\beta) = FDTC(\gamma) = k/m$. The fractional Dehn twist coefficient is multiplicative under exponentiation so we know $FDTC(\beta^N) = \frac{kN}{m}$ and $FDTC(\gamma^M) = \frac{kM}{m}$ but $\beta^N = \gamma^M$ so we must have that $M = N$. Finally we know that Nth roots are unique up to conjugation in the braid group [GM03] so that β and γ are conjugate.

□

The spectral sequence from the annular Khovanov homology of an annular link L to the knot Floer homology of the lift of the annular axis in $\Sigma(L)$ is defined with $\mathbb{Z}/2\mathbb{Z}$ coefficients. At times, however, we will work with annular Khovanov homology over $\mathbb{C}$ because with these coefficients annular Khovanov homology has the structure of an $\mathfrak{sl}_2(\mathbb{C})$ representation [GLW18, Proposition 3].

For the readers convenience we recall, from [GLW18, Proposition 14], that;

$$\mathrm{AKh}^i(\widehat{\beta_n}, \mathbb{C}) = \begin{cases} V_{(n)}\{n-1\} & \text{for } i = 0 \\ V_{(n-2)}\{n+1\} & \text{for } i = 1 \\ 0 & \text{otherwise} \end{cases} \tag{4.9}$$

Here $V_{(m)}$ is the $(m+1)$-dimensional irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$. Now, we will need to study annular Khovanov homology with coefficients with $\mathbb{Z}/2$. Note that $\mathrm{rank}(\mathrm{AKh}(\beta_n;\mathbb{Z}/2)) \geq \mathrm{rank}(\mathrm{AKh}(\beta_n;\mathbb{C})) = 2n$. Now, $T(2,n)$ can be thought of as a 2-periodic knot with quotient β_n. It follows from [SZ18] that $\mathrm{rank}(\mathrm{Kh}(T(2,n);\mathbb{Z}/2)) \geq \mathrm{rank}(\mathrm{AKh}(\beta_n;\mathbb{Z}/2))$. Indeed, $\mathrm{rank}(\mathrm{Kh}(T(2,n);\mathbb{Z}/2)) = 2n$ by the universal coefficient theorem and [K$^+$00, Proposition 26] so in fact $\mathrm{rank}(\mathrm{AKh}(\beta_n;\mathbb{Z}/2)) = 2n$ and the above description of Annular Khovanov homology is equally valid for $\mathbb{Z}/2$ coefficients.

We require the following Lemma;

Lemma 4.3.4. Let $L \subseteq A \times I \subseteq S^3$ be an annular link. If $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{AKh}(\widehat{\beta_n}, \mathbb{Z}/2\mathbb{Z})$ then L is an unknot in S^3.

Proof. We first compute $\mathrm{AKh}(L, \mathbb{C})$ from $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})$. Throughout, we will use that the dimension of annular Khovanov homology over $\mathbb{C}$ can be no larger than that over $\mathbb{Z}/2\mathbb{Z}$. Because L is a n-braid closure, $\mathrm{AKh}(L, \mathbb{C})$ must contain a weight n irreducible $\mathfrak{sl}_2(\mathbb{C})$ representation in grading $i = 0$, of dimension n. Thus $\mathrm{AKh}(L, \mathbb{C})$ must consist only of this representation in grading $i = 0$ because $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})$ has dimension n in homological grading 0. We therefore have that all of the generators in grading $i = 1$ for $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})$ must correspond to generators of $\mathrm{AKh}(L, \mathbb{C})$, for it not, they would correspond to 2-torsion in $\mathrm{AKh}(L, \mathbb{Z})$, but the torsion contributes dimension in two different homological gradings by the universal coefficient theorem.

A simple computation of annular Khovanov homology verifies that L is not the trivial braid. Thus by [BG15, Theorem 3.1], we know that the differential ∂_- on $\mathrm{AKh}(L, \mathbb{C})$ inducing the spectral sequence to $\mathrm{Kh}(L)$ must send the highest weight generator in the grading $i = 0$ to something non-zero. The only generator in the correct quantum grading is the highest weight generator in the grading $i = 1$ so that must be the image of the highest weight generator in the grading $i = 0$ under ∂_-. The action ∂_- is part of the action of $\mathfrak{sl}_2(\wedge)$ on $\mathrm{AKh}(L, \mathbb{C})$ and commutes up to sign with the lowering operator f [GLW18, Theorem 1]. This means that the image of ∂_- is spanned by all generators in grading $i = 1$. Thus $\mathrm{Kh}(L)$ is dimension 2, and L is the unknot. □

Notice that the proof of the above Lemma indeed determines the differential ∂_- inducing the spectral sequence from $\mathrm{AKh}(L)$ to the Khovanov homology of the unknot.

Lemma 4.3.5. Let $L \subseteq A \times I \subseteq S^3$ be an annular link. If $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{AKh}(\widehat{\beta_{2n}}, \mathbb{Z}/2\mathbb{Z})$ then $\widehat{\mathrm{HFK}}(\tilde{U}, \Sigma(L)) \cong \widehat{\mathrm{HFK}}(T(2, 2n))$.

As with everything else in this section, the proof of this Lemma appears in [BM20]. We have expanded the argument slightly here, though it is the same in essence.

Proof. For the proof of this lemma we identify $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})$ with the E_2 page of a link surgery spectral sequence – an object in Floer homology discussed in [OS05b] – and use the Maslov gradings on the Floer theory side to deduce that the spectral sequence to $\widehat{\mathrm{HFK}}(-\widetilde{U})$, collapses on the E_2-page.

Let D be a diagram for the annular link L. Suppose D has n crossings. $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})$ can be identified as the E_2 page of a spectral sequence from a complex graded over a cube $I \in \{0,1\}^n$, with, vertices given by $\widehat{\mathrm{HFK}}(\tilde{B}_I) \cong W^l \otimes V^m$, where $\tilde{B}_I$ is the lift of the braid axis B in the double branched cover of the link given by resolving L by I, and l is the number of components of D_I that link non-trivially with B and m is the number of components of D_I which do not. Here $W \cong \mathbb{F}_{1/2}[1/2] \oplus \mathbb{F}_{-1/2}[-1/2]$, while $V \cong \mathbb{F}_{1/2}[0] \oplus \mathbb{F}_{-1/2}[0]$, where the subscript denotes the Maslov grading of the generator while, $[x]$ denotes the Alexander grading. The Alexander grading is identified with half of the Annular grading on Annular Khovanov homology, while for each I the relative Maslov grading of any two generators agrees with half relative quantum grading of the generators.

The differential on this complex which induces the spectral sequence to the knot Floer homology of the double branched cover of the braid axis is given by the maps induced by the cobordism maps taking $\widehat{\mathrm{HFK}}(\tilde{B}_I)$ to $\widehat{\mathrm{HFK}}(\tilde{B}_{I'})$ of $I < I'$ in the lexicographic ordering. Annular Khovanov homology is then identified with the E_2 page of this spectral sequence. Note that we can recover the relative Maslov grading on the Floer homology side in each individual homological grading. Our goal is to recover the absolute Maslov grading on the Floer homology side.

On the Floer homology side the differential ∂_- giving the spectral sequence from $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})$ to $\mathrm{Kh}(L, \mathbb{Z}/2\mathbb{Z})$ – where we view $\mathrm{Kh}(L, \mathbb{Z}/2\mathbb{Z})$ as the E_2 page of the spectral sequence to the double branched cover of L as in [OS05b] – is the sum of components which lower the Maslov index by 1. Thus ∂_- lowers the Maslov grading by one. Using the remark following Lemma 4.3.4, this implies that for $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})$, generators in the same annular grading correspond on the Floer theoretic side to elements of the same relative Maslov grading. Since elements in the same Alexander grading have the same Maslov grading, the spectral sequence to $\widehat{\mathrm{HFK}}(-\widetilde{U})$ collapses as all differentials preserve the Alexander grading and decrease the Maslov grading.

It follows that $\widehat{\mathrm{HFK}}(-\tilde{U}) \cong \widehat{\mathrm{HFK}}(T(2,2n))$ up to a shift in the Maslov grading. To upgrade this the relative Maslov grading statement to an absolute Maslov grading statement, notice that there are only two generators which survive in the spectral sequence from $\mathrm{AKh}(L, \mathbb{Z}/2)$ to $\mathrm{Kh}(L, \mathbb{Z}/2\mathbb{Z})$, namely the generators that sit in the k gradings $-2n$ and $2-2n$. These generators must correspond under the Floer homology interpretation to generators in Maslov gradings 0 and 1 respectively. This determines the Maslov gradings of the generators under the Floer theoretic interpretation of $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})$.

The claim $\widehat{\mathrm{HFK}}(\widetilde{U}) \cong \widehat{\mathrm{HFK}}(T(2,2n))$ then follows from the fact that $\widehat{\mathrm{HFK}}(-\widetilde{U}) \cong (\widehat{\mathrm{HFK}}(\widetilde{U}))^*$ with the appropriate change in gradings.

□

We can now prove Theorem 4.3.1;

Proof of Theorem 4.3.1. Suppose L is an annular link such that $\mathrm{AKh}(L) \cong \mathrm{AKh}(\widehat{\beta_{2n}})$. Lemma 4.3.5 implies that $\widehat{\mathrm{HFK}}(\tilde{U}, \Sigma(L)) \cong \widehat{\mathrm{HFK}}(T(2,2n))$, whence $\tilde{U}$ is $T(2,2n)$ by Theorem 4.0.1, which is fibered of genus n. Up to isotopy fibered link exteriors have unique fibrations by Seifert surfaces – for instance see [EN85, Chapter 1.4]. Note that the exterior of a link may fiber in different ways if one does not require the fibers to be Seifert surfaces. The monodromies of fibered links are unique up to conjugation. The monodromy of a fibered link in $\Sigma(L)$ is the image of a braid representing L in $\mathrm{Mod}(S^2_{2n})$ under the Birman-Hilden correspondence. Finally, by Proposition 4.3.2

conjugate monodromies must come from conjugate braids so L must be isotopic to $\widehat{\beta_{2n}}$. $\square$

Chapter 5

Almost L-space knots

As we saw in Chapter 3, L-space knots are knots which admit Dehn surgeries to 3-manifolds with Heegaard Floer homology of minimal rank. In this chapter we study "almost L-space knots", which are knots which admit large Dehn surgeries to 3-manifolds with Heegaard Floer homology of next-to-minimal rank. Our main result in this chapter is a classification of the $\mathrm{CFK}^\infty(-)$ type of almost L-space knots. As corollaries we show that almost L-space knots satisfy various strong topological properties, including some recently given by Baldwin-Sivek [BS22]. We also give some new cable link detection results.

The following terminology was introduced by Baldwin-Sivek [BS22].

Definition 5.0.1. Let Y be a rational homology sphere. We call Y an *almost L-space* if $\mathrm{rank}(\widehat{\mathrm{HF}}(Y)) = |H_1(Y;\mathbb{Z})| + 2$.

We note that there is no rational homology sphere with $\mathrm{rank}(\widehat{\mathrm{HF}}(Y)) = |H_1(Y;\mathbb{Z})| + 1$, as the decatigorification from $\widehat{\mathrm{HF}}(Y)$ to $|H_1(Y)|$ respects parity. Almost L-spaces are thus the rational homology spheres for which Inequality 3.1 is "almost tight". There is a question of whether or not this is quite the "correct" definition for almost L-spaces. See Section 5.4 for some discussion.

Definition 5.0.2. [BS22] A knot K in S^3 is called an *almost L-space knot* if there exists an $n \geq 2g(K) - 1$ such that $\text{rank}(\widehat{\text{HF}}(S^3_n(K))) = n + 2$.

As we will seem there are a number of equivalent definitions of almost L-space knots. The reason to include the condition $n \geq 2g(K) - 1$ is that if K is an L-space knot then $\text{rank}(\widehat{\text{HF}}(S^3_{2g(K)-2}(K))) = 2g(K)$.

This chapter is organised as follows; in Section 5.1 we survey our results, in Section 5.2 we discuss the chain complexes which arise as the CFK^∞-type of almost L-space knots, in Section 5.4 we give some alternate characterizations of almost L-spaces and prove Proposition 5.4.4 and Proposition 5.4.6. In Sections 5.5, 5.6, and 5.7 we prove Theorem 5.1.1. In Section 5.8 we give the applications.

5.1 Summary of Results

We turn our attention now to knot Floer homology. Our main result is a classification of $\text{CFK}^\infty(-)$ for almost L-space knots;

Theorem 5.1.1. Let K be an almost L-space knot. Then $\text{CFK}^\infty(K)$ has the filtered chain homotopy type of one of the following complexes;

1. A staircase complex direct sum a box complex.

2. An almost staircase complex.

The definitions of the complexes referred to in this theorem are deferred to Section 5.2. They are illustrated, however, in Figures 5.1, 5.2, 5.3 and 5.4.

We note in contrast that an L-space knot has CFK^∞ given by a staircase complex, as proven by Ozsváth-Szabó [OS05a].

It is straightforward to construct examples of almost L-space knots with $\text{CFK}^\infty(-)$ of type 2 by taking $(2, 2g(K) - 3)$-cables of L-space knots. Examples of knots with CFK^∞ of type 1 – and indeed type 2 after a change of basis – include the figure eight, the mirror of 5_2, $T(2,3)\#T(2,3)$, 10_{139}, and $12n_{725}$. There are many almost staircase

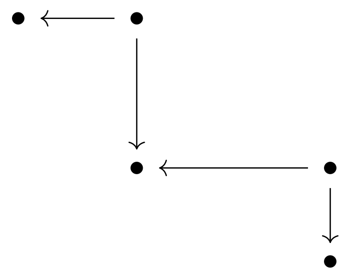

Figure 5.1: A staircase complex. The horizontal direction indicates the U grading and the vertical direction indicates the A grading.

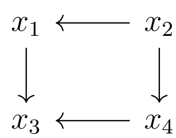

Figure 5.2: A box complex. The horizontal direction indicates the U grading and the vertical direction indicates the A grading. Note the arrows are of length one.

complexes which do not arise as the complexes of $(2, 2g(K) - 3)$-cables of L-space knots or as box plus staircase complexes. The author is unaware if any such complex arises as $\mathrm{CFK}^\infty(K)$ for some K. Indeed, the author is likewise unaware of an answer to the following question;

Question 5.1.2. Which staircase complexes arise as $\mathrm{CFK}^\infty(K)$ for some K?

We prove Theorem 5.1.1 by adapting the techniques in homological algebra Ozsváth-Szabó used to classify the CFK^∞ type of L-space knots. With Theorem 5.1.1 at hand we can obtain the following result;

Proposition *5.4.4*. Let K be a knot of genus g. K is an almost L-space knot if and only if $\mathrm{rank}(\widehat{\mathrm{HF}}(S^3_{p/q}(K))) = p + 2q$ for all $p/q \geq 2g - 1$.

This is an analogue of the result that if K a knot of genus g then K is an L-space knot if and only if $\mathrm{rank}(\widehat{\mathrm{HF}}(S^3_{p/q}(K))) = p$ for all $p/q \geq 2g - 1$. We prove this result with the aid of Hansleman-Rasmussen-Watson's immersed curve interpretation of the surgery formula in bordered Floer homology [HRW16].

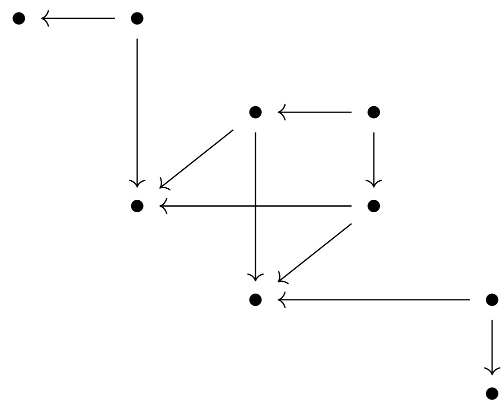

Figure 5.3: An almost staircase complex of type 2. The horizontal direction indicates the U grading and the vertical direction indicates the A grading.

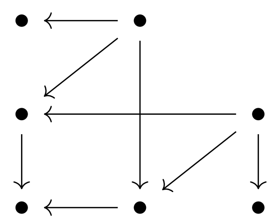

Figure 5.4: An almost staircase complex of type 1. The horizontal direction indicates the U grading and the vertical direction indicates the A grading.

We can also give a sharpen Theorem 5.1.1 somewhat by the following proposition;

Proposition *5.4.6.* Suppose K is an almost L-space knot with the CFK^∞ type of a box plus staircase complex. Then $\widehat{\mathrm{HFK}}(K, 0)$ is supported in a single Maslov grading.

Here $\widehat{\mathrm{HFK}}(K)$ is a weaker version of knot Floer homology that can be recovered from $\mathrm{CFK}^\infty(K)$. This proposition is proven using involutive knot Floer homology [HM17] and Sarkar's basepoint moving map [Sar15]. This result can perhaps be though of as a relative version of a result of Hanselman-Kutluhan-Lidman concerning the geography problem for $\mathrm{HF}^+(Y, \mathfrak{s})$ – another version of Heegaard Floer homology – for $\mathfrak{s}$ a self conjugate *spin*^{c} structure, with $\widehat{HF}(Y, \mathfrak{s})$ of next to minimal rank [HKL19].

A number of topological properties of L-space knots can be deduced from the classification of their CFK^∞-type – for example that they are strongly-quasi-positive and fibered. Likewise Theorem 5.1.1 allows us to show that almost L-space knots satisfy or almost satisfy various strong topological properties.

Corollary *5.8.1* ([BS22]). Suppose K is a genus one almost L-space knot. Then K is $T(2,-3)$, the figure eight knot or the mirror of the knot 5_2.

This result is the analogue of the fact that the only genus one L-space knot is $T(2,3)$.

Corollary *5.8.2*. The only composite almost L-space knot is $T(2,3)\#T(2,3)$.

This is the analogue of Baldwin-Vela-Vick's result that L-space knots are prime [BVV18, Corollary 1.4]

Corollary *5.8.3* ([BS22]). The mirror of 5_2 is the only almost L-space knot which is not fibered.

This is the analogue of the fact that L-space knots are fibered.

Corollary *5.8.4*. Suppose K is an almost L-space knot for which $|\tau(K)| < g_3(K)$. Then K is the figure eight knot.

Here $\tau(-)$ is a knot invariant due to Ozsváth-Szabó [OS04c]. Note that $|\tau(K)| \leq g_3(K)$ for all K. This result is the analogue of the fact that for L-space knots K, $\tau(K) = g(K)$.

Corollary 5.1.3 ([BS22]). The only almost L-space knot that is not strongly quasi-positive is the figure eight knot.

This is the analogue of the fact that L-space knots are strongly quasi-positive.

In a different direction, Theorem 5.1.1 allows us to recover a result of Hedden;

Corollary *5.8.6* ([Hed07, Hed11]). Suppose K is a knot and $\text{rank}(\widehat{\text{HFK}}(K_n)) = n+2$. Then K is an L-space knot and $n = 2g(K) - 1$.

Here K_n indicates the core of n-surgery on K. We again prove this with the aid of Hanselman-Rasmussen-Watson's immersed curve invariant. From here we can obtain some link detection results for cables of L-space knots using the same arguments as were used by the author and Dey for similar purposes in [BD22a]. Let $K_{p,q}$ denote the (p, q)-cable of K. Here p indicates the longitudinal wrapping number while q indicates the meridional wrapping number. We have the following;

Proposition *5.8.7.* Suppose K is an L-space knot. If a 2-component link L satisfies $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(K_{2,4g(K)-2})$. Then L is a $(2, 4g(K)-2)$-cable of an L-space knot K' such that $\widehat{\mathrm{HFK}}(K') \cong \widehat{\mathrm{HFK}}(K)$.

Here we orient $(2, 2n)$-cables as the boundary of annuli. Moreover we have a stronger result for *link Floer homology*, an invariant due to Ozsváth-Szabó [OS08a];

Proposition *5.8.8.* Suppose K is a L-space knot and L satisfies $\widehat{\mathrm{HFL}}(L) \cong \widehat{\mathrm{HFL}}(K_{m,2mg(K)-m})$. Then L is a $(m, 2mg(K)-m)$-cable of an L-space knot K' with $\widehat{\mathrm{HFK}}(K') \cong \widehat{\mathrm{HFK}}(K)$.

In fact, the conclusion that K' is an L-space knot in the above two Propositions follows from the fact that $\widehat{\mathrm{HFK}}(K) \cong \widehat{\mathrm{HFK}}(K')$, as is also true for the corresponding results in [BD22a]. This observation follows either from examining the immersed curve invariant or a result of Lidman-Moore-Zibrowius [LMZ20, Lemma 2.7].

As a corollary we have the following detection results;

Corollary 5.1.4. Knot Floer homology detects $T(2,3)_{2,2}$ and $T(2,5)_{2,6}$ amongst two component links.

Here these two links are again oriented as the boundary of annuli. The corresponding result for Link Floer homology is the following;

Corollary 5.1.5. Link Floer homology detects $T(2,3)_{m,2m}$ and $T(2,5)_{m,6m}$.

Note that $T(2,3)$ and $T(2,5)$ are L-space knots. The above two results thus follow directly from Propositions 5.8.7 and 5.8.8, Ghiggini's result that knot Floer homology detects $T(2,3)$ [Ghi08], and Farber-Reinoso-Wang's result that knot Floer homology detects $T(2,5)$ [FRW22].

We also give a characterization of links with the same link Floer homology type of the (m, mn)-cables of almost L-space knots for large n, generalizing work of the author and Dey [BD22a, Theorem 3.1];

We conclude this summary with some questions;

Question 5.1.6. Is there a version of the L-space conjecture for almost L-spaces?

The L-space conjecture posits an equivalent characterization of L-spaces in terms of both an orderability condition on their fundamental groups, and whether or not they admit taut foliations [BGW13].

Given a property that L-space knots exhibit, it is natural to ask if almost L-space knots too exhibit that property. For example, L-space knots do not have essential Conway spheres by a result of Lidman-Moore-Zibrowius [LMZ20], so it is natural to ask the following;

Question 5.1.7. Do almost L-space knots have essential Conway spheres?

5.2 Chain Complexes for almost L-space knots

Theorem 5.1.1 states that the chain complexes of almost L-space knot admit particularly simple models. That is, there exist particularly simple models of their chain homotopy type. Throughout this chapter we shall use i to indicate the U-grading on CFK^∞ and j to indicate the Alexander grading. To make the statement of Theorem 5.1.1 we need the following definitions;

Definition 5.2.1. A *staircase complex* is a set of generators $\{x_i\}_{1\leq i\leq N} \cup \{y_i\}_{1\leq i\leq N+1}$ where $N \in \mathbb{Z}^{\geq 0}$ such that x_i and y_i only differ in i coordinate while x_i and y_{i+1} only differ in j coordinate, and the non-trivial differentials are given by $\partial x_i = y_i + y_{i+1}$ for $1 \leq i \leq N$.

An example of such a complex is shown in Figure 5.1.

Definition 5.2.2. Let $N \in \mathbb{Z}^{\geq 0}$. A *type* 1 *almost staircase complex* is a complex admitting a basis $\{x_i\}_{0<|i|\leq N+1} \cup \{y_i\}_{0<|i|\leq N} \cup \{z\}$ such that;

1. x_i and y_{i-1} only differ in i coordinate while x_i and y_i only differ in j coordinate.

2. Suppose x_{-1} is of bigrading (a, b). Then z is of bigrading $(a-1, b)$ and x_1 is of bigrading $(a-1, b+1)$.

3. The non-trivial components of the differential are given by;

   (a) $\partial y_i = x_i + x_{i+1}$ for $i \neq \pm 1$,

   (b) $\partial y_{\pm 1} = x_{\pm 2} + x_1 + x_{-1}$,

   (c) $\partial x_{\pm 1} = z$.

An example of such a complex is shown in Figure 5.4. The $N = 0$ case corresponds to the case of the left handed trefoil.

Definition 5.2.3. Let $N \in \mathbb{Z}^{\geq 1}$. A *type* 2 *almost staircase complex* is a complex admitting a basis $\{x_i\}_{0<|i|\leq N} \cup \{y_i\}_{0<|i|\leq N} \cup \{z\}$ such that;

1. For $i < 0$ x_i and y_i only differ in i coordinate while x_{i-1} and y_i only differ in j coordinate. For $i > 0$ x_i and y_i only differ in j coordinate while x_i and y_{i+1} only differ in i coordinate

2. Suppose y_{-1} is of bigrading (a, b). Then z is of bigrading $(a, b+1)$ and y_1 is of bigrading $(a-1, b+1)$.

3. The non-trivial components of the differential are given by;

   (a) $\partial y_i = x_i + x_{i+1}$ for $i < 0$,

   (b) $\partial y_i = x_i + x_{i-1}$ for $i > 0$,

   (c) $\partial y_{\pm 1} = x_{-1} + x_1$,

(d) $\partial z = y_0 + y_1$.

An example of such a complex is shown in Figure 5.4. Almost staircases of type 1 arise as $(2, 2g(K) - 3)$-cables of L-space knots while almost staircase complexes of type 2 do not. This can be deduced using Hanselman-Watson's cabling formula for knot Floer homology in terms of immersed curves [HW19]. Indeed, the author does not know the answer to the following question;

Question 5.2.4. Do there exist knots with the CFK^{∞}-type of almost staircase complexes of type 2?

The remaining complex we have to define is the following;

Definition 5.2.5. A *box complex* is a complex with of generators x_3, x_4, x_1, x_2 with coordinates $(0,0), (0,1), (1,0)$ and $(1,1)$ respectively, up to overall shift in bigrading, differentials given by;

1. $\partial x_2 = x_1 + x_4$
2. $\partial x_2 = \partial x_4 = x_3$.
3. 0 otherwise.

An example of such a complex is shown in Figure 5.2. With these definitions at hand, the statement of Theorem 5.1.1 is made rigorous. We note however that there is a degree of overlap between complexes of type one and complexes of type two in the statement of Theorem 5.1.1. Namely, in light of proposition 5.4.6, box plus staircase complexes for knots K with $\mathrm{rank}(\widehat{\mathrm{HFK}}(K, 1)) = 2$ can arise as almost staircase complexes. Note that the mirror of 5_2, $T(2,3)\#T(2,3)$, 10_{139} and $12n_{725}$ all satisfy this property.

5.3 The Hendricks-Manolescu Involution

Each Heegaard diagram (Σ, α, β) for S^3 induces a symmetry on S^3 given by exchanging the α and β curves and changing the orientation of Σ. Hendricks-Manolescu built a version of Heegaard Floer homology called "Involutive knot Floer homology" which accounts for this symmetry [HM17]. This invariant has proven to be strictly stronger than Heegaard Floer homology in various settings. For example, one can prove using involutive techniques that that the figure eight knot is not slice, whereas there is no such proof in Heegaard Floer homology.

The key ingredient in Involutive knot Floer homology is an involution ι.

Theorem 5.3.1 ([HM17])**.** $\mathrm{CFK}^\infty(K)$ can be endowed with a grading preserving, skew filtered quasi-isomorphism $\iota : \mathrm{CFK}^\infty(K) \to \mathrm{CFK}^\infty(K)$.

Of course, we need to define some of the terms occurring in the statement of this theorem. Recall that $\mathrm{CFK}^\infty(K)$ can be viewed as a bi-filtered complex.

Definition 5.3.2. A map f on a bi-filtered chain complex $\mathcal{F}$ is *skew-filtered* if $f(\mathcal{F}_{i,j}) \subseteq \mathcal{F}_{j,i}$.

Definition 5.3.3. Let $(C, \mathcal{F})$ be a filtered chain complex. f is a *quasi-isomorphism* if f induces an isomorphism on the associated graded complexes.

Theorem 5.3.4 (Proposition 6.3 [HM17])**.** The quasi isomorphism type of $(\mathrm{CFK}^\infty(K), \iota)$ is an invariant of K.

Theorem 5.3.5. [HM17] ι^2 is filtered chain homotopy equivalent to ζ.

Here ζ is Sarkar's basepoint pushing map, which we discussed in Section 4.1. We will use the fact that $\mathrm{CFK}^\infty(K)$ admits maps ι and ζ to show that certain chain complexes do not arise as the CFK^∞-type of knots.

5.4 Some Remarks on Almost L-spaces and almost L-space knots

In this section we give some equivalent definitions of almost L-spaces and almost L space knots. First we give a definition of almost L-spaces in terms of of HF^+.

Proposition 5.4.1. Let Y be a rational homology sphere. The following conditions are equivalent;

1. Y is an almost L-space.
2. $\mathrm{rank}(\widehat{\mathrm{HF}}(Y,\mathfrak{s})) = 1$ aside from for a unique Spin^c structure $\mathfrak{s}_0$ for which $\mathrm{rank}(\widehat{\mathrm{HF}}(Y,\mathfrak{s}_0)) = 3$.
3. $\mathrm{HF}^+(Y,\mathfrak{s}) \cong \tau^+$ aside from in a unique Spin^c structure $\mathfrak{s}_0$ for which $\mathrm{HF}^+(Y,\mathfrak{s}_0) \cong \tau^+ \oplus \mathbb{Z}[U]/U^n$ for some n.
4. $\mathrm{HF}^+_{\mathrm{red}}(Y,\mathfrak{s}) \cong 0$ aside from in a unique Spin^c structure $\mathfrak{s}_0$ for which $\mathrm{HF}^+_{\mathrm{red}}(Y,\mathfrak{s}_0) \cong \mathbb{Z}[U]/[U^n]$ for some n.

Here τ^+ indicates a tower $\mathbb{F}[U,U^{-1}]/U$ and $\mathrm{HF}^+_{\mathrm{red}}(Y,\mathfrak{s})$ is the submodule of $\mathrm{HF}^+_{\mathrm{red}}(Y,\mathfrak{s})$ generated by elements that are not in the image of U^n for sufficiently large n. The proof is routine.

Proof. $1 \iff 2$ is just our definition of L-space.

To see that $2 \implies 3$, note that in general $\mathrm{HF}^+(Y,\mathfrak{s}) = \tau^+ \oplus \mathrm{HF}^+_{\mathrm{red}}(Y,\mathfrak{s})$. Now considering the long exact sequence induced by the short exact sequence;

$$0 \longrightarrow \widehat{\mathrm{CF}}(Y,\mathfrak{s}) \longrightarrow \mathrm{CF}^+(Y,\mathfrak{s}) \xrightarrow{U} \mathrm{CF}^+(Y,\mathfrak{s}) \longrightarrow 0$$

we see that $\mathrm{HF}^+(Y,\mathfrak{s}) \cong \tau^+$ unless $\mathfrak{s} = \mathfrak{s}_0$. In the latter case observe that for each summand of $\mathrm{HF}^+_{\mathrm{red}}(Y,\mathfrak{s})$ we find two generators of $\widehat{\mathrm{HF}}(Y,\mathfrak{s}_0)$. It follows that

$\mathrm{HF}^+_{\mathrm{red}}(Y,\mathfrak{s}) \cong \mathbb{Z}[U]/U^n$ for some $n \geq 0$. $3 \implies 4$ follows from the definition. $4 \implies 2$ follows immediately from the long exact sequence used previously.

□

A similar statement can be made for HF^-;

Proposition 5.4.2. Let Y be a rational homology sphere. The following conditions are equivalent;

1. Y is an almost L-space.
2. $\mathrm{HF}^-(Y,\mathfrak{s}) \cong \tau^-$ aside from in a unique Spin^c structure $\mathfrak{s}_0$ for which $\mathrm{HF}^-(Y,\mathfrak{s}_0) \cong \tau^+ \oplus \mathbb{Z}[U]/U^n$ for some n.
3. $\mathrm{HF}^-_{\mathrm{red}}(Y,\mathfrak{s}) \cong 0$ aside from in a unique Spin^c structure $\mathfrak{s}_0$ for which $\mathrm{HF}^-_{\mathrm{red}}(Y,\mathfrak{s}_0) \cong \mathbb{Z}[U]/[U^n]$ for some n.

Here τ^- indicates a tower $\mathbb{F}[U]$ and $\mathrm{HF}^-_{\mathrm{red}}(Y,\mathfrak{s})$ is the U torsion submodule of $\mathrm{HF}^-(Y,\mathfrak{s})$. The proof is again routine.

Proof. This follows just as in the case of the previous proposition but applying the following short exact sequence at the chain level;

$$0 \longrightarrow \mathrm{CF}^-(Y,\mathfrak{s}) \xrightarrow{U} \mathrm{CF}^-(Y,\mathfrak{s}) \longrightarrow \widehat{CF}(Y,\mathfrak{s}) \longrightarrow 0$$

□

Remark 5.4.3. Note that one might reasonably have given a stronger definition of almost L-spaces as rational homology spheres with, say, $\mathrm{HF}^+(Y;\mathbb{Z}) \cong \tau^+ \oplus \mathbb{Z}$. It is not clear to the author if this would be a better definition.

We now give the following alternate characterisation of almost L-space knots;

Proposition 5.4.4. Let K be a knot of genus g. K is an almost L-space knot if and only if $\mathrm{rank}(\widehat{\mathrm{HF}}(S^3_{p/q}(K))) = p + 2q$ for all $p/q \geq 2g-1$.

We find it convenient to prove this proposition by way of Hanselman-Rasmussen-Watson's immersed curve interpretation of bordered Floer homology [HRW16, HRW18]. This invariant can be thought of as assigning to each knot K a closed multi-curve $\Gamma(K)$ in the infinite strip $[0,1]\times\mathbb{R}$, punctured at $\{0\}\times\mathbb{Z}$. Pulling $\gamma(K)$ tight we obtain a collection of straight line segments. This is called a *singular pegboard diagram* in [HRW16]. There is a unique such segment which is not vertical. Let m denote its slope. Let n be the number of vertical line segments, counted with multiplicity. Note that $n-m$ is even. We will apply the following proposition of Hanselman;

Proposition 5.4.5 (Hanselman [Han22])**.** Let K be a knot with m and n as above

$$\text{rank}(\widehat{\text{HF}}(S^3_{p/q}(K))) = |p-qm| + n|q|.$$

The proof of this result amounts to a count of intersection points. We now prove Proposition 5.4.4.

Proof of Proposition 5.4.4. We first prove the forward direction. If K is an almost L-space knot, Theorem 5.1.1 implies that $m=2g(K)-1$, $n=2g(K)+1$. Proposition 5.4.5 immediately implies that $\text{rank}(\widehat{\text{HF}}(S^3_{p/q}(K))) = p+2q$ for all $p/q \geq 2g(K)-1$.

For the opposite direction suppose that there exist p,q such that $\frac{p}{q}\geq 2g(K)-1$ and $\text{rank}(\widehat{\text{HF}}(S^3_{p/q}(K))) = p+2q$. We first show that K is an L-space knot or an almost L-space knot. Note that Proposition 5.4.5 implies that, with m,n as above $p+2q = n|q|+|mq-p|$. Since K cannot be the unknot U as $\text{rank}(\widehat{\text{HF}}(S^3_{p/q}(U))) = p$, we have that $2g(K)-1>0$ and may take $p,q\geq 0$, so that $p+2q = nq+|mq-p|$. Suppose $m>\frac{p}{q}$. Then $2\dfrac{p}{q}+2 = n+m$, so that $2m+2\geq n+m$ and $m+2>n$. But $n-m\in 2\mathbb{Z}^{\geq 0}$, so that $n=m$ in which case K is an L-space knot and $\frac{p}{q}=m-1$. Otherwise we have that $n=m+2$. It follows in turn that $\text{rank}(\widehat{\text{HF}}(S^3_s(K))) = s+2$ for any sufficiently large s so that K is an almost L-space knot. □

We conclude this section with the following proposition, taking as given Theorem 5.1.1;

Proposition 5.4.6. Suppose K is an almost L-space knot with the CFK^∞-type of a box plus staircase complex. Then $\widehat{\mathrm{HFK}}(K, 0)$ is supported in a single Maslov grading.

Proof. Suppose K is an almost L-space knot with the CFK^∞-type of a box plus staircase complex.

We first show that $\widehat{\mathrm{HFK}}(K, 0)$ is supported in a single grading. Consider Hendricks-Manolescu's involution ι on $\mathrm{CFK}^\infty(K)$ [HM17]. Recall that ι^2 yields Sarkar's basepoint pushing map [Sar15]. Thus ι^2 is determined up to filtered chain homotopy by the CFK^∞-type of K by a result of Zemke [Zem17, Corololary C]. Consider the components of ι and Sarkar's basepoint pushing map acting on the box complex. Endow the box complex with the generators x_1, x_2, x_3, x_4 as in Definition 5.2.5. Let e denote $U^{-1}x_3$. Sarkar's basepoint pushing map is given by $x_1 \mapsto x_1$, $x_4 \mapsto x_4$, $e \mapsto e$ and $x_2 \mapsto x_2 + e$. Suppose that z is not of the same Maslov grading as e and x_2. Observe that the component of $\iota(x_2)$ in (U, A)-grading $(0, 0)$ is given by either $e + x_2$ or x_2, while $\iota(e) = e$. Neither of these maps square to Sarkar's basepoint pushing map, so we have a contradiction and that z is of the same Maslov grading as e and x_2. $\square$

5.5 CFK^∞ of almost L-space knots as a bigraded vector space without Maslov gradings

In this section we determine CFK^∞ as a bigraded vector space for almost L-space knots without Maslov grading. We apply similar techniques as used by Ozsváth-Szabó [OS05a] in the proof of the corresponding result for L-space knots.

Recall from Section 2.3 that $\widehat{\mathrm{HF}}(S^3_n(K))$ admits a $\mathbb{Z}/n$ grading. We first determine the grading in which $\mathrm{rank}(\widehat{\mathrm{HF}}(S^3_n(K), [i])) = 3$.

Lemma 5.5.1. Suppose K is an almost L-space knot and $n \geq 2g(K)-1$ is an integer. Then $\operatorname{rank}(\widehat{\mathrm{HF}}(S^3_n(K)),[i])$ is 1 unless $[i]=0$, in which case it is 3.

Proof. The set of Spin^c structures admit a conjugation action which induces an isomorphism on $\widehat{\mathrm{HF}}(S^3_n(K))$. This action sends $\widehat{\mathrm{HF}}(S^3_n(K),[i])$ to $\widehat{\mathrm{HF}}(S^3_n(K),[-i])$. Since there is a unique Spin^c structure in which $\widehat{\mathrm{HF}}(S^3_n(K))$ is not one, it follows that this Spin^c structure is self conjugate. There are potentially two such Spin^c structures, namely 0 and $\frac{n}{2}$ if n is even. We show that the latter case is impossible. Since $n \geq 2g(K)-1$ we have that $\frac{n}{2} \geq g(K)$. However, it follows immediately from Ozsváth-Szabó's surgery formula [OS08b] that $\widehat{\mathrm{HF}}(S^3_n(K);[i])$ is of rank 1 for $i \geq g(K)$, a contradiction. ∎

We note in passing that if K is an L-space knot then $\operatorname{rank}(\widehat{\mathrm{HF}}(S^3_{2g(K)-2}(K)),[g(K)-1])=3$.

Before proceeding we introduce some notation. CFK^{∞} is a $\mathbb{Z}\oplus\mathbb{Z}$ graded complex, where the grading (i,j) indicates a U-grading of j and an Alexander grading of i. We denote by $C(f(i,j)=0)$ the subcomplex of the associated graded consisting of generators of grading (i,j) satisfying $f(i,j)=0$.

Set $X_m=\{i\leq 0, j=m\}$, $Y_m=\{i=0, j\leq m-1\}$. Let UX_m be the complex generated over $\{i<0, j=-1\}$.

Following, Ozsváth-Szabó's approach in the L-space knot setting [OS05a, Section 3], observe that we have a pair of short exact sequences;

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & C\{UX_m\} & \longrightarrow & C(UX_m \cup Y\} & \longrightarrow & C\{Y_m\} & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & C\{X_m \cup Y_m\} & & \\
 & & & & & & \downarrow & & \\
 & & & & & & C\{X\} & & \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}
$$

This yields a pair of exact triangles in homology shown in Figure 5.5;

$$
\begin{array}{ccccccccc}
H_*(UX_m) & & \xleftarrow{\quad c \quad} & & H_*(Y_m) & & \xleftarrow{\quad d \quad} & & H_*(X_m) \\
 & \searrow^{e} & & \nearrow^{b} & & \searrow^{f} & & \nearrow^{a} & \\
 & & H_*(UX_m \cup Y_m) & & & & H_*(X_m \cup Y_m) & &
\end{array}
$$

Figure 5.5: A pair of exact triangles we will use repeatedly in this section and the next.

As noted by Ozsváth-Szabó [OS05a, Section 3], the composition of the two horizontal maps is zero.

We first extract a lemma from work of Ozsváth-Szabó [OS05a, Section 3].

Lemma 5.5.2. Suppose $H_*(X \cup Y) \cong \mathbb{F}$, $H_*(UX \cup Y) \cong \mathbb{F}^3$. Then $1 \leq \operatorname{rank}(H_*(X)) \leq 2$.

Proof. Suppose $H_*(X) \cong \mathbb{F}^n$. In this specific context the exact triangles from Figure 5.5 yield;

$$
\begin{array}{ccccccccc}
\mathbb{F}^n & & \xleftarrow{\quad c \quad} & & H_*(Y) & & \xleftarrow{\quad d \quad} & & \mathbb{F}^n \\
 & \searrow & & \nearrow^{b} & & \searrow & & \nearrow^{a} & \\
 & & \mathbb{F}^3 & & & & \mathbb{F} & &
\end{array}
$$

If a is injective then $H_*(Y) \cong \mathbb{F}^{n-1}$, d has kernel of rank 1, b must have image of rank 1, and c must have kernel of rank 1. Since the composition of the two horizontal maps must be trivial it follows that $1 \leq n \leq 2$.

If a is trivial then d is injective and $H_*(Y) \cong \mathbb{F}^{n+1}$ whence b has image of rank 2 and c has kernel of rank 2. It follows that the image of $c \circ d$ is of rank $n-1$ or $n-2$, whence $n = 1$ or $n = 2$ since the composition of the two maps is trivial.

□

Lemma 5.5.3. Suppose $H_*(X \cup Y) \cong \mathbb{F}^3$, $H_*(UX \cup Y) \cong \mathbb{F}$. Then $1 \leq \operatorname{rank}(H_*(X)) \leq 2$.

Proof. This follows from the proof of the previous lemma, after dualizing, noting that we are working over a field. □

Of course we are interested in computing $H_*(0, m)$ rather than $H_*(X_m)$. We now relate these quantities.

Lemma 5.5.4. We have the following;

1. Suppose $\operatorname{rank}(H_*(X_m)) = 2$. Then $\operatorname{rank}(H_*(0, m))$ is one of $\operatorname{rank}(H_*(\{i < 0, j = m\})) - 2, \operatorname{rank}(H_*(\{i < 0, j = m\}))$ or $\operatorname{rank}(H_*(\{i < 0, j = m\})) + 2$.

2. Suppose $\operatorname{rank}(H_*(X)_m) = 1$. Then $\operatorname{rank}(H_*(0, m))$ is either $\operatorname{rank}(H_*(\{i < 0, j = m\})) - 1$ or $\operatorname{rank}(H_*(\{i < 0, j = m\})) + 1$.

3. Suppose $\operatorname{rank}(H_*(X_m)) = 0$. Then $\operatorname{rank}(H_*(0, m)) = \operatorname{rank}(H_*(\{i < 0, j = m\}))$.

Proof. There is a short exact sequence;

$$0 \longrightarrow C\{i < 0, j = m\} \longrightarrow C\{i \leq 0, j = m\} \longrightarrow C(0, m) \longrightarrow 0$$

giving the exact triangle on homology shown in Figure 5.6.

The result follows directly.

□

We proceed now to compute the rank of $H_*(0, n)$ for each n.

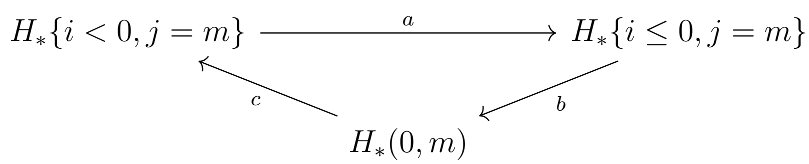

Figure 5.6: Another exact triangle we will use repeatedly in this section and the next.

Lemma 5.5.5. Suppose K is an almost L-space knot. Then $\operatorname{rank}(\widehat{\mathrm{HFK}}(K, A)) \leq 1$ for $|A| > 1$.

Proof. Given Lemma 5.5.1, this follows directly from work of Ozsváth-Szabó in the setting in which K is an L-space knot as opposed to an almost L-space knot [OS05a, Section 3]. ☐

Lemma 5.5.6. Suppose K is an almost L-space knot. Then $\operatorname{rank}(H_*(i < 0, j = 1)) = 1$ or 0

Proof. Again this follows directly from work of Ozsváth-Szabó in the setting in which K is an L-space knot as opposed to an almost L-space knot [OS05a, Section 3] given Lemma 5.5.1. ☐

It thus remains to determine $H_*(\{(0, j) : |j| \leq 1\})$.

To do so we combining Lemmas 5.5.2, 5.5.4 and 5.5.5 and obtain the following;

1. Suppose $\operatorname{rank}(H_*(i < 0, j = 1)) = 0$. Then $\operatorname{rank}(H_*(i \leq 0, j = 1)) = \operatorname{rank}(H_*(0, 1))$ is 1 or 2.

2. Suppose $\operatorname{rank}(H_*(i < 0, j = 1)) = 1$. Then;

   (a) If $\operatorname{rank}(H_*(i \leq 0, j = 1)) = 2$ then $\operatorname{rank}(H_*(0, 1))$ is either 1 or 3.

   (b) If $\operatorname{rank}(H_*(i \leq 0, j = 1)) = 1$ then $\operatorname{rank}(H_*(0, 1))$ is either 0 or 2.

We seek to exclude the case that $\operatorname{rank}(H_*(0, 1)) = 3$.

Lemma 5.5.7. Suppose K is an almost L-space knot. Then $\operatorname{rank}(H_*(0, 1)) \neq 3$

Proof. If $\text{rank}(H_*(0,1)) = 3$ then $\text{rank}(H_*(i \leq 0, j = 1)) = \text{rank}(H_*\{i < 0, j = 0\}) = 2$. We have the following cases;

1. $\text{rank}(H_*\{i \leq 0, j = 0\}) = 1$ so that $\text{rank}(H_*(0,0)) = 1$ or 3 by Lemma 5.5.4.

2. $\text{rank}(H_*\{i \leq 0, j = 0\}) = 2$ so that $\text{rank}(H_*(0,0)) = 0$ 2 or 4 by Lemma 5.5.4, a contradiction since this rank should be odd.

It thus remains to exclude the case that $\text{rank}(H_*(i \leq 0, j = 1)) = 2, \text{rank}(H_*(0,1)) = 3, \text{rank}(H_*\{i \leq 0, j = 0\}) = 1$. This is straightforward as $H_*(i \leq 0, j = -1) \cong H_*(i = -1, j \leq 0) \cong H_*(j = 0, i \leq 1)$, which is of rank zero or 1, as can be seen by applying the exact sequences in Figure 5.5 to determine $H_*(Y_2)$. Applying the exact triangle in Figure 5.6 we find that that $\text{rank}(H_*(0,-1))$ is $0, 1$ or 2, a contradiction, since it is supposed to be 3.

□

We also exclude the case that $H_*(0,1) \cong 0$.

Lemma 5.5.8. Suppose K is an almost L-space knot. Then $H_*(0,1) \ncong 0$

We note that if we counted all knots which admit a positive L-space surgery as almost L-space knots then this Lemma would be false. In particular there there exist L-space knots K with $H_*(0,1) \cong 0$. For any such K, $\text{rank}(\widehat{\text{HF}}(S^3_{2g(K)-2}(K))) = 2g(K)$.

Proof. Suppose $H_*(0,1) \cong 0$. Then we must have that $H_*(i \leq 0, j = 1) \cong H_*(i < 0, j = 0) \cong \mathbb{F}$. We have two cases according to Lemma 5.5.3. Applying the exact triangle 5.6 in each instance we have that;

1. If $\text{rank}(H_*(j = 0, i \leq 0)) = 1$ then $\text{rank}(H_*(0,0))$ is 0 or 2, both of which are even, a contradiction.

2. If $\text{rank}(H_*(j = 0, i \leq 0)) = 2$ then $\text{rank}(H_*(j = -1, i < 0)) = 2$. Since $\text{rank}(H_*(j = -1, i \leq 0)) = \text{rank}(H_*(i = -1, j \leq 0)) \leq 1$ as in the previous lemma, it follows that $H_*(0, -1)$ is either 1, 2 or 3, a contradiction.

$\square$

We are thus left with the cases;

1. $\text{rank}(H_*(i < 0, j = 1)) = 0$ and $\text{rank}(H_*(i \leq 0, j = 1)) = \text{rank}(H_*(0, 1))$ is 1 or 2

2. $\text{rank}(H_*(i < 0, j = 1)) = 1$ and $\text{rank}(H_*(i \leq 0, j = 1) = 2$ and $\text{rank}(H_*(0, 1))$ is 1.

We now compute $H_*(0, 0)$, again by cases;

1. $\text{rank}(H_*(0, 1)) = 1$. Then either;
   (a) $\text{rank}(H_*(i < 0, j = 0)) = 2$ and $\text{rank}(H_*(0, 1))$ is $0, 1, 2$ or 3. The even cases are excluded as $H_*(0, 0)$ must be of odd rank.
   (b) If $\text{rank}(H_*(i < 0, j = 0)) = 1$ then $\text{rank}(H_*(0, 1))$ is $0, 1, 2$ or 3. The even cases are excluded as before.

2. $\text{rank}(H_*(0, 1)) = 2$. Then either;
   (a) $\text{rank}(H_*(i < 0, j = 0)) = 2$ and $\text{rank}(H_*(0, 1))$ is $0, 1, 2$ or 3. The even cases are excluded as before.
   (b) If $\text{rank}(H_*(i < 0, j = 0)) = 1$ then $\text{rank}(H_*(0, 1))$ is $0, 1, 2$ or 3. The even cases are excluded as before.

In sum we have the following possibilities; $\widehat{\text{HFK}}(K, 1) \cong \mathbb{F}$ in which case $\widehat{\text{HFK}}(K, 0) \cong \mathbb{F}$ or $\mathbb{F}^3$; or $\widehat{\text{HFK}}(K, 1) \cong \mathbb{F}^2$ in which case $\widehat{\text{HFK}}(K, 0) \cong \mathbb{F}$ or $\mathbb{F}^3$.

5.6 Maslov Gradings

We now proceed to compute the Maslov gradings of the bigraded vector spaces produced in the previous section. We proceed again by cases, following the argument given by Ozsváth-Szabó for their corresponding step in the L-space knot case. Essentially this amounts to keeping track of the Maslov gradings in the exact triangles shown in Figure 5.5 and 5.6. Specifically we apply the fact that the maps f, a, e, b in Figure 5.5 preserve the Maslov grading, while maps c and d each lower it by 1. Likewise we use the fact that in Figure 5.6 the maps a and b preserve the Maslov grading while map c decreases it by 1.

We assume throughout this section that the genus of the almost L-space knot in question is at strictly greater than one. We are safe in this assumption as genus 1 almost L-space knots are rank at most two in their maximal Alexander grading and the only such knots are $T(2, -3)$, the mirror of 5_2 and the figure eight knot by results of Ghiggini [Ghi08] and Baldwin-Sivek [BNS22].

Lemma 5.6.1. Suppose K is an almost L-space knot of genus at least two and that Maslov grading, m of the generator x of lowest Alexander grading > 1, while the Alexander grading is A. We have that;

1ai) $\widehat{\mathrm{HFK}}(K, 1) \cong \mathbb{F}_{m+1}$ and $\widehat{\mathrm{HFK}}(K, 0) \cong \mathbb{F}_m$.

1aii) $\widehat{\mathrm{HFK}}(K, 1) \cong \mathbb{F}_d$ and $\widehat{\mathrm{HFK}}(K, 0) \cong \mathbb{F}_{m-1} \oplus \mathbb{F}_{d-1} \oplus \mathbb{F}_{d-1}$

1bi) $\widehat{\mathrm{HFK}}(K, 1) \cong \mathbb{F}_{m-1} \oplus \mathbb{F}_{m-2}$ and $\widehat{\mathrm{HFK}}(K, 0) \cong \mathbb{F}_{m-3}$

1bii) $\widehat{\mathrm{HFK}}(K, 1) \cong \mathbb{F}_a \oplus \mathbb{F}_{m-1}$ and $\widehat{\mathrm{HFK}}(K, 0) \cong \mathbb{F}_{a-1} \oplus \mathbb{F}_{m-2} \oplus \mathbb{F}_{a-1}$

2ai) $\widehat{\mathrm{HFK}}(K, 1) \cong \mathbb{F}_{m+1-2A}$ and $\widehat{\mathrm{HFK}}(K, 0) \cong \mathbb{F}_{m-2A}$

2aii) $\widehat{\mathrm{HFK}}(K, 1) \cong \mathbb{F}_b$ and $\widehat{\mathrm{HFK}}(K, 0) \cong \mathbb{F}_{m-2A+1} \oplus \mathbb{F}_{b-1} \oplus \mathbb{F}_{b-1}$

2bi) $\widehat{\mathrm{HFK}}(K, 1) \cong \mathbb{F}_{m-2A+1} \oplus \mathbb{F}_{m-2A}$, $\widehat{\mathrm{HFK}}(K, 0) \cong \mathbb{F}_{m-2A+1}$

2bii) $\widehat{\mathrm{HFK}}(K,1) \cong \mathbb{F}_b \oplus \mathbb{F}_{m-2A+3}$, $\widehat{\mathrm{HFK}}(K,0) \cong \mathbb{F}_{b-1} \oplus \mathbb{F}_{b-1} \oplus \mathbb{F}_{m-2A+2}$

Proof. We proceed by cases;

1. Suppose $H_*(i<0, j=1) \cong 0$

   (a) Suppose $H_*(0,1) \cong \mathbb{F}_d$. It follows that $H_*(i \leq 0, j=1) \cong \mathbb{F}_d$, whence in turn $H_*(i<0, j=0) \cong \mathbb{F}_{d-2}$.

      i. Suppose $H_*(0,0) \cong \mathbb{F}_a$. Then $H_*(i \leq 0, j=0) \cong \mathbb{F}_b \oplus \mathbb{F}_c$. Then $a=b$ and $d-2=c$. It follows that $H_*(i<0, j=-1) \cong \mathbb{F}_{a-2} \oplus \mathbb{F}_{d-4}$ It follows that $d-3=a-2$. Indeed, $H_*(i \leq 1, j=-1) \cong \mathbb{F}_{d-4}$. We then find that $d-4-2(A-1) = m-2A-1$, so that $d=m+1$.

      ii. Suppose $H_*(0,0) \cong \mathbb{F}_a \oplus \mathbb{F}_b \oplus \mathbb{F}_c$. Then without loss of generality $d-2=c-1$ and $H_*(i \leq 0, j=0) \cong \mathbb{F}_a \oplus \mathbb{F}_b$, so that $H_*(i<0, j=-1) \cong \mathbb{F}_{a-2} \oplus \mathbb{F}_{b-2}$. It follows without loss of generality that $d-3=b-2$. Indeed, $H_*(i \leq 1, j=-1) \cong \mathbb{F}_{a-2}$. We then find that $a-2-2(A-1) = m-2A-1$, so that $a=m-1$.

   (b) Suppose $H_*(0,1) \cong \mathbb{F}_a \oplus \mathbb{F}_b$. Then $H_*(i \leq 0, j=1) \cong \mathbb{F}_a \oplus \mathbb{F}_b$.

      i. If $H_*(0,0) \cong \mathbb{F}_c$ then $H_*(i \leq 0, j=0) \cong \mathbb{F}_{b-2}$, $c-1=a-2$. It follows that $H_*(i<0, j=-1) \cong \mathbb{F}_{b-4}$. Thus $a-3=b-4$. As before we find that $a=m-1$.

      ii. If $H_*(0,0) \cong \mathbb{F}_c \oplus \mathbb{F}_d \oplus \mathbb{F}_e$ then without loss of generality $e-1 = a-2, d-1=b-2$ and $H_*(i \leq 0, j=0) \cong \mathbb{F}_c$. It follows that $c-2=b-3$ or $a-3$. Suppose $c-2=b-3$. Then we find that $a-2-2(A-1) = m-2A-1$ so that $a=m-1$. Suppose $c-2=a-3$ then we find that $b=m-1$.

2. Suppose $H_*(i<0, j=1) \cong \mathbb{F}_a$. Then $a = m-2(A-1)$. Whatsmore;

(a) Suppose $H_*(0,1) \cong \mathbb{F}_b$. Then $H_*(i \leq 0, j = 1) \cong \mathbb{F}_a \oplus \mathbb{F}_b$ and $H_*(i < 0, j = 0) \cong \mathbb{F}_{a-2} \oplus \mathbb{F}_{b-2}$

i. Suppose $H_*(0,0) \cong \mathbb{F}_c$. Suppose $a-2 = c-1$ and $H_*(i \leq 0, j = 0) \cong \mathbb{F}_{b-2}$. It follow that $H_*(i < 0, j = -1) \cong \mathbb{F}_{b-4}$, a contradiction. Thus $b-2 = c-1$, and $H_*(i \leq 0, j = 0) \cong \mathbb{F}_{a-2}$. It follows that $b-3 = a-4$.

ii. Suppose $H_*(0,0) \cong F_c \oplus \mathbb{F}_d \oplus \mathbb{F}_e$. Then without loss of generality $c-1 = a-2, d-1 = b-2$ and $H_*(i \leq 0, j = 0) \cong \mathbb{F}_e$. It follows that $e-2$ is $b-3$.

(b) Suppose $H_*(0,1) \cong \mathbb{F}_b \oplus \mathbb{F}_c$. It follows without loss of generality that $c-1 = a$ and $H_*(i \leq 0, j = 1) \cong \mathbb{F}_b$ while $\mathbb{F}_{b-2} \cong H_*(i < 0, j = 0)$.

i. Suppose $H_*(0,0) \cong \mathbb{F}_d$. Then $H_*(i \leq 0, j = 0) \cong \mathbb{F}_{b-2} \oplus \mathbb{F}_d$ and $H_*(i < 0, j = 0) \cong \mathbb{F}_{b-4} \oplus \mathbb{F}_{d-2}$. It follows that $b-3 = d-2$, $a-3 = d-2$.

ii. Suppose $H_*(0,0) \cong \mathbb{F}_d \oplus \mathbb{F}_e \oplus \mathbb{F}_f$. It follows that $f-1 = d-2$ without loss of generality. Similarly $H_*(i \leq 0, j = 0 \cong \mathbb{F}_d \oplus \mathbb{F}_e)$. Thus $H_*(i < 0, j = -1) \cong \mathbb{F}_{d-1} \oplus \mathbb{F}_{e-1}$. Thus $\{d-2, e-2\} = \{b-3, c-3\}$ i.e. without loss of generality $d = b-1, e = c-1$

□

5.7 The filtrered chain homotopy type of CFK^{∞}

In this section we seek to determine the filtered chain homotopy type of CFK^{∞} of almost L-space knots. We have 8 cases to deal with according to Lemma 5.6.1, although we will see that there is a certain amount of degeneracy amongst these cases.

We again assume throughout this section that the genus of the almost L-space knot in question is at strictly greater than one. We are safe in this assumption as genus 1 almost L-space knots are rank at most two in their maximal Alexander grading and the only such knots are $T(2,-3)$, the mirror of 5_2 and the figure eight knot by results of Ghiggini [Ghi08] and Baldwin-Sivek [BNS22].

Lemma 5.7.1. Let K be an almost L-space knot of genus strictly greater than 1. Then there exists an integer d such that either;

1. $H_*(i=0, j>1) \cong \mathbb{F}_0$ and $H_*(i=0, j<-1) \cong \mathbb{F}_d$ and $H_*(|i| \leq 1) \cong \mathbb{F}_{d+1}$
2. $H_*(i=0, j>1) \cong \mathbb{F}_0 \oplus \mathbb{F}_d$ and $H_*(i=0, j<-1) \cong 0$ and $H_*(|i| \leq 1) \cong \mathbb{F}_{d-1}$

Proof. $H_*(i=0, j>1)$ can be computed using the isomorphisms $H_*(i=0, j<1) \cong H_*(j=0, i<1) \cong H_*(j=1, i \leq 0)$, while $H_*(i=0, j<-1)$ can be computed using the isomorphisms $H_*(i=0, j<-1) \cong H_*(j=0, i<-1) \cong H_*(j=1, i \leq 0)$ up to appropriate grading shifts.

To compute $H_*(i=0, |j| \leq 1)$ observe that there is a short exact sequence;

$$0 \longrightarrow C(i=0, j \leq 1) \longrightarrow C(i=0) \longrightarrow C(i=0, j>1) \longrightarrow 0$$

giving an exact triangle on homology;

$$\begin{array}{ccc} H_*\{i=0, j \leq 1\} & \longrightarrow & H_*(i=0) \cong \mathbb{F}_0 \\ & \nwarrow \quad \swarrow & \\ & H_*(i=0, j>1) & \end{array}$$

and another short exact sequence;

$$0 \longrightarrow C\{i=0, j<-1\} \longrightarrow C\{i=0, j \leq 1\} \longrightarrow C(\{i=0, |j| \leq 1) \longrightarrow 0$$

giving an exact triangle on homology;

$$\begin{array}{ccc} H_*\{i=0, j<-1\} & \longrightarrow & H_*\{i=0, j \leq 1\} \\ & \nwarrow \quad \swarrow & \\ & H_*(i=0, |j| \leq 1) & \end{array}$$

For grading reasons this determines the chain complexes $C(x, y : |y-x| > 1)$.

□

We seek to determine the rest of the chain complex.

Lemma 5.7.2. Suppose K is an almost L-space knot and $\mathrm{rank}(H_*(0,0)) = 1$. Then $\mathrm{CFK}^\infty(K)$ is an almost staircase complex.

Proof. We know that $H_*(i = 0) \cong \mathbb{F}_0$ and that $\mathrm{rank}(H_*(0,i)) \leq 1$ for $i \neq \pm 1$, $1 \leq \mathrm{rank}(H_*(0,\pm 1)) \leq 2$. Whatsmore, in $\widehat{\mathrm{HFK}}(K)$ the (M,A)-gradings of the 5 generators of smallest A-grading are either $(0,d),(1,d{+}1),(-1,d{-}1),(A,d),(-A,d-2A)$ if d is odd or $(0,d),(1,d+1),(-1,d-1),(-A,d),(A,d+2A)$ if d is even. We have a number of cases;

1. If d is even then;

    (a) if $A > 1$ then $H_*(i = 0, j \leq 1) \cong 0$ and $H_*(i = 0, j \geq A) \cong \mathbb{F}_0$. This forces $C(i = 0)$ to be of the desired form, perhaps with the addition of components of the differential from $(0,0)$ to $(0,-A-1)$ – if there exists a generator of this grading – and from $(0,-1)$ to $(0,-A')$, where A' is the smallest integer $A' > A+1$ for which $C(0,-A')$ is non-trivial. $C(j = 0)$ is determined similarly, up to the addition of two components of the differential. In CFK^∞ the resulting additional 4 components of the differential can be removed by a filtered chain homotopy. In order that $\partial^2 = 0$ there must be two diagonal components to the differential on $\mathrm{CFK}^\infty(K)$, as shown in Figure 5.3.

    (b) if $A = 1$ then the fact that $H_*(i = 0) \cong \mathbb{F}_0$, $H_*(i = 0, j \leq 1) \cong \mathbb{F}_{d+2}$, $H_*(i = 0, j \leq -1) \cong \mathbb{F}_d$, $H_*(i = 0, j < -1) \cong 0$ determines $C(i = 0)$, perhaps with the addition of components of the differential from $(0,0)$ to $(0,-2)$ – if there exists a generator of this grading – and from th generator of Maslov grading $d-1$ in bigrading $(0,-1)$ to $(0,-A')$, where A' is the smallest integer $A' > 2$ for which $C(0,-A)$ is non-trivial. $C(j = 0)$ is determined similarly, up to the addition of two components of the differential. In CFK$^\infty$ the resulting additional 4 components of the differential can be removed by a filtered chain homotopy. In order that $\partial^2 = 0$ there must be two diagonal components to the differential on CFK$^\infty(K)$, as shown in Figure 5.3.

2. If d is odd then;

   (a) if $A > 1$ then $H_*(i = 0, j \leq 1) \cong \mathbb{F}_{d-1}$ and $H_*(i = 0, j > 1) \cong \mathbb{F}_0 \oplus \mathbb{F}_d$. This once again determines $C(i = 0)$, perhaps up to the addition of two components of the differential one from a generator of bigrading $(0, A+1)$ – if such a generator exists – to the generator of grading $(0, 0)$ and from $(0, A')$ to $(0, 1)$ where A' is the smallest $A' > A + 1$ such that $C(0, A')$ is non-trivial. $C(j = 0)$ is determined similarly, up to the addition of two components of the differential. In CFK$^\infty$ the resulting additional 4 components of the differential can be removed by a filtered chain homotopy. In order that $\partial^2 = 0$ there must be two diagonal components to the differential on CFK$^\infty(K)$, as shown in Figure 5.4.

   (b) if $A = 1$ then the fact that $H_*(i = 0) \cong H_*(i = 0, j > 1) \cong \mathbb{F}_0$, $H_*(i = 0, j \leq 1) \cong 0$, $H_*(i = 0, j \leq 1) \cong 0$ determines $C(i = 0)$ as being of the desired form perhaps with addition of two unwanted components. One of these components is from the generator of bigrading $(0, 2)$ – if it exists – to the generator of grading $(0, 0)$, the other is a component from from $(0, A')$ to the generator in bigrading $(0, 1)$ of Maslov grading $d + 1$ where A' is the smallest $A' > 2$ such that $C(0, A')$ is non-trivial. $C(j = 0)$ is similarly determined. In CFK$^\infty$ the resulting additional 4 components of the differential can be removed by a filtered chain homotopy. In order that $\partial^2 = 0$ there must be two diagonal components to the differential on

$\mathrm{CFK}^\infty(K)$, as shown in Figure 5.4.

□

To deal with the case that $\mathrm{rank}(H_*(0,0)) = 3$ we first understand the behavior of the complex near the diagonal;

Lemma 5.7.3. Suppose K is an almost L-space knot and $\mathrm{rank}(H_*(0,0)) = 3$. Then up to a filtered change of basis $C_*(\{|j-i| \leq 1\})$ is the direct sum of a box complex and a staircase complex.

Proof. We proceed by cases according to $\mathrm{rank}(H_*(0,1))$.

Suppose $H_*(0,1) \cong \mathbb{F}_a, H_*(0,0) \cong \mathbb{F}_{a-1}^2 \oplus \mathbb{F}_d, H_*(0,-1) \cong \mathbb{F}_{a-2}$. Note that these Maslov gradings are determined by Lemma 5.6.1. Let x, y_1, y_2, z, w be the respective generators. Note that there is a unique from ∂ can take on $C(i = 0|j| \leq 1)$ such that $H_*(i = 0, |j| \leq 1) \cong \mathbb{F}_d$, up to a basis change in $C(0,0)$. It follows from here that $(C(i, j : |j-i| \leq 1\}), \partial)$ is determined up to the addition of diagonal components of ∂. The only way to add diagonal components is to have diagonal components from w to $U^{n+1}x, U^n w$ for some n or vice versa. A filtered change of basis removes these components.

Suppose now that $H_*(0,1) \cong \mathbb{F}_m \oplus \mathbb{F}_d, H_*(0,0) \cong \mathbb{F}_{m-1}^2 \oplus \mathbb{F}_{d-1}$. Then we have that $H_*(0,-1) \cong \mathbb{F}_{m-2} \oplus \mathbb{F}_{d-2}$. Note that these Maslov gradings are determined by Lemma 5.6.1. Let x_1 be the generator of $H_*(0,1)$ of Maslov grading m, y_1 be the generator of $H_*(0,1)$ of Maslov grading d, x_2, x_3 be the generators of $H_*(0,0)$ of Maslov grading $m-1$, y_2 be the generator of $H_*(0,0)$ of Maslov grading $d-1$, x_4 be the generator of $H_*(0,-1)$ of Maslov grading $m-2$ and y_3 be the generator of $H_*(0,-1)$ of Maslov grading $d-2$.

Suppose $m-1 \notin \{d, d-1, d-2\}$. Consider the restriction of the differential to $C(i = 0, |j| \leq 1)$. Then after a basis change we may take $\partial x_1 = x_2$. Since $m-1 \neq d, d-1$, we have $\partial x_3 = x_4$. The remaining component of the differential is then determined. Specifically, after mirroring we may assume that y_1 is the generator that persists and the remaining component of the differential is $\partial y_2 = y_3$.

We then have that in $C(|j-i| \leq 0)$ the y generators form a single staircase and the x generators form boxes. The question remains over whether or not there are diagonal components. By inspection these come in pairs and can be removed by a filtered change in basis.

We now deal individually with the cases $m = d+1$, $m = d$ and $m = d-1$;

1. $m = d+1$

   (a) Suppose the Maslov grading $d-2$ generator is not the one which persists to $H_*(i=0, |j| \leq 1)$. Then there is a component of ∂ from the Maslov index $d-1$ generator to the Maslov index $d-2$ generator. We have the following subcases;

      i. Suppose the generator of $H_*(0,1)$ of Maslov index d generator does not persist to $H_*(i=0, |j| \leq 1)$. Then it has a component to the generator of $H_*(0,-1)$ of Maslov index $d-1$. The Maslov index $d+1$ generator must have a component to one of the generators of $H_*(0,0)$ of Maslov index d. Such a ∂ clearly cannot be extended to $C(|j-i|leq1)$ a contradiction.

      ii. Suppose the generator of $H_*(0,1)$ of Maslov index d persists. After a change of basis this determines the vertical components of the differential and it is readily observed that $C(|j-i| \leq 1)$ is the direct sum of a staircase and a box complex.

   (b) Suppose the Maslov index $d-2$ generator does persist to $H_*(i=0, |j| \leq 1)$. This is impossible because it is of the wrong Maslov grading.

2. $m = d$. In this case the chain complex is thin and so splits as a direct sum of boxes and staircases by work of Petkova [Pet13, Lemma 7].

3. $m = d - 1$

   (a) Suppose the Maslov index $d-3$ generator is not the one which persists to $H_*(i=0, |j| \leq 1)$. After a change of basis we may assume the differential has a component from a single Maslov index $d-2$ generator to it. We have the following cases;

      i. Suppose the other generator of $H_*(0,0)$ of Maslov grading $d-2$ does not persist to $H_*(i=0, |j| \leq 1)$. Then there is a component of the differential from the the generator of $H_*(0,1)$ of Maslov index $d-1$ to it and indeed the whole vertical complex is determined. Indeed, the whole complex is seen to be a staircase complex plus a box complex. There can be no diagonal components to the differential for for grading reasons.

      ii. Suppose the generator of $H_*(0,0)$ of Maslov index $d-2$ persists to $H_*(i=0, |j| \leq 1)$. Then the vertical components of ∂ are determined by the Maslov gradings. It is readily observed that this does not extend to a differential on $C(|j-i| \leq 1)$.

□

This determines the complex up to addition of additional arrows between $C(i,j : |j-i| \leq 1), C(i,j : j-i > 1), C(i,j : i-j > 1)$. In order that $H_*(i=0) \cong H_*(j=0) \cong \mathbb{F}_0$ it is readily seen that the staircases from $C(|j-i| > 1)$ and $C(|j-i| < 1)$ connect to form a large staircase. We now show that up to filtered chain homotopy there are no diagonal components;

Proposition 5.7.4. Suppose K is an almost L-space knot and $\mathrm{rank}(H_*(0,0)) = 3$. Then $\mathrm{CFK}^{\infty}(K)$ is the direct sum of a box complex and a staircase complex.

Before proving this result we find it convenient to broaden our definition of staircase complexes to include their duals. We proceed with this in mind.

Proof. We have shown that $\mathrm{CFK}^\infty(K)$ contains a box complex and a staircase complex. Call them B and C respectively. We need to show that there is no component of ∂ from C to B after a filtered change of basis. The fact that there is no component from C to B then follows by dualizing the complex and the fact we have broaden our definition of staircase complexes.

Suppose towards a contradiction that there is some component of ∂ from C to B. Consider the generators of lowest j-grading. Amongst these consider that of the highest i grading. There is a unique such generator, call it z.

Consider the components of ∂z in C which has the lowest j grading. Amongst these consider those with the lowest i grading.

Let x_1, x_2, x_3, x_4 be the generators of C ordered as in figure 5.2.

Suppose this generator is x_4. In order that $\partial^2 = 0$, we must have that ∂z has a length one horizontal component to a generator z' such that $\partial z'$ whose differential has a component x_3. The filtered change of basis $z' \mapsto z' + x_4$ removes both unwanted components of the differential.x

We proceed now to the case that the generator is x_3. Performing the filtered change of basis $z \mapsto z + x_3$ removes both unwanted components of the differential.

Suppose this generator is x_2. In order that $\partial^2 = 0$ we have that there is a generator z' with the same j grading such that $\langle z, z' \rangle \neq 0$ and that there is a component of the differential from z' to x_4. In fact, again in order that $\partial^2 = 0$, we must also have a component of ∂ from z' to x_1. Performing a filtered change of basis $z' \mapsto z' + x_2$ removes the unwanted components of the differential.

Suppose this generator is x_1. Again we must have a generator z' such that there is a component of the differential from z to z' and there is a component of ∂ from z' to x_3. The filtered change of basis $z' \mapsto z' + x_1$ then removes these unwanted components of the differential.

□

5.8 Applications

In this section we prove the applications advertised in the chapter summary.

Corollary 5.8.1. Suppose K is a genus one almost L-space knot. Then K is $T(2,-3)$, the figure eight knot, or the mirror of the knot 5_2.

Proof. Theorem 5.1.1 implies that if K is an almost L-space knot then the rank of $\widehat{\mathrm{HFK}}(K)$ in Alexander grading 1 is at most two. The result then follows immediately from Baldwin-Sivek's classification of nearly fibered knots [BNS22] – i.e. genus one knots with knot Floer homology of maximal Alexander grading of rank two – and Ghiggini's classification of genus 1 knots with knot Floer homology of rank one in their maximal Alexander grading [Ghi08]. □

Corollary 5.8.2. The only composite almost L-space knot is $T(2,3)\#T(2,3)$.

Proof. Let K be an almost L-space knot. Knots of genus at most 1 are prime. If $g(K)>2$ then $\mathrm{rank}(\widehat{\mathrm{HFK}}(K,g(K)-1))=\mathrm{rank}(\widehat{\mathrm{HFK}}(K,g(K)-1))=1$ by a result of Baldwin-Vela-Vick [BVV18, Theorem 1.1]. It follows that K cannot be prime just as it does in the L-space case [BVV18, Corllary 1.4].

Suppose $g(K)=2$ and $K=K_1\#K_2$. Then since $\widehat{\mathrm{HFK}}(K)\cong\widehat{\mathrm{HFK}}(K_1)\otimes\widehat{\mathrm{HFK}}(K_2)$ and $\widehat{\mathrm{HFK}}(K)$ is trivial in Alexander gradings ≥ 2 and of Maslov grading 0 in Alexander grading 2, it follows that K_1,K_2 are both genus 1 [OS05a], fibered [Ghi08] and strongly quasi-positive [Hed10]. It follows that $K_1=K_2=T(2,3)$. It is readily checked that $T(2,3)\#T(2,3)$ is indeed an almost L-space knot. □

Corollary 5.8.3 ([BS22])**.** The mirror of 5_2 is the only almost L-space knot which is not fibered.

Proof. If K is an almost L-space knot with $g(K)>1$ then the rank of the knot Floer homology of K in the maximal Alexander grading is 1 by Theorem 5.1.1. It follows that K is fibered by work of Ghiggini [Ghi08] and Ni [Ni07] □

Corollary 5.8.4. Suppose K is an almost L-space knot for which $|\tau(K)| < g_3(K)$. Then K is the figure eight knot.

Proof. This follows immediately from Theorem 5.1.1 and Corollary 5.8.1. ☐

Corollary 5.8.5 ([BS22])**.** The only almost L-space knot that is not strongly quasi-positive is the figure eight knot.

Proof. This again follows immediately from Theorem 5.1.1 and Corollary 5.8.1. ☐

We now recover a result of Hedden;

Corollary 5.8.6 ([Hed07, Hed11])**.** Suppose K is a knot and $\text{rank}(\widehat{\text{HFK}}(K_n)) = n+2$. Then K is an L-space knot and $n = 2g(K) - 1$.

Recall here that K_n denotes the core of n-surery on K. Again we find it helpful to think of surgery in terms of Hanselman-Rasmussen-Watson's immersed curve invariants.

Proof. The immersed curve invariants of almost L-space knots are determined by 5.1.1. It follows that if K is an almost L-space knot then $\text{rank}(\widehat{\text{HFK}}(K_n)) > n+2$. Indeed, we have that if K is neither an L-space knot nor an almost L-space knot then $\text{rank}(\widehat{\text{HFK}}(K_n)) \geq \text{rank}(\widehat{\text{HF}}(S^3_n(K))) > n+2$. Likewise it follows immediately from the classification of immersed curves for L-space knots that if K is an L-space knot then $\text{rank}(\widehat{\text{HFK}}(K_n)) = n+2$ if and only if $n = 2g(K) - 1$. In sum we have that $\text{rank}(\widehat{\text{HFK}}(K_n)) = n+2$ if and only if K is an L-space knot and $n = 2g(K) - 1$. ☐

Proposition 5.8.7. Suppose K is an L-space knot. If a 2 component link L satisfies $\widehat{\text{HFK}}(L) \cong \widehat{\text{HFK}}(K_{2,4g(K)-2})$. Then L is a $(2, 4g(K)-2)$-cable of an L-space knot K' such that $\widehat{\text{HFK}}(K') \cong \widehat{\text{HFK}}(K)$.

Proof of Proposition 5.8.7. Applying 5.8.6, this result follows as in the proof of [BD22a, Theorem 4.1]. ☐

Proposition 5.8.8. Suppose K is an L-space knot $\widehat{\mathrm{HFL}}(L) \cong \widehat{\mathrm{HFL}}(K_{m,2mg(K)-m})$. Then L is a $(2, 2mg(K)-m)$ cable of an L-space knot K' with $\widehat{\mathrm{HFK}}(K') \cong \widehat{\mathrm{HFK}}(K)$.

Proof of Proposition 5.8.8. Applying 5.8.6, this result follows as in the proof of [BD22a, Theorem 3.1]. □

We now give a characterizations of cables of (m, mn) cables of almost L-space knots for sufficiently large m.

Proposition 5.8.9. Let K be an almost L-space knot. Suppose L is a link such that $\widehat{\mathrm{HFL}}(L) \cong \widehat{\mathrm{HFL}}(K_{m,mn})$ with $m > 2g(K) - 1$. Then L is the (m, mn)-cable of an almost L-space knot K' such that $\widehat{\mathrm{HFK}}(K') \cong \widehat{\mathrm{HFK}}(K)$.

Proof. Suppose K is as in the statement of the theorem. The same argument as used by the author and Dey in [BD22a, Theorem 3.1] implies that L is the (m, mn)-cable of some knot K' such that $\mathrm{rank}(\widehat{\mathrm{HFK}}(K'_m)) = m + 4$ and $\Delta_{K'}(t) = \Delta_K(t)$.

We now show that K' is either an L-space knot or an almost L-space knot. Let γ be the immersed curve invariant of K'. Applying Proposition 5.4.5, we have that $m + 4 \geq \mathrm{rank}(\widehat{\mathrm{HF}}(S^3_m(K')) = |m - a| + b \geq m$ where b is the number of vertical components in a singular pegboard diagram for γ counted with multiplicity and a is the slope of the unique segment which is not vertical. Suppose $n < a$. Then $a + b \leq 2m + 4 < 2a + 2$. $b - a$ is an even non-negative integer. It follows that $b = a + 2$ or $b = a$. If $b = a$ then K' is an L-space knot, as shown by an application of Proposition 5.4.5. If $b = a + 2$ then K' is an almost L-space knot as shown by an application of Proposition 5.4.5. If $m \geq a$ then we have that $b - a \leq 4$. However, this yields only one new case, namely $b - a = 4$. In this case it can be observed from the immersed curve formula for $\widehat{\mathrm{HFK}}(K_m)$ that $\mathrm{rank}(\widehat{\mathrm{HFK}}(K_m)) \geq m + 6$.

Suppose now that K' is an L-space knot. Then $m = 2g(K') - 2$, as can be seen from Proposition 5.4.5. Since $\widehat{\mathrm{HFL}}(K'_{m,mn})$ and $\widehat{\mathrm{HFL}}(K_{m,mn})$, $K_{m,mn}$ and $K'_{m,mn}$ have the same genus. Applying work of Gabai [Gab86], Neumann-Rudolph [NR87] and

Rudolph [Rud02], it in turn follow that K and K' have the same genus, contradicting $m > 2g(K) - 1$.

Thus K' is an almost L-space knot. It remains to show that $\widehat{\mathrm{HFK}}(K') \cong \widehat{\mathrm{HFK}}(K)$. Observe that since K and K' share the same Alexander polynomial. Theorem 5.1.1 and Proposition 5.4.6 imply that $\Delta_K(t)$ determines $\widehat{\mathrm{HFK}}(K)$ for almost L-space knots, completing the proof. □

Corollary 5.8.10. Link Floer homology detects the (m, mn)-cables of $T(2, -3)$, the figure eight knot and the mirror of 5_2 for $n > 1$.

Proof. Let K be one of the knots in the statement of the proposition. Suppose L is a link such that $\widehat{\mathrm{HFL}}(L) \cong \widehat{\mathrm{HFL}}(K_{m,mn})$ for some $n > 1$. Proposition 5.8.9 implies that L is as (m, mn) cable of an almost L-space knot K' with the same $\widehat{\mathrm{HFK}}(-)$ type as K. Proposition 5.8.1 shows that there are only 3 genus one almost L-space knots, namely those listed in the statement of the proposition. They are each distinguished by their Alexander polynomials, concluding the proof. □

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