ESSAYS ON APPORTIONMENT METHODS FOR AFFIRMATIVE ACTION

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Abstract

This collection of two essays in market design examines the designs of affirmative action policies.

In the first chapter, “Affirmative Action in Two Dimensions: A Multi-Period Apportionment Problem”, we study affirmative action policies that take the form of reserved seats or positions and apply at two levels simultaneously. For instance, in India, beneficiary groups are entitled to their proportion of reserved seats in public universities at both university and at department levels. We theoretically and empirically document the shortcomings of existing solutions. We propose a method with appealing theoretical properties and empirically demonstrate advantages over the existing solutions using recruitment advertisement data from India. Our problem also suggests possible extensions in the theory of apportionment (translating electoral votes into parliamentary seats).

In the second chapter, “Impartial Rosters for Affirmative Action’’, we present an answer to this question for the case where all positions are homogeneous. Devising methods is particularly necessary when the number of seats is small. For instance, a university appoints at most one assistant professor of economics every year, while the country’s affirmative action policy has more than one beneficiary group. To ensure that, over a period of time, each beneficiary group respects the spirit of an affirmative action policy, India devised a tool called roster. We present a theory of designing rosters to argue that only a few rosters can be considered impartial in that they do not favor some beneficiaries over others. We provide a method that constructs the set of impartial rosters. We show that the existing roster of India is not one of them and favors categories with a larger proportion of seats relative to the smaller ones.

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Chapter 1

Affirmative Action in Two Dimensions: A Multi-Period Apportionment Problem

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1.1 Introduction

In many countries affirmative action policies take the form of reserved seats or positions, for which only eligible candidates compete. For instance, in India beneficiary groups are entitled to their proportion of reserved seats in government jobs and publicly funded institutions. However, because of the indivisible nature of positions, the policy prescribed percentage of seats can almost never be met in practice. The fractional seats that arise in literal calculation nearly always need to be adjusted in some manner to yield whole numbers. The question then becomes: what are the ideal whole number counterparts of an affirmative action policy prescribed fractional seats?

The problem gets more complex when positions are heterogeneous. For instance, faculty positions (say assistant professors) in a university are listed under various departments.[1] Each faculty position, therefore, simultaneously represents two units, a department

[1]Bureaucrats of a country are posted in different states.

and the university. If both the department and the university adhere to the affirmative action policy, both must reserve the prescribed percentage of seats. In this paper we ask, how many seats *should* the departments and the university reserve in such cases? We term this novel problem as the problem of *reservations in two dimensions*.

An ideal solution to the problem of reservations in two dimensions should ensure that in each period as well as over time the seat allocations stay "close" to the prescribed fractional seats both (i) at the department level, and (ii) at the university level. However, delivering this is not easy. In fact each of the two solutions seen in practice in India fails to do so. Both existing solutions use a tool called *roster* that lays down the number of positions to be reserved for every number of total positions.[2] The debate in India revolves around whether the individual departments should follow the roster, or whether the university as a whole should follow it. If the departments follow the roster the solution fails to deliver the benefit of reservations at the university level. Whereas if the university as a whole follows the roster, the reserved positions could get allocated to merely a few departments in the university.

The problem with existing solutions is that they do not account for the interdependence of the departments and the university in calculating reserved seats. The reason is that each solution either operates at the department level or the university level, but not at both simultaneously. This is the main source of various shortcomings that we will document in Section 1.4. Not surprisingly, both these solutions are met with several petitions and protests leading to subsequent and frequent changes in the law. The noteworthy debates about these solutions (from Indian courts to public protests) that inspired us to write this paper are summarized in Section 1.2. We wish to design a tool, similar to a roster, that satisfactorily deals with the problem of reservations in two dimensions.

Reservations in two dimensions give rise to matrix problems, with input data in the form of a fair share table X. Its entries x_{ij} signify the fraction of seats beneficiary j is entitled in

[2]See Figure 1.17 and Figure 1.18 for the sequence in which the beneficiary groups take turns in claiming a position in India.

department i as per the affirmative action policy. The rows represent the first subdivision of the university into departments. The columns accommodate the several beneficiaries, and therefore present a second subdivision. The university is assumed to be broken down either way, providing department sizes as row sums, and overall (university level) beneficiary claims as column sums. The task is to find a two-way apportionment, with seat allocations (whole numbers, not fractions!) $\bar{x}_{ij}$ summing row-wise to the pre-specified row sums, while remaining "as near as may be" to the fractional seats x_{ij}.

The fair share table would be the ideal seat allocations if only the seats were divisible. Therefore, it is natural to consider integral seat allocations with entries that are rounded to an adjacent integer of entries of the fair share table as an ideal solution. That is, ideal seat allocations $\bar{x}_{ij}$ would consist of entries x_{ij} of the fair share table rounded up or down to the nearest integer. In fact, this is one of the most appealing and natural apportionment ideas, known as *staying within the quota* (see Balinski and Young (2010) for a fascinating discussion). The problem of reservations in two dimensions can therefore be viewed as a rounding problem of translating a matrix of fair shares to a matrix of seat allocations obtained by rounding the fair shares up or down.

Such matrix problems are not unique to the implementation of affirmative action policies. Biproportional apportionment methods introduced by Balinski and Demange (1989a,b) deal with such problems while translating electoral votes into parliamentary seats. Controlled rounding procedures introduced by Cox and Ernst (1982) also deal with such matrix problems in maintaining census data anonymity. What is unique about the problem we analyze in the affirmative action context is their multi-period aspect. For example, a department with only one new faculty position each year cannot reserve the position for the same beneficiary group each year. In such cases, to ensure that each beneficiary group gets its prescribed percentage of positions over a period of time, the beneficiary groups must take turns in claiming positions. Matrix problems with such multi-period considerations are unique to reservations in two dimensions.

The aim of this paper is twofold. The first objective is to present a comprehensive evaluation of existing solutions in light of staying within the quota property and the multi-period considerations. We do so theoretically in Section 1.4 and empirically in Section 2.5. The second objective is to check whether a solution exists to the problem of reservations in two dimensions that stays "close" to the prescribed fractional seats both (i) at the department level, and (ii) at the university level. The answer is affirmative.

Our first results that deal with the problem of reservations in two dimensions without the multi-period considerations are straightforward. The rounding problem has an elegant solution, called *controlled rounding*, that stays within quota and is simple enough to be implemented by hand. The technique was introduced by Cox (1987) to make slight perturbations in two-dimensional census data to ensure confidentiality of aggregate statistics while maintaining a good approximation of the original data. Adaptation of Cox's controlled rounding technique to our problem is summarized in Section 2.7. In addition to providing a solution that stays within quota, Cox's controlled rounding procedure provides an *unbiased* lottery solution, that is, entries of the fair share matrix are rounded up or down so that ex-ante positive and negative biases balance to yield zero bias.

The main theoretical contributions of our article address the multi-period problem of reservations in two dimensions, and are presented in Section 1.5.2. We show that there does not exist a solution for the problem of reservations in two dimensions that stays within quota at both the university and the department level simultaneously (Proposition 2). We give an even stronger result: There does not exist a solution for which the reservation table deviates from the fair share table bounded by a finite number (Proposition 3). These results justify the struggle in figuring out a solution in real-life practice as discussed in Section 1.2. Since the two constraints that staying within quota property imposes cannot be satisfied simultaneously, we ask: can these constraints be satisfied *approximately*? By approximately we mean, the probability of violating that constraint is exponentially decreasing with the size of the constraint. The answer is affirmative.

The main results of the article, stated in Theorem 1 and Theorem 2, show that there exists an unbiased solution that stays within quota at the department level and approximately stays within quota at the university level. The proof of Theorem 1 involves constructing a lottery solution that stays within quota at the department level and is unbiased at both the department and the university levels. An overview of the proof is presented in Section 1.5.2. The key technique is to design a procedure that takes the fractions of reservations and generates a roster that lays down the number of positions to be reserved for every number of total positions. For a roster, the staying within quota constraint regulates the cumulative number of positions for each category. Since there could be many rosters that would stay within quota, the procedure generates a *random roster* by assigning each solution roster a probability. Our solution to the problem of reservation in two dimensions assigns a roster to each department adhering to the probabilities dictated by the procedure.

The procedure of constructing a random roster is built around a network flow algorithm that takes a flow network as input and randomly constructs another flow network with fewer fractional flows as its output. By iterative application of this algorithm, a flow network with integral flows is generated. The random flow network has the following two properties: the expected value of each flow after the next iteration is the same as its current value and each constraint (imposed by the stay within quota property) remains satisfied. Since each flow network with integral flows can be mapped to a roster, this procedure generates a random roster. We next show, in Theorem 2, that the approximation errors are small. We do so by applying the multiplicative form of Chernoff concentration bounds to our solution in order to prove that, in addition to staying within quota at the department level, the solution approximately stays within quota at the university level. Moreover, we show that our bounds on the approximation errors are tight.

Lastly, in Section 2.5, we present an empirical case study of a two dimensional reservations problem from India using recruitment advertisement data. The objective is twofold. The first objective is to document the shortcomings of existing procedures empirically. In particular, we highlight the severity of the problem by documenting the instances and magnitude of violations. Our second objective is to quantify the performance of our proposed solutions. We do so by running simulations on the recruitment data, thus creating reservation tables per the procedures advocated in this article, and comparing the outcomes with existing (advertised) solutions.

1.1.1 Contributions with respect to the Related Literature

With a recent surge of research interest in implementation of affirmative action schemes, unnoticed issues in implementation of nation-wide affirmative action policies are coming to light. A considerable number of recent papers have documented such shortcomings and have also proposed practical alternatives to better implement such policies (Abdulkadiroğlu and Sönmez (2003), Kojima (2012), Hafalir et al. (2013), Ehlers et al. (2014), Echenique and Yenmez (2015), Dur et al. (2018), Dur et al. (2019), Sönmez and Yenmez (2019); Sönmez and Yenmez (2022a) and many others). Ours is another paper in this class. While the focus of the contemporary market design literature has been the design and analysis of assignment mechanisms given reserved seats and quotas, our paper looks at another side of affirmative action schemes: how many seats to reserve?

Distributing indivisible objects among a group of claimants in proportion to their claims, known as the *apportionment problem*, is the center point of the seminal work of Young (1995) and Balinski and Young (2010). The two-dimensional version, the *biproportional apportionment problem*, gives rise to similar matrix problems as ours, but has been investigated in a different context (Gassner (1988), Balinski and Demange (1989a), Balinski and Demange (1989b), Maier et al. (2010), Lari et al. (2014)).[3] In their context of translating electoral votes into parliamentary seats, the foremost criteria for desirability of a solution is "proportionality". However, in our context (of affirmative action) searching for the "closest" solution is better suited. More importantly, the multi-period constraints that the

[3]See Pukelsheim (2017) for detailed results and insights on biproportional apportionment problems.

problem reservation in two dimensions introduces have not been featured in the literature on biproportional apportionment problem.

Lastly, our paper is related to the literature on rounding techniques. The *controlled rounding* procedure introduced in Cox (1987) suffices to solve the problem of reservations in two dimensions for a special case of the model (see Proposition 1). For the general case, our rounding approach is similar to the ones developed in the literature on *approximation algorithms* from computer science (Ageev and Sviridenko (2004), Gandhi et al. (2006) and others). These techniques are not new to market designers. The literature on implementation of random and therefore fractional assignments solves such problems in the presence of a very rich "bihierarchical" structure on the set of constraints (Budish et al. (2013), Pycia and Ünver (2015), and Akbarpour and Nikzad (2020)). In particular, Budish et al. (2013) and Akbarpour and Nikzad (2020) build implementation methods for random allocation mechanisms based on techniques from deterministic and randomized rounding developed in Edmonds (2003) and Gandhi et al. (2006). Our constraints, in addition to following a "bihierarchical" structure, also extend in the time dimension in order to accommodate the multi-period considerations. It is this multi-period aspect of our problem that renders existing solutions inadequate. A rounding procedure for a multi-period model with a "bihierarchical" constraint structure (upper and lower quotas at the department level and approximate constraints at the university level) is a theoretical contribution of our paper (Theorem 1 and Theorem 2).

1.2 Motivating Debate from India

The 1950 Constitution of India provides a clear basis for positive discrimination in favor of disadvantaged groups, in the form of *reservation policies*. India's reservation policies mandate exclusive access to a fixed percentage of government jobs and seats in publicly funded institutions to the members of Scheduled Castes (SC, 15%), Scheduled Tribes (ST, 7.5%), Other Backward Classes (OBC, 27%) and Economically Weaker Sections (EWS, 10%). For the sake of transparency, the number of reserved seats for each category are explicitly and publicly advertised in advance of any admissions or recruitment cycle.

The procedures used to calculate the number of reserved seats in various settings are also explicit and public. But they have nowhere been more contentious than in the case of universities. Unlike other government jobs, for the same faculty position in a university (say assistant professor), the eligibility and selection criteria changes with the department. Thus the faculty positions in different departments are not interchangeable across a university. Each faculty position, therefore, simultaneously represents two units, a department and the university, where each unit is subject to the reservation policy. It is this feature of faculty positions that led to complications which made all three arms of the Indian government – the executive, the judiciary and the legislative – intervene.

The Executive. In August 2006, the University Grants Commission (UGC) issued *Guidelines for Strict Implementation of Reservation Policy of the Government in Universities* to all government educational institutions in India.[4,5] Through this document the UGC prohibited the practice of treating *department as the unit* for application of the reservation scheme, that is, for calculating the proportion of seats to be reserved (see clause 6(c) in the guidelines). Instead, UGC mandated *university as the unit* for the purpose of reservation. That is, the positions in a university shall be clubbed together across departments as three separate categories: professors, associate professors (or readers), and assistant professors (or lecturers), for the application of the rule of reservation (see clause 8(a)(v) in the guidelines). However, UGC's order was challenged in the court.

[4]UGC is a statutory autonomous organization responsible for implementation of policy of the Central Government in the matter of admissions as well as recruitment to the teaching and non-teaching posts in central universities, state universities and institutions which are deemed to be universities.

[5]Document last accessed on 12 June 2021 at `https://www.ugc.ac.in/pdfnews/7633178_English.pdf`

The Judiciary. In April 2017, the Allahabad High Court allowed a petition demanding reservations in faculty posiitons treating department as the unit, and quashed clauses 6(c) and 8(a)(v) of the UGC Guidelines of 2006.[6] The court argued that treating the university as the unit "would be not only impracticable, unworkable but also unfair and unreasonable" for the following two reasons stated in the judgment:

> Merely because Assistant Professor, Reader, Associate Professor, and Professor of each subject or the department are placed on the same pay-scale, but their services are neither transferable nor they are in competition with each other. It is for this reason also that clubbing of the posts for the same level treating the University as a 'Unit' would be completely unworkable and impractical. It would be violative of Article 14 and 16 of the Constitution.
>
> If the University is taken as a 'Unit' for every level of teaching and applying the roster, it could result in some departments/subjects having all reserved candidates and some having only unreserved candidates. Such proposition again would be discriminatory and unreasonable. This, again, would be violative of Article 14 and 16 of the Constitution.

Following the court order, universities advertised vacancies with a sharp fall in the number of reserved positions. This is apparent in the case of Banaras Hindu University, presented in Table 1.1, where the number of unreserved seats increased from 1188 under government's quashed solution to 1562 under court's proposed solution.[7] The reason was that many departments had a small number of faculty positions (fewer than six). Given that each department followed the same fixed sequence in which categories take turns in claiming a position, the court's solution led to a small number of positions for the reserved categories at the university level.[8] This sparked a series of teachers' unions led protests across India.

[6]Judgement last accessed on 12 June 2021 at https://indiankanoon.org/doc/177500970/

[7]Last accessed on 12 June 2021 at https://indianexpress.com/article/explained/hrd-ministry-ordinance-teacher-quota-university-prakash-javadekar-5616157/

[8]See Figure 1.17 and Figure 1.18 for the sequence in which the beneficiary groups take turns in claiming a position in India.

Table 1.1: NUMBER OF RESERVED POSITIONS IN BANARAS HINDU UNIVERSITY

	University as a Unit (Government's Solution)					Department as a Unit (Court's Solution)				
Position	General	SC	ST	OBC	Total	General	SC	ST	OBC	Total
Professor	197	38	18	0	253	250	3	0	0	253
Associate Professor	410	79	39	0	528	500	25	3	0	528
Assistant Professor	581	172	86	310	1149	812	91	26	220	1149
Total	1188	289	143	310	1930	1562	119	29	220	1930

Notes: Data shared in government's Special Leave Petition filed in the Supreme Court of India.

The Legislative. The protests compelled the government to file a petition in the Supreme Court against the Allahabad High Court verdict. "How can the post of professor of Anatomy be compared with the professor of Geography? Are you clubbing oranges with apples?" questioned the Supreme Court rejecting the appeal and terming the Allahabad high court judgment as "logical".[9] Facing a huge aggrieved vote bank, three days prior to announcement of Lok Sabha election, in March 2019, the government promulgated an ordinance that considered the university as the unit. This ordinance is now an Act of Parliament, and therefore the law in India.[10]

Today, university is the unit for application of the reservation scheme. The court's objection that "it could result in some departments/subjects having all reserved candidates and some having only unreserved candidates" inspired us to write this paper.

1.3 Model and the Primitives

We provide a model in this section to formulate the problem of reservation in two dimensions. Since our primary application is the reservation of teaching positions in Indian universities, the terminology used is appropriate for that application.

[9] Last accessed on 12 June 2021 at `https://main.sci.gov.in/supremecourt/2019/5495/5495_2019_Order_27-Feb-2019.pdf`

[10] Last accessed on 12 June 2021 at `http://egazette.nic.in/WriteReadData/2019/206575.pdf`

1.3.1 Model

A **problem of reservation in two dimensions** in period $t \in \{1, 2, \dots, T\}$ is a quadruple $\Lambda^t = (\mathcal{D}, \mathcal{C}, \boldsymbol{\alpha}, (\mathbf{q}^s)_{s=1}^t)$. $\mathcal{D}$ and $\mathcal{C}$ are finite sets of **departments** and **categories** where $m := |\mathcal{D}| \geq 2$ and $n := |\mathcal{C}| \geq 2$. The **reservation scheme** is defined by a vector of fractions $\boldsymbol{\alpha} = [\alpha_j]_{j \in \mathcal{C}}$. For each category $j \in \mathcal{C}$, $\alpha_j \in (0, 1)$ fraction of vacancies are to be reserved so that $\sum_{j \in \mathcal{C}} \alpha_j = 1$. $\mathbf{q}^s = [q_i^s]_{i \in \mathcal{D}}$ represents the **vector of vacancies** associated with the departments in period $s \in \{1, 2, \dots, t\}$. Let $Q_i^t := \sum_{s \leq t} q_i^s$ denote **period-t cumulative sum of vacancies in department** i.

A **period-t fair share table** for problem Λ^t is a two-way table

$$X^t = \begin{array}{c|c} (x_{ij}^t)_{m \times n} & (x_{i,n+1}^t)_{m \times 1} \\ \hline (x_{m+1,j}^t)_{1 \times n} & (x_{m+1,n+1}^t)_{1 \times 1} \end{array}$$

with rows indexed by $i \in \mathcal{D} \cup \{m+1\}$ and columns by $j \in \mathcal{C} \cup \{n+1\}$, such that internal entries $x_{ij}^t = \alpha_j Q_i^t$ for all $i \in \mathcal{D}$ and $j \in \mathcal{C}$, row total entries $x_{i,n+1}^t = Q_i^t$ for all $i \in \mathcal{D}$, column total entries $x_{m+1,j}^t = \alpha_j \sum_{i \in \mathcal{D}} Q_i^t$ for all $j \in \mathcal{C}$, and grand total entry $x_{m+1,n+1}^t = \sum_{i \in \mathcal{D}} Q_i^t$. Fair shares specify the fraction of seats a category is entitled to receive as per the reservation scheme until period t. The internal entry x_{ij}^t represents the **period-t fair share for category j in department i**. The **period-t fair share for a category c_j in the university** is denoted by column total entry $x_{m+1,j}^t$. The grand total entry $x_{m+1,n+1}^t$ represents the cumulative sum of vacancies at the university.

For instance, consider a problem $\Lambda^2 = (\{d_1,\ d_2\}, \{c_1,\ c_2\}, \boldsymbol{\alpha} = [0.1, 0.9], (\mathbf{q}^1, \mathbf{q}^2) = ([9, 8], [17, 7]))$. Figure 1.1 illustrates its period-1 and period-2 fair share tables. There are two departments $\mathcal{D} = \{d_1,\ d_2\}$, corresponding to rows in the tables, and two categories $\mathcal{C} = \{c_1,\ c_2\}$, corresponding to columns. The reservation scheme reserves 10% positions in the university for members of category c_1. In period-1, department d_1 has 9 and department d_2 has 8 positions, represented by the column 3 of X^1. In period-2, department d_1 has 17 and department d_2 has 7 positions. Therefore, period-2 cumulative sums of vacancies in departments d_1 and d_2 are 26 and 15, represented by the column 3 of X^2. The first column of table X^1 (X^2) represents the period-1 (period-2) fair shares associated with the category c_1 and the second column represents the period-1 (period-2) fair shares associated with category c_2. The first row of X^1 (X^2) represents the period-1 (period-2) fair shares associated with the department d_1 and the second row represents the period-1 (period-2) fair shares associated with department d_2.

Figure 1.1: FAIR SHARE TABLES

$$X^1 = \begin{array}{cc|c} 0.9 & 8.1 & 9 \\ 0.8 & 7.2 & 8 \\ \hline 1.7 & 15.3 & 17 \end{array}$$

(a) PERIOD-1 FAIR SHARE TABLE

$$X^2 = \begin{array}{cc|c} 2.6 & 23.4 & 26 \\ 1.5 & 13.5 & 15 \\ \hline 4.1 & 36.9 & 41 \end{array}$$

(b) PERIOD-2 FAIR SHARE TABLE

A two-way table is **additive** if entries add along the rows and columns to all corresponding totals. A **period-t reservation table** for the problem Λ^t is a $(m+1) \times (n+1)$ non-negative integer two-way table $\bar{X}^t = (\bar{x}^t_{ij})$, with rows indexed by $i \in \mathcal{D} \cup \{m+1\}$ and columns by $j \in \mathcal{C} \cup \{n+1\}$, such that $\bar{X}^t$ is additive and $\bar{x}^t_{i,n+1} = x^t_{i,n+1}$ for all $i \in \mathcal{D}$. The internal entry $\bar{x}^t_{ij}$ represents the **period-t reservation for category j in department i**. The **period-t reservation for a category j in the university** is denoted by column total entry $\bar{x}^t_{m+1,j}$. We denote by $\bar{\mathcal{X}}$ the set of reservation tables.

A **period-t sequence of fair share tables** for the problem Λ^t is a sequence of two-way tables $Y^t = (X^1, \ldots, X^t)$, where table X^s is the period-s fair share table for all $s \in \{1, 2, \ldots, t\}$. We denote by $\mathcal{Y}^t$ the set of all period-t sequences of fair share tables. Given a sequence of tables Y^t, if $Y^t = (Y^{t-1}, X^t)$, then we say that Y^t **follows** Y^{t-1}.

1.3.2 Deterministic Solutions and Properties

A **deterministic solution** $R : \cup_{s=1}^T \mathcal{Y}^s \to \bar{\mathcal{X}}$ maps each sequence of fair share tables to a reservation table such that, for any $Y^t \in \cup_{s=1}^T \mathcal{Y}^s$,

1. $R(Y^t)$ is a period-t reservation table, and

2. $R(Y^t) \geq R(Y^{t-1})$ for all Y^t that follow Y^{t-1}.[11]

Part 2 of definition incorporates the idea that reservations are irreversible. We denote by $\mathcal{R}^T$ the set of deterministic solutions for reservation problems of length T.

For instance, Figure 1.2 illustrates two possible deterministic solutions for the problem depicted in Figure 1.1.

Figure 1.2: TWO DETERMINISTIC SOLUTIONS

$$X^1 = \begin{array}{cc|c} 0.9 & 8.1 & 9 \\ 0.8 & 7.2 & 8 \\ \hline 1.7 & 15.3 & 17 \end{array}$$

(a) PERIOD-1 FAIR SHARE TABLE

$$X^2 = \begin{array}{cc|c} 2.6 & 23.4 & 26 \\ 1.5 & 13.5 & 15 \\ \hline 4.1 & 36.9 & 41 \end{array}$$

(b) PERIOD-2 FAIR SHARE TABLE

$$R_1(Y^1) = \begin{array}{cc|c} 1 & 8 & 9 \\ 1 & 7 & 8 \\ \hline 2 & 15 & 17 \end{array}$$

(c) PERIOD-1 RESERVATION TABLE

$$R_1(Y^2) = \begin{array}{cc|c} 3 & 23 & 26 \\ 1 & 14 & 15 \\ \hline 4 & 37 & 41 \end{array}$$

(d) PERIOD-2 RESERVATION TABLE

$$R_2(Y^1) = \begin{array}{cc|c} 0 & 9 & 9 \\ 0 & 8 & 8 \\ \hline 0 & 17 & 17 \end{array}$$

(e) PERIOD-1 RESERVATION TABLE

$$R_2(Y^2) = \begin{array}{cc|c} 3 & 23 & 26 \\ 1 & 14 & 15 \\ \hline 4 & 37 & 41 \end{array}$$

(f) PERIOD-2 RESERVATION TABLE

We denote by $R(y^t)$ and x^t the internal and totals entries of $R(Y^t)$ and X^t, respectively. The ideal solution would be the fair share table if we were allowed to reserve fractional seats. Therefore, it is natural to consider integral seat allocations with entries rounded to an adjacent integer of the fair share table entries as an ideal solution. We next formulate this idea.

A deterministic solution R **stays within quota** if, for any Y^t,

1. R **stays within department quota**: each internal entry $R(y^t) = \lceil x^t \rceil$ or $\lfloor x^t \rfloor$, and

2. R **stays within university quota**: each total entry $R(y^t) = \lceil x^t \rceil$ or $\lfloor x^t \rfloor$.

[11]The relation "is greater than or equal to", denoted "$\geq$", compares tables entry-wise; that is, $X \geq X'$ if, for all $(1 \leq i \leq m+1,\ 1 \leq j \leq n+1)$, $x_{ij} \geq x'_{ij}$.

Our property formulates the idea that a deterministic solution should not deviate from its cumulative fair share by more than one seat. In this way, everyone gets either the ceiling of its cumulative fair share or the floor of its cumulative fair share.[12] There are two dimension of staying within quota: (1) each internal entry $R(y^t_{ij})$ $(1 \leq i \leq m,\ 1 \leq j \leq n)$ is either x^t_{ij} rounded up or rounded down and (2) each total entry $R(y^t_{m+1,j})$ $(1 \leq j \leq n)$ is either $x^t_{m+1,j}$ round up or rounded down. If a solution satisfies the former one for any problem, we say that it stays within department quota. If a solution satisfies the later one for any problem, we say that it stays within university quota. For instance, in Figure 1.2, the solution R_1 stays within both department and university quota; however, the solution R_2 stays within department quota only.

1.3.3 Lottery Solutions and Properties

Randomization is the most natural and common mechanism to use in resource allocation problems when in doubt which of two or more agents should get an indivisible object. We next introduce a function to adapt this idea.

A **lottery solution** is a probability distribution ϕ over the set of deterministic solutions, where $\phi(R)$ denotes the probability of solution R. We denote by φ^T the set of lottery solutions for reservation problems of length T.

For any sequence of fair share tables Y^t, a lottery solution ϕ induces a **period-t expected reservation table** $E_\phi(Y^t) := \sum_R \phi(R)R(Y^t)$. The internal entry (i, j) in this table represents the expected fraction of seats that category j receives at department i under ϕ. The column total entry $(m + 1, j)$ represents the expected fraction of seats that category j receives in the university under ϕ.

Our next two properties make sure that in expectation a lottery solution always achieves the fair shares as well as in implementation it picks a reservation table that is as close as to

[12] For any $x \in \mathbb{R}$, $\lfloor x \rfloor$ and $\lceil x \rceil$ are the largest integer no larger than x, i.e., floor of x, and the smallest integer no smaller than x, i.e., ceiling of x, respectively.

fair shares for each departments in every period.

Definition 1. *A lottery solution ϕ is unbiased if, for any $Y^t \in \cup_{s=1}^{T} \mathcal{Y}^s$,*

$$E_\phi(Y^t) = X^t.$$

This property formulates the idea that a lottery solution should implement the fair share tables in an expected sense; that is, for any Y^t, $\sum_R \phi(R)R(Y^t) = X^t$. An unbiased lottery solution promotes ex-ante "fairness". Such solutions, on the other hand, may result in an "unfair" outcome ex-post, in which one category receives all seats, while others receive none. In other words, the ex-post outcome can differ greatly from the fair share tables. To avoid this, we next extend the staying within quota property to lottery solutions.

Definition 2. *A lottery solution ϕ* ***stays within quota*** *if, for any R such that $\phi(R) > 0$,*

1. *R stays within department quota, and*

2. *R stays within university quota.*

We study lottery solutions ϕ that only pick deterministic solutions that stay within quota. There are two dimension of staying within quota. We say that a lottery solution stays within department quota if it only gives positive probabilities to deterministic solutions that stays within department quota. We say that a lottery solution stays within university quota if it only gives positive probabilities to deterministic solutions that stays within university quota.

1.4 Solutions from India and their shortcomings

There are two solutions seen in practice in India, the Government's solution and the Court's solution. Both solutions use a tool called roster to determine the number of positions to be reserved. Formally, a **roster** $\sigma : \{1, 2, \dots\} \to \mathcal{C}$ is an ordered list over the set of categories $\mathcal{C}$. A roster assigns each position a category so that for any number of total positions, the number of positions to be reserved are clearly laid out. Since only a few seats might arise every period, the objective of maintaining a roster is to ensure that, over a period of time, each category gets its affirmative action policy prescribed percentage of seats.

Maintaining rosters is central to implementation of reservations in India.[13] It makes uniform and transparent implementation of the reservation policy across various government departments possible. However, maintaining rosters for educational institutions raises additional complications. Does each department in a university maintain its own roster? Or does the university as a whole maintain a roster? These questions gave rise to two solutions in India.

Before illustrating the solutions, we first introduce an example that makes the solutions easier to comprehend. The example will also be sufficient to demonstrate the various shortcomings of the two solutions.[14]

Example 1. *Consider a problem* $\Lambda^3 = (\{d_1,\ d_2,\ d_3,\ d_4\}, \{c_1,\ c_2\}, \boldsymbol{\alpha} = [1/3, 2/3], (\mathbf{q}^1, \mathbf{q}^2, \mathbf{q}^3) = ([2,1,2,1],[2,1,2,1],[2,1,2,1]))$. *Figure 1.3 illustrates its period-1, period-2, and period-3 fair share tables. The reservation scheme reserves 1/3 of the positions in the university for members of category* c_1. *Each period, department* d_1, d_2, d_3, *and* d_4 *have 2, 1, 2, and 1 positions, respectively. Therefore, period-2 cumulative sums of vacancies in departments are 4, 2, 4, and 2, respectively. And, period-3 cumulative sums of vacancies in departments are 6, 3, 6, and 3, respectively. The roster is*

$$\sigma(k) = \begin{cases} c_1, & \textit{if } k \textit{ is a multiple of } 3 \\ c_2, & \textit{otherwise} \end{cases}$$

[13]See Figure 1.17 and Figure 1.18 for the rosters prescribed by Government of India.

[14]An example with two categories and two department is also sufficient to demonstrate the shortcomings. Example 1 is constructed so that it not only illustrates the shortcomings of the both solutions, but it also demonstrates the differences between the Court's and the Government's solutions.

Figure 1.3: FAIR SHARE TABLES

$$X^1 = \begin{array}{cc|c} 2/3 & 4/3 & 2 \\ 1/3 & 2/3 & 1 \\ 2/3 & 4/3 & 2 \\ 1/3 & 2/3 & 1 \\ \hline 2 & 4 & 6 \end{array}$$

(a) PERIOD-1 FAIR SHARE TABLE

$$X^2 = \begin{array}{cc|c} 4/3 & 8/3 & 4 \\ 2/3 & 4/3 & 2 \\ 4/3 & 8/3 & 4 \\ 2/3 & 4/3 & 2 \\ \hline 4 & 8 & 12 \end{array}$$

(b) PERIOD-2 FAIR SHARE TABLE

$$X^3 = \begin{array}{cc|c} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \\ \hline 6 & 12 & 18 \end{array}$$

(c) PERIOD-3 FAIR SHARE TABLE

We will see that the choice of the roster in Example 1 is not the source of the shortcomings of the Government's and Court's solutions. The source of problem is that they do not account for interdependence of the departments and the university in calculating reserved seats.

1.4.1 Government's Solution and its Shortcomings

The Government's solution treats the *university as the unit*. That is, positions across all departments are pooled together and the roster is maintained at the university level.

For the problem in Example 1, in period-1, department d_1 has two positions: The number of positions reserved for department d_1 is determined by the 1st and 2nd positions in the roster (i.e., $\sigma(1) = c_2,\ \sigma(2) = c_2$). Department d_2 has one position: The number of positions reserved for department d_2 is determined by the 3th position in the roster (i.e., $\sigma(3) = c_1$).[15] Department d_3 has two positions: The number of positions reserved for department d_3 is determined by the 4th and 5th positions in the roster (i.e., $\sigma(4) = c_2,\ \sigma(5) = c_2$). Department d_4 has one position: The number of positions reserved for department d_4 is determined by the 6th position in the roster (i.e., $\sigma(6) = c_1$). The period-1 reservation table is illustrated by $R_G(Y^1)$ in Figure 1.4.

In period-2, department d_1 has two positions: The number of positions reserved for department d_1 is determined by the 7th and 8th positions in the roster (i.e., $\sigma(7) = c_2,\ \sigma(8) = c_2$). Department d_2 has one position: The number of positions reserved for department d_2

[15]When pooling positions across departments, a fixed order over departments is required to apply to the roster. In India, the alphabetic order over departments is used.

is determined by the 9th positions in the roster (i.e., $\sigma(9) = c_1$). Department d_3 has two positions: The number of positions reserved for department d_3 is determined by the 10th and 11th positions in the roster (i.e., $\sigma(10) = c_2,\ \sigma(11) = c_2$). Department d_4 has one position: The number of positions reserved for department d_4 is determined by the 12th position in the roster (i.e., $\sigma(12) = c_1$). The period-2 reservation table is illustrated by $R_G(Y^2)$ in Figure 1.4. We apply this solution for the next period. The period-3 reservation table is illustrated by $R_G(Y^3)$ in Figure 1.4.

Figure 1.4: COURT'S AND GOVERNMENT'S SOLUTION

$$X^1 = \begin{array}{cc|c} 2/3 & 4/3 & 2 \\ 1/3 & 2/3 & 1 \\ 2/3 & 4/3 & 2 \\ 1/3 & 2/3 & 1 \\ \hline 2 & 4 & 6 \end{array}$$

(a) PERIOD-1 FAIR SHARE TABLE

$$X^2 = \begin{array}{cc|c} 4/3 & 8/3 & 4 \\ 2/3 & 4/3 & 2 \\ 4/3 & 8/3 & 4 \\ 2/3 & 4/3 & 2 \\ \hline 4 & 8 & 12 \end{array}$$

(b) PERIOD-2 FAIR SHARE TABLE

$$X^3 = \begin{array}{cc|c} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \\ \hline 6 & 12 & 18 \end{array}$$

(c) PERIOD-3 FAIR SHARE TABLE

$$R_G(Y^1) = \begin{array}{cc|c} 0 & 2 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 0 & 1 \\ \hline 2 & 4 & 6 \end{array}$$

(d) PERIOD-1 RESERVATION TABLE

$$R_G(Y^2) = \begin{array}{cc|c} 0 & 4 & 4 \\ 2 & 0 & 2 \\ 0 & 4 & 4 \\ 2 & 0 & 2 \\ \hline 4 & 8 & 12 \end{array}$$

(e) PERIOD-2 RESERVATION TABLE

$$R_G(Y^3) = \begin{array}{cc|c} 0 & 6 & 6 \\ 3 & 0 & 3 \\ 0 & 6 & 6 \\ 3 & 0 & 3 \\ \hline 6 & 12 & 18 \end{array}$$

(f) PERIOD-3 RESERVATION TABLE

$$R_C(Y^1) = \begin{array}{cc|c} 0 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \\ \hline 0 & 6 & 6 \end{array}$$

(g) PERIOD-1 RESERVATION TABLE

$$R_C(Y^2) = \begin{array}{cc|c} 1 & 3 & 4 \\ 0 & 2 & 2 \\ 1 & 3 & 4 \\ 0 & 2 & 2 \\ \hline 2 & 10 & 12 \end{array}$$

(h) PERIOD-2 RESERVATION TABLE

$$R_C(Y^3) = \begin{array}{cc|c} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \\ \hline 6 & 12 & 18 \end{array}$$

(i) PERIOD-3 RESERVATION TABLE

Period-3 reservation for category c_1 in department d_1 and department d_3 is 0, however, the fair share is 2 positions. Moreover, period-3 reservation for category c_1 in department d_2 and department d_4 is 3, however, the fair share is 1 position. Therefore, the Government's solution R_G does not stay within department quota. Moreover, in Example 1, if the departments had the same number of positions for the next periods, department d_1 and department d_3 would not reserve any seats for category c_1, and department d_2 and department

d_4 would not reserve any seats for category c_2.

Two shortcomings of the Government's solution R_G are revealed by Example 1:

1. The Government's solution R_G does not stay within quota.

2. The Government's solution R_G allows for large deviations in seat allocations from fair shares at the department level.

Essentially, Example 1 shows that treating university as the unit can lead to outcomes that fail to follow the reservation policy at the department level.[16]

1.4.2 Court's Solution and its Shortcomings

The Court's solution treats *department as the unit*. That is, positions are not pooled across departments. Instead, each department independently maintains a roster.

For the problem in Example 1, in period-1, department d_1 has two positions: The number of positions reserved for department d_1 is determined by the 1st and 2nd positions in its roster (i.e., $\sigma(1) = c_2,\ \sigma(2) = c_2$). Department d_2 has one position: The number of positions reserved for department d_2 is determined by the 1st position in its roster (i.e., $\sigma(1) = c_2$). Department d_3 has two positions: The number of positions reserved for department d_3 is determined by the 1st and 2nd positions in its roster (i.e., $\sigma(1) = c_2,\ \sigma(2) = c_2$). Department d_4 has one position: The number of positions reserved for department d_4 is determined by the 1st position in its roster (i.e., $\sigma(1) = c_2$). The period-1 reservation table is illustrated by $R_C(Y^1)$ in Figure 1.4.

In period-2, department d_1 has two positions: The number of positions reserved for department d_1 is determined by the 3th and 4th positions in its roster (i.e., $\sigma(3) = c_1,\ \sigma(4) = c_2$). Department d_2 has one position: The number of positions reserved for department d_2 is determined by the 2nd positions in its roster (i.e., $\sigma(2) = c_1$). Department d_3 has two

[16]In fact, in Proposition 3, we show that for any solution that stays within university quota, the deviations in seat allocations from fair shares at the department level can not be limited by a fixed number.

positions: The number of positions reserved for department d_3 is determined by the 3th and 4th positions in its roster (i.e., $\sigma(3) = c_1,\ \sigma(4) = c_2$). Department d_4 has one position: The number of positions reserved for department d_4 is determined by the 2nd position in its roster (i.e., $\sigma(2) = c_1$). The period-2 reservation table is illustrated by $R_C(Y^2)$ in Figure 1.4. We apply this solution for the next period. The period-3 reservation table is illustrated by $R_C(Y^3)$ in Figure 1.4.

Period-1 reservation for category c_1 in the university is 0, however, the fair share is 2 positions. Moreover, period-2 reservation for category c_1 in the university is 2, however, the fair share is 4 positions. Therefore, the Court's solution R_C does not stay within university quota. Moreover, in Example 1, if there were 4 more departments d_5, d_6, d_7, and d_8, with the same number of positions as department d_1, d_2, d_3, and d_4, respectively, period-1 reservation for category c_1 in the university would still be 0. And, period-2 reservation for category c_1 in the university would still be 4 while the fair share was 8 positions.

Two shortcomings of the Court's solution R_C are revealed by Example 1:

1. The Court's solution R_C does not stay within quota.

2. The Court's solution R_C allows for large deviations in seat allocations from fair shares at the university level.

Essentially, Example 1 shows that treating department as the unit can lead to outcomes that fail to follow the reservation policy at the university level.

1.5 Designing Reserves in Two-Dimensions: Results

1.5.1 Single Period Results

One way to approach the problem of reservation in two dimensions is to ignore the time dimension, that is, the problem can be treated as an independent problem in each period.[17]

[17] This is analogous to biproportional apportionment problems. In some proportional electoral systems with more than one constituency the number of seats must be allocated to parties within territorial constituen-

In that case, a lottery solution that is unbiased and stays within quota always exists.

Proposition 1. *There exists a lottery solution $\phi \in \varphi^1$ that is unbiased and stays within quota.*

The proof, presented in Section 2.7, uses an adaptation of the Cox (1987) controlled rounding procedure to construct a unbiased lottery solution that stays within quota. By Proposition 1, any period-1 fair share table is implemented by a lottery solution that only gives positive probability to period-1 reservation tables that do not deviate from fair shares by more than one seat. The following corollary directly follows Proposition 1.

Corollary 1. *There exists a deterministic solution $R \in \mathcal{R}^1$ that stays within quota.*

Corollary 1 implies that for any problem of length $T = 1$, there always exists a reservation table that stays within quota. That is, there is a satisfactory solution to the problem of reservation in two dimensions if in each period the problem is treated independently.

1.5.2 Multi Period Results

Treating each period's problem independently can lead to adverse outcomes over time. In particular, since integer seat allocations differ from the fair share tables in every period, accumulation of these differences can result in large deviation from fair shares over time. We next show this issue in an example.

Example 2. *Consider a problem depicted in the following fair share table, with two departments d_1, d_2 having 2 and 7 positions, respectively, and two categories c_1, c_2, and the reservation scheme vector $\boldsymbol{\alpha} = [0.1, 0.9]$.*

The following deterministic solution stays within quota, but it does not give any positions to category c_1.

cies, as well as, the number of seats that each party has to receive at a national level.

$$X = \begin{array}{cc|c} 0.2 & 1.8 & 2 \\ 0.7 & 6.3 & 7 \\ \hline 0.9 & 8.1 & 9 \end{array}$$

(a) FAIR SHARE TABLE

$$R(X) = \begin{array}{cc|c} 0 & 2 & 2 \\ 0 & 7 & 7 \\ \hline 0 & 9 & 9 \end{array}$$

(b) RESERVATION TABLE

Example 2 suggests that not reserving any seats is a solution that stays within quota. For instance, a university can repeatedly apply this solution to each period's problem and does not reserve a single seat.

In general case, a lottery solution ϕ that treats each period's problem independently rounds up or rounds down each fair share with some probabilities. Therefore, in the range of the lottery solution ϕ, there exists an outcome that rounds down a particular entry in every period. That is, the lottery solution ϕ can result in seat allocations with sizeable deviations from fair shares.

We next examine how our single period results extend to the multi-period problem. We first show that for every problem, a deterministic solution that stays within quota does not always exists.

Proposition 2. *There does not exist a deterministic solution $R \in \mathcal{R}^T$ that stays within quota for $T > 1$.*

Proposition 2 implies that for every problem of length $T > 1$, unlike single period, solutions deviate from fair shares by more than one seat. It also implies that it is impossible to both stay within university quota and stay within department quota. We next generalize staying within quota property to allow for some differences in fair shares and seat allocations.

A **bias of a deterministic solution** R at Y^t is a two-way table *bias*$(R(Y^t))$, with each entry *bias*$(R(y^t)) := R(y^t) - x^t$. The bias of a solution is the difference between the solution and the fair share table. With this definition, a solution stays within quota if, for any Y^t, each entry $|\text{bias}(R(y^t))| < 1$, that is, for any problem, the bias of the solution is always less than 1 in absolute value. Our next property allows a solution to deviate from fair shares up to a constant number.

Definition 3. *A deterministic solution $R \in \mathcal{R}^T$ has a* ***finite bias*** *if there exists a constant $b > 0$ such that, for any $Y^t \in \cup_{s=1}^T \mathcal{Y}^s$,*

$$|bias(R(y^t))| < b.$$

One might be tempted to think that there would be solutions that allow for larger deviations in seat allocations from fair shares at the department level but stays within university quota. We show that such solutions do not exist.

Proposition 3. *There does not exist a deterministic solution $R \in \mathcal{R}^T$ that has a finite bias and stays within university quota for $T > 1$.*

The proof is in Section 2.7. Proposition 2 is a corollary of Proposition 3. By Proposition 2 we learn that any procedure that stays within university quota cannot stay within department quota. By Proposition 3 we learn that any procedure that stays within university quota can lead to departments to grow in size over time without reserving a single seat.

Proposition 2 and Proposition 3 have a stronger implication: there is no deterministic solution to the problem of reservation in two dimensions that stays within quota. This negative result provides yet another reason to use lottery solutions to address the problem of reservation in two dimensions.

We next present the main existence result: the set of lottery solutions that are unbiased and stay within department quota is non-empty.

Theorem 1. *There exists a lottery solution ϕ that is unbiased and stays within department quota.*

A formal proof of Theorem 1 is presented in Section 2.7. The proof utilizes a network flow to construct a lottery over rosters. Each department is then assigned a roster drawn independently from the constructed lottery. This two-step procedure induces a lottery solution, denoted ϕ^* and defined formally in Section 2.7. The lottery solution is shown to be unbiased and stays within department quota, that is, each category gets (i) ex-ante its fair share, and (ii) ex-post its fair share either rounded up or down in every department.[18]

Theorem 1 implies that there is a lottery solution that ensures that each department sticks to the reservation scheme while the university, as a whole, respects the fair shares in an expected sense. By Proposition 3, however, we know that such solutions can result in biases greater than one at the university level. To show that our lottery solution limits the probability of these occurrences, we modify the staying within university quota property.

We denote the outcome of a lottery solution ϕ at a sequence of fair share tables Y^t by the random variable Z^t and its entries by z^t_{ij}. The deviation of the outcome of lottery solution ϕ for a category $j \in \mathcal{C}$ in the university is $z^t_{m+1,j} - x^t_{m+1,j}$. This random variable measures the deviation of the seat allocation at the university level from its fair share.

Definition 4. *A lottery solution ϕ* ***approximately stays within university quota*** *if, for any Y^t, for any category $j \in \mathcal{C}$ and for any $b > 0$, we have*

$$Pr(z^t_{m+1,j} - x^t_{m+1,j} \geq b) \leq e^{-\frac{b^2}{3x^t_{m+1,j}}} \ ,$$

$$Pr(z^t_{m+1,j} - x^t_{m+1,j} \leq -b) \leq e^{-\frac{b^2}{2x^t_{m+1,j}}} \ .$$

We establish probabilistic concentration bounds on the deviations for our lottery solution ϕ^* and show that ϕ^* approximately stays within university quota.

Theorem 2. *The lottery solution ϕ^* is unbiased, stays within department quota, and approximately stays within university quota.*

Theorem 2 follows from a Chernoff-type concentration bound. We establish the proba-

[18]One can show that the set of lottery solutions that are unbiased and stay within university quota is also non-empty. However, staying within the department quota property better suits our applications because one goal of affirmative policies is to increase diversity in all sub-units (departments and university as a whole), and the smallest sub-units in our setup are departments.

bility bounds in a fashion similar to Gandhi et al. (2006). By this property, the probability of deviating from university quota by a value greater than b decays exponentially with b^2. Therefore, there is a procedure that ensures that each department obeys the reservation scheme, while the university as a whole approximately follows the reservation scheme.

We next show that the bounds in Definition 4 are tight (up to a multiplicative constant in the exponent) and thus rules out any improvement of the deviation of the seat allocation at the university level from its fair share.

Proposition 4. *Consider a lottery solution that is unbiased, stays within department quota and limits the probability of deviation of the seat allocation at university level in following way: for any Y^t, for any category $j \in \mathcal{C}$ and for any $b > 0$, the lottery satisfies*

$$Pr(z^t_{m+1,j} - x^t_{m+1,j} \geq b) \leq f(x^t_{m+1,j}, b)\,,$$

$$Pr(z^t_{m+1,j} - x^t_{m+1,j} \leq -b) \leq f(x^t_{m+1,j}, b)\,.$$

Then, there exists a constant $k > 0$ such that for any $b > 0$,

$$\lim_{x^t_{m+1,j} \to \infty} \frac{e^{-\frac{b^2}{x^t_{m+1,j}}k}}{f(x^t_{m+1,j}, b)} = 0\,.$$

Proposition 4 shows that there exists a constant $k > 0$ such that any lottery that is unbiased and stays within department quota can approximately stays within university quota (in the sense of Definition 4) with a probabilistic guarantee no better than $e^{-\frac{b^2}{x^t_{m+1,j}}k}$. A proof of Proposition 4 is presented in Section 2.7.

1.6 Empirical Study of Reservation in Two Dimensions

Here we present a comprehensive evaluation of recruitment advertisements to highlight the severity of shortcomings in the existing solutions and to reflect the benefits of adopting our proposed solutions. Specifically, we evaluate the general quality of the advertised two-way apportionments with respect to the instances and magnitude of quota violations, and present the advantage our proposed solution exhibits.

Our data comprises 60 advertisements released in the following five recruitment settings where two-dimensional reservation problems are seen in practice.

1. Assistant Professors of University of Delhi
2. Officers of Indian Administrative Services
3. Officers of Indian Forest Services
4. Officers of Indian Police Services
5. Assistants of Reserve Bank of India

In the preceding sections we presented and analyzed the problem in the context of a university. Therefore, we will continue to use the same terminology for all advertisements. The term *departments* refers to departments in a university for the assistant professors advertisements. However, for other advertisements the departments correspond to the states (in India) where an officer or an assistant shall be recruited. Similarly, the term *university* corresponds to the country (India) in the latter advertisements.

An overview of the recruitment advertisement data is presented in Table 1.2. The advertisements provide a variety of two-dimensional reservation problems with the number of departments varying from 8 to 50; the number of vacancies in a department varying from 1 to 30; and the number of vacancies in the university varying from 21 to 1000. The advantage of using data from different institutions is that the variety of procedures used at these institutions help highlight the robustness of shortcomings we discussed in Section 1.4.

Table 1.2: OVERVIEW OF RECRUITMENT ADVERTISEMENTS

		Departments		Dept. Vacancies		Total Vacancies	
Institution	Ads	Avg.	Min-Max	Avg.	Min-Max	Avg.	Min-Max
University of Delhi	23	19.7	8-50	4.2	1-10	94.8	21-405
Indian Administrative Services	15	24.7	24-26	9.47	5-15	148.9	87-180
Indian Forest Services	7	25.1	24-26	3.6	3-4	95.4	78-110
Indian Police Services	8	25.3	24-26	12.8	10-16	150.1	148-153
Reserve Bank of India	7	17	17-17	23.7	13-30	648.1	500-1000

1.6.1 Single Period Analysis

First consider the problem of reservations as a single period problem. Thus in this subsection each advertisement is treated as an independent single period two-dimensional reservations problem. In line with our theoretical analysis, we use the department and university quota violations in judging the quality of solutions advertised.

Table 1.3 shows that the instances of both the department quota and the university quota violations are pervasive in the advertised solutions of all the institutions. The percentage of instances of violations, obtained by dividing the number of violations that occurred by the maximum number of violations possible, is an informative summary measure. Based on this measure, the probability that a typical category would witness a department quota violation in a typical department ranges from 0.08 in University of Delhi to 0.59 in Reserve Bank of India. The probability that a typical category would witness a university quota violation ranges from 0.18 in India Forest Services to 0.93 in Reserve Bank of India.

In order to provide a complete picture of the severity of shortcomings, we present the magnitude of bias (in cases of quota violation) in Table 1.3. The magnitude of bias is the absolute value of bias as defined in Section 1.5.2. At the department level, this measure shows that, in case of quota violation, the average deviation from fair shares for a typical category ranges from 1.3 in University of Delhi to 4.1 in Reserve Bank of India. At the university level, this measure shows that, in case of quota violation, the average deviation

Table 1.3: SINGLE PERIOD QUOTA VIOLATIONS – STATISTICS

	Instances of Violations				Magnitude of Bias	
	Avg.	Min-Max	Total	Percentage	Avg.	Min-Max
University of Delhi						
Department Quota	6.8	0-24	156	8%	1.3	1-4
University Quota	2.6	1-5	60	59.4%	2.7	1-13
Indian Administrative Services						
Department Quota	28.9	2-48	434	29.2%	1.8	1-6.5
University Quota	1.9	0-4	28	46.7%	3.9	1-6.9
Indian Forest Services						
Department Quota	17.6	8-24	123	17.5%	1.5	1-2.9
University Quota	0.7	0-2	5	17.9%	1.4	1.3-1.5
Indian Police Services						
Department Quota	32.8	27-38	262	32.5%	1.8	1-5.1
University Quota	2.9	1-4	23	71.9%	2.3	1.2-5.3
Reserve Bank of India						
Department Quota	40.1	34-49	281	59%	4.1	1-35.8
University Quota	3.7	3-4	26	92.9%	20.8	2.5-60.6

from fair shares for a typical category ranges from 1.4 in Indian Forest Services to 20.8 in Reserve Bank of India.

As a single period problem, the two-dimensional reservations problem has been shown to admit an elegant solution called *controlled rounding* that stays within quota (see section Section 1.5.1). If each reservation problem were to be treated independently, adopting controlled rounding procedure for making reservation tables would lead no quota violations. Therefore making it possible to achieve simultaneously the prescribed percentage of reservations at both the department and the university level in single period problems (as shown in Proposition 1).

1.6.2 Multi Period Analysis

In Section 1.2, with emphasis on maintaining rosters, the intent of India's policymakers is clear. In the face of the indivisibility of seats, their policies aim to achieve the prescribed percentage of reservations not in a single period but over time. Therefore, analysis of the recruitment data is incomplete without checking whether the quota and biases cancel out and consequently disappear over time. For this purpose we need to look at sequences of consecutive advertisements that share the same set of departments and the same reservation policy. There are seven such sequences in our data.

Results from the last period of these seven sequences of consecutive advertisements in Table 1.4 show that the single period violations are not cancelling over time, rather they are adding up. Both the instances of violations and the magnitude of bias are now higher than the numbers reported in Table 1.3 for single period problems. The probability that a typical category would witness a department quota violation in a typical department ranges from 0.36 in Indian Forest Services to 0.88 in Reserve Bank of India. The probability that a typical category would witness a university quota violation ranges from 0.50 in Indian Forest Services to 1 in Reserve Bank of India. At the department level, in case of quota violation, the average deviation from fair shares for a typical category ranges from 1.8 in Indian Forest Services to 11.7 in Reserve Bank of India. At the university level, in case of quota violation, the average deviation from fair shares for a typical category ranges from 1.5 in Indian Forest Services to 83.2 in Reserve Bank of India.

The findings suggest that the problem worsens with time in that there are more instances of violations and larger deviations from policy prescribed percentage of reservations. This is not surprising given the negative results presented in Proposition 2 and Proposition 3. However, the scope of improvement is clear. Theorem 1 and Theorem 2 show that there exists an unbiased solution that stays within quota at the department level and approximately stays within quota at the university level. A comparison of this proposed solution with the existing solution is the point of our next simulation exercise. For this exercise we

Table 1.4: MULTI PERIOD QUOTA VIOLATIONS – STATISTICS

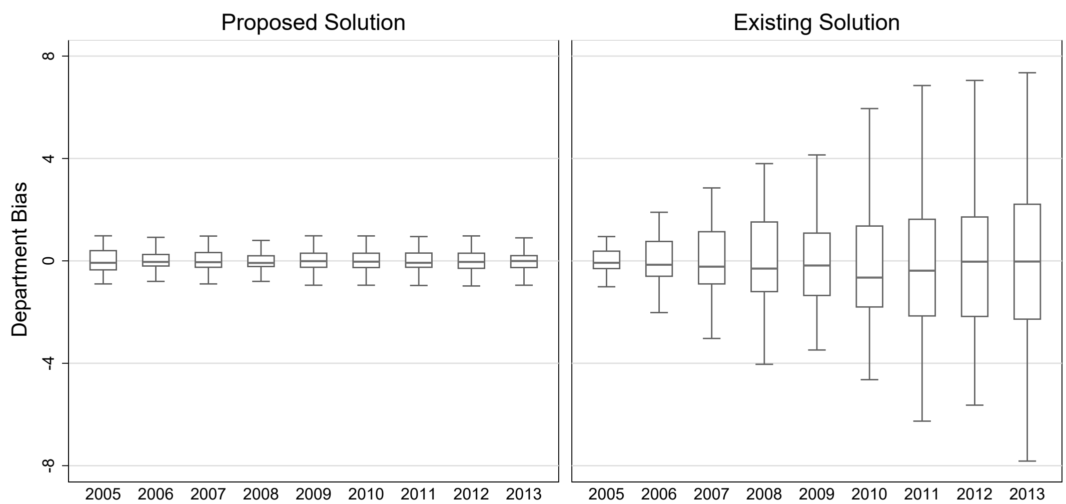
	Instances of Violations		Magnitude of Bias	
	Total	Percentage	Avg.	Min-Max
Indian Administrative Services: 2005 to 2013				
Department Quota	76	79.2%	3.5	1-14.1
University Quota	4	100%	16.6	5.2-28
Indian Administrative Services: 2014 to 2018				
Department Quota	79	75.9%	4.1	1-18.1
University Quota	2	50%	4	3.5-4.4
Indian Forest Services: 2011 to 2013				
Department Quota	35	36.5%	1.8	1-3.1
University Quota	2	50%	1.5	1.2-1.8
Indian Forest Services: 2015 to 2018				
Department Quota	54	51.9%	2.8	1.1-6.9
University Quota	3	75%	3.1	2-4.9
Indian Police Services: 2010 to 2011				
Department Quota	45	46.9%	2.2	1-4.2
University Quota	4	100%	2.6	1.7-3.5
Indian Police Services: 2014 to 2018				
Department Quota	73	70.2%	3.4	1.1-12.6
University Quota	3	75%	5	2-7.7
Reserve Bank of India: 2012 to 2017				
Department Quota	60	88.2%	11.7	1-35.5
University Quota	4	100%	83.2	10-166.4

will consider the longest sequence of consecutive advertisements in our data: the advertisement of Indian Administrative Services from 2005 to 2013.

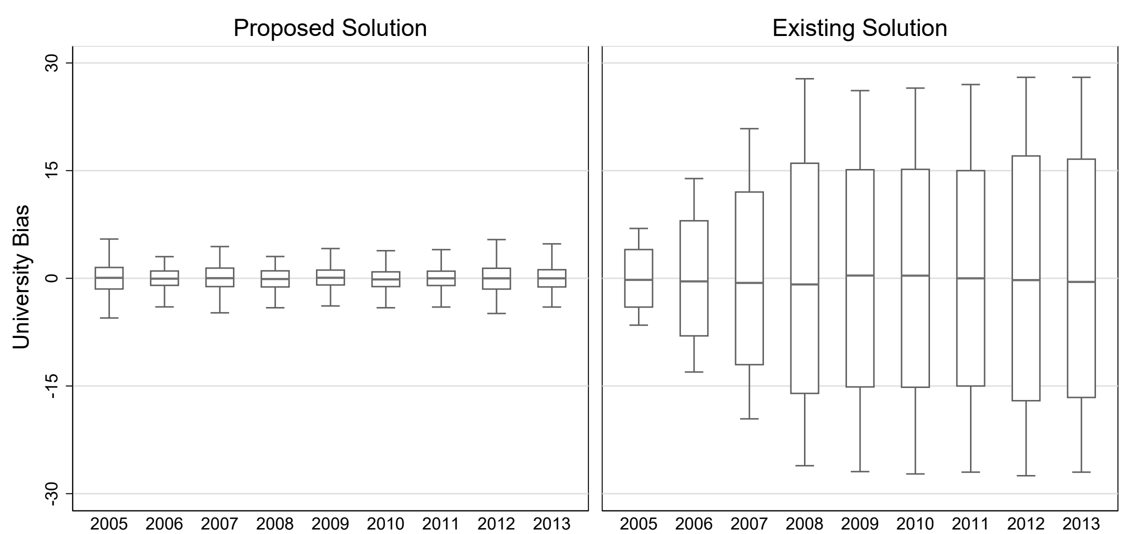
The objective of the simulation exercise is to compare the evolution of bias over time under the existing solution with the solution proposed in this paper. For this purpose, we simulate a set of 50 advertisements adhering to the proposed solution and plot the bias at each time period in Figure 1.6. The top-left panel shows that, for the proposed solution's advertisements, the department bias stays well within the $[-1, 1]$ interval, that is, there are no quota violations at the department level. In contrast, under the existing (advertised) solution presented in the top-right panel, the bias accumulates over time at the department level. The bottom-left panel shows that though the university violations occur under the proposed solution, the bias does not add up over time. The significance is apparent when one compares it to the evolution of bias under the existing solution presented in the bottom-right panel.

Figure 1.6: BIASES OF PROPOSED AND EXISTING SOLUTIONS

(a) DEPARTMENT BIAS OVER TIME



(b) UNIVERSITY BIAS OVER TIME



Note: Box plots show medians, quartiles, and adjacent values of bias distributions over time.

1.7 Conclusion

This paper has offered an analysis of two-dimensional reservation problems using the theory of apportionment and rounding problems. We have theoretically and empirically documented the shortcomings of existing solutions and proposed a solution with demonstrable advantages over the existing solutions. From a broader perspective, even though our search for quality solutions is limited to the *staying within quota* property, the analysis here can be viewed as illustrative of substantial scope for improvement in existing procedures for two-dimensional reservation problems.

Our approach is obviously limited and the problem is open to several alternative approaches that deserve extra work. A particular one that deserves mention is the error minimization approach that has yielded a class of methods to solve biproportional apportionment problems (Ricca et al. (2012) and Serafini and Simeone (2012)). These methods take a fractional matrix as the target (fair share table in our case) and solve a constrained optimization problem where the objective corresponds to a measure of the error between the solution and the target matrix. Such an approach may pave the way to a richer study of defining and finding appealing solutions to two-dimensional reservation problems.

Our problem also suggests possible extensions in the theory of apportionment. We believe that the multi-period considerations introduced in this paper could be worth exploring in the classic biproportional apportionment problem context of translating electoral votes into parliamentary seats.

1.8 Appendix

Proof of Theorem 1

The proof is constructive and has two parts. We first define the Roster-Finding Algorithm, which takes a reservation scheme vector as inputs and generates a random roster as an output, that is, a lottery over rosters.We then assign the random roster to each department independently. The random roster is constructed such that if every department follows it, the induced solution stays within the department quota. We denote this solution as our lottery solution ϕ^*. We, lastly, show that the lottery solution ϕ^* is unbiased.

Proof of Theorem 1. Let $\mathcal{C}$ be the set of categories and $\boldsymbol{\alpha} = [\alpha_j]_{j\in\mathcal{C}}$ be the reservation scheme. Let P represent the given reservation scheme as a $k \times n$ two-way table, where the rows denote the index of the seats and the columns denote the categories. The internal entry p_{ij} equals to α_j for every (i, j). Let assume that for each column, entries sum up to an integer (if there is a common multiplier for fractions in the reservation scheme vector, then such k exists).[19] The output of the algorithm will be an integral table that define how a department reserves its positions over time, i.e., a roster. We next construct a set of constraints that bounds the elements of the table P.

For each constraints K, let $\underline{p_K}$ and $\bar{p_K}$ be the floor and ceiling of the constraint. That is, $\underline{p_K} = \lfloor \sum_{(i,j)\in K} p_{ij} \rfloor$ and $\bar{p_K} = \lceil \sum_{(i,j)\in K} p_{ij} \rceil$. We will consider tables P' that satisfying, for each K,

$$\underline{p_K} \leq \sum_{(i,j)\in K} p'_{ij} \leq \bar{p_K}.$$

We have three types of constraints. Internal constraints make sure that each internal entry can be either 1 or 0. Row sums are required to be one since every position is assigned to exactly one category. Column constraints make sure that difference between cumulative some of positions given to a category and cumulative fair shares is less than one.

Let $\mathcal{K}_I$ be the internal constraints, i.e., $0 \leq p'_{ij} \leq 1$ for every (i, j). Let $k_{ij} := \{(i, j)\}$ denote such constraint. Let $\mathcal{K}_R$ be the set of row constraints, i.e., $\sum_{j\in\mathcal{C}} p'_{ij} = 1$ for every i. Let $R_i := \{(i, j)|j \in \mathcal{C}\}$ denote such constraint. Let $\mathcal{K}_C$ be the set of column constraints, i.e., $\lfloor \sum_{i\leq l} p_{ij} \rfloor \leq \sum_{i\leq l} p'_{ij} \leq \lceil \sum_{i\leq l} p_{ij} \rceil$ for every $2 \leq l \leq n$ and $j \in \mathcal{C}$. Let $C_{lj} :=$

[19] The generalization to non-integer sums is made by constructing an extended table P' in a way that is equivalent to P except the last row. The last row of P' is generated by taking 1- fractional part of the column totals (similar to how the extended table is created in the algorithm given for proof of Proposition 1).

$\{(i,j)|i \leq l\}$ denote such constraint.

We next create a flow network. The set of vertices consists of the source, the sink, vertices for each $k \in \mathcal{K}_I$, each $R \in \mathcal{K}_R$, and for each $C \in \mathcal{K}_C$. The following rule governs the placement of directed edges:

1. A directed edge from source C_{nj} for every $j \in \mathcal{C}$.

2. A directed edge from C_{lj} to k_{lj} and C_{l-1j} for every $l \geq 3$ and $j \in \mathcal{C}$.

3. A directed edge from C_{2j} to k_{2j} and k_{1j} for every $j \in \mathcal{C}$.

4. A directed edge from k_{ij} to R_i for every (i,j).

5. A directed edge from R_i to sink for every i.

Note that the constraint structure for $\mathcal{K}_C \cup \mathcal{K}_I$ and $\mathcal{K}_R \cup \mathcal{K}_I$ are hierarchical. A set of constraints $\mathcal{K}$ is **hierarchical** if, for every pair of constraints K' and K'', we have that $K' \subset K''$ or $K'' \subset K'$ or $K' \cap K'' = \emptyset$.

We next associate flow with each edge. Notice that there is only one incoming edge for each vertex $K \in \mathcal{K}_C \cup \mathcal{K}_I$. And, there is only one outgoing edge for each vertex $K \in \mathcal{K}_R \cup \mathcal{K}_I$. Observe that it is because of the hierarchical sets of constraints. Therefore, it is sufficient to associate incoming flows for each vertex $K \in \mathcal{K}_C \cup \mathcal{K}_I$ and outgoing flows for each vertex $K \in \mathcal{K}_R \cup \mathcal{K}_I$. For each vertex $K \in \mathcal{K}_C \cup \mathcal{K}_I$, the incoming flow is equal to $\sum_{(i,j)\in K} p_{ij}$. For each vertex $K \in \mathcal{K}_R \cup \mathcal{K}_I$, the outgoing flow is equal to $\sum_{(i,j)\in K} p_{ij}$. Furthermore, the flow association ensures that the amount of incoming flow is equal to the amount of outgoing flow for each vertex.

Notice that we map table P with the constraint structures to a flow network. In addition, the mapping is injective. As long as the constraints are still satisfied after the transformation, every transformation in the flow network can be mapped back to table P.

Definition 5. *We call the pair of tables (P^1, P^2) a* ***decomposition*** *of table P, if*

1. *there exists* $\beta \in (0,1)$ *such that* $P = \beta P^1 + (1-\beta)P^2$*,*
2. *for each constraint* K*,* $\underline{p_K} \leq \sum_{(i,j)\in K} p^l_{ij} \leq \bar{p_K}$ *for* $l = 1, 2$*, and*
3. *table* P^1 *and* P^2 *have more number of integral entries than table* P*.*

The following constructive algorithm has two parts. We first find a cycle of fractional edges in the network flow. We then alter the flow of edges in two different ways until one edge becomes integral. It will provide us a decomposition of table P.

Roster-Finding Algorithm

Repeat the following as long as the flow network contains a fractional edge:

Step 1: Choose any edge that has fractional flow. Since the total inflow equals to total outflow for each vertex, there will an adjacent edge that has fractional flow. Continue to add new edges with fractional flows until a cycle is formed.

Step 2: Modify the flows in the cycle in two ways to create P^1 and P^2:

1. First way: the flow of each forward edge is increased and the flow of each backward edge is decreased at the same rate until at least one flow reaches an integer value. Record the amount of adjustment as d_-. Map back the resulting flow network to a two way table. Denote the table as P^1.
2. Second way: the flow of each forward edge is decreased and the flow of each backward edge is increased at the same rate until at least one flow reaches an integer value. Record the amount of adjustment as d_+. Map back the resulting flow network to a two way table. Denote the table as P^2.
3. Set $\beta = \frac{d_-}{d_- + d_+}$.
4. The pair of tables (P^1, P^2) is a decomposition of table P, where $P = \beta P^1 + (1-\beta)P^2$.

The algorithm creates a lottery over integral two-way tables that share the same constraint structure as table P.[20] Assume that $\bar{P}$ is an integral table constructed by the algo-

[20] Moreover, the expected table equals to table P.

rithm, and its compound probability is γ. We construct a roster by each of these integral tables as follows. For each internal entry of table $\bar{P}$, if $\bar{p}_{ij} = 1$ then assign $\sigma(i) = c_j$. We next assign probability γ to roster σ. Thus, we obtain a random roster.

Notice that the expected number seats for each category j in the first q seats equals to $q\alpha_j$ for $q = 1, 2, \ldots$. We next create the induced lottery solution ϕ^* for the problem of reservation in two dimensions as follows. We assign the random roster to each department. Each department then reserves positions according to the roster realized from the lottery. For example, if roster σ is realized for department i then, the number of positions reserved in department i in period-1 is determined by $\sigma(1), \ldots, \sigma(q_i^1)$. The number of positions reserved in department i in period 2 is determined by $\sigma(q_i^1 + 1), \ldots, \sigma(q_i^1 + q_i^2)$.

We next show that the lottery solution ϕ^* is unbiased. Given the lottery solution ϕ^* and a a sequence of fair share tables $Y^t = (X^1, \ldots, X^t)$, we denote the outcome of the lottery solution by the random variable Z^t. We know that the expected number of positions reserved to category j in department i until period-t is $E(z_{ij}^t) = \sum_{s \leq t} q_i^s \alpha_j$. Moreover, the internal entry x_{ij}^t of fair share table X^t also equals to $\sum_{s \leq t} q_i^s \alpha_j$. Thus, the lottery solution ϕ^* is unbiased.

This proves the theorem. ■

An example for Theorem 1

To make the Roster-Finding Algorithm easier to understand and show the whole procedure that constructs the lottery solution ϕ^*, we show an example.

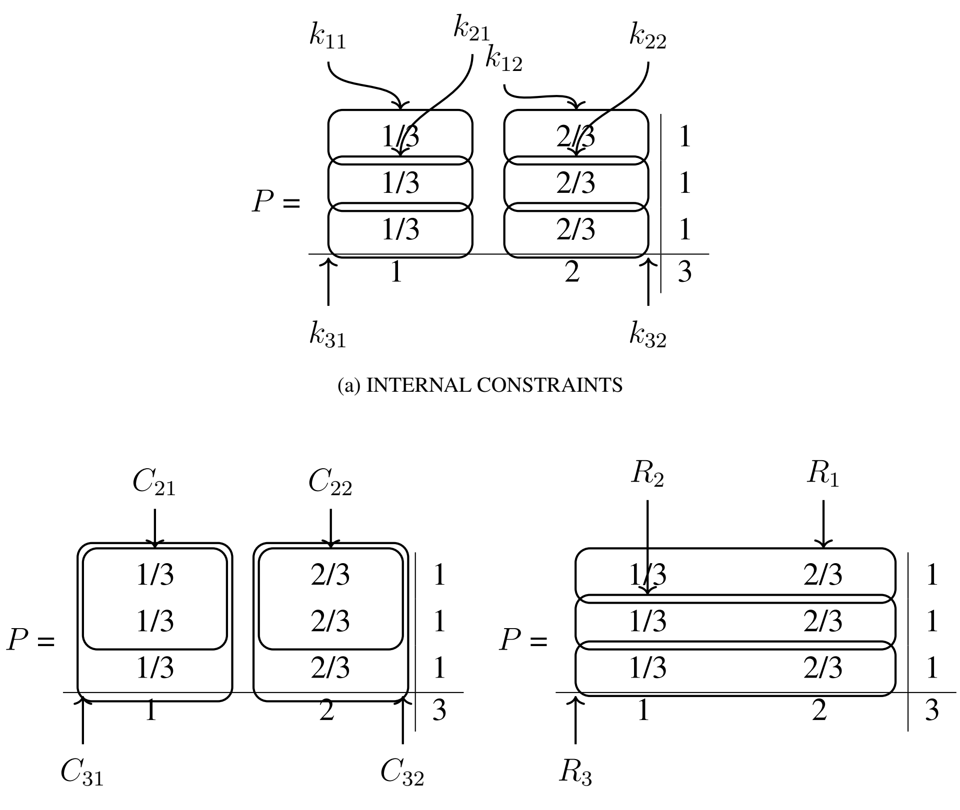
Consider a university where there are two categories $\mathcal{C} = \{c_1, c_2\}$ and the reservation scheme is $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) = (1/3, 2/3)$. Suppose we wish to implement the reservation scheme in a problem of reservation in two dimensions. We represent the given reservation scheme as a two-way table P, where the rows denote the index of the positions and the columns denote the categories. Each internal entry $p_{ij} = \alpha_j$. The output of the algorithm will be an integral table that define how a department reserves its positions over time, i.e., a roster.

There are three positions for easy illustration.[21] However, this method works for more general (total $3k$ positions, where $k = 1, 2, \ldots$) cases. The example table P is

$$P = \begin{array}{cc|c} 1/3 & 2/3 & 1 \\ 1/3 & 2/3 & 1 \\ 1/3 & 2/3 & 1 \\ \hline 1 & 2 & 3 \end{array}$$

Figure 1.7 illustrates the constraint structure. Column constraints are $C_{31} = \{k_{11}, k_{21}, k_{31}\}$, $C_{21} = \{k_{11}, k_{21}\}$, $C_{32} = \{k_{12}, k_{22}, k_{32}\}$, and $C_{22} = \{k_{12}, k_{22}\}$, and row constraints are $R_1 = \{k_{11}, k_{12}\}$, $R_2 = \{k_{21}, k_{22}\}$, and $R_3 = \{k_{31}, k_{32}\}$.

Figure 1.7: CONSTRAINT STRUCTURE OF THE EXAMPLE P



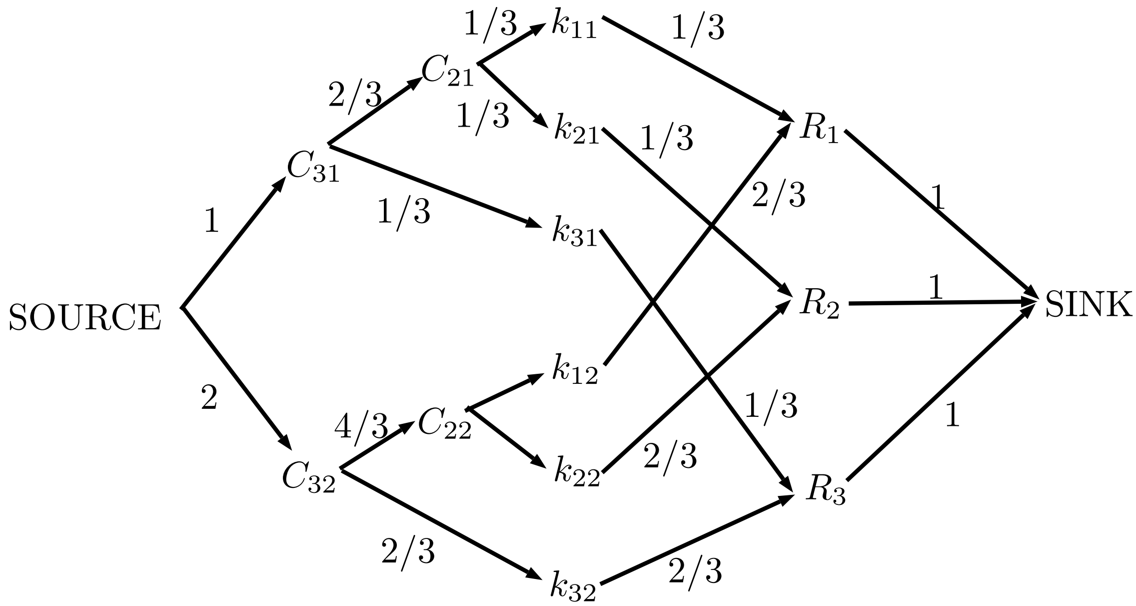
(a) INTERNAL CONSTRAINTS

(b) COLUMN CONSTRAINTS

(c) ROW CONSTRAINTS

[21]Using the common multiple of the fractions in the reservation scheme, three in our case, also helps to understand.

Figure 1.8: FLOW NETWORK REPRESENTATION OF THE EXAMPLE P

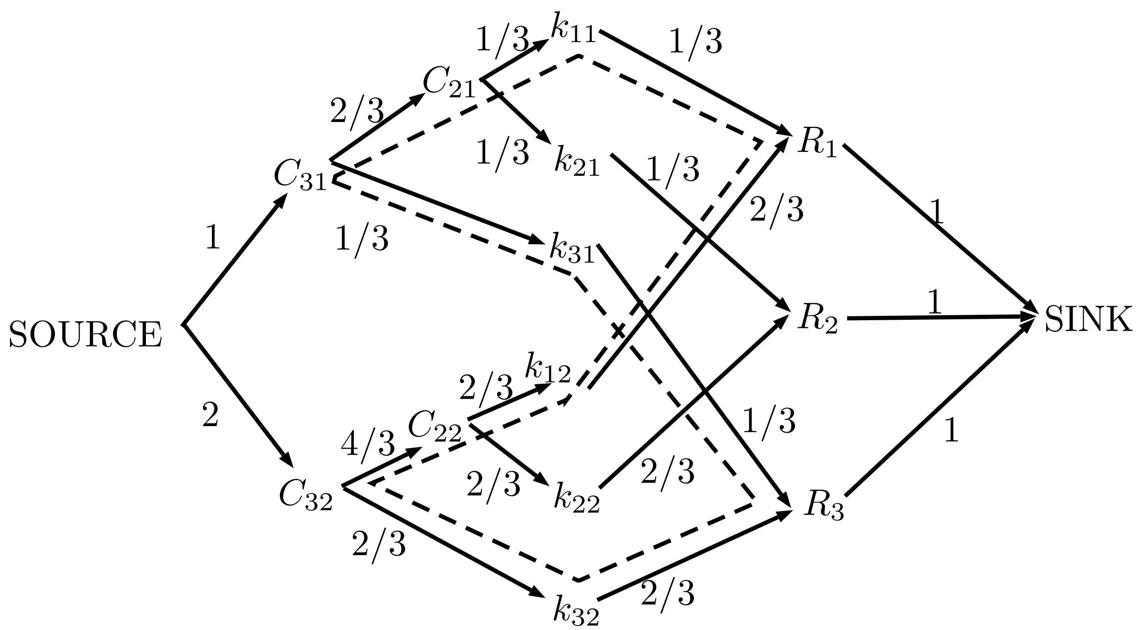


The two-way table P with the constraints is then represented as a network flow. Starting from the source, the flows first pass through the sets in column constraints, which are arranged in descending order of set-inclusion. That is, for example, $C_{31} \supset C_{21} \supset k_{11}$. This explains the flow network on the left side of Figure 1.8, where the numbers on the edges represent the flows. The flows then proceed along the directed edges that represent the set-inclusion tree, eventually reaching the singleton sets. That is, for example, $k_{11} \subset R_1$. This explains the flow network on the right side of Figure 1.8.

In the flow network, note that the flow associated with each edge reflects the totals of elements in the corresponding set. And, the flow arriving at each vertex equals the flow leaving that vertex. Now we are ready to present the algorithm. The algorithm will conserve these two properties while constructively find new flow network with fewer fractional elements.

We first identify a cycle of edges with fractional flows. Choosing any fractional edge, say (C_{31}, k_{31}), we find another fractional edge that is neighbor to k_{31}. If a vertex has a fractional edge then it has to have another fractional edge: since total inflow equals to outflow for every vertices(except source and sink), we would have a contradiction. We continue to add new fractional edges until we form a cycle. In our example, the cycle of fractional edges is $C_{31} \rightarrow^{1/3} k_{31} \rightarrow^{1/3} R_3 \leftarrow^{2/3} k_{32} \leftarrow^{2/3} C_{32} \rightarrow^{4/3} C_{22} \ldots \leftarrow^{2/3} C_{31}$. We illustrates this cycle in Figure 1.9 with dashed lines.

Figure 1.9: AN EXAMPLE OF CYCLE WITH FRACTIONAL EDGES



Next, we alter the cycle's edge flows. We first increase the flow of each forward edge while decreasing the flow of each backward edge at the same time until at least one flow reaches an integer value. A table P_1 is created as a result of the resulting network flow. In the example, flows along all forward edges increase from 2/3 to 1, 1/3 to 2/3, and 4/3 to 5/3, while flows along all backward edges decrease from 1/3 to 0 and 2/3 to 1/3. The adjustment is $d_+ = 1/3$. Next, the flows of the edges in the cycle are readjusted in the opposite direction, increasing those with backward edges and lowering those with forward edges in an analogous way, resulting in a new table P_2. In the example, flows along all forward edges decrease from 2/3 to 1/3, 1/3 to 0, and 4/3 to 1, while flows along all backward edges increase from 1/3 to 2/3 and 2/3 to 1. The adjustment is $d_- = 1/3$.

Now, we can decompose P into these two tables, i.e., $P = \frac{d_-}{d_- + d_+} P_1 + \frac{d_+}{d_- + d_+} P_2 = \frac{1}{2} P_1 + \frac{1}{2} P_2$. The algorithm picks P_1 with probability 0.5 and P_2 with probability 0.5. We reiterate the decomposition process until no fractions left.

At each iteration, at least one fraction in P is converted to an integer, while all current integers remain constant. Each fraction must appear in at least one iteration. As a result, the process must converge to an integer table in less iterations than the initial number of fractions in table P.

Since only the fractions along one cycle in the flow network are modified in each iteration, the expected change at this iteration for entries not on this cycle is 0, i.e., the expected change in corresponding entries in P is 0. For those fractional edges that are modified, the probabilities are picked so that the expected adjustment in each iteration is 0.

Fractional edges that are adjusted multiple times will have a variety of intermediate adjustment probabilities, but because our procedure keeps the expected change at 0 in each iteration, the compound probabilities will also keep the expected change at 0.

Proof of Theorem 2

Here, we prove that lottery solution ϕ^* in Theorem 1 approximately stays university quota. In words, the lottery solution ϕ^* is designed in such a way such that it hardly ever round up (or round down) most of the entries in each column of X^t. We show the approximately staying university quota property by proving two lemmas. We first show that entries of each column of Z^t are "independent". We next prove the approximately staying university quota by applying Chernoff concentration bounds.

Lemma 1. *For any subset of $S \subset \{1, 2, \ldots, m\}$ and any $j \in \{1, 2, \ldots, n\}$, we have*

$$\mathit{Pr}\Big[\bigwedge_{i\in S} z_{ij}^t = \lceil x_{ij}^t \rceil\Big] = \prod_{i\in S} \mathit{Pr}\Big[z_{ij}^t = \lceil x_{ij}^t \rceil\Big]\,,$$

$$\mathit{Pr}\Big[\bigwedge_{i\in S} z_{ij}^t = \lfloor x_{ij}^t \rfloor\Big] = \prod_{i\in S} \mathit{Pr}\Big[z_{ij}^t = \lfloor x_{ij}^t \rfloor\Big]\,.$$

Proof. Notice that the random roster is assigned to each department independently. Consequently, for any pair (i, i'), random variables z_{ij}^t and $z_{i'j}^t$ become independent, which proves the lemma. ■

Lemma 2. *For any subset of $S \subset \{1, 2, \ldots, m\}$ and any $j \in \{1, 2, \ldots, n\}$ with $\sum_{i \in S} x_{ij}^t = \mu$, and for any $\epsilon > 0$, we have*

$$Pr\Big[\sum_{i \in S} z_{ij}^t - \mu > \epsilon\mu\Big] \leq e^{-\mu\frac{\epsilon^2}{3}} ,$$

$$Pr\Big[\sum_{i \in S} z_{ij}^t - \mu < -\epsilon\mu\Big] \leq e^{-\mu\frac{\epsilon^2}{2}} .$$

Proof. We begin by recalling a result of Chernoff et al. (1952), which demonstrates that the independence property has the following large deviations result. Chernoff bounds are well-known concentration inequalities that limit the deviation of a weighted sum of Bernoulli random variables from their mean. We now use the multiplicative form of Chernoff concentration bound.

Theorem 3. *Chernoff bound: Let $A_1, A_2, \ldots, A_m$ be m independent random variables taking values in $\{0, 1\}$. Let $\mu = \sum_{i=1}^m E[A_i]$. Then, for any $\epsilon \geq 0$,*

$$Pr\Big[\sum_{i=1}^m A_i \geq (1+\epsilon)\mu\Big] \leq e^{-\mu\frac{\epsilon^2}{3}} ,$$

$$Pr\Big[\sum_{i=1}^m A_i \leq (1-\epsilon)\mu\Big] \leq e^{-\mu\frac{\epsilon^2}{2}} .$$

The random variable z_{ij}^t can take two values, either $\lceil x_{ij}^t \rceil$ or $\lfloor x_{ij}^t \rfloor$. If we subtract the fix number $\lfloor x_{ij}^t \rfloor$ from z_{ij}^t, then we obtain a Bernoulli distribution. Lemma 1 says that the set of random variables in each column of Z^t are independent, which means Chernoff concentration bounds hold for each column of Z^t. ■

Proof of Theorem 2. We can now prove Theorem 2. In Lemma 2, if we choose $S =$

$\{1, \ldots, m\}$, then $\sum_{i=1}^{m} x_{ij}^t = x_{m+1,j}^t$. This fact along with Lemma 2 yields our result for Theorem 2. ■

Proof of Proposition 4

Proof. For any $x_{m+1,j}^t := \mu > 0$ and any constant $b := \epsilon\mu$, we construct a problem instance. For the rest of the proof we fix category j, μ, and ϵ. This instance contains n departments, m categories. The vacancies are as follows: $q_i^s = 0$ vacancies for all $s < t$ and $q_i^t = 1$ for any $i \in \mathcal{D}$. Choose a constant $\underline{\epsilon}, \bar{\epsilon} \in (0, 1)$ such that $\epsilon \in (\underline{\epsilon}, \bar{\epsilon})$. Choose $\alpha \in (0, 1/(1 + \bar{\epsilon}))$ such that μ/α is an integer. Let $m = \mu/\alpha$. For category j, α fraction of vacancies are to be reserved. Note that, by definition, $x_{ij}^t = \alpha$ for all $i \in \mathcal{D}$.

Consider a lottery solution that is unbiased and stays within department quota. Let z_{ij}^t denote the the outcome of such lottery for category j in department i. Note that, by definition of such lottery, $Pr(z_{ij}^t = 1) = \alpha$ and $Pr(z_{ij}^t = 0) = 1 - \alpha$ must fold for all $i \in \mathcal{D}$. And, by definition, the random variable $z_{m+1,j}^t = \sum_{i=1}^{m} z_{i,j}^t$ is a sum of independent Bernoulli trials. Hence, $z_{m+1,j}^t$ has a binomial distribution. That is,

$$Pr(z_{m+1,j}^t = c) = \binom{m}{c} \alpha^c (1 - \alpha)^{m-c}.$$

Let $B_\alpha(m, \lambda)$ be the (upper) tail of the binomial distribution from λm to m. That is,

$$B_\alpha(m, \lambda) = \sum_{c=\lambda m}^{m} \binom{m}{c} \alpha^c (1 - \alpha)^{m-c}$$

where λm is an integer and $\alpha < \lambda < 1$. When $\lambda = (1 + \epsilon)\alpha$, by definition, the probability of $z_{m+1,j}^t$ is at least $b + x_{m+1,j}^t = (1 + \epsilon)\mu$ is

$$B := Pr(z_{m+1,j}^t \geq (1 + \epsilon)\mu) = B_\alpha(m, (1 + \epsilon)\alpha).$$

The goal is to show that B is at least $e^{-\mu\epsilon^2 l}$, where $l > 0$ is a constant independent of μ

and ϵ. This would imply that $f(\mu, \epsilon\mu) \geq e^{-\mu\epsilon^2 l}$. Hence, setting k to be any constant larger than l would prove the proposition.

To show lower bounds on the tail distribution, we use the following lemma.

Lemma 3. *Ahle (2017). When $\lambda \geq 0.5$,*

$$B_\alpha(m, \lambda) \geq \frac{1}{\sqrt{2m}} e^{-mH(\lambda;\alpha)}$$

where $H(\lambda; \alpha) = \lambda \log \frac{\lambda}{\alpha} + (1-\lambda) \log \frac{1-\lambda}{1-\alpha}$.

Applying this lemma for $m = \mu/\alpha$ and $\lambda = (1+\epsilon)\alpha$ implies:

$$\begin{aligned} B &\geq \frac{1}{\sqrt{2\mu/\alpha}} e^{-\frac{\mu}{\alpha} H((1+\epsilon)\alpha;\alpha)} \\ &= \frac{1}{\sqrt{2\mu/\alpha}} e^{-\frac{\mu}{\alpha}[(1+\epsilon)\alpha \log(1+\epsilon) + (1-(1+\epsilon)\alpha) \log \frac{1-(1+\epsilon)\alpha}{1-\alpha}]} \\ &= \frac{1}{\sqrt{2\mu/\alpha}} e^{-\mu[(1+\epsilon)\log(1+\epsilon) + \frac{1-(1+\epsilon)\alpha}{\alpha} \log(1-\alpha\epsilon/(1-\alpha))]} \\ &\geq \frac{1}{\sqrt{2\mu/\alpha}} e^{-\mu[(1+\epsilon)\epsilon + \frac{1-(1+\epsilon)\alpha}{1-\alpha}\epsilon)]} && (1.1) \\ &= \frac{1}{\sqrt{2\mu/\alpha}} e^{-\mu\epsilon^2(1+\frac{1}{\epsilon}+\frac{1-(1+\epsilon)\alpha}{(1-\alpha)\epsilon})} && (1.2) \end{aligned}$$

where (1.1) holds since $\log(1+\epsilon) < \epsilon$ and $\log(1-\alpha\epsilon/(1-\alpha)) < -\frac{\alpha\epsilon}{1-\alpha}$ for all $\epsilon \in (0,1)$.

The proof is complete when we observe that the right-hand side of (1.2) is larger than $e^{-\mu\epsilon^2 l}$ for any $l \geq 1 + 2/\underline{\epsilon}$ and sufficiently large μ.[22]

■

[22]The proof is symmetric for the lower tail since $Pr(z^t_{m+1,j} \leq (1-\epsilon)\mu) = B_{1-\alpha}(m, (1-\epsilon)\alpha)$

Proof of Proposition 1

In this section, we present the complete proof of Proposition 1. The proof is an adaptation of the procedure of Cox (1987).[23]

Proof. We present a constructive proof of Proposition 1 using following algorithm. The rounding algorithm takes a fair share table as input and generates a (random) reservation table as output. To make the algorithm easier to understand, after each step we demonstrate the algorithm on an example depicted in Figure 1.10.

Rounding Algorithm

Step 1: Given a fair share table X, we construct an extended table V by adding an extra row to table X. The last row of V is generated by taking 1 - fraction part of the column totals of table X.

In our example, shown in Figure 1.10, table V is equivalent to table X except the last row. Adding this extra row makes the column totals integers.

Figure 1.10: STEP 1 OF PROCEDURE

$$X = \begin{array}{ccc|c} 0.5 & 0.5 & 1 & 2 \\ 0.25 & 0.25 & 0.5 & 1 \\ 0.75 & 0.75 & 1.5 & 3 \\ \hline 1.5 & 1.5 & 3 & 6 \end{array}$$

(a) FAIR SHARE TABLE

$$V = \begin{array}{ccc|c} 0.5 & 0.5 & 1 & 2 \\ 0.25 & 0.25 & 0.5 & 1 \\ 0.75 & 0.75 & 1.5 & 3 \\ 0.5 & 0.5 & 0 & 1 \\ \hline 2 & 2 & 3 & 7 \end{array}$$

(b) EXTENDED TABLE

We focus on the internal entries of table V. The procedure involves iterative adjustment of the fractions in table V until all fractions have been replaced by integers.

Step 2: If table V contains no fractions, then skip to Step 8.

Step 3: Choose any fraction v_{ij} in table V. At (i, j) begin an alternating row-column (or column-row) path of fractions. A cycle will be formed (all edges

[23] An alternative proof utilizes network flow approach, very similar to the one in proof Theorem 1. However, for its simplicity and ease of use by hand, we show a modified version of the procedure of Cox (1987).

Figure 1.11: STEP 3 OF PROCEDURE

$$V = \begin{array}{c|c} \begin{pmatrix} 0.5 & 0.5 & 1 \\ 0.25 & 0.25 & 0.5 \\ 0.75 & 0.75 & 1.5 \\ 0.5 & 0.5 & 0 \end{pmatrix} & \begin{array}{c} 2 \\ 1 \\ 3 \\ 1 \end{array} \\ \hline \begin{array}{ccc} 2 & 2 & 3 \end{array} & 7 \end{array}$$

fractions).

In our example, shown in Figure 1.11, the cycle of fractions is $(i_1, j_1) \rightarrow (i_1, j_2) \rightarrow (i_2, j_2) \rightarrow (i_2, j_3) \rightarrow (i_3, j_3) \rightarrow (i_3, j_2) \rightarrow (i_4, j_2) \rightarrow (i_4, j_1) \rightarrow (i_1, j_1)$.

Step 4: Modify the cycle. First, raise the odd edges and reduce the even edges at the same rate until at least one edge reaches an integer value. The resulting table then gives rise to a table V_1.

In our example, the odd edges rise by 0.5 and even edges reduce by 0.5 ($d_+ = 0.5$). The resulting table V_1 is shown in Figure 1.12.

Step 5: Next, readjust the edges in the cycle in the reverse direction, raising even edges and reducing odd edges in an analogous manner, which gives rises to another table V_2.

In our example, the even edges rise by 0.25 and odd edges reduce by 0.25 ($d_- = 0.25$). The resulting table V_2 is shown in Figure 1.12.

Step 6: Select either V_1 or V_2 with probabilities $p_1 = \frac{d_-}{d_- + d_+}$ and $\frac{d_+}{d_- + d_+}$, respectively.

Figure 1.12: STEP 6 OF PROCEDURE

$$V_1 = \begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0.25 & 0.75 & 0 & 1 \\ 0.75 & 0.25 & 2 & 3 \\ 0 & 1 & 0 & 1 \\ \hline 2 & 2 & 3 & 7 \end{array} \qquad V_2 = \begin{array}{ccc|c} 0.25 & 0.75 & 1 & 2 \\ 0.25 & 0 & 0.75 & 1 \\ 0.75 & 1 & 1.25 & 3 \\ 0.75 & 0.25 & 0 & 1 \\ \hline 2 & 2 & 3 & 7 \end{array}$$

In our example, table V is decomposed into table V_1 and table V_2 where $V = \frac{1}{3}V_1 + \frac{2}{3}V_2$. There are few fraction elements in both tables.

Step 7: Reiterate Step 6 until no fractional elements left.

Step 8: Delete the last row of the table and report it as the outcome of the algorithm.

The algorithm must end in finite steps (at most the number of fractions in share table V) and, at the end we must have an integer table.

Lemma 4. *The outcome of the Rounding Algorithm stays within quota.*

Proof. In Step 4 and 5, after each adjustment the row and column sums remains the same. Moreover, after adjustments every element v_{ij} in table V always remains less than or equal to $\lceil v_{ij} \rceil$ and greater than or equal to $\lfloor v_{ij} \rfloor$. Therefore, the outcome of the algorithm will stay within quota. ■

Lemma 5. *The Rounding Algorithm satisfies the following property: For any iteration and for any entry of the table,*

$$E(v_{ij}|V) = v_{ij}$$

Proof. Note that in Step 4, v_{ij} raises by d_+ and in Step 5, it reduces by d_-. In Step 6, the probabilities of raising and decreasing are assigned as $\frac{d_-}{d_-+d_+}$ and $\frac{d_+}{d_-+d_+}$. Therefore, the expected adjustment will be $d_+\frac{d_-}{d_-+d_+} + d_-\frac{d_+}{d_-+d_+} = 0$. ■

In words, Lemma 5 proves that entries of the fair share table X are rounded up or down so that ex-ante positive and negative biases balance to yield zero bias.

Lemma 4 and Lemma 5 prove Proposition 1. ■

Proof of Proposition 2 and Proposition 3

Since Proposition 2 is a special case of Proposition 3, we prove the latter. We prove the proposition by contradiction.

Proof. Suppose a deterministic solution R stays within university quota. We show an example of a problem of reservation in two dimensions that the solution R can not have a finite bias. That is, for any constant $b > 0$, there exist a Y^t and an internal entry y^t such that $|\text{bias}(R(y^t))| > b$.

Example 3. *Consider a problem with three departments $d_1, d_2,$ and d_3, two categories c_1, c_2, the reservation scheme vector $\boldsymbol{\alpha} = [0.5, 0.5]$. The departments $d_1, d_2,$ and d_3 have $\mathbf{q}^1 = [0, 0, 1]$ positions in period-1 and $\mathbf{q}^2 = [1, 0, 0]$ positions in period-2.*

Notice that staying within university quota is equivalent to reserving exactly k positions for c_1 and c_2 in every 2k cumulative sum of vacancies in the university, where $k = 1, 2, 3, \ldots$. In period-1, department d_3 can reserve positions to either categories. Without loss of generality, we assume that it reserves 1 position for c_1. In period-2, since there are 2 cumulative sum of vacancies in the university, there should be exactly 1 position reserved for c_1. Department d_1 should reserve 1 position for category c_2. The period-1 and period-2 reservation tables are shown in Figure 1.13.

Figure 1.13: PERIOD-1 AND PERIOD-2 RESERVATION TABLES

$$X^1 = \begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.5 & 0.5 & 1 \\ \hline 0.5 & 0.5 & 1 \end{array}$$

(a) PERIOD-1 FAIR SHARE TABLE

$$X^2 = \begin{array}{cc|c} 0.5 & 0.5 & 1 \\ 0 & 0 & 0 \\ 0.5 & 0.5 & 1 \\ \hline 1 & 1 & 2 \end{array}$$

(b) PERIOD-2 FAIR SHARE TABLE

$$R(Y^1) = \begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \\ \hline 1 & 0 & 1 \end{array}$$

(c) PERIOD-1 RESERVATION TABLE

$$R(Y^2) = \begin{array}{cc|c} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \\ \hline 1 & 1 & 2 \end{array}$$

(d) PERIOD-2 RESERVATION TABLE

If departments have $\mathbf{q}^3 = [0, 0, 1]$ positions in period-3, department d_3 can reserve its position to either categories. These two cases are show in Figure 1.15.

Case 1: We assume that the solution is $R = R_1$. If the departments have $\mathbf{q}^4 = [1, 0, 0]$ positions in period-4, department d_1 should reserve 1 position for category c_2. Otherwise, the solution R would violate staying within university quota property. Period-4 fair share table and the period-4 reservation table are illustrated by X_1^4 and $R_1(X_1^4)$ in Figure 1.15.

Figure 1.14: TWO CASES FOR PERIOD-3 RESERVATION TABLES

$$X^3 = \begin{array}{cc|c} 0.5 & 0.5 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 2 \\ \hline 1.5 & 1.5 & 3 \end{array}$$

(a) PERIOD-3 FAIR SHARE TABLE

$$R_1(Y^3) = \begin{array}{cc|c} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 2 & 0 & 2 \\ \hline 2 & 1 & 3 \end{array}$$

(b) PERIOD-3 RESERVATION TABLE

$$R_2(Y^3) = \begin{array}{cc|c} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 2 \\ \hline 1 & 2 & 3 \end{array}$$

(c) PERIOD-3 RESERVATION TABLE

Case 2: We assume that the solution is $R = R_2$. If the departments have $\mathbf{q}^4 = [0, 1, 0]$ positions in period-4, department d_2 should reserve 1 position for category c_1. Otherwise, the solution R would violate staying within university quota property. Period-4 fair share table and the period-4 reservation table are illustrated by X_2^4 and $R_2(X_2^4)$ in Figure 1.15.

Figure 1.15: TWO CASES FOR PERIOD-4 RESERVATION TABLES

$$X_1^4 = \begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 1 & 1 & 2 \\ \hline 2 & 2 & 4 \end{array}$$

(a) CASE 1: PERIOD-4 FAIR SHARE TABLE

$$R_1(X_1^4) = \begin{array}{cc|c} 0 & 2 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 2 \\ \hline 2 & 2 & 4 \end{array}$$

(b) CASE 1: PERIOD-4 RESERVATION TABLE

$$X_2^4 = \begin{array}{cc|c} 0.5 & 0.5 & 1 \\ 0.5 & 0.5 & 1 \\ 1 & 1 & 2 \\ \hline 2 & 2 & 4 \end{array}$$

(c) CASE 2: PERIOD-5 FAIR SHARE TABLE

$$R_2(X_2^4) = \begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \\ \hline 2 & 2 & 4 \end{array}$$

(d) CASE 2: PERIOD-4 RESERVATION TABLE

If departments have $\mathbf{q}^5 = [0, 0, 1]$ positions in period-3, department d_3 can reserve its position to either categories. These two cases are show in Figure 1.16.

In Example 3 for each case, period-5 reservation for category c_1 in department d_1 is 0 and period-5 reservation for category c_2 in department d_2 is 0. We can extend these example for more periods analogously. The idea is following. In each period, the university has only one position. Department d_3 has always one position in odd periods and in the following period either department d_1 or department d_2 has one position according to these following cases.

- Case I: If department d_3 reserves 1 position to category c_1, department d_1 has one

Figure 1.16: TWO CASES FOR PERIOD-5 RESERVATION TABLES

$$X_1^5 = \begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 1.5 & 1.5 & 3 \\ \hline 2.5 & 2.5 & 5 \end{array}$$

(a) PERIOD-5 FAIR SHARE TABLE

$$R_{1.1}(Y_1^5) = \begin{array}{cc|c} 0 & 2 & 2 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \\ \hline 3 & 2 & 5 \end{array}$$

(b) PERIOD-5 RESERVATION TABLE

$$R_{1.2}(Y_1^5) = \begin{array}{cc|c} 0 & 2 & 2 \\ 0 & 0 & 0 \\ 2 & 1 & 3 \\ \hline 2 & 3 & 5 \end{array}$$

(c) PERIOD-5 RESERVATION TABLE

$$X_2^5 = \begin{array}{cc|c} 0.5 & 0.5 & 1 \\ 0.5 & 0.5 & 1 \\ 1.5 & 1.5 & 3 \\ \hline 2.5 & 2.5 & 5 \end{array}$$

(d) PERIOD-5 FAIR SHARE TABLE

$$R_{2.1}(Y_2^5) = \begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \\ \hline 3 & 2 & 5 \end{array}$$

(e) PERIOD-5 RESERVATION TABLE

$$R_{2.2}(Y_2^5) = \begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \\ \hline 2 & 3 & 5 \end{array}$$

(f) PERIOD-5 RESERVATION TABLE

position in the next period.

- Case II: If department d_3 reserves 1 position to category c_2, department d_2 has one position in the next period.

In case I, department d_1 should reserve 1 position for category c_2, otherwise, solution would violate staying university quota property. In case II, department d_2 should reserve 1 position for category c_1, otherwise, solution would violate staying university quota property. Example 3 shows that if a solution stays within university quota, departments can grow in size without giving a seat to one category, i.e., the solution violates finite bias.[24] This proves the proposition. ■

[24] An example for any number of categories and departments can be constructed in a similar way. Example 3 is constructed so that it not only illustrates the failure, but it also demonstrates any solution can fail to have finite bias in all categories.

1.9 Tables and Figures

Figure 1.17: 200-POINT ROSTER PRESCRIBED BY THE GOVERNMENT OF INDIA

FOR DIRECT RECRUITMENT

Model Roster of Reservation with reference to posts for Direct recruitment on All India Basis by Open Competition

Sl. No. of Post	Share of Entitlement				Category for which the posts should be earmarked
	SC @15%	ST @7.5%	OBC @27%	EWS @10%	
1	0.15	0.08	0.27	0.10	UR
2	0.30	0.15	0.54	0.20	UR
3	0.45	0.23	0.81	0.30	UR
4	0.60	0.30	1.08	0.40	OBC-1
5	0.75	0.38	1.35	0.50	UR
6	0.90	0.45	1.62	0.60	UR
7	1.05	0.53	1.89	0.70	SC-1
8	1.20	0.60	2.16	0.80	OBC-2
9	1.35	0.68	2.43	0.90	UR
10	1.50	0.75	2.70	1.00	**EWS-1**
11	1.65	0.83	2.97	1.10	UR
12	1.80	0.90	3.24	1.20	OBC-3
13	1.95	0.98	3.51	1.30	UR
14	2.10	1.05	3.78	1.40	ST-1
15	2.25	1.13	4.05	1.50	SC-2
16	2.40	1.20	4.32	1.60	OBC-4
17	2.55	1.28	4.59	1.70	UR
18	2.70	1.35	4.86	1.80	UR
19	2.85	1.43	5.13	1.90	OBC-5
20	3.00	1.50	5.40	2.00	SC-3
21	3.15	1.58	5.67	2.10	**EWS-2**
22	3.30	1.65	5.94	2.20	UR
23	3.45	1.73	6.21	2.30	OBC-6
24	3.60	1.80	6.48	2.40	UR
25	3.75	1.88	6.75	2.50	UR
26	3.90	1.95	7.02	2.60	OBC-7
27	4.05	2.03	7.29	2.70	SC-4
28	4.20	2.10	7.56	2.80	ST-2
29	4.35	2.18	7.83	2.90	UR
30	4.50	2.25	8.10	3.00	OBC-8
31	4.65	2.33	8.37	3.10	**EWS-3**

Source: `https://dopt.gov.in/sites/default/files/ewsf28fT.PDF`[25]

[25]Last accessed on 27 July 2022.

Figure 1.18: 13-POINT ROSTER PRESCRIBED BY THE GOVERNMENT OF INDIA

FOR DIRECT RECRUITMENT

Roster for Direct Recruitment otherwise than through Open Competition for cadre strength upto 13 posts

Cadre Strength	Initial Recruit-ment	Replacement No.												
		1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th	11th	12th	13th
1	UR	UR	UR	OBC	UR	UR	SC	OBC	UR	**EWS**	UR	OBC	SC	ST
2	UR	UR	OBC	UR	UR	SC	OBC	UR	**EWS**	UR	OBC	SC	ST	
3	UR	OBC	UR	UR	SC	OBC	UR	**EWS**	UR	OBC	SC	ST		
4	OBC	UR	UR	SC	OBC	UR	**EWS**	UR	OBC	SC	ST			
5	UR	UR	SC	OBC	UR	**EWS**	UR	OBC	SC	ST				
6	UR	SC	OBC	UR	**EWS**	UR	OBC	SC	ST					
7	SC	OBC	UR	**EWS**	UR	OBC	SC	ST						
8	OBC	UR	**EWS**	UR	OBC	SC	ST							
9	UR	**EWS**	UR	OBC	SC	ST								
10	**EWS**	UR	OBC	SC	ST									
11	UR	OBC	SC	ST										
12	OBC	SC	ST											
13	SC	ST												

Source: `https://dopt.gov.in/sites/default/files/ewsf28fT.PDF`[26]

[26] Last accessed on 27 July 2022.

Chapter 2

Impartial Rosters for Affirmative Action

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2.1 Introduction

In many countries affirmative action policies prescribe a proportion of seats and jobs in publicly funded institutions to various beneficiary groups (Sönmez and Yenmez (2022b)). Since seats are indivisible, imaginative methods are used to help achieve the objectives of affirmative action in practice. Devising such methods is particularly necessary when the number of seats is small. For instance, a university may appoint at most one assistant professor of economics every year, while the country's affirmative action policy may have more than one beneficiary group. To ensure that, over a period of time, each beneficiary group gets its affirmative action policy prescribed percentage of seats, India devised a tool called *roster*.[1]

India's publicly announced *roster* details a sequence of length two hundred in which five beneficiary groups of an affirmative action policy take turns in claiming seats. A roster allocates the indivisible seats to beneficiaries by fixing the order of rotation in claiming

[1]For India's roster, see Figure 2.8 and Figure 2.9, or visit `https://dopt.gov.in/sites/default/files/ewsf28fT.PDF`, last accessed on 27 July 2022.

seats. Indian affirmative action scheme, *the reservation policy*, determines the number of reserved seats in a roster, but the sequence of taking turns is not fixed by the legislation, and is therefore up to the designer. We argue that the current Indian roster favors beneficiaries entitled to larger proportions of seats by delaying the seats for beneficiaries entitled to a smaller proportion (see Section 2.5).

The reservation policy is formulated after taking into account many socio-economic-political factors that led to the need for affirmative action. For hiring and admissions in large numbers, the reservation policy reflects years of progress that resulted in such policies. Maintaining rosters would be unnecessary if there were enough seats to reserve exact proportions for all beneficiaries. However, in practice the seats arrive over time in small numbers, therefore rosters are maintained and the delay in reserving seats occurs naturally. The delay being associated with the size of proportion of reservation is nothing but an unjustified additional layer of partiality against some beneficiary groups, partiality introduced by the problem of indivisibility of seats.

Consider, for example, a reservation policy with two beneficiary groups, R and B, entitled to 20% and 80% seats, respectively. A roster of size 20 would detail a sequence of length 20 in which the two groups would take turns in claiming their 4 and 16 seats, respectively. Which group should receive the first seat? and which group should receive the second? and so on. And the question is: Why? In this paper we present a theory of designing rosters to argue that only a few rosters can be considered impartial in that they do not favor some beneficiaries over others.

The paper is organized as follows. Section 2.2 presents our formulation of the roster construction problem. Section 2.3 argues for notions of impartiality, characterizes the set of *impartial rosters*, and lists their properties. Section 2.4 details a method of finding all impartial rosters of a roster construction problem. Section 2.5 scrutinizes the Indian roster while contrasting it with an impartial roster. The paper concludes with a discussion with respect to the related literature in Section 2.6. All proofs are relegated to Section 2.7.

2.2 Formulation

A **roster construction problem** is a tuple $\Lambda = (\mathcal{C}, \boldsymbol{\alpha}, n)$. $\mathcal{C}$ is a finite set of **categories** where $m := |\mathcal{C}| \geq 2$. The **reservation policy** is defined by a vector of fractions $\boldsymbol{\alpha} = [\alpha_j]_{j \in \mathcal{C}}$. For each category $j \in \mathcal{C}$, $\alpha_j \in (0, 1)$ fraction of positions are to be reserved so that $\sum_{j \in \mathcal{C}} \alpha_j = 1$. The **size of the roster** is n where n is a positive integer. Throughout the paper, we fix a set of categories $\mathcal{C}$ and a reservation policy $\boldsymbol{\alpha}$. We assume that $\alpha_j n \in \mathbb{N}$ for all $j \in \mathcal{C}$.

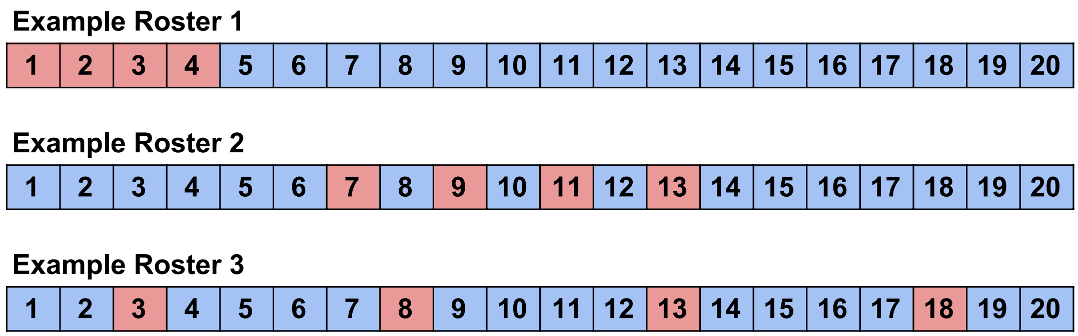
A **roster** $R_n : \{1, \dots, n\} \rightarrow \mathcal{C}$ maps each position to a category such that, $|R_n^{-1}(c)| = \alpha_c n$ for all $j \in \mathcal{C}$. We denote by $\mathcal{R}_n$ the set of rosters. The definition incorporates the idea that the total number of positions a roster assigns to a category is the same as the proportion given by the reservation policy; that is, all categories must get their quantum of reserved positions once n positions are filled. Therefore, we can restrict our attention to only those rosters where this is possible.

We next introduce an example that makes the notion of roster easier to comprehend. There are two categories for easy illustration. The example will also be sufficient to present the various aspects of designing rosters in upcoming chapters.

Example 1. *Consider a problem* $\Lambda = (\{R,\ B\}, \boldsymbol{\alpha} = [0.2, 0.8], 20)$. *There are two categories* $\mathcal{C} = \{R,\ B\}$, *represented by red and blue colors. The reservation policy reserves 20% positions for members of category* R. *The size of the roster is* $n = 20$. *Therefore, the number of positions given to the category* R *and* B *is 4 and 16, respectively. Figure 2.1 illustrates three possible rosters for the problem. For instance, Example Roster 1 is*

$$R_n(k) = \begin{cases} R, & \textit{if } k \in \{1, 2, 3, 4\} \\ B, & \textit{otherwise} \end{cases}$$

Figure 2.1: EXAMPLE ROSTERS



2.3 Impartial Rosters

The goal of this section is to argue for notions of impartiality, characterize the set of impartial rosters, and list their properties. For this purpose, we first need to describe the distribution of seats for a category.

The **cumulative distribution of seats for category** j under roster R_n is

$$F_j(t) = \frac{|\{s \in R_n^{-1}(j) \ | \ s \leq t\}|}{|R_n^{-1}(j)|}, \quad \text{for} \quad t \in \{0, \ldots .n\}.$$

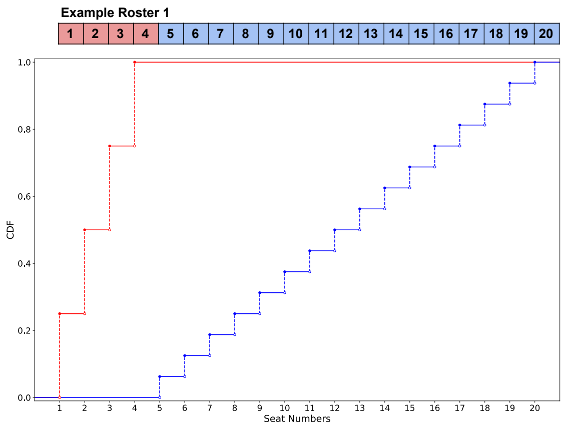
These cumulative distribution functions measure the fraction of seats a category is received as per the reservation policy until position t; that is, the number of seats given to a category until position t over the total number of seats given to a category.

For instance, Figure 2.2 illustrates cumulative distribution of seats for the rosters depicted in Figure 2.1.

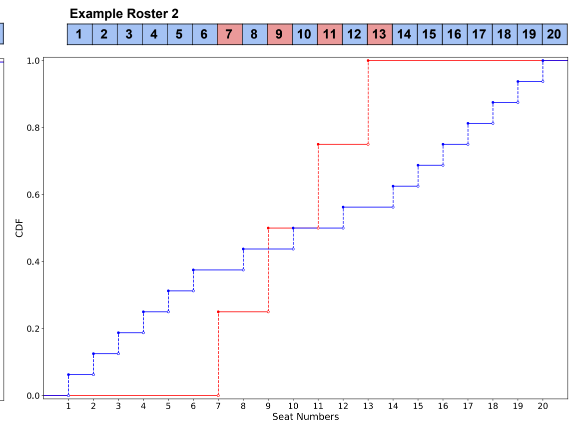
2.3.1 Impartial in Frequencies

The task at hand is to devise criteria of partiality. Since seats are indivisible, at every position in the roster, between any two categories, there will always be a certain partiality that gives one of the categories a slight advantage over the other. It is straightforward to say that for any pair of categories $i, j \in \mathcal{C}$, category i is **favored relative** to category j at position t under roster R_n, if $F_i(t) > F_j(t)$. We denote by $\#|F_i(t) < F_j(t)|$ the number

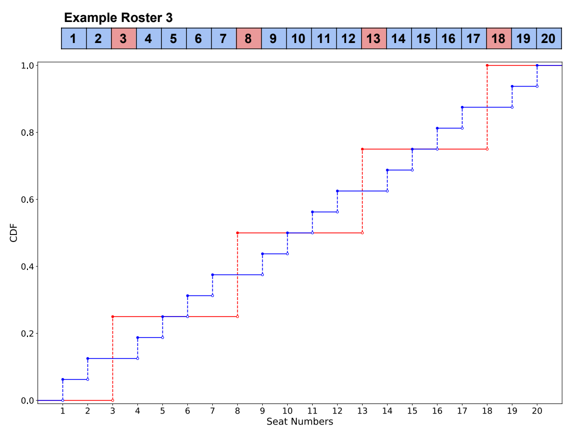
Figure 2.2: DISTRIBUTION OF SEATS



(a) EXAMPLE ROSTER 1



(b) EXAMPLE ROSTER 2



(c) EXAMPLE ROSTER 3

of positions category i is favored relative to category j. One measure of partiality is to count such instances. Impartiality demands that a roster should not favor any category at a majority of positions.

Definition 1. *A roster R_n is **impartial in frequencies** if for any pairs of categories $i, j \in \mathcal{C}$ with $\alpha_i \neq \alpha_j$ and for any $t \in \{1, \ldots .n\}$*

$$\#|F_i(t) > F_j(t)| = \#|F_i(t) < F_j(t)|.$$

This property formulates the idea that a roster should implement the reservation policy without favoring any category in terms of occasions; that is, for any two category i, j with different reservation policy, $|\{t \in \{1, \ldots, n\} | F_i(t) < F_j(t)\}| = |\{t \in \{1, \ldots, n\} | F_i(t) > F_j(t)\}|$.

For example, in Figure 2.2, the Example Roster 1 is not impartial in frequencies since category R is favored relative to category B at all positions. Example Roster 2 and Example Roster 3 are impartial in frequencies. Such rosters, on the other hand, may result in large differences between distributions of seats across categories, in which one category receives most initial seats while the others receive few. Therefore a reasonable measure of partiality is the distance between the distributions of seats.

2.3.2 Impartial in Distance

Impartiality demands that this distance should be minimized. The question that remains is: what distance function should be optimized? We discuss numerous reasonable measures and show that they characterize the same set of rosters.

For any pair of categories $i, j \in \mathcal{C}$, **distance between distributions of seats for category** i **and category** j **at position** t under roster R_n is $|F_i(t) - F_j(t)|$. For example, in Figure 2.2, Example Roster 3 has a smaller distance between distributions for category R and category B compared to Example Roster 2 for all positions.

For any pair of categories $i, j \in \mathcal{C}$, **weighted distance between distributions of seats for category** i **and category** j **at position** t under roster R_n is $\alpha_i \alpha_j |F_i(t) - F_j(t)|$.

The **partiality of roster at position** t under roster R_n is $\sum_{i \neq j} (\alpha_i \alpha_j)^2 (F_i(t) - F_j(t))^2$. This measures the weighted distance between the distribution of seats across categories at position t.

The **overall partiality of roster** R_n is

$$\sum_{t=1}^{n} \sum_{i \neq j} (\alpha_i \alpha_j)^2 (F_i(t) - F_j(t))^2.$$

Overall partiality measures the total weighted distance between distributions of seats across categories. Categories with higher reservation policies have a bigger impact on this measure.

Definition 2. *A roster R_n^* is* ***weakly more impartial*** *than roster R_n if overall partiality of roster R_n^* is strictly smaller than overall partiality of roster R_n.*

Definition 3. *A roster R_n^** ***strongly more impartial*** *than roster R_n if*

1. *for any position t, partiality of roster R_n^* at position t is smaller than partiality R_n at position t, and*

2. *there is at least one position where partiality of roster R_n^* is strictly smaller.*

It follows from the definitions that if R_n^* is strongly more impartial than R_n, then R_n^* is also weakly more impartial than R_n. Our strongly more impartial notion generates a partial ordering on rosters, while the weakly more impartial notion generates a linear ordering on rosters.

A roster R_n^* **minimizes overall partiality** if there does not exist a roster that is weakly more impartial than roster R_n^*; that is, it minimizes the total weighted distance between distributions of seats across categories. We denote $\mathcal{I}_n^w$ by the set of rosters that minimizes overall partiality.

A roster R_n^* **minimizes partiality** if there does not exist a roster that is strongly more impartial than roster R_n^*; that is, it minimizes the distance between distributions of seats across categories. We denote $\mathcal{I}_n^s$ by the set of rosters that minimizes partiality. It follows from the definitions that $\mathcal{I}_n^s$ is subset of $\mathcal{I}_n^w$.

Theorem 1. *The set of rosters that minimizes overall partiality is the same as the set of rosters that minimizes partiality.*

Theorem 1 implies that if a roster minimizes overall partiality, it will also minimize partiality.

2.3.3 Impartial in achieving the Uniform Ideal

The uniform distribution would be the ideal seat allocation for all categories if only the seats were divisible. Therefore, distance from the uniform distribution is another reasonable measure of partiality. We denote by $U(t)$ uniform distribution; that is, for any $t \in \{1, \dots, n\}$, $U(t) = t/n$. The distance between the distribution of seats for category j and the uniform distribution is $|(F_j(t) - U(t)|$.

The **non-uniformity of roster at position** t under roster R_n is $\sum_{j \in \mathcal{C}} \alpha_j (F_j(t) - U(t))^2$. This measures the weighted distance between the distribution of seats and the uniform distribution at position t.

The **overall non-uniformity of roster** R_n is

$$\sum_{t=1}^{n} \sum_{j \in \mathcal{C}} \alpha_j (F_j(t) - U(t))^2.$$

Overall non-uniformity measures the total weighted distance between the uniform distribution and the distribution of seats across categories. Categories with higher reservation policies have a bigger impact on this measure.

Definition 4. *A roster R_n^* is **weakly more uniform** than roster R_n if overall non-uniformity of roster R_n^* is strictly smaller than overall non-uniformity of roster R_n.*

Definition 5. *A roster R_n^* is **strongly more uniform** than roster R_n if*

1. *for any position t, non-uniformity of roster R_n^* at position t is smaller than non-uniformity of roster R_n at position t, and*
2. *there is at least one position where non-uniformity of roster R_n^* is strictly smaller.*

It follows from the definitions that if R_n^* is strongly more uniform than R_n, then R_n^* is also weakly more uniform than R_n. Our strongly more uniform notion generates a partial ordering on rosters, while the weakly more uniform notion generates a linear ordering on rosters.

A roster R_n^* **achieves overall uniform spread** if there does not exist a roster that is weakly more uniform than roster R_n^*; that is, it minimizes the weighted distance between the distribution of seats and the uniform distribution across categories. We denote $\mathcal{U}_n^w$ by the set of rosters that achieves overall uniform spread.

A roster R_n^* **achieves uniform spread** if there does not exist a roster that is weakly more uniform than roster R_n^*; that is, it minimizes the distance between the distribution of seats and the uniform distribution across categories. We denote $\mathcal{U}_n^s$ by the set of rosters that achieves uniform spread. It follows from the definitions that $\mathcal{U}_n^s$ is subset of $\mathcal{U}_n^w$.

Theorem 2. *The set of rosters that achieves overall uniform spread is the same as the set of rosters that achieves uniform spread.*

Theorem 2 implies that if a roster minimizes overall uniform spread, it will also minimize uniform spread.

2.3.4 Impartial Rosters

Theorem 3. *The set of rosters that achieves overall uniform spread is the same as the set of rosters that minimizes overall partiality.*

Theorem 3 implies that the four notions we introduced to compare the impartiality of rosters lead to the same set of rosters. We denote $\mathcal{I}_n^*$ by such set of rosters; that is, $\mathcal{I}_n^* := \mathcal{I}_n^s = \mathcal{I}_n^w = \mathcal{U}_n^s = \mathcal{U}_n^w$. A roster R_n is **impartial** if $R_n \in \mathcal{I}_n^*$.

Theorem 4. *If a roster R_n^* is impartial, then roster R_n^* is impartial in frequencies.*

Theorem 4 implies that an impartial roster implements the reservation policy without favoring any category in terms of occasion.

Next we detail several desirable properties of impartial rosters. Given categories $\mathcal{C}$ and reservation policy $\boldsymbol{\alpha}$, let s denote the size of the smallest roster possible; that is, s is the lowest common denominator for reservation policy, i.e., $s =_{m \in \mathbb{N}} \alpha_j m \in \mathbb{N}$ for any $j \in \mathcal{C}$.

Definition 6. *A roster R_n is **proportionality-preserving** if $R_n = (R_s^1, \ldots, R_s^k)$, where $k = n/s$ and $R_s^i \in \mathcal{R}_s$ for any $i \in \{1, \ldots, k\}$.*

The definition incorporates the idea that the total number of positions a roster assigns to a category is the same as the proportion given by the reservation policy for up to all viable positions; that is, all categories must get their quantum of reserved positions once every multiple of s positions is filled.

Proposition 1. *A roster R_n^* is impartial if and only if $R_n^* = (R_s^1, \ldots, R_s^k)$ where $R_s^i \in \mathcal{I}_s^*$ for any $i \in \{1, \ldots, k\}$.*

Proposition 1 implies that an impartial roster is proportionality-preserving, and its smaller components are smaller impartial rosters. That is, the set of impartial rosters of size n can be constructed by permuting n/s number of impartial rosters of size s.

Definition 7. *A roster R_n is **balanced** if for any pair of categories i, j with $\alpha_i = \alpha_j$ and for any $t \in \{0, \ldots .n\}$,*

$$|\{s \in R_n^{-1}(i) \mid s \leq t\}| - |\{s \in R_n^{-1}(j) \mid s \leq t\}| \leq 1.$$

The definition simply states that whenever two categories have the same reservation policy, then the number of seats given to them up to any position should not differ by more than one seat.

Proposition 2. *If a roster R_n^* is impartial, then roster R_n^* is balanced.*

2.4 Constructing Impartial Rosters

We show, in Section 2.3, that our four notions to compare the impartiality of rosters lead to the same set of rosters, which is the set of impartial rosters. For practical purposes, knowing the fact that impartial rosters exist is not sufficient. In this section, we provide an iterative algorithm that constructs the set of impartial rosters. We first introduce the notion of representing rosters as a staircase. We then define the Ideal Staircase Generating Algorithm, which takes a reservation policy and number of positions as inputs and creates a set of staircases as an output.

2.4.1 Roster as Staircase

We denote by x_j^t the number of seats given to category j until position t under roster R_n; that is, $x_j^t := |\{s \in R_n^{-1}(c) \mid s \leq t\}|$. We denote by n_j the total number of seats given to category j; that is, $n_j := \alpha_j n$. The **staircase representation of roster** R_n is

$$\mathbf{x^t} = (x_1^t, \ldots, x_j^t, \ldots, x_m^t), \quad \text{for} \quad t \in \{0, \ldots, n\}.$$

These m-dimensional points measure the number of seats a category receives per the reservation policy until position t.

We denote $\mathbf{e_j}$ by the **standard unit vector in the direction of the j-th axis**; that is, vector with j-th component equals 1 and all other components equal 0.

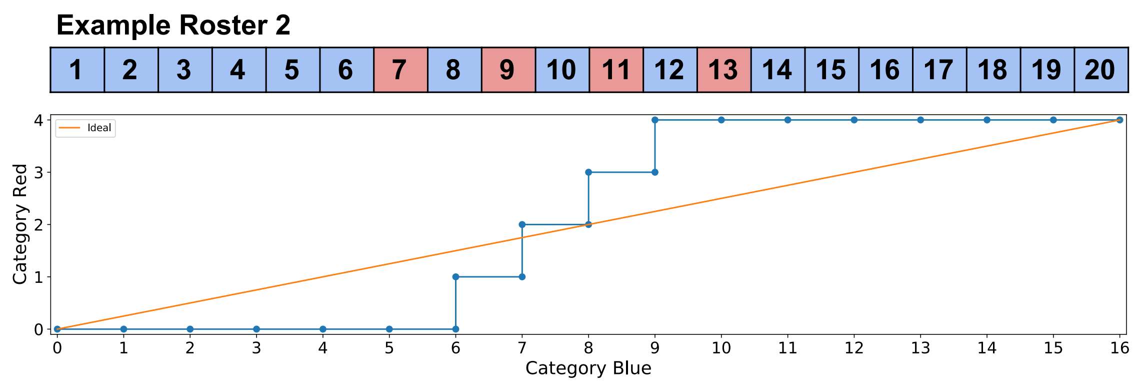
Given two consecutive points $\mathbf{x^{t-1}}$ and $\mathbf{x^t}$, if $\mathbf{x^t} = \mathbf{x^{t-1}} + \mathbf{e_j}$, then we say that staircase **moves to direction j at step** t; that is, $x_j^t = x_j^{t-1} + 1$. Note that for staircase representation of a roster, we can move to only one direction at any step.

In this representation, the **ideal fractional line** is the line connecting origin $(0, \ldots, 0)$ and $(n_1, \ldots, n_j, \ldots, n_m)$. The ideal fractional line would be the ideal seat allocation if only the seats were divisible. This line is described by the vector

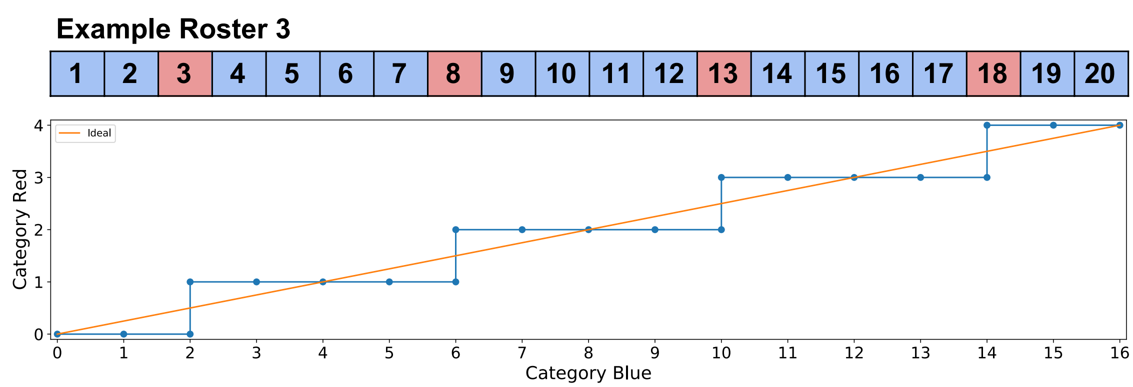
$$\mathbf{u} =< n_1, \ldots, n_j, \ldots, n_m > .$$

For instance, Figure 2.3 illustrates staircase representation for the rosters depicted in Figure 2.1. In Figure 2.3, the ideal fractional line is the line connecting origin and $< 16, 4 >$.

Figure 2.3: STAIRCASE REPRESENTATION



(a) EXAMPLE ROSTER 2



(b) EXAMPLE ROSTER 3

2.4.2 Ideal Staircase

In the staircase representation, the **euclidean distance between point x^{t} and the ideal fractional line**, is defined as the shortest distance between point $\mathbf{x}^{\mathbf{t}}$ and any point on the line. It is the length of the line segment that is perpendicular to the fractional line and passes through the point $\mathbf{x}^{\mathbf{t}}$; that is,

$$d_{stair}(\mathbf{x}^{\mathbf{t}}, \mathbf{u}) = ||\mathbf{x}^{\mathbf{t}} - (\mathbf{x}^{\mathbf{t}} \cdot \mathbf{u})\frac{\mathbf{u}}{||\mathbf{u}||_2}||_2$$

where $||\cdot||_2$ is the Euclidean norm.

The following iterative algorithm has two parts. At each step t, for each staircase, we first find the set of directions the staircase can move such that the Euclidean distance between the next point $\mathbf{x^t}$ and the ideal fractional line is minimum. The staircase next moves to such directions and resulting in a set of staircases.

Ideal Staircase Generating Algorithm

Step 0: For $t \in \{0, \dots, n\}$, initialize $\mathcal{S}_t$ to empty set. Add $\{\mathbf{x^0} = (0, \dots, 0)\}$ to $\mathcal{S}_0$.

Repeat the following for $t \in \{1, \dots, n\}$.

Step t: Repeat the following for any staircase $\{\mathbf{x^0}, \dots, \mathbf{x^{t-1}}\}$ in $\mathcal{S}^{t-1}$.

Part 1: Find the set of directions M that staircase can move such that it minimizes the Euclidean distance between the next point $\mathbf{x^t}$ and the ideal fractional line. That is,

$$M =_j d_{stair}(\mathbf{x^{t-1}} + \mathbf{e_j}, \mathbf{u}).$$

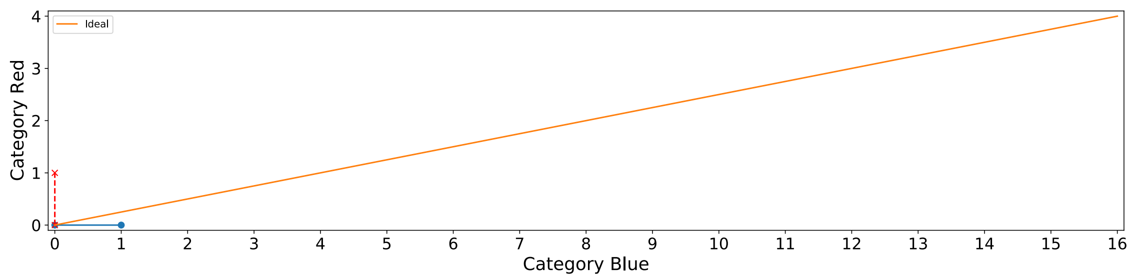
Part 2: For each direction j in M, add $\{\mathbf{x^0}, \dots, \mathbf{x^{t-1}}, \mathbf{x^{t-1}} + \mathbf{e_j}\}$ to $\mathcal{S}_t$.

The algorithm creates a set of staircases in each step. Each staircase in the last step $\mathcal{S}_n$ represents a roster. We call such rosters as **ideal staircase rosters**. We denote by $\mathcal{S}_n^*$ the set of ideal staircase rosters.

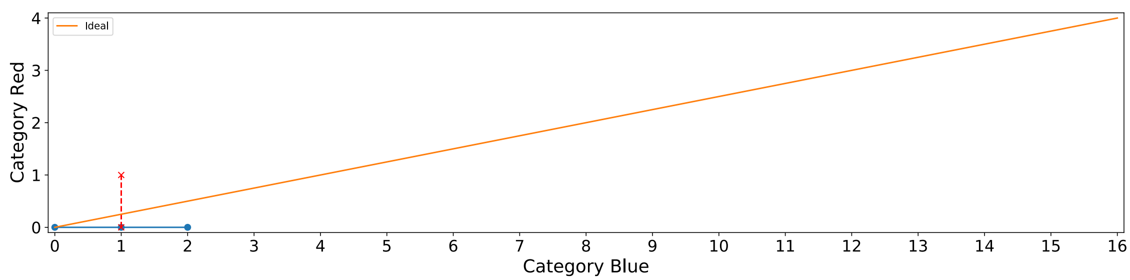
To make the Ideal Staircase Generating Algorithm easier to understand and show the whole procedure that constructs the set of rosters, we show an example.

Consider the roster construction problem Example 1. The number of positions given to the category R and B is 4 and 16, respectively. The category B is represented by the x-axis (1st axis), and the category R is represented by the y-axis (2nd axis). The ideal fractional line is the line connecting the origin and $(4, 16)$. This line is described by the vector $\mathbf{u} =< 4, 16 >$. We start from $\mathcal{S}_0 = \{\{\mathbf{x^0} = (0, \dots, 0)\}\}$. Figure 2.4 illustrates Ste 1, 2, 3, and 4 of the Ideal Staircase Generating Algorithm. At step 1, staircase moves to right (direction 1) since $d_{stair}((1, 0), \mathbf{u}) < d_{stair}((0, 1), \mathbf{u})$. We add $\{(0, 0), (1, 0)\}$ to $\mathcal{S}_1$. At Step 2, staircase moves to right (direction 1) since $d_{stair}((2, 0), \mathbf{u}) < d_{stair}((1, 1), \mathbf{u})$.

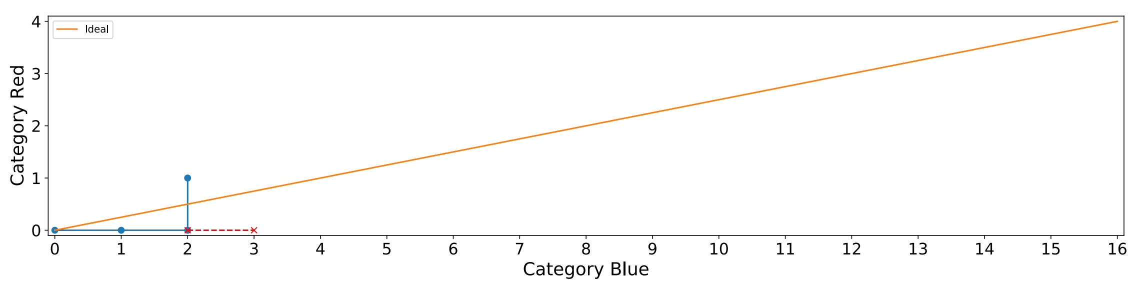
Figure 2.4: CONSTRUCTING IDEAL STAIRCASE



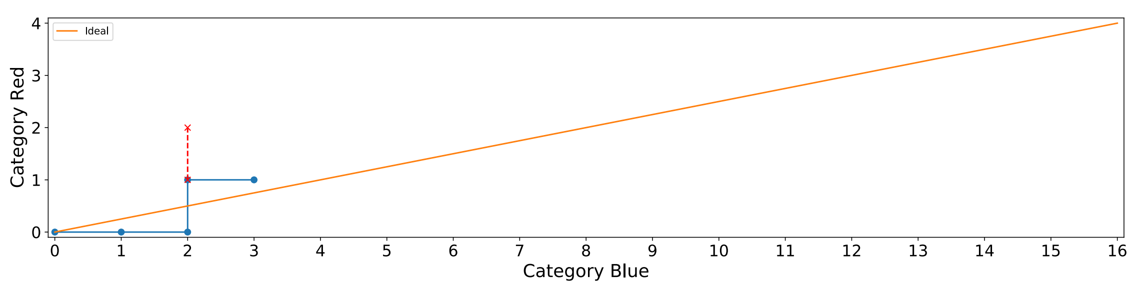
(a) STEP 1



(b) STEP 2



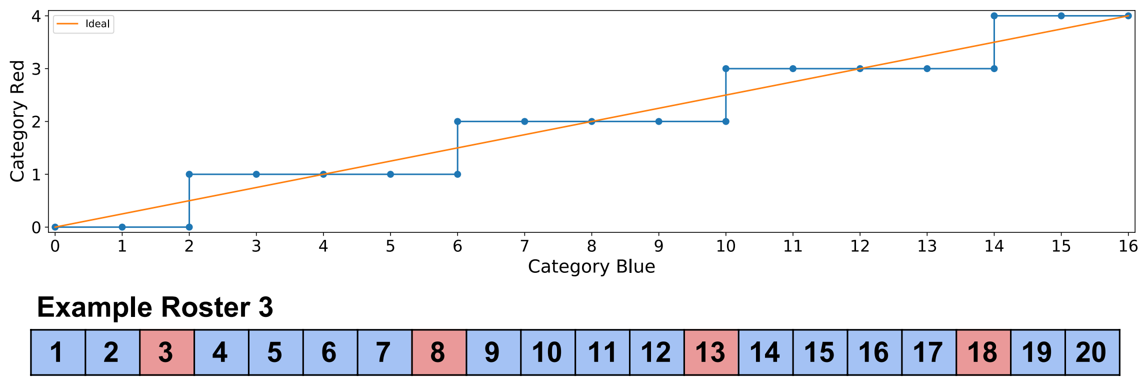
(c) STEP 3



(d) STEP 4

We add $\{(0,0),(1,0),(2,0)\}$ to $\mathcal{S}_2$. At Step 3, staircase moves to up (direction 2) since $d_{stair}((2,1),\mathbf{u}) < d_{stair}((3,0),\mathbf{u})$. We add $\{(0,0),(1,0),(2,0),(2,1)\}$ to $\mathcal{S}_3$. At Step 4, staircase moves to right (direction 1) since $d_{stair}((3,1),\mathbf{u}) < d_{stair}((2,2),\mathbf{u})$. We add $\{(0,0),(1,0),(2,0),(2,1),(3,1)\}$ to $\mathcal{S}_4$. Figure 2.5 illustrates the final output of the algorithm.

Figure 2.5: IDEAL STAIRCASE



Theorem 5. *A roster R_n^* is impartial if and only if roster R_n^* is an ideal staircase roster.*

Theorem 5 implies that the set of impartial rosters can be computable, and there exists a fast algorithm that generates them in linear time.

2.5 The Indian Roster

$\Lambda^{\texttt{IN}} = (\{UR, OBC, SC, EWS, ST\}, \boldsymbol{\alpha} = [0.405, 0.27, 0.15, 0.10, 0.075], 200)$ is the Indian roster construction problem. It consists of five categories of seats – Unreserved (UR), Other Backward Classes (OBC), Scheduled Castes (SC), Economically Weaker Sections (EWS), and Scheduled Tribes (ST). The reservation policy dictates the division of the 200 seats in a roster among the five categories – 81 *UR*, 54 *OBC*, 30 *SC*, 20 *EWS*, and 15 *ST*. However, the positions each category is assigned in a roster is left up to the designer.

The designers in India's case belong to the executive branch of the government. They detail a method for making rosters in the *Annexure I to Office Memorandum No. 36012/2/96-Estt(Res)* dated July 2, 1997 as following:[2]

[2]For *Annexure I to Office Memorandum No. 36012/2/96-Estt(Res)* visit `https://documents.`

> `The method for making roster is to multiply each post by the prescribed percentages of reservations for the different reserved categories. The point at which the multiple for a community obtains a complete number or oversteps the number is to be reserved for that community -- while taking care to space out the different reserved categories evenly. Thus, at point number 15, in the roster at Annexure-II both OBC and SC get entitled. However, since the earlier reserved point has gone to OBC, point no. 15 has been reserved for SC and point no. 16 for OBC.`

In all Indian government institutions, rosters are constructed and maintained as per the above mentioned method. Including the most recent revision detailed in *Office memorandum No.36039/1/2019-Estt (Res)* dated January 31, 2019 issued by the Department of Personnel and Training (Ministry of Personnel, Public Grievances, and Pensions, Government of India).[3] It is hard to improve upon the transparency rosters provide in the nationwide implementation of the reservation policy. Moreover, they do achieve the goal of reserving seats in a manner such that over a sufficiently long period of time the prescribed percentage of reserved is met by each institution. Yet the choice of the roster construction method can and has been scrutinized for the delay associated with the arrival of reserved seats.

Gupta and Thorat (2019) and Gupta (2018) criticize the current roster method for delaying reserved seats, causing a meager representation of some reserved category candidates. They write:

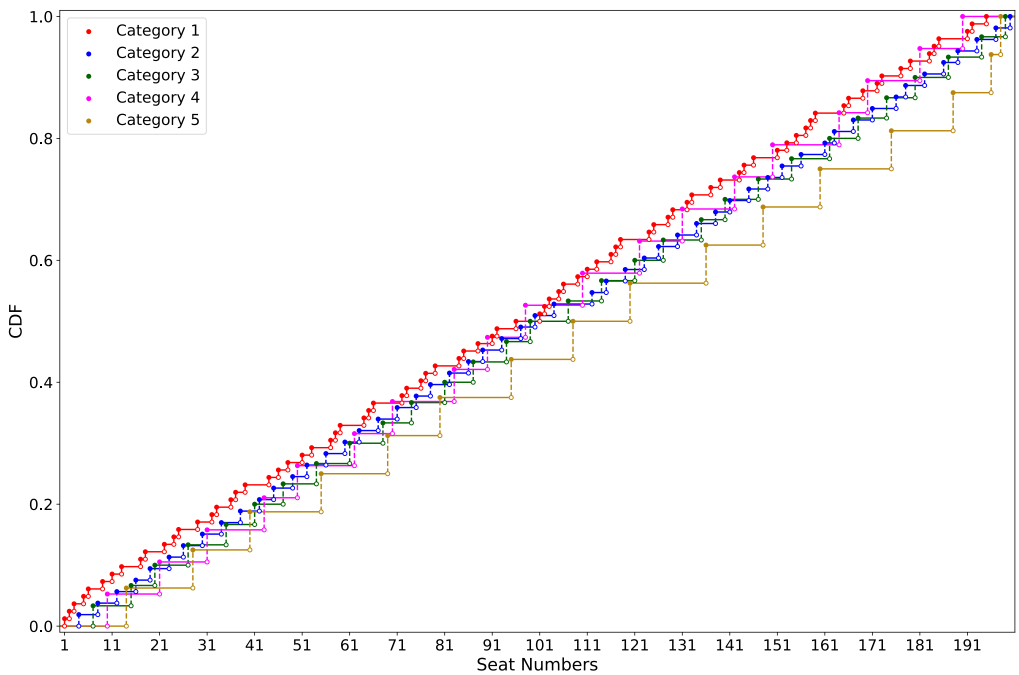
> `A mathematical juggling has been used by policymakers to reduce the constitutionally mandated reservation for the deprived sections.`

The complaint becomes apparent on simple eyeballing of the top panel of Figure 2.6 which plots the cumulative seat distribution under the current Indian roster. The figure shows that the current roster favors categories with a larger proportion of seats relative to

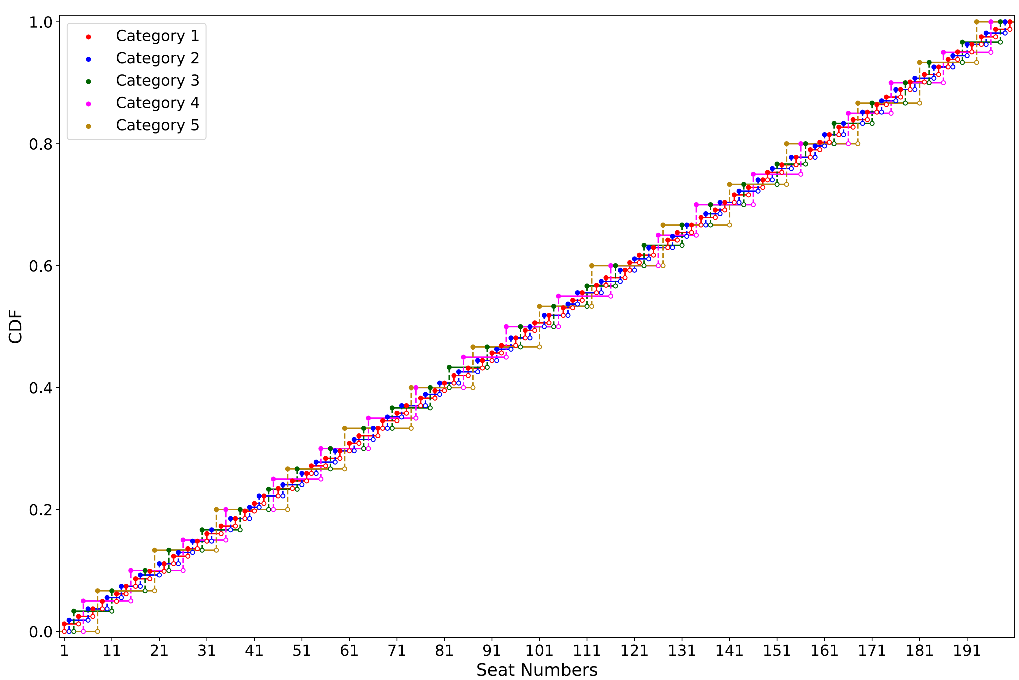
`doptcirculars.nic.in/D2/D02adm/OM%20dated%202%207%2097BsMyq.pdf`, last accessed on 27 July 2022..

[3]For *Office memorandum No.36039/1/2019-Estt (Res)* visit `https://dopt.gov.in/sites/default/files/ewsf28fT.PDF`, last accessed on 27 July 2022..

Figure 2.6: DISTRIBUTION OF SEATS



(a) CDF for the Indian Roster



(b) CDF for Impartial Roster of $\Lambda^{\texttt{IN}}$

the smaller ones. For instance, compare category 1 (UR), which has the largest proportion of seats ($\alpha_{UR} = 0.405$) with category 5 (ST), which has the smallest proportion of seats

($\alpha_{ST} = 0.075$). Throughout the roster we can see that, $F_1(t) > F_5(t)$. That is, category UR is *favored relative* to category ST at all positions $t < 200$. Similar comparisons can be made between all pairs of categories.

Such complaints do not arise under our impartial roster. The bottom panel of Figure 2.6 plots the cumulative seat distribution for an impartial roster of $\Lambda^{\texttt{IN}}$. In an impartial roster, seats are spread "as even as possible" without favoring any category over the other at any point in the roster.

2.6 Discussion with respect to the Related Literature

A considerable number of recent studies have documented unnoticed issues in implementation of nation-wide affirmative action policies, and have offered practical alternatives for better implementation of such policies (see Abdulkadiroğlu and Sönmez (2003), Kojima (2012), Hafalir et al. (2013), Ehlers et al. (2014), Echenique and Yenmez (2015), Aygün and Turhan (2017), Dur et al. (2019), Aygun and Bó (2021), Sönmez and Yenmez (2022b); Sönmez and Yenmez (2019) among others). Ours is another paper in this class. While the focus of the contemporary market design literature has been the design and analysis of assignment mechanisms given reserved seats and quotas, our paper (also Kurino and Hatakeyama (2022) and Evren and Khanna (2022)) looks at another side of affirmative action schemes: how many seats to reserve?

The idea of rosters is similar to that of precedence orders according to which institutions prioritize individual slots above others (as in Kominers and Sönmez (2016), and Dur et al. (2018)), but in a world where seats arrive sequentially, over time in small numbers (Abizada and Bó (2021)). A serious limitation of using rosters is that it is not possible to differentiate between vertical reservations, horizontal reservations, or any form of reservations (see Sönmez and Yenmez (2022b)). For instance, a roster cannot allocate all positions of a beneficiary group at the very end just as it is done in the static implementation of vertical reservations in India. On top of that, the treatment of a horizontal reservation is not any different than that of a vertical reservation in a roster.

To accommodate a richer variety of forms of reservations – vertical reservations, horizontal reservations, horizontal reservations within vertical reservations – a deviation from rosters is required. Dynamic implementation of the reservation policy is recommended, that allows reserving a seat early in time, but also allows that particular seat to be considered unreserved at a later point depending on the history of seat allocation (see Pathak et al. (2020) and Aygün and Turhan (2020)). The advantage of this method over rosters is obvious; the mandated vertical and horizontal reservations can be implemented at any point in time. However, it is hard to quash the use of rosters. Both the legislators and their electorates greatly value the transparency rosters provide in the implementation of the reservation policy.

Distributing indivisible objects among a group of claimants in proportion to their claims is famously known as the *apportionment problem* (and is the center point of the seminal work of Young (1995)). A roster construction problem can be framed as a sequence of apportionment problems. In fact, all symmetric, weakly proportional, and house monotone apportionment methods discussed in Balinski and Young (2010) can be used in constructing admissible rosters. Moreover, the ideal staircase defined in Section 2.4.2 belongs to the class of rank-index apportionment methods by definition and is shown to resemble the outcomes of the well-known Webster's method of apportionment in Section 2.7.

2.7 Appendix

In order to prove Theorem 1, Theorem 2, Theorem 3, we first present the connections between the four notions to compare the impartiality of rosters. We prove a series of lemmas and finally show that all notions lead to the same set of rosters.

Let x_j^t be the number of seats given to category j until point t. Note that $\sum_j x_j^t = t$

for $t \in \{1, \ldots, n\}$. Let n_j be the total number of seats in the roster for category j, that is, $n_j = x_j^n = \alpha_j n$.

With these definitions, the distribution of seats for category j at point t is

$$F_j(t) = \frac{x_j^t}{n_j}$$

and the location at the staircase at point t is

$$\mathbf{x^t} = (x_1^t, \ldots, x_j^t, \ldots, x_j^t).$$

We denote by $d_{partial}(\mathbf{x^t})$ **weighted distance between distributions of accross seats at point $\mathbf{x^t}$**; that is,

$$d_{partial}(\mathbf{x^t}) = \sqrt{\sum_{i \neq j} (\alpha_i \alpha_j)^2 (F_i(t) - F_j(t))^2}.$$

Lemma 1. *Given a roster R_n and its stair case representation $\mathbf{x^t}$ for $t \in \{0, \ldots, n\}$,*

$$d_{stair}(\mathbf{x^t}, \mathbf{u}) = K d_{partial}(\mathbf{x^t})$$

where $K = \frac{\sqrt{n}}{\sqrt{\sum_j \alpha_j^2}}$.

Proof. In the staircase, the ideal fractional line is the line connecting the origin and $(n_1, \ldots, n_m)$. The line is represented by the normalized vector

$$\mathbf{u} = < \frac{\alpha_1}{A^{1/2}}, \ldots, \frac{\alpha_j}{A^{1/2}}, \ldots, \frac{\alpha_m}{A^{1/2}} >,$$

where $A = \sum_j \alpha_j^2$. With this definition, the Euclidean norm for $\mathbf{u}$ is $||\mathbf{u}||_2 = 1$.

The euclidean distance between $\mathbf{x^t}$ and the ideal fractional line is defined as the shortest distance between point $\mathbf{x^t}$ and any point on the line. It is the length of the line segment that is perpendicular to the fractional line and passes through the point $\mathbf{x^t}$:

$$d_{stair}(\mathbf{x^t}, \mathbf{u}) = ||\mathbf{x^t} - (\mathbf{x^t} \cdot \mathbf{u})\frac{\mathbf{u}}{||\mathbf{u}||_2}||_2$$

where $|| \cdot ||_2$ is the Euclidean norm.

Note that

$$(\mathbf{x^t} \cdot \mathbf{u}) = \sum_j \frac{x_j^t \alpha_j}{A^{1/2}}$$

and

$$\mathbf{x^t} - (\mathbf{x^t} \cdot \mathbf{u})\frac{\mathbf{u}}{||\mathbf{u}||_2} =< x_1^t - \alpha_1 \frac{B}{A}, \dots , x_j^t - \alpha_j \frac{B}{A}, \dots , x_m^t - \alpha_m \frac{B}{A} >,$$

where $B = \sum_j x_j^t \alpha_j$. The distance between $\mathbf{x^t}$ and $\mathbf{u}$, $d_{stair}(\mathbf{x^t}, \mathbf{u})$, becomes

$$d_{stair}(\mathbf{x^t}, \mathbf{u}) = \left(\sum_j (\frac{x_j^t A - \alpha_j B}{A})^2 \right)^{1/2} = \frac{1}{A^{1/2}} \left(\frac{1}{A} \sum_j (x_j^t A - \alpha_j B)^2 \right)^{1/2}.$$

Note that,

$$
\begin{aligned}
\frac{1}{A}\sum_j (x_j^t A - \alpha_j B)^2 &= \frac{1}{A}\left(A^2 \sum_j {x_j^t}^2 - 2AB\sum_j x_j^t \alpha_j + B^2 \sum_j \alpha_j^2\right) \\
&= \frac{1}{A}\left(A^2 \sum_j {x_j^t}^2 - 2AB^2 + B^2 A\right) \\
&= \frac{1}{A}\left(A^2 \sum_j {x_j^t}^2 - AB^2\right) \\
&= A\sum_j {x_j^t}^2 - B^2 \\
&= \sum_j \alpha_j^2 \sum_j {x_j^t}^2 - (\sum_j x_j^t \alpha_j)^2 \\
&= \sum_i \alpha_i^2 \sum_i (x_i^t)^2 - (\sum_i (x_i^t \alpha_i)^2 + 2\sum_{i\neq j} \alpha_i \alpha_j x_i^t x_j^t) \\
&= \sum_{i\neq j} (\alpha_j x_i^t - \alpha_i x_j^t)^2 \\
&= \sum_{i\neq j} \alpha_i^2 \alpha_j^2 (\frac{x_i^t}{\alpha_i} - \frac{x_j^t}{\alpha_j})^2 \\
&= n\sum_{i\neq j} \alpha_i^2 \alpha_j^2 (\frac{x_i^t}{n_i} - \frac{x_j^t}{n_j})^2 \\
&= n\sum_{i\neq j} \alpha_i^2 \alpha_j^2 (F_i(t) - F_j(t))^2 \\
&= n(d_{partial}(\mathbf{x^t}))^2.
\end{aligned}
$$

■

Lemma 2. *The following integer programming problems have the same set of solutions*

1.

$$\min_{x^t} d_{stair}(\mathbf{x^t}, \mathbf{u}) \textit{ s.t. } \sum_j x_j^t = t \textit{ and } \mathbf{x^t} \geq 0 \textit{ integer}$$

2.

$$\min_{x^t} d_{partial}(\mathbf{x^t}) \textit{ s.t. } \sum_j x_j^t = t \textit{ and } \mathbf{x^t} \geq 0 \textit{ integer}$$

Proof. Note that in Lemma 1, we show that

$$d_{stair}(\mathbf{x^t}, \mathbf{u}) = K d_{partial}(t)$$

Therefore, the objective functions are the monotonic transformation of each other; that is, they preserve the order of comparing $\mathbf{x^t}$ and will have the same set of solutions.

■

We denote by $d_{uni}(\mathbf{x^t})$ **weighted distance between the distribution of seats and the uniform distribution at point** $\mathbf{x^t}$; that is,

$$d_{uni}(\mathbf{x^t}) = \sum_j \alpha_j (F_j(t) - U(t))^2.$$

Note that,

$$\begin{aligned}
d_{uni}(\mathbf{x^t}) &= \sum_j \alpha_j (F_j(t) - U(t))^2 \\
&= \sum_j \alpha_j (\frac{x_j^t}{\alpha_j} - \frac{t}{n})^2 \\
&= \frac{1}{n^2} \sum_j \frac{{x_j^t}^2}{\alpha_j} - \frac{1}{n^2} \sum_j 2x_j^t t + \frac{1}{n^2} \sum_j \alpha_j t^2 \\
&= \frac{1}{n^2} \sum_j \frac{{x_j^t}^2}{\alpha_j} - \frac{2t^2}{n^2} + \frac{t^2}{n^2} \\
&= \frac{1}{n^2} \sum_j \frac{{x_j^t}^2}{\alpha_j} - \frac{t^2}{n^2}.
\end{aligned}$$

Therefore, minimizing $d_{uni}(\mathbf{x^t})$ is equivalent to minimizing $\sum_j \frac{{x_j^t}^2}{\alpha_j}$.

Lemma 3. *For any* $\mathbf{x^t} = (x_1^t, \dots, x_j^t, \dots, x_m^t)$*, for any pair of* i, j *with* $x_i^t > 0$*,*

$$d_{uni}(\mathbf{x^t}) \le d_{uni}(\mathbf{x^t} + e_j - e_i) \iff \frac{x_i^t - 0.5}{\alpha_i} \le \frac{x_j^t + 0.5}{\alpha_j}.$$

Proof. Using the fact that $d_{uni}(\mathbf{x^t}) = \frac{1}{n^2}\sum_j \frac{{x_j^t}^2}{\alpha_j} - \frac{t^2}{n^2}$,

$$\begin{aligned} d_{uni}(\mathbf{x^t}) \le d_{uni}(\mathbf{x^t} + e_j - e_i) \quad &\iff \quad \frac{{x_i^t}^2}{\alpha_i} + \frac{{x_j^t}^2}{\alpha_j} \le \frac{(x_i^t - 1)^2}{\alpha_i} + \frac{(x_j^t + 1)^2}{\alpha_j} \\ &\iff \quad \frac{{x_i^t}^2}{\alpha_i} - \frac{(x_i^t - 1)^2}{\alpha_i} \le \frac{(x_j^t + 1)^2}{\alpha_j} - \frac{{x_j^t}^2}{\alpha_j} \\ &\iff \quad \frac{x_i^t - 0.5}{\alpha_i} \le \frac{x_j^t + 0.5}{\alpha_j}. \end{aligned}$$

■

Lemma 4. *The following sets are equivalent*

1.

$$_{\mathbf{x^t}} d_{uni}(\mathbf{x^t}) \textit{ s.t. } \sum_j x_j^t = t \textit{ and } \mathbf{x^t} \ge 0 \textit{ integer}$$

2.

$$\{\mathbf{x^t} \mid \textit{for any } i,j \textit{ with } x_i^t > 0, \frac{x_i^t - 0.5}{\alpha_i} \le \frac{x_j^t + 0.5}{\alpha_j}; \sum_j x_j^t = t \textit{ and } \mathbf{x^t} \ge 0 \textit{ integer } \}$$

Proof. Lemma 3 implies that if $\mathbf{x^t}$ minimizes $d_{uni}(\mathbf{x^t})$ then the following inequalities hold.

$$\frac{x_i^t - 0.5}{\alpha_i} \le \frac{x_j^t + 0.5}{\alpha_j} \text{ for any } i,j \text{ with } x_i^t > 0$$

Therefore, if $\mathbf{x^t}$ is in the former set, then it is also in the latter set.

Suppose $\mathbf{y^t}$ is in the latter set but not in the former; that is,

$$\frac{y_i^t - 0.5}{\alpha_i} \le \frac{y_j^t + 0.5}{\alpha_j} \text{ for any } i,j \text{ with } y_i^t > 0 \text{ .}$$

We denote by $H = \{j | x_j^t > y_j^t\}$ the set of categories in $\mathbf{x^t}$ that has more number of seats. We denote by $L = \{j | x_j^t < y_j^t\}$ the set of categories in $\mathbf{x^t}$ that has less number of seats. We denote by $h_j := x_j^t - y_j^t$ for $j \in H$. We denote by $l_j := y_j^t - x_j^t$ for $j \in L$. Since $\sum_j y_j^t = \sum_j x_j^t = t$, we have $\sum_{j \in H} h_j = \sum_{j \in L} l_j > 0$.

Using the inequalities for $\mathbf{x^t}$ and $\mathbf{y^t}$ for $i \in L$ and $j \in H$, we have

$$\frac{2y_i^t - l_j}{\alpha_i} \leq \frac{2y_j^t + h_j}{\alpha_j} \text{ for any } i \in L \text{ and } j \in H\ .$$

If we calculate the summation of such inequities, we find that

$$\sum_{j \in L} \frac{l_j(2y_j^t - l_j)}{\alpha_j} \leq \sum_{j \in H} \frac{h_j(2y_j^t + h_j)}{\alpha_j}.$$

Note that,

$$\begin{aligned}
\sum_j \frac{{x_j^t}^2}{\alpha_j} - \sum_j \frac{{y_j^t}^2}{\alpha_j} &= \sum_j \left(\frac{{x_j^t}^2}{\alpha_j} - \frac{{y_j^t}^2}{\alpha_j}\right) \\
&= \sum_{j \in H} \left(\frac{{x_j^t}^2}{\alpha_j} - \frac{{y_j^t}^2}{\alpha_j}\right) - \sum_{j \in L} \left(\frac{{y_j^t}^2}{\alpha_j} - \frac{{x_j^t}^2}{\alpha_j}\right) \\
&= \sum_{j \in H} \frac{h_j(2y_j^t + h_j)}{\alpha_j} - \sum_{j \in L} \frac{l_j(2y_j^t - l_j)}{\alpha_j} \\
&\geq 0.
\end{aligned}$$

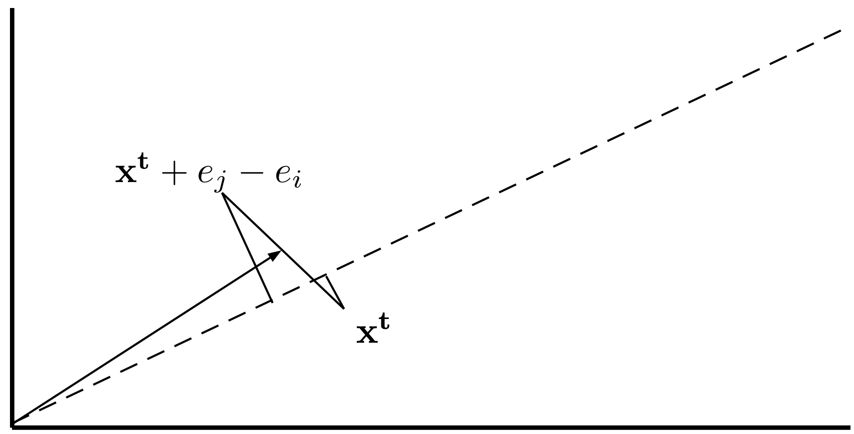
This contradicts the assumption that $\mathbf{y^t}$ does not belong to the first set; that is, if a $\mathbf{y^t}$ is in the second set then $\mathbf{y^t}$ must be in the first set. ∎

Lemma 5. *For any $\mathbf{x^t} = (x_1^t, \ldots, x_j^t, \ldots, x_m^t)$, for any pair of i, j with $x_i^t > 0$,*

$$d_{stair}(\mathbf{x^t}, \mathbf{u}) \leq d_{stair}(\mathbf{x^t} + e_j - e_i, \mathbf{u}) \iff \frac{x_i^t - 0.5}{\alpha_i} \leq \frac{x_j^t + 0.5}{\alpha_j}.$$

Proof. In the staircase representation, the euclidean distance between point $\mathbf{x^t}$ and the ideal fractional line, is the shortest distance between point $\mathbf{x^t}$ and any point on the line. It is the length of the line segment that is perpendicular to the fractional line and passes through the

Figure 2.7: STAIRCASE ILLUSTRATION



point $\mathbf{x^t}$. Since only two dimension changes in m-dimensional point $\mathbf{x^t}$, we can reduce the staircase representation those two dimensions. If distance from $\mathbf{x^t}$ is smaller than distance from $\mathbf{x^t} + e_j - e_i$, then the middle point of these two points, $\mathbf{x^t} + 0.5e_j - 0.5e_i$ lays on the same side as the point $\mathbf{x^t} + e_j - e_i$. That is,

$$\frac{x_j^t + 0.5}{x_i^t - 0.5} \geq \frac{\alpha_j}{\alpha_i}.$$

■

Lemma 6. *The following sets are equivalent*

1.

$${}_{\mathbf{x^t}} d_{stair}(\mathbf{x^t}, \mathbf{u}) \textit{ s.t. } \sum_j x_j^t = t \textit{ and } \mathbf{x^t} \geq 0 \textit{ integer}$$

2.

$$\{\mathbf{x^t} \mid \textit{for any } i, j \textit{ with } x_i^t > 0, \frac{x_i^t - 0.5}{\alpha_i} \leq \frac{x_j^t + 0.5}{\alpha_j}; \sum_j x_j^t = t \textit{ and } \mathbf{x^t} \geq 0 \textit{ integer } \}$$

Proof. Lemma 5 implies that if $\mathbf{x^t}$ minimizes $d_{stair}(\mathbf{x^t}, \mathbf{u})$ then the following inequalities hold.

$$\frac{x_i^t - 0.5}{\alpha_i} \leq \frac{x_j^t + 0.5}{\alpha_j} \text{ for any } i, j \text{ with } x_i^t > 0$$

Therefore, if $\mathbf{x^t}$ is in the former set, then it is also in the latter set.

Suppose $\mathbf{y^t}$ is in the latter set but not in the former; that is,

$$\frac{y_i^t - 0.5}{\alpha_i} \leq \frac{y_j^t + 0.5}{\alpha_j} \text{ for any } i, j \text{ with } y_i^t > 0 \ .$$

We denote by $H = \{j|x_j^t > y_j^t\}$ the set of categories in $\mathbf{x^t}$ that has more number of seats. We denote by $L = \{j|x_j^t < y_j^t\}$ the set of categories in $\mathbf{x^t}$ that has less number of seats. We denote by $h_j := x_j^t - y_j^t$ for $j \in H$. We denote by $l_j := y_j^t - x_j^t$ for $j \in L$. Since $\sum_j y_j^t = \sum_j x_j^t = t$, we have $\sum_{j \in H} h_j = \sum_{j \in L} l_j > 0$.

Using the inequalities for $\mathbf{x^t}$ and $\mathbf{y^t}$ for $i \in L$ and $j \in H$, we have

$$\frac{2y_i^t - l_j}{\alpha_i} \leq \frac{2y_j^t + h_j}{\alpha_j} \text{ for any } i \in L \text{ and } j \in H \ .$$

If we calculate the summation of such inequities, we find that

$$\sum_{j \in L} \frac{l_j(2y_j^t - l_j)}{\alpha_j} \leq \sum_{j \in H} \frac{h_j(2y_j^t + h_j)}{\alpha_j}.$$

Note that,

$$\begin{aligned}
\sum_j \frac{{x_j^t}^2}{\alpha_j} - \sum_j \frac{{y_j^t}^2}{\alpha_j} &= \sum_j \left(\frac{{x_j^t}^2}{\alpha_j} - \frac{{y_j^t}^2}{\alpha_j}\right) \\
&= \sum_{j \in H} \left(\frac{{x_j^t}^2}{\alpha_j} - \frac{{y_j^t}^2}{\alpha_j}\right) - \sum_{j \in L} \left(\frac{{y_j^t}^2}{\alpha_j} - \frac{{x_j^t}^2}{\alpha_j}\right) \\
&= \sum_{j \in H} \frac{h_j(2y_j^t + h_j)}{\alpha_j} - \sum_{j \in L} \frac{l_j(2y_j^t - l_j)}{\alpha_j} \\
&\geq 0.
\end{aligned}$$

This contradicts the assumption that $\mathbf{y^t}$ does not belong to the first set; that is, if a $\mathbf{y^t}$ is in the second set then $\mathbf{y^t}$ must be in the first set. ■

Lemma 7. *The following integer programming problems have the same set of solutions*

1.

$$\min_{\mathbf{x^t}} d_{stair}(\mathbf{x^t}, \mathbf{u}) \textit{ s.t. } \sum_j x_j^t = t \textit{ and } \mathbf{x^t} \geq 0 \textit{ integer}$$

2.

$$\min_{\mathbf{x^t}} d_{uni}(\mathbf{x^t}) \textit{ s.t. } \sum_j x_j^t = t \textit{ and } \mathbf{x^t} \geq 0 \textit{ integer}$$

Proof. It follows from Lemma 4 and Lemma 6.

■

Let $\mathcal{Z}^t := \{\mathbf{x^t} \mid$ for any i, j with $x_i^t > 0$, $\frac{x_i^t - 0.5}{\alpha_i} \leq \frac{x_j^t + 0.5}{\alpha_j}$; $\sum_j x_j^t = t$ and $\mathbf{x^t} \geq 0$ integer $\}$. We denote $\mathbf{e_k}$ by the **standard unit vector in the direction of the k-th axis**; that is, vector with k-th component equals 1 and all other components equal 0.

Lemma 8. *If $\mathbf{x^t} \in \mathcal{Z}^t$ and $\mathbf{y^{t-1}} \in \mathcal{Z}^{t-1}$, there exist a category k such that $\mathbf{x^t} = \mathbf{y^{t-1}} + e_k$.*

Proof. Contrary, suppose there does not exist such category k. Since $\sum_l x_l^t = t$ and $\sum_l y_l^t = t - 1$, there must be categories i and j such that $y_j^{t-1} \geq x_j^t + 1$ and $x_i^t \geq y_i^{t-1} + 2$.

Note that,

$$\frac{(y_j^{t-1} - 1) + 0.5}{\alpha_j} \geq \frac{x_j^t + 0.5}{\alpha_j} \geq \frac{x_i^t - 0.5}{\alpha_i} \geq \frac{(y_i^{t-1} + 2) - 0.5}{\alpha_i}$$

Here, the first inequality is due to $y_j^{t-1} \geq x_j^t + 1$. The second inequality is due to $\mathbf{x^t} \in \mathcal{Z}^t$ and $x_i^t > 0$. The last inequality due to $x_i^t \geq y_i^{t-1} + 2$. This imply that $\frac{y_j^{t-1} - 0.5}{\alpha_j} \geq \frac{y_i^{t-1} + 1.5}{\alpha_i} > \frac{y_i^{t-1} + 1}{\alpha_i}$.

This contradicts the assumption that $\mathbf{y^{t-1}} \in \mathcal{Z}^{t-1}$. ■

Proof of Theorem 1

Proof. The overall partiality of roster R_n is

$$\sum_{t=1}^{n}\sum_{i\neq j}(\alpha_i\alpha_j)^2(F_i(t)-F_j(t))^2=\sum_{t=1}^{n}(d_{partial}(\mathbf{x^t}))^2$$

where $\mathbf{x^t}$ for $t \in \{1,\dots,n\}$ is the staircase representation of R_n. Here, $\mathbf{x^t}-\mathbf{x^{t-1}}=e_k$ for some $k \in \mathcal{C}$.

Lemma 2, Lemma 6, and Lemma 8 together imply that if roster R_n minimizes the overall partiality, then for the staircase representation of R_n, i.e., $\mathbf{x^t}$ for $t \in \{1,\dots,n\}$, each $\mathbf{x^t}$ is a solution for integer programming problem

$$\min_{\mathbf{x^t}} d_{partial}(\mathbf{x^t}) \text{ s.t. } \sum_j x_j^t = t \text{ and } \mathbf{x^t} \geq 0 \text{ integer.}$$

Therefore, if a roster minimizes overall partiality, it will also minimize partiality.

■

Proof of Theorem 2

Proof. The overall non-uniformity of roster R_n is

$$\sum_{t=1}^{n}\sum_{j\in\mathcal{C}}\alpha_j(F_j(t)-U(t))^2=\sum_{t=1}^{n}d_{uni}(\mathbf{x^t}).$$

where $\mathbf{x^t}$ for $t \in \{1,\dots,n\}$ is the staircase representation of R_n. Here, $\mathbf{x^t}-\mathbf{x^{t-1}}=e_k$ for some $k \in \mathcal{C}$.

Lemma 4, Lemma 7, and Lemma 8 together imply that if roster R_n minimizes the non-uniformity, then for the staircase representation of R_n, i.e., $\mathbf{x^t}$ for $t \in \{1,\dots,n\}$, each $\mathbf{x^t}$

is a solution for integer programming problem

$$\min_{\mathbf{x^t}} d_{uni}(\mathbf{x^t}) \text{ s.t. } \sum_j x_j^t = t \text{ and } \mathbf{x^t} \geq 0 \text{ integer.}$$

Therefore, if a roster minimizes overall uniform spread, it will also minimize uniform spread.

■

Proof of Theorem 3

Proof. Proof of Theorem 1 and Proof of Theorem 2 imply that

1. $\mathcal{I}_n^s = \mathcal{I}_n^w$
2. $\mathcal{U}_n^s = \mathcal{U}_n^w$

where $\mathcal{I}_n^w$ is the set of rosters that minimizes overall partiality, $\mathcal{I}_n^s$ is the set of rosters that minimizes partiality, $\mathcal{U}_n^w$ is the set of rosters that achieves overall uniform spread, and $\mathcal{U}_n^s$ is the set of rosters that achieves uniform spread.

It follows from Lemma 2 and Lemma 7, that they are the same set of rosters. ■

2.8 Tables and Figures

Figure 2.8: MODEL ROSTER OF RESERVATION WITH REFERENCE TO POSTS

FOR DIRECT RECRUITMENT

Model Roster of Reservation with reference to posts for Direct recruitment on All India Basis by Open Competition

Sl. No. of Post	Share of Entitlement				Category for which the posts should be earmarked
	SC @15%	ST @7.5%	OBC @27%	EWS @10%	
1	0.15	0.08	0.27	0.10	UR
2	0.30	0.15	0.54	0.20	UR
3	0.45	0.23	0.81	0.30	UR
4	0.60	0.30	1.08	0.40	OBC-1
5	0.75	0.38	1.35	0.50	UR
6	0.90	0.45	1.62	0.60	UR
7	1.05	0.53	1.89	0.70	SC-1
8	1.20	0.60	2.16	0.80	OBC-2
9	1.35	0.68	2.43	0.90	UR
10	1.50	0.75	2.70	1.00	**EWS-1**
11	1.65	0.83	2.97	1.10	UR
12	1.80	0.90	3.24	1.20	OBC-3
13	1.95	0.98	3.51	1.30	UR
14	2.10	1.05	3.78	1.40	ST-1
15	2.25	1.13	4.05	1.50	SC-2
16	2.40	1.20	4.32	1.60	OBC-4
17	2.55	1.28	4.59	1.70	UR
18	2.70	1.35	4.86	1.80	UR
19	2.85	1.43	5.13	1.90	OBC-5
20	3.00	1.50	5.40	2.00	SC-3
21	3.15	1.58	5.67	2.10	**EWS-2**
22	3.30	1.65	5.94	2.20	UR
23	3.45	1.73	6.21	2.30	OBC-6
24	3.60	1.80	6.48	2.40	UR
25	3.75	1.88	6.75	2.50	UR
26	3.90	1.95	7.02	2.60	OBC-7
27	4.05	2.03	7.29	2.70	SC-4
28	4.20	2.10	7.56	2.80	ST-2
29	4.35	2.18	7.83	2.90	UR
30	4.50	2.25	8.10	3.00	OBC-8
31	4.65	2.33	8.37	3.10	**EWS-3**

Source: `https://dopt.gov.in/sites/default/files/ewsf28fT.PDF`[4]

[4] Last accessed on 27 July 2022.

Figure 2.9: ROSTER FOR DIRECT RECRUITMENT FOR CADRE STRENGTH UP TO 13 POSTS

FOR DIRECT RECRUITMENT

Roster for Direct Recruitment otherwise than through Open Competition for cadre strength upto 13 posts

Cadre Strength	Initial Recruit-ment	Replacement No.												
		1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th	11th	12th	13th
1	UR	UR	UR	OBC	UR	UR	SC	OBC	UR	**EWS**	UR	OBC	SC	ST
2	UR	UR	OBC	UR	UR	SC	OBC	UR	**EWS**	UR	OBC	SC	ST	
3	UR	OBC	UR	UR	SC	OBC	UR	**EWS**	UR	OBC	SC	ST		
4	OBC	UR	UR	SC	OBC	UR	**EWS**	UR	OBC	SC	ST			
5	UR	UR	SC	OBC	UR	**EWS**	UR	OBC	SC	ST				
6	UR	SC	OBC	UR	**EWS**	UR	OBC	SC	ST					
7	SC	OBC	UR	**EWS**	UR	OBC	SC	ST						
8	OBC	UR	**EWS**	UR	OBC	SC	ST							
9	UR	**EWS**	UR	OBC	SC	ST								
10	**EWS**	UR	OBC	SC	ST									
11	UR	OBC	SC	ST										
12	OBC	SC	ST											
13	SC	ST												

Source: `https://dopt.gov.in/sites/default/files/ewsf28fT.PDF`[5]

[5] Last accessed on 27 July 2022.

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