Effective Equidistribution on Hilbert Modular Varieties

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We compute effective error rates for the equidistribution of translates of diagonal orbits on Hilbert modular varieties. The translation is determined by n real parameters and our results require the assumption that all parameters are non-zero. The error rate is given in explicit polynomial terms of the translation parameters and Sobolev type norms of the test functions. The effective equidistribution is applied to give counting estimates for binary quadratic forms of square discriminant over real number rings.

Contents

with thanks to my parents and to Jasmine

Chapter 1

Introduction

Dynamics on homogeneous spaces is of classical interest and has strong connections with number theory. Two fundamental ideas are that of mixing and the equidistribution of orbit translates. For both topics, foundational work was laid by Ratner in [Rat87] and in her series exploring unipotent orbits [Rat90b] [Rat90a] [Rat91]. The orbit equidistribution question in particular often has immediate and intrinsic number theoretic applications. Specifically, orbit equidistribution can provide a way to asymptotically count lattice points [EM93] [DRS93] [KM98].

The two dynamical phenomenon just mentioned are very much intertwined. One can view equidistribution as a "singular version" of mixing, and this view yields an intuitive proof strategy that works for finite volume orbits. Making this more explicit, let G be a connected semi-simple Lie group and $\Gamma \subset G$ a lattice so $X = \Gamma\backslash G$ is of finite measure. The measure μ on X is inherited from the Haar measure on G, and so we have a μ-invariant G-action X given by $\Gamma x \mapsto \Gamma x g$. For $f_1, f_2 \in L^2(X, \mu)$ we call this action *mixing* if

$$\int_X f_1(x) f_2(xg)\, d\mu \to \int_X f_1(x)\, d\mu \cdot \int_X f_2(x)\, d\mu = \mu(f_1)\, \mu(f_2)$$

as $g \to \infty$ (in the sense of leaving compact sets). Ineffective mixing (that is without explicit error terms) in this fairly general setting (by comparison to the setting we will consider) is a well known consequence of the Howe-Moore theorem [HM79]. The G action above is of course a representation of G on $L^2(X, \mu)$ given by $(\pi(g) \circ f)(x) = f(xg)$, with the fixed subspace of π consisting of locally constant functions. If f_1 and f_2 are orthogonal to the subspace of locally constant functions then mixing implies that $\langle \pi(g) \circ f_1, f_2 \rangle \to 0$ (as $g \to \infty$). Thus mixing is an example of the decay of matrix coefficients, a phenomenon for which effective results are well established in more generality than used here, see [Rat87], [HC66], [Ven10] and [Oh02].

For equidistribution we consider a closed subgroup $H \subset G$ with its Haar measure dh and the quotient $H_\Gamma = (H \cap \Gamma)\backslash H$. We could then consider the measure $\mu_H(f) = \int_{H_\Gamma} f(h) dh$ and the translates by $g \in G$ given by $\mu_{Hg}(f) = \int_{H_\Gamma} f(hg) dh$.

We say the translates of the H-orbit *equidistribute* if

$$\mu_{Hg}(f) \to \int_X f(x)\, d\mu$$

as $g \to \infty$ in some sense. One could view $\mu_H(f)$ as integration of f over X multiplied against a singular distribution supported on the orbit H_Γ. If the orbit is of finite volume, so $\int_{H_\Gamma} 1dh < \infty$, equidistribution then could be seen as analogous to mixing of f and this singular distribution. To illustrate, suppose Ξ_H is a distribution on X such that

$$\int_X f(x)\Xi_H(x)\, d\mu = \int_H f(h)\, dh$$

and normalized so that $\int_X \Xi_H(x)d\mu = 1$. Then "mixing" would say that

$$\int_X f(x)\Xi_H(xg)\, d\mu = \int_{H_\Gamma} f(x_0 hg)\, dh \to \int_X f(x)\, d\mu \cdot \int_X \Xi(x)\, d\mu = \int_X f(x)\, d\mu$$

Of course, Ξ_H is not in $L^2(X)$, so mixing results don't apply, however this analogy provides intuition into how equidistribution might follow from mixing. By approximating Ξ_H by a smooth function supported on a small neighborhood of the orbit of H, we could then apply mixing to this smoothed orbit. This works so long as translates of this thickened orbit remain as a smooth approximation of the translated orbit. Eskin and McMullen in [EM93] were able to do this for finite volume orbits via their *wavefront lemma*, which essentially states that the translation of this thickened orbit remains a smooth approximation of the translated orbit.

The wavefront lemma unfortunately doesn't work for orbits of infinite volumes and obtaining equidistribution results in this setting, both effective and ineffective, has been the focus of more recent research. In the work of Shapira and Zheng [SZ19] they refine the notion of weak* convergence and describe, ineffectively, the limiting distribution for translates of diagonal orbits on the quotients

$$SL(d, \mathbb{Z})\backslash SL(d, \mathbb{R}).$$

In [OS13], [KK17] and [KK20], equidistribution for diagonal orbits on $\Gamma\backslash SL_2(\mathbb{R})$ is given effectively. In particular, in [KK17], soft, dynamic methods give effective equidistribution. This in turn is applied to a diophantine counting problem as well as to weighted second moments of $GL(2)$-automorphic L-functions. While not as immediate as the finite case, effective decay of matrix coefficients is an important step in the process still, and the exponent of the error rate for equidistribution ultimately derives from the exponent for mixing.

1.1 Equidistribution of Diagonal Orbits

In this thesis we extend the method in [KK17] from $G = SL_2(\mathbb{R})$ to the group $G = SL_2(\mathbb{R})^n$ for arbitrary n. However, we limit ourselves to only considering the lattices Γ that arise from a totally real number field L (of degree n) as

$SL_2(\mathscr{O})$, where $\mathscr{O}$ is the ring of integers of L (the details of this will be made explicit). We then consider translates of the diagonal orbits in $\Gamma\backslash G$. Additionally, the test-functions we consider are K-invariant (where $K \subset G$ is the subgroup $K = SO(2)^n$), thus the equidistribution takes place on the Hilbert modular variety $\Gamma\backslash G/K$. A generic translate, in this context, can be understood by analyzing translation by a tuple of unipotent matrices:

$$n_T = \left(\begin{pmatrix} 1 & T_1 \\ 0 & 1 \end{pmatrix}, \cdots, \begin{pmatrix} 1 & T_n \\ 0 & 1 \end{pmatrix} \right).$$

We are able to obtain an effective error term for equidistribution under the assumption that $|N(T)| = \prod |T_i| > 0$. For the error term to be smaller than the main terms it is not sufficient that $N(T) \to \infty$ as in fact we need the large T_i to effectively approach ∞ slightly faster than the small T_i are approaching 0. We let A be the diagonal subgroup of $(SL_2(\mathbb{R}))^n$ and define

$$A_\Gamma = (A \cap \Gamma)\backslash A. \tag{1.1}$$

Then we have

Theorem 1.1. *There exists a constant $\eta > 0$ (determined from the spectral gap of Γ) such that for any K-invariant Schwartz test function Ψ and any $\varepsilon > 0$*

$$\int_{A_\Gamma} \Psi(an_T)\, da = \mu(\Psi) 2^{n-1} \mathscr{R}\sqrt{\mathscr{D}} \sum_{j=1}^{n} \log(1 + T_j^2) + \mu_E(\Psi) + O_\Psi\left(\prod_{T_j \le 1} T_j^{-\frac{1}{n+1} - \varepsilon} \prod_{T_j > 1} T_j^{-\eta} \right)$$

where $\mathscr{R}$ and $\mathscr{D}$ are the regulator and discriminant of L and μ_E is a regularized Eisenstein distribution, the precise definition of which will follow.

The dependence of the error term on Ψ will be made explicit in terms of Sobolev-like norms, and the constant η is derived from already established effective decay of matrix coefficients [Ven10] and [Oh02]. If η_M is an exponent for effective mixing on $\Gamma\backslash G$, then our equidistribution exponent is

$$\eta = \frac{\eta_M}{(n+1)(n+2)}.$$

From the work of Blomer and Brumley estimating the spectral gap of $\Gamma\backslash G$, [BB11], we have the estimate $\eta_M = 1/2 - 7/64 - \varepsilon = 25/64 - \varepsilon$, independent of the number ring L.

We will conclude this section with a note on proof strategy. A crucial step in the proof of Theorem 1.1 will be establishing the decay of Fourier coefficients, to which chapter 4 is dedicated. In the case $\mathscr{O} = \mathbb{Z}$, one can imagine a function ψ on $SL_2(\mathbb{Z})\backslash\mathbb{H}$ where $\mathbb{H}$ is the hyperbolic plane

$$\mathbb{H} = \{z = x + iy \in \mathbb{C} : y > 0\}.$$

Then ψ has a Fourier expansion from the periodicity in the real component: $\psi(x+iy) = \psi(x+1+iy)$, and we have the expansion

$$\psi(x+iy) = \sum_{m\in\mathbb{Z}} a_\psi(m;y)e^{2\pi imx}$$

As shown in [KK17], these Fourier coefficients $a_\psi(m;y)$ decay to zero as $y \to 0$. We use the higher dimensional analogue of this. To do this we use the higher dimensional analogue of the equidistribution of low-lying horocyclic segments. This is done, these segments being compact, by the intuitive method described earlier: the horocyclic segment is thickened and then mixing can be applied.

1.2 An Application: Counting Quadratic Forms

A standard application of such an equidistribution result is the asymptotic counting of lattice points, as done in [EM93] and [KK17], and is directly related to a classic question in Diophantine analysis of counting the integer points of an affine variety, as in [DRS93]. One can consider a variety V defined by integral polynomials $f_i \in \mathbb{Z}[x_1, \cdots x_m]$:

$$V = \{x \in \mathbb{C}^m : f_i(x) = 0, i = 1, \cdots m\}.$$

The desire is to understand the asymptotics as $T \to \infty$ of the function

$$N(T,V) = \{m \in V(\mathbb{Z}) : \|m\| \leq T\}.$$

For the number ring $\mathscr{O}$, we consider an $\mathscr{O}$-analog to this question: let $f_i \in \mathscr{O}[x_1, \cdots x_m]$ and define, for a real embedding σ of $\mathscr{O}$, the variety

$$V_\sigma := \{x \in \mathbb{C}^m : \sigma(f_i)(x) = 0, i = 1, \cdots m\}$$

so that we can consider the variety $V = \prod_\sigma V_\sigma$. Then we can define the set

$$V(\mathscr{O}) = \{\vec{\alpha} \in \mathscr{O}^m : \sigma_j(\vec{\alpha}) \in V_{\sigma_i}\}$$

and consider the analogous asymptotics of the function

$$N(T,V) = \{\vec{\alpha} \in V(\mathscr{O}) : \|\vec{\alpha}\| \leq T\}$$

for an appropriately defined norm $\|\cdot\|$. Analogous to the counting results in [KK17], our result allows the counting of quadratic forms of a square discriminant, and the counting follows the pattern established in [EM93], [DRS93] and [KK17]. Let $f_\alpha \in \mathscr{O}[x_1,x_2,x_3]$ be defined as $x_2^2 - 4x_1x_3 - \alpha$ and define $V_\alpha = \{x \in \mathbb{R}^3 : f_\alpha(x) = 0\}$. A quadratic form $ax^2 + bxy + cy^2$ of discriminant $d = b^2 - 4ac$ can be viewed as living on the variety

$$W_d := V_{\sigma_1(d)} \times V_{\sigma_2(d)} \times \cdots \times V_{\sigma_n(d)}$$

via the map

$$(a,b,c) \mapsto (\sigma_1(a), \sigma_1(b), \sigma_1(c), \cdots, \sigma_n(a), \sigma_n(b), \sigma_n(c)) \in \mathbb{R}^{3n}.$$

We will call the $\mathscr{O}$-points of W_d the image of the above map, so we define

$$W_d(\mathscr{O}) = \{(a,b,c) \in \mathscr{O}^3 : (a^{(i)}, b^{(i)}, c^{(i)}) \in V_{d^{(i)}}\}. \tag{1.2}$$

Then Γ acts on $W_d(\mathscr{O})$ and our equidistribution result allows us to asymptotically count the Γ-orbit points that lie inside a norm-ball of a given radius for a K-invariant norm on W_d. From [Efr87] we know that $W_d(\mathscr{O})$ decomposes into finitely many orbits allowing us to count each orbit separately and sum over the orbits.

Explicitly, for $(a,b,c) \in W_d(\mathscr{O})$, $\gamma \in SL_2(\mathscr{O})$ we will write $(a,b,c)^\gamma$ to indicate its image under the action of γ. We denote the stabilizer of (a,b,c) by $H_{(a,b,c)} = \{\gamma \in SL_2(\mathscr{O}) : (a,b,c)^\gamma = (a,b,c)\}$. Then we can construct the counting function

$$N_{(a,b,c)}(R) = \#\{\gamma \in H_{(a,b,c)} \backslash SL_2(\mathscr{O}) : \|(a,b,c)^\gamma\| < R\} \tag{1.3}$$

For a given square discriminant $d = \omega^2$, we will show that $\{(0,\omega,\tau) : \tau \in \mathscr{O}/(\omega)\}$ comprises a full set of representatives for the classes of forms. Thus we are interested in the counting function

$$\mathscr{N}_\omega(R) = \sum_{\tau \in \mathscr{O}/(\omega)} N_{(0,\omega,\tau)}(R) \tag{1.4}$$

and have the following result

Theorem 1.2. *For $\omega \in \mathscr{O}$, there exists constants K_1 and $K_2 = K_2(\omega)$, $\eta > 0$ such that*

$$\mathscr{N}_\omega(R) = V_n(R)(K_1 \log(R) + K_2 + O(R^{-\eta}))$$

where $V_n(R) = \Gamma(n/2+1)^{-1}\pi^{n/2}R^n$ is the volume of an n-dimensional sphere of radius R.

Chapter 2

Background and Notation

We will say $f(x) = O(g(x))$ if there is some $M > 0$ such that $|f(x)| < M|g(x)|$ for all x in the domain of both functions. If the bound only holds for a reduced domain of x it will be stated explicitly. We will also use the notation $\ll$: we will write

$$f(x) \ll g(x) \iff f(x) = O(g(x)).$$

If simultaneously $f(x) \ll g(x)$ and $g(x) \ll f(x)$ then we write

$$f(x) \asymp g(x).$$

A subscript on O or $\ll$ indicates something on which the implied constant may depend, as in the statement of theorem 1.1. Without any subscript the implied constant is allowed to depend on L.

Fourier series will play a crucial role throughout so to make notation more concise we will use the following definition

$$e(x) = \exp(2\pi i x). \tag{2.1}$$

2.1 Real Number Rings

Totally real number rings play a critical role in these results and so we record here some of the elementary properties that will be needed for the analysis ahead. See [Mar18] for a thorough reference on the topic.

Definition 2.1. *If $L/\mathbb{Q}$ is a finite field extension, we say L is a* totally real *extension if for every embedding $\sigma : L \to \mathbb{C}$ we have that $\sigma(L) \subset \mathbb{R}$. We say $\alpha \in L$ is an* integer *of L if there is a monic polynomial $f \in \mathbb{Z}[x]$ such that $f(\alpha) = 0$. The set of integers is denoted $\mathscr{O}_L$ and form a ring. We call $\mathscr{O}_L$ a* real number ring *if L is a totally real extension of $\mathbb{Q}$.*

Throughout this thesis we will consider L fixed and denote the degree of the extension by $[L : \mathbb{Q}] = n$. As L will be considered fixed, we drop the subscript

L and just refer to the integers of L by $\mathscr{O}$. Of critical importance will be the structure of the units of $\mathscr{O}$. We begin by recalling the definition of the norm of elements of L.

Definition 2.2. *Let $\sigma_1, \cdots, \sigma_n$ be the n distinct embeddings of L into $\mathbb{R}$. We will adopt the notation that $\alpha^{(i)} = \sigma_i(\alpha)$. Then for $\alpha \in L$ the* norm *of α is defined to be*

$$N(\alpha) = \prod_{i=1}^{n} \sigma_i(\alpha).$$

Similarly the trace *of α is defined as*

$$tr(\alpha) = \sum_{i=1}^{n} \alpha^{(i)}.$$

For convenience we will use the notation $N(y)$ and $tr(y)$ for tuples generically. So for $y = (y_1, \cdots y_n) \in \mathbb{R}^n$ we let $N(y) = \prod_j y_j$ and $tr(y) = \sum_j y_j$.

Thus if $\alpha \in L$ then $N(\alpha) \in \mathbb{Q}$, and $\alpha \in \mathscr{O}$ implies $N(\alpha) \in \mathbb{Z}$. We can then define

Definition 2.3. *An element $u \in \mathscr{O}$ is called a* unit *if $|N(u)| = 1$. We let U denote the group of units.*

The units U form a group under multiplication, the structure of this group plays a critical role in our analysis.

Theorem 2.4 (Structure Theorem)**.** *The group U is the product of a cyclic group and a free $\mathbb{Z}$-module of rank $n-1$. The cyclic group is $\{-1, 1\}$. A basis for free part will be called a set of* fundamental units

For proof see [Mar18], Chapter 5. We will implicitly associate the element $\alpha \in L$ and the vector $(\sigma_1(\alpha), \cdots, \sigma_n(\alpha)) \in \mathbb{R}^n$ with the understanding that an ordering to the embeddings has been fixed. Thus for $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ we will denote $\langle \alpha, x \rangle = \sum_i \alpha^{(i)} x_i$. In this vein, we will use the notation $\|\alpha\|_\infty$ to mean $\max_i\{|\alpha^{(i)}|\}$. Using this association we have that $\mathscr{O} \subset \mathbb{R}^n$ is a lattice. For the next definition we will make a basis of U explicit: let $(\epsilon_1, \cdots, \epsilon_{n-1})$ be a fundamental set of units. Then we can construct the matrix

$$U = \begin{bmatrix} 1 & \log|\epsilon_1^{(1)}| & \cdots & \log|\epsilon_{n-1}^{(1)}| \\ 1 & \log|\epsilon_1^{(2)}| & \cdots & \log|\epsilon_{n-1}^{(2)}| \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \log|\epsilon_1^{(n)}| & \cdots & \log|\epsilon_{n-1}^{(n)}| \end{bmatrix} \tag{2.2}$$

Definition 2.5. *The* discriminant *of $\mathscr{O}$, denoted $\mathscr{D}$, is the square of the volume of the fundamental domain of $\mathscr{O}$. Furthermore the* regulator *of $\mathscr{O}$, denoted $\mathscr{R}$, is defined to be* $\det(U)/n$. *This is independent of the choice of fundamental units.*

We now give with proof some elementary facts that will be needed later.

Lemma 2.6. *For any nonzero ideal $J \subset \mathscr{O}$ and any $u \in U$ there is an integer $m \in \mathbb{Z}$ such that*

$$u^m - u^{-m} \in J$$

Proof. By the finiteness of $\mathscr{O}/J$ we have that for any $u \in U$ there is some $m \in \mathbb{Z}$ such that $u^{2m} \equiv 1 \mod J$. Thus $u^{2m} - 1 \in J$. And we have that

$$u^{2m} - 1 \in J \implies (u^{2m} - 1)u^{-m} = u^m - u^{-m} \in J$$

□

We now consider the dual lattice to $\mathscr{O}$, which we denote $\mathscr{O}^*$. This is defined by

$$\mathscr{O}^* = \{\alpha^* \in \mathbb{R}^n : Tr(\alpha^*\alpha) \in \mathbb{Z}\ \forall\, \alpha \in \mathscr{O}\}. \tag{2.3}$$

We note the following fact about $\mathscr{O}^*$:

Lemma 2.7. *$\mathscr{O}^*$ is a fractional ideal of $\mathscr{O}$, and as such is also realized as a lattice in $\mathbb{R}^n$. Thus there is some smallest integer $m_L \in \mathbb{Z}$ such that $m_L\mathscr{O}^* = J_L \subset \mathscr{O}$ is an ideal.*

Given a smooth function $f : \mathbb{R}^n \to \mathbb{C}$ such that $\forall\, \alpha \in \mathscr{O}$, $f(x) = f(x + \alpha)$, we can write it as its Fourier series

$$f(x) = \sum_{\alpha^* \in \mathscr{O}^*} a_f(\alpha^*) e(\langle \alpha^*, x \rangle) = \sum_{\alpha \in J_L} a_f(\alpha/m_L) e(\langle \alpha, x \rangle / m_L)$$

It will often be useful to consider the function $\log : L \to \mathbb{R}^n$ given by

$$\log(\alpha) = (\log(|\alpha^{(1)}|), \cdots, \log(|\alpha^{(n)}|)). \tag{2.4}$$

Note that if $u \in U$ then $\log(u)$ lies in the hyperplane orthogonal to the vector $(1, \cdots, 1)$.

Now we consider the equivalence relation on elements of $\mathscr{O}$ (or $\mathscr{O}^*$) given by U. Namely, for $\alpha, \beta \in \mathscr{O}$ we say $\alpha \sim \beta$ if $\exists u \in U$ such that $\alpha = u\beta$ (likewise for $\alpha^*, \beta^* \in \mathscr{O}^*$) and denote by $[\alpha]$ (resp. $[\alpha^*]$) the equivalence class. Note that since $N(\alpha\beta) = N(\alpha)N(\beta)$ then $N([\alpha])$ is well-defined. We have

Lemma 2.8. *There exists C_1 such that any $[\alpha]$ (or $[\alpha^*]$) has a representative α_0 such that for all $i \in \{1, \cdots, n\}$*

$$C_1^{-1} |N(\alpha)|^{1/n} \leq |\alpha_0^{(i)}| \leq C_1 |N(\alpha)|^{1/n}$$

Proof. Let P denote the hyperplane in $\mathbb{R}^n$ that is orthogonal to the vector $(1, \cdots, 1)$, and for $x \in \mathbb{R}^n$ let x_P denote its orthogonal projection onto P. By theorem 2.4 $\log(U)$ forms a lattice in P. Let D denote a fundamental domain of this lattice. Then for any $\alpha \in \mathscr{O}$ there is some $u \in U$ such that $(\log(\alpha) + \log(u))_P \in D$. D is finite, so there is some C_1 such that for all $x \in D$, $\|x\|_\infty \leq C_1$. Then we simply note that $\log(\alpha)_P = \log(\alpha/|N(\alpha)|^{1/n})$, and we are done. □

An immediate consequence of lemma 2.8 is a similar result but for the action of square units on the set $[\alpha^*]$. We have the following corollary:

Corollary 2.9. *If we consider the action only of square units (u^2 with $u \in U$) on the set $[\alpha^*]$, then there are 2^{n-1} equivalence classes inside $[\alpha^*]$. Furthermore there is a constant $C = C(L)$ such that each class has a representative $\tilde{\alpha}_0^*$ such that for all $i \in \{1, \cdots, n\}$*

$$C^{-1}|N(\alpha^*)|^{1/n} \leq |(\tilde{\alpha}^*)^{(i)}| \leq C|N(\alpha^*)|^{1/n}$$

We call the set of such representatives $\{\tilde{\alpha}^*\}$.

Proof. For a given α^*, we consider the set $\{u^2\alpha^* : u \in U\}$. Letting $\{\epsilon_1, \cdots, \epsilon_{n-1}\}$ denote the set of fundamental units, there is an element $\tilde{\alpha}^*$ of $\{u^2\alpha^* : u \in U\}$ such that

$$\tilde{\alpha}^* = (\epsilon_1^{e_1} \cdots \epsilon_{n-1}^{e_{n-1}})\alpha_0^*$$

where α_0^* is the representative from Lemma 2.8 and with $e_j \in \{0, 1\}$. Then we set

$$C = C_1 \exp\left(\sum_{j=1}^{n-1} \|\log(\epsilon_j)\|_\infty\right)$$

and our result follows from Lemma 2.8. □

We now give a lemma that will allow us to use $N(\alpha^*)$ as a bound for the decay of Fourier coefficients.

Lemma 2.10. *Let $\Omega^* = (\beta_1, \cdots \beta_n)$ be an integral basis of $\mathscr{O}^*$. For $\alpha^* \in \mathscr{O}^*$ let $\tilde{\alpha}^*$ be the representative from Corollary 2.9 and let $m = (m_1, \cdots, m_n)$ be the coefficients of $\tilde{\alpha}^*$ in the basis Ω^*. Then we have that*

$$\prod_{m_i \neq 0} |m_i| \ll N(\alpha^*)$$

where the implied constant depends only on Ω^*.

Proof. First note that $N(\alpha^*) = N(\tilde{\alpha}^*)$. Now let

$$D^* = \begin{pmatrix} \beta_1^{(1)} & \cdots & \beta_1^{(n)} \\ \vdots & \ddots & \vdots \\ \beta_n^{(1)} & \cdots & \beta_n^{(n)} \end{pmatrix}$$

so $m = D^*\tilde{\alpha}^*$. We then observe that

$$\prod_{m_i \neq 0} |m_i| \leq (\|m\|_\infty)^n$$

and likewise

$$\min_i \{|(\tilde{\alpha}^*)^{(i)}|\}^n \leq N(\alpha^*).$$

Now since $\tilde{\alpha}^* = (D^*)^{-1}m$ we have that $\|m\|_\infty \ll \|\tilde{\alpha}^*\|_\infty$ where the constant only depends on the Eigenvalues of D^*. Thus we have for $i = 1, \cdots, n$

$$C^{-2}N(\alpha^*)^{1/n} \leq |(\tilde{\alpha}^*)^{(i)}| \leq C^2 N(\alpha^*)^{1/n}$$

Thus we have that

$$\prod_{m_i \neq 0} |m_i| \ll |N(\alpha^*)|$$

□

When $\mathscr{O} = \mathbb{Z}$ and thus $\mathscr{O}^* = \mathbb{Z}$, the proof of equidistribution in [KK17] requires the estimate (evident from Abel summation) that

$$\sum_{\substack{m \in \mathbb{Z} \\ m > M}} m^{-k} \ll M^{1-k}.$$

We record here the generalization we will need of this:

Lemma 2.11. *For $k > 1$, we have the following bound:*

$$\sum_{|N([\alpha^*])| > M} |N(\alpha^*)|^{-k} = O(M^{1-k}).$$

Proof. We begin with the function $\theta(m) = \#\{[\alpha] : |N(\alpha)| = m\}$ and note that $|N(\alpha)| = |\mathscr{O}/(\alpha)|$ so $\theta(m) = \#\{(\alpha) : |\mathscr{O}/(\alpha)| = m\}$. Now recall the Dedekind zeta function

$$\zeta_L(s) = \sum_{I \subseteq \mathscr{O}} |\mathscr{O}/I|^{-s}$$

converges for $\Re(s) > 1$. If $\mathscr{O}$ is a principal ideal domain then we have

$$\zeta_L(s) = \sum_{(\alpha) \subseteq \mathscr{O}} |\mathscr{O}/(\alpha)|^{-s} = \sum_{m \in \mathbb{N}} \Theta(m) m^{-s}$$

however in general we have that

$$\sum_{m \in \mathbb{N}} \Theta(m) m^{-s} \leq \zeta_L(s).$$

Regardless we have that the sum $\sum_{m \in \mathbb{N}} \Theta(m) m^{-s}$ converges for $\Re(s) > 1$ so the Wiener-Ikahara Taubarian theorem tells us that

$$\sum_{m=0}^{m=M} \theta(m) = O(M).$$

We then apply Abel summation and obtain

$$\sum_{|N([\alpha])| > M} |N(\alpha)|^{-k} = \sum_{n > M} \theta(n) n^{-k} = O(M^{1-k})$$

Recalling the constant m_L defined in Lemma 2.7, we can conclude this proof by noting

$$\sum_{|N([\alpha^*])| > M} |N(\alpha^*)|^k = \sum_{\substack{|N([\alpha])| > M(m_L)^{-n} \\ [\alpha] \in J}} |N(\alpha/m_L)|^k \ll \sum_{|N([\alpha])| > M} |N(\alpha)|^k$$

□

2.2 Hilbert Modular Group and Variety

As we associated the set $\mathscr{O}$ to a lattice in $\mathbb{R}^n$ we can associate the set of matrices $SL_2(\mathscr{O})$ to a lattice inside $SL_2(\mathbb{R})^n$ via the n embeddings from $\mathscr{O} \to \mathbb{R}$. We will denote by $\mathbb{H}$ the upper half plane $\mathbb{H} = \{x + iy : y > 0\}$. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{R})$ we have the action on $\mathbb{H}$ given by fractional linear transformation

$$g \cdot z \mapsto \frac{az + b}{cz + d}. \tag{2.5}$$

We will denote the Hilbert modular group, $PSL_2(\mathscr{O})$, by Γ. By applying the fractional linear transform on each component, Γ acts on the n-fold Cartesian product of $\mathbb{H}$, $\mathbb{H} \times \cdots \times \mathbb{H}$. We will denote this space $\mathbb{H}^n$. This action is irreducible in the sense that if $\gamma \in \Gamma$ acts as the identity in any component of $(z_1, \cdots, z_n) \in \mathbb{H}^n$ then γ is the identity. Furthermore, any irreducible, non-uniform lattice of $PSL_2(\mathbb{R})^n$ is conjugate to a group commensurable with a Hilbert modular group (see [Gee88] and [Efr87]). The quotient of this action $\Gamma\backslash\mathbb{H}^n$ is called a Hilbert modular variety. This generalizes the classical construction of the modular curve $SL_2(\mathbb{Z})\backslash\mathbb{H}$.

Recalling the unipotent, orthogonal, and diagonal subgroups of $SL_2(\mathbb{R})$, we will let N, A, and K represent the Cartesian products of these subgroups. So in explicit coordinates we have

$$\begin{aligned}
N &= \left\{ n_x = \left(\begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix}, \cdots, \begin{pmatrix} 1 & x_n \\ 0 & 1 \end{pmatrix} \right) : x \in \mathbb{R}^n \right\} \\
A &= \left\{ a_y = \left(\begin{pmatrix} y_1^{1/2} & 0 \\ 0 & y_1^{-1/2} \end{pmatrix}, \cdots, \begin{pmatrix} y_n^{1/2} & 0 \\ 0 & y_n^{-1/2} \end{pmatrix} \right) : y \in (\mathbb{R}^+)^n \right\} \\
K &= \left\{ k_\theta = \left(\begin{pmatrix} \cos(\theta_1) & \sin(\theta_1) \\ -\sin(\theta_1) & \cos(\theta_1) \end{pmatrix}, \cdots, \begin{pmatrix} \cos(\theta_n) & \sin(\theta_n) \\ -\sin(\theta_n) & \cos(\theta_n) \end{pmatrix} \right) : \theta \in [0, 2\pi)^n \right\}.
\end{aligned} \tag{2.6}$$

We denote the element of N with coordinates $x = (x_1, \cdots x_n)$ by n_x, and similarly a_y and k_θ. The subgroups above are the subgroups for the Iwasawa decomposition of $SL_2(\mathbb{R})^n = NAK$. The Haar measures, in these coordinates, is $dn = dx = dx_1 dx_2 \cdots dx_n$, $da = dy/N(y)$ and $dk = d\theta$. This gives us the volume form on $\Gamma\backslash\mathbb{H}^n$

$$\frac{dx\, dy}{N(y)^2}.$$

As with the classical identification of $\mathbb{H}$ with the quotient $PSL_2(\mathbb{R})/SO(2)$, we can associate $\mathbb{H}^n$ with $PSL_2(\mathbb{R})^n/K$. If we associate $g \in PSL_2(\mathbb{R})^n$ with the image of $(i, \cdots, i)$ under the above action we get the explicit coordinates

$$n_x a_y k_\theta \mapsto (x_1 + iy_1, \cdots, x_n + iy_n) \in \mathbb{H}^n.$$

With this association we can view a K-invariant function Ψ that is periodic with respect to Γ as a function on $\Gamma\backslash\mathbb{H}^n$. We will refer to these coordinates on $\Gamma\backslash G/K$ as the Iwasawa coordinates.

2.3 Eisenstein Series and Cuspidal Coordinates

Let $\sim$ be the relation on $\mathscr{O}^2$ given by $(\alpha_1, \alpha_2) \sim (\lambda\alpha_1, \lambda\alpha_2)$ for all $\lambda \in \mathscr{O}\backslash\{0\}$ and let $\mathbb{P}(\mathscr{O}) = \mathscr{O}^2/\sim$. The fractional linear transform from (2.5) maps $\mathbb{P}(\mathscr{O})$ to itself. A *cusp* is an orbit under this map, and these correspond to ideal classes in $\mathscr{O}$ (see [Gee88]). The orbit of $(1, 0)$, associated to the fundamental ideal class, we call the cusp "at infinity" and denote by $\infty = (\infty, \cdots, \infty)$. We denote by $\Gamma_\infty \subset \Gamma$ the stabilizer of ∞, which consist of upper triangular matrices in $SL_2(\mathscr{O})$ (see [Efr87]). While the methods employed here are soft and do not rely on explicit spectral decomposition, we will need to use some basic properties of the family of automorphic Eisenstein series that that give us the continuous spectrum of $L^2(\Gamma\backslash\mathbb{H}^n)$ (see [Efr87]). Recall that the Laplacian on $L^2(\Gamma\backslash\mathbb{H}^n)$ is

$$\Delta_j = y_j^2\left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2}\right) \quad j = 1, \cdots, n.$$

By an eigenfunction of this operator we mean a function which is a simultaneous eigenfunction of each Δ_i. Recall that for $L = \mathbb{Q}$ we have

$$E(z, s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} y(\gamma \cdot z)^s$$

which satisfies the eigenvalue relation $\Delta E(z, s) + s(s-1)E(z, s) = 0$. One could construct a simultaneous eigenfunction of each Δ_j by simply taking the product over each factor and obtain $E(\vec{z}, \vec{s}) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \prod_i y_i(\gamma \cdot z)^{s_i}$. However this is not a function on $\Gamma\backslash\mathbb{H}^n$ as it is not automorphic with the action of $(A \cap \Gamma) \subset \Gamma_\infty$. For the classical case where $L = \mathbb{Q}$, the diagonal matrices belonging to Γ are trivial, but for a totally real L of degree n, we have the non-trivial units giving us an $n-1$ dimensional free $\mathbb{Z}$-module structure to $A \cap \Gamma$ (from theorem 2.4). This essentially relates the values of the n different s_i's in the naive definition by $n-1$ linear equations leaving us with a single complex variable in the end. Before stating the proper definition for our Eisenstein series we will introduce the Efrat's "cuspidal" coordinates, which make it convenient to describe the fundamental domain of $(A \cap \Gamma)\backslash A$.

We recall the construction from [Efr87]. Let $\epsilon_1, \cdots, \epsilon_{n-1}$ be a basis for U, then we have the coordinate transform matrix U defined in (2.2) and the inverse

$$U^{-1} = \begin{bmatrix} 1/n & \cdots & 1/n \\ e_1^{(1)} & \cdots & e_n^{(1)} \\ \vdots & & \vdots \\ e_1^{(n-1)} & \cdots & e_n^{(n-1)} \end{bmatrix}. \tag{2.7}$$

Then we can define our "cuspidal" coordinates on A as the following

$$Y_0 = \prod_j y_j = N(y)$$
$$Y_k = \frac{1}{2}\sum_{j=1}^n e_j^{(k)} \log y_j \quad k = 1, \cdots, n-1.$$

We can express the standard y_i coordinates in terms of the cuspidal coordinates as follows:

$$y_k = Y_0^{1/n} \sum_{j=1}^{n-1} \left(\epsilon_j^{(k)}\right)^{2Y_j}.$$

We will abuse notation slightly and let $Y : (\mathbb{R}^+)^n \to (\mathbb{R}^+)^n$ also denote function converting from the Iwasawa coordinates for A to the cuspidal coordinates:

$$Y(y) = (N(y), \frac{1}{2}\sum_{j=1}^{n} e_j^{(1)} \log y_j, \cdots, \frac{1}{2}\sum_{j=1}^{n} e_j^{(n-1)} \log y_j). \tag{2.8}$$

The Haar measure on A expressed in our various coordinate is as follows:

$$da = \frac{dy}{N(y)} = \frac{dY}{Y_0}.$$

Under these coordinates the fundamental domain for the action of $A \cap \Gamma$ on A is the following (see [Efr87])

$$\{(Y_0, \cdots Y_{n-1}) | Y_0 \in (0, \infty), (Y_1, \cdots, Y_{n-1}) \in [0,1]^{n-1}\}$$

It is also useful to make new coordinates for N. Let $\omega_1, ...\omega_n$ be an integral basis for $\mathscr{O}$ then we define the matrix

$$D = \begin{bmatrix} \omega_1^{(1)} & \cdots & \omega_1^{(n)} \\ \vdots & \ddots & \vdots \\ \omega_n^{(1)} & \cdots & \omega_n^{(n)} \end{bmatrix}. \tag{2.9}$$

We want to use the column vectors as a basis for our x_j coordinates, so we let $X = (X_1, \cdots, X_n) = D^{-1}x$. Then we have

$$dx = \det(D)^{-1} dX.$$

With respect to these coordinates, a fundamental domain of $\Gamma_\infty \backslash \mathbb{H}^n$ can be expressed as

$$\{(X, Y) \in \mathbb{R}^{2n-1} \times \mathbb{R}^+ : X \in [0,1]^n, Y_1, \cdots Y_{n-1} \in [0,1], Y_0 \in \mathbb{R}^+\}.$$

The cuspidal coordinates make the formulation of Eisenstein series particularly convenient. For each $m \in \mathbb{Z}^{n-1}$ we can define the series

$$E(z, s, m) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} Y_0(\gamma z)^s \exp\left(2\pi i \sum_{q=1}^{n-1} m_q Y_q(\gamma z)\right) \tag{2.10}$$

We now record a particularly useful identity for computing $\langle \Psi, E(\cdot, s, m)\rangle$. We define

$$Y' = (Y_1, \cdots Y_{n-1}) \tag{2.11}$$

and by a slight abuse of notation

$$Y'(z) = (Y_1(z), \cdots Y_{n-1}(z)) \tag{2.12}$$

(all the cuspidal coordinates except Y_0) to make the expression more compact.

Lemma 2.12. *For $\Psi : \Gamma\backslash\mathbb{H}^n \to \mathbb{C}$ of sufficiently fast decay,*

$$\langle \Psi, E(\cdot, s, m)\rangle = \int_{\Gamma_\infty\backslash\mathbb{H}^n} \Psi(z)\ Y_0(z)^s\ e(\langle m, Y'(z)\rangle)\ \frac{dx\,dy}{N(y)^2}$$

Proof. This is simply a matter of unraveling the definition. Letting H be a fundamental domain of $\Gamma\backslash\mathbb{H}^n$ we have

$$\begin{aligned}\langle \Psi, E(\cdot, s, m)\rangle &= \int_H \Psi(z)\ E(z,s,m)\ \frac{dx\,dy}{N(y)^2} \\ &= \int_H \Psi(z) \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} Y_0(\gamma z)^s e\left(\sum_{q=1}^{n-1} m_q Y_q(\gamma z)\right) \frac{dx\,dy}{N(y)^2} \\ &= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \int_H \Psi(z) Y_0(\gamma z)^s e\left(\sum_{q=1}^{n-1} m_q Y_q(\gamma z)\right) \frac{dx\,dy}{N(y)^2} \\ &= \int_{\Gamma_\infty\backslash\mathbb{H}^n} \Psi(z)\ Y_0(z)^s\ e(\langle m, Y'(z)\rangle)\ \frac{dx\,dy}{N(y)^2}.\end{aligned}$$

□

From [Efr87] we have the following fact about the residues of the Eisenstein series:

Lemma 2.13. *For $m \neq 0$, $E(z,s,m)$ has no poles for $\Re(s) > 1/2$. For $m = 0$ there is a single, simple pole at $s = 1$ and the residue is given by $\frac{2^{n-1}\mathscr{R}\sqrt{\mathscr{D}}}{vol(\Gamma\backslash\mathbb{H}^n)}$.*

With Lemma 2.13 we can define the "regularized" Eisenstein series, $\tilde{E}$, in the following way:

$$\tilde{E}(z,s,m) = \begin{cases} E(z,s,m) - \frac{2^{n-1}\mathscr{R}\sqrt{\mathscr{D}}}{vol(\Gamma\backslash\mathbb{H}^n)(s-1)} & m = 0 \\ E(z,s,m) & m \neq 0 \end{cases} \tag{2.13}$$

and from Lemma 2.13 we have that $\tilde{E}(z,s,m)$ is regular at $s = 1$ for each m.

2.4 Effective Mixing

As indicated in the introduction, a key piece for our equidistribution result is the effective decay of matrix coefficients. The decay of matrix coefficients has been handled in more generality than we need for our results, see [Ven10] [Oh02] [Rat87]. However, to get the precise statement for $SL_2(\mathbb{R})^n$ in terms of Sobolev norms, we follow the approach used by Venkatesh in [Ven10], where he showed the result for the $n = 1$ case.

First we make explicit the definition of Sobolev norms that we will use. We fix a basis $\mathscr{B} = X_1, \cdots, X_{3n}$ for the Lie algebra $\mathfrak{g}$ of G. Then for $f \in C^\infty(\Gamma\backslash G)$ we define the L^p, order-d Sobolev norm of f to be

$$S_{p,d}(f) := \sum_{\mathrm{ord}(\mathscr{D})\leq d} \|\mathscr{D}f\|_p$$

where $\mathscr{D}$ ranges over monomials in $\mathscr{B}$ of order at most d. We begin by considering only functions f_1 and f_2 of finite K-type. The goal is to establish an effective rate for smooth, compactly supported functions in $L^2(\Gamma\backslash G)$ orthogonal to a constant function. The subscript M in what follows simply stands for "mixing." We will show the following.

Theorem 2.14. *For Ψ, $\Phi \in C_c^\infty(\Gamma\backslash G)$ such that each is orthogonal to a constant function and for a_y defined in* (2.6)*, there is a positive constant η_M such that*

$$|\langle a_y \circ \Phi, \Psi\rangle| \ll S_{2,n}(\Psi)S_{2,n}(\Phi)N(y)^{\eta_M}$$

To prove this, we follow the same path as Venkatesh in [Ven10] where he did this for the $n = 1$ case. From Oh ([Oh02]) we have the decay for finite K-type functions expressed using the Harish-Chandra function Ξ_G, which we can compute explicitly for a_y as

$$\Xi_{PSL_2(\mathbb{R})^n}(a_y) = \int_0^\pi \cdots \int_0^\pi \prod_i (y_i \cos^2(\theta_i) + y_i^{-1}\sin^2(\theta_i))^{-1/2}\, d\theta_1 \cdots d\theta_n$$

and which we can estimate as

$$\Xi_{PSL_2(\mathbb{R})^n}(a_y) \ll \prod_i \left(y_i + y_i^{-1}\right)^{-1/2}. \tag{2.14}$$

Then from [Oh02] we have

Theorem 2.15 (Oh)**.** *Let (π, V) be a representation of $SL_2(\mathbb{R})^n$ with a strong spectral gap and let $f_1, f_2 \in V$ be of finite K-type and orthogonal to a constant. Then there is a constant $0 < \Theta < 1$ such that for all $y \in (\mathbb{R}^+)^n$*

$$\langle \pi(a_y)f_1, f_2\rangle \ll (\dim(Kf_1)\dim(Kf_2))^{1/2}\, \Xi_{PSL_2(\mathbb{R})^n}(a_y)^{1-\Theta}.$$

From (2.14) we can make the crude estimate that $\Xi_{PSL_2(\mathbb{R})^n}(a_y) \ll N(y)^{1/2}$. With $V = L^2(\Gamma\backslash G)$ we have an explicit bound available due to Blomer and Brumley in [BB11], which gives us the estimate $\eta_M = 25/64 - \varepsilon$ for any $\varepsilon > 0$. What remains to prove Theorem 2.14 is to replace the K-dimension with the Sobolev norms. We follow the method of Venkatesh from [Ven10]. We expand the f_i (assumed to be of finite K-type still) as a sum of elements that transform under finite order elements of K: $f = \sum_{\vec{a}\in\mathbb{Z}^n} f^{(a_1,\cdots,a_n)}$. Then we have that $\dim(\pi(K)\cdot f^{(\vec{a})}) = 1$ so we have

$$\left\langle \pi(a_y)\cdot f_1, f_2\right\rangle \ll \left(\sum_{\vec{a}\in\mathbb{Z}^n} \|f_1^{(\vec{a})}\|_2\right)\left(\sum_{\vec{a}\in\mathbb{Z}^n} \|f_2^{(\vec{a})}\|_2\right) N(y)^{\eta_M}.$$

We then need to bound the sum $\sum_{\vec{a}\in\mathbb{Z}^n} \|f^{(\vec{a})}\|_2$. We will use the following lemma.

Lemma 2.16.

$$\sum_{\vec{a}\in\mathbb{Z}^n} \|f^{(\vec{a})}\|_2 \ll \|f\|_2^{1/(n+1)} S_{2,n}(f)^{n/(n+1)}$$

Proof. For $M \in \mathbb{N}$ we have that

$$\begin{aligned}\sum_{\vec{a}\in\mathbb{Z}^n} \|f^{(\vec{a})}\|_2 &= \sum_{\|\vec{a}\|_\infty < M} \|f^{(\vec{a})}\|_2 + \sum_{\|\vec{a}\|_\infty \geq M} \|f^{(\vec{a})}\|_2 \frac{\prod(1+|a_i|)}{\prod(1+|a_i|)} \\ &\ll \left(\sum_{\vec{a}\in\mathbb{Z}^n} \|f^{(\vec{a})}\|_2^2\right)^{1/2} M^{n/2} + \left(\sum_{\vec{a}\in\mathbb{Z}^n} \|f^{(\vec{a})}\|_2^2 \prod(1+|a_i|)^2\right)^{1/2} M^{-1/2}.\end{aligned}$$

We then let

$$M = \left[\left(\sum_{\vec{a}\in\mathbb{Z}^n} \|f^{(\vec{a})}\|_2^2\right)^{-1} \left(\sum_{\vec{a}\in\mathbb{Z}^n} \|f^{(\vec{a})}\|_2^2 \prod(1+|a_i|)^2\right)\right]^{\frac{1}{n+1}}$$

and have our result upon noting that

$$S_{2,n}(f) \ll \left(\sum_{\vec{a}\in\mathbb{Z}^n} \|f^{(\vec{a})}\|_2^2 \prod(1+|a_i|)^2\right)^{1/2}.$$

□

We have so far shown our desired result for K-finite functions. Then 2.14 follows for smooth f_i by the density of K-finite functions.

2.5 Total Variation

In addition to Sobolev norms, which naturally require some degree of smoothness, we will also make use of total variation. We record what we need in this section, and refer the reader to [AFP00] for a thorough reference. We first recall the definition. For E an open subset of $\mathbb{R}^n$, $C_c^1(E, \mathbb{R}^n)$ the set of C^1 functions from E to $\mathbb{R}^n$ with compact support and $\psi \in C_c^1(E)$ we let $\|\phi\|_\infty$ be the essential supremum norm of the function $\|\phi\|_2 : E \to \mathbb{R}$. Then for $f \in L^1(E)$ the total variation of f is defined to be

$$\operatorname{Var}(f) := \sup_{\phi\in C_c^1(E)} \left\{\int_E f(x)\operatorname{div}(\phi)(x)dx : \|\phi\|_\infty \leq 1\right\}.$$

If the function f is smooth then the above definition coincides with the more intuitive definition of $\operatorname{Var}(f) = \int_E |\nabla(f)|dx$. Let $Y \subset E$ be a $n-1$ dimensional manifold of finite volume, and suppose ∇f is defined on $E\backslash Y$ but is discontinuous on Y with magnitude given by a function $h \in L^1(Y)$. Then we have (from [AFP00])

$$\operatorname{Var}(f) = \int_{E\backslash Y} |\nabla f|\, dx + \int_{Y\cap E} |h(y)|\, dy.$$

In the process of proving equidistribution we will need the following construction. Let $\mathscr{B}_\delta$ be a collection of disjoint boxes of side-length δ that cover E. We define

$$S_{B,\delta}(f) = \sum_{B\in\mathscr{B}_\delta} \sup_{x_1,x_2\in B} \{|f(x_1) - f(x_2)|\}\delta^{n-1}$$

and we then have the following

Lemma 2.17.

$$S_{B,\delta}(f) \ll Var(f).$$

Proof. Intuitively, we can view $S_{B,\delta}$ as a Riemann sum of the distributional derivative of f. Explicitly, we will let μ_n be the n-dimensional Lebesgue measure and μ_{n-1} the $n-1$ dimensional Lebesgue measure on Y. We will begin by summing over $B \in \mathscr{B}_\delta$ such that $B \cap Y = \emptyset$. Over a n-dimensional box of side length δ we estimate the differences using the gradient of f. We have

$$\max_{x_1,x_2\in B}\{|f(x_1) - f(x_2)|\} \leq \max_{x\in B} |\nabla f(x)|\delta\sqrt{n}.$$

Then we can view the sum of the value $\max_{x\in B} |\nabla f(x)|\delta\sqrt{n}$ over the boxes as an upper Riemann integral of the function $|\nabla f|$. We then have

$$\sum_{B\cap Y=\emptyset} \max_{x\in B} |\nabla f(x)|\delta^n = \int_{E\setminus Y} |\nabla f|\, d\mu_n + O(S_{2,\infty}(f)\mu(D)\delta).$$

Then along Y we can view $\sum_{B\cap Y\neq\emptyset} \max_{x_1,x_2\in B}\{|f(x_1) - f(x_2)|\}\delta^{n-1}$ as a Riemann integral of the magnitude of the discontinuity function h, since $\#\{B : B \cap Y \neq \emptyset\} \asymp |Y|\delta^{n-1}$. Thus we have

$$\sum_{B\cap Y\neq\emptyset} \max_{x_1,x_2\in B}\{f(x_1) - f(x_2)\}\delta^{n-1} \asymp \int_Y |h|\, d\mu_{n-1}$$

so overall we have that

$$S_{B,\delta} \ll \int_{E\setminus Y} |\nabla f|\, d\mu_n + \int_Y |h|\, d\mu_{n-1}.$$

This corresponds precisely to the absolutely continuous and singular parts of the total variation of f, with our function f belonging to $SB\mathrm{Var}(E)$ given our assumptions (see [AFP00]). Thus we can conclude that

$$S_{B,\delta}(f) \ll \mathrm{Var}(f).$$

□

In particular we can let χ_Y be an indicator function for the set Y with boundary, ∂Y, a smoothly embedded $n-1$ dimensional manifold of finitely many connected components. Then letting $|\partial Y|$ be the $n-1$ dimensional volume of the boundary we have

$$S_{B,\delta}(\chi_A) \ll |\partial Y| + O_Y(\delta). \tag{2.15}$$

Chapter 3

Equidistribution Results

In this chapter we state our equidistribution results in the explicit coordinates defined in Chapter 2. The integral over a generic translate is given by $\int_{A_\Gamma} \Psi(ag)da$ where $g \in SL_2(\mathbb{R})^n$ and A_Γ is defined in (1.1). However, expressing elements in the ANK decomposition, we have $g = a_y n_x k_\theta$. Thus after changing coordinates on A, a generic translate can be understood by analysis of the translate by elements of N. In the Iwasawa coordinates, the translated orbit, An_T, is given by $(T_1 y_1 + iy_1, \cdots T_n y_n + iy_n)$ where $y \in (\mathbb{R}^+)^n$. As we will need to assume $N(T) \neq 0$, we can re-scale via the coordinate transform $y_i \mapsto y_i/T_i$ giving us $(y_1 + iy_1/T_1, \cdots, y_n + iy_n/T_n)$. For convenience we will let

$$z_k(y, T) = y_k + iy_k/T_k = y_k(1 + i/T_k)$$

and, in cuspidal coordinates

$$Z_k(Y, T) = (1 + i/T_k)Y_0^{1/n} \sum_{j=1}^{n-1} \left(\epsilon_j^{(k)}\right)^{2Y_j}.$$

For convenience, we will refer to the tuple in the following way

$$Z_T(Y) = (Z_1(Y, T), \cdots, Z_n(Y, T)).$$

Along these lines, we define the function Z to convert between the cuspidal coordinates and the Iwasawa coordinates:

$$Z(X, Y) = (x_1 + iy_1, \cdots, x_n + iy_n) = x + iy$$

where $x = XD$ (D defined in (2.9)) and $y_k = Y_0^{1/n} \sum_{j=1}^{n-1} \left(\epsilon_j^{(k)}\right)^{2Y_j}$ as previously described.

3.1 Involutional Symmetry

For

$$\omega = \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \in \Gamma$$

we have the involutional symmetry $\Psi(\omega g) = \Psi(g)$. We use this to reduce our problem to one of a one-sided orbit. Under the action of ω we have that $Z_k(Y, T) \mapsto -1/Z_k(Y, -T)$. This can be realized via the following coordinate transformation:

$$\begin{aligned} Y_0 &\mapsto Y_0^{-1} \prod_{j=1}^{n} (1 + T_j^{-2}) \\ Y_k &\mapsto -Y_k + \frac{1}{2} \sum_{j=1}^{n} e_j^{(k)} \log(1 + T_j^{-2}). \end{aligned}$$

Letting C be a positive constant (to be determined) if we restrict Y_0 to the ray $[C, \infty)$ and apply the above coordinate transform we get:

$$\begin{aligned} &\int_{C}^{\infty} \int_{I^{n-1}} \Psi(Z_T(Y)) \frac{dY}{Y_0} \\ &= \int_{0}^{C^{-1} \prod_j (1+T_j^{-2})^{-1}} \int_{I^{n-1}} \Psi(\cdots, -1/Z_k(Y, -T), \cdots) \frac{dY}{Y_0} \\ &= \int_{0}^{C^{-1} \prod_j (1+T_j^{-2})^{-1}} \int_{I^{n-1}} \Psi(Z_{-T}(Y)) \frac{dY}{Y_0}. \end{aligned}$$

Then defining

$$C(T) = \prod_{i=1}^{n} (1 + T_i^{-2})^{-1/2} \tag{3.1}$$

we have

$$\int_{A_\Gamma} \Psi(a \cdot n_T)\, da = \int_{C(T)}^{\infty} \int_{I^{n-1}} \Psi(Z_T(Y)) \frac{dY}{Y_0} + \int_{C(T)}^{\infty} \int_{I^{n-1}} \Psi(Z_{-T}(Y)) \frac{dY}{Y_0}. \tag{3.2}$$

We will treat only the first integral explicitly but make no assumption on the signs of the T_i. Thus we need to only consider the "one-sided" orbit where Y_0 ranges over $[C(T), \infty)$. We define now using these explicit coordinate the measure that we will analyze:

$$\mu_T(\Psi) = \int_{C(T)}^{\infty} \int_{I^{n-1}} \Psi(Z_T(Y)) \frac{dY}{Y_0}. \tag{3.3}$$

We can now state our equidistribution result in explicit coordinates, and define the distribution μ_E. Recall, $\mathscr{R}$ and $\mathscr{D}$ are defined in Definition 2.5 and $\tilde{E}$ is the regularized Eisenstein series from (2.13). We then have

Theorem 3.1. *There exists a Sobolev type norm $S_{\mathfrak{S}}$ and a constant $\eta > 0$ such that for any $T = (T_1, \cdots, T_n)$ with $N(T) \neq 0$ and any $\varepsilon > 0$ we have*

$$\mu_T(\Psi) = \mu(\Psi)2^{n-2}\mathscr{R}\sqrt{\mathscr{D}}\sum_{j=1}^{n}\log(1+T_j^2) + \langle \Psi, \tilde{E}(\cdot, 1, 0)\rangle + O\left(S_{\mathfrak{S}}(\Psi)\prod_{|T_j|\leq 1}|T_j|^{-\frac{1}{n+1}-\varepsilon}\prod_{|T_j|>1}|T_j|^{-\eta}\right).$$

Then equation (3.2) along with Theorem 3.1 implies Theorem 1.1. To prove Theorem 3.1 we will split Ψ into two parts, the part that gives us the main terms, and the part that results in the error term. To do so we first decompose Ψ as a Fourier series.

3.2 Fourier Decomposition

The action of the subgroup of Γ

$$N \cap \Gamma = \left\{ \left(\begin{array}{cc} 1 & k \\ 0 & 1 \end{array} \right) \middle| k \in \mathscr{O} \right\}$$

on a function periodic in Γ gives periodicity in a (real n-dimensional) lattice in the x coordinates. Recall $\mathscr{O}^*$ is the dual lattice of $\mathscr{O}$, defined in (2.3). We thus have the Fourier decomposition

$$\Psi(x+iy) = \sum_{\alpha^* \in \mathscr{O}^*} a_\Psi(\alpha^*; y)e(\langle \alpha^*, x\rangle)$$

and we can define

$$\Psi^\perp(x+iy) = \sum_{\alpha^* \neq 0} a_\Psi(\alpha^*; y)e(\langle \alpha^*, x\rangle). \tag{3.4}$$

We can now split $\mu_T(\Psi)$ into a main term and error term:

$$\mu_T(\Psi) = \underbrace{\int_{C(T)}^{\infty}\int_{I^{n-1}} a_\Psi(0;y)\,\frac{dY}{Y_0}}_{\mathscr{M}_T(\Psi)} + \underbrace{\int_{C(T)}^{\infty}\int_{I^{n-1}} \Psi^\perp(Z_T(Y))\,\frac{dY}{Y_0}}_{\mathscr{E}_T(\Psi)}. \tag{3.5}$$

We than have the following two theorems. First the decay of $\mathscr{E}_T(\Psi)$:

Theorem 3.2. *With $\mathscr{E}_T(\Psi)$ defined as above, there is a constant $\eta > 0$ and a Sobolev like norm $\mathscr{S}_{\mathfrak{S}}$ such that for all $T = (T_1, \cdots, T_n)$ with $N(T) \neq 0$, and any $\varepsilon > 0$ we have*

$$\mathscr{E}_T(\Psi) \ll S_{\mathfrak{S}}(\Psi)\prod_{|T_j|\leq 1}|T_j|^{-\frac{1}{n+1}-\varepsilon}\prod_{|T_j|>1}|T_j|^{-\eta}.$$

Secondly the equidistribution of $\mathscr{M}_T(\Psi)$:

Theorem 3.3. *For all $T = (T_1, \cdots, T_n)$ we have*

$$\mathcal{M}_T(\Psi) = \mu(\Psi) 2^{n-2} \mathcal{R} \sqrt{\mathcal{D}} \log \prod_j (1 + T_j^2) + \langle \Psi, \tilde{E}(\cdot, 1, 0) \rangle + O(\prod (1 + T_j^2)^{-1/4} \|\Psi\|_2)$$

Theorems 3.2 and 3.3 together with (3.5) imply Theorem 3.1.

3.3 Equidistribution of Partial Orbits

We can also achieve equidistribution by translating certain subsets of $(A \cap \Gamma) \backslash A$. Let $B \subset \mathbb{T}^{n-1}$ be a set of non-zero measure $|B|$, with a boundary ∂B a smooth manifold of dimension $n-2$, with finitely many connected components and of finite measure $|\partial B|$. When $n = 2$ the situation is rather simple as these conditions require B to be a finite collection of intervals, and $|\delta B|$ is just 2 times the number of intervals. We can now define the measure that arises by restricting the orbit we translate by the set B:

$$\mu_{T,B}(\Psi) = \frac{1}{|B|} \int_{C(T)}^{\infty} \int_B \Psi(Z_T(Y)) \frac{dy}{N(y)}. \tag{3.6}$$

For efficiency of notation we will let

$$R(T) = 2^{n-2} \mathcal{R} \sqrt{\mathcal{D}} \log \prod_j (1 + T_j^2)$$

and we have the following equidistribution of this restricted orbit:

Theorem 3.4. *Under the same assumptions as Theorem 3.1 we have*

$$\mu_{T,B}(\Psi) = \mu(\Psi) R(T) + \langle \Psi, \tilde{E}(\cdot, 1, 0) \rangle + O\left(S_{\mathfrak{S}}(\Psi) \left[\frac{1 + |\partial B|}{|B|} \right]^{\frac{n}{n+1}} \prod_{|T_j| \leq 1} |T_j|^{-\frac{1}{n+1} - \varepsilon} \prod_{|T_j| > 1} |T_j|^{-\eta} \right).$$

Note that Theorem 3.4 allows the set B to shrink sufficiently slowly as T grows. With Theorem 3.4 as motivation, we will actually show something slightly more general. We introduce a function ϕ on A that is constant in Y_0, and so ϕ is a function only of Y' (defined in (2.11)). We restrict ϕ to be of unit integral so

$$\int_{[0,1]^{n-1}} \phi(Y') \, dY' = 1. \tag{3.7}$$

Explicitly let $\hat{\phi}$ be a function on $\mathbb{T}^{n-1} = [0,1)^{n-1}$ with $\int_{\mathbb{T}^{n-1}} \hat{\phi} = 1$. We then can define the function ϕ on $\Gamma \backslash \mathbb{H}^n$ by taking $\phi(z) = \hat{\phi}(Y_1(z), \cdots, Y_{n-1}(z))$. Abusing notation slightly, we will define norms on ϕ to be that norm on $\hat{\phi}$. So we define

$\|\phi\|_p$ to be $\|\hat{\phi}\|_p$, and likewise define $S_{p,d}(\phi) := S_{p,d}(\hat{\phi})$ and by $\mathrm{Var}(\phi) := \mathrm{Var}(\hat{\phi})$. We now define the measure which encompasses both μ_T and $\mu_{T,B}$:

$$\mu_{T,\phi}(\Psi) = \int_{C(T)}^{\infty} \int_0^1 \cdots \int_0^1 \Psi(Z_T(Y))\phi(Z_T(Y)) \, \frac{dY}{Y_0}. \tag{3.8}$$

To reduce $\mu_{T,\phi}$ to $\mu_{T,B}$ we let

$$\hat{\phi} = |B|^{-1}\chi_B \tag{3.9}$$

where χ_B is the indicator function of the set B. To reduce to μ_T we take ϕ to be identically 1. Then 3.4 will follow from the following generalizations of Theorems 3.5 and 3.6 which generalize Theorems 3.2 and 3.3 respectively. We define

$$\mathscr{E}_{\phi,T}(\Psi) = \int_{C(T)}^{\infty} \int_{I^{n-1}} \Psi^{\perp}(Z_T(Y))\phi(Z_T(Y)) \, \frac{dY}{Y_0}.$$

We will now explicitly define our Sobolev type norms. We let

$$S_D(\Psi,\phi) = \|\Psi\|_\infty \mathrm{Var}(\phi) + \|\phi\|_\infty S_{\infty,1}(\Psi) \tag{3.10}$$

and using the constant C_F defined in (4.13) we define

$$S_{\mathfrak{S}}(\Psi,\phi) = \left(\|\Psi\|_\infty^{1-\frac{1}{2C_F+1}} S_F(\Psi)^{\frac{1}{2C_F-1}} S_D(\Psi,\phi)^n \right)^{\frac{1}{n+1}}. \tag{3.11}$$

Then we have the following decay of $\mathscr{E}_{\phi,T}(\Psi)$:

Theorem 3.5. *For all T such that $N(T) \neq 0$, there exists a constant $\eta > 0$ such that for any $\varepsilon > 0$ we have*

$$\mathscr{E}_{\phi,T}(\Psi) \ll S_{\mathfrak{S}}(\Psi,\phi) \prod_{|T_j|\leq 1} |T_j|^{-\frac{1}{n+1}-\varepsilon} \prod_{|T_j|>1} |T_j|^{-\eta}.$$

Similarly we define

$$\mathscr{M}_{\phi,T}(\Psi) = \int_{C(T)}^{\infty} \int_{I^{n-1}} a_\Psi(0;y)\phi(Y) \, \frac{dY}{Y_0}$$

and we have the following equidistribution:

Theorem 3.6.

$$\mathscr{M}_{\phi,T}(\Psi) = \mu(\Psi)R(T) + \langle \Psi, \tilde{E}(\cdot,1,0)\rangle + O\left(\prod_j (1+T_j^2)^{-1/4} \|\Psi\|_2 \|\phi\|_2 \right).$$

Note that the main term of $\mathscr{M}_{\phi,T}(\Psi)$ is independent of ϕ because we have assumed ϕ is of unit integral in (3.7). By taking ϕ to be identically 1, Theorems 3.5 and 3.6 reduce to Theorems 3.2 and 3.3. Furthermore, to arrive at Theorem 3.4 we define ϕ as in (3.9) and we have from (2.15) that $\mathrm{Var}(\phi) \asymp \frac{|\partial B|}{|B|}$ and we can compute directly that $\|\phi\|_2 = |B|^{-1/2}$.

Chapter 4

Decay of Fourier Coefficients

To prove Theorem 3.5 we first need to show decay of the Fourier coefficients. For $\alpha^* \in \mathscr{O}^*$ we can consider the set $[\alpha^*] = \{u\alpha^* | u \in U\}$. Then for a smooth function Ψ we can define the supremum of Fourier coefficients over the set $[\alpha^*]$:

$$\tilde{a}_\Psi([\alpha^*]; y) := \sup_{u \in U}\{|a_\Psi(u\alpha^*; y)|\}.$$

The goal of this section is then to prove the following decay of $\tilde{a}_\Psi$; the subscript F in what follows stands for "Fourier coefficients."

Theorem 4.1. *There exists positive constants η_F and C_F (depending on L) and a Sobolev type norm S_F such that uniformly for all $y \in (\mathbb{R}^+)^n$ and all $\alpha^* \in \mathscr{O}^*$ we have*

$$\tilde{a}_\Psi([\alpha^*], y) \ll S_F(\Psi) N(\alpha^*)^{C_F} N(y)^{\eta_F}.$$

We will prove Theorem 4.1 in several steps. We will first show decay relative to a basis: let $\Omega = \{\omega_1, \cdots, \omega_n\}$ be a basis for $\mathscr{O}^*$ as a $\mathbb{Z}$-module. We will consider this fixed and constants can implicitly depend on the choice of basis. We then have

Theorem 4.2. *Let $\alpha^* \in \mathscr{O}^*$ and let $(m_1, \cdots, m_n) \in \mathbb{Z}^n$ be the Ω-coefficients of α^*. Then there exists constants $0 < \eta_F < 1$ and $C_F > 0$ (depending on L) and a Sobolev type norm S_F such that*

$$|a_\Psi(\alpha^*, y)| \ll S_F(\Psi) N(y)^{\eta_F} \prod_{m_i \neq 0} |m_i|^{C_F}.$$

Theorem 4.1 will follow from Theorem 4.2 after establishing the following lemma.

Lemma 4.3. *For any $\alpha^* \in \mathscr{O}^*$ and $y \in (\mathbb{R}^+)^n$ there exists some $u \in U$ and some $y' \in (\mathbb{R}^+)^n$ such that*

$$N(y') = N(y)$$
$$a_\Psi(\alpha^*; y) = a_\Psi(u^2\alpha^*; y')$$
$$\prod_{m_i \neq 0} |m_i| \ll |N(\alpha^*)|$$

where $(m_1, \cdots, m_n) \in \mathbb{Z}^n$ *are the* Ω*-coefficients of* $u^2\alpha^*$*. The implied constant does not depend on* α^**.*

Proof of Lemma 4.3. First we will show that for any $u \in U$, $\alpha^* \in \mathscr{O}^*$ we have

$$a_\Psi(u^2\alpha^*; y) = a_\Psi(\alpha^*; y \cdot \mathrm{diag}(u^2))$$

where

$$\mathrm{daig}(\alpha) = \begin{pmatrix} \alpha^{(1)} & 0 & \cdots & 0 \\ 0 & \alpha^{(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha^{(n)} \end{pmatrix}. \tag{4.1}$$

Let $\mathscr{F}$ be a fundamental domain of the action of $N \cap \Gamma$ on $\mathbb{R}^n$ (the action being translation from (2.5)). By definition we have

$$a_\Psi(u^2\alpha^*; y) = \int_{\mathscr{F}} \Psi(x + iy)\, e(-\langle u^2\alpha^*, x\rangle)\, dx.$$

We then make the change of variables $x' = x \cdot \mathrm{diag}(u^2)$ and $dx' = dx$ since $\det(\mathrm{diag}(u^2)) = N(u^2) = 1$. We let $\mathscr{F}'$ be the transformed domain of integration under this coordinate transform. While this new region of integration is not necessarily the same as before it is still a fundamental domain of the $N \cap \Gamma$ action. We also will use the fact that $a_{u^2} \in \Gamma$ so $\Psi(a_{u^2} \cdot z) = \Psi(z)$. Note that the square is necessary as $a_u \notin \Gamma$ if u is not a square in $\mathscr{O}$ (due to the square roots in the definition of a_y in (2.6)). Thus we have

$$\begin{aligned} a_\Psi(u^2\alpha^*; y) &= \int_{\mathscr{F}'} \Psi(x' \cdot \mathrm{diag}(u^{-2}) + iy))\, e(-\langle \alpha^*, x'\rangle)\, dx' \\ &= \int_{\mathscr{F}'} \Psi(x' + iy \cdot \mathrm{diag}(u^2)))\, e(-\langle \alpha^*, x'\rangle)\, dx' \\ &= a_\Psi(\alpha^*; y \cdot \mathrm{diag}(u^2)). \end{aligned}$$

We note that $N(y \cdot \mathrm{diag}(u^2)) = N(y)$. Letting m be the Ω^*-coefficients of $u^2\alpha^*$, using Corollary 2.9 and Lemma 2.10 we have that there is a $u \in U$ such that $\prod_{m_i \neq 0} |m_i| \ll |N(\alpha^*)|$. ☐

Combining Lemma 4.3 with Theorem 4.2 we obtain for all $u \in U$ the bound

$$a_\Psi(u\alpha^*; y) \ll S_F(\Psi) N(y)^{\eta_F} N(\alpha^*)^{C_F}$$

with the implied constant not depending on u or α^*. Thus the bound holds also for $\tilde{a}_\Psi([\alpha^*], y)$. Thus all that remains to prove Theorem 4.1 is the proof of Theorem 4.2, to prove which we will first need the equidistribution of low-lying "horocyclic patches", given below.

4.1 Equidistribution of horocyclic patches

We now show the effective equidistribution of low-lying horocyclic patches. A "horocyclic patch" is defined in the following way. Let $P \subset \mathbb{R}^n$ be a compact rectangular subset. That is, up to a linear change of coordinates, P is a product of intervals: $P = I_1 \times \cdots \times I_n$. We can construct the horocyclic patch $P(y) \subset \mathbb{H}^n$ by

$$P(y) = \{x + iy \in \mathbb{H}^n : x \in P\}.$$

We let $|P|$ be the volume of the set P. In what follows the subscript H stands for "horocyclic." We have the following lemma.

Lemma 4.4. *There exist positive constants η_H and C_H, both functions of Γ, such that for all $y \in (\mathbb{R}^+)^n$*

$$\frac{1}{|P|}\int_{P(y)} \Psi(x + iy)\, dx = \int_{\Gamma\backslash\mathbb{H}^n} \Psi\, dg + O\left(S_H(\Psi) \cdot |P|^{-C_H} \cdot N(y)^{\eta_H}\right) \tag{4.2}$$

Proof. We begin by noting that the bound is trivial as $N(y) \to \infty$, so we can assume $N(y)$ is bounded away from ∞. Furthermore, for any $u \in U$ we have that $\Psi(a_{u^2}z) = \Psi(z)$, which, as in the proof of Lemma 4.3, gives use the following symmetry:

$$\begin{aligned}\frac{1}{|P|}\int_{P(y)} \Psi(x + iy)\, dx &= \frac{1}{|P|}\int_{P(y)} \Psi(x \cdot \mathrm{diag}(u^2) + iy \cdot \mathrm{diag}(u^2))\, dx \\ &= \frac{1}{|P|}\int_{P(y)\cdot\mathrm{diag}(u^2)} \Psi(x + iy \cdot \mathrm{diag}(u^2))\, dx\end{aligned} \tag{4.3}$$

where the final equality uses the fact that $|\det(\mathrm{diag}(u^2))| = 1$. The set $P(y) \cdot \mathrm{diag}(u^2)$ is also a rectangular set of volume $|P|$, so we can assume that each y_i is bounded above. We begin by constructing a smoothed indicator function of $P(y)$: let $\rho : \mathbb{R} \to \mathbb{R}$ be a smooth, non-negative function with support on $(-1/2, 1/2)$ and $\int_{\mathbb{R}} \rho(x)dx = 1$. For the positive real parameter δ we will define $\rho_\delta(x) = \frac{1}{\delta}\rho(x/\delta)$. We can then define the function $\tilde{\rho}_\delta : \mathbb{R}^n \to \mathbb{R}$ by

$$\tilde{\rho}_\delta(x) = \rho_\delta\left(\sqrt{\sum_j x_j^2}\right).$$

We are now ready to define a smooth indicator function. We will use the $NA\bar{N}$ decomposition on each factor as our coordinates, where $\bar{N}$ consists of the lower triangular unipotent matrices. This gives us the coordinates

$$n_x a_t \bar{n}_{\bar{x}} = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} t^{1/2} & 0 \\ 0 & t^{-1/2} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ \bar{x} & 1 \end{bmatrix}$$

in each component near the identity. Letting $\mathbb{1}_P$ be the indicator function for the set $P \subset \mathbb{R}^n$, we can define the smoothed indicator function, ξ_δ on G:

$$\xi_\delta(g) = c \prod_{j=1}^{n} \rho_\delta(\log(t_j)) \cdot \prod_{j=1}^{n} \rho_\delta(\bar{x}_j) \cdot \left(\tilde{\rho}_\delta \star \frac{1}{|P|}\mathbb{1}_P\right)(x)$$

where c is a constant so we have $\int_G \xi_\delta = 1$ (independent of P). As ξ_δ has compact support we can automorphize ξ_δ by summing over Γ to get

$$\Xi_\delta(g) = \sum_{\gamma \in \Gamma} \xi_\delta(\gamma g).$$

Then we can estimate $S_{\infty,n}(\Xi_\delta) \ll \delta^{-n-1}|P|^{-1}$ and the volume of the support can be estimated as $|P|\delta^n$. We can then estimate $S_{n,2}(\Xi_\delta) \ll \sqrt{S_{\infty,n}(\Xi_\delta)^2|P|\delta^n}$ and we have

$$S_{2,n}(\Xi_\delta) \ll \delta^{-n/2-1}|P|^{-1/2}. \tag{4.4}$$

By applying Theorem 2.14, we obtain

$$\mathscr{C} = \langle a(y) \circ \Psi, \Xi_\delta \rangle = \int_{\Gamma \backslash G} \Psi \, dg + O(S_{2,n}(\Psi)\delta^{-n/2-1}|P|^{-1/2}N(y)^{\eta_M}). \tag{4.5}$$

We can also directly compare the matrix coefficient with the integral over $P(y)$. As $\bar{n}_{\bar{x}} a_y = a_y \bar{n}_{y\bar{x}}$, we have that $\mathscr{C}$ is bounded by the average of Ψ over a δ-thickened indicator of $P(y)$. Thus we can estimate $\mathscr{C}$ in the following way:

$$\mathscr{C} = \frac{1}{|P|} \int_{P(y)} \Psi(x + iy) \, dx + O(\delta S_{\infty,1}(\Psi)). \tag{4.6}$$

Now, letting

$$\delta = \left(|P|^{-1/2} S_{\infty,1}(\Psi)^{-1} S_{2,n}(\Psi) N(y)^{\eta_M}\right)^{\frac{2}{n+4}}$$

we then combine (4.5) and (4.6) and obtain Lemma 4.4 with

$$\begin{aligned} S_H(\Psi) &= S_{\infty,1}(\Psi)^{\frac{n+2}{n+4}} S_{2,n}(\Psi)^{\frac{1}{2n+1}} \\ \eta_H &= \frac{2\eta_M}{n+4} \\ C_H &= \frac{1}{n+4}. \end{aligned} \tag{4.7}$$

□

4.2 Proof of Theorem 4.2

We are now ready to prove the decay relative to a basis:

Proof of theorem 4.2. We begin by applying the coordinate transform defined by the matrix D given in (2.9): we let $X = (X_1, \cdots, X_n) = D^{-1}x$. In these coordinates the action of $N \cap \Gamma$ is that of unit translation in each coordinate direction. Then for $\alpha^* \in \mathscr{O}^*$ and $m = D^{-1}((\alpha^*)^{(1)}, \cdots, (\alpha^*)^{(n)}) \in \mathbb{Z}^n$ we have

$$a_\Psi(\alpha^*; y) = a_\Psi(Dm; y) = \frac{1}{\det(D)} \int_{I^n} \Psi(x + iy) e(-\langle m, X \rangle) dX.$$

We will begin by discretizing our space along each X_i direction that corresponds to a non-zero m_i. We will divide each such axis into $J_i \in \mathbb{N}$ equal segments bounded by the points $X_i^{j_i} = j_i/J_j$ where $j_i \in \{0, J_i\}$. If $m_i = 0$ then we take $J_i = 1$. Then we can re-write our integral as a sum of integrals over the patches defined by this descretization. Let $P_{(j_1,\cdots,j_n)}$ be the patch $[(j_1 - 1)/J_1, j_1/J_1] \times \cdots \times [(j_n - 1)/J_n, j_n/J_n]$ then we have

$$a_\Psi(Dm; y) = \frac{1}{|D|} \sum_{(j_1,\cdots j_n)=(1,\cdots,1)}^{(J_1,\cdots,J_n)} \int_{P_{(j_1,\cdots,j_n)}} \Psi(DX + iy)\, e(-\sum_{i=1}^{n} m_i X_i)\, dX. \tag{4.8}$$

Within each patch we can estimate

$$e(-\sum_{i=1}^{n} m_i X_i) = e(-\sum_{i=1}^{n} m_i \frac{j_i}{J_i}) + O(\sum_{i=1}^{n} \frac{m_i}{J_i}).$$

The term $e(-\sum_{i=1}^{n} m_i \frac{j_n}{J_n})$ is constant inside the integrals so we re-factor (4.8) to obtain

$$\begin{aligned} &\sum_{(j_1,\cdots,j_n)} e(-\sum_{i=1}^{n} m_i \frac{j_n}{J_n}) \int_{P_{(j_1,\cdots,j_n)}} \Psi(DX + iy)\, dX \\ &\qquad + O\left(\sum_{(j_1,\cdots,j_n)} \int_{P_{(j_1,\cdots,j_n)}} |\Psi(DX + iy)| \sum_{i=1}^{n} \frac{m_i}{J_i}\, dX \right). \end{aligned} \tag{4.9}$$

The error term of (4.9) can be recombined into a single integral and then simply bounded by

$$\int_{I^n} |\Psi(x + iy)| \cdot (\sum_{i=1}^{n} \frac{m_i}{J_i})\, dX \ll \|\Psi\|_\infty (\sum_{i=1}^{n} \frac{m_i}{J_i}). \tag{4.10}$$

We then use Lemma 4.4 to estimate the main term of (4.9). The volume of al the $P_{(j_1,\cdots,j_n)}$ is $(J_1 \cdot J_2 \cdots J_n)^{-1}$ so we have

$$\int_{P_{(j_1,\cdots,j_n)}} \Psi(x + iy)\, dX = |P| \int_{\Gamma \backslash \mathbb{H}^n} \Psi(\vec{z})\, d\mu + O\left(S_H(\Psi) N(J)^{C_H - 1} N(y)^{\eta_H}\right). \tag{4.11}$$

Now $\sum e(-\sum_{i=1}^{n} m_i X_i^{j_i}) = 0$ since the sum is over roots of unity. Thus only the error term of (4.11) is left and together with (4.10) we obtain

$$|a_\Psi(Dm; y)| \ll S_H(\Psi) N(J)^{C_H} N(y)^{\eta_H} + \|\Psi\|_\infty (\sum_{i=1}^{n} \frac{m_i}{J_i}). \tag{4.12}$$

What remains is to determine the values of the J_i. For $m_j \neq 0$ we let

$$J_j = \left\lceil m_j \left(\prod_{m_i \neq 0}^{n} |m_i|^{-C_H} \cdot N(y)^{\eta_H} \cdot \|\Psi\|_\infty \cdot S_H(\Psi)^{-1} \right)^{\frac{1}{1+nC_H}} \right\rceil$$

and $J_j = 1$ otherwise. Putting these choices for J_i into (4.12) we have

$$|a_\Psi(Dm; y)| \ll S_F(\Psi) \left(N(y)\right)^{\eta_F} \left(\prod_{m_i \neq 0}^{n} m_i \right)^{C_F}$$

with

$$\begin{aligned} S_F(\Psi) &= \|\Psi\|_\infty^{\dfrac{nC_H}{1+nC_H}} S_H(\Psi)^{\dfrac{1}{1+nC_H}} \\ \eta_F &= \frac{\eta_H}{1+nC_H} \\ C_F &= \frac{C_H}{1+nC_H} \end{aligned} \tag{4.13}$$

and this concludes the proof of Theorem 4.1. □

Chapter 5

Proof of Equidistribution

5.1 Proof of Theorem 3.5

This proof will proceed in two major parts, first we will use the Fourier decomposition of Ψ to write $\mathscr{E}_{\phi,T}(\Psi)$ in terms of the Fourier coefficients $\tilde{a}$, then we will use the decay of Fourier coefficients to finish the proof. Before we begin we need some setup. We will be approximating regions of A as the union of small rectangular sets. We will do this in explicit coordinates, so we let $r \in \mathbb{N}^n$ and define the rectangular region

$$B(r) := \{y \in (\mathbb{R}^+)^n : y_i \in [r_i, r_i + 1],\ i \in \{1, n\}\}.$$

We want to shrink these regions as necessary so for $0 < \delta < 1$ we consider the sets $\delta B(r) = \{\delta^{-1} y \in B(r) : y \in (\mathbb{R}^+)^n\}$ and note that "smallest" value in this rectangular region is $\delta r = (\delta r_1, \cdots, \delta r_n)$. We will then integrate over regions of A by summing over integrals over these rectangular sets that intersect with the region of interest. Before doing that we record a useful estimate.

Lemma 5.1. *For any non-zero $\alpha^* \in \mathscr{O}^*$, and δr bounded away from 0, we have*

$$\sum_{u \in U} \int_{\delta B(r)} e(\langle y, u\alpha^* \rangle) \frac{dy}{N(y)} \ll N(\alpha^*)^{-1} \log(|N(\alpha^*)|)^{n-1}. \tag{5.1}$$

Before proving this we note that a more delicate bound could be had by allowing dependence on δ. However, the nontrivial complication this adds later on does not change the factor of $N(\alpha^*)^{-1}$, which is the consequential aspect of this estimate.

Proof of Lemma 5.1. We first observe that for any $1 > \delta > 0$, $m \in \mathbb{R}$ and x_0 bounded away from 0 and ∞ we have crudely that

$$\int_{x_0}^{x_0(1+\delta)} e(mx) \frac{dx}{x} \ll \min\{|m|^{-1}, 1\}. \tag{5.2}$$

Again, a more delicate bound would be replace 1 with δ, however this adds non-trivial complication to the following computations and doesn't yield a meaningful improvement to our estimations. We then apply (5.2) to each of the iterated integrals of the integral over $\delta B(r)$:

$$\int_{\delta B(r)} e(\langle y, u\alpha^* \rangle) \frac{dy}{N(y)} = \prod_{j=1}^{n} \int_{r_j\delta}^{(r_j+1)\delta} e\left(y_j (\alpha^*)^{(j)} u^{(j)}\right) \frac{dy_j}{y_j}. \tag{5.3}$$

We will be somewhat crude in our application of (5.2) and either estimate each integral as $O(|(\alpha^*)^{(i)}|^{-1})$, giving us the final estimate of $O(N(\alpha^*)^{-1})$, or estimate all but one of them as $O(1)$, giving us the final estimate of $O(\|u\alpha^*\|_\infty^{-1})$. This gives us the following:

$$\int_{\delta B(r)} e(\langle y, u\alpha^* \rangle) \frac{dy}{N(y)} \ll \min\{N(\alpha^*)^{-1}, \|u\alpha^*\|_\infty^{-1}\}. \tag{5.4}$$

As we will be summing over all $u \in U$ we can assume without loss of generality that $\alpha^* = \alpha_0^*$, the representative of $[\alpha^*]$ from Lemma 2.8. We then split the sum over U into two parts: a finite sum over u for which $\|\log u\|_\infty \leq R$ and an infinite sum over u for which $\|\log u\|_\infty > R$, where R is some constant that will be decided later. Then the left hand side of (5.1) can be written as

$$\sum_{\|\log u\|_\infty \leq R} \int_{\delta B(r)} e(\langle y, u\alpha_0^* \rangle) \frac{dy}{N(y)} + \sum_{\|\log u\|_\infty > R} \int_{\delta B(r)} e(\langle y, u\alpha_0^* \rangle) \frac{dy}{N(y)}. \tag{5.5}$$

For the sum over $\|\log u\|_\infty \leq R$ we bound the integral by $N(\alpha^*)^{-1}$ and obtain

$$\sum_{\|\log u\|_\infty < R} \int_{\delta B(r)} e(\langle y, u\alpha_0^* \rangle) \frac{dy}{N(y)} \ll R^{n-1} N(\alpha^*)^{-1} \tag{5.6}$$

where the R^{n-1} comes from the volume of the region $\|\cdot\|_\infty \leq R$ in the hyperplane of units.

For the second sum in (5.5) we bound the integral by $\|u\alpha_0^*\|_\infty^{-1}$. By Lemma 2.8 we have that $|(\alpha_0^*)^{(i)}| \gg N(\alpha)^{1/n}$ for each $(\alpha_0^*)^{(i)}$. Additionally we note that if $\|\log u\|_\infty = r$ then $\max_i\{\log |u^{(i)}|\} \geq r/(n-1)$ and together we have that

$$\|u\alpha_0^*\|_\infty^{-1} \leq e^{-r/(n-1)} N(\alpha_0^*)^{-1/n}. \tag{5.7}$$

Then we apply (5.7) to the second sum of (5.5) and we have

$$\begin{aligned}
\sum_{\|\log u\|_\infty > R} \int_{\delta B(r)} e(\langle y, u\alpha_0^* \rangle) \frac{dy}{N(y)} &\ll \sum_{\|\log u\|_\infty > R} \exp(-\|\log u\|_\infty/(n-1)) |N(\alpha^*)|^{-1/n} \\
&\ll |N(\alpha^*)|^{-1/n} \sum_{r > R} r^{n-1} \exp(-r/(n-1)) \\
&\ll |N(\alpha^*)|^{-1/n} \int_R^\infty r^{n-1} \exp(-r/(n-1))\, dr \\
&\ll |N(\alpha^*)|^{-1/n} R^{n-1} \exp(-R/(n-1)).
\end{aligned}$$

By setting $R = (n-1)^2 \log(|N(\alpha^*)|^{1/n})$ we get our desired result. Note that in the case that $n = 1$, there is no sum as there is no free part to the group of units so Lemma 5.1 is true simply from (5.2). $\square$

Expressed in the cuspidal coordinates, the region of integration of $\mu_{T,\phi}$ is

$$\{Y \in (\mathbb{R}^+)^n : Y_0 \in [C(T), \infty),\ Y_j \in [0,1] \text{ when } j \neq 0\}$$

(with $C(T)$ defined in (3.1)). Noting that $C(T) < 1$, we will define the following subset of this region

$$H = \{Y \in (\mathbb{R}^+)^n : Y_0 \in [1,2],\ Y_j \in [0,1] \text{ when } j \neq 0\}$$

and define

$$\mathscr{B}(H, \delta) := \{r \in \mathbb{Z}^n : Y(\delta B(r)) \cap H \neq \emptyset\} \tag{5.8}$$

where $Y : (\mathbb{R}^+)^n \to (\mathbb{R}^+)^n$ is the coordinate conversion function defined in (2.8).

5.1.1 Using Fourier Decomposition

We will make use of the following abuse of notation: for two n-tuples $x = (x_1, \cdots, x_n)$ and $T = (T_1, \cdots, T_n)$ we will write $x/T = (x_1/T_1, \cdots x_n/T_n)$. We are now ready to show the following Lemma, which is the first step of proving Theorem 3.5.

Lemma 5.2. *Let $S_D(\Psi, \phi) = \|\Psi\|_\infty Var(\phi) + \|\phi\|_\infty S_{\infty,1}(\Psi) + \|\Psi \cdot \phi\|_\infty$. Then for δ_k decaying sufficiently fast with k we have*

$$\mathscr{E}_{\phi,T}(\Psi) \ll \sum_{k=\lfloor \log_2(C(T)) \rfloor}^{\infty} \Bigg[\delta_k S_D(\Psi, \phi) + 2^{-k} k^{n-1} \sum_{r \in \mathscr{B}(H, \delta_k)} \sum_{[\alpha^*] \neq 0} \frac{\tilde{a}_\Psi(\alpha^*; 2^{k/n} \delta_k r/T)}{N(\alpha^*)} \log(|N(\alpha^*)|)^{n-1} \Bigg]$$

Proof of Lemma 5.2. We begin by subdividing $[C(t), \infty)$ into dyadic intervals in the following way:

$$\mathscr{E}_{\phi,T}(\Psi) = \sum_{k=\lfloor \log_2(C) \rfloor}^{\infty} \int_{2^k}^{2^{k+1}} \int_{I^{n-1}} \Psi^\perp(Z_T(Y)) \phi(Z_T(Y)) \frac{dY}{Y_0}.$$

We then apply the change of variables $y_i \mapsto y_i 2^{k/n}$, which corresponds to $Y_0 \mapsto 2^k Y_0$ and $Y_i \mapsto Y_i$ (stays fixed) for $i = 1, \cdots, n-1$. We adapt the notation $Z_T(Y)$ to make this easier to express: we define

$$Z_T^k(Y) = Z_T(2^k Y_0, Y_1, \cdots, Y_{n-1}).$$

Likewise, for using the Iwasawa coordinates we let

$$z_T^k(y) = 2^{k/n} z_T(y)$$

and we have

$$\mathscr{E}_{\phi,T}(\Psi) = \sum_{k=\lfloor \log_2(C(T)) \rfloor}^{\infty} \int_1^2 \int_{I^{n-1}} \Psi^{\perp}(Z_T^k(Y))\phi(Z_T^k(Y)) \frac{dY}{Y_0}. \tag{5.9}$$

We will replace the region H by the union of the boxes $\delta_k B(r)$ where $r \in \mathscr{B}(H, \delta_k)$. Doing so introduces some error. This error can be bounded by the error introduced by expanding H by δ_k in all directions, which in turn can bounded by $\delta_k \|\Psi \cdot \phi\|_\infty$. We have:

$$\begin{aligned}\int_H \Psi^{\perp}(Z_T^k(Y))\phi(Z_T^k(Y)) \frac{dY}{Y_0} = \sum_{r \in \mathscr{B}(H,\delta_k)} \int_{\delta_k B(r)} \Psi^{\perp}(z_T^k(y))\phi(z_T^k(y)) \frac{dy}{N(y)} \\ + O(\delta_k \|\Psi \cdot \phi\|_\infty).\end{aligned} \tag{5.10}$$

We will first focus on

$$\mathscr{E}_k' = \sum_{r \in \mathscr{B}(H,\delta_k)} \int_{\delta_k B(r)} \Psi^{\perp}(z_T^k(y))\phi(z_T^k(y)) \frac{dy}{N(y)}. \tag{5.11}$$

We make the approximation that the imaginary part if $z_T^k(y)$ is constant within each $\delta_k B(r)$. So for $y \in \delta_k B(r)$, instead of evaluating our functions at

$$z_T^k(y) = 2^{k/n}(\cdots, y_j + iy_j/T_j, \cdots)$$

we keep the imaginary part constant at the value $\delta_k r$ and evaluate them at

$$z_T^k(y, r, \delta_k) := 2^{k/n}(\cdots, y_j + i\delta_k r_j/T_j, \cdots).$$

For Ψ we make the estimate that for $y \in \delta_k B(r)$

$$\Psi(z_T^k(y)) = \Psi(z_T^k(y, r, \delta_k)) + O(S_{\infty,1}(\Psi)\delta_k).$$

For ϕ we will be a little less crude and make the following estimation:

$$|\phi(z_T^k(y))| \leq |\phi(z_T^k(y, r, \delta_k))| + \sup_{y \in \delta_k B(r)} |\phi(z_T^k(y)) - \phi(z_T^k(y, r, \delta_k))|.$$

For the sake of being concise we will denote

$$D_{\delta_k B(r)}(\phi, k, T) = \sup_{y \in \delta_k B(r)} |\phi(z_T^k(y)) - \phi(z_T^k(y, r, \delta_k))|. \tag{5.12}$$

Then we can estimate the product $\Psi \cdot \phi$ in the following way:

$$\begin{aligned}\Psi(z_{k,T}(y))\phi(z_{k,T}(y)) =& \Psi(z_T^k(y, r, \delta_k))\phi(z_T^k(y, r, \delta_k)) \\ &+ O(\|\Psi\|_\infty D_{\delta_k B(r)}(\phi, k, T) + \|\phi\|_\infty(\phi) S_{\infty,1}(\Psi)\delta_k).\end{aligned} \tag{5.13}$$

Then combining (5.13) and (5.11) we get

$$\mathcal{E}'_k = \sum_{r\in\mathcal{B}(H,\delta_k)} \int_{\delta_k B(r)} \Psi(z_T^k(y,r,\delta_k))\phi(z_T^k(y,r,\delta_k)) \frac{dy}{N(y)} + O\left(\sum_{r\in\mathcal{B}(H,\delta_k)} \int_{\delta_k B(r)} \|\Psi\|_\infty D_{\delta_k B(r)}(\phi,k,T) + \|\phi\|_\infty(\phi) S_{\infty,1}(\Psi)\delta_k \frac{dy}{N(y)}\right). \tag{5.14}$$

From Lemma 2.17 we have that

$$\sum_{r\in\mathcal{B}(H,\delta_k)} D_{\delta_k B(r)}(\phi,k,T)\delta_k^{n-1} \ll \mathrm{Var}(\phi). \tag{5.15}$$

Now, combining (5.15) with (5.14) and (5.10) we have

$$\int_H \Psi^\perp(Z_T^k(Y)) \frac{dY}{Y_0} = \sum_{r\in\mathcal{B}(H,\delta_k)} \left[\int_{\delta_k B(r)} \Psi^\perp(z_T^k(y,r,\delta_k))\phi(z_T^k(y,r,\delta_k)) \frac{dy}{N(y)}\right] + O(\delta_k S_D(\Psi,\phi))$$

with $S_D(\Psi,\phi)$ the Sobolev type norm given in (3.10). Now we have discretized the imaginary part of the argument and separated the resulting error. We refer to these parts by

$$\mathcal{E}_B(k) = \sum_{r\in\mathcal{B}(H,\delta_k)} \left[\int_{\delta_k B(r)} \Psi^\perp(z_T^k(y,r,\delta_k))\phi(z_T^k(y,r,\delta_k)) \frac{dy}{N(y)}\right] \tag{5.16}$$

and

$$\mathcal{E}_{\mathrm{disc}}(k) = O(\delta_k S_D(\Psi,\phi)).$$

We now focus on $\mathcal{E}_B(k)$ and express $\Psi^\perp$ as its Fourier expansion:

$$\Psi^\perp(x+iy) = \sum_{\alpha^*\neq 0} a_\Psi(\alpha^*;y)e(\langle x,\alpha^*\rangle).$$

The imaginary part of $z_T^k(y,r,\delta_k)$ is $(\cdots, 2^{k/n}\delta_k r_j/T_j, \cdots)$ which we write as $2^{k/n}\delta_k r/T$, so we have

$$\Psi^\perp(z_T^k(y,r,\delta_k)) = \sum_{\alpha^*\neq 0} a_\Psi(\alpha^*; 2^{k/n}\delta_k r/T)e(\langle 2^{k/n}y,\alpha^*\rangle). \tag{5.17}$$

We note now that inside a given $\delta_k B(r)$, only the expression $e(\langle y,\alpha^*\rangle)$ is not constant. So combining (5.17) and (5.16) we have

$$\begin{aligned}
\mathcal{E}_B(k) &= \sum_{r\in\mathcal{B}(H,\delta_k)} \phi(z_T^k(y,r,\delta_k)) \sum_{\alpha^*\neq 0} a_\Psi(\alpha^*; 2^{k/n}\delta_k r/T) \int_{\delta_k B(r)} e(\langle 2^{k/n}y,\alpha^*\rangle) \frac{dy}{N(y)} \\
&= \sum_{r\in\mathcal{B}(H,\delta_k)} \phi(z_T^k(y,r,\delta_k)) \sum_{[\alpha^*]\neq 0} \sum_{u\in U} a_\Psi(u\alpha^*; 2^{k/n}\delta_k r/T) \int_{\delta_k B(r)} e(\langle 2^{k/n}y,u\alpha^*\rangle) \frac{dy}{N(y)} \\
&\leq \sum_{r\in\mathcal{B}(H,\delta_k)} \phi(z_T^k(y,r,\delta_k)) \sum_{[\alpha^*]\neq 0} \tilde{a}_\Psi(\alpha^*; 2^{k/n}\delta_k r/T) \sum_{u\in U} \left|\int_{\delta_k B(r)} e(\langle 2^{k/n}y,u\alpha^*\rangle) \frac{dy}{N(y)}\right|.
\end{aligned}$$

We can then apply Lemma 5.1 and we have

$$\begin{aligned}\mathscr{E}_B(k) &\ll 2^{-k}k^{n-1}\sum_{r\in\mathscr{B}(H,\delta_k)}\phi(Z_{k,T}(Y,r))\sum_{[\alpha^*]\neq 0}\frac{\tilde{a}_\Psi(\alpha^*;2^{k/n}\delta_k r/T)}{N(\alpha^*)}\log(|N(\alpha^*)|)^{n-1}\\ &\ll 2^{-k}k^{n-1}\|\phi\|_\infty\sum_{r\in\mathscr{B}(H,\delta_k)}\sum_{[\alpha^*]\neq 0}\frac{\tilde{a}_\Psi(\alpha^*;2^{k/n}\delta_k r/T)}{N(\alpha^*)}\log(|N(\alpha^*)|)^{n-1}.\end{aligned} \tag{5.18}$$

This concludes the proof upon noting that

$$\mathscr{E}_{\phi,T}(\Psi) = \sum_{k=\lfloor\log_2(C(T))\rfloor}^{\infty}(\mathscr{E}_B(k)+\mathscr{E}_{\mathrm{disc}}(k)). \tag{5.19}$$

□

5.1.2 Applying Decay of Fourier Coefficients

Now we use decay of Fourier coefficients to finish the proof of Theorem 3.5. We will apply decay of Fourier coefficients to the estimate for $\mathscr{E}_B(k)$ from (5.18) and can complete the

proof of Theorem 3.5. We will begin by focusing on on the inner sum of (5.18):

$$\sum_{[\alpha^*]\neq 0}\frac{\tilde{a}_\Psi(\alpha^*;2^{k/n}\delta_k r/T)}{N(\alpha^*)}\log(|N(\alpha^*)|)^{n-1}.$$

We split the sum into two parts: a finite sum over $0 < |N(\alpha^*)| \leq M$ and an infinite sum over $|N(\alpha^*)| > M$:

$$\begin{aligned}S_{>M}(k) &:= \sum_{|N([\alpha^*])|>M}\frac{\tilde{a}_\Psi(\alpha^*;2^{k/n}\delta_k r/T)}{N(\alpha^*)}\log(|N(\alpha^*)|)^{n-1}\\ S_{\leq M}(k) &:= \sum_{0<|N([\alpha^*])|\leq M}\frac{\tilde{a}_\Psi(\alpha^*;2^{k/n}\delta_k r/T)}{N(\alpha^*)}\log(|N(\alpha^*)|)^{n-1}.\end{aligned}$$

We will begin by applying the Cauchy-Schwartz inequality to $S_{>M}(k)$. We have

$$S_{>M}(k) \leq \left(\sum_{N([\alpha^*])>M}\tilde{a}_\Psi(\alpha^*;2^{k/n}\delta_k r/T)^2\right)^{1/2}\left(\sum_{N([\alpha^*])>M}\frac{\log|N(\alpha^*)|^{2n-2}}{N(\alpha^*)^2}\right)^{1/2}.$$

Then we apply Lemma 2.11 to obtain

$$\left(\sum_{N([\alpha^*])>M}\frac{\log|N(\alpha^*)|^{2n-2}}{N(\alpha^*)^2}\right) \ll M^{-1}\log(M)^{2n-2}$$

and conclude

$$S_{>M}(k) \ll \|\Psi\|_\infty M^{-1/2}\log(M)^{n-1}. \tag{5.20}$$

Then to estimate $S_{\leq M}(k)$ we apply the decay of Fourier coefficients from Theorem 4.1 and obtain

$$\begin{aligned} S_{\leq M}(k) &\ll S_F(\Psi)2^{\eta_F k} \sum_{N([\alpha^*])<M} |N(\alpha^*)|^{C_F-1} \log |N(\alpha^*)|^{n-1} |N(\delta_k r/T)|^{\eta_F} \\ &\ll 2^{\eta_F k} S_F(\Psi)|N(T)|^{-\eta_F} N(\delta_k r)^{\eta_F} M^{C_F} \log(M)^{n-1} \end{aligned} \tag{5.21}$$

Letting $M = (2^{-\alpha k}\|\Psi\|_\infty S_F(\Psi)^{-1}|N(T)|^{\eta_F})^{1/(C_F+1/2)}$ we combine (5.20) and (5.21) and we obtain

$$S_{>M}(k) + S_{\leq M}(k) \ll \|\Psi\|_\infty^{1-\frac{1}{2C_F+1}} S_F(\Psi)^{\frac{1}{2C_F-1}} |N(T)|^{-\frac{\eta_F}{2C_F+1}} \log(T)^{n-1} 2^{\frac{k\eta_F}{2C_F+1}}.$$

To make our equations more manageable we will define

$$S(\Psi, T) = \|\Psi\|_\infty^{1-\frac{1}{2C_F+1}} S_F(\Psi)^{\frac{1}{2C_F-1}} |N(T)|^{-\frac{\eta_F}{2C_F+1}} \log(T)^{n-1}$$

and we have

$$\begin{aligned} \mathscr{E}_B(k) &\ll 2^{-k(1-\frac{\eta_F}{2C_F+1})} k^{(n-1)} \sum_{r\in\mathscr{B}(H,\delta_k)} \prod (\delta_k r)_j^{-1} S(\Psi, T) \\ &\ll 2^{-k(1-\frac{\eta_F}{2C_F+1})} k^{(n-1)} S(\Psi, T)(\delta_k)^{-n}. \end{aligned}$$

Now in total we have

$$\mathscr{E}_B(k) + \mathscr{E}_{\text{disc}}(k) \ll 2^{-k(1-\frac{\eta_F}{2C_F+1})} k^{(n-1)} S(\Psi, T)(\delta_k)^{-n} + \delta_k S_D(\Psi, \phi).$$

We then take

$$\delta_k = \left(2^{-k(1-\frac{\eta_F}{2C_F+1})} k^{(n-1)} S(\Psi, T) S_D(\Psi, \phi)^{-1}\right)^{\frac{1}{n+1}}$$

which gives us

$$\mathscr{E}_B(k) + \mathscr{E}_{\text{disc}}(k) \ll 2^{-k(1-\frac{\eta_F}{2C_F+1})/(n+1)} k^{\frac{n-1}{n+1}} \left(S(\Psi, T) S_D(\Psi, \phi)^n\right)^{\frac{1}{n+1}}.$$

Putting this into (5.19) we get

$$\begin{aligned} \mathscr{E} &\ll \left(S(\Psi, T) S_D(\Psi, \phi)^n\right)^{\frac{1}{n+1}} \sum_{k=\lfloor \log_2(C(T)) \rfloor}^{\infty} 2^{-k(1-\frac{\eta_F}{2C_F+1})/(n+1)} k^{\frac{n-1}{n+1}} \\ &\ll \left(S(\Psi, T) S_D(\Psi, \phi)^n\right)^{\frac{1}{n+1}} C(T)^{-(1-\frac{\eta_F}{2C_F+1})/(n+1)}. \end{aligned}$$

Recall from (3.1) that we defined $C(T) = \prod_{i=1}^n (1 + T_i^{-2})^{-1/2}$. We then estimate

$$C(T)^{-1} = \prod_j \frac{\sqrt{1 + T_j^2}}{|T_j|} \ll \prod_{|T_j|\leq 1} |T_j|^{-1}$$

so $N(T)C(T)^{-1} \asymp \prod_{T_i>1} T_i$ and we have proved theorem 3.5 with

$$\eta = \frac{\eta_F}{(2C_F+1)(n+1)}$$
$$S_{\mathfrak{S}}(\Psi,\phi) = (\|\Psi\|_\infty^{1-\frac{1}{2C_F+1}} S_F(\Psi)^{\frac{1}{2C_F-1}} S_D(\Psi,\phi)^n)^{\frac{1}{n+1}}. \tag{5.22}$$

The powers of $\log(T)$ present in $S(\Psi,T)$ result in the exponent of $-\frac{1}{n+1}-\varepsilon$ for the $T_i \leq 1$. For the $T_i > 1$, they are raised to a power η that is bounded strictly below a value, so the powers of $\log(T)$ do not affect this. Note that when ϕ is identically 1, we have that $S_D(\Psi,\phi) \ll S_{1,\infty}(\Psi)$ so this also concludes the proof of Theorem 3.2. □

5.2 Proof of Theorem 3.6

The final step now is to show that our main term $\mathscr{M}_{\phi,T}(\Psi)$ equidistributes.

Proof of Theorem 3.6. Let $\mathscr{F}$ be a fundamental domain of the action of $\Gamma \cap N$ on $\mathbb{R}^n$ (where the action is translation from (2.5)). Then, using the Fourier expansion of ϕ we have the following computation

$$\begin{aligned}
\mathscr{M}_{\phi,T}(\Psi) &= \int_{C(T)}\int_{I^{n-1}} a_\Psi(0;y(Y)/T)\phi(Y)\,\frac{dY}{Y_0} \\
&= \int_{C(T)}\int_{I^{n-1}} \phi(Y)\int_{\mathscr{F}} \Psi(Z(X,Y))\,dX\frac{dY}{Y_0} \\
&= \int_{C(T)}\int_{I^{n-1}} \sum_{m\in\mathbb{Z}^n} a_\phi(m)\int_{\mathscr{F}} \Psi(Z(X,Y))\,dX\frac{dY}{Y_0}.
\end{aligned}$$

In order to apply Lemma 2.12 we use the indicator function of the set $[C(T),\infty)$. We define

$$h_T(Y_0) = \begin{cases} Y_0 & Y_0 \geq N(T)^{-1}C(T) = \prod_j (1+T_j^2)^{-1/2} \\ 0 & \text{otherwise} \end{cases}.$$

We then have the Mellin transform pair for σ such that $\Re(\sigma) > 1$:

$$\begin{aligned}
\hat{h}_T(s) &= \int_0^\infty h_T(Y_0)^{-s-1}dY_0 = \frac{\prod_j (1+T_j^2)^{\frac{1}{2}(s-1)}}{s-1} \\
h_T(Y_0) &= \frac{1}{2\pi i}\int_{(\sigma)} \hat{h}_T(s)Y_0^s ds.
\end{aligned}$$

Then we have

$$\begin{aligned}
\mathscr{M}_{\phi,T}(\Psi) &= \sum_{m\in\mathbb{Z}^{n-1}} a_\phi(m) \int_{\Gamma_\infty\backslash\mathbb{H}^n} \Psi(Z(X,Y))\; h_T(Y_0)\; e(\langle m, Y\rangle)\; dX\frac{dY}{Y_0^2} \\
&= \sum_{m\in\mathbb{Z}^{n-1}} a_\phi(m) \int_{\Gamma_\infty\backslash\mathbb{H}^n} \Psi(Z(X,Y)) \left(\frac{1}{2\pi i}\int_{(\sigma)} \hat{h}_T(s) Y_0^s\; ds\right) e(\langle m, Y\rangle)\; dX\frac{dY}{Y_0^2} \\
&= \sum_{m\in\mathbb{Z}^{n-1}} a_\phi(m) \frac{1}{2\pi i}\int_{(\sigma)} \hat{h}_T(s) \int_{\Gamma_\infty\backslash\mathbb{H}^n} \Psi(Z(X,Y)) Y_0^s\; e(\langle m, Y\rangle)\; dX\frac{dY}{Y_0^2} ds \\
&= \sum_{m\in\mathbb{Z}^{n-1}} a_\phi(m) \frac{1}{2\pi i}\int_{(\sigma)} \hat{h}_T(s)\langle \Psi, E(\cdot, s, m)\rangle\; ds.
\end{aligned}$$

Next we want to shift the contour of integration to $\sigma = 1/2$. To do so we make use of the regularized Eisenstein series given in (2.13). Since $\langle \Psi, \frac{1}{\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)}\rangle = \mu(\Psi)$ and

$$\frac{1}{2\pi i}\int_{(\sigma)} \frac{\prod_j (1+T_j^2)^{\frac{1}{2}(s-1)}}{(s-1)^2}\; ds = \log \prod_j (1+T_j^2)^{\frac{1}{2}}$$

we have

$$\begin{aligned}
\mathscr{M}_{\phi,T}(\Psi) = a_\phi(0)\mu(\Psi) 2^{n-2} \mathscr{R}\sqrt{\mathscr{D}} \log \prod_j (1+T_j^2) \\
+ \frac{1}{2\pi i} \sum_{m\in\mathbb{Z}^{n-1}} a_\phi(m) \int_{(\sigma)} \hat{h}_T(s) \langle \Psi, \tilde{E}(\cdot, s, 0)\rangle\; ds.
\end{aligned}$$

From (3.7) we have that $a_\phi(0) = 1$. Now we shift the contour to $\sigma = 1/2$ and pick up the residue from the simple pole at $s = 1$. For the case where $L = \mathbb{Q}$ there is the possibility of finitely many other exceptional poles in the region $(1/2, 1)$, as is handled in [KK17], but for $n \geq 2$ the only exceptional pole is at $s = 1$ when $m = 0$ ([Efr87]). The shift yields

$$\begin{aligned}
\mathscr{M}_{\phi,T}(\Psi) = \mu(\Psi) 2^{n-2} R\sqrt{D} \log \prod_j (1+T_j^2) + \langle \Psi, \tilde{E}(\cdot, 1, 0)\rangle \\
+ \frac{1}{2\pi i} \sum_{m\in\mathbb{Z}^{n-1}} a_\phi(m) \int_{(1/2)} \hat{h}_T(s) \langle \Psi, \tilde{E}(\cdot, s, m)\rangle\; ds.
\end{aligned}$$

Using the bound (evident from spectral decomposition)

$$\sum_{m\in\mathbb{Z}^{n-1}} \int_{1/2} |\langle \Psi, E(\cdot, s, m)\rangle|^2\; ds \leq \|\Psi\|^2$$

we can then use the Cauchy–Schwarz inequality repeatedly to estimate

$$\sum_{m\in\mathbb{Z}^{n-1}} a_\phi(m) \int_{(1/2)} \hat{h}_T(s)\langle\Psi, \widetilde{E}(\cdot,s,m)\rangle\, ds$$

$$\leq \left[\sum_{m\in\mathbb{Z}^{n-1}} |a_\phi(m)|^2 \sum_{m\in\mathbb{Z}^{n-1}} |\int_{(\frac{1}{2})} \hat{h}_T(s)\langle\Psi, \widetilde{E}(\cdot,s,m)\rangle\, ds|^2\right]^{\frac{1}{2}}$$

$$\leq \|\phi\|_2 \sum_{m\in\mathbb{Z}^{n-1}} \left[\int_{(\frac{1}{2})} |\hat{h}_T(s)|^2\, ds\right]^{\frac{1}{2}} \left[\int_{(\frac{1}{2})} |\langle\Psi, \widetilde{E}(\cdot,s,m)\rangle|^2\, ds\right]^{\frac{1}{2}}$$

$$\leq \|\phi\|_2 \|\Psi\|_2 \prod (1+T_j^2)^{-1/4}$$

and we are done. □

Chapter 6

Counting Quadratic Forms of Square Discriminant

Before proving Theorem 1.2 we will use our equidistribution result to count cosets of lattice points inside the cone-like sets $C_d(R)$ defined in (6.1). These sets are set-wise invariant under the action of A, and as $A \cap \Gamma$ is in general not trivial, we will be counting the number of cosets of $(A \cap \Gamma)$ that lie inside these sets. To relate this to quadratic forms we decompose the set of forms into finitely many Γ orbits. Then for a given orbit we establish the appropriate correspondence between the counting of quadratic forms and the counting of the aforementioned cosets. The strategy used here is standard and follows the pattern of other counting results such as [EM93], [OS13] and [KK20].

6.1 Counting Lattice Cosets

For $R \in \mathbb{R}^+$, $d = \omega^2$ with $\omega \in \mathscr{O}$ we define the cone-like set $C_d(R) \subset \mathbb{H}^n$ given by

$$C_d(R) = \{(z_1, \cdots, z_n) \in \mathbb{H}^n : \sum_{i=1}^{n} d^{(i)} \frac{\Re(z_i)^2}{\Im(z_i)^2} < R^2\}. \tag{6.1}$$

These sets, while infinite in volume are still well rounded in a certain sense:

Lemma 6.1. *Let B_δ be a ball of radius δ. For $g \cdot i \in B_\delta$, $R > \sqrt{nTr(d)}$, $\gamma \in \Gamma$ we have that*

$$(\gamma) \cdot i \in C_R \implies (\gamma g) \cdot i \in C_{R(1+10\delta)}$$

Proof. Let $g \cdot i = (\alpha_1 + i\beta_1, \dots, \alpha_n + i\beta_n)$. Then $g \cdot i \in B_\delta$ implies then that $|\alpha_i| \leq \delta$ and $|\beta_i - 1| \leq 2\delta$. Letting $\gamma \cdot i = (x_1 + iy_1, \cdots x_n + iy_n)$ we then have that $(\gamma g) \cdot i = (\cdots, x_j + y_j\alpha_j + y_j\beta_j i, \cdots) = (\cdots, x_j' + iy_j', \cdots)$. Then we have that

$$\frac{(x_j')^2}{(y_j')^2} = \frac{x_j^2 + 2x_jy_j\alpha_j + y_j^2\alpha_j^2}{\beta_j^2 y_j^2}$$

We then have two cases (for each $1 \le j \le n$). If $x_j \ge y_j$ then we can rearrange and obtain

$$\frac{(x_j')^2}{(y_j')^2} = \frac{x_j^2}{y_j^2} \frac{1 + 2\frac{y_j}{x_j}\alpha_j + \frac{y_j^2}{x_j^2}\alpha_j^2}{\beta_j^2} \le \frac{x_j^2}{y_j^2}(1 + 7\delta).$$

On the other hand if $x_j < y_j$ then we have that

$$\begin{aligned}
\frac{x_j^2 + 2x_j y_j \alpha_j + y_j^2 \alpha_j^2}{\beta_j^2 y_j^2} &= \frac{x_j^2}{y_j^2}\frac{1}{\beta_j^2} + 2\frac{x_j}{y_j \beta_j} + \frac{\alpha_j^2}{\beta_j^2} \\
&\le \frac{x_j^2}{y_j^2}(1 + 3\delta) + 2\delta(1 + 2\delta) + \delta^2(1 + 3\delta) \\
&\le \frac{x_j^2}{y_j^2}(1 + 3\delta) + 3\delta.
\end{aligned}$$

Thus in total we have that

$$\begin{aligned}
\sum_{j=1}^{n} D^{(j)} \frac{x_j'^2}{y_j'^2} &\le R^2(1 + 7\delta) + 3nTr(D)\delta \\
&= R^2(1 + 7\delta + 3\frac{nTr(D)}{R^2}\delta) \le R^2(1 + 10\delta)
\end{aligned}$$

□

The set $C_d(R) \subset \mathbb{H}^n$ is fixed (set-wise) by the action of A so we wish to count equivalence classes of lattice points inside $C_d(R)$, where the equivalence relation is the action of $A \cap \Gamma$. We define

$$\mathcal{N}_{C_d}(R) := \#\{\gamma \in (\Gamma \cap A)\backslash\Gamma : \gamma \cdot z \in C_d(R)\}.$$

We will show that

Theorem 6.2. *For constants $\kappa_1 = \kappa_1(n)$ and $\kappa_2 = \kappa_2(n, \omega)$ and all $R > d$*

$$\mathcal{N}_{C_d}(R) = |N(\omega)| V_n(R) \left(\kappa_1 \log(R) + \kappa_2 + O(R^{\frac{-n\eta}{1+n+9/2}}) \right)$$

We will prove theorem 6.2 in two steps. We construct a periodic indicator function of the set $C_d(R)$. Letting $\chi_{C_d(R)}(z)$ be the indicator function of $C_d(R)$, we construct

$$\Xi_{d,R} := \sum_{\gamma \in (A \cap \Gamma)\backslash\Gamma} \chi_{C_d(R)}(\gamma \cdot z).$$

Letting $\Psi = \sum_{\gamma \in \Gamma} \delta(\gamma \cdot z)$ (where δ is the standard delta distribution supported on $(i, \cdots, i)$) we have that

$$\mathcal{N}_{C_d}(R) = \langle \Xi_{d,R}, \Psi \rangle$$

with the standard inner product in $L^2(\Gamma\backslash\mathbb{H}^n)$. To apply Theorem 1.1 to this situation we need to smooth Ψ. For some $\rho > 0$ we define $\psi_\rho : \mathbb{H}^n \to \mathbb{R}$ be a smooth bump function supported on the product of balls of radius ρ about i. We let $\Psi_\rho = \sum_{\gamma\in\Gamma} \psi_\rho(\gamma \cdot z)$. Then we have the smooth counting function given by $\langle \Xi_{d,R}, \Psi_\rho\rangle$. Using Lemma 6.1, the well roundedness of the set $C_d(R)$ implies that

$$\langle \Xi_{d,R(1-10\rho)}, \Psi_\rho\rangle \leq \mathscr{N}_{C_d}(R) \leq \langle \Xi_{d,R(1+10\rho)}, \Psi_\rho\rangle.$$

Thus we will apply Theorem 1.1 to estimate the smooth counting function $\langle \Xi_{d,R}, \Psi_\rho\rangle$. We have

Lemma 6.3. *Let $\omega \in \mathscr{O}$ and $d = \omega^2$. We define*

$$\beta_d(R) = \int_{\|T\|_2\leq R} \sum_{i=1}^{n} \log(1 + T_i^2 d^{(i)})\, dT.$$

Then with the functions $\Xi_{d,R}$, Ψ_ρ defined as above, we have for $R > \sqrt{nTr(d)}$

$$\langle \Xi_{d,R}, \Psi_\delta\rangle = |N(\omega)|V_n(R)\left(2^{n-1}\mathscr{R}\sqrt{D}\frac{\beta_d(R)}{V_n(R)} + 2E(i,1,0) + O(R^{\frac{-n\eta}{1+n+9/2}})\right).$$

Proof. We have

$$\begin{aligned}
\langle \Xi_{d,R}, \Psi_\delta\rangle &= \int_{\Gamma\backslash\mathbb{H}^n} \Xi_{d,R}(z)\Psi_\delta(z)\, d\mu \\
&= \int_{\Gamma\backslash\mathbb{H}^n} \sum_{\gamma\in(A\cap\Gamma)\backslash\Gamma} \chi_{C_d(R)}(\gamma\cdot z)\Psi_\delta(z)\, d\mu \\
&= \sum_{\gamma\in(A\cap\Gamma)\backslash\Gamma} \int_{\Gamma\backslash\mathbb{H}^n} \chi_{C_d(R)}(\gamma\cdot z)\Psi_\delta(\gamma\cdot z)\, d\mu \\
&= \int_{A_\Gamma\backslash\mathbb{H}^n} \chi_{C_d(R)}(z)\Psi_\delta(z)\, d\mu \\
&= \int_{\sum d^{(i)}T_i^2<R^2} \int_{A_\Gamma} \Psi_\delta(a_y n_T) d\mu_y(y)\, dT \\
&= |N(\omega)| \int_{\|T\|_2\leq R} 2\mu_{(T_i\omega^{(i)})}(\Psi_\delta)\, dT.
\end{aligned}$$

The final equality is achieved by the coordinate transformation $T_i \mapsto T_i/\omega^{(i)}$ (recall $d = \omega^2$). We then apply theorem 1.1 to the innermost integral. The error term is then bounded by a constant times

$$S_{\mathfrak{S}}(\Psi_\delta) \int_0^R \int_{\|T\|_2=r} \prod_{|T_i|\leq 1} |T_i|^{-\frac{1}{n+1}-\varepsilon} \prod_{T_i>1} |T_i|^{-\eta}\, dT\, dr.$$

Examining just the inner integral we have

$$\int_{\|T\|_2=r} \prod_{|T_i|\leq 1} |T_i|^{-\frac{1}{n+1}-\varepsilon} \prod_{T_i>1} |T_i|^{-\eta}\, dT \ll r^{-n\eta} \int_{\|T\|_2=1} \prod_{|T_i|\leq 1} |T_i|^{-\frac{1}{n+1}-\varepsilon} \prod_{T_i>1} |T_i|^{-\eta}\, dT.$$

This last integral is finite since $\frac{1}{1+n} + \varepsilon < 1$. Thus our error term is bounded by

$$S_{\mho}(\Psi_\delta) R^{n(1-\eta)}.$$

For the main term we have a geometric factor

$$\beta_d(R) = \int_{\|T\|_2 \le R} \sum_{i=1}^{n} \log(1 + T_i^2 d^{(i)})\, dT \tag{6.2}$$

We can estimate $S_{\mho}(\Psi_\delta) \ll \delta^{-n-9/2}$ and $\langle \Psi_\delta, \widetilde{E}(\cdot, 1, 0)\rangle = \widetilde{E}(i, 1, 0) + O(\delta)$ and thus by setting

$$\delta = R^{\frac{-n\eta}{1+n+9/2}}$$

we get our result. □

Lemma 6.4. *There are constants $\kappa_1 = \kappa_1(n)$ and $\kappa_2 = \kappa_2(n, d)$ such that for all $R > 0$ we have*

$$\beta_d(R) = \kappa_1 R^n \log(R) + \kappa_2 R^n + O(R^{n-1}).$$

This error term is then swallowed by the error term above.

Proof. From (6.2) we see that $\beta_d(R)$ is symmetric in each T_i. Thus we will first estimate for a single T_i the integral $\int_{\|T\|_2 \le R} \log(1 + T_i^2 d^{(i)}) dT$. We pass to spherical coordinates. Let $T_1 = r\cos(\phi)$, $T_2 = r\cos(\phi)\sin(\theta_1)$ etc. We will only consider the most convenient coordinate, which we have made T_1. We have

$$\int_0^R \int_0^{2\pi} \cdots \int_0^{2\pi} \int_0^{\pi} \log(1 + r^2\cos^2(\phi) d^{(1)})\, r^{n-1}\ \sin^{n-2}(\phi)\ \sin^{n-3}(\theta_1) \\ \cdots \sin(\theta_{n-2})\, d\phi\, d\theta_1 \cdots\ d\theta_{n-1}\, dr.$$

After the change of variables $x = \cos(\phi)$ the inner integral becomes

$$\int_{-1}^{1} \log(1 + r^2 x^2 d^{(1)})(1 - x^2)^{\frac{n-2}{2}}\, dx \\ = (2\log(r) + \log(d^{(1)})) \int_{-1}^{1} (1 - x^2)^{\frac{n-2}{2}}\, dx + \int_{-1}^{1} \log(r^{-2}/d^{(1)} + x^2)(1 - x^2)^{\frac{n-2}{2}}\, dx.$$

Then we can estimate

$$\int_{-1}^{1} \log(r^{-2}/d^{(1)} + x^2)(1 - x^2)^{\frac{n-2}{2}}\, dx = \int_{-1}^{1} \log(x^2)(1 - x^2)^{\frac{n-2}{2}}\, dx + O(r^{-1}).$$

We then sum over the $d^{(i)}$ and integrate over $r \in [0, R]$ and we have our result. □

Combining the above lemmas proves Theorem 6.2.

6.2 Counting Quadratic Forms

We consider binary quadratic forms with coefficients in $\mathcal{O}$:

$$Q_{a,b,c}(x,y) = ax^2 + bxy + cy^2 = (x,y)\begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}(x,y)^t.$$

For convenience we will denote a quadratic form simply by the triple (a,b,c). We consider the two quadratic forms (a,b,c) and (a',b',c') congruent if there is a matrix $\gamma \in SL_2(\mathcal{O})$ such that

$$\begin{bmatrix} a' & b'/2 \\ b'/2 & c' \end{bmatrix} = \gamma \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \gamma^t.$$

This gives us an equivalence relation among quadratic forms, and we will write $(a',b',c') = (a,b,c)^\gamma$. This action of $SL_2(\mathcal{O})$ on the set of quadratic forms can be expressed as a matrix acting on the tuple (a,b,c), which gives us an embedding of $SL_2(\mathcal{O})/\{\pm I\}$ into SO_{b^2-4ac} (where SO_{b^2-4ac} is the generalized special orthogonal group that leaves the ternary quadratic form $b^2 - 4ac$ fixed). Explicitly, for $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we have

$$\iota(\gamma) = \begin{bmatrix} a^2 & 2ab & b^2 \\ ac & bc + ad & bd \\ c^2 & 2cd & d^2 \end{bmatrix}$$

so that $(a,b,c)^\gamma = (a,b,c) \cdot \iota(\gamma)$. For a given discriminant d, the number of equivalence classes is known to be finite (see [Efr87]). For $d = \omega^2$ a perfect square (over $\mathcal{O}$) it can be counted directly (in much the same way that this was counted for forms over $\mathbb{Z}$ in [KK20]). First, a form of square discriminant factors (not necessarily uniquely) into linear forms

$$(a,b,c)(x,y) = (Ax + By)(Cx + Dy).$$

We can represent this form by the matrix $M \in M_{2,\omega}(\mathcal{O})$

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

and then the discriminant is given by $(AD-BC)^2$. Furthermore it can be verified by direct computation that for $\gamma \in \Gamma$ then $(a,b,c)^\gamma$ corresponds to $M\gamma$. Thus the equivalence classes of quadratic forms correspond precisely to the equivalence classes of $M_{2,\omega}(\mathcal{O})/SL_2(\mathcal{O})$, which we will now count.

Lemma 6.5. *For $d = \omega^2$, with $\omega \in \mathcal{O}$, there are $|\mathcal{O}/(\omega)|$ equivalence classes of quadratic forms of discriminant d and the full set of representatives is given by $\{\tau x^2 + \omega xy | \tau \in \mathcal{O}/(\omega)\}$.*

Proof. First we will show that any matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M_{2,\omega}$ is equivalent to a lower triangular matrix. The upper right entree will be $rA + pB$ after multiplication by $\gamma \in SL_2(\mathscr{O})$ where (r) and (p) are comaximal. We know there exist comaximal principal ideals R and P such that $(A)R = (B)P$. We can let $R = (r)$ and $P = (p)$. Thus $Ar = Bxp$ for some x and $Ayr = Bp$ for some y implying that x and y are units. Thus there exist r and p such that $Ar = -Bp$ with (r) and (p) comaximal. Thus there exists $\gamma \in SL_2(\mathscr{O})$ such that

$$M\gamma = \begin{bmatrix} A' & 0 \\ B' & D' \end{bmatrix}.$$

We can drop the primes and consider the matrix $\begin{bmatrix} A & 0 \\ B & D \end{bmatrix}$. This matrix corresponds to the the form $Ax(Bx + Dy) = ABx^2 + ADxy$ with discriminant $(AD)^2$. Without loss of generality we can assume $A = 1$ and thus $d = \omega$. Then we have a full set of representatives given by the set of matrices

$$\left\{ \begin{bmatrix} 1 & 0 \\ \tau & \omega \end{bmatrix} \middle| \tau \in \mathscr{O}/(\omega) \right\}.$$

□

For each class of quadratic forms of discriminant ω^2 we will get the same dynamical situation, as the stabilizer will be conjugate to A, the intersection with Γ will be have the same structure: a free abelian group of order $n-1$, see [Efr87]. In fact, we can compute the stabilizer explicitly. For discriminant $d = \omega^2$, the class of $(0, \omega, 0)$ is stabilized by A. For the class represented by $(0, \omega, \tau)$ the stabilizer consists of the matrices

$$H_\tau = \left\{ \begin{pmatrix} y^{1/2} & \frac{\tau}{2\omega}(y^{1/2} - y^{-1/2}) \\ 0 & y^{-1/2} \end{pmatrix} \middle| y \in \mathbb{R}^+ \right\}.$$

By Lemma 2.6, for any unit u, there is some power such that $(u^k - u^{-k})\beta \in (2\omega)$. Thus $H_\tau \cap \Gamma$ has the same structure as $A \cap \Gamma$. Thus for a set of fundamental units $u_1, \cdots u_{n-1}$ there is a set of integers $k_1(\tau), \cdots k_{n-1}(\tau)$ such that the matrices

$$\begin{pmatrix} u_i^{k_i} & \frac{\tau}{2\omega}(u_i^{k_i} - u_i^{-k_i}) \\ 0 & u_i^{-k_i} \end{pmatrix}$$

generate $H_\tau \cap \Gamma$. Similarly if we can decompose a generic element g into the product $h_g n_g k_g$ where $h_g \in H_\tau$, $n_g \in N$ and $k_g \in K$. Then letting

$$h_\tau(y) = \begin{pmatrix} y^{1/2} & \frac{\tau}{2\omega}(y^{1/2} - y^{-1/2}) \\ 0 & y^{-1/2} \end{pmatrix}$$

we have that $h_\tau(y)n_T \cdot i = iy + y(T + \frac{\tau}{2\omega}) - \frac{\tau}{2\omega}$ so for a function periodic in Γ, we have that $\Psi(h_\tau(y)n_T) = \Psi(n_{-\frac{\tau}{2\omega}} a_y n_{T+\frac{\tau}{2\omega}})$. If we define the shifted function

$$\Psi_{\frac{\tau}{2\omega}}(g) := \Psi(n_{-\frac{\tau}{2\omega}} g)$$

then we can express integrating Ψ over a translated obit of H_τ with the integral of $\Psi_{\frac{\tau}{2\omega}}$ over a translated orbit of A. Thus the only change we need to make to our equidistribution result in this case is to account for the larger fundamental domain of $H_\tau \cap \Gamma$ acting on H_τ. Changing to cuspidal coordinates we can express this fundamental domain by

$$\begin{aligned} &Y_0 \in \mathbb{R}^+ \\ &Y_j \in [0, k_j], \quad j \in \{1, n-1\} \end{aligned}$$

So we have the following lemma

Lemma 6.6. *Letting $N(k(\tau)) = \prod_j k_j(\tau)$ and $n_{T'} = n_{T-\tau/2\omega}$ we have that*

$$\int_{H_\Gamma} \Psi(hn_T)\, dh = N(k)\mu_{T'}(\Psi_{\frac{\tau}{2\omega}}).$$

Since $\mu(\Psi) = \mu(\Psi_{\frac{\tau}{2\omega}})$, we can apply our equidistribution result to the above integral and we that

$$\begin{aligned} \int_{H_\Gamma} \Psi(hn_T)\, dh =& \mu(\Psi) 2^{n-1} N(k) \mathscr{R}\sqrt{\mathscr{D}} \sum \log(1 + (T_j')^2) + 2\langle \Psi, \tilde{E}(\cdot, 1, 0)\rangle \\ &+ O_\Psi\left(\prod_{(T_j')\le 1} (T_j')^{-\frac{1}{n+1}-\varepsilon} \prod_{T_j > 1} (T_j')^{-\eta} \right). \end{aligned}$$

This gives us an analogous counting result for translated cones

$$C_d^\tau(R) = \{(z_1, \cdots, z_n) \in \mathbb{H}^n : \sum_{i=1}^n d^{(i)} \frac{(\Re(z_i) + \tau^{(i)}/2\omega^{(i)})^2}{\Im(z_i)^2} < R^2\}$$

and the shifted counting functions

$$\mathscr{N}_{C_d}^\tau(R) = \#\{\gamma \in \Gamma : \gamma \cdot i \in C_d^\tau(R)\}.$$

By applying Lemma 6.6 to the same setup as with Theorem 6.2 we get the following.

Theorem 6.7. *There exists constants $\kappa_1 = \kappa_1(n)$ and $\kappa_2 = \kappa_2(n, \omega, \tau)$ so that for $R > \sqrt{nTr(d)}$ we have the bound*

$$\mathscr{N}_{C_d}^\tau(R) = |N(\omega)| V_n(R) \left(\kappa_1 \log(R) + \kappa_2 + O(R^{\frac{-n\eta}{1+n+9/2}}) \right).$$

Now for a quadratic form (a, b, c), we give it the norm

$$\|(a, b, c)\| = \sqrt{2Tr(a^2) + Tr(b^2) + 2Tr(c^2)}.$$

This norm is K invariant and thus our equidistribution result lets us count forms under this norm. For $d = \omega^2$, with $\omega \in \mathscr{O}$ we define

$$\mathscr{N}_\omega(R) = \{(a, b, c) \in \mathscr{O}^3 : b^2 - 4ac = \omega^2,\ \|(a, b, c)\| \leq R\}.$$

We will estimate $\mathscr{N}_\omega(R)$ by considering each class separately. We define

$$\mathscr{N}_{(a,b,c)}(R) = \#\{\gamma \in SL_2(\mathscr{O}) : \|(a, b, c)^\gamma\| < R\}$$

and we have

$$\mathscr{N}_\omega(R) = \sum_{\tau \in \mathscr{O}/(\omega)} \mathscr{N}_{(0,\omega,\tau)}(R).$$

We can readily compute the norm of $(0, \omega, 0)^\gamma$. The norm is K-invariant so we can let $\gamma = a_y n_x$. The form $(0, \omega, 0)$ is stabilized by A so we have simply that $(0, \omega, 0)^{n_x} = (0, \omega, x\omega)$. Then $\|(0, \omega, x\omega)\| = \sqrt{Tr(\omega^2) + 2Tr(x^2\omega^2)}$. Thus we have

$$\|(0, \omega, 0)^\gamma\| \leq R \iff \gamma \in C_{\omega^2}\left(\sqrt{\frac{1}{2}(R^2 - Tr(\omega^2))}\right)$$

Similarly for the quadratic form $(0, \omega, \tau)$ we have that

$$\|(0, \omega, \tau)^\gamma\| \leq R \iff \gamma \in C^\tau_{\omega^2}\left(\sqrt{\frac{1}{2}(R^2 - Tr(\omega^2))}\right)$$

Summing over all τ in a set of representatives for $\mathscr{O}/(\omega)$ we arrive at Theorem 1.2.

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