RIBBON COBORDISMS

Marius Huber

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Marius Huber

Advisor: Joshua Evan Greene, Ph.D.

We study ribbon cobordisms between 3-manifolds, i.e. rational homology cobordisms that admit a handle decomposition without 3-handles. We first define and study the more general notion of quasi-ribbon cobordisms, and analyze how lattice-theoretic methods may be used to obstruct the existence of a quasi-ribbon cobordism between two given 3-manifolds. Building on this and on previous work of Lisca, we then determine when there exists such a cobordism between two connected sums of lens spaces. In particular, we show that if an oriented rational homology sphere Y admits a quasi-ribbon cobordism to a lens space, then Y must be homeomorphic to $L(n, 1)$, up to orientation-reversal. As an application, we classify ribbon χ-concordances between connected sums of 2-bridge links. Lastly, we show that the notion of ribbon rational homology cobordisms yields a partial order on the set consisting of aspherical 3-manifolds and lens spaces, thus providing evidence towards a conjecture formulated by Daemi, Lidman, Vela-Vick and Wong.

Contents

List of Figures

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Chapter 1

Introduction

1.1 Background and context

It is a common theme in low-dimensional topology to study manifolds and links inside them by investigating which manifolds they can be realized as the boundary of. For instance, given a knot $K \subset S^3$, the smooth 4-genus $g_4(K)$ is defined as the minimal genus among all surfaces that are smoothly and properly embedded in B^4 and whose boundary is K. This can be regarded as a measure of complexity of a knot, and has been the subject of intense study. By definition, any slice knot K, i.e. one with $g_4(K) = 0$, bounds a smoothly and properly embedded disk $D^2 \subset B^4$, and removing a small enough regular neighborhood (in B^4) of a point $p \in D^2$ yields a concordance from the unknot U to K. That is, U and K cobound an annulus that is smoothly and properly embedded in $S^3 \times [0, 1]$. Thus, one may regard concordance as a measure of similarity of two knots, as measured by g_4.

The study of both the smooth 4-genus and knot concordance translates into the language of 3- and 4-manifolds as follows. Given any smoothly and properly embedded surface $S \subset B^4$, one may form the double cover of B^4 branched along S to obtain a 4-manifold W with boundary $\Sigma_2(S^3, K)$, the double cover of S^3 branched along K. It is a classical fact that, if S happens to be a disk, then W is a *rational homology* 4-ball, i.e. $H_*(W;\mathbb{Q}) \cong H_*(B^4;\mathbb{Q})$. Similarly, if $C \subset S^3 \times [0,1]$ is a concordance from a knot K to another knot K', then $W = \Sigma_2(S^3 \times [0,1], C)$ – which is a cobordism from $\Sigma_2(S^3, K)$ to $\Sigma_2(S^3, K')$ – has the property that inclusion of either boundary component into W induces an isomorphism on the level of rational homology. Such cobordisms are referred to as *rational homology cobordisms*. Thus, similarly to the case of knots, whether or not a 3-manifold (resp. a pair of 3-manifolds) bounds a rational homology 4-ball (resp. cobounds a rational homology cobordism) can be regarded as a measure of complexity (resp. similarity) of 3-manifolds. Moreover, thanks to the construction involving branched double covers that we just described, any statement about the non-existence of a rational homology ball (resp. rational homology cobordism) bounded by a particular manifold (resp. pair of manifolds) translates into a statement about the non-sliceness (resp. non-concordance) of a particular knot (resp. pair of knots). Thus, classifying manifolds up to rational homology cobordism becomes a natural and interesting problem to investigate. Indeed, the question of which rational homology 3-spheres bound rational homology 4-balls appears as Problem 4.5 on Kirby's List [Kir78], where it is attributed to Andrew Casson.

While this question seems too broad to admit a concise answer in general, there are many rational homology 3-spheres that are known to bound or not to bound rational homology 4-balls; see e.g. [CH81] for some of the earliest examples. Amazingly, in the case of spherical 3-manifolds, a complete answer to the above question is known [CP21, Theorem 1.1]. This result heavily relies on work by Paolo Lisca [Lis07a, Lis07b], who in a series of two remarkable papers completely classified lens spaces (and connected sums thereof) up to rational homology cobordism. The starting point of Lisca's work is the observation that by capping off a putative rational homology W cobordism from Y_1 to Y_2 by gluing a 4-manifold to either end, one obtains an embedding of the intersection lattices of the pieces used to cap off W into the intersection lattice of the newly formed closed 4-manifold. In the case where the latter is definite and hence, by Donaldson's celebrated Diagonalization Theorem, isometric to the standard Euclidean lattice, one obtains a very concrete lattice embedding obstruction to Y_1 and Y_2 being rational homology cobordant. This allowed Lisca to classify homology cobordant lens spaces through a purely combinatorial, albeit involved, analysis of embeddings of certain integral lattices into the standard Euclidean lattice. As an application of that classification, Lisca furthermore completely classified 2-bridge links up to concordance, and, in particular, determined which of them are slice.

1.2 Summary of results

A concordance $C \subset S^3 \times [0,1]$ from K to K' is said to be *ribbon*, if it has no local maxima with respect to the second coordinate on $S^3 \times [0,1]$. A *ribbon knot* is one that can be obtained via a ribbon concordance starting at the unknot. Lisca not only determined which 2-bridge links are slice, but also verified that each of those is indeed ribbon, thus verifying the Slice-Ribbon Conjecture for 2-bridge links. It thus remained an open problem to determine when there exists a ribbon concordance from a 2-bridge link to another.

As explained in the previous section, the double cover of $S^3 \times [0,1]$ branched along a concordance between two links K and K' yields a rational homology cobordism between $\Sigma_2(S^3, K)$ and $\Sigma_2(S^3, K')$. In the case where the concordance happens to be ribbon, this rational homology cobordism enjoys the additional property that it admits a handle decomposition relative to $\Sigma_2(S^3, K)$ that does not use any 3-handles. Such cobordisms are said to be *ribbon*, and were defined and studied in [DLVVW20]. In the first non-introductory chapter of this dissertation we generalize the notion of ribbon cobordisms to that of *quasi-ribbon* cobordisms. These are cobordisms that look like ribbon cobordisms through the lens of integral homology. We analyze the (co-)homological properties of such cobordisms and illustrate to what extent lattice-theoretic methods can be used to obstruct their existence.

Quasi-ribbon cobordisms between lens spaces

It turns out that the homological properties of a quasi-ribbon cobordism that reflect the absence of 3-handles in a ribbon cobordism translate into a lattice-theoretic condition that allows one to refine the embedding obstruction used by Lisca. This refined obstruction is strong enough to completely determine when there exists a quasi-ribbon cobordism from one lens space to another.

Theorem 1.2.1. Suppose that $L(p_1, q_1)$ admits a quasi-ribbon cobordism to $L(p_2, q_2)$. Then, up to simultaneous orientation reversal of $L(p_1, q_1)$ and $L(p_2, q_2)$, one of the following holds:

1. $L(p_1, q_1) \cong L(p_2, q_2)$;
2. $L(p_1, q_1) \cong L(n, 1)$ and $p_2/q_2 \in \mathcal{F}_n$, for some $n \geq 2$; or
3. $L(p_1, q_1) \cong S^3$ and $p_2/q_2 \in \mathcal{R}$.

Conversely, in each of these cases $L(p_1, q_1)$ admits a quasi-ribbon cobordism to $L(p_2, q_2)$.

Here, $\mathcal{R}$ and $\mathcal{F}_n$, $n \geq 2$, are sets of rational numbers defined in Lisca's work. As a consequence of the above result, we obtain a complete classification of pairs of 2-bridge links that can be related by a ribbon χ-concordance; we refer the reader to Section 4.2 for a precise definition of ribbon χ-concordance, but point out that this notion coincides with the usual notion of concordance in the case of knots.

Corollary 1.2.2. Let $p_i, q_i \in \mathbb{Z}$ be coprime, $i = 1, 2$. Then, possibly after replacing both $K(p_1, q_1)$ and $K(p_2, q_2)$ by their mirror images, we have that $K(p_1, q_1)$ admits a ribbon χ-concordance to $K(p_2, q_2)$ if and only if one of the following holds:

1. $K(p_1, q_1) \simeq K(p_2, q_2)$;

2. $K(p_1, q_1) \simeq K(n, 1)$ and $p_2/q_2 \in \mathcal{F}_n$, for some $n \geq 2$; or

3. $K(p_1, q_1) \simeq U$ and $p_2/q_2 \in \mathcal{R}$.

Similarly to Lisca's work, the proofs of Theorem 1.2.1 and Corollary 1.2.2 can be pushed further to determine when there exists a quasi-ribbon cobordism (resp. ribbon χ-concordance) from a connected sum of lens spaces to another (resp. from a connected sum of 2-bridge links to another); see Theorem 4.1.2 (resp. Corollary 4.2.3).

Ribbon cobordisms as a partial order

A key difference between the notions of ribbon concordance and general concordance is that only the latter is an obviously symmetric notion; similarly for ribbon cobordism and general rational homology cobordism. Indeed, Cameron Gordon conjectured that ribbon concordance is an asymmetric notion, and hence yields a partial order on the set of knots in S^3 [Gor81, Conjecture 1.1]. This conjecture was famously resolved by Ian Agol [Ago22], leaving open the corresponding version for 3-manifolds [DLVVW20, Conjecture 1.1] which asserts that ribbon cobordism is a partial order on the set of homeomorphism classes of closed, connected, oriented 3-manifolds. Building on the argument used by Agol to prove Gordon's conjecture, we show that the aforementioned conjecture holds true when restricted to the class of aspherical 3-manifolds. Combined with our results pertaining to ribbon cobordisms between lens spaces, this allows us to deduce the following. Note that we do not require the manifolds involved to be closed.

Theorem 1.2.3. Let Y_1 and Y_2 be compact, oriented, 3-manifolds, possibly with boundary, such that there exists a ribbon cobordism W_i from Y_i to Y_j, $\{i, j\} = \{1, 2\}$. If Y_i is either aspherical or a lens space, $i = 1, 2$, then $Y_1 \cong Y_2$.

1.3 Organization

In Chapter 2, we generalize the notion of ribbon cobordisms to that of quasi-ribbon cobordism and study the (co-)homological properties of such cobordisms. In Chapter 3, we proceed to analyze how lattice-theoretic methods can be used to obstruct the existence of quasi-ribbon cobordisms. In Chapter 4, we use these methods to determine when there exists a quasi-ribbon cobordism from a lens space to another, extend this result to the case of connected sums of lens spaces, and deduce the corresponding corollaries about ribbon χ-concordances between 2-bridge links and connected sums thereof. In Chapter 5, we prove our results in support of the conjectured partial order on the set of 3-manifolds. Lastly, in Chapter 6, we close by raising some questions and giving a conjectural answer to the question as to when there exists a quasi-ribbon cobordism from a prism manifold to another.

Conventions and notation

Throughout this dissertation, all manifolds are assumed to be oriented, so that $-Y$ stands for the oriented manifold obtained from Y by reversing orientation, and we write $Y \cong Y'$ to indicate that Y and Y' are related by an orientation-preserving homeomorphism. By a handle decomposition of a 4-dimensional cobordism W from Y to Y' we mean that W is built from $Y \times [0,1]$ by attaching 1-, 2- and 3-handles, where the attaching region of each handle is supported in $\mathrm{int}(Y) \times \{1\}$, or in the boundary of previously attached handles. In particular, if ∂Y is non-empty, the attaching regions of the handles of W avoid $\partial Y \times \{1\} \subset Y \times \{1\}$, and $\partial Y \cong \partial Y'$. In order to minimize the number of signs in Chapter 4, we go by the definition that $L(p,q)$ is the oriented 3-manifold obtained by performing p/q-framed Dehn surgery along the unknot $U \subset S^3$. Finally, the mirror image of a link $K \subset S^3$ is denoted by $\overline{K}$, and we write $K \simeq K'$ to denote that the links K and K' are isotopic.

Chapter 2

Ribbon cobordisms and homology

In this chapter, we collect some basic homological facts about ribbon rational homology cobordisms. These technical results will allow us to formulate a lattice-theoretic obstruction to the existence of such cobordisms. Recall the following definition.

Definition 2.0.1. Let Y_1 and Y_2 be rational homology 3-spheres. A rational homology cobordism W from Y_1 to Y_2 is said to be *ribbon* if W admits a handle decomposition relative to $Y_1 \times I$ that uses 1- and 2-handles only. If such a cobordism exists, we write $Y_1 \leq Y_2$.

Remark. In the remainder of this paper, we will refer to ribbon rational homology cobordisms simply as *ribbon cobordisms*.

We will see in Section 2.1 that the handle structure of a ribbon cobordism is reflected in the vanishing of certain relative integral homology groups. Indeed, since the methods used in Chapter 4 rely solely on these homological properties of ribbon cobordisms, we define and analyze this family of cobordisms that, through the lens of integral homology, look like ribbon cobordisms. Moreover, with the goal of pulling lattices into the picture, we also study the (co-)homology of the closed 4-manifolds that are formed by capping off such a cobordism with 2-handlebodies.

In this chapter – unless stated otherwise – all homology groups are to be understood with integral coefficients and, moreover, all maps between (co-)homology groups are to be understood to be induced by inclusion.

2.1 The (co-)homology of a (quasi-)ribbon cobordism

To motivate the following proposition, we recall that, if W is a ribbon cobordism from Y_1 to Y_2, then the inclusions of Y_1 and Y_2 into W induce injective and surjective maps, respectively, on first integral homology [DLVVW20, Lemma 3.1].

Proposition 2.1.1. Let $W\colon Y_1 \to Y_2$ be a rational homology cobordism. Then the following are equivalent.

1. $\iota_1\colon H_1(Y_1) \to H_1(W)$ is injective and $\iota_2\colon H_1(Y_2) \to H_1(W)$ is surjective.
2. $H_1(W, Y_2) = 0$.
3. $H_2(W, Y_1) = 0$.

Proof. First, observe that, by Lefschetz duality for triples [Hat02, Theorem 3.43] and universal coefficients, we have that

$$\begin{aligned} H_2(W, Y_1) &\cong H^2(W, Y_2) \\ &\cong \mathrm{Hom}(H_2(W, Y_2), \mathbb{Z}) \oplus \mathrm{Ext}(H_1(W, Y_2), \mathbb{Z}) \\ &\cong H_1(W, Y_2). \end{aligned}$$

The last equality follows from the fact that, because W is a rational homology cobordism, $|H_i(W, Y_k)| < \infty$ for all $i \in \mathbb{Z}$ and $k \in \{1, 2\}$. Hence, (2) and (3) are equivalent.

Consider now the following portion of the long exact sequence in homology of the pair (W, Y_k), where $k \in \{1, 2\}$:

$$H_2(W, Y_k) \longrightarrow H_1(Y_k) \xrightarrow{\iota_k} H_1(W) \longrightarrow H_1(W, Y_k)$$

If $H_1(W, Y_2) = 0$, then, by what we have shown so far, we also have $H_2(W, Y_1) = 0$. Thus exactness of the sequence implies injectivity and surjectivity of ι_1 and ι_2, respectively.

Conversely, if (1) holds, so that ι_2 is surjective, the above exact sequence above with $k = 2$ implies that $H_1(W, Y_2) = 0$. □

In light of the above, we make the following definition.

Definition 2.1.2. A rational homology cobordism $W : Y_1 \to Y_2$ is *quasi-ribbon* if any of the conditions from Proposition 2.1.1 holds.

The nomenclature is justified by the following.

Proposition 2.1.3. Any ribbon rational homology cobordism is quasi-ribbon.

Proof. If $W : Y_1 \leq Y_2$, then $H_1(W, Y_2) = 0$, since $-W$ is a rational homology cobordism from Y_2 to Y_1 which admits a handle decomposition that does not use any 1-handles. □

Before moving on to the cohomological side of things, we formulate a simple criterion for a cobordism to be quasi-ribbon that relies just on the orders of the cohomology groups involved. We include the following lemma for completeness and for future reference, and omit its proof. Both statements are an easy consequence of the long exact sequence in homology of the pair $(W, \partial W)$; see [CG86, Lemma 3].

Lemma 2.1.4. Suppose that Y is a rational homology 3-sphere and W is a rational homology 4-ball such that $\partial W = Y$. Then $|H_1(Y)| = m^2$, for some $m \geq 1$, and the image of $H_1(Y) \to H_1(W)$ has order m.

More generally, if $W\colon Y_1 \to Y_2$ is a rational homology cobordism from a rational homology 3-sphere to another, then $|H_1(\partial W)| = m^2$, for some $m \geq 1$, and the image of $H_1(\partial W) \to H_1(W)$ has order m.

Remark 2.1.5. It is not hard to see that the above lemma remains true if one reverses the direction of all maps and replaces all first homology groups with the corresponding second cohomology groups.

Proposition 2.1.6. Let Y_1 and Y_2 be rational homology spheres, and suppose that $W\colon Y_1 \to Y_2$ is a quasi-ribbon cobordism. Then

$$|H_1(Y_2)| = u \cdot |H_1(W)| = u^2 \cdot |H_1(Y_1)|,$$

for some $u \geq 1$.

Proof. Set $p_i = |H_1(Y_i)|$ and $w = |H_1(W)|$. By definition of a quasi-ribbon cobordism, we have that

$$H_1(Y_1) \hookrightarrow H_1(W) \twoheadleftarrow H_1(Y_2).$$

It follows that $p_1|w$ and $w|p_2$, i.e. $p_2 = uw = uvp_1$ for some $u, v \geq 1$. By Lemma 2.1.4, we must have that $p_1p_2 = m^2$, for some $m \geq 1$. Thus, $p_1p_2 = uvp_1^2 = m^2$, so uv must be a perfect square. Moreover, by Lemma 2.1.4 again, and by surjectivity of $H_1(Y_2) \to H_1(W)$, we have that $w = \sqrt{p_1p_2} = p_1\sqrt{uv}$, and hence that $v = \frac{w}{p_1} = \sqrt{uv}$. It follows that $u = v$, which finishes the proof. $\square$

We conclude this section by inspecting the maps on cohomology that are induced by inclusion of the boundary components into a quasi-ribbon cobordism. It turns out that one must require Y_1 and Y_2 to be rational homology spheres in order to have a precise characterization of a cobordism being quasi-ribbon in cohomological terms.

Proposition 2.1.7. If $W\colon Y_1 \to Y_2$ is quasi-ribbon, then $\rho_1\colon H^2(W) \to H^2(Y_1)$ is surjective and $\rho_2\colon H^2(W) \to H^2(Y_2)$ is injective. If Y_1 and Y_2 are rational homology spheres, then the converse holds.

Proof. Suppose that W is quasi-ribbon, and consider the following portion of the long exact sequence in cohomology of the pair (W, Y_k), where $k \in \{1, 2\}$:

$$H^1(Y_k) \longrightarrow H^2(W, Y_k) \longrightarrow H^2(W) \xrightarrow{\rho_k} H^2(Y_k) \longrightarrow H^3(W, Y_k) \tag{2.1}$$

By Lefschetz duality for triples and the fact that W is quasi-ribbon, we have that

$$H^3(W, Y_1) \cong H_1(W, Y_2) = 0,$$

and

$$H^2(W, Y_2) \cong H_2(W, Y_1) = 0.$$

Hence exactness of (2.1) implies that ρ_1 and ρ_2 are surjective and injective, respectively.

Assume now that Y_k is a rational homology sphere, $k \in \{1, 2\}$ and that ρ_1 and ρ_2 are surjective and injective, respectively. Then $H^1(Y_2) = 0$, and thus, by exactness of (2.1) with $k = 2$ and injectivity of ρ_2, it follows that $H^2(W, Y_2) = 0$. By Lefschetz duality for triples, $H_2(W, Y_1) \cong H^2(W, Y_2) = 0$, which implies that W is quasi-ribbon. $\square$

2.2 The cohomology of a capped off quasi-ribbon cobordism

In the following, given compact, closed, oriented 3-manifolds Y_1 and Y_2, let X_1 and X_2 be 2-handlebodies[1] such that $\partial X_1 = -Y_1$ and $\partial X_2 = Y_2$. Moreover, given any

[1] A 2-handlebody is defined to be a 4-manifold that is obtained by attaching finitely many 2-handles to B^4 along some link in $S^3 = \partial B^4$.

cobordism $W\colon Y_1 \to Y_2$, let $Z = X_1 \cup_{Y_1} W \cup_{Y_2} X_2$ denote the closed 4-manifold obtained by capping off W with X_1 and X_2. We now analyze how surjectivity and injectivity of ρ_1 and ρ_2, respectively, translate into this setting. Let us first address surjectivity of $\rho_1\colon H^2(W) \to H^2(Y_1)$.

Proposition 2.2.1. Let $W\colon Y_1 \to Y_2$ be a rational homology cobordism. Then $\rho_1\colon H^2(W) \to H^2(Y_1)$ is surjective iff $r_1\colon H^2(Z) \to H^2(X_1)$ is surjective.

Proof. Suppose first that $r_1\colon H^2(Z) \to H^2(X_1)$ is surjective, and consider the following commutative diagram:

$$\begin{tikzcd} H^2(Z) \arrow[r] \arrow[d, "r_1"] & H^2(W) \arrow[d, "\rho_1"] & \\ H^2(X_1) \arrow[r, "\delta"] & H^2(Y_1) \arrow[r] & H^3(X_1, Y_1) \end{tikzcd}$$

Here, the horizontal maps stem from the long exact sequences in cohomology of the respective pairs. By Poincaré duality and the fact that X_1 is a 2-handlebody, we have that

$$H^3(X_1, Y_1) \cong H_1(X_1) = 0,$$

which implies that δ is surjective. It follows that $\delta \circ r_1$ is surjective, and hence, by commutativity of the diagram, that ρ_1 is surjective.

Conversely, suppose that $\rho_1\colon H^2(W) \to H^2(Y_1)$ is surjective. Define $W_1 = X_1 \cup_{Y_1} W$, and observe that $r_1\colon H^2(Z) \to H^2(X_1)$ factors as

$$H^2(Z) \longrightarrow H^2(W_1) \longrightarrow H^2(X_1). \tag{2.2}$$

Moreover, by excision and Poincaré duality,

$$H^3(Z, W_1) \cong H^3(X_2, Y_2) \cong H_1(X_2) = 0.$$

Hence, by the long exact sequence in cohomology of the pair (Z, W_1), the first map in (2.2) is surjective, and it thus suffices to show that $H^2(W_1) \to H^2(X_1)$ is surjective as well. To that end, consider the following commutative diagram, where the rows stem from the long exact sequences in cohomology of the respective pairs, and the vertical maps are the natural restriction maps.

$$\begin{tikzcd}
H^2(W_1) \arrow[r] \arrow[d] & H^2(X_1) \arrow[r] \arrow[d] & H^3(W_1, X_1) \arrow[r, "\alpha"] \arrow[d, "\varepsilon"', "\cong"] & H^3(W_1) \arrow[r] \arrow[d] & H^3(X_1) \arrow[d] \\
H^2(W) \arrow[r, "\rho_1"] & H^2(Y_1) \arrow[r] & H^3(W, Y_1) \arrow[r, "\beta"] & H^3(W) \arrow[r] & H^3(Y_1)
\end{tikzcd}$$

Note that ε is an isomorphism by excision. Now, by exactness of the bottom row, surjectivity of ρ_1 implies that β is injective, and hence that $\beta \circ \varepsilon$ is injective. It follows from commutativity of the diagram that α is injective, and hence, by exactness of the top row, that $H^2(W_1) \to H^2(X_1)$ is surjective, as desired. $\square$

Before analyzing injectivity of $\rho_1 \colon H^2(W) \to H^2(Y_1)$, we need the following lemma. Recall that, by Remark 2.1.5, $|H^2(Y_1 \amalg Y_2)|$ is a perfect square.

Lemma 2.2.2. If Y_1 and Y_2 are rational homology spheres, then $H_2(X_1 \amalg X_2)$ maps injectively into $H_2(Z)$, and, moreover, we have that

$$[H_2(Z)/\mathrm{Tors} : H_2(X_1 \amalg X_2)] = m,$$

where m is such that $|H^2(Y_1 \amalg Y_2)| = m^2$.

Proof. Set $Y = Y_1 \amalg Y_2$ and $X = X_1 \amalg X_2$. Because Y is the disjoint union of two rational homology spheres, and because X is the disjoint union of two 2-handlebodies, the Mayer-Vietoris sequence associated to the decomposition $Z = X \cup_Y W$ takes the form

$$0 \longrightarrow H_2(X) \oplus H_2(W) \longrightarrow H_2(Z) \longrightarrow H_1(Y) \longrightarrow H_1(W),$$

which shows that the map $H_2(X) \to H_2(Z)$ is injective. Moreover, it follows from inspecting the long exact sequence in homology of the pair (Z, W), together with the fact that $H_2(Z, W) \cong H_2(X, \partial X)$ (by excision) is torsion-free, that $H_2(W)$ maps isomorphically onto the torsion subgroup of $H_2(Z)$. By [CG86, Lemma 3], the image of the rightmost map in the exact sequence above has order m, and the claim follows. □

Proposition 2.2.3. Let Y_1 and Y_2 be rational homology spheres, and suppose that $W\colon Y_1 \to Y_2$ is a rational homology cobordism. Suppose further that $|H^2(Y_2)| = u^2 \cdot |H^2(Y_1)|$, for some $u \geq 1$. If $\rho_2\colon H^2(W) \to H^2(Y_2)$ is injective, then $H_2(Z)$ is torsion-free. If $u > 1$, the converse holds.

Proof. In the following, given a map f, we let $R(f) = |\mathrm{Im}(f)|$ denote the order of the image of f. Let us assume for the moment that we have established that $H_2(Z)$ being torsion-free is equivalent to $|H^2(W)| = pu$. Set $p = |H^2(Y_1)|$ and consider the natural restriction map $\rho\colon H^2(W) \to H^2(\partial W)$. We have that $H^2(\partial W) \cong H^2(Y_1) \oplus H^2(Y_2)$, so that we can write $\rho = \rho_1 \oplus \rho_2$, and, moreover, we have $R(\rho) = \max\{R(\rho_1), R(\rho_2)\}$. On the other hand, since Y_1 and Y_2 are rational homology spheres, $R(\rho) = \sqrt{p \cdot pu^2} = pu$, by Remark 2.1.5. This implies that $|H^2(W)| \geq pu$ and, moreover, that $pu = \max\{R(\rho_1), R(\rho_2)\}$. Now, if ρ_2 is injective, we must have that $|H^2(W)| = pu$, since otherwise $\max\{R(\rho_1), R(\rho_2)\} \geq R(\rho_2) > pu$. By the equivalence that we assumed, this implies that $H_2(Z)$ is torsion-free.

Conversely, if $u > 1$ and $|H^2(W)| = pu$, then $pu = \max\{R(\rho_1), R(\rho_2)\}$ implies that $R(\rho_2) = pu$, which, in turn, means that ρ_2 is injective.

It remains to show that $H_2(Z)$ being torsion-free is equivalent to $|H^2(W)| = pu$. Define $X = X_1 \amalg X_2$, and consider the following portion of the long exact sequence in homology of the pair (Z, X):

$$H_3(Z, X) \longrightarrow H_2(X) \longrightarrow H_2(Z) \longrightarrow H_2(Z, X) \longrightarrow H_1(X) \qquad (2.3)$$

By excision, Poincaré duality, universal coefficients and the fact that W is a rational homology cobordism from a rational homology sphere to another, we have that

$$\begin{aligned} H_3(Z,X) &\cong H_3(W,\partial W) \\ &\cong H^1(W) \\ &\cong \mathrm{Hom}(H_1(W),\mathbb{Z}) \oplus \mathrm{Ext}(H_0(W),\mathbb{Z}) = 0. \end{aligned}$$

Moreover, since X is the disjoint union of two 2-handlebodies, we have that $H_1(X) = 0$. Hence exactness of (2.3) implies that

$$H_2(Z,X) \cong H_2(Z)/H_2(X). \tag{2.4}$$

Observe now that, by excision and Poincaré duality,

$$H_2(Z,X) \cong H_2(W,\partial W) \cong H^2(W),$$

so that, by (2.4),

$$H^2(W) \cong H_2(Z)/H_2(X).$$

Note now that, because X is the disjoint union of two 2-handlebodies, $H_2(X)$ is torsion-free, and hence, by (2.3), must map into the torsion-free part $H_2^F(Z)$ of $H_2(Z)$. It follows that

$$H_2(Z)/H_2(X) = (H_2^F(Z) \oplus H_2^T(Z))/H_2(X) = (H_2^F(Z)/H_2(X)) \oplus H_2^T(Z),$$

where $H_2^T(Z)$ denotes the torsion part of $H_2(Z)$. The first direct summand has cardinality pu by Lemma 2.2.2, and it follows that

$$|H^2(W)| = [H_2(Z)\colon H_2(X)] = pu \cdot |H_2^T(Z)|.$$

Hence $H_2(Z)$ is torsion-free iff $|H^2(W)| = pu$, as claimed. $\square$

To summarize, we obtain the following (co-)homological characterization of quasi-ribbon cobordisms between rational homology spheres. This result will be the key ingredient in the next chapter, where we will translate the (co-)homological properties of a quasi-ribbon cobordism into lattice-theoretic terms.

Theorem 2.2.4. Let Y_1 and Y_2 be rational homology 3-spheres, and suppose that $W\colon Y_1 \to Y_2$ is a rational homology cobordism. Moreover, let $Z = X_1 \cup_{Y_1} W \cup_{Y_2} X_2$ denote the closed 4-manifold obtained by capping off W with 2-handlebodies X_1 and X_2. If W is quasi-ribbon, then

1. $|H_1(Y_2)| = u^2 \cdot |H_1(Y_1)|$, for some $u \geq 1$;

2. $r_1\colon H^2(Z) \to H^2(X_1)$ is surjective; and

3. $H_2(Z)$ is torsion-free.

If $u > 1$, the converse holds.

Proof. Suppose first that W is quasi-ribbon. By Proposition 2.1.6, it follows that (1) holds. Moreover, $\rho_1\colon H^2(W) \to H^2(Y_1)$ and $\rho_2\colon H^2(W) \to H^2(Y_2)$ are surjective and injective, respectively, by Proposition 2.1.7, so (2) and (3) follow from Propositions 2.2.1 and 2.2.3, respectively.

Conversely, suppose $u > 1$ and that (1)–(3) hold. Then, combining Propositions 2.2.1 and 2.2.3, it follows that $\rho_1\colon H^2(W) \to H^2(Y_1)$ and $\rho_2\colon H^2(W) \to H^2(Y_2)$ are surjective and injective, respectively, which, by Proposition 2.1.7, implies that W is quasi-ribbon. $\square$

Chapter 3

Quasi-ribbon cobordisms and lattices

3.1 Preliminaries on lattices

An *integral lattice* is a free Abelian group Λ endowed with a symmetric, bilinear pairing $\langle \cdot, \cdot \rangle \colon \Lambda \times \Lambda \to \mathbb{Z}$. We usually write $x \cdot y$ to mean $\langle x, y \rangle$, $x, y \in \Lambda$. The *rank* of a lattice is defined as the rank of its underlying Abelian group. A matrix $\Gamma = \{\gamma_{ij}\}_{i,j=1}^{n}$ is called a *Gram matrix* for an integral lattice Λ if $\gamma_{ij} = x_i \cdot x_j$, for some basis $\{x_1, \ldots, x_n\}$ of Λ. We say that an integral lattice is *positive definite* (resp. *negative definite*) if any (and therefore all) of its Gram matrices is. Lastly, the *discriminant* $\operatorname{disc}(\Lambda)$ of an integral lattice Λ is defined to be $|\det(\Gamma)|$, where Γ is any Gram matrix for Λ.

Two lattices Λ_1, Λ_2 are *isometric* if there exists an isomorphism $\varphi \colon \Lambda_1 \to \Lambda_2$ that preserves the bilinear pairings, and we write $\Lambda_1 \cong \Lambda_2$. Moreover, we say that Λ_1 and Λ_2 are *stably isometric* (denoted by $\Lambda_1 \simeq \Lambda_2$) if $\Lambda_1 \cong \Lambda_2 \oplus \mathbb{Z}^k$ or $\Lambda_1 \oplus \mathbb{Z}^k \cong \Lambda_2$ for some $k \geq 0$, where $\mathbb{Z}^n$ denotes the standard Euclidean lattice with basis $\{e_1, \ldots, e_n\}$ and pairing given by $e_i \cdot e_j = \delta_{ij}$. To any integral lattice Λ one can associate its *dual* *lattice* $\Lambda^* := \{\xi \in \Lambda \otimes \mathbb{Q} \mid \langle \xi, x \rangle \in \mathbb{Z} \text{ for all } x \in \Lambda\}$, where $\langle \cdot, \cdot \rangle$ is extended to a $\mathbb{Q}$-valued pairing on $\Lambda \otimes \mathbb{Q}$ by $\mathbb{Q}$-bilinearity.

Given a sublattice Λ of M, we define the *orthogonal complement* of Λ in M as $\Lambda^\perp = \{x \in \mathrm{M} \mid x \cdot y = 0 \text{ for all } y \in \Lambda\}$.[1] Finally, we say that a sublattice $\Lambda \subset \mathrm{M}$ is *primitive* if the quotient M/Λ is a free Abelian group. We establish two equivalent characterizations of primitivity that will be used later.

Lemma 3.1.1. Let M be an integral lattice and $\Lambda \subset \mathrm{M}$ a sublattice. Then the following are equivalent.

1. M/Λ is torsion-free.

2. The natural restriction map $r \colon \mathrm{M}^* \to \Lambda^*$ is surjective.

3. $(\Lambda \otimes \mathbb{Q}) \cap \mathrm{M} = \Lambda$.

Proof. (1)⇔(2): Set $n = \mathrm{rk}\mathrm{M}$, $m = \mathrm{rk}\Lambda$, and represent the embedding $\Lambda \subset \mathrm{M}$ by a matrix $A \in \mathbb{Z}^{n\times m}$ with respect to some bases of M and Λ. Then, after endowing M^* and Λ^* with their respective dual bases, the restriction $r \colon \mathrm{M}^* \to \Lambda^*$ is represented by the matrix $A^\top \in \mathbb{Z}^{m\times n}$. Now, by Smith normal form, $A^\top$ is surjective iff $A^\top = P[E_m|0]Q$ for some invertible $P \in \mathbb{Z}^{m\times m}, Q \in \mathbb{Z}^{n\times n}$, where E_m denotes the identity matrix of size $m \times m$. Thus, $A^\top$ is surjective iff $A = Q^\top \left[\frac{E_m}{0}\right] P^\top$, which, by Smith normal form again, is in turn equivalent to M/Λ being torsion-free.

(1)⇔(3): Suppose $(\Lambda \otimes \mathbb{Q}) \cap \mathrm{M} = \Lambda$, and assume for the sake of a contradiction that M/Λ contains torsion. Then there exists some $x \in \mathrm{M} \setminus \Lambda$ with the property that $kx \in \Lambda$ for some $k \geq 2$, which implies that $x \in (\Lambda \otimes \mathbb{Q}) \cap \mathrm{M} \subset \Lambda$, a contradiction.

Conversely, suppose that M/Λ is torsion-free. Because $\Lambda \subset (\Lambda \otimes \mathbb{Q}) \cap \mathrm{M}$, it suffices to show that $(\Lambda \otimes \mathbb{Q}) \cap \mathrm{M} \subset \Lambda$. Pick $x \in (\Lambda \otimes \mathbb{Q}) \cap \mathrm{M}$. Then there exists $k \geq 2$ such that $kx \in \Lambda \cap \mathrm{M} = \Lambda$. Since M/Λ is free, we must therefore have $x \in \Lambda$, too. □

[1] Throughout this Chapter, M denotes a capitalized μ, whereas M stands for a capitalized m, as usual.

Remark 3.1.2. For later reference, we point out that, using characterization (3) from Lemma 3.1.1, it is easily verified that the orthogonal complement $\Lambda^{\perp}$ to any sublattice $\Lambda \subset \mathrm{M}$ is a primitive sublattice, and, moreover, that the only full-rank primitive sublattice of a lattice M is, in fact, M itself.

3.2 Lattices and 4-manifolds

In this subsection, we briefly recall how integral lattices relate to 4-manifolds and 3-manifolds bounding them (for details, see e.g. [Sav12, Sections 5.1 and 6.1] and [GS99, Section 1.2]). Given any compact, oriented 4-manifold X, the intersection form defines a symmetric bilinear pairing

$$Q_X\colon (H_2(X;\mathbb{Z})/\mathrm{Tors}) \otimes (H_2(X;\mathbb{Z})/\mathrm{Tors}) \to \mathbb{Z},$$

and thus endows the torsion-free part of $H_2(X;\mathbb{Z})$ with the structure of an integral lattice

$$\Lambda_X = (H_2(X;\mathbb{Z})/\mathrm{Tors}, Q_X).$$

Moreover, if we abusively let Q_X denote a Gram matrix for Λ_X, the dual lattice is given by

$$\Lambda_X^* = (H^2(X;\mathbb{Z})/\mathrm{Tors}, Q_X^{-1}).$$

Note that the pairing on Λ_X^* is, in general, a $\mathbb{Q}$-valued pairing. It is a $\mathbb{Z}$-valued pairing if $\mathrm{disc}(\Lambda_X) = |\det(Q_X)| = 1$ (which is equivalent to ∂X being a possibly empty disjoint union of integer homology spheres). For a given compact, closed, oriented 3-manifold Y, let X be a 2-handlebody such that $\partial X = Y$. Note that, given a description of Y as integer surgery along some n-component link in $L \subset S^3$, such an X can be obtained by attaching 4-dimensional 2-handles to $S^3 = \partial B^4$ along the link L, with framing given by the surgery coefficients. Endowing the intersection lattice

$\Lambda = (H_2(X;\mathbb{Z}), Q_X)$ with the basis consisting of the surfaces obtained by capping off each of the cores of the 2-handles of X with a Seifert surface for its attaching curve, the Gram matrix of Λ in this basis coincides with the linking matrix $\mathcal{L}$ of L. If, in addition, $\partial X = Y$ is a rational homology sphere, the long exact sequence in cohomology of the pair (X, Y) takes the form

$$0 \longrightarrow H_2(X;\mathbb{Z}) \xrightarrow{\mathcal{L}} H^2(X;\mathbb{Z}) \longrightarrow H^2(Y;\mathbb{Z}) \longrightarrow 0,$$

where we have replaced the first cohomology group with its Poincaré dual. Indeed, one can check that, endowing $H_2(X;\mathbb{Z})$ and $H^2(X;\mathbb{Z})$ with the basis described above and its Hom-dual basis, respectively, the first map in the exact sequence above is represented by $\mathcal{L}$. It follows that the isomorphism type of $H^2(Y;\mathbb{Z}) \cong H_1(Y;\mathbb{Z})$ can be recovered as the quotient of $H^2(X;\mathbb{Z})$ by $H_2(X;\mathbb{Z})$ under the map $\mathcal{L}$. In particular, the order of $H^2(Y;\mathbb{Z}) \cong H_1(Y;\mathbb{Z})$ is given by $|\det(\mathcal{L})| = \mathrm{disc}(\Lambda)$.

3.3 Obstructing quasi-ribbon cobordisms using integral lattices

In this subsection, we investigate how the homological properties of a quasi-ribbon cobordism that we found in Chapter 2 translate into lattice-theoretic terms.

More precisely, let Y_1 and Y_2 be rational homology 3-spheres, and suppose that $W\colon Y_1 \to Y_2$ is any rational homology cobordism. There is an associated closed, oriented 4-manifold

$$Z = X_1 \cup_{Y_1} W \cup_{Y_2} X_2,$$

which is obtained by capping off the cobordism W with 2-handlebodies X_1 and X_2 that can be obtained as described in Subsection 3.2. More precisely, we require $\partial X_1 = -Y_1$ and $\partial X_2 = Y_2$, so that we can glue X_1 and X_2 to the boundary components of

W via orientation-preserving homeomorphisms. Inspecting the long exact sequence in homology of the pair $(Z, X_1 \amalg X_2)$ shows that the map induced by inclusion

$$\varphi\colon H_2(X_1 \amalg X_2;\mathbb{Z}) \to H_2(Z;\mathbb{Z})$$

is injective. Letting Λ_k and Λ_Z denote the intersection lattices $(H_2(X_k;\mathbb{Z}), Q_{X_k})$, and $(H_2(Z;\mathbb{Z})/\mathrm{Tors}, Q_Z)$, respectively, $k = 1, 2$, it follows that we obtain an embedding of integral lattices

$$\varphi\colon \Lambda_1 \oplus \Lambda_2 \hookrightarrow \Lambda_Z.$$

We are now in a position to state the lattice-theoretic reformulation of Theorem 2.2.4.

Theorem 3.3.1. Let Y_1 and Y_2 be rational homology 3-spheres, and suppose that $W\colon Y_1 \to Y_2$ is a rational homology cobordism. Moreover, let $\varphi\colon \Lambda_1 \oplus \Lambda_2 \hookrightarrow \Lambda_Z$ denote the lattice embedding from above. If W is quasi-ribbon, then $H_2(Z;\mathbb{Z})$ is torsion-free, and

1. $\mathrm{disc}(\Lambda_2) = u^2 \cdot \mathrm{disc}(\Lambda_1)$, for some $u \geq 1$; and
2. $\varphi(\Lambda_1) = \varphi(\Lambda_2)^{\perp}$.

If $u > 1$, the converse holds.

Proof. It suffices to show that conditions (1) and (2) are equivalent to the conditions (1) and (2) from Theorem 2.2.4. As explained in Subsection 3.2, we have that

$$\mathrm{disc}(\Lambda_k) = |H_1(Y_k;\mathbb{Z})|,$$

for $k = 1, 2$, which implies that condition (1) from above is equivalent to condition (1) from Theorem 2.2.4. Furthermore, as noted in Subsection 3.2, $\Lambda_Z^* = H^2(Z;\mathbb{Z})$, and hence Lemma 3.1.1 implies that condition (2) from Theorem 2.2.4 is equivalent to $\Lambda_1 \subset \Lambda_Z$ being a primitive sublattice. Observe that $\varphi(\Lambda_1) \subset \varphi(\Lambda_2)^\perp$ is a full-rank sublattice. Thus, by Remark 3.1.2, $\Lambda_1 \subset \Lambda_Z$ being a primitive sublattice is, in turn, equivalent to condition (2) from above. $\square$

Remark 3.3.2. If both Λ_1 and Λ_2 are positive definite, then Z is a closed, orientable, smooth 4-manifold, and hence, by [Don87, Theorem 1], $\Lambda_Z \cong \mathbb{Z}^n$, as integral lattices.

Note that in Theorem 3.3.1, we left condition (3) from Theorem 2.2.4 unchanged; we argue that this property cannot, in fact, be translated into a precise lattice-theoretic counterpart. More precisely, if one wishes to obstruct a given rational homology cobordism $W\colon Y_1 \to Y_2$ from being quasi-ribbon, we may as well assume that

$$|H_1(Y_2;\mathbb{Z})| = u^2 \cdot |H_1(Y_1;\mathbb{Z})|,$$

for some $u \geq 1$, since otherwise Theorem 3.3.1 already excludes the possibility of W being quasi-ribbon (and because it is typically easy to check whether or not this condition holds). Let us further assume that, in fact, W satisfies conditions (1) and (3) from Theorem 3.3.1. It follows that W being quasi-ribbon is equivalent to $H_2(Z;\mathbb{Z})$ being torsion-free. But, since the intersection lattice Λ_Z is insensitive to the torsion part of $H_2(Z;\mathbb{Z})$, it seems unlikely that one can translate injectivity of ρ_2 (which condition (3) from Theorem 2.2.4 stems from) into a property of the lattice embedding $\varphi\colon \Lambda_1 \oplus \Lambda_2 \hookrightarrow \Lambda_Z$.

Nevertheless, as we will see in the following chapter, Theorem 3.3.1 (combined with the machinery set up in [Lis07a, Lis07b]) is enough to completely determine when there exists a quasi-ribbon cobordism from a connected sum of lens spaces to another.

Finally, let us sketch how one can derive further lattice-theoretic obstructions in the case where $|H_1(Y_1;\mathbb{Z})| = |H_1(Y_2;\mathbb{Z})|$ (and Y_i is a rational homology sphere, $i = 1, 2$, as before). Suppose that $W\colon Y_1 \to Y_2$ is a quasi-ribbon cobordism. It follows from Proposition 2.1.7 that both ρ_1 and ρ_2 are isomorphism, and, by [DLVVW20, Lemma 3.2], W is, in fact, an *integral* homology cobordism (indeed, [DLVVW20, Lemma 3.2] is proved in the case where W is a ribbon cobordism, but the exact same proof goes through in the case of a quasi-ribbon cobordism). This implies that there exists a bijection between $\mathrm{Spin}^c(Y_1)$ and $\mathrm{Spin}^c(Y_2)$ which preserves the d-invariants. In fact, since $\mathrm{Spin}^c(Y_k)$ forms a torsor over $H_1(Y_k;\mathbb{Z})$, $k=1,2$, there exists an isomorphism of torsors $\varphi\colon \mathrm{Spin}^c(Y_1) \to \mathrm{Spin}^c(Y_2)$ such that

$$d(Y_1,\mathfrak{s}) = d(Y_2,\varphi(\mathfrak{s})),$$

for all $\mathfrak{s} \in \mathrm{Spin}^c(Y_1)$. If X_k happens to be sharp[2], the d-invariants of Y_k can be computed combinatorially from Λ_k, $k=1,2$. This yields a lattice-theoretic obstruction to there existing a quasi-ribbon cobordism from a rational homology sphere to another one with the same order of first homology. Moreover, if Y_k is a Heegaard Floer L-space, then (up to a multiplicative constant) the Casson-Walker invariant $\lambda_{\mathrm{CW}}(Y_k)$ can be computed as the arithmetic mean of the d-invariants of Y_k, $k=1,2$; see [Rus05, Theorem 5.3.3.]. In this case, by the preceding discussion, it follows that if $W\colon Y_1 \to Y_2$ is quasi-ribbon, then the Casson-Walker invariants of Y_1 and Y_2 must coincide. We summarize this discussion in the following.

Proposition 3.3.3. Let $W\colon Y_1 \to Y_2$ be a quasi-ribbon cobordism, where Y_1 and Y_2 are rational homology 3-spheres such that $|H_1(Y_1;\mathbb{Z})| = |H_1(Y_2;\mathbb{Z})|$. Then the set of d-invariants of Y_1 coincides with that of Y_2 (counted with multiplicity). Moreover, if Y_1 and Y_2 are L-spaces, we have that

$$\lambda_{\mathrm{CW}}(Y_1) = \lambda_{\mathrm{CW}}(Y_2).$$

Remarks 3.3.4.

[2]See e.g. [Gre15, Definition 2.1] for a precise definition.

1. It is clear that the obstruction involving λ_{CW} is a priori less general and weaker than the one involving d-invariants. Nevertheless, the former might be easier to apply in practice because there exist closed formulae for the Casson-Walker invariant for many families of 3-manifolds (see e.g. [Ném05, Section 2.4.2], [BNOV18, Section 2.1] and [Ras04, Section 2.3]), and, moreover, it does not rely on the existence of sharp 4-manifolds bounded by Y_1 and Y_2.

2. The obstruction coming from d-invariants can be extended to one that applies in the case where $|H_1(Y_1;\mathbb{Z})| < |H_1(Y_2;\mathbb{Z})|$; see Section 6.2.

Chapter 4

Quasi-ribbon cobordisms between lens spaces

4.1 Introduction

In this chapter, we use Theorem 3.3.1 from the previous chapter to determine when there exists a ribbon cobordism from a connected sum of lens spaces to another. As an application, we furthermore determine when there exists a χ-concordance from a connected sum of 2-bridge links to another.[1]

Before stating our results, we recall that the question of when two connected sums of lens spaces cobound a rational homology cobordism that is not necessarily ribbon has been completely answered by Lisca in a series of two papers [Lis07a, Lis07b]. More precisely, Lisca defines a set of rational numbers $\mathcal{R}$ with the property that the lens space $L(p, q)$, $p > q > 0$, bounds a rational homology ball (or, equivalently, is rational homology cobordant to S^3) if and only if $p/q \in \mathcal{R}$.[2] In [Lis07b], Lisca

[1] The content of this chapter is that of [Hub21].

[2] Although we will not use the precise form of $\mathcal{R}$, we remark that its original definition [Lis07a, Definition 1.1] is incomplete; see e.g. the footnote in [Lec12, p. 247]

moreover defines the sets

$$\mathcal{F}_n = \left\{ \frac{nm^2}{nmk+1} \;\middle|\; m > k > 0, \gcd(m,k) = 1 \right\} \subset \mathbb{Q},\ n \geq 2,$$

which are in turn characterized by the fact that any lens space $L(p,q)$, $p > q > 0$, with $p/q \in \mathcal{F}_n$ is rational homology cobordant to $L(n,1)$, $n \geq 2$. The sets $\mathcal{R}$ and $\mathcal{F}_n$, $n \geq 2$, are the building blocks of Lisca's classification of connected sums of lens spaces up to rational homology cobordism; we refer the reader to [Lis07b, Theorem 1.1] for a precise statement. In order to demonstrate the aforementioned properties of the sets $\mathcal{R}$ and $\mathcal{F}_n$, $n \geq 2$, Lisca exhibits explicit *ribbon* rational homology cobordisms from S^3 (resp. $L(n,1)$) to any lens space $L(p,q)$ with $p/q \in \mathcal{R}$ (resp. $p/q \in \mathcal{F}_n$). Building on the machinery that Lisca sets up to prove his classification, we prove that *any* ribbon rational homology cobordism from one lens space to another must, in fact, emanate from either S^3, or from a lens space of the form $L(n,1)$, for some $n \geq 2$. Indeed, we completely determine when there exists a ribbon cobordism between two lens spaces.

Theorem 4.1.1. Suppose that $L(p_1,q_1) \leq L(p_2,q_2)$. Then, up to simultaneous orientation reversal of $L(p_1,q_1)$ and $L(p_2,q_2)$, one of the following holds:

1. $L(p_1,q_1) \cong L(p_2,q_2)$;
2. $L(p_1,q_1) \cong L(n,1)$ and $p_2/q_2 \in \mathcal{F}_n$, for some $n \geq 2$; or
3. $L(p_1,q_1) \cong S^3$ and $p_2/q_2 \in \mathcal{R}$.

Conversely, in each of these cases $L(p_1,q_1) \leq L(p_2,q_2)$ holds.

Remarks.

1. Because the proof of Theorem 4.1.1 uses Theorem 3.3.1 as the obstruction to $L(p_1,q_1) \leq L(p_2,q_2)$, the above result remains true when the hypothesis is

weakened to there existing a quasi-ribbon cobordism from $L(p_1, q_1)$ to $L(p_2, q_2)$. In other words, it follows that if there exists a quasi-ribbon cobordism from a lens space L_1 to another lens space L_2, then, in fact, $L_1 \leq L_2$. The same is true for Theorem 4.1.2 below.

2. Theorem 4.1.1 remains true when $L(p_1, q_1)$ is replaced by any oriented rational homology sphere Y. Indeed, if Y admits a ribbon cobordism to a lens space, then $\pi_1(Y)$ is finite cyclic by [DLVVW20, Proposition 1.14], and hence, by Geometrization, Y is a lens space, too.

3. Theorem 4.1.1 can thus be interpreted as saying that the only rational homology spheres that admit a ribbon cobordism to a lens space are the lens spaces of the form $L(n, 1)$, for some $n \geq 1$.

4. While Theorem 4.1.1 makes no statement about the uniqueness of ribbon cobordisms between lens spaces, it follows from combining [Lis07b, Lemma 3.5] with [BBL16, Corollary 1.3] that cases (2) and (3) above can be realized by a ribbon cobordism which uses just one 1-handle, and which is thus of minimal complexity, in the terminology of [AGL18].

Based on Theorem 4.1.1, we determine which pairs of connected sums of lens spaces cobound a ribbon cobordism. Before stating the result, we make the following observation. Suppose that W and W' are ribbon cobordisms from Y_1 to Y_2 and from Y_1' to Y_2', respectively, so that W can be built by attaching 1- and 2-handles to $Y_1 \times I$, and similarly for W'. By attaching corresponding 1- and 2-handles to $(Y_1 \times I) \natural (Y_1' \times I)$ outside of the region where the boundary connected sum takes place, we obtain a ribbon cobordism W'' from $Y_1 \# Y_1'$ to $Y_2 \# Y_2'$.

Theorem 4.1.2. Suppose that $Y_1 \leq Y_2$, where Y_i is a finite connected sum of lens spaces, $i = 1, 2$. Then there exists a ribbon cobordism W from Y_1 to Y_2 that can be decomposed as a boundary connected sum of ribbon cobordisms in such a way that each summand is homeomorphic to one between the following ordered pairs (up to simultaneous orientation reversal of the pair):

1. $(L(p,q), L(p,q))$, $p/q > 1$;
2. $(L(n,1), L(p,q))$, $p/q \in \mathcal{F}_n$, for some $n \geq 2$;
3. $(S^3, L(p,q))$, $p/q \in \mathcal{R}$;
4. $(S^3, L(p,p-q)\#L(p,q))$;
5. $(S^3, L(n,n-1)\#L(p,q))$, $p/q \in \mathcal{F}_n$, for some $n \geq 2$;
6. $(S^3, L(p_1,p_1-q_1)\#L(p_2,q_2))$, $p_i/q_i \in \mathcal{F}_n$, $i = 1,2$, for some $n \geq 2$; or
7. $(S^3, L(p_1,q_1)\#L(p_2,q_2))$, $p_i/q_i \in \mathcal{F}_2$, $i = 1,2$.

Conversely, if (Y_1, Y_2) is any of the pairs from (1)–(7), then $Y_1 \leq Y_2$ holds.

From Theorems 4.1.1 and 4.1.2 we derive two corollaries concerning the concordance of 2-bridge links. The proofs of those corollaries rely on the fact that any lens space $L(p,q)$ arises as the double cover of S^3 branched along the 2-bridge link $K(p,q)$ (see Subsection 4.2).

Outline of the proof

We give an overview of the argument we use to prove Theorems 4.1.1 and 4.1.2. Suppose that W is a ribbon cobordism from $L(p,q)$ to $L(r,s)$, so that $\partial W = -L(p,q) \amalg L(r,s)$. Both $-L(p,q) \cong L(p,p-q)$ and $L(r,s)$ bound positive definite plumbings $X(p,p-q)$ and $X(r,s)$, respectively (see the paragraph preceding Definition 4.3.1), with intersection lattices given by the linear lattices $\Lambda(p/(p-q))$ and $\Lambda(r/s)$, respectively (see Definition 4.3.1). We thus obtain a full-rank isometric embedding of lattices $\varphi\colon \Lambda(p/(p-q)) \oplus \Lambda(r/s) \hookrightarrow \mathbb{Z}^N$, where $\mathbb{Z}^N$ denotes the standard positive definite Euclidean lattice. As we have already seen in Subsection 3.3, the fact that W is a ribbon cobordism translates into the condition $\varphi(\Lambda(p/(p-q))) = \varphi(\Lambda(r/s))^\perp$. Furthermore, it follows from Lisca's combinatorial machinery set up in [Lis07a, Lis07b], that the subset $S \subset \mathbb{Z}^N$ consisting of the images of the standard basis elements of $\Lambda(p/(p-q)) \oplus \Lambda(r/s)$ can be obtained from a certain 'minimal' such subset by repeatedly applying an operation called 2-final expansion (see Subsection 4.4). In Lemma 4.4.2, we show that the stable isometry type of $\varphi(\Lambda(r/s))^\perp$, in fact, remains unchanged under 2-final expansions, i.e. $\varphi(\Lambda(r/s))^\perp$ only changes by adding orthogonal direct summands isometric to $\mathbb{Z}^N$. Combined with Lisca's classification of lens spaces bounding rational balls, and standard facts about linear lattices, this allows us to deduce Theorem 4.1.1.

The proof of Theorem 4.1.2 relies on Theorem 4.1.1, combined with the fact that if a connected sum of lens spaces bounds a rational homology ball, then the corresponding embedding of the orthogonal direct sum of the intersection lattices of the connected summands into $\mathbb{Z}^N$ can be decomposed into smaller embeddings which involve at most two of the direct summands (Lemma 4.4.3).

4.2 Applications to the χ-concordance of 2-bridge links

Recall that the family of 2-bridge links can be parametrized by pairs of coprime integers $p, q \in \mathbb{Z}$ in such a way that $K(p, p-q) \simeq \overline{K(p,q)}$, $\Sigma_2(S^3, K(p,q)) \cong L(p,q)$, and, moreover, that $K(p,q)$ is a knot precisely when p is odd (see e.g. [BZ85, Chapter 12]). Inspired by [DO12, Definition 2], we make the following definition.

Definition 4.2.1. Let $L_0, L_1 \subset S^3$ be links, and let $C \subset S^3 \times I$ be a properly embedded surface satisfying $L_i = C \cap S^3 \times \{i\}$, $i = 0, 1$. We say that C is a ribbon

χ-concordance from L_0 to L_1 if $\chi(C) = 0$ and C has no local maxima with respect to the second coordinate of $S^3 \times I$. If such a concordance exists, we write $L_0 \leq_\chi L_1$.

Note that in the case where both L_0 and L_1 are knots, this notion coincides with the usual notion of ribbon concordance. Furthermore, we remark that our definition is more general than [DO12, Definition 2]. Indeed, that definition requires some decorations on the components of the links involved, which are necessary because connected sum is not well-defined for links, and, eventually, to endow the set of χ-concordance classes with a group structure. The statement of Corollary 4.2.3, however, holds true regardless of how one chooses to form the connected sum of the links involved. Furthermore, [DO12, Definition 2] requires a χ-concordance to have no closed components, which becomes redundant as soon as we demand that C be ribbon. Finally, we point out that we do not require C to be orientable. In fact, as will become apparent in the proof of Corollary 4.2.2, this flexibility is the reason why we choose to go by this definition of concordance, as opposed to one that requires a concordance between links to be a disjoint union of annuli. Nevertheless, it is easily verified that the double cover of $S^3 \times I$ branched along a ribbon χ-concordance from L_0 to L_1 is a ribbon cobordism from $\Sigma_2(S^3, L_0)$ to $\Sigma_2(S^3, L_1)$.

Corollary 4.2.2. Let $p_i, q_i \in \mathbb{Z}$ be coprime, $i = 1, 2$. Then, possibly after replacing both $K(p_1, q_1)$ and $K(p_2, q_2)$ by their mirror images, we have $K(p_1, q_1) \leq_\chi K(p_2, q_2)$ if and only if one of the following holds:

1. $K(p_1, q_1) \simeq K(p_2, q_2)$;
2. $K(p_1, q_1) \simeq K(n, 1)$ and $p_2/q_2 \in \mathcal{F}_n$, for some $n \geq 2$; or
3. $K(p_1, q_1) \simeq U$ and $p_2/q_2 \in \mathcal{R}$.

Proof. If $K(p_1, q_1) \leq_\chi K(p_2, q_2)$, then we have $L(p_1, q_1) \leq L(p_2, q_2)$. Therefore, one of (1)–(3) from Theorem 4.1.1 holds, and the claim follows.

Conversely, it is clear that $K(p_1,q_1) \leq_\chi K(p_2,q_2)$ holds if $K(p_1,q_1) \simeq K(p_2,q_2)$. Suppose that $K(p_1,q_1) \simeq K(n,1)$ and $p_2/q_2 \in \mathcal{F}_n$, for some $n \geq 2$. By [Lis07b, Lemma 3.5], there exists a ribbon move turning $K(p_2,q_2)$ into a split link consisting of $K(n,1)$ and an unknot. Capping off the unknot with a disk yields the desired ribbon χ-concordance. If $p_2/q_2 \in \mathcal{R}$, then, by [Lis07a, Theorem 1.2], $K(p_2,q_2)$ bounds a properly embedded ribbon surface $C \subset B^4$ that is homeomorphic either to a disk or to the disjoint union of a disk with a Möbius band, depending on whether $K(p_2,q_2)$ is a knot or a link, respectively. In either case, $\chi(C) = 1$, so by removing a small disk from C we obtain a ribbon χ-concordance from U to $K(p_2,q_2)$. $\square$

Before stating the analogous corollary to Theorem 4.1.2, we observe that if C and C' are ribbon χ-concordances from K_1 to K_2 and from K_1' to K_2', respectively, where $K_i, K_i' \subset S^3$ are links, $i = 1,2$, we can sum C and C' together along properly embedded intervals $J \subset C$, $J' \subset C'$ that are transverse to $S^3 \times \{t\} \subset S^3 \times I$ for all $t \in I$. This yields a ribbon χ-concordance C'' from $K_1 \# K_1'$ to $K_2 \# K_2'$.

Corollary 4.2.3. Suppose that $K_1 \leq_\chi K_2$, where K_i is a finite connected sum of 2-bridge links, $i = 1,2$. Then there exists a ribbon χ-concordance C from K_1 to K_2 that can be decomposed as a sum of ribbon χ-concordances in such a way that each summand is (possibly after mirroring) a ribbon χ-concordance between one of the following ordered pairs:

1. $(K(p,q), K(p,q))$, $p/q > 1$;

2. $(K(n,1), K(p,q))$, $p/q \in \mathcal{F}_n$, for some $n \geq 2$;

3. $(U, K(p,q))$, $p/q \in \mathcal{R}$;

4. $(U, K(p,p-q) \# K(p,q))$;

5. $(U, K(n,n-1) \# K(p,q))$, $p/q \in \mathcal{F}_n$, for some $n \geq 2$;

6. $(U, K(p_1, p_1 - q_1)\#K(p_2, q_2))$, $p_i/q_i \in \mathcal{F}_n$, $i = 1, 2$, for some $n \geq 2$; or

7. $(U, K(p_1, q_1)\#K(p_2, q_2))$, $p_i/q_i \in \mathcal{F}_2$, $i = 1, 2$.

Conversely, if (K_1, K_2) is any of the pairs from (1)–(7), then $K_1 \leq K_2$ holds.

Proof. Let $C \subset S^3 \times I$ be a ribbon χ-concordance as in the statement of the theorem, so that $W = \Sigma_2(S^3 \times I, C)$ is a ribbon cobordism from Y_1 to Y_2, where $Y_i = \Sigma_2(S^3, K_i)$ is a connected sum of lens spaces of corresponding parameters, $i = 1, 2$. By Theorem 4.1.2, we may assume that W is a boundary connected sum of the ribbon cobordisms listed there. Together with the fact that $\Sigma_2(S^3, K\#K') \cong \Sigma_2(S^3, K)\#\Sigma_2(S^3, K')$, this implies that C must be of the desired form.

Conversely, it suffices to show that each of the cases (1)–(7) can be realized by a ribbon χ-concordance. Cases (1)–(3) follow from Corollary 4.2.2. For case (4), note that $K(p, p-q) \amalg K(p, q) \simeq \overline{K(p, q)} \amalg K(p, q) \subset S^3$ bounds a disjoint union of m annuli, properly embedded in B^4, where m equals either 1 or 2, depending on whether $K(p, q)$ is a knot or a link, respectively. Moreover, this disjoint union of annuli can be chosen not to have any local maxima with respect to the radial distance function on B^4. It follows that $K(p, p - q)\#K(p, q)$ bounds a disjoint union of a disk and, possibly, an annulus with the same property. Puncturing the disk yields a ribbon χ-concordance from U to $K(p, p-q)\#K(p, q)$. For case (5), note that, by [Lis07b, Lemma 3.5] again, $K(p, q)$ can be turned into $K(n, 1)$ by the reverse of a ribbon χ-concordance, which, using case (4), shows that $U \leq_\chi K(n, n-1)\#K(n, 1) \leq_\chi K(n, n-1)\#K(p, q)$. The remaining cases are handled similarly. $\square$

4.3 Linear lattices

In Subsection 3.2, we reviewed how the intersection of a 2-handlebody X relates to the 3-manifold $Y = \partial X$. In the case where $Y = L(p, q)$, $p > q > 0$, one can choose

X to be the plumbing along a linear graph with weights $a_1, \ldots, a_n$, where the $a_i \geq 2$ are such that

$$[a_1, \ldots, a_n]^- := a_1 - \cfrac{1}{a_2 - \cfrac{1}{\cdots - \cfrac{1}{a_n}}} = \frac{p}{q}.$$

We denote this plumbing by $X(p, q)$.

Definition 4.3.1. A lattice Λ is a *linear lattice* if it admits a basis $\{v_1, \ldots, v_n\}$ such that

$$v_i \cdot v_j = \begin{cases} a_i, & \text{if } i = j, \\ 0 \text{ or } 1, & \text{if } |i - j| = 1, \\ 0, & \text{if } |i - j| > 1, \end{cases} \tag{4.1}$$

for some $a_1, \ldots, a_n \geq 2$. In the case where $v_i \cdot v_j = 1$ whenever $|i - j| = 1$, we write $\Lambda = \Lambda(a_1, \ldots, a_n)$ or $\Lambda = \Lambda(p/q)$, where $p/q = [a_1, \ldots, a_n]^-$. Moreover, we will write $\Lambda(\ldots, 2^{[k]}, \ldots) = \Lambda(\ldots, \underbrace{2, \ldots, 2}_{k}, \ldots)$.

Remark 4.3.2. We state without proof two facts about linear lattices that we implicitly use throughout this chapter.

1. Given a linear lattice $\Lambda(p/q)$, $p/q > 1$, the integers $a_1, \ldots, a_n$ are uniquely determined by the conditions $a_i \geq 2$, $i = 1, \ldots, n$, and $[a_1, \ldots, a_n]^- = p/q$ (see e.g. [GS99, Section 5.2]). Moreover, it is easily verified that $\Lambda(p/q)$, $p/q > 1$, is a positive definite lattice.

2. For $p/q > 1$, the intersection lattice of $X(p, q)$ is isometric to $\Lambda(p/q)$. In fact, $\partial X(p, q) \cong \partial X(r, s)$ if and only if $\Lambda(p/q) \cong \Lambda(r/s)$ (see e.g. [Sav12, Section 1.5], and [Ger95, Theorem 3] and [Gre13, Proposition 3.6]).

In what follows, we will always endow a linear lattice Λ with the standard basis $\{v_1, \ldots, v_n\}$ satisfying (4.1).

4.4 Embeddings of linear lattices

We now briefly recall a few definitions from [Lis07a] and [Lis07b] that will play a role later in this chapter.

Definition 4.4.1. Let $S = \{v_1, \dots, v_N\} \subset \mathbb{Z}^N$.

1. S is called a *linear subset* if its elements satisfy

$$v_i \cdot v_j = \begin{cases} a_i, & \text{if } i = j, \\ 0 \text{ or } 1, & \text{if } |i-j| = 1, \\ 0, & \text{if } |i-j| > 1, \end{cases}$$

   for some $a_i \geq 2$.

2. If S is a linear subset, its *intersection graph* is defined as the graph with one vertex for each element $v_i \in S$ and an edge (v_i, v_j) precisely if $v_i \cdot v_j = 1$. The number of connected components of the intersection graph is denoted by $c(S)$.

3. Two elements $v, w \in \mathbb{Z}^N$ are said to be *linked* if there exists an index $k \in \{1, \dots, N\}$ such that $e_k \cdot v \neq 0$ and $e_k \cdot w \neq 0$. Moreover, a linear subset S is called *irreducible* if for any $v, w \in S$ there exist $v_1, \dots, v_m \in S$ such that $v_1 = v$, $v_m = w$, and v_i and v_{i+1} are linked, $i = 1, \dots, m-1$. If S is not irreducible, it is called *reducible*.

4. Let $S = \{v_1, \dots, v_N\}$ be a linear subset such that $|v_i \cdot e_j| \leq 1$ for all $i, j \in \{1, \dots, N\}$, and suppose that there exist $h, s, t \in \{1, \dots, N\}$ such that $v_t \cdot v_t > 2$ and $e_h \cdot v_j \neq 0$ if and only if $j \in \{s, t\}$. Define $S' \subset \langle e_1, \dots, e_{h-1}, e_{h+1}, \dots, e_N \rangle$ by $S' := S \setminus \{v_s, v_t\} \cup \{v_t - (e_h \cdot v_t) e_h\}$. Then S' is said to be obtained from S by a *contraction*, and, conversely, S is said to be obtained from S' by an *expansion*.

5. If, in addition to the hypotheses of (4), both the vertices of the intersection graph of S corresponding to v_s and v_t have degree 1, and, moreover, $v_s \cdot v_s = 2$, we say that S' is obtained from S by a 2-*final contraction*, and that S is obtained from S' by a 2-*final expansion*.

6. Let $S' = \{v_1, \ldots, v_N\} \subset \mathbb{Z}^N$ be a linear subset and suppose that there exists $1 < t < N$ such that $C' = \{v_{t-1}, v_t, v_{t+1}\}$ is a connected component of the intersection graph of S' satisfying $v_{t-1} \cdot v_{t-1} = v_{t+1} \cdot v_{t+1} = 2$, $v_t \cdot v_t > 2$ and $\{i \mid v_i \cdot e_j \neq 0\} = \{t-1, t, t+1\}$, for some $1 \leq j \leq N$. Let $S \subset \mathbb{Z}^M$ be a subset of cardinality $M \geq N$ obtained from S' by applying a sequence of 2-final expansions to the connected component C' of S'. Then the connected component C of of the intersection graph of S that naturally corresponds to C' is said to be a *bad component* of S. The number of bad components of S is denoted by $b(S)$.

As an example illustrating the above definition, consider $S = \{v_1, v_2, v_3, v_4, v_5\} \subset \mathbb{Z}^5$ where $v_1 = e_1 - e_2$, $v_2 = e_3 - e_4 + e_5$, $v_3 = e_1 + e_2 + e_3$, $v_4 = e_3 + e_4$ and $v_5 = e_4 + e_5$. It is easily checked that S is an irreducible linear subset of $\mathbb{Z}^5$, and, moreover, that its intersection graph is the disjoint union of a single isolated vertex and the linear graph on four vertices. Indeed, $\langle S \rangle \cong \Lambda(2) \oplus \Lambda(3,3,2,2) \subset \mathbb{Z}^5$ as an integral lattice with pairing induced by that of $\mathbb{Z}^5$. Observe that $v_2 \cdot v_2 = 3 > 2$ and that $e_5 \cdot v_j \neq 0$ if and only if $j \in \{2, 5\}$. Hence, in the notation of part (4) of Definition 4.4.1, we can choose $h = 5$, $s = 5$ and $t = 2$, and we see that S admits a contraction to the linear subset $S' = \{v_1, v_2 - (e_5 \cdot v_2)e_5, v_3, v_4\} = \{e_1 - e_2, e_3 - e_4, e_1 + e_2 + e_3, e_3 + e_4\} \subset \mathbb{Z}^4$ (it is not hard to see that it is generally true that if S' is obtained from a linear subset S by a contraction or an expansion, then S' is a linear subset as well). Note that the vertices of the intersection graph of S corresponding to v_2 and v_5 are the two vertices of degree one. Moreover, we have $v_5 \cdot v_5 = 2$, so that the contraction $S \searrow S'$ we just exhibited is, in fact, a 2-final contraction. Finally, observe that $\{e_3 - e_4, e_1 + e_2 + e_3, e_3 + e_4\} \subset S'$

forms a bad component of S' (here, $j = 3$ in the notation of part (6) of Definition 4.4.1). Hence, $\{v_2, v_3, v_4, v_5\} \subset S$ is a bad component of S, by definition. The notion of a bad component will be crucial in the proof of Theorem 4.1.1. Loosely speaking, if $S \subset \mathbb{Z}^N$ is a linear subset corresponding to an isometric embedding $\Lambda \hookrightarrow \mathbb{Z}^N$, then a bad component C of S corresponds to a direct summand of Λ of the form $\Lambda(p/q)$ with $p/q \in \mathcal{F}_n$, for some $n \geq 2$. In the above example, the bad component $\{v_2, v_3, v_4, v_5\}$ of S corresponds to the direct summand $\Lambda(3,3,2,2) = \Lambda(18/7)$, and $18/7 \in \mathcal{F}_2$.

We conclude this subsection by proving two lemmas that will be used in the proofs of the main theorems, the first of which deals with the orthogonal complement to a bad component of a linear subset.

Lemma 4.4.2. Let $S = \{v_1, \ldots, v_N\} \subset \mathbb{Z}^N$ be a linear subset and suppose that the intersection graph of S has a bad component $C = \{v_{t-1}, v_t, v_{t+1}\}$, so that $v_t \cdot v_t = m+1$, for some $m \geq 2$. Then, with respect to some orthonormal basis of $\mathbb{Z}^n$, we have that

$$C = \langle e_{m+1} + e_{m+2}, e_1 + \cdots + e_{m+1}, e_{m+1} - e_{m+2} \rangle \subset \mathbb{Z}^N.$$

Moreover, if $C' \subset \mathbb{Z}^{N+K}$ is a linear subset that is obtained from C by a sequence of K 2-final expansions, $K \geq 0$, then $\langle C' \rangle^{\perp} \simeq \Lambda(m/(m-1))$.

Proof. For the first part, note that, by definition of a bad component, the coefficients of v_i are at most 1 in absolute value, $i \in \{t-1, t, t+1\}$. We may thus assume that $v_t = e_1 + \cdots + e_{m+1}$. Using the facts $\{i \mid v_i \cdot e_j \neq 0\} = \{t-1, t, t+1\}$ for some $1 \leq j \leq N$ and $v_{t-1} \cdot v_{t+1} = 0$, it is then readily checked that, up to change of orthonormal basis of $\mathbb{Z}^N$, it must be the case that $v_{t-1} = e_{m+1} + e_{m+2}$ and $v_{t+1} = e_{m+1} - e_{m+2}$ (where we assumed, without loss of generality, that $j = m+1$).

For the second part, if $K = 0$, so that $C' = C$, it is easily verified that

$$\langle C \rangle^{\perp} = \langle e_1 - e_2, \ldots, e_{m-1} - e_m, e_{m+3}, \ldots, e_N \rangle \simeq \Lambda(2^{[m-1]}) \cong \Lambda(m/(m-1)).$$

Consider the case where $K > 0$, and suppose that C is given as above. It is easy to see that there are only two ways of applying a 2-final expansion to C, which (up to change of orthonormal basis of $\mathbb{Z}^N$) are given by replacing

$$\langle e_{m+1} + e_{m+2}, e_1 + \cdots + e_{m+1}, e_{m+1} - e_{m+2} \rangle \subset \mathbb{Z}^N$$

by either

$$\langle e_{m+2} + e_{m+3}, e_{m+1} + e_{m+2}, e_1 + \cdots + e_{m+1}, e_{m+1} - e_{m+2} + e_{m+3} \rangle \subset \mathbb{Z}^{N+1}$$

or

$$\langle e_{m+1} + e_{m+2} + e_{m+3}, e_1 + \cdots + e_{m+1}, e_{m+1} - e_{m+2}, -e_{m+2} + e_{m+3} \rangle \subset \mathbb{Z}^{N+1},$$

and similarly for potential subsequent 2-final expansions. Hence

$$\langle C' \rangle^{\perp} = \langle e_1 - e_2, \ldots, e_{m-1} - e_m, e_{m+K+3}, \ldots, e_{N+K} \rangle \simeq \Lambda(2^{[m-1]}) \cong \Lambda(m/(m-1)),$$

and the claim follows. □

The following lemma pertaining to reducible linear subsets will be used in the proof of Theorem 4.1.2, where it will allow us to reduce a full-rank embedding of a linear lattice into smaller embeddings. We point out that this is essentially a reformulation of [Lis07b, Lemma 5.5]. To clarify the hypotheses of the lemma, note that if a connected sum of lens spaces $L(p_1, q_1) \# \cdots \# L(p_n, q_n)$ bounds a rational ball W, we not only obtain a full-rank isometric embedding

$$\Lambda(p_1/q_1) \oplus \cdots \oplus \Lambda(p_n/q_n) \hookrightarrow \mathbb{Z}^N$$

, but, by considering $-W$, we additionally obtain a full-rank isometric embedding

$$\Lambda(p_1/(p_1-q_1)) \oplus \cdots \oplus \Lambda(p_n/(p_n-q_n)) \hookrightarrow \mathbb{Z}^{N'}.$$

We remark that the existence of both embeddings is necessary for the conclusion of the lemma to hold.

Lemma 4.4.3. Let $\Lambda = \Lambda(p_1/q_1) \oplus \cdots \oplus \Lambda(p_n/q_n)$ be a linear lattice, set $\Lambda' = \Lambda(p_1/(p_1-q_1)) \oplus \cdots \oplus \Lambda(p_n/(p_n-q_n))$, and suppose that there exist full-rank isometric embeddings $\varphi\colon \Lambda \hookrightarrow \mathbb{Z}^N$ and $\varphi'\colon \Lambda' \hookrightarrow \mathbb{Z}^{N'}$.

Then, after possibly permuting $p_1/q_1, \ldots, p_n/q_n$ and switching the roles of Λ and Λ', φ can be decomposed as $\varphi = \varphi_1 \oplus \widetilde{\varphi}$ in such a way that the linear subset $S_1 \subset \mathbb{Z}^N$ corresponding to φ_1 is irreducible, and φ_1 and $\widetilde{\varphi}$ are full-rank isometric embeddings of one of the following forms:

1. $\varphi_1\colon \Lambda(p_1/q_1) \hookrightarrow \mathbb{Z}^{N_1}$ and $\widetilde{\varphi}\colon \Lambda(p_2/q_2) \oplus \cdots \oplus \Lambda(p_n/q_n) \hookrightarrow \mathbb{Z}^{N_2}$, $N_1 + N_2 = N$;
2. $\varphi_1\colon \Lambda(p_1/q_1) \oplus \Lambda(p_2/q_2) \hookrightarrow \mathbb{Z}^{N_1}$ and $\widetilde{\varphi}\colon \Lambda(p_3/q_3) \oplus \cdots \oplus \Lambda(p_n/q_n) \hookrightarrow \mathbb{Z}^{N_2}$, $N_1 + N_2 = N$.

Proof. Let $S \subset \mathbb{Z}^N$ be the linear subset corresponding to φ and write $S = S_1 \cup \cdots \cup S_m$, where the S_i are the maximal irreducible subsets of S, so that each S_i corresponds to the orthogonal direct sum of some of the $\Lambda(p_1/q_1), \ldots, \Lambda(p_n/q_n)$, $i \in \{1, \ldots, m\}$. Since no $v \in S_i$ is linked to any $w \in S_j$, $i \neq j$, φ can be decomposed as an orthogonal direct sum of isometric embeddings $\varphi = \varphi_1 \oplus \cdots \oplus \varphi_m$ such that the linear subset corresponding to φ_i is $S_i \subset \mathbb{Z}^N$, $i \in \{1, \ldots, m\}$. Moreover, we can view each φ_i as a full-rank isometric embedding into $\mathbb{Z}^{N_i} \subset \mathbb{Z}^N$, where $\mathbb{Z}^{N_i} = \langle e_i \mid v \cdot e_i \neq 0 \text{ for some } v \in S_i \rangle$. Indeed, we have $N_i = |S_i|$, $i \in \{1, \ldots, m\}$, by linear independence of the elements of S (cf. [Lis07a, Remark 2.1]). By [Lis07b, Lemma 5.3], we may assume that, after possibly reordering the $S_1, \ldots, S_m$ and the $p_1/q_1, \ldots, p_n/q_n$ and switching the roles of Λ and Λ', the quantity $I(S_1) + b(S_1)$ is negative (we refer the reader to [Lis07a, Definition 2.3] for the definition of $I(S)$). Thus, [Lis07b, Proposition 4.10] implies that $c(S_1) \leq 2$, that is, the domain of φ_1 consists of at most two orthogonal direct summands of Λ. It follows that φ_1 may be assumed to be of the form $\varphi_1 \colon \Lambda(p_1/q_1) \hookrightarrow \mathbb{Z}^{N_1}$, or $\varphi_1 \colon \Lambda(p_1/q_1) \oplus \Lambda(p_2/q_2) \hookrightarrow \mathbb{Z}^{N_1}$. Setting $\widetilde{\varphi} = \varphi_2 \oplus \cdots \oplus \varphi_m$, we have that $\varphi = \varphi_1 \oplus \widetilde{\varphi}$ is of the desired form. $\square$

4.5 The proof of Theorem 4.1.1

The main ingredient to the proof of Theorem 4.1.1 is the following proposition, which essentially deals with the case where the linear subset $S \subset \mathbb{Z}^N$ coming from a ribbon cobordism contains bad components.

Proposition 4.5.1. Let W be a ribbon cobordism from $L(p, q)$ to $L(r, s)$, where $r/s \in \mathcal{F}_n$, for some $n \geq 2$, and $p \neq r$. Suppose further that the linear subset $S \subset \mathbb{Z}^N$ associated to the corresponding embedding $\varphi \colon \Lambda(p/(p-q)) \oplus \Lambda(r/s) \hookrightarrow \mathbb{Z}^N$ is irreducible. Then we must have that $L(p, q) \cong L(n, 1)$.

Proof. Set $L_1 = L(p, q)$, $L_2 = L(r, s)$ and $\Lambda_1 = \Lambda(p/(p-q))$, $\Lambda_2 = \Lambda(r/s)$. Write $S = S_1 \cup S_2$, where $S_i \subset \mathbb{Z}^N$ is the linear subset corresponding to $\varphi(\Lambda_i)$, so that $\varphi(\Lambda_i) = \langle S_i \rangle$, $i = 1, 2$.

Since $r/s \in \mathcal{F}_n$, for some $n \geq 2$, [Lis07b, Theorem 1.1] implies that we must have that either $p/q = n/1$, or that either p/q or $p/(p-q)$ belongs to $\mathcal{F}_n$ (where $p/(p-q) \in \mathcal{F}_n$ can only happen if $n = 2$). If $p/q = n/1$ we are done, so we assume that L_1 is not homeomorphic to a lens space of the form $L(n, 1)$. In what follows, we determine the stable isometry type of $\varphi(\Lambda_1) = \varphi(\Lambda_2)^{\perp}$, which, by Remark 4.3.2, determines the oriented homeomorphism type of L_1.

Since $c(S) = 2$, it follows from [Lis07b, Lemma 5.2] that $b(S) \in \{0, 1, 2\}$. If $b(S) = 0$, then by the first subcase of the proof in [Lis07b, p. 2160], we have that

$L_1 \cong L_2$, which contradicts our assumption on p and r. Therefore, we must have that $b(S) \in \{1, 2\}$. Suppose first that $b(S) = 1$, so that either S_1 or S_2 is a bad component. Using the facts $L_1 \not\cong L(n, 1)$ and $p \neq r$, it follows from the second subcase of the proof in [Lis07b, p. 2160], that, after possibly replacing W by $-W$, the bad component of S is, in fact, S_2 and, moreover, that S_2 admits a sequence of 2-final contractions $S_2 \searrow \cdots \searrow C$, where $C = \{v_{t-1}, v_t, v_{t+1}\}$ with $v_t \cdot v_t = n + 1$. Thus, by Lemma 4.4.2, we have that $\varphi(\Lambda_2)^\perp = \langle S_2 \rangle^\perp \simeq \Lambda(n/(n-1))$. It follows that $\varphi(\Lambda_1) = \varphi(\Lambda_2)^\perp \simeq \Lambda(n/(n-1))$, which implies that $L_1 \cong L(n, 1)$.

It remains to address the case where $b(S) = 2$, so that both S_1 and S_2 are bad components. Then, by the third subcase of the proof in [Lis07b, p. 2162], we must have that $r/s \in \mathcal{F}_2$, and S_2 admits a sequence of 2-final contractions $S_2 \searrow \cdots \searrow C$, where $C = \{v_{t-1}, v_t, v_{t+1}\}$ with $v_t \cdot v_t = 3$. By the argument used in the previous case, it follows that $L_1 \cong L(2, 1)$. □

Proof of Theorem 4.1.1. We first show that the conditions (1)–(3) are necessary. Set $L_1 = L(p, q)$, $L_2 = L(r, s)$ and $\Lambda_1 = \Lambda(p/(p-q))$, $\Lambda_2 = \Lambda(r/s)$, and let W be a ribbon cobordism from L_1 to L_2. By Theorem 3.3.1 and the remark following it, we obtain a full-rank isometric embedding $\varphi\colon \Lambda_1 \oplus \Lambda_2 \hookrightarrow \mathbb{Z}^N$ such that $\varphi(\Lambda_1) = \varphi(\Lambda_2)^\perp$. Let S denote the corresponding linear subset, so that $\langle S \rangle = \varphi(\Lambda_1 \oplus \Lambda_2)$.

Suppose first that $S \subset \mathbb{Z}^n$ is irreducible. It follows from first case of the proof in [Lis07b, p. 2160] that in this case, either $L_1 \cong L_2$, or that (after possibly switching to $-W$ instead of W) at least one of p/q and r/s belongs to $\mathcal{F}_n$, for some $n \geq 2$. If $L_1 \cong L_2$, case (1) of Theorem 4.1.1 holds, whereas if $r/s \in \mathcal{F}_n$, then Proposition 4.5.1 implies that $L_1 \cong L(n, 1)$, and case (2) of Theorem 4.1.1 holds. If $p/q \in \mathcal{F}_n$, then, by the main theorem of [Lis07b], either $L_2 \cong L(n, 1)$ or $r/s \in \mathcal{F}_n$. In the former case, however, by definition of $\mathcal{F}_n$, we have that $|H_1(L_1; \mathbb{Z})| > |H_1(L_2; \mathbb{Z})|$, which contradicts Proposition 2.1.6, and we must thus have $r/s \in \mathcal{F}_n$. But then, using Proposition 4.5.1 again, it follows that $L_1 \cong L(p, q) \cong L(n, 1)$, which is a contradiction, since $n/1 \notin \mathcal{F}_n$ for any $n \geq 2$.

It remains to deal with the case where $S \subset \mathbb{Z}^n$ is reducible. In this case we can write $S = S_1 \cup S_2$, where S_1 and S_2 are the maximal irreducible linear subsets that correspond to $\varphi(\Lambda_1)$ and $\varphi(\Lambda_2)$, respectively. Indeed, by Lemma 4.4.3, S cannot decompose into three or more maximal irreducible subsets. Using the irreducibility of S_1 and the fact that $\varphi(\Lambda_1) = \varphi(\Lambda_2)^{\perp}$, it follows that $\varphi(\Lambda_1) \subset \mathbb{Z}^{N_1}$ is a primitive full-rank sublattice, where $\mathbb{Z}^{N_1} = \langle e_i \mid v \cdot e_i \neq 0 \text{ for some } v \in S_1 \rangle \subset \mathbb{Z}^N$. By Remark 3.1.2, we thus have that $\varphi(\Lambda_1) \cong \mathbb{Z}^{N_1}$, which implies that $L_1 \cong S^3$ and, consequently, $L_2 \cong L(r, s)$ with $r/s \in \mathcal{R}$. Thus, case (3) of Theorem 4.1.1 holds.

Conversely, suppose that L_1, L_2 are lens spaces such that case (1), (2) or (3) holds. In case (1), we can choose the product cobordism $L_1 \times [0, 1]$ to verify that $L_1 \leq L_2$. In cases (2) and (3), $L_1 \leq L_2$ follows from [Lis07b, Lemma 3.5] and [Lis07a, Theorem 1.2], respectively. □

4.6 The proof of Theorem 4.1.2

In this subsection, we prove Theorem 4.1.2. While to a large extent it is a consequence of Theorem 4.1.1, we will need the following additional result.

Proposition 4.6.1. Let $Y = L_1 \# L_2$, where L_i is a lens space that does not bound a rational homology ball, $i = 1, 2$. If Y bounds a rational homology ball, then Y must be (possibly orientation-reversingly) homeomorphic to one of the following:

1. $L(p, p-q) \# L(p, q)$, $p/q > 1$;
2. $L(n, n-1) \# L(p, q)$, $p/q \in \mathcal{F}_n$ for some $n \geq 2$;
3. $L(p_1, p_1 - q_1) \# L(p_2, q_2)$, $p_i/q_i \in \mathcal{F}_n$, $i = 1, 2$, for some $n \geq 2$; or
4. $L(p_1, q_1) \# L(p_2, q_2)$, $p_i/q_i \in \mathcal{F}_2$, $i = 1, 2$.

Conversely, if Y is homeomorphic to one of the manifolds in (1)–(4), then Y bounds a *ribbon* rational homology ball.

Proof. The first half of the statement is an immediate consequence of the main theorem of [Lis07b].

Conversely, suppose that $Y \cong L(p, p-q)\#L(p,q)$, $p/q > 1$, and note that in this case Y is the double cover of S^3 branched along the link $K(p,p-q)\#K(p,q) = \overline{K(p,q)}\#K(p,q)$. As in the proof of Corollary 4.2.3, we have that $\overline{K(p,q)}\#K(p,q)$ bounds a (possibly disconnected) properly embedded surface $C \subset B^4$ with $\chi(C) = 1$, and such that C has no local maxima with respect to the radial distance function on B^4. It follows that Y bounds a ribbon rational homology ball. The remaining cases are handled similarly (cf. also the proof of Corollary 4.2.3). □

Proof of Theorem 4.1.2. Let W be a ribbon cobordism from $Y_1 = L_1\#\cdots\#L_I$ to $Y_2 = M_1\#\cdots\#M_J$, where the $L_1,\dots,L_I$ and $M_1,\dots,M_J$ are lens spaces. By Theorem 3.3.1 and the remark following it, we obtain a full-rank isometric embedding of linear lattices

$$\varphi\colon \Lambda_1 \oplus \cdots \oplus \Lambda_I \oplus \mathrm{M}_1 \oplus \cdots \oplus \mathrm{M}_J \hookrightarrow \mathbb{Z}^N,$$

such that

$$\varphi(\mathrm{M}_1 \oplus \cdots \oplus \mathrm{M}_J)^\perp = \varphi(\mathrm{M}_1)^\perp \oplus \cdots \oplus \varphi(\mathrm{M}_J)^\perp = \varphi(\Lambda_1) \oplus \cdots \oplus \varphi(\Lambda_I). \tag{4.2}$$

Here, Λ_i and M_j correspond to L_i and M_j, respectively, $i = 1,\dots,I$, $j = 1,\dots,J$. As in the proof of Proposition 4.6.1, it suffices to determine the stable isometry types of the lattices $\Lambda_1,\dots,\Lambda_I,\mathrm{M}_1,\dots,\mathrm{M}_J$. After possibly replacing W by $-W$, we may apply Lemma 4.4.3, so that φ decomposes as $\varphi = \varphi_1 \oplus \widetilde{\varphi}$, where $\varphi_1\colon \Lambda \to \mathbb{Z}^{N_1} \subset \mathbb{Z}^N$ is a full-rank isometric embedding into a direct summand of $\mathbb{Z}^N$, such that the corresponding linear subset $S_1 \subset \mathbb{Z}^{N_1}$ is irreducible, and Λ is one of the following:

1. Λ_i for some $i \in \{1, \ldots, I\}$;
2. $\Lambda_i \oplus \Lambda_{i'}$, for some $i, i' \in \{1, \ldots, I\}$, $i \neq i'$;
3. M_j for some $j \in \{1, \ldots, J\}$;
4. $\mathrm{M}_j \oplus \mathrm{M}_{j'}$, for some $j, j' \in \{1, \ldots, J\}$, $j \neq j'$; or
5. $\Lambda_i \oplus \mathrm{M}_j$, for some $i \in \{1, \ldots, I\}$, $j \in \{1, \ldots, J\}$.

For cases (1) and (2), note that, by (4.2), $\varphi(\Lambda)$ is a direct summand of the primitive sublattice $\varphi(\mathrm{M}_1 \oplus \cdots \oplus \mathrm{M}_J)^\perp \subset \mathbb{Z}^N$. Thus, $\varphi(\Lambda) \subset \mathbb{Z}^{N_1}$ is a full-rank primitive sublattice, which, by Remark 3.1.2, implies that $\Lambda \cong \mathbb{Z}^{N_1} \simeq 0$. It follows that $L_i \cong S^3$ in case (1), and $L_i \# L_{i'} \cong S^3$ in case (2), so we may discard these connected summands from Y_1.

In case (3), we have a full-rank isometric embedding $\varphi_1 \colon \mathrm{M}_j \to \mathbb{Z}^{N_1}$, which by the main theorem of [Lis07a] implies that $M_j \cong L(p, q)$ with $p/q \in \mathcal{R}$, and case (3) of Theorem 4.1.2 holds.

Similarly, in case (4) it follows that $M_j \# M_{j'} \cong L(p_1, q_1) \# L(p_2, q_2)$, such that $L(p_1, q_1) \# L(p_2, q_2)$ bounds a rational homology ball, but, since S_1 is irreducible, $L(p_k, q_k)$ does not, $k = 1, 2$. By Proposition 4.6.1, $M_j \# M_{j'}$ bounds a ribbon rational homology ball, and must be homeomorphic to one of the manifolds listed there, so one of the cases (4)–(7) from Theorem 4.1.2 holds.

Finally, in case (5), we have that $\varphi_1 \colon \Lambda_i \oplus \mathrm{M}_j \hookrightarrow \mathbb{Z}^{N_1}$ is a full-rank isometric embedding which, by (4.2) again, has the property that $\varphi_1(\Lambda_i) = \varphi_1(\mathrm{M}_j)^\perp$. Since S_1 is irreducible, we can apply the first half of the proof of Theorem 4.1.1 to φ_1 to conclude that case (1) or case (2) of Theorem 4.1.2 must hold.

We can now apply the above procedure to $\widetilde{\varphi}$ and then iterate it, where at each step we may have to replace e.g. $\widetilde{\varphi}$ by $\widetilde{\varphi}'$, where $\widetilde{\varphi}'$ is obtained from $\widetilde{\varphi}$ as in the statement of Lemma 4.4.3. Since at each step we discard a non-zero number of direct summands of $\Lambda_1 \oplus \cdots \oplus \Lambda_I \oplus \mathrm{M}_1 \oplus \cdots \oplus \mathrm{M}_J$, this process terminates. $\square$

Chapter 5

Ribbon cobordisms as a partial order

5.1 Introduction

In the this chapter, we partially prove the following conjecture that asserts that the notion of ribbon cobordisms gives rise to a partial order among 3-manifolds.[1]

Conjecture 5.1.1 (Daemi-Lidman-Vela-Vick-Wong [DLVVW20, Conjecture 1.1])**.** The preorder on the set of homeomorphism classes of closed, connected, oriented 3-manifolds given by ribbon cobordisms is a partial order.

This conjecture can be seen as a 3-manifold version of Gordon's conjecture that ribbon concordance gives a partial order on the set of knots in S^3 [Gor81, Conjecture 1.1], which was famously proved by Agol [Ago22]. As speculated by Agol, it is natural to wonder whether the techniques used to prove Gordon's Conjecture could be used to make progress towards proving Conjecture 5.1.1. In this chapter, we do exactly that, and show that Conjecture 5.1.1 holds for the class of aspherical 3-manifolds (recall that a 3-manifold Y is aspherical if $\pi_k(Y) = 0$ for all $k \geq 2$, or, equivalently, if

[1] The content of this chapter is that of [Hub22].

Y is irreducible and has infinite fundamental group). Combining this with previous work of the author [Hub21] on ribbon cobordisms between lens spaces, we obtain the following. Note that we do not require the manifolds involved to be closed.

Theorem 5.1.2. Let Y_1 and Y_2 be compact, oriented, 3-manifolds, possibly with boundary, such that there exists a ribbon cobordism W_i from Y_i to Y_j, $\{i, j\} = \{1, 2\}$. If Y_i is either aspherical or a lens space, $i = 1, 2$, then $Y_1 \cong Y_2$.

The above result is a rather direct consequence of the following two more technical results. To put these into context, recall that if Y_1 and Y_2 are compact 3-manifolds and if W is a ribbon cobordism from Y_1 to Y_2, one has the following diagram of maps induced by inclusion ([DLVVW20, Theorem 1.14] and [Gor81, Lemma 3.1]):

$$\pi_1(Y_1) \hookrightarrow \pi_1(W) \twoheadleftarrow \pi_1(Y_2)$$

Hence, if Y is a 3-manifold with finite fundamental group, and if W is a ribbon cobordism from Y to itself, then the inclusion of either boundary component into W induces an isomorphism of fundamental groups. Our main technical result states that this remains true if Y has infinite fundamental group, provided that Y is aspherical.

Theorem 5.1.3. Let Y be a compact, oriented, aspherical 3-manifold, possibly with boundary, and suppose that W is a ribbon cobordism from Y_1 to Y_2, where $Y_i \cong Y$, $i = 1, 2$. Then the inclusion of Y_i into W induces an isomorphism $\pi_1(Y_i) \cong \pi_1(W)$, $i = 1, 2$.

As a consequence, we obtain the following result concerning pairs of 3-manifolds with the property that there exists a ribbon cobordism in either direction.

Theorem 5.1.4. Let Y_1, Y_2 be compact, oriented, aspherical 3-manifolds, $i = 1, 2$, possibly with boundary, and suppose that there exists a ribbon cobordism W_i from Y_i to Y_j, $\{i, j\} = \{1, 2\}$. Then the inclusion of Y_i into W_j induces an isomorphism

$\pi_1(Y_i) \cong \pi_1(W_j)$, $i,j = 1,2$. In particular, there exists an orientation-preserving homotopy equivalence $f\colon (Y_i, \partial Y_i) \to (Y_j, \partial Y_j)$, $\{i,j\} = \{1,2\}$.

We conclude by pointing out that essentially the same results were independently found by Friedl, Misev and Zentner [FMZ22].

5.2 Proofs of results

The following lemma is used in the algebro-geometric portion of the proof of Theorem 5.1.3, which, in turn, is virtually the same as the proof of [Ago22, Theorem 1.2]. We provide a proof of the lemma for completeness, but also to highlight the use of residual finiteness of fundamental groups of 3-manifolds.

Lemma 5.2.1. Suppose Γ is a residually finite group, and let $\gamma \in \Gamma \setminus \{1\}$. Then there exists $n > 0$ and a homomorphism $\rho\colon \Gamma \to \mathrm{SO}(n)$ such that $\rho(\gamma) \neq 1$.

Proof. We first show that any finite group embeds into $\mathrm{SO}(n)$ for some $n > 0$. For this, recall that the symmetric group on n elements S_n is generated by the $n-1$ transpositions $\tau_{i,i+1} = (i, i+1)$, $i = 1, \ldots, n-1$. For $i = 1, \ldots, n-1$, define $\varphi_{(i,i+1)}\colon \mathbb{R}^n \to \mathbb{R}^n$ by

$$\varphi_{(i,i+1)}(x_0, \ldots, x_i, x_{i+1}, \ldots, x_n) = (-x_0, \ldots, x_{i+1}, x_i, \ldots, x_n).$$

One can check that $\varphi_{(i,i+1)} \in \mathrm{SO}(n+1)$ for all $i = 1, \ldots, n-1$, and hence the above assignment defines an embedding of S_n into $\mathrm{SO}(n+1)$. Since any finite group embeds into S_n for some $n > 0$, it follows that the same holds with S_n replaced by $\mathrm{SO}(n)$.

Now, let Γ be residually finite and $\gamma \in \Gamma$ non-trivial. By definition of residual finiteness, there exists a finite group G and a surjection $q_\gamma\colon \Gamma \to G$ such that $q_\gamma(\gamma) \neq 1$. Postcomposing q_γ with an embedding of G into $\mathrm{SO}(n)$, for some $n > 0$, yields the claim. $\square$

Proof of Theorem 5.1.3. Suppose that W is a ribbon cobordism as in the statement of Theorem 5.1.3. By [DLVVW20, Theorem 1.14], we have that

$$\pi_1(Y_1) \xhookrightarrow{i_1} \pi_1(W) \xleftarrow{i_2} \pi_1(Y_2), \tag{5.1}$$

where i_k is the map induced by inclusion $\iota_k \colon Y_k \to W$, $k = 1, 2$, and $\pi_1(Y_1) \cong \pi_1(Y_2)$. As in [Ago22], for $n > 0$ and a manifold X, let $R_n(X) = R_n(\pi_1(X))$ denote the representation variety of $\pi_1(X)$ to $\mathrm{SO}(n)$. By [DLVVW20, Proposition 1.15], we then have that

$$R_n(Y_1) \xleftarrow{r_1} R_n(W) \xhookrightarrow{r_2} R_n(Y_2), \tag{5.2}$$

where r_k is the restriction map, $k = 1, 2$. As shown in [Ago22], r_1 is obtained by projection of $R_n(W)$ onto the subspace spanned by the coordinates corresponding to $\pi_1(Y_1)$ (regarded as a subgroup of $\pi_1(W)$) and hence is a polynomial map, and, moreover, $R_n(Y_1)$ and $R_n(Y_2)$ are related by a polynomial isomorphism. Precomposing this isomorphism with r_1, one obtains a surjective polynomial map $\varphi \colon R_n(W) \to R_n(Y_2)$. By the argument given in [Ago22], r_2, in fact, embeds $R_n(W)$ into $R_n(Y_2)$ as a real algebraic subset. This allows one to show that $i_2 \colon \pi_1(Y_2) \to \pi_1(W)$ is injective as follows. Given $\gamma \in \pi_1(Y_2) \setminus \{1\}$, one can, using residual finiteness of 3-manifold groups (which follows from [Thu82, Theorem 3.3] and Geometrization) and Lemma 5.2.1, find $n > 0$ and a representation $\rho \in R_n(Y_2)$ with the property that $\rho(\gamma) \neq 1$. By the above, $R_n(W) \subset R_n(Y_2)$ is an algebraic subset that admits a surjective polynomial map to $R_n(Y_2)$ (namely, φ). Thus, [Ago22, Lemma A.2] implies that $R_n(W) = R_n(Y_2)$, and it follows that the representation ρ is the restriction of some representation $\rho' \in R_n(W)$. Hence, $\rho'(i_2(\gamma)) = (r_2(\rho'))(\gamma) = \rho(\gamma) \neq 1$, which implies that $i_2(\gamma)$ is non-trivial. It follows that i_2 is injective and hence an isomorphism.

It remains to show that i_1 is an isomorphism. To show this, we adapt an argument used in the proof of [DLVVW20, Proposition 9.2]. Set $\overline{W} = -W$, so that $\overline{W}$ is a rational homology cobordism from Y_2 to Y_1 which is built from $Y_2 \times I$ by attaching 2- and 3-handles. By what we have shown so far, the map $\pi_1(Y_2) \to \pi_1(\overline{W})$ induced by inclusion is an isomorphism, which implies that each of the 2-handles of $\overline{W}$ is attached to $Y_2 \times I$ along a null-homotopic curve in $Y_2 \times \{1\}$, and hence each attaching curve bounds an immersed disk in $Y_2 \times \{1\}$. Let $\overline{W}_2$ denote the space obtained by attaching just the 2-handles of $\overline{W}$ to $Y_2 \times I$. Define a map $\rho_2 \colon \overline{W}_2 \to Y_2$ as follows. First, shrink each of the 2-handles to its core, then map each core to the disk in $Y_2 \times \{1\}$ bounded by its attaching curve via a map that is the identity on the attaching curve itself, and, finally, apply the obvious deformation retraction of $Y_2 \times I$ onto $Y_2 = Y_2 \times \{0\}$. The obstruction to extending ρ_2 over the 3-handles of $\overline{W}$ lies in $H^3(\overline{W}, \overline{W}_2; \pi_2(Y_2))$ (see e.g. [Hat02, Proposition 4.72]). Since we assumed Y_2 to be aspherical, this group vanishes, and ρ_2 extends to a retraction $\rho \colon \overline{W} \to Y_2$. Indeed, by definition of a ribbon cobordism between manifolds with boundary, we have that $\partial Y_1 = \partial Y_2 \times \{1\} \subset \overline{W}$, and it follows that ρ is the identity in a neighborhood of $\partial Y_2 \times \{1\} \subset \overline{W}$. Letting $\rho_* \colon \pi_1(\overline{W}) \to \pi_1(Y_2)$ denote the map induced by ρ, it follows that $\rho_* \circ i_2 = (\rho \circ \iota_2)_* = (\mathrm{id}_{Y_2})_* = \mathrm{id}_{\pi_1(Y_2)}$, which implies that ρ_* is an isomorphism, because i_2 is. Consider now the map $f = \rho \circ \iota_1 \colon Y_1 \to Y_2$. Since $H_3(W;\mathbb{Z}) \cong H_3(Y_2;\mathbb{Z}) \cong \mathbb{Z}$, and because ρ is a retraction, ρ induces an isomorphism on the level of third integral homology. Similarly, the inclusion $\iota_1 \colon Y_1 \to W$ induces an isomorphism of third integral homology groups. It follows that the map induced by f sends the relative fundamental class $[Y_1, \partial Y_1] \in H_3(Y_1, \partial Y_1; \mathbb{Z})$ to $[Y_2, \partial Y_2] \in H_3(Y_2, \partial Y_2; \mathbb{Z})$. That is, f is an orientation-preserving degree one map. Now, by (5.1), i_1 is injective, and it follows that $f_* \colon \pi_1(Y_1) \to \pi_1(Y_2)$ is injective, because ρ_* is an isomorphism. By [Ron92, Lemma 1.2], f_* is also surjective, and hence an isomorphism. It follows that $i_1 \colon \pi_1(Y_1) \to \pi_1(W)$ is an isomorphism, as desired. □

Proof of Theorem 5.1.4. Let W be the composition of the cobordisms W_1 and W_2, i.e. $W = W_1 \cup_{Y_2} W_2$, so that W is a ribbon cobordism from Y_1 to itself. Letting

$h_i \colon \pi_1(W_i) \to \pi_1(W)$ denote the map induced by inclusion of W_i into W, $i = 1, 2$, and using [DLVVW20, Theorem 1.14], we obtain the following diagram of maps induced by inclusion on the level of fundamental groups.

$$\begin{tikzcd}
\pi_1(Y_1) \arrow[r, hook, "i_1^1"] \arrow[drr, bend right, "\cong"'] & \pi_1(W_1) \arrow[dr, "h_1"] & \pi_1(Y_2) \arrow[l, two heads, "i_2^1"'] \arrow[r, hook, "i_1^2"] & \pi_1(W_2) \arrow[dl, "h_2"'] & \pi_1(Y_1) \arrow[l, two heads, "i_2^2"'] \arrow[dll, bend left, "\cong"] \\
& & \pi_1(W) & &
\end{tikzcd} \tag{5.3}$$

The fact that the maps $h_1 \circ i_1^1$ and $h_2 \circ i_2^2$ are isomorphisms follows from Theorem 5.1.3. This immediately implies that i_2^2 is injective, and hence an isomorphism. Switching the roles of W_1 and W_2, we see that i_2^1 is an isomorphism as well. Moreover, using the fact that W_2 is a ribbon cobordism, it follows by an argument similar to the one used to show injectivity of i_1^1 and i_1^2 (see e.g. the proof of [DLVVW20, Proposition 2.1]) that h_1 is injective. Note that, for that argument to apply, we need the fact that $\pi_1(W_1)$ is residually finite; but $\pi_1(W_1) \cong \pi_1(Y_2)$ via i_2^1, and $\pi_1(Y_2)$, being the fundamental group of a compact 3-manifold, is residually finite. Now, since $h_1 \circ i_1^1$ is an isomorphism, h_1 is also surjective and hence an isomorphism, which implies that i_1^1 is an isomorphism, too. Switching the roles of W_1 and W_2, we see that h_2 and i_2^1 are isomorphisms as well. Hence all maps in (5.3) are isomorphisms.

It remains to prove the existence of the claimed homotopy equivalences. To that end, note that, because all horizontal maps in (5.3) are isomorphisms, we can apply the argument from the last paragraph of the proof of Theorem 5.1.3 to W_1 to obtain an orientation-preserving degree one map $f \colon Y_1 \to Y_2$ that induces an isomorphism of fundamental groups. Since we assumed Y_1 and Y_2 to be aspherical, it follows that f induces an isomorphism on all homotopy groups and hence is a homotopy equivalence by Whitehead's theorem (see e.g. [Hat02, Theorem 4.5]). Moreover, by construction of the retraction ρ from the proof of Theorem 5.1.3, we have that $f(\partial Y_1) \subset \partial Y_2$ and, indeed, that f fixes ∂Y_1 pointwise. It follows that $f \colon (Y_1, \partial Y_1) \to (Y_2, \partial Y_2)$ is a homotopy equivalence, as desired. The above argument applied to $W = W_2$ yields a homotopy equivalence going in the other direction. $\square$

We are now in a position to prove the main result of this chapter.

Proof of Theorem 5.1.2. Observe that, by (5.3), Y_1 is a lens space iff Y_2 is. Assume first that Y_1, and hence, by definition of ribbon cobordism, also Y_2, is closed. If both Y_1 and Y_2 are lens spaces, the claim is a straightforward consequence of [Hub21, Theorem 1.2], so we may assume that both Y_1 and Y_2 are aspherical. By Theorem 5.1.4, there exists an orientation-preserving homotopy equivalence $f\colon Y_1 \to Y_2$. Note that $\pi_1(Y_2)$ is not just infinite, but torsion-free by [AFW15, (C.3)], and hence the Borel Conjecture in dimension three [KL09, Theorem 0.7] implies that f is homotopic to a homeomorphism. This homeomorphism must be orientation-preserving, because this property is preserved under homotopy, and it follows that $Y_1 \cong Y_2$.

It remains to address the case where Y_1, and hence also Y_2, has non-empty boundary. By what we assumed, it follows that both Y_1 and Y_2 are aspherical. By Theorem 5.1.4, there exists an orientation-preserving homotopy equivalence $f\colon (Y_1, \partial Y_1) \to (Y_2, \partial Y_2)$. Since Y_i has non-empty boundary, and hence is Haken, $i = 1, 2$, f is homotopic to a homeomorphism from Y_1 to Y_2 by [Wal68, Corollary 6.5]. As before, this homeomorphism must be orientation-preserving, because f was, and it follows that $Y_1 \cong Y_2$. $\square$

Chapter 6

Outlook

We conclude this dissertation by raising and discussing some questions.

6.1 Ribbon vs. quasi-ribbon

Clearly, a rational homology cobordism can contain 3-handles, but still be quasi-ribbon. Nevertheless, it is natural to ask the following.

Question. If $W\colon Y_1 \to Y_2$ is quasi-ribbon, does it follow that $Y_1 \leq Y_2$?

As noted in the remarks following 4.1.1, the results from Chapter 4 answer this question in the affirmative if Y_2 is a connected sum of lens spaces.

If $Y_1 \cong S^3$, the above question asks whether any rational homology 3-sphere Y that bounds rational homology 4-ball W with $H_2(W;\mathbb{Z}) = 0$ (one might call this a quasi-ribbon ball) actually bounds a ribbon ball. This question, in turn, can be seen as a "weaker" version of the (hard) question as to whether any integral homology sphere that bounds an integral homology ball actually bounds a ribbon ball.

6.2 Correction terms

At the end of Subsection 3.3 we sketched how d-invariants can be used to provide potentially stronger lattice-theoretic obstructions to a pair of 3-manifolds cobounding a quasi-ribbon cobordism, provided that the orders of the first integral homology groups of the two manifolds agree. The same idea can be applied in the case where the orders of first homology differ. More precisely, let Y_1 and Y_2 be rational homology 3-spheres, and suppose that W is a quasi-ribbon cobordism from Y_1 to Y_2. Setting $|H^2(Y_1;\mathbb{Z})| = p$, we have that $|H^2(W;\mathbb{Z})| = pu$ and $|H^2(Y_2;\mathbb{Z})| = pu^2$, for some $u > 1$, by Remark 2.1.5. Moreover, because the restriction maps $\rho_1\colon H^2(W;\mathbb{Z}) \to H^2(Y_1;\mathbb{Z})$ and $\rho_2\colon H^2(W;\mathbb{Z}) \to H^2(Y_2;\mathbb{Z})$ are surjective and injective, respectively, it follows that there exists a torsor homomorphism from $\rho_2(\mathrm{Spin}^c(W)) \subset \mathrm{Spin}^c(Y_2)$ to $\mathrm{Spin}^c(Y_1)$ that preserves the d-invariant. Namely, this u-to-one map is given by first extending a given $\mathfrak{s} \in \rho_2(\mathrm{Spin}^c(W))$ across W, and then restricting that extension to Y_1. On the other hand, since Y_1 and Y_2 are rational homology cobordant, we have that $-Y_1 \# Y_2$ bounds a rational homology ball. Assuming that each of $-Y_1$ and Y_2 bounds a positive definite sharp 4-manifold X_k, $k = 1, 2$, so that we can form the positive-definite, closed 4-manifold $Z = X_1 \cup_{Y_1} W \cup_{Y_2} X_2$, it follows that the corresponding lattice embedding

$$\varphi\colon \Lambda_1 \oplus \Lambda_2 \to \Lambda_Z \cong \mathbb{Z}^n$$

has the property that every element of $\mathrm{coker}(\varphi) = \mathbb{Z}^n/\mathrm{Im}(\varphi)$ has a representative in $\{0,1\}^n$; see [GJ11, Theorem 3.6] for details. Note that this obstruction does not take into account the fact that we started out with a *quasi-ribbon* cobordism, and hence does not reflect the existence of the u-to-one torsor homomorphism of Spin^c-structures described above. Surjectivity of ρ_1, however, implies the existence of such a map. Indeed, as in the proof of Proposition 2.2.3, and because we assumed $u > 1$, we must have that $\mathrm{Im}(\rho_2) \subset \mathrm{Spin}^c(Y_2)$ has order pu. Surjectivity of ρ_1 then implies that the extend-then-restrict map $\mathrm{Spin}^c(Y_2) \supset \mathrm{Im}(\rho_2) \to \mathrm{Spin}^c(Y_1)$ is u-to-one. In conclusion, it appears that – at least in the case where the orders of first integral homology of Y_1 and Y_2 differ – the conditions

1. $\varphi(\Lambda_1) = \varphi(\Lambda_2)^\perp$; and

2. every element of $\mathrm{coker}(\varphi)$ has a representative in $\{0,1\}^n$

is as far as one can push lattice theoretic methods to obstruct the existence of a quasi-ribbon cobordism from Y_1 to Y_2. It is hence natural to ask the following.

Question. Does there exist a pair of rational homology spheres Y_1 and Y_2 that passes the obstruction given by (1) and (2) above, yet there does not exist a quasi-ribbon cobordism from Y_1 to Y_2?

In the same vein, one may ask the following question pertaining to quasi-ribbon cobordisms between manifolds whose orders of first integral homology coincide (cf. Proposition 3.3.3 and the discussion preceding it).

Question. Does there exist a pair of L-spaces Y_1 and Y_2 with $b_1(Y_i) = 0$, $i = 1, 2$, such that Y_1 and Y_2 are integral homology cobordant and $\lambda_{\mathrm{CW}}(Y_1) = \lambda_{\mathrm{CW}}(Y_2)$, yet there does not exist a quasi-ribbon cobordism from Y_1 to Y_2?

6.3 Ribbon cobordisms between prism manifolds

While classifying all pairs of 3-manifolds that cobound a (quasi-)ribbon cobordism appears to be an intractable task in general, one might hope to obtain such a classification for the case where the 3-manifolds in question happen to be spherical. This would give a refined statement of [CP21, Theorem 1.1]. More precisely, recall that a 3-manifold Y is spherical if it is of the form $Y = S^3/\Gamma$, where $\Gamma \leq \mathrm{SO}(4)$ is a finite subgroup acting freely on S^3. The class of spherical 3-manifolds is divided into five subclasses, and, moreover, any spherical 3-manifold can be realized as a Seifert fibered rational homology sphere (see e.g. [CP21, Section 2], and the references therein). Each of the five subclasses of spherical 3-manifolds corresponds to a particular type of integral lattice that arises as the intersection lattice of the 2-handlebodies bounded by the spherical 3-manifolds. Hence, one obtains a lattice embedding problem for each of the five subclasses.

From this perspective, the results from Chapter 4 can be seen as a partial answer to the question of when there exists a ribbon cobordism between two spherical 3-manifolds. In what follows, we give a conjectural answer to the corresponding question for the class of spherical 3-manifolds consisting of lens spaces and prism manifolds.

Definition 6.3.1. Given a pair of coprime integers p and q, where $p \geq 1$, the prism manifold $P(p, q)$ is defined to be the 3-manifold depicted in Figure 6.1.[1]

Remark 6.3.2. As one can check, $P(1, q)$ is actually homeomorphic to the lens space $L(4q, 2q - 1)$. For this reason, the parameter p in the definition above is sometimes required to be greater than 1. However, we choose to allow for the possibility $p = 1$ for ease of exposition of what is to follow.

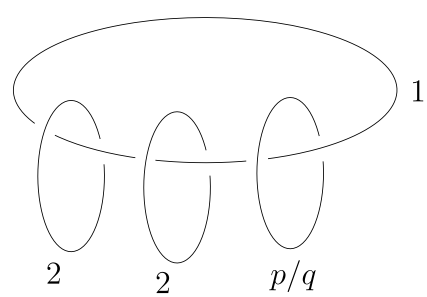

Figure 6.1: Surgery description for $P(p, q)$.

As with any Seifert manifold, one can expand p/q into a continued fraction as in Subsection 4.3 in order to obtain an integer surgery description of $P(p, q)$, which in

[1]This orientation convention is opposite to the one used in e.g. [BHM+20]. This is in accordance with the fact that we used the opposite of the standard orientation for lens spaces in Chapter 4.

turn can be succinctly described as in Figure 6.2, where $[a_1, \ldots, a_n]^- = p/q$, $a_i \geq 2$ for $i = 1, \ldots, n$.

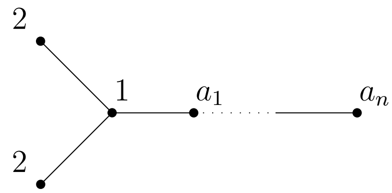

Figure 6.2: Integer surgery description for $P(p, q)$.

The weighted graph Γ in Figure 6.2 moreover describes a link as follows. For each vertex v of Γ, take an (untwisted) annulus $S^1 \times [0, 1]$ and introduce $w(v)$ half-twists, where $w(v)$ denotes the weight of the vertex v. Then, form a connected surface by plumbing together two of the twisted bands in the collection precisely when the corresponding vertices are connected by an edge in Γ. The boundary of the resulting surface forms a link $L \subset S^3$, and, moreover, the double cover of S^3 branched along L is precisely the prism manifold described by that same tree. Using this perspective, we show the following.

Proposition 6.3.3. For any prism manifold $P(p, q)$,

$$-L(q, p) \leq P(p, q).$$

Remarks.

1. This result immediately yields the classification of prism manifolds up to rational homology cobordism as stated in [CP21]. Moreover, it implies that all prism manifolds of the form $P(p + qk, q)$, $k \in \mathbb{Z}$, can be obtained via a ribbon cobordism emanating from the same lens space $-L(q, p)$.

2. We point out that the above result is a special case of [Lec12, Lemma 3.1]. However, we choose to include the following proof because it additionally shows

that the 2-bridge link corresponding to $-L(q,p)$ admits a ribbon concordance to the Montesinos link corresponding to $P(p,q)$.

Proof. Consider a surgery description for $P(p,q)$ given by a graph Γ as in Figure 6.2. The corresponding link L is shown in Figure 6.3a, where T is some tangle representing the plumbing of the twisted bands that correspond to the vertices with labels $a_1,\ldots,a_n$. Note that the bottommost half twist corresponds to the unique trivalent vertex of Γ. This link is isotopic to the one shown in Figure 6.3b. Performing a band move on this link as indicated transforms the link into the link depicted in Figure 6.3c. Note that the resulting unknot can be slid off of the rest of the link, and what remains is the link L' corresponding to the plumbing of twisted bands along the linear graph Δ shown in Figure 6.3d. It follows that $L' \leq L$.

Moreover, thinking of Δ as a surgery description for a 3-manifold, it follows that the corresponding 3-manifold Y admits a ribbon cobordism to $P(p,q)$. Since Δ is a linear graph, Y is homeomorphic to a lens space. Indeed, using $[a_1,\ldots,a_n]^- = p/q$, we have that

$$[0,a_1,\ldots,a_n]^- = 0 - \frac{1}{[a_1,\ldots,a_n]^-} = -\frac{q}{p},$$

which implies that $Y \cong -L(q,p)$, as desired.[2] □

Now, it follows e.g. from inspecting the Seifert invariants of a prism manifold that $P(p,q)$ is homeomorphic to a lens space if and only if $p=1$, in which case

$$P(1,q) \cong L(4q, 2q-1).$$

Thus, Proposition 6.3.3 at once yields infinitely many pairs of prism manifolds that cobound a ribbon cobordism. Indeed, we conjecture the following analogue to Theorem 4.1.1. Its validity has been verified in a large number of cases through a computer

[2]Recall that, by the conventions used in this dissertation, $L(p,q)$ is defined as p/q-surgery along the unknot in S^3.

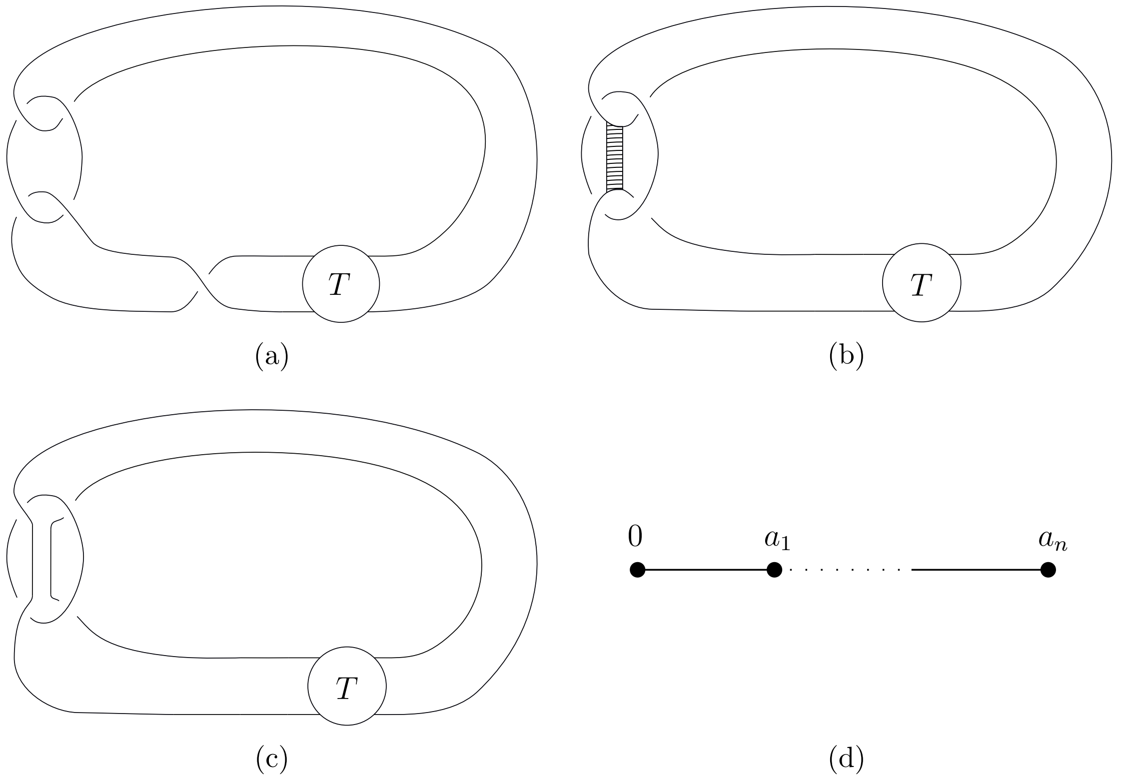

Figure 6.3: Illustration of $-L(q,p) \leq P(p,q)$.

search using SoaPy [Hub].

Conjecture 6.3.4. Let P_1 and P_2 be prism manifolds, and suppose that $P_1 \leq P_2$. Then, up to simultaneous orientation reversal of P_1 and P_2, one of the following holds:

1. $P_1 \cong P_2$;

2. $P_1 \cong P(1,n)$ and $P_2 \cong P(2n+1+4nk, 4n)$, for some $n \geq 2$ and $k \in \mathbb{Z}$; or

3. $P_1 \cong P(1,1)$ and $P_2 \cong P(q-p+qk, q)$ for $q/p \in \mathcal{F}_4$ and $k \in \mathbb{Z}$.

Conversely, in each of these cases $P_1 \leq P_2$ holds.

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