ESSAYS IN MARKET DESIGN

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Abstract

The Market Design approach, which involves the creation of markets with desirable properties, has been successfully applied to study a wide range of real-world economic problems. The market design approach is helpful in scenarios where money can't be used as a medium of exchange to facilitate transactions. The allocation of school/college seats to students, assigning residency positions to physicians, cadet-branch matching, and exchange of organs like kidneys and liver are some s problems that have been successfully studied using the market design approach. Typically, the market design approach concerns with the setting up of two-sided markets with agents on each side of the market having preferences over each other or agents on one side and objects (school seats, military branches, public health goods like beds, ventilators, etc.) on the other side with agents having preferences over the objects and objects having priority over the agents. Priority ranking of

agents can be considered an entitlement ranking where agents with higher priority have the right for the object compared to the agent with lower priority. The insights from the matching theory are then used to create a mechanism that matches agents with agents or objects for the given set of preferences/priority ranking satisfying desirable properties. Primary among these properties is stability, an equilibrium concept for matching. Stable matching ensures that matched agents/objects on the two sides of the market do not have an incentive to break up their respective matching and form a better matching for themselves. In the market design problem of matching agents to objects, stability ensures that the agent's priority for objects is not violated. Other properties include strategy-proofness, where agents do not have an incentive to misreport their preferences. Strategy-proof mechanisms are simple and ensure that high-information agents cannot game the system at the expense of low-information agents.

The priority ranking thus used in matching agents to objects has been subject to much criticism. The underlying process that generates the priority rankings can be inherently discriminatory. Exam scores are used to generate the priority ranking in allocating school seats to students. In the New York City school system, there has been a growing call for abolishing exams since it is considered to favor students with more resources. Similarly, the priority system used in the exchange of organs like kidneys and liver and triage allocation of scarce resources and services like hospital beds, vaccines, and ventilators has received much criticism. Triage protocols are developed with a utilitarian notion of maximum benefit given the constraints. This can result in people with better access to health care resources being better positioned under a triage protocol than those with lesser access.

The dissertation comprises two essays, a joint work with Kenzo Imamura where I study the pairwise kidney-exchange problem and a ventilator sharing problem where I study the triage allocation of ventilator slots under sharing.

In the first essay, I consider the problem of allocating ventilator slots for sharing under a triage protocol that generates the priority order. The triage protocol is considered discriminatory since patients with better access to health care through their life cycle have a better chance to be placed ahead in the order when compared with patients with lesser access to healthcare services. I consider the allocation of ventilator slots under a system of reserves, where slots are set-aside for types of patients to address the shortcoming of the triage protocol. Sharing is possible between patients who are compatible. In addition to addressing the shortcomings of the generated priority order, I focus on the question of what does respecting the generating priority order in a sharing environment mean.

In the second essay, we consider the pairwise-kidney exchange problem, where incompatible patient donor pairs are matched with each other subject to patient donor pairs being compatible with each other and acceptable to each other under a priority order. The priority order is generated using a composite score which includes variables like the area of patient donor location, and post-transplant medical survivability, among other factors. In response to the concerns, two mechanisms have been developed in the literature for pairwise-kidney exchange, a mechanism that facilitates pairwise-kidney exchange under a strict priority order and an egalitarian mechanism that doesn't have a priority ordering among compatible patients. Owing to the utilitarian nature of priority order ranking, the egalitarian mechanism has not been considered for adoption. We develop a compromise mechanism between the egalitarian mechanism and the mechanism which respects strict priority order. We show that the compromise mechanism carries forward nice properties like strategy-proofness, which incentivizes each patient-donor pair to reveal their complete set of compatible patient-donor pairs and bridges the concern of a need for priority order with egalitarianism.

The predominant literature in Matching theory considers matching agents with agents/objects under a priority order considering all agents to be equal and the priority ordering to be the only difference in consideration among agents. My dissertation contributes to the matching literature where different agents can vary in ways other than the priority ordering and we try to find solutions that strives to address the inequity . I thank my advisors for their generous advice and feedback in shaping my dissertation.

Contents

1 Ventilator Sharing - Access and Equity **1**

1.1 Introduction 1

1.2 Model 12

1.3 Allocation of Ventilators 14

1.4 Matching with Reserves: 16

1.5 Priorities and Maximal matching 58

1.6 Equity and Maximal Matching 78

1.7 Dynamics of Ventilator Allocation 82

1.8 Conclusion 83

2 Pairwise Kidney Exchange - Critique and Extension **84**

2.1 Introduction 84

2.2 Literature Review 88

2.3 Generalized Pairwise Kidney Exchange [1] 90

2.3.1 Gallai-Edmonds Decomposition 94

2.3.2 Induced two-sided matching market 97

2.3.3 Stochastic Exchange 98

2.4 Weighted Egalitaraian Mechanism 102

2.4.1 Preliminaries 102

2.4.2 Allocation rule 102

2.4.3 Mechanism 104

2.5 Priority and Weighted Egalitarian mechanism 109

2.6 Conclusion 111

[1] For continuity, We will be borrowing notations and use preliminary analysis from Roth Alvin et al. (2005)

List of Figures

1.1 **Baitler, Kallet, Kacmarek et.al.(2020)** 3

1.2 (White Douglas and Lo., 2020) 81

Chapter 1

Ventilator Sharing - Access and Equity

1.1 Introduction

The Covid-19 pandemic has placed tremendous strain on public health infrastructure, forcing the implementation of triage protocols for resources like ventilators, ICU beds, and vaccines. The state ofNewYorkwas hit hard during the first wave of the covid-19 pandemic. During March-April 2020, the city of New York recorded

20,900 deaths(Cov) more than the expected number of deaths for the same period. With the rise in the number of infections and fearing a surge in demand for public health services like mechanical ventilation, the state of New York requested the Food and Drug Administration(FDA) to allowventilator sharing. On March 25, 2020, the Food and Drug Administration(FDA) issued an Emergency Use Authorization(EUA) for ventilator splitters to facilitate ventilator sharing between patients. On March 26, 2020, the state of New York allowed its hospitals to treat two coronavirus patients with a single ventilator(Lorenzo et al., 2008). On the same day, the New York-Presbyterian Hospital released protocols for ventilator sharing between two patients. The protocol allows ventilator sharing between two eligible patients if the differences in Driving pressure, Respiratory rate, and Positive End Expiratory Pressure(PEEP) are within acceptable limits.

With growing global interconnection, the Covid-19 pandemic cannot be considered a tail event, and the occurrence of future influenza pandemics at an increasing frequency cannot be ruled out. Also, the requirements for public health services like mechanical ventilation cannot be reduced with the increase in the production of more ventilators.

There are approximately 62,000 full-featured ventilators(Lewis et al., 2010; Association, 2005) available in the United States with an additional 10,000 - 20,000(An-

Parameter	Acceptable Limit in Either Patient	Acceptable Difference between Patients (Patient A – Patient B)
Anticipated time needing invasive ventilation, h	72 or higher	—
V_T, ml/kg PBW	4–8	—
Driving pressure (ΔP = plateau pressure – PEEP), cm H_2O	5–16	0–6*
Respiratory rate, breaths/min	12–30	0–8
PEEP, cm H_2O	5–18	0–5
$F_{I_{O_2}}$, %	21–60	—
pH	7.30 or higher	—
Oxygen saturation, %	92–100	—
Ventilator titration	No recent major changes as judged clinically	—
Neuromuscular blockade	No contraindication to initiation if not already receiving	—
Respiratory infectious status	Both patients have same respiratory pathogen	None
Asthma or COPD	No severe baseline disease nor current exacerbation	—
Hemodynamic stability	No rapid vasopressor increase	—

Definition of abbreviations: COPD = chronic obstructive pulmonary disease; PBW = predicted body weight; PEEP = positive end-expiratory pressure.
If patients do not meet all criteria, pairing them on a single ventilator is not recommended. Further details are provided in the full protocol (*see* the online supplement). PBW denotes predicted body weight in kg, calculated for males as PBW = 50 + 2.3 [height (inches) – 60] and for females as PBW = 45.5 + 2.3 [height (inches) – 60].
Acceptable differences for between-patient parameters were specified only for driving pressure, respiratory rate, PEEP, and respiratory infectious status.
*Between-patient difference in driving pressure is the most important parameter to minimize in assessing potential compatibility of two patients.

Figure 1.1: **Baitler, Kallet, Kacmarek et.al.(2020)**

drew and Fink, February 29 2020) ventilators in Strategic National Stockpile. In addition, there are 98,000 ventilators that can provide basic functions during an emergency. During a pandemic, the limiting factor may not be the availability of ventilators but of respiratory therapists(Emanuel Ezekiel and Wertheimer., 2006) to operate them safely over three shifts a day. In 2018, the community hospitals employed around 76,000 full-time respiratory therapists. Regulations on patient safety and employee management also reduce the availability of respiratory therapists. California law requires one respiratory therapist for every four ventilated patients. Under such law, approximately 75,000 would be required to care for 100,000 ventilated patients over three shifts of the day(Emanuel Ezekiel and Wertheimer., 2006).

Using projections from the ”Spanish Flu” pandemic of 1918-19, a 2007 study by (Vawter Dorothy et al., 2007) estimate that a severe influenza pandemic will result in 10 million hospitalizations with 750,000 patients requiring mechanical ventilation. Even if the production of ventilators picks up, the human capital constraints can result in rationing of ventilators during the course of a severe pandemic.During a disaster, Lorenzo et al. (2008) note that physical, monetary, legal, and human capital constraints exist in providing mechanical ventilation to all patients and demonstrated ventilator sharing in four adult-human-sized sheep for 12 hours. Similarly, a study of multiple simulated patients on a single ventilator has been carried out by G and CB (2006). Ventilator sharing have for long been considered a viable alternative to combat the surge in demand during the pandemic.

Any mechanism that facilitates the sharing of ventilators between two compatible patients must consider the shortcomings of triage protocols in achieving equity and the need to achieve competing objectives on the part of respective governments tackling the public health crisis.

The triage allocation of mechanical ventilators were developed in response to a request from the Department of Health and Human Services(HHS)(Pan), by the Center for Disease Control and Prevention in 2011(CDC), providing a conceptual framework to state public health officials. The document outlines several ethical principles to be considered for the process and implementation of triage protocols by federal, state, local, and tribal health officials. These ethical principles include maximizing net benefits, maximizing lives and number of life years saved; instrumental valuation, prioritization of people who are viewed essential to prevent social disintegration during a pandemic; life cycle principle, prioritizing individuals based on various life cycle stages achieved. It is also suggested that multiple ethical principles be combined to produce a composite priority score for allocation to reflect the diverse ethical considerations underlying triage protocols.

Based on the conceptual framework, states have developed their guidelines. The state of New York(Howard et al., 2015) issued guidelines suggests prioritizing patients to save the maximum number of lives, while Pennsylvania(Pen) and Colorado(Col) guidelines prioritizes patients based on saving the maximum number of lives and life years, incorporating multiple ethical principles. The use of single/multiple ethical principles produces broad categories of patients. Tiebreakers are used to resolve allocation between patients within the same category. For example, Colorado guidelines suggest the use of a tiered approach for breaking ties. A lower tier is used to break the tie if patients are tied in all upper tiers of considerations. In Colorado guidelines, the top tier prioritizes patients based on their short-term and long-termseverity of illness, followed by pediatric patients, health careworkers, and first responders, followed by pregnant patients, sole caregivers. If ties between patients could not be resolved using the above criteria, the ventilators are allocated randomly between the patients.

Bioethicists(Marylou et al.; Williams, 1997; Ramsey, 1970; Rescher, 1969) and economists(Pathak Parag et al., 2020) have long argued about the problems of integrating these multiple ethical principles on a single priority scale. (Piscitello et al., 2020) note that, of the 26 states which have published their guidelines, 11 have recommended exclusion criteria for ventilator allocation. Exclusion criteria are meant to exclude patients from consideration for ventilator allocation. Beyond exclusion criteria, the triage protocol can fall short in two more ways. The triage protocol might be insufficient in addressing equity, being blind to category-specific claims of patients. (Reyes, 2020) notes that Blacks, Hispanics, Indigenous people, and Pacific Islanders have suffered disproportionate death rates from Covid-19 compared to other ethnic groups. Using state-level estimates,(Karaca-Mandic et al., 2021) show disproportionate hospitalization of Blacks and Hispanic people from Covid-19, while Azar Kristen et al. (2020) find that in a large health system in Northern California, Black people are 2.7 times more likely to be hospitalized after controlling for age, sex, comorbidities, and income compared to non-Hispanic whites. Mahajan Uma and Larkins-Pettigrew. (2020) show the disproportional impact of Covid through a correlational analysis on racial demographics and deaths in 2886 counties across the US. The triage protocols implemented by the states might be insufficient in promoting instrumental value. While issuing guidelines for triage protocols, New York did not prioritize health workers and first responders for tiebreakers fearing the unavailability of ventilators to other people in some areas. During the course of a pandemic, states might want to incentivize health care workers, first responders, caregivers, and essential workers to help save more lives and preserve order. The adopted guidelines can fall short in meeting this requirement.

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(Pathak Parag et al., 2020) highlight these shortcomings of the adopted triage protocols and propose a system of reserves as a compromise solution. A system of reserves sets aside ventilators for each type specific claims, available only for patients belonging to those type. Reserves have been used as a solution in response to type specific claims in the allocation of scarce medical resources. For example, Massachusetts pledged to allocate additional 20% vaccines to communities that have experienced disproportionate COVID-19 burden and high social vulnerability(MAp). New Hampshire allocated 10% of the state's available vaccine supply to geographical areas that are highly vulnerable to COVID-19(NHp). California required all vaccine providers to set aside 40% of their total appointment capacity for statewide prioritized groups. Outside the field of medicine, reserves have been used in the allocation of school seats and jobs. Chicago school system reserves 60% of a school's total seats for allocation based on the applicant's neighborhood tier(Dur et al., 2020). In India, horizontal reservation uses a system of reserves in providing job opportunities under affirmative action (Sonmez and Yenmez., 2019b,a).

In section II we define the ventilator allocation and sharing problem as a Non-Transferable Utility(**NTU**) problem with a system of reserves to overcome the shortcomings of triage protocol. Patients get 1 utility if they get a ventilator and 0 utility if they don't. While it is clear what respecting priorities should be for ventilator allocation problems without sharing, compatibility requirements in sharing a ventilator results in matching lower priority patients while keeping higher priority patients unmatched. We introduce the definition of **weakly respects priorities** which specifies that a lower priority patient can be allocated a ventilator slot if an unmatched higher priority patient cannot be provided a ventilator slot in the absence of lower priority patients. Using these definitions we then investigate sequential reserve matching, which processes ventilators based on a sequential ordering of types. We show that sequential reserve matching need not be maximal in allocating ventilator slots to patients belonging to the reserved type and introduce **Matching Algorithm with Reserves**. We demonstrate that the matching algorithm with reserves weakly respects priorities and it is maximal in allocating ventilator slots to patients belonging to at least one of the reserved types. In ventilator sharing problems, lower priority patients can be matched and relatively higher priority patients can be omitted due to compatibility issues. We show that the matching chosen by **Matching Algorithm with Reserves** maximizes the sum of priority scores compared to other matches which are identical with respect to patients who are allocated ventilator machines and threshold for allocating ventilator machine/slots to patients on the priority order.

(Beitler et al., 2020) study the safety and implementation of the New York Presbyterian Hospital protocols for ventilator sharing on four pairs of patients. Safe implementation of ventilator sharing carries potential risks and they recommend ventilator sharing to be restricted to centers with appropriate expertise. American health care system is patient-centric and respects patients' autonomy in decision making. Even though public health emergencies like disasters or pandemics can call for the suspension of these rights, the risks involved in the ventilator sharing problem can be hard to ignore. Hospital systems can ask for voluntary disclosure of compatible patients with whom patients can share a ventilator. In this regard, we analyze the incentives provided by the mechanism which implements the matching algorithm with reserves. We show that the mechanism is dominant strategy incentive compatible, incentivizing patients to fully disclose their list of compatible patients.

Even though weakly respecting priorities is an intuitive notion of what respecting priorities should be in matching problems with sharing, the matchings which satisfy weakly respects priorities need not maximize the number of patients who can be allocated a ventilator slot. In section III, we define **priority compliance** which is weaker than weakly respecting priorities . We then introduce **Modified Matching Algorithm with Reserves** which is priority compliant and is a compromise between matching greedily, maximizing the number of patients who are allocated a ventilator slot, and the intuitive notion of weakly respecting priorities. We discuss the incentive properties of **Modified Matching Algorithm with Reserves** and show that they provide dominant strategy incentive for patients to fully disclose the list of compatible patients.

Following Budish (2011), we define the notion of satisfying equity for the ventilator sharing problem. We show that under a random priority order the **Modified Matching Algorithm with Reserves** satisfies equity and allocates ventilator slots to the maximum number of patients. Using the conclusions from random priority order mechanism implementing **Modified Matching Algorithm with Reserves**, we explore an alternative way to the system of reserves for matching patients with ventilator slots taking into account their type-specific claims. Since patients are allocated ventilator slots dynamically, we then discuss matching in dynamic settings.

Though the focus of the paper is on ventilator sharing, the problem can be extended to study similar problems like public housing allocation with sharing, and public provision of shareable services like broadband connection to remote locations. In a 2014 study, Kermit et al. (2014) find that in households aged over 50, 10.2 million households(nearly one-sixth) were moderately cost-burdened [1], while nearly 9.6 million households were severely cost-burdened [2]. In the coming years, the demographic shift towards the older population is expected to increase homelessness. According to **Annual Homelessness Assessment Report(2016)** presented to Congress, there were 67,000 people aged 62 years and older who were homeless

[1] Moderately cost-burdened households pay more than 30 percent of income for housing, including utilities

[2] Severely cost-burdened households pay more than 50 percent of income for housing, including utilities

compared to 45,451 older people in 2007 an increase of 48.2%. Senior individuals are often house rich and cash poor. In such a scenario counties and cities can adopt house sharing programs, sharing a house or an apartment between two or more unrelated people to keep the cost of fighting homelessness down and overcome the lack of affordable housing units. The cities and counties can use a non-discriminatory compatibility questionnaire which can be used by participants to screen suitable partners for sharing a house or an apartment.

1.2 Model

- Let $\mathbf{P}$ denote the set of patients who are eligible for a ventilator.
- Let $\mathbf{T}$ denote the set of types under which patients are eligible to get priority access for a ventilator.
- Patients can belong to one or more of the types t, $t \in \mathbf{T}$. Let r_t be the set of slots reserved for type t.
- Let $u \in \mathbf{T}$ represent the type under which all patients are eligible.
- Let $\tau : \mathbf{P} \to 2^{\mathbf{T}}$ be the function which assigns each patient to an element of $2^{\mathbf{T}}$,

the power set of types.

- Let q be the total number of ventilators. There are up to $2q$ possible slots which are to be allocated among patients of different types.

- Let $r \equiv \{r_{t_1}, r_{t_2}, ..., r_{t_n}\}$ be the reserve system. Each r_{t_i} is the set of slots reserved for type t_i. For any type t_i, $\sum_{i=1}^{i=n} r_{t_i} \leq q$.

- A patient is eligible for ventilator sharing if their respiratory statistics are acceptable.

- Two patients can share ventilators among themselves only if they are eligible and difference between key respiratory statistics is within acceptable limit.

- Let $\mathscr{G} = \{(p_i, p_j) \mid p_i, p_j \in \mathbf{P} \cup \{\emptyset\}, p_i$ compatible with $p_j\}$.

- A type assignment function $\mu : \mathbf{P} \rightarrow \mathbf{T} \cup \{\emptyset\}$ that assigns every patient to a type or $\emptyset$ such that for all types t, $t \in \mathbf{T}$, $|\mu^{-1}(t)| \leq 2q$.

- If a patient $p \in \mathbf{P}$ doesn't have access to a ventilator slot, then $\mu(p) = \emptyset$.

- A sharing function $\phi : \mathbf{P} \rightarrow \mathbf{P}$ which assigns for every patient the patient whom they are sharing ventilator with. If $\phi(p_i) = p_j$, then $\phi(p_j) = p_i$.

- A matching is a (μ, ϕ) pair.

- A matching (μ, ϕ) is feasible if,

  (i) For every $t \in \mathbf{T} \setminus \{u\}$, $|\mu^{-1}(t)| \leq r_t$.

  (ii) If $\mu(p) \neq \emptyset$, then $(p, \phi(p)) \in \mathscr{G}$.

  (iii) If $\mu(p) = \emptyset$, then $\phi(p) = \emptyset$

- Let $\mathscr{M}$ be the set of feasible matchings.

- A ventilator sharing problem is a list $(\mathbf{P}, \mathscr{G}, \mathbf{T}, r, \tau)$

1.3 Allocation of Ventilators

A general theory of reserves comprises of a set of types, a quota for each of the types and priority ordering for each of the type which can rank patients differently across different types. To analyze the allocation of ventilators and shared slots we will concern ourselves with a baseline priority order since, in most triage problems and in most practical applications patients are ranked based on a baseline priority order. For example, 19 states use baseline priority ranking based on Sequential

Organ Failure Assessment(SOFA) scores (Piscitello et al., 2020) for allocation of ICU beds and ventilators.

Let π be the baseline priority order over patients. $p_1 \, \pi \, p_2$, implies that patient p_1 is strictly preferred to patient p_2. Suppose patient p_1 and p_2 belong to type t (i.e.) $t \in \tau(p_1)$ and $t \in \tau(p_2)$ then,

$$p_1 \, \pi \, p_2 \iff p_1 \, \pi_t \, p_2$$

where π_t is the induced priority ordering over patients for type t.

Baseline priority order is developed with consideration for SOFA score, comorbidities and life cycle experiences. Prioritisation based on SOFA score is done to save the most number of lives since SOFA scores are considered to be indicator of each patients health. The triage problem in its current form does not include considerations for compatible patients who are willing to share ventilators with other patients. Thus, the triage problem can involve a trade-off between saving healthy patients and saving the most number of patients.

Example 1. *There are three patients* $\mathbf{P} = \{p_1, p_2, p_3\}$ *,a single type* $\mathbf{T} = \{u\}$ *and a single ventilator to be allocated under the unreserved type . The baseline priority order* *patients is given by*

$$p_1 \, \pi \, p_2 \, \pi \, p_3$$

The compatible patients who can share ventilator is $\mathscr{G} = \{(p_2, p_3\}$.

There are two possible matches with,

$$\mu^{-1} = \{p_1\} \, and \, \mu^{'-1} = \{p_2, p_3\}$$

Matching μ *respects the baseline priority order* π *while matching* $\mu^{'}$ *matches the most number of patients.*

1.4 Matching with Reserves:

We will now consider the problem of matching patients with reserves under a system of reserves $r \equiv \{r_{t_1}, r_{t_2}, ..., r_{t_n}\}$. The reserves on the set of machines can be considered as a set aside, guaranteeing patients from the reserved categories a minimum of compatible machines. In this regard, the following definition will be helpful.

Let δ^{μ} refers to the set of patients who are matched to a ventilator slot under the matching μ to a type other than u and who belong to that type.

$$\delta^{\mu} = \{p \mid \mu(p) = t, t \in \tau(p) \, \text{and} \, t \in \mathbf{T} \setminus \{u\}\}$$

While applying a system of reserves, we can encounter a situation where all the patients belonging to a particular type have been considered for matching and there exists unused reserves of slots. In those scenarios we can follow either a Hard Reserve implementation or a Soft Reserve implementation.

Hard Reserve: A baseline priority order (π) follows a hard reserve system, if for any type $t \in \mathbf{T} \setminus \{u\}$, and for any patient p with $t \notin \tau(p)$,

$$\emptyset \, \pi_t \, p$$

In a hard reserve system of reserve implementation, if reserve slots of a particular type are under utilized, then they are not made available for other patients.

Soft Reserve: A baseline priority order (π) follows a soft reserve system, if for any type $t \in \mathbf{T} \setminus \{u\}$, and for any patients p, q, r with $t \in \tau(p)$, $t \notin \tau(q)$ and $t \notin \tau(r)$

$$p \; \pi_t \; q \; \pi_t \; \emptyset$$

$$p \; \pi_t \; r \; \pi_t \; \emptyset$$

$$q \; \pi_t \; r \iff q \; \pi \; r$$

In a soft reserve system of reserve implementation, if reserve slots of a particular type are under utilized, then they are made available for other patients following the baseline priority order.

Definition 1. A matching $(\mu, \phi) \in \mathscr{M}$, **complies with eligibility requirements** if for any patient $p \in \mathbf{P}$ and for any type $t \in \mathbf{T}$,

$$\mu(p) = t \Rightarrow p \; \pi_t \; \emptyset$$

A patient is allocated a ventilator slot under a type t, only if the patient is eligible to receive a ventilator under category t.

Definition 2. A matching $(\mu,\phi) \in \mathcal{M}$ is **non-wasteful** if for any patient $p \in \mathbf{P}$ and for any type $t \in \mathbf{T}$ if,

$$\mu(p) = \emptyset, \text{ and } p\ \pi_t\ \emptyset$$

then there exists no matching $(\mu^{'},\phi^{'}) \in \mathcal{M}$ such that for $q \in \mathbf{P} \setminus \{p\}$

$$\mu'(q) = \mu(q) \text{ and } \mu'(p) = t$$

A matching in $\mathcal{M}$ is **non-wasteful** if a patient can be allocated a ventilator slot under reserve constraints without unmatching already matched patients.

For any patient p and a matching μ let, $\mathcal{U}_p^{\delta}(t)$ represent the set of patients who are more preferred to patient p with respect to type t under π and who are matched to a ventilator slot under the type t to which patient p belongs. Mathematically,

$$\mathcal{U}_p^{\mu}(t) = \{q \in \mathbf{P} \mid q\ \pi_t\ p\ \&\ \mu(q) = t\}$$

Definition 3. A matching $(\mu,\phi) \in \mathcal{M}$, **respects type priorities for slots** if for any patient $p \in \mathbf{P}$ and $t \in \tau(p)$,$t \in \mathbf{T} \setminus \{u\}$, if

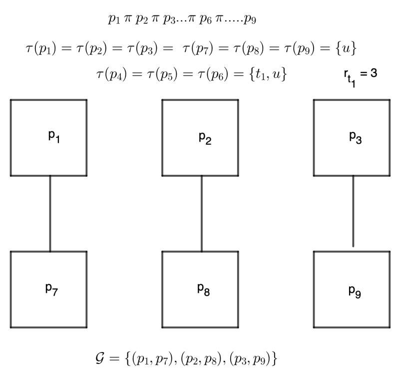
$$\mu(p) = \emptyset \Rightarrow |\mathcal{U}_p^{\mu}(t)| = r_t$$

For type $t = u$,

$$\mu(p) = \emptyset \Rightarrow |\mathcal{U}_p^{\mu}(u)| \geq q - \sum_{t \in \mathbf{T} \setminus \{u\}} r_t$$

In a ventilator sharing problem there are q guaranteed slots. If an eligible patient isn't matched to a ventilator slot then it should be the case that, the guaranteed slots for the types for which the patients is eligible have been assigned to higher priority patients in the same type. Note that, the definition mentions respecting type priorities over q slots and not $2q$ possible slots

Having reserves over 2q slots can lead to undesirable allocations in terms of beneficiary assignment. Consider the following example. There are 9 patients, 3 ventilators and two types $\mathbf{T} = \{t_1, u\}$. Let $r_{t_1} = 3$. Consider a ventilator sharing problem with a soft reserve system of reserve implementation.



From the above example the number of beneficiaries of type t_1 who were able to access ventilator slot is 0 even though there are 3 slots reserved for the type. Having reserves over $2q$ slots can lead to undesirable allocations from a policy point of view. The problem can be resolved if there exists a priorities over type space and slots are allocated based on the priority ordering of types. The techniques developed in this paper can be used to choose desirable allocations for the reserves over $2q$ slots problem. In the paper I try to implement reserves over q slots.

Definition 4. A matching $(\mu, \phi) \in \mathcal{M}$ **respects priorities** if for patients $p, q \in \mathbf{P}$ and for any type $t \in \mathcal{T}$

$$\mu(p) = t \text{ and } \mu(q) = \emptyset \Rightarrow p \, \pi_t \, q$$

Let $\mathcal{M}^{rp}$ be the set of feasible matchings which respects priorities.

In the problem of sharing ventilators, the ability to match a lower priority patient to a ventilator slot is contingent on finding a suitable patient with whom a ventilator can be shared. On the basis of compatibility, a lower priority patient can be matched to a ventilator slot compared to a relatively higher priority patient. This motivates a more permissive definition of a matching respecting priorities.

For any patient p and a matching μ let, $\mathscr{L}_p^{\mu^{-1}(t)}$ represent the set of patients who are not preferred to patient p with respect to type t under π and who are matched to a ventilator slot under the type t to which patient p belongs. Mathematically,

$$\mathscr{L}_p^{\mu^{-1}(t)} = \{q \mid p\, \pi_t\, q \,\&\, \mu(q) = t\}$$

$$\mathscr{L}_p^{\mu^{-1}} = \bigcup_{t \in \tau(p)} \mathscr{L}_p^{\mu^{-1}(t)}$$

Definition 5. Consider a ventilator sharing problem with a system of reserves. If a matching $(\mu,\phi) \in \mathscr{M}$, has $\mu(p) = \emptyset$ and $\mathscr{L}_p^{\mu^{-1}} \neq \emptyset$, then the matching (μ,ϕ) **weakly respects priorities** if there exists no matching $(\mu^{'},\phi^{'}) \in \mathscr{M}$ such that, for all $q \in \mathbf{P} \setminus \mathscr{L}_p^{\mu^{-1}}$

$$\mu^{'}(q) = \mu(q)$$

and $\mu^{'}(p) \neq \emptyset$.

In a matching problem with sharing, a matching respects priorities if freeing the slots to which a patient is eligible by means of priority ordering doesn't help the patient to get matched. Let $\mathcal{M}^{wrp}$ be the set of feasible matchings which weakly respects priorities.

Proposition 1. *For any ventilator sharing problem* $(\mathbf{P}, \mathcal{G}, \mathbf{T}, r, \tau)$*,*

$$\mathcal{M}^{rp} \subseteq \mathcal{M}^{wrp}$$

If $\mathcal{G} = \emptyset$ *then,*

$$\mathcal{M}^{rp} = \mathcal{M}^{wrp}$$

Proof. Take any matching $(\mu, \phi) \in \mathcal{M}^{rp}$. If $(\mu, \phi) \in \mathcal{M}^{rp}$ then for any patient p , $\mathcal{L}_p^{\mu^{-1}} = \emptyset$. $(\Rightarrow)$ $(\mu, \phi) \in \mathcal{M}^{wrp}$.

If $\mathcal{G} = \emptyset$ then, take any matching $(\mu, \phi) \in \mathcal{M}^{wrp}$. Suppose there exists a patient p with $\mu(p) = \emptyset$ with $\mathcal{L}_p^{\mu^{-1}} \neq \emptyset$. Since $\mathcal{G} = \emptyset$, all patients are allocated ventilators which aren't shared with other patients.

Let $\mu(q) = t$. Since $q \in \mathscr{L}_p^{\mu^{-1}}$, $p\, \pi_t\, q$. Construct a matching μ' as follows.

$$\mu'(q) = \emptyset,\ \mu'(p) = t$$

For all patients $r \in \mathbf{P} \setminus \{p, q\}$

$$\mu'(r) = \mu(r)$$

Matching μ' allocates a ventilator to patient p which was allocated to patient q while matching patient p to type t to which patient q was matched.

($\Rightarrow$) Matching $(\mu, \phi) \notin \mathscr{M}^{wrp}$. Contradiction.

$\blacksquare$

The following algorithm can be used to choose a Pareto efficient matching which respects priorities. I will build the algorithm considering the case of a single type u and all the patients belonging to that type.

For the algorithm, at every step we will consider the induced graph (E, V) where V is the set of patients who have been matched to the algorithm up to that step.

There exists an edge $e \in E$ between two vertices p_1 and p_2, if there exists patients p_1 and p_2 who are compatible with each other.

Matching Algorithm:

Step 0: Let $\phi^0(p) = \emptyset$ for $p \in \mathbf{P}$. Since no patient is matched with any ventilator, no patient is sharing their ventilator with any other patient. $\mu^0(p) = \emptyset$ for $p \in \mathbf{P}$.

Step 1: Set $\phi^1(p) = \phi^0(p)$ Let q be the number of available ventilators which are to be allocated. Choose the patient p_1 who is the most preferred patient remaining according to baseline priority order. Match the patient to an empty ventilator and reduce the count of available ventilators by 1 to $q-1$. $\phi^1(p) = \phi^0(p)$ for all $p \in \mathbf{P}$.

$\mu^1(p) = \mu^0(p)$ for all $p \in \mathbf{P} \setminus \{p_1\}$

$\mu^1(p_1) = u$

$\vdots$

All the available ventilators have been allocated by the end of step q.

Step q+1: Set $\phi^{q+1}(p) = \phi^q(p)$. There are no available ventilators, yet to be allocated. Choose the patient p_{q+1} who is the $(q+1)^{th}$ most preferred patient in the baseline priority order. Check if an already matched patient p_i can be made to share a ventilator with another already matched patient p_j. If so, make patients p_i

and p_j share a ventilator and allocate the empty ventilator to patient p_{q+1}.

Update $\phi^{q+1}(p_i) = p_j$ and $\phi^{q+1}(p_j) = p_i$.

For patients $p \in \mathbf{P} \setminus \{p_i, p_j\}$, $\phi^{q+1}(p) = \emptyset$

$\mu^{q+1}(p) = \mu^q(p)$ for all $p \in \mathbf{P} \setminus \{p_{q+1}\}$

$\mu^{q+1}(p_{q+1}) = u$

Suppose there exists no already matched patient p_i who can be made to share with another already matched patient p_j . Check if patient p_{q+1} can be made to share a ventilator with another already matched patient p_i. Make an already matched patient p_i share a ventilator with patient p_{q+1} , if they are compatible.

$\phi^{q+1}(p_i) = p_{q+1}$

$\phi^{q+1}(p_{q+1}) = p_i$.

For all other patients $p \in \mathbf{P} \setminus \{p_i, p_{q+1}\}$, $\phi^{q+1}(p) = \emptyset$.

If patient p_{q+1} is not compatible with any of the patients then patient p_{q+1} is unmatched and $\mu^{q+1} = \mu^q$.

$\phi^{q+1}(p) = \phi^q(p)$

$\vdots$

Step q+k: There are no available ventilators, yet to be allocated. Choose the patient p_{q+k} who is the $(q+k)^{th}$ preferred patient from baseline priority order. Check if there exists a chain of patients $[\emptyset - p_1^m] - [p_2^m - \phi^{q+k}(p_2^m)] - [p_3^m - \phi^{q+k}(p_3^m)] .... - [p_k^m - \emptyset]$ with the following properties.

$$\mu^{k-1}(p_i^m) \neq \emptyset \quad i \in \{1, 2, 3..., n\}$$

p_1^m can share a ventilator with p_2^m and $\phi^{q+k}(p_i^m)$ is compatible with p_{i+1}^m for $i > 1$ and ϕ^{q+k} is updated along the chain by turning edges in the chain belonging to matching to not belonging to the matching and vice versa. If such a chain exists, then it is an augmenting path. Allocate the ventilator of patient p_1^m to patient p_{q+k}^m and make patient p_i^m share ventilator with $\phi^{q+k}(p_{i-1}^m)$. We will call such a chain a Ventilator Freeing Chain(**VFC**).

$\mu^{q+k}(p) = \mu^{q+k-1}(p)$ for all $p \in \mathbf{P} \setminus \{p_{q+k}\}$

$\mu^{q+k}(p_{q+k}) = u$

Suppose there exists no such chain. Check if there exists a chain of patients $[\emptyset - p_1^m] - [p_2^m - \phi(^{q+k}(p_2^m)] - [p_3^m - \phi^{q+k}(p_3^m)] .... - [p_k^m - \phi^{q+k}(p_k^m)]$ such that

$$\mu^{k-1}(p_i^m) \neq \emptyset \quad i \in \{1,2,3...,n\}$$

Patient p_1^m is compatible with patient p_2^m and for all $i > 1$ patient $\phi^{q+k}(p_i^m)$ is compatible with patient p_{i+1}^m. ϕ^{q+k} is updated along the chain by turning edges in the chain belonging to matching to not belonging to the matching and vice versa. We will call such a chain a Ventilator Slot Freeing Chain(**VSFC**). Check if there exists a **VSFC** exists such that patient p_{q+k} can be matched, (i.e)$\phi^{q+k}(p_k^m)$ is compatible with p_{q+k}. If such a chain exists, then it is an augmenting path.

If such a chain exists patient p_1^m is made to share ventilator with patient p_2^m and for all $i > 1$ patient $\phi^{q+k}(p_i^m)$ are made to share ventilator with patient p_{i+1}^m and ϕ^{q+k} is updated along the chain. $\phi^{q+k}(p_k^m)$ is allocated the ventilator of patient p_1^m and patient p_{q+k} shares ventilator with patient $\phi^{q+k}(p_k^m)$.

Suppose there exists no such chain, then patient p_{q+k} is unmatched.

$\phi^{q+k}(p) = \phi^{q+k-1}(p)$.

$\mu^{q+k} = \mu^{q+k-1}$

Proceed with the next patient in baseline priority order.

$\vdots$

The algorithm terminates in $|\mathbf{P}|$ steps.

(μ, ϕ) where $\mu = \mu^{|\mathbf{P}|}, \phi = \phi^{|\mathbf{P}|}$ is the matching chosen by the algorithm.

Theorem 1. *The matching algorithm chooses a Pareto efficient matching that weakly respects priorities.*

Proof. **Claim:** Let $\mathbf{P}^{k-1}$ be the set of patients matched by step $k-1$. Let p_k be the patient chosen by step k of the algorithm. Suppose p_k is not matched by step k. Then there exists no matching where for any patient $p \in \mathbf{P}^{k-1} \cup \{p_k\}$, $\mu(p) \neq \emptyset$.

Consider any step $l, l > k$. Let p_l be the patient chosen by step l of the algorithm.

There are no available ventilators, yet to be allocated. If there are available ventilators which can be allocated, then patient p_k is allocated a ventilator and is matched by step k of the algorithm. Contradiction.

Lemma 1. *For steps $l > k$, there exists no chain of patients $[\emptyset - p_1^m] - [p_2^m - \phi^l(p_2^m)] - [p_3^m - \phi^l(p_3^m)] \ldots . - [p_n^m - \emptyset]$ with the following properties*

p_1^m is compatible with p_2^m and $\phi^l(p_i^m)$ is compatible with p_{i+1}^m for $i > 1$ and ϕ^l is updated along the chain by turning edges in the chain belonging to matching to not belonging to the matching and vice versa.

Proof. We will establish the lemma using induction.

Since patient p_k is not matched to a ventilator slot by the end of step k, there exists no chain of patients $[\emptyset - p_1^m] - [p_2^m - \phi^k(p_2^m)] - [p_3^m - \phi^k(p_3^m)]\ldots. - [p_n^m - \emptyset]$ with the following properties

$$\mu^k(p_i^m) \neq \emptyset \quad i \in \{1, 2, 3\ldots, n\}$$

p_1^m is compatible with p_2^m and $\phi^k(p_i^m)$ is compatible with p_{i+1}^m for $i > 1$ in step $k+1$.

Suppose patient p_{k+1} is matched by the end of step $k+1$. Then there doesn't exist a chain $[\emptyset - p_1^m] - [p_2^m - \phi^{k+1}(p_2^m)] - [p_3^m - \phi^{k+1}(p_3^m)\ldots. - [p_n^m - \emptyset]$ with

$$\mu(p_i^m) \neq \emptyset \quad i \in \{1, 2, 3\ldots, n\}$$

p_1^m is compatible with p_2^m and $\phi^{k+1}(p_i^m)$ is compatible with p_{i+1}^m for $i > 1$ by the end of step $k+1$ and ϕ^{k+1} is updated along the chain by turning edges in the chain belonging to matching to not belonging to the matching and vice versa.

Suppose not. Suppose there exists such a chain by the end of step $k+1$. Then construct a chain as follows. For any i, if $[p_i^m - \phi^{k+1}(p_i^m)] = [p_i^m - \phi^k(p_i^m)]$ then retain the vertex as part of the chain.

If $[p_i^m - \phi^{k+1}(p_i^m)] \neq [p_i^m - \phi^k(p_i^m)]$ then replace $[p_i^m - \phi^{k+1}(p_i^m)]$ with $[p_i^m - \phi^k(p_i^m)] - [\phi^{k+1}(\phi^k(p_i^m)) - \phi^k(\phi^{k+1}(\phi^k(p_i^m)))] - \ldots[\phi^{k+1}(\ldots(\phi^k(p_i^m))) - \phi^k(\ldots\phi^{k+1}(\phi^k(p_i^m)))]$ where $\phi^k(\ldots\phi^{k+1}(\phi^k(p_i^m))) = \phi^{k+1}(p_i^m)$.

If $\phi^{k+1}(p_i^m) = p_{k+1}$ then stop when $\phi^k(\ldots\phi^{k+1}(\phi^k(p_i^m))) = \emptyset$.

($\Rightarrow$) There exists a chain of patients, $[\emptyset - p_1^m] - [p_2^m - \phi^k(p_2^m)] - [p_3^m - \phi^k(p_3^m)] \ldots. - [p_n^m - \emptyset]$ by the end of step k with

$$\mu(p_i^m) \neq \emptyset \quad i \in \{1, 2, 3\ldots, n\}$$

p_1^m is compatible with p_2^m and $\phi^k(p_i^m)$ is compatible with p_{i+1}^m for $i > 1$ and ϕ^k is updated along the chain by the end of step k by turning edges in the chain belonging to matching to not belonging to the matching and vice versa. If such a chain exists, then it is an augmenting path. Contradiction.

By inductive argument, there exists no chain of patients $[\emptyset - p_1^m] - [p_2^m - \phi^{l-1}(p_2^m)] - [p_3^m - \phi^{l-1}(p_3^m)\ldots. - [p_n^m - \emptyset]$ with

$$\mu(p_i^m) \neq \emptyset \quad i \in \{1,2,3\ldots,n\}$$

p_1^m is compatible with p_2^m and $\phi^{l-1}(p_i^m)$ is compatible with p_{i+1}^m for $i > 1$ and ϕ^{l-1} is updated along the chain by the end of step $l-1$ by turning edges in the chain belonging to matching to not belonging to the matching and vice versa.

Suppose patient p_l is matched by the end of step l. Then there doesn't exist a chain $[\emptyset - p_1^m] - [p_2^m - \phi^{l}(p_2^m)] - [p_3^m - \phi^{l}(p_3^m)\ldots. - [p_n^m - \emptyset]$ with

$$\mu(p_i^m) \neq \emptyset \quad i \in \{1,2,3\ldots,n\}$$

p_1^m is compatible with p_2^m and $\phi^{l}(p_i^m)$ is compatible with p_{i+1}^m for $i > 1$ and ϕ^{l} is updated along the chain by the end of step l by turning edges in the chain belonging to matching to not belonging to the matching and vice versa.

Suppose not. Suppose there exists such a chain. Then construct a chain as follows. For any i, if $[p_i^m - \phi^l(p_i^m)] = [p_i^m - \phi^{l-1}(p_i^m)]$ then retain the vertex as part of the chain.

If $[p_i^m - \phi^l(p_i^m)] \neq [p_i^m - \phi^{l-1}(p_i^m)]$ then replace $[p_i^m - \phi^l(p_i^m)]$ with $[p_i^m - \phi^{l-1}(p_i^m)] - [\phi^l(\phi^{l-1}(p_i^m)) - \phi^{l-1}(\phi^l(\phi^{l-1}(p_i^m)))] - ...[\phi^l(...(\phi^{l-1}(p_i^m))) - \phi^{l-1}(...\phi^l(\phi^{l-1}(p_i^m)))]$ where $\phi^{l-1}(...\phi^l(\phi^{l-1}(p_i^m))) = \phi^l(p_i^m)$.

If $\phi^l(p_i^m) = p_l$ then stop when $\phi^{l-1}(...\phi^l(\phi^{l-1}(p_i^m))) = \emptyset$.

($\Rightarrow$) There exists a chain of patients, $[\emptyset - p_1^m] - [p_2^m - \phi^{l-1}(p_2^m)] - [p_3^m - \phi^{l-1}(p_3^m)].... - [p_n^m - \emptyset]$ by the end of step $l-1$ with

$$\mu(p_i^m) \neq \emptyset, \quad i \in \{1,2,3...,n\}$$

p_1^m is compatible with p_2^m and $\phi^{l-1}(p_i^m)$ is compatible with p_{i+1}^m for $i > 1$ in step l and ϕ^{l-1} is updated along the chain by turning edges in the chain belonging to matching to not belonging to the matching and vice versa. Contradiction.

■

Lemma 2. *For steps $l > k$, there exists no chain of patients $[\emptyset - p_1^m] - [p_2^m - \phi^l(p_2^m)] - [p_3^m - \phi^l(p_3^m)].... - [p_k^m - \phi^l(p_k^m)]$ with the following properties.*

$$\mu^{l-1}(p_i^m) \neq \emptyset \quad i \in \{1,2,3...,n\}$$

$\phi^l(p_k^m)$ is compatible with p_k. Patient p_1^m is compatible with patient p_2^m and for all $i > 1$ patient $\phi^l(p_i^m)$ is compatible with patient p_{i+1}^m and ϕ^l is updated along the chain by turning edges in the chain belonging to matching to not belonging to the matching and vice versa.

Proof. Patient p_k was not matched to a ventilator slot by the end of step k. There exists no chain of patients $[\emptyset - p_1^m] - [p_2^m - \phi^k(p_2^m)] - [p_3^m - \phi^k(p_3^m)].... - [p_n^m - \phi^k(p_n^m)]$ with the following properties

$$\mu^k(p_i^m) \neq \emptyset \quad i \in \{1,2,3...,n\}$$

p_1^m is compatible with p_2^m and $\phi^k(p_i^m)$ is compatible with p_{i+1}^m for $1 < i < n$ and $\phi^k(p_n^m)$ is compatible with patient p_k in step $k+1$.

Suppose patient p_{k+1} is matched by the end of step of $k+1$. Then there doesn't exist a chain $[\emptyset - p_1^m] - [p_2^m - \phi^{k+1}(p_2^m)] - [p_3^m - \phi^{k+1}(p_3^m).... - [p_n^m - \phi^{k+1}(p_n^m]$ with

$$\mu(p_i^m) \neq \emptyset \quad i \in \{1, 2, 3..., n\}$$

p_1^m is compatible with p_2^m and $\phi^{k+1}(p_i^m)$ is compatible with p_{i+1}^m for $1 < i < n$ and $\phi^{k+1}(p_n^m)$ is compatible with patient p_k by the end of step $k+1$.

Suppose not. Suppose there exists such a chain. Then construct a chain as follows. For any i, if $[p_i^m - \phi^{k+1}(p_i^m)] = [p_i^m - \phi^k(p_i^m)]$ then retain the vertex as part of the chain.

Case i: $\phi^{k+1}(p_i^m) \neq p_{k+1}$. If $[p_i^m - \phi^{k+1}(p_i^m)] \neq [p_i^m - \phi^k(p_i^m)]$ then replace $[p_i^m - \phi^{k+1}(p_i^m)]$ with $[p_i^m - \phi^k(p_i^m)] - [\phi^{k+1}(\phi^k(p_i^m)) - \phi^k(\phi^{k+1}(\phi^k(p_i^m)))] - ...[\phi^{k+1}(...(\phi^k(p_i^m))) - \phi^k(...\phi^{k+1}(\phi^k(p_i^m)))]$ where $\phi^k(...\phi^{k+1}(\phi^k(p_i^m))) = \phi^{k+1}(p_i^m)$.

Case ii: $\phi^{k+1}(p_i^m) = p_{k+1}$. Stop when $\phi^k(...\phi^{k+1}(\phi^k(p_i^m))) = \emptyset$.

($\Rightarrow$)There exists a chain of patients, $[\emptyset - p_1^m] - [p_2^m - \phi^k(p_2^m)] - [p_3^m - \phi^k(p_3^m)].... - [p_n^m - \phi^k(p_n^m)]$ by the end of step k with

$$\mu(p_i^m) \neq \emptyset \quad i \in \{1, 2, 3..., n\}$$

p_1^m is compatible with p_2^m and $\phi^k(p_i^m)$ is compatible with p_{i+1}^m for $1 < i < n$ and $\phi^k(p_n^m)$ is compatible with p_k in step $k+1$. Contradiction.

(or)There exists a chain of patients, $[\emptyset - p_1^m] - [p_2^m - \phi^k(p_2^m)] - [p_3^m - \phi^k(p_3^m)] .... - [p_n^m - \emptyset]$ by the end of step k with

$$\mu(p_i^m) \neq \emptyset \quad i \in \{1, 2, 3..., n\}$$

p_1^m is compatible with p_2^m and $\phi^k(p_i^m)$ is compatible with p_{i+1}^m for $i > 1$ in step $k+1$. Contradiction.

By inductive argument, there exists no chain of patients $[\emptyset - p_1^m] - [p_2^m - \phi^{l-1}(p_2^m)] - [p_3^m - \phi^{l-1}(p_3^m) .... - [p_n^m - \phi^{l-1}(p_n^m)]$ with

$$\mu(p_i^m) \neq \emptyset \quad i \in \{1, 2, 3..., n\}$$

p_1^m is compatible with p_2^m and $\phi^{l-1}(p_i^m)$ is compatible with p_{i+1}^m for $1 < i < n$ and $\phi^{l-1}(p_n^m)$ is compatible with patient p_k by the end of step $l-1$.

Suppose patient p_l is matched by the end of step l.Then there doesn't exist a chain $[\emptyset - p_1^m] - [p_2^m - \phi^l(p_2^m)] - [p_3^m - \phi^l(p_3^m) .... - [p_n^m - \phi^l(p_n^m)]$ with

$$\mu(p_i^m) \neq \emptyset \quad i \in \{1,2,3...,n\}$$

p_1^m is compatible with p_2^m and $\phi^l(p_i^m)$ is compatible with p_{i+1}^m for $1 < i < n$ and $\phi^l(p_n^m)$ is compatible with patient p_k by the end of step l.

Suppose not. Suppose there exists such a chain. Then construct a chain as follows. For any i, if $[p_i^m - \phi^l(p_i^m)] = [p_i^m - \phi^{l-1}(p_i^m)]$ then retain the vertex as part of the chain.

Case i: $\phi^{k+1}(p_i^m) \neq p_{k+1}$ If $[p_i^m - \phi^l(p_i^m)] \neq [p_i^m - \phi^{l-1}(p_i^m)]$ then replace $[p_i^m - \phi^l(p_i^m)]$ with $[p_i^m - \phi^{l-1}(p_i^m)] - [\phi^l(\phi^{l-1}(p_i^m)) - \phi^{l-1}(\phi^l(\phi^{l-1}(p_i^m)))] - ...[\phi^l(...(\phi^{l-1}(p_i^m))) - \phi^{l-1}(...\phi^l(\phi^{l-1}(p_i^m)))]$ where $\phi^{l-1}(...\phi^l(\phi^{l-1}(p_i^m))) = \phi^l(p_i^m)$.

Case ii: $\phi^{k+1}(p_i^m) = p_{k+1}$ Stop when $\phi^{l-1}(...\phi^l(\phi^{l-1}(p_i^m))) = \emptyset$.

$(-)$There exists a chain of patients, $[\emptyset - p_1^m] - [p_2^m - \phi^{l-1}(p_2^m)] - [p_3^{l-1} - \phi^{l-1}(p_3^m)]....-$ $[p_n^{l-1} - \phi^{l-1}(p_n^m)]$ by the end of step $l-1$ with

$$\mu(p_i^m) \neq \emptyset \quad i \in \{1,2,3...,n\}$$

p_1^m is compatible with p_2^m and $\phi^{l-1}(p_i^m)$ is compatible with p_{i+1}^m for $1 < i < n$ and $\phi^{l-1}(p_n^m)$ is compatible with p_k in step $k+1$. Contradiction.

(or)There exists a chain of patients, $[\emptyset - p_1^m] - [p_2^m - \phi^{l-1}(p_2^m)] - [p_3^m - \phi^{l-1}(p_3^m)] .... - [p_n^m - \emptyset]$ by the end of step k with

$$\mu(p_i^m) \neq \emptyset \quad i \in \{1,2,3...,n\}$$

p_1^m is compatible with p_2^m and $\phi^{l-1}(p_i^m)$ is compatible with p_{i+1}^m for $i > 1$ in step $k+1$. Contradiction. ■

From the Lemmas (1) & (2) , it can be concluded that if p_k is not matched by step k, then there exists no matching where for any patient $p \in \mathbf{P}^{k-1} \cup \{p_k\}$, $\mu(p) \neq \emptyset$. ■

Once empty ventilators are exhausted by matching with patients, the ability to match a patient to an empty ventilator slot depends on the existence of chains. Each patient is a vertex in the algorithm. Since a chain can include a vertex more than once, the number of chains that needs to be checked for the existence of a matching with the chosen patient, grows exponentially as more and more patients are considered for a ventilator slot. Thus, the complexity of finding a Pareto efficient match by the above procedure increases with the increase in number of patients.

Simple Chains: A chain is simple if all its vertices are distinct.

A chain in the above algorithm is simple, if a patient is included at most once as part of the chain.

Proposition 2. *There exists a chain such that patient p_{q+k} is matched by step $q+k$ of the algorithm. Then there exists a simple chain such that patient p_{q+k} is matched by step $q+k$.*

Proof. There exists a chain of patients $[\emptyset - p_1^m] - [p_2^m - \phi^{q+k-1}(p_2^m)] - [p_3^m - \phi^{q+k-1}(p_3^m)] \ldots. - [p_k^m - \phi^{q+k-1}(p_k^m)]$ with the following properties.

$$\mu^{k-1}(p_i^m) \neq \emptyset, \quad i \in \{1,2,3...,n\}$$

$\phi^{q+k-1}(p_k^m)$ is compatible with p_{q+k}. Patient p_1^m is compatible with patient p_2^m and for all $i > 1$ patient $\phi^{q+k-1}(p_i^m)$ is compatible with patient p_{i+1}^m.

By the algorithm, patient p_1^m shares a ventilator with p_2^m and for all $i > 1$ patient $\phi^{q+k-1}(p_i^m)$ shares ventilator with patient p_{i+1}^m by end of step k. Patient $\phi^{q+k-1}(p_k^m)$ shares the ventilator vacated by patient p_1^m with patient p_{q+k} and ϕ^{q+k} is updated along the chain.

Let $\mathbf{P}^{q+k-1}$ be the set of patients matched by step $q+k-1$. Since no patient who is matched by end of step $q+k-1$ is waitlisted by end of step $q+k$, construct the simple chain as follows.

$$[\emptyset - p_1^m] - [\phi^{q+k}(p_1^m) - \phi^{q+k-1}(\phi^{q+k}(p_1^m))] - [\phi^{q+k}(\phi^{q+k-1}(\phi^{q+k}(p_1^m))) - \phi^{q+k-1}(\phi^{q+k}(\phi^{q+k-1}(\phi^{q+k}(p_1^m)))].... - [\phi^{q+k}(......(p_1^m)) - \phi^{q+k-1}(......(p_1^m))]. \qquad \blacksquare$$

We now consider the allocation of ventilator slots under multiple types and a system of reserves. A common way of allocating objects under a system of reserves is by using sequential reserve matchings. In sequential reserve matchings ventilator slots are allocated to types based on the sequential order over the types. Sequential reserve matchings have been employed in allocation of school seats and in the allocation of jobs under affirmative action(Sonmez and Yenmez., 2019b,a). Even though sequential reserve matchings are intuitive and easy to apply, they can produce undesirable matchings when patients can fall under more than one category. Consider the following example

Example 2. *There are 3 patients* $\mathbf{P} = \{p_1, p_2, p_3\}$. *There are three types,* $\mathbf{T} = \{t_1, t_2, u\}$. $\tau(p_1) = \{t_1, t_2, u\}$, $\tau(p_2) = \{u\}$, $\tau(p_3) = \{t_1, u\}$. *The patients aren't compatible with each other,* $\mathcal{G} = \emptyset$. *There are 4 ventilator slots to be allocated and one ventilator slot is reserved for* t_1 *and* t_2. *The reserves are to be processed between types* t_1 *and* t_2 *based* *on the order of precedence $\rhd$, $t_1 \rhd t_2$ with a soft reserve system of implementation. The baseline order of priority π over patients is given by*

$$p_1 \, \pi \, p_2 \, \pi \, p_3$$

It can be seen that under the sequential order of processing reserves $t_1 \rhd t_2$, $\mu(p_1) = t_1, \mu(p_2) = u$, which allocates the ventilator slot reserved under type t_2 to patient p_2 even though p_2 does not belong to type t_2. The matching allocates one slot to a beneficiary belonging to a type . But there exists a matching μ^*, with $\mu^*(p_1) = t_2$ and $\mu^*(p_3) = t_1$ which maximizes the allocation of ventilator slots to beneficiaries belonging to types.

The above example suggests that applying a sequential reserve matching need not maximize the allocation of ventilator slots to patients belonging to different types. To overcomes this, we will adopt the **Matching Algorithm With Reserves**.

For the **Matching Algorithm with Reserves** we will consider two graphs. Graph (E,V), where vertices V is the set of patients. There exists an edge $e \in E$ between two vertices v_i and v_j, if there exists patients p_i and p_j who are compatible with each other.

Directed Graph $(\mathscr{E}^k, \mathscr{V})$ induced by step k, where vertices $\mathscr{V}$ is the set of types. There exists a directed edge $e \in \mathscr{E}^k$ from vertex v_i to v_j, if there exists patients p_i who is matched under type t_i by step k and $t_j \in \tau(p_i)$. In words, there exists a directed edge from type t_i to t_j if there exists a patient who is matched under t_i by step k and who is eligible to be matched under type t_j.

A simple directed chain $t_1 \to t_2 \to ...t_i \to ... \to t_n$ for a patient p is called a Type Slot Freeing Chain (**TSFC**) if $u \notin \{t_1, t_2, ...t_n\}$, $t_1 \in \tau(p)$ and $r_{t_n} > 0$.

Matching Algorithm with Reserves:

Let $\mu^0(p) = \emptyset$ for all patients $p \in \mathbf{P}$.

A patient is **eligible** for ventilator machine under a type t if $t \in \tau(p)$.

Step 1: Choose the most prioritized patient p_1 according to baseline priority order. Check if there exists a **TSFC** for patient p_1 in the directed graph $(\mathscr{E}^0, \mathscr{V})$, with $t_1 \in \tau(p_1)$. If such a type exists match p_1 to an empty ventilator under type t_1. $\mu^1(p_1) = t_1$

$\mu^1(p) = \mu^0(p), \forall p \in \mathbf{P} \setminus \{p_1\}$. Reduce r_{t_1} by 1.

Step 2: Choose the second most prioritized patient p_2 from baseline priority order. Check if there exists a **TSFC** for patient p_2 in the directed graph $(\mathscr{E}^2, \mathscr{V})$, with $t_1 \in \tau(p_2)$. If such a type exists match p_2 to the slot of an empty ventilator machine under the type t_1 and update matching μ^2 along **TSFC** by matching a patient matched to t_1 under μ^1 to t_2. Reduce r_{t_2} by 1.Update μ^2 accordingly.

$\vdots$

$\vdots$

Step k: Choose the k^{th} most prioritized patient p_k from baseline priority order. Check if there exists a **TSFC** in the directed graph $(\mathscr{E}^{k-1}, \mathscr{V})$, with $t_1 \in \tau(p_k)$. If such a type exists match p_k to the slot of an empty ventilator machine under the type t_1 and update match μ^k along **TSFC** by matching a patient matched to t_i under μ^{k-1} to t_i under μ^k such that the patient is eligible to be matched under t_i . Reduce r_{t_n} by 1. Update μ^k accordingly.

Repeat the steps until r_{t_i} is reduced to 0 for every $t_i \in \mathbf{T}$ or all the patients are checked to be matched under one of the types. If there are ventilator slots under types which are yet to be allocated after exhausting the entire set of patients, then there exists types with more ventilator slots than patients. The ventilator slots will remain unallocated if hard reserve system of reserve implementation is followed.

In case of soft reserve system of reserve implementation, allocate the ventilators to unmatched patients under respective types based on baseline priority order. Repeat the steps with unmatched patients, starting with the most preferred unmatched patient.

⋮

⋮

Step l+1: Choose the most prioritized patient p_{l+1} according to the baseline priority order who is unmatched. If there exists no such patient matching $\mu^l = \mu$ is the matching chosen by the algorithm.

Check if there exist a **VFC** in the graph (E^l, V^l) , where V^l is the set of patients matched up to step l. There exists an edge between matched patients if two patients are compatible. If there exists a **VFC**, use the **VFC** and allocate an empty ventilator to patient p_{l+1}. Match p_{l+1} to unreserved category u.

If there doesn't exist a **VFC**, check if there exists a **VSFC** with respect to the graph (E^l, V^l) such that patient p_{l+1} can be matched , where V^l is the set of patients matched up to step l. There exists an edge between matched patients if two patients are compatible. If there exists a **VSFC** such that p_{l+1} can be matched, use the **VSFC** and allocate an empty ventilator slot to patient $p_l + 1$.Match p_{l+1} with unreserved category u. Update μ^{l+1} accordingly.

If there doesn't exist a **VFC** and **VSFC** such that patient p_{l+1} can be matched, then patient p_{l+1} is unmatched by the end of step $l+1$.

⋮

⋮

Step n: Choose patient p_n who is next in baseline priority order to patient p_{n-1} and who is unmatched.

If there exists no slots to match patient p_n then matching $\mu^{n-1} = \mu$ is the matching chosen by the algorithm.

Check if there exist a **VFC** in the graph (E^{n-1}, V^{n-1}) , where V^l is the set of patients matched up to step $l-1$. There exists an edge between matched patients if two patients are compatible. If there exists a **VFC**, use the **VFC** and allocate an empty ventilator to patient p_n. Match patient p_n under the unreserved category u. Update μ^n.

If there doesn't exist a **VFC**, check if there exists a **VSFC** such that patient p_{n-1} can be matched with respect to the graph (E^{n-1}, V^{n-1}) . If there exists a **VSFC** with respect to patient p_{n-1}, use the **VSFC** and allocate an empty ventilator slot to patient p_n. Match patient p_n under unreserved category u. Update μ^n accordingly.

If there doesn't exist a **VFC** and **VSFC** with respect to patient p_{n-1}, then patient p_n is unmatched by the end of step n.

⋮

⋮

The algorithm ends when all the slots are matched or all the unmatched students are processed for matching with empty slots.Let (μ^{mar}, ϕ^{mar}) be the matching chosen by the matching algorithm with reserves.

Let,

$$\delta^{\mu^{mar}} = \{p \mid \mu^{mar}(p) = t, t \in \tau(p) \text{and} \, t \in \mathbf{T} \setminus \{u\}\}$$

$\delta^{\mu^{mar}}$ contains the set of patients matched to slots reserved under types.

For any matching (μ^{mar}, ϕ^{mar}) , let μ_p be,

- the lowest π-priority patient with the property that every weakly higher π patient has been matched under μ, if some patient is unmatched under μ
- $\emptyset$ if all patients are matched under μ

For any matching (μ^{mar}, ϕ^{mar}) and associated δ^{μ}, let $\delta^{\mu}_{p_t}$ be,

- the lowest π-priority patient which is matched to a slot under some type $t \in \mathbf{T} \setminus \{u\}$
- $\emptyset$ if all patients are matched under μ

Theorem 2. *For any ventilator sharing problem ($\mathbf{P}, \mathscr{G}, \mathbf{T}, r, \tau$) with a hard reserve or a soft reserve system of reserve implementation,*

(i) *The matching (μ^{mar}, ϕ^{mar}) complies with eligibility requirements, is non-wasteful and weakly respects priorities.*

(ii) *(μ^{mar}, ϕ^{mar}) respects type priorities for slots.*

(iii) *For any feasible matching (μ, ϕ) which respects type priorities for slots,*

$$|\delta^{\mu^{mar}}| \geq |\delta^{\mu}|$$

The number of patients beneficiaries belonging to the types and who are provided a slot is weakly greater under the matching chosen by the algorithm than the number of beneficiaries who are provided a slot under their types for any matching which respects type priorities for slots.

(iv) *For any feasible matching (μ, ϕ) with,*

$$|\delta^{\mu^{mar}}| = |\delta^{\mu}|$$

$$\delta_p^{\mu^{mar}} \, \pi \, \delta_p^{\mu}$$

Proof. (i)At each step of the algorithm patients are matched to type for which they are eligible. Hence the matching chosen by the matching algorithm with reserves complies with eligibility requirements.

Suppose (μ^{mar}, ϕ^{mar}) is not non-wasteful. Then there exists a patient p with

$$\mu^{mar}(p) = \emptyset, \text{ and } p \; \pi_t \; \emptyset$$

and a matching μ'

$$\mu'(p') = \mu(p) \text{ and } \mu'(p') = t$$

Let k be the last step in which patient p was considered for matching. The above condition implies that there exists a **VSFC** with respect to patient p and a ventilator slot which patient p can be matched. Contradiction.

Suppose the matching (μ^{mar}, ϕ^{mar}) does not weakly respects priorities. There exists a matching $(\mu^{'}, \phi^{'}) \in \mathscr{M}$ and patient p such that, for all $q \in \mathbf{P} \setminus \mathscr{L}_p^{\mu^{mar}}$

$$\mu^{'}(q) = \mu^{mar}(q)$$

and $\mu^{'}(p) \neq \emptyset$.

Let k be the step at which patient p was considered for matching under matching algorithm with reserves. Then patient p is eligible to be matched since there exists a feasible matching $(\mu^{'}, \phi^{'})$ with $\mu^{'}(p) \neq \emptyset$.

The patient p isn't matched by the end of step k by the algorithm. By theorem 1, there doesn't exists any feasible matching such that patients in $\mathbf{P} \setminus (\mathscr{L}_p^{\mu^{mar}} \cup \{p\})$ are all matched. Contradiction

(ii)Suppose the matching (μ^{mar}, ϕ^{mar}) does not respect type priorities for slots. Then there exists a patient p and a type t, $t \in \tau(p)$

$$\mu^{mar}(p) = \emptyset \text{ and } |\mathscr{U}_p^{\mu^{mar}}(t)| < r_t$$

If a patient is matched by end of a step then the patient is matched by the end of algorithm.

Let k be the step at which patient p was considered for matching. Then there exists a **TSFC** such that patient p can be matched to type t. This implies that, patient p is matched by end of step k. Patient p is matched by the end of algorithm. Contradiction.

(iii) Consider any matching (μ, ϕ). Then δ^{μ} be the set of students matched to types other than unreserved type and who belong to the type they are matched to.

Suppose,

$$|\delta^{\mu^{mar}}| < |\delta^{\mu}|$$

Let $\delta^{diff} \equiv (\delta^{\mu} \setminus \delta^{\mu^{mar}}$ be the set of patients matched under δ and not under δ^{mar}.

Then there exists a type $t \in \{T\} \setminus \{u\}$ such that, there exists no patient $p \in \delta^{diff}$ who can be matched under type t and,

$$|\delta_t^{\mu^{mar}}| < |\delta_t^{\mu}|$$

Let $\mathbf{T}_{ud}$ represent the set of such underdemanded types.

Starting with such a type $t \in \mathbf{T}_{ud}$, we can construct directed chain $t \to t_1$ such that there exists a patient p with $\mu^{mar}(p) = t_1$, $\mu(p) = t$ and

$$|\delta_{\{t,t_1\}}^{\mu^{mar}}| < |\delta_{\{t,t_1\}}^{\mu}|$$

Let $\mathbf{T}_1$ represent the set of such types such that for every patient p, $\mu^{mar}(p) = t_1$, $\mu(p) = t$, $t_1 \in \mathbf{T}_1$ and $t \in \mathbf{T}_{ud}$

Suppose there doesn't exist such a type $t_1 \in T_1$. Then,

$$|\delta^{\mu^{mar}}| < |\delta^{\mu}|$$

which is a contradiction.

Similarly there exists a type $t_2 \neq \{t, t_1\}$ and a directed chain $t \to t_1 \to t_2$ such that there exists a patient p with $\mu^{mar}(p) = t_2$, $\mu(p) = t_1$ and

$$|\delta^{mar}_{\{t,t_1,t_2\}}| < |\delta_{\{t,t_1,t_2\}}|$$

But $\mathbf{T} \setminus \{u\}$ is finite. This implies that, there exists a patient $p \in \delta^{diff}$ and a step k in the matching algorithm with reserves such that, there exists a **TSFC** and patient p is matched under δ^{mar} by the end of step k. Contradiction. ■

After the initial allocation of ventilators to patients based on a system of reserves, successive patients are provided access to ventilators depending on the ability of already matched patients to share ventilator amongst them and with unmatched patients.

For any matching μ, let μ_p be,

- the lowest π-priority patient with the property that every weakly higher π patient has been matched under μ, if some patient is unmatched under μ
- $\emptyset$ if all patients are matched under μ

For any matching μ, let $\mu^o = (p_1, p_2, ..., p_{|\mu^{-1}(\mathbf{T})|})$ be an ordered sequence of matched patients such that $p_i \, \pi \, p_j$ iff $i < j$.

For any pair of matching $\mu, \nu \in \mathscr{M}$, let $\mu^o = (p_1, p_2, ..., p_{|\mu^{-1}(\mathbf{T})|})$ and $\nu^o = (q_1, q_2, ..., q_{|\nu^{-1}(\mathbf{T})|})$. Matching μ **order dominates** matching ν, $\mu \geq \nu$, if $|\mu^{-1}| \geq |\nu^{-1}|$ and for all i, $p_i \overline{\pi} q_i$.

Note that, **order dominates**($\geq$) is a partial order on the set of matchings $\mathscr{M}$.

Theorem 3. *Consider matchings (μ^{mar}, ϕ^{mar}) and (μ, ϕ), $\mu^{mar} \neq \mu$ with their corresponding $\delta^{\mu^{mar}}$ and δ^{μ}. Suppose $\mu_p^{mar} = \mu_p$ and $\delta^{\mu^{mar}} = \delta^{\mu}$. Then $\mu^{mar} \succ \mu$.*

Proof. Let $\mathscr{L}^{\mu_p^{mar}}$ contain the set of patients who are less preferred to μ_p^{mar} and who aren't allocated a ventilator slot.

$$\mathscr{L}^{\mu_p^{mar}} = \{p \mid \mu_p^{mar} \, \pi \, p \,\&\, \mu^{mar}(p) = \emptyset\}$$

Note that for any matching $\mu \in \mathscr{M}$ with $\mu_p = \mu_p^{mar}$ and $\delta^{\mu^{mar}} = \delta^{\mu}$,

$$\mu^{mar}|_{\mathbf{P}\setminus\mathscr{L}^{\mu_p^{mar}}} \succeq \mu|_{\mathbf{P}\setminus\mathscr{L}^{\mu_p^{mar}}}$$

We will prove the claim via induction.

Let $p^* \in \mathscr{L}^{\mu_p^{mar}}$ be the most prioritized patient such that $\mu^{mar}(p^*) = \emptyset$. By Theorem 1, for all matching $\mu \in \mathscr{M}$ with $\mu_p = \mu_p^{mar}$ and $\delta^{\mu} = \delta^{\mu mar}$, $\mu(p^*) = \phi$. This implies,

$$\mu^{mar}|_{\mathbf{P}\setminus\mathscr{L}^{p^*}} \succeq \mu|_{\mathbf{P}\setminus\mathscr{L}^{p^*}}$$

Let $\hat{p} \in \mathscr{L}^{\mu_p^{mar}}$. For all patients p, with $p\,\overline{\pi}\,\hat{p}$

$$\mu^{mar}|_{\mathbf{P}\setminus\mathscr{L}^{p}} \succeq \mu|_{\mathbf{P}\setminus\mathscr{L}^{p}}$$

Let $\hat{p}_m$ be the most prioritized patient in $\mathscr{L}^{\hat{p}}$. It suffices to show that,

$$\mu^{mar}|_{\mathbf{P}\setminus\mathscr{L}^{\hat{p}_m}} \succeq \mu|_{\mathbf{P}\setminus\mathscr{L}^{\hat{p}_m}}$$

If $|\,\mu^{mar}|_{\mathbf{P}\setminus\mathscr{L}^{\hat{p}}}\,| > |\,\mu|_{\mathbf{P}\setminus\mathscr{L}^{\hat{p}}}\,|$, it immediately implies that

$$\mu^{mar}|_{\mathbf{P}\setminus\mathscr{L}^{\hat{p}_m}} \succeq \mu|_{\mathbf{P}\setminus\mathscr{L}^{\hat{p}_m}}$$

Suppose $|\mu^{mar}|_{\mathbf{P}\setminus\mathscr{L}^{\hat{p}}}| = |\mu|_{\mathbf{P}\setminus\mathscr{L}^{\hat{p}}}|$. It is then enough to show that,

$$\mu(\hat{p}_m) \neq \emptyset \Rightarrow \mu^{mar}(\hat{p}_m) \neq \emptyset$$

Let,

$$\beta = \{p \in \mathscr{L}^{\mu_{\mathbf{P}}} \setminus \mathscr{L}^{\hat{p}} | p \in \mu^{-1} \,\&\, p \notin (\mu^{mar})^{-1}\}$$

β contains the set of patients who are less preferred to μ_p^{mar} who are matched in μ and not in μ^{mar}.

$\mu(\hat{p}_m) \neq \emptyset$. Then there exists a chain of patients matched in $\mu|_{\mathbf{P}\setminus\mathscr{L}^{\hat{p}}}$, $[p_1^m - \phi(p_1^m)] - \ldots - [p_i^m - \phi(p_i^m)] - \ldots .[p_n^m - \emptyset]$ such that $\hat{p^m}$ is compatible with p_1^m, $\phi(p_{i-1}^m)$ is compatible with p_i^m for $i > 1$ and for any j, $p_j^m \in \beta \implies \phi(p_j^m) \notin \beta$ and vice versa.

Suppose not. Suppose both $p_j^m, \phi(p_j^m) \in \beta$. Then consider the restriction of μ, $\mu|_{\mathbf{P}\setminus\mathscr{L}^{\mu_p}}$ .Let $\hat{\mu_p}$ be the most prioritized patient in $\mathscr{L}^{\mu_p}$. There exists a **VFC** or **VSFC** such that $\hat{\mu_p}$ can be provided a ventilator slot and $\hat{\mu_p}$ can be matched in $\mu|_{\mathbf{P}\setminus\mathscr{L}^{\hat{\mu_p}}}$.

Since $\mu^{mar}|_{\mathbf{P}\setminus\mathscr{L}^{\mu_p^{mar}}} = \mu|_{\mathbf{P}\setminus\mathscr{L}^{\mu_p}}$, there exists a **VFC** or **VSFC** such that $\mu_p^{\hat{mar}} = \hat{\mu_p}$ can be matched to a restriction of μ^{mar}, $\mu^{mar}|_{\mathbf{P}\setminus\mathscr{L}^{\mu_p^{mar}}}$. But $\mu_p^{\hat{mar}} \notin \mu^{mar}$ and μ^{mar} weakly respects priorities. Contradiction.

Let ϕ^{μ} be the function which keeps track of patients who are sharing ventilator in the matching $\mu|_{\mathbf{P}\setminus\mathscr{L}^{\hat{p}m}}$

Suppose $\mu^{mar}(\hat{p}_m) = \emptyset$. Then $|\mu^{mar}|_{\mathbf{P}\setminus\mathscr{L}^{\hat{p}m}}| < |\mu|_{\mathbf{P}\setminus\mathscr{L}^{\hat{p}m}}|$.

Let $\phi^{\mu^{mar}}$ be the function which keeps track of patients who are sharing ventilator in the matching $\mu^{mar}|_{\mathbf{P}\setminus\mathscr{L}^{\hat{p}m}}$.

Since $|\mu^{mar}|_{\mathbf{P}\setminus\mathscr{L}^{\hat{p}m}}| < |\mu|_{\mathbf{P}\setminus\mathscr{L}^{\hat{p}m}}|$, there exists a patient $p_1 \in \mathbf{P} \setminus \mathscr{L}^{\mu_{p_1}^{mar}}$ such that $\phi^{\mu(p_1)} \neq \emptyset$ and $\phi^{\mu^{mar}}(p_1) = \emptyset$.

We can construct a chain as follows. Let [$\emptyset$ - p_n] - [$\phi^{\mu^{mar}}(p_{n-1})$ - p_{n-1}]-...-[$\phi^{\mu^{mar}}(p_{i-1})$ - p_{i-1}]....-[$\phi^{\mu^{mar}}(p_2)$ - p_2]-[p_1 - $\emptyset$] where $\phi^{\mu}(p_n) = \tilde{p}$, $\tilde{p} \in \beta \cup \{\hat{p}_m\}$, and $\phi^{\mu}(p_i) = \phi^{\mu^{mar}}(p_{i-1})$.

If $\tilde{p} \in \beta$ then there exists a patient who could have been matched by the step of the algorithm at which the patient is chosen and isn't matched by the end of step. Contradiction.

If $\tilde{p} = \hat{p}_m$ then patient $\hat{p}_m$ is matched under μ^{mar} . Contradiction. ■

Ventilator sharing discussed so far can carry significant risks to patients sharing ventilators. On February 9, 2021 the FDA on its letter to Health Care providers(FDA) noted the challenges in ventilator sharing. Ventilator sharing requires continual balancing of respiratory mechanics between co-vented patients; requires paralysis and deep sedation to prevent asynchrony; can cause pendelluft [3], resulting in lung injury; increased complexity in decision making; and logistical problems with lack of ventilator alarms to alert individual ventilation problems. Even though we want to maximize the number of lives saved by providing ventilator access during a public health emergency, due to increased risks hospital systems might want to take consent for ventilator sharing from their patients. To encourage patients to share ventilator, hospitals can provide sufficient leeway in letting patients choose set of compatible patients for ventilator sharing. Thus any algorithm for ventilator sharing must provide incentives for patients to reveal their full set of compatible patients to maximize the number of patients matched. The following proposition shows that any mechanism using **Matching Algorithm with Reserves** provides dominant strategy incentives to reveal their full set of compatible patients

Proposition 3. *Consider the mechanism which implements Matching Algorithm with Reserves for any given ventilator sharing problem* $(\mathbf{P}, \mathscr{G}, \mathbf{T}, r, \tau)$. *The mechanism is*

[3] Swinging air from one co-vented patient to another

strategyproof.

Proof. For any patient $p_i \in \mathbf{P}$ and $\mathcal{G}$, let $\mathcal{G}_{p_i}$ be the set of patients who are compatible with patient p_i. Note that, for any pair of patients $p_i, p_j \in \mathbf{P}$, $(p_i, p_j) \in \mathcal{G}$ iff $p_j \in \mathcal{G}_{p_i}$ and $p_i \in \mathcal{G}_{p_j}$.

For any patient $p_i \in \mathbf{P}$ and a $\mathcal{G}$, let $\mathcal{G}_{-p_i} = (\mathcal{G}_{p_1}, \mathcal{G}_{p_2}, ...\mathcal{G}_{p_{i-1}}, \mathcal{G}_{p_{i+1}}, ...\mathcal{G}_{p_n})$ be the list of compatible patients chosen by patients other than p_i.

Let $\hat{\mathcal{G}}_{p_i}$ be the actual set of patients who are compatible for patient p_i. Let k be the step at which patient p_i is chosen to be matched in the algorithm. Take any $\mathcal{G}'_{-p_i}$. We will consider three cases.

Case i: $k < q$ or patient p_i is allocated a ventilator slot under soft reserves system of reserve implementation. Then irrespective of the set of compatible patients chosen by patient p_i, patient p_i is matched to a ventilator slot. Hence patient p_i is matched to a ventilator slot if patient p_i reveals $\hat{\mathcal{G}}_{p_i}$ as the set of compatible patients.

Case ii: There exists a **VFC** such that patient p_i is matched by end of step k in the algorithm. Then irrespective of the set of compatible patients chosen by patient p_i, patient p_i is matched to a ventilator slot. Hence patient p_i is matched to a ventilator slot if patient p_i reveals $\hat{\mathcal{G}}_{p_i}$ as the set of compatible patients.

Case iii: There exists no ventilator to be allocated and there exists no **VFC**. Then for any $\mathscr{G}'_{p_i} \subseteq \hat{\mathscr{G}}_{p_i}$, if there exists **VSFC** such that patient p_i is matched by end of step k, then there exists a **VSFC** under $\hat{\mathscr{G}}_{p_i}$ such that patient p_i is matched by end of step k. ■

1.5 Priorities and Maximal matching

For a ventilator sharing problem $(\mathbf{P}, \mathscr{G}, \mathbf{T}, r, \tau)$, let $\mathscr{M}^c \subseteq \mathscr{M}$ be the set of matches which complies with eligibility requirements and respects type priorities for slots. The matching chosen by the matching algorithm with reserves $(\mu^{mar}, \phi^{mar}) \in \mathscr{M}^c$, need not be the maximal matching in $\mathscr{M}^c$. Consider the following example.

Example 3. *There are 5 patients,* $\mathbf{P} = \{p_1, p_2, p_3, p_4, p_5\}$*. There are 2 ventilators, a single type* u*, and* $\mathscr{G} = \{(p_1, p_2), (p_1, p_4), (p_4, p_5)\}$*. The priority order* π *over the set of patients* $\mathscr{G}$ *is given by,*

$$p_1 \, \pi \, p_2 \, \pi \, p_3 \, \pi \, p_4 \, \pi \, p_5$$

There are two matches which complies with eligibility requirements and respects type *priorities for ventilator slots.*

$$\mu^{-1} = \{p_1, p_2, p_3\} \, and \, \mu'^{-1} = \{p_1, p_2, p_4, p_5\}$$

μ is chosen by the matching algorithm with reserves while μ' assigns most ventilator slots to most patients .

The result can be extended to matchings which weakly respects priorities. In a matching which weakly respects priorities, a ventilator made available by letting an already matched compatible patient share ventilator with another patient, be provided to a high priority patient irrespective of the patient's ability to share the ventilator.

Definition 6. A matching $(\mu, \phi) \in \mathscr{M}$ is **priority compliant**, if for any pair of patients p, q and any type t with $p \, \pi_t \, q$,

$$\mu(q) = t \; \& \; \mu(t) = \emptyset$$

then there doesn't exist a matching $(\mu', \phi') \in \mathscr{M}$ such that $\mu'(r) = \mu(r)$ for all $r \in \mathbf{P} \setminus \{p, q\}$ and

$$\mu'(p) = t \;\&\; \mu'(q) = \emptyset$$

A matching is priority compliant if unmatching a lower priority patient, doesn't make it possible for a higher priority patient to be assigned a ventilator slot under the same type.

Let $\mathscr{M}^{pc}$ be the set of matchings which are priority compliant.

Proposition 4. *For any ventilator sharing problem* $(\mathbf{P}, \mathscr{G}, \mathbf{T}, r, \tau)$,

$$\mathscr{M}^{rp} \subseteq \mathscr{M}^{wrp} \subseteq \mathscr{M}^{pc}$$

If $\mathscr{G} = \emptyset$ *then,*

$$\mathscr{M}^{rp} = \mathscr{M}^{wrp} = \mathscr{M}^{pc}$$

Proof. Let $(\mu, \phi) \in \mathscr{M}^{pc}$. If $\mathscr{G} = \emptyset$, then there doesn't exists patients $p, q \in \mathbf{P}$ and a type t with $p \, \pi_t \, q$ and

$$\mu(q) = t \text{ and } \mu(p) = \emptyset$$

($\Rightarrow$) Matching (μ,ϕ) respects priorities. $(\mu,\phi) \in \mathscr{M}^{rp}$.

Let $(\mu,\phi) \in \mathscr{M}^{wrp}$. Then, there exists no matching $(\mu',\phi') \in \mathscr{M}$ such that, for all $r \in \mathbf{P} \setminus \mathscr{L}_p^{\mu}$

$$\mu'(r) = \mu(r) \text{ and } \mu'(p) \neq \emptyset$$

($\Rightarrow$) There exists no patient q, type t with $p\,\pi_t\,q$, and a matching $(\mu',\phi') \in \mathscr{M}$ such that for $r \in \mathbf{P} \setminus \{p,q\}$

$$\mu'(r) = \mu(r)$$

$$\mu(p') = t\ ,\mu(p) = \emptyset \text{ and } \mu'(p) = t, \mu'(q) = \emptyset$$

($\Rightarrow$) Matching (μ,ϕ) is priority complinat. $(\mu,\phi) \in \mathscr{M}^{pc}$.

■

Definition 7. A matching $(\mu,\phi) \in \mathscr{M}$ is **maximal**, if $|\mu^{-1}| \geq |\mu'^{-1}|$, for all $(\mu',\phi') \in \mathscr{M}$.

Proposition 5. *There exists a maximal matching which is priority compliant.*

Proof. Let $(\mu,\phi) \in \mathscr{M}$ be a maximal matching. Suppose (μ,ϕ) is not priority compliant. Then we can find a priority compliant maximal matching as follows.

1. **Step 1:** Choose the unmatched patient with highest priority p_1. Check if there exists a chain such that $[p_1^m - \phi(p_1^m)] - [p_2^m - \phi(p_2^m] - \ldots - [p_i^m - \phi(p_i^m)] - \ldots.[p_n^m - \phi(p_n^m)]$, where for all $j \in \{1,2,3,\ldots n\}$, p_j^m and $\phi(p_j^m)$ are patients who are sharing a ventilator with respect to matching μ, p_1 can share ventilator with patient p_1^m, for all $j \in \{1,2,3,\ldots n\}$ $\phi(p_{j-1}^m)$ can share a ventilator with p_j^m and $p_1 \, \pi_t \, \phi(p_n^m)$, $\mu(\phi(p_n^m)) = t$.

   If there exists such a chain then there exists a matching μ^1 such that $\mu^1(p) = \mu(p)$ for all $p \in \mathbf{P} \setminus \{p_1, \phi(p_n^m)\}$ and

   $$\mu^1(p_1) = t \;\&\; \mu^1(\phi(p_n^m)) = \emptyset$$

   If there exists such a chain, choose matching μ^1 and repeat the step. If there exists no such chain, then retain matching μ and repeat the step with unmatched patient p_2 who is proritized more than the other unmatched patients excluding p_1.

   $\vdots$

The process ends in a finite number of steps, since there are a finite number of patients. ■

Modified Matching Algorithm with Reserves

Step 0: Allocate the q ventilators as in matching algorithm with reserves. Let μ^0 and be the matching obtained by allocating empty ventilators to patients.

Let $\alpha = \{p \mid \mu^0(p) \neq \emptyset\}$ be the set of patients chosen to fill the q guaranteed slots.

Step 1: All the empty/available ventilators are allocated. Check if there exists a Ventilator Freeing Chain(**VFC**). If there exists a **VFC**, make patients share ventilator according to **VFC**, making the ventilator available for future patients. Exhaust **VFCs** so that, there exists no more ventilators which can be made available for patients who are yet to be matched. Let q_f be the count of ventilators freed by making patients share ventilator with each other and exhausting **VFCs**. $\mu^1 = \mu^0$

Step 2: There exists a patient in $(\mu^1)^{-1}$ who isn't sharing a ventilator with another patient. If there doesn't exist such a patient proceed to step k Choose the most prioritized unmatched patient p_2, $p_2 \in \mathbf{P} \setminus ((\mu^1)^{-1})$. Check if there exists a Ventilator Slot Freeing Chain(**VSFC**) such that patient p_2 can be matched. If there exists a

VSFC such that p_2 can be matched, make patients share ventilator according to **VSFC** allocating ventilator slot to patient p_2. Update μ^2. If no such **VSFC** with respect to patient p_2 exists, then patient p_2 is unmatched.$\mu^2 = \mu^1$.

Step 3: There exists a patient in $(\mu^2)^{-1}$ who isn't sharing a ventilator with another patient. If there doesn't exist such a patient proceed to step k. Choose the most prioritized unmatched patient $p_3 \in \mathbf{P} \setminus ((\mu^2)^{-1} \cup \{p_2\})$. Check if there exists a Ventilator Slot Freeing Chain(**VSFC**) such that patient p_3 can be matched. If there exists a **VSFC** such that patient p_3 can be matched, make patients share ventilator according to **VSFC** allocating ventilator slot to patient p_3. Update μ^3. If no such **VSFC** exists such that patient p_3 can be matched then patient p_3 is unmatched.$\mu^3 = \mu^2$.

$\vdots$

Step k: All the patients in $(\mu^{k-1})^{-1}$ are sharing a ventilator with another patient or $\mathbf{P} \setminus ((\mu^{k-1})^{-1} \cup \{p_2, p_3, ..., p_{k-1}\}) = \emptyset$. Choose the most preferred unmatched patient p_k, $p_k \in \mathbf{P} \setminus ((\mu^{k-1})^{-1})$. Match patient p_k, temporarily to one of the q_f available ventilators.

$$\mu_{te}^k(p) = \mu^{k-1}(p), \, p \in \mathbf{P} \setminus \{p_k\}$$

$$\mu_{te}^{k}(p_k) = u$$

where μ^{k_t} is the temporary matching at step k. Let

$$\mathscr{L}_{p_k} = \{p' \mid p_k \, \pi \, p' \,\&\, \mu_{te}^{k}(p') = \emptyset\}$$

Choose the most preferred patient $p_1^l \in \mathscr{L}_{p_k}$. Check if there exists a Ventilator Slot Freeing Chain(**VSFC**) in the matching $\mu^{k_{te}}$ such that patient p_1^l can be matched. If there exists such a chain then

$$\mu^{k}(p) = \mu_{te}^{k}(p),\, p \in \mathbf{P} \setminus \{p_1^l\}$$

$$\mu^{k}(p_1^l) = u$$

.

Proceed with step $k+1$ if there exists an available ventilator. If there exists no available ventilator then,

$$\mu = \mu^k, \phi = \phi^k$$

is the matching chosen by the algorithm.

If there doesn't exist a **VSFC** such that p_1^l can be matched, choose p_2^l, the most preferred patient in $\mathscr{L}_{p_k}$ excluding p_1^l, $p_2^l \in \mathscr{L}_{p_k} \setminus \{p_1^l\}$. Repeat the above steps with the temporary matching $\mu^{k_t}(p)$.

If there exists no patient $p_j^l \in \mathscr{L}_{p_k}$ such that there exists a **VSFC** matching p_j^l then patient p_k is unmatched.

Step k+1: Choose the most preferred unmatched patient p_{k+1}, $p_{k+1} \in \mathbf{P} \setminus ((\mu^k)^{-1} \cup \{p_k\})$. Match patient p_{k+1}, temporarily to one of the available ventilators.

$$\mu_{te}^{k+1}(p) = \mu^k(p),\, p \in \mathbf{P} \setminus \{p_{k+1}\}$$

$$\mu_{te}^{k+1}(p_{k+1}) = u$$

where μ^{k+1_t} is the temporary matching at step k. Let

$$\mathscr{L}_{p_{k+1}} = \{p' \mid p_{k+1} \, \pi \, p' \, \& \, \mu^k_{te}(p') = \emptyset\}$$

Choose the most preferred patient $p^l_1 \in \mathscr{L}_{p_{k+1}}$. Check if there exists a Ventilator Slot Freeing Chain(**VSFC**) in the matching μ^{k+1_t} such that patient p^l_1 can be matched. If there exists such a chain then

$$\mu^{k+1}(p) = \mu^{k+1}_{te}(p),\ p \in \mathbf{P} \setminus \{p^l_1\}$$

$$\mu^{k+1}(p^l_1) = u$$

.

Proceed with step $k+2$ if there exists an available ventilator. If there exists no available ventilator then,

$$\mu = \mu^{k+1}, \phi = \phi^{k+1}$$

is the matching chosen by the algorithm.

If there doesn't exist a **VSFC**, choose p^l_2, the most preferred patient in $\mathscr{L}_{p_{k+1}}$ excluding p^l_1, $p^l_2 \in \mathscr{L}_{p_{k+1}} \setminus \{p^l_1\}$. Repeat the above steps with the temporary matching

$\mu^{k+1_t}(p)$.

If there exists no patient $p_j^l \in \mathscr{L}_{p_{k+1}}$ such that there exists a **VSFC** matching p_j^l then patient p_{k+1} is unmatched.

$\vdots$

Step m: If $\mathbf{P} \setminus ((\mu^{m-1})^{-1} \cup \{p_k, p_{k+1}, ...p_{m-1}\}) = \emptyset$, then choose p_m the most preferred unmatched patient, and allocate an available ventilator to p_m.

$$\mu^m(p) = \mu^{m-1}(p),\ p \in \mathbf{P} \setminus \{p_m\}$$

$$\mu^m(p_m) = u$$

If there exists an available ventilator, proceed to step $m+1$ and repeat step m. If there doesn't exist any available ventilator then,

$$\mu = \mu^m, \phi = \phi^m$$

is the matching chosen by the algorithm

$\vdots$

Let ($\mu^* = \mu$, $\phi^* = \phi$) be the matching chosen by the modified matching algorithm with reserves and $\delta^{\mu^*} = \{p \mid \mu^*(p) = t, t \in \tau(p) \text{ and } t \in \mathbf{T} \setminus \{u\}\}$.

Let $\mathcal{M}^\alpha$ be the set of matchings such that, for any matching $(\mu, \phi) \in \mathcal{M}^\alpha$, the set of patients chosen to fill the q guaranteed slots is α

Theorem 4. *For any ventilator sharing problem ($\mathbf{P}, \mathcal{G}, \mathbf{T}, r, \tau$) with a hard reserve or a soft reserve system of reserve implementation,*

(i) The matching (μ^*, ϕ^*) chosen by ***MMAR*** *complies with eligibility requirements, is non-wasteful and priority compliant.*

(ii) (μ^*, ϕ^*) respects type priorities for slots.

(iii) For any matching (μ, ϕ) which respects type priorities for slots,

$$|\delta^{\mu^*}| \geq |\delta^{\mu}|$$

The number of patients beneficiaries belonging to the types and who are provided a slot is weakly greater under the matching chosen by the algorithm than the number of beneficiaries who are provided a slot under their types for any matching which respects type priorities for slots.

(iv) *(μ^*,ϕ^*) is $\mathscr{M}^\alpha$ maximal. For any matching $(\mu,\phi) \in \mathscr{M}^\alpha$, $|\mu^*| \geq |\mu|$.*

(v) *For any $\mathscr{M}^\alpha$ maximal matching (μ,ϕ), $\mu_p \, \overline{\pi} \, \mu^*_p$*

Proof. (i)Following the steps of **MMAR**, it can be seen that (μ^*,ϕ^*) complies with eligibility requirements and is non-wasteful.

Suppose (μ^*,ϕ^*) is not priority compliant. Then there exists patients p,q and a type t with $p \, \pi_t \, q$ with,

$$\mu^*(q) = t \;\&\; \mu^*(p) = \emptyset$$

and a matching $(\mu,\phi) \in \mathscr{M}$ such that $\mu(r) = \mu^*(r)$, for all $r \in \mathbf{P} \setminus \{p,q\}$ and

$$\mu(p) = t \;\&\; \mu(q) = \emptyset$$

Note that $|\mu| = |\mu^*|$.

Let ϕ^μ be the function which maps every patient to the patient with whom they share their ventilator with under matching (μ,ϕ).

Since (μ^*,ϕ^*) respects type priorities of slots, there doesn't exist a patient q who is allocated a slot under type t, and has lower priority than an unmatched patient p under type t. Similarly under **MMAR** algorithm, no free ventilator is allocated to patient under type u and has lower priority to an unmatched patient under type u.

Suppose patient p was chosen to be matched by step k of **MMAR** algorithm. Let ϕ^k be the function which maps every patient to the patient with whom they share their ventilator by step k of **MMAR** algorithm .

Patient p is matched temporarily to one of the q_f available ventilators. There exists a patient $p_i^l \in \mathscr{L}_p$, such that there exists a **VSFC**, $[p_i^l - \phi(p_i^l)] - [\phi^k(\phi(p_i^l)) - \phi(\phi^k(\phi(p_i^l)))] - ......[\phi^k(....\phi(p_i^l)) - p]$ and patient p is matched by the end of step k. Contradiction.

Suppose patient p wasn't chosen to be matched to one of available ventilators under **MMAR**. Then patient p' also wasn't chosen to be matched under **MMAR** algorithm. Let m be the step at which patient p' was matched under **MMAR** algorithm. There exists a **VSFC**, $[p - \phi(p)] - [\phi^k(\phi(p)) - \phi(\phi^k(\phi(p)))] - ......[\phi^k(....\phi(p)) - \phi(\phi^k(....\phi(p)))]$ such that patient p can be matched over patient p' by step m of **MMAR** algorithm. Contradiction.

$(\Rightarrow)$ (μ^*, ϕ^*) is priority compliant.

Conclusions (ii) and (iii) follow from theorem 2.

(iv) For any matching $(\mu,\phi) \in \mathscr{M}^{\alpha}$, let q_s^{μ} represent the number of patients who are guaranteed a slot under α and who do not share the ventilator machine with any other patient under matching (μ,ϕ). Then for any matching $(\mu,\phi) \in \mathscr{M}^{\alpha}$, $q_s^{\mu^*} \geq q_s^{\mu}$.

Suppose not. Suppose $q_s^{\mu^*} < q_s^{\mu}$. Then there exists a patient p and a **VSFC** reducing $q_s^{\mu^*}$, which is a contradiction considering the steps of **MMAR** algorithm.

Suppose matching (μ^*,ϕ^*) isn't maximal in $\mathscr{M}^{\alpha}$. Then there exists a matching $\mu \in \mathscr{M}^{\alpha}$ such that $|\mu^{-1}| > |\mu^{*^{-1}}|$. Then there exists patients $p,q \in \mu^{-1}$ and $p,q \notin \mu^{*-1}$ such that patients p,q share ventilator with each other.

Let ϕ be the function which maps every patient to the patient with whom they share their ventilator with under matching μ.

Let ϕ^* be the function which maps every patient to the patient with whom they share their ventilator with under matching μ^*

Suppose not.Then there are two possible cases.

(a)Suppose there exists only patient p such that, $p \in \mu^{-1}$ and $p \notin \mu^{*-1}$.

Then there exists a patient $p' \in \mu^{*-1}$ with $\phi^*(p') = \emptyset$ and a **VSFC**, $[p - \phi(p)] - [\phi^*(\phi(p)) - \phi^{\mu}(\phi^*(\phi(p)))] - \ldots - [\phi^*(\ldots(\phi^*(\phi(p)) - p']$ such that patient p was matched under **MMAR** algorithm. Contradiction.

(b)Suppose there exists no patients $p,q \in \mu^{-1}$ and $p,q \notin \mu^{*-1}$ such that patients p,q do not share ventilator with one another. $\phi^*(p) \neq q$. Then there exists a patient $r \in \mu^{*-1}$ with $\phi^*(r) = \emptyset$ and a **VSFC**, $[p - \phi(p)] - [\phi^*(\phi(p)) - \phi(\phi^*(\phi(p)))] - \ldots - [\phi^*(\ldots(\phi^*(\phi(p)) - p']$ such that patient p was matched under **MMAR** algorithm. Contradiction.

Since $q_s^{\mu^*} \geq q_s^{\mu}$, the existence of patients $p,q \in \mu^{-1}$ and $p,q \notin \mu^{*-1}$ such that patients p,q share ventilator with each other implies that there exists a ventilator among the q_f free ventilators under **MMAR** algorithm, which was allocated to a single patient p' instead of patients p,q. Contradiction.

(v) Let $(\phi^{-1})^*|_{\alpha}(\emptyset)$ capture the set of patients who are allocated a ventilator slot respecting type priorities ($\alpha \neq \emptyset$) and who do not share ventilator with any other patient under (μ^*,ϕ^*). Then for any maximal matching $(\mu,\phi) \in \mathcal{M}^{\alpha}$,

$$|(\phi^{-1})^*|_{\alpha}(\emptyset)| \leq |(\phi^{-1})|_{\alpha}(\emptyset)|$$

Suppose not. Then there exists a patient p such that $\delta^*(p) \neq \emptyset$ and $p \in (\phi^{-1})^*|_{\delta^{*-1}(\mathbf{T})}(\emptyset)$ and $p \notin (\phi^{-1})|_{\delta^{*-1}(\mathbf{T})}(\emptyset)$. Let $\phi^{\mu}(p) = \tilde{p}$. Then by **MMAR**, it cannot be that $\mu^*(\tilde{p}) = \emptyset$. If $\mu^*(\tilde{p}) = \emptyset$, then $\tilde{p}$ could have been matched with patient p under the steps of **MMAR** algorithm. Contradiction.

If $\tilde{p} \in (\phi^{-1})^*|_{\delta^{*-1}(\mathbf{T})}(\mathbf{P})$, then we can construct a chain backwards as follows. $[p-\tilde{p}]-[\phi^{\mu^*}(\tilde{p})-\phi^{\mu}(\phi^{\mu^*}(\tilde{p}))]-[\phi^{\mu^*}(\phi^{\mu}(\phi^{\mu^*}(\tilde{p})))-\phi^{\mu}(\phi^{\mu^*}(\phi^{\mu}(\phi^{\mu^*}(\tilde{p}))))]-....$ till $\phi^{\mu}(...\phi^{\mu^*}(\phi^{\mu}(\phi^{\mu^*}(\tilde{p})))) \notin (\phi^{-1})^*|_{\alpha}(\mathbf{P})$. Such a chain exists, since the number of patients are finite. $(\Rightarrow)$ Patient p shares a ventilator under **MMAR**. Contradiction.

Let k be the first step in which a most preferred patient p_k is chosen to be matched temporarily to a freed ventilator. Following the previous result, at the beginning of step k for any maximal matching $(\mu,\phi) \in \mathscr{M}^{\alpha}$ we have

$$|(\phi^{-1})^*|_{\alpha}(\emptyset)| \leq |(\phi^{-1})|_{\alpha}(\emptyset)|$$

If $|(\phi^{-1})^*|_{alpha}(\emptyset)| < |(\phi^{-1})|_{\alpha}(\emptyset)|$, then patient p_k is matched by the end of algorithm since (μ,ϕ) and (μ^*,ϕ^*) are maximal matching in $\mathscr{M}^{\alpha}$.

Suppose $|(\phi^{-1})^*|_{\alpha}(\emptyset)| = |(\phi^{-1})|_{\alpha}(\emptyset)|$. If $\mu(p_k) \neq \emptyset$ and $\phi^{\mu}(p_k) = \emptyset$ then patient p_k can be matched to a freed ventilator under μ^*, since (μ,ϕ) and (μ^*,ϕ^*) are maximal matches in $\mathscr{M}^{\alpha}$. If $\mu(p_k) \neq \emptyset$ and $\phi^{\mu}(p_k) \neq \emptyset$, then given the steps of **MMAR** algorithm we can construct a chain backwards as follows, $[p_k - \phi^{\mu}(p_k)] - [\phi^{\mu^*}(\phi^{\mu}(p_k)) - \phi^{\mu}(\phi^{\mu^*}(\phi^{\mu}(p_k)))]-[\phi^{\mu^*}(\phi^{\mu}(\phi^{\mu^*}(\phi^{\mu}(p_k))))-\phi^{\mu}(\phi^{\mu^*}(\phi^{\mu}(\phi^{\mu^*}(\phi^{\mu}(p_k)))))]-....$ till $\mu^{*^k}\phi^{\mu}(...\phi^{\mu^*}(\phi^{\mu}(\phi^{\mu^*}(\phi^{\mu}(p_k)))$ $\emptyset$ where μ^{*^k} is the matching chosen by step k of **MMAR** algorithm. This implies,

$$|(\phi^{-1})^*|_{\alpha}(\emptyset)| \leq |(\phi^{-1})|_{\alpha\}}(\emptyset)|$$

Suppose the conclusion holds up to step $k+t-1$ of **MMAR** algorithm. By the beginning of step $k+t$ of **MMAR** algorithm we have,

$$|(\phi^{-1})^*|_{\alpha\cup\{p_k,p_{k+1},\ldots,p_{k+t-1}\}}(\emptyset)| \leq |(\phi^{-1})|_{\alpha\cup\{p_k,p_{k+1},\ldots,p_{k+t-1}\}}(\emptyset)|$$

If $|(\phi^{-1})^*|_{\alpha\cup\{p_k,p_{k+1},\ldots,p_{k+t-1}\}}(\emptyset)| \leq |(\phi^{-1})|_{\alpha\cup\{p_k,p_{k+1},\ldots,p_{k+t-1}\}}(\emptyset)|$, then patient p_{k+t} is matched by the end of algorithm since (μ,ϕ) and (μ^*,ϕ^*) are maximal matching in $\mathscr{M}^{\alpha}$.

Suppose $|(\phi^{-1})^*|_{\alpha\cup\{p_k,p_{k+1},\ldots,p_{k+t-1}\}}(\emptyset)| \leq |(\phi^{-1})|_{\alpha\cup\{p_k,p_{k+1},\ldots,p_{k+t-1}\}}(\emptyset)|$. If $\mu(p_{k+t}) \neq \emptyset$ and $\phi^{\mu}(p_{k+t}) = \emptyset$ then patient p_{k+t} can be matched to a freed ventilator under (μ^*,ϕ^*), since (μ,ϕ) and (μ^*,ϕ^*) are maximal matches in $\mathscr{M}^{\delta^*}$. If $\mu(p_{k+t}) \neq \emptyset$ and $\phi^{\mu}(p_{k+t}) \neq \emptyset$, then given the steps of **MMAR** algorithm we can construct a chain backwards as follows, $[p_{k+t}-\phi^{\mu}(p_{k+t})]-[\phi^{\mu^*}(\phi^{\mu}(p_{k+t}))-\phi^{\mu}(\phi^{\mu^*}(\phi^{\mu}(p_{k+t})))]-[\phi^{\mu^*}(\phi^{\mu}(\phi^{\mu^*}(\phi^{\mu}(p_{k+t}))))-\phi^{\mu}(\phi^{\mu^*}(\phi^{\mu}(\phi^{\mu^*}(\phi^{\mu}(p_{k+t})))))]-\ldots.$ till ${\mu^*}^{k+t}\phi^{\mu}(\ldots\phi^{\mu^*}(\phi^{\mu}(\phi^{\mu^*}(\phi^{\mu}(p_{k+t}))))) = \emptyset$ where ${\mu^*}^{k+t}$ is the matching chosen by step $k+t$ of **MMAR** algorithm.

$(\Rightarrow)$ For any maximal matching $(\mu,\phi) \in \mathscr{M}^{\alpha}$, $\mu_p \,\overline{\pi}\, \mu_p^*$. ■

The algorithm allocates ventilator based on priority order and categorical claims. Once allocated, ventilators are freed by making already matched patients share ventilator with each other. Patients who cannot share ventilator with already matched patients, are then checked with other unmatched patients to see if they can share their ventilator with them, or if the unmatched patients can initiate a chain freeing already matched patients to share with them. Patients who aren't allocated a ventilator slot after these initial steps, are allocated one of the freed ventilators if they are able to share the ventilator with another unmatched patient or if there exists another unmatched patient who can initiate a chain, freeing a matched patient to be share a ventilator with the patient who is chosen to be matched albeit temporarily based on the priority order. In this algorithm, there is a bargaining problem between patients for other patients with whom they can share a ventilator with. Patients who are allocated ventilators based on their priority orders have first access to other patients who are allocated a ventilator and patients who are allocated a ventilator and don't share ventilator with other already matched patients are provided first access to unmatched patients for sharing ventilator with them. Similarly, among unmatched patients access to other unmatched patients are provided based on their priority order.

Proposition 6. *Consider the mechanism which implements Modified Matching Algo-*

rithm with Reserves for any given ventilator sharing problem $(\mathbf{P}, \mathscr{G}, \mathbf{T}, r, \tau)$. *The mechanism is strategyproof.*

Proof. For any patient $p_i \in \mathbf{P}$ and $\mathscr{G}$, let $\mathscr{G}_{p_i}$ be the set of patients who are compatible with patient p_i. Note that, for any pair of patients $p_i, p_j \in \mathbf{P}$, $(p_i, p_j) \in \mathscr{G}$ iff $p_j \in \mathscr{G}_{p_i}$ and $p_i \in \mathscr{G}_{p_j}$.

Let $\hat{\mathscr{G}}_{p_i}$ be the actual set of patients who are compatible for patient p_i. Let k be the step at which patient p_i is chosen to be matched in the algorithm. Take any $\mathscr{G}'_{-p_i}$. We will consider three cases.

Case i: $k < q$ or patient p_i is allocated a ventilator under soft reserves system of reserve implementation. Then irrespective of the set of compatible patients chosen by patient p_i, patient p_i is matched to a ventilator slot. Hence patient p_i is matched to a ventilator slot if patient p_i reveals $\hat{\mathscr{G}}_{p_i}$ as the set of compatible patients.

Case ii: Step t, $t > q$ and patient p_i is chosen for the second time by step t for allocating the freed ventilator. Then irrespective of the set of compatible patients chosen by patient p_i, patient p_i is matched to a ventilator slot. Hence patient p_i is matched to a ventilator slot if patient p_i reveals $\hat{\mathscr{G}}_{p_i}$ as the set of compatible patients.

Case iii Patient p_i is chosen to be matched temporarily by some step l. Then for any $\mathscr{G}'_{p_i} \subseteq \hat{\mathscr{G}}_{p_i}$, if patient p_i is allocated a ventilator slot by end of step l, then patient pi is allocated a ventilator slot if patient p_i reveal $\hat{\mathscr{G}}_{p_i}$ as the set of compatible patients. ■

MMAR is incentive compatible, incentivizing people to reveal their true set of compatible patients. Unlike **MAR**, **MMAR** maximizes the number of patients who are matched to ventilator slots conditional on the initial distribution.

1.6 Equity and Maximal Matching

Definition 8. For the problem of allocating q ventilators over n patients, the random mechanism allocating ventilators to patients satisfies equity if for any patient p, $u_p \geq \min\{\frac{q}{n}, 1\}$

Theorem 5. *Consider the ventilator sharing problem* $(\mathbf{P}, \mathscr{G}, \mathbf{T} = \{u\})$. *For every ventilator sharing problem with a single type, a random mechanism satisfies equity and maximizes the expected number of patients matched iff it is a random priority mechanism implementing* ***MMAR****.*

Proof. (⇒)Let $\mathscr{U} = (u_{p_1}, u_{p_2}, u_{p_3}, ..., u_{p_n})$ be the vector of utilities of patients obtained under any random mechanism which satisfies equity and maximizes the number of patients matched for every ventilator sharing problem $(\mathbf{P}, \mathscr{G}, \mathbf{T} = \{u\})$. For any deterministic mechanism in the support of the random mechanism, let λ represent the probability with which the mechanism is chosen by the random mechanism and ρ represent the utility vector chosen by the mechanism. There are finite deterministic mechanisms in the support of random mechanism and any utility vector chosen by the random mechanism can be broken down as,

$$(u_{p_1}, u_{p_2}, ..., u_{p_i}, ..., u_{p_n}) = \lambda_1 \rho_1 + \lambda_2 \rho_2 + ... + \lambda_i \rho_i + ... + \lambda_t \rho_t$$

$$\sum_i \lambda_i = 1 \text{ and } \rho_{ij} \in \{0, 1\}$$

For each ρ there exists at least one priority order such that the outcome vector can be obtained. For example for each ρ, patients who are allocated a ventilator slot can be placed in priority above patients who didn't obtain ventilator slots. Since the random mechanism satisfies equity for every problem, the random mechanism should have uniform support and the probability of selecting a mechanism where a patient p is one of the top q prioritised patients is, $\frac{q}{|\mathbf{P}|}$.

From theorem 4, for any given α, algorithm **MMAR** is $\mathscr{M}^{\alpha}$ maximal. Therefore for any random priority mechanism implementing **MMAR**

$$\sum_{p_i} u_{p_i} = \sum_{p_i} u_{p_i}^{\text{MMAR}}$$

where $\mathscr{U}^{\text{MMAR}} = (u_{p_1}^{\text{MMAR}}, u_{p_2}^{\text{MMAR}}, u_{p_3}^{\text{MMAR}}, \ldots, u_{p_n}^{\text{MMAR}})$ is the utility vector chosen by random priority mechanism implementing **MMAR**.

($\Leftarrow$) Random priority mechanism implementing **MMAR** satisfies equity and maximizes the expected number of patients matched. The conclusion follows from **MMAR** algortihm and Theorem 4. ■

Theorem 5 shows that ventilator sharing problem with single category instead of multiple categories and allocating ventilator machines/slots using a random priority mechanism implementing **MMAR** algorithm produces an equitable allocation maximizing the expected number of patients matched. The triage protocols for allocation of ventilators use a priority ranking based on a single dimensional score and reserves are used as a compromise in response to categorical claims. Consider the scoring system, the variants of which are used in a multi-principle priority point system.

Illustration of a Multi-Principle Strategy to Allocate Ventilators during a Public Health Emergency

Principle	**Specification**	**Point System***			
		1	**2**	**3**	**4**
Save the most lives	Prognosis for short term survival (SOFA** score)	SOFA score <6	SOFA score 6-9	SOFA score 10-12	SOFA score >12
Save the most years of life	Prognosis for long term survival (Medical assessment of comorbidities)	No comorbidities which limit long term survival	Minor comorbidities with small impact on long term survival	Major comorbidities with substantial impact on long term survival	Severe comorbidities; likely death within 1 year
Life cycle principle**	Prioritize those who have had the least chance to live through life's stages. (Age in years)	age 12-40	ages 41-60	ages 61-74	age >75

*Those with the lowest cumulative score would receive the highest priority to receive mechanical ventilation and critical care services.

**SOFA= Sequential Organ Failure Assessment

***Pediatric populations may need to be considered separately, since their small size may require the use of different mechanical ventilators and personnel.

Figure 1.2: (White Douglas and Lo., 2020)

The multi-principle triage protocol prioritises patients based on a single score which is a combination of patient's short term and long term prognosis. Often, categorical claims for ventilators can arise due to discrimination in access to ventilators. Patients belonging to categories with claims can be disproportionately under represented in such a scoring system if there is lack of early access to health care and health services which can improve short term prognosis, and lack of access to health care and health services in general which can affect their long term prognosis. While reserves can provide a compromise in providing access to categories, we can also find correlations of categories with priority scores while controlling for other factors and add it to their priority scores to include the categorical claims and use a single category to allocate ventilator machines/slots. This can reduce the categories and from the above theorem it can make the allocation more egalitarian, maximizing the expected number of patients matched.

1.7 Dynamics of Ventilator Allocation

Ventilator allocation and sharing have been discussed as a static problem even though the problem is actually dynamic. Patients arrive and leave over time and the needs of patients changes over time. (Piscitello et al., 2020) notes that out of 26 states which have published guidelines for triage, 22 of them discuss ventilator reallocation. All 22 guidelines are in support of withdrawal of mechanical ventilation, if demanded by triage protocol. Most of the guidelines mention setting up of triage committees which will determine eligibility of patients for ventilator allocation. In such a scenario, dynamic ventilator allocation problem can be considered as static problem at every time period when the triage committee makes a decision. One inconvenience of this process is that patients might have to be moved around whenever the triage committee reevaluates patients eligibility and priority. (R. et al.) suggests maintaining of reserve ventilators which can be used to prepare patients for ventilator sharing and for weaning off patients of ventilators. Maintenance of these reserve stock of ventilators can be helpful in moving patients around, though this can be cumbersome.

1.8 Conclusion

This paper is an attempt at studying sharing economy, where object can be shared by two persons and the allocation of object to a person depends on their priority ranking. The model studied assumes homogeneity of objects wherein participants are indifferent between the objects which are being provided. Under homogeneity, it is shown that pareto-efficient matching respecting different notions of priority exists. One immediate extension of the problem is to analyze the same problem with heterogeneous objects. Heterogeneous objects are more relevant in public housing allocation with sharing since people can have varied preferences over different housing units. We hope that the methods discussed and definitions proposed would serve as a good definition in taking the discussion forward.

Chapter 2

Pairwise Kidney Exchange - Critique and Extension

2.1 Introduction

As of May 2018, there are 95,139 candidates in waiting list for a compatible kidney. From 2006-2017, on average 12.6 candidates are removed daily from the waiting list due to death while 7.6 candidates are removed daily from the list for being too sick to transplant [1]. In 1984, the National Organ Transplant Act was passed, which outlawed the sale of human organs and also established the formation of Organ Procurement and Transplantation Network(OPTN). OPTN maintains candidates waiting list and matches cadaver, live donor kidneys to candidates based on a Kidney Allocation System(KAS)[2] based on various criteria.

Alongside the establishment of OPTN, there were proposals to setup a national databse for incompatible donor recipient pairs (Rapaport Felix, 1986; Ross et al., 1997). In 2000, the transplantation community issued a consensus statement indicating that exchange between incompatible donor recipient pairs is "ethically acceptable". (Abecassis et al., 2000). Since then economists have designed theoretical exchange markets satisfying various desirable normative properties. Roth Alvin et al. (2004) show how to organize multiple-pair exchanges efficiently, while providing agents with the right incentives.Using tissue type statistics of Caucasian population, they illustrate how live organ donations can be increased from 54% up to 91% if multiple-pair exchanges are feasible and up to 75% if only paired-exchanges are feasible.

[1] https://optn.transplant.hrsa.gov/data
[2] https://optn.transplant.hrsa.gov/media/1235/kas_faqs.pdf

Taking into account logistics and incentive constraints (exchanges have to be carried out simultaneously since a donor can withdraw consent after her intended recipient has received a kidney), Roth Alvin et al. (2005) proposed a pairwise kidney exchange market. They constrcuted two mechanisms for allocation of kidneys, priority mechanism - a deterministic mechanism when there is a well defined ordinal priority ranking over agents and an egalitarian mechanism - a stochastic mechanism when all the participating agents have an equal claim for available compatible kidney. They show how the above mentioned mechanisms can be used to organize efficient exchanges while providing strategy-proof incentives to agents.

Since participants can have different waiting times ,variable access to compatible kidneys due to biological reasons and other disparities, to construct a true egalitarian mechanism it makes sense to offer weights to participating agents. Weights added as an parameter to the participating agents in our paper will have the following Aristotelian interpretation(Row, 2002). Under Pareto efficiency, agents will have claim for a compatible kidney, where agent's strength of claim is proportional to the weight awarded.

The aim of this paper is threefold. First, we will introduce weights as an additional parameter to the participating agents in a pairwise kidney matching problem and using above interpretation of weights we will generalize the egalitarian mechanism of Roth Alvin et al. (2005) to a weighted egalitarian mechanism. Second, we will construct the dual of the mechanism design problem which is the constrained allocation problem and using theories of distributive justice will show how it is equivalent to very well known solution concepts. The reason for constructing the dual of the problem is that if agents have dominant strategy incentives, then it is easier to study fairness of the allocation mechanism through the normative characterization of the dual of the problem. Finally, we will place the egalitarian , weighted egalitarian and priority mechanism in a continuum which will help us better understand the changing incentives for agents.

The paper will be organized as follows. Section[2] will contain Literature review followed by generalization of egalitarian mechanism to weighted egalitarian mechanism in Section[3]. In Section[4], we will construct the dual of the mechanism discussing various normative properties of the equivalent allocation mechanisms. In Section[5] we will place the egalitarian, weighted egalitarian and priority mechanism in a continuum and discuss the changing incentives to agents.This we believe, will provide us with a deeper perspective for policy analysis.

2.2 Literature Review

This paper intersects with the literature on transplantation,graph theory, mechanism design and distributive justice. This paper borrows the basic problem structure from Roth Alvin et al. (2005). To be precise, like the original paper

1. Only pairwise exchange between incompatible donor recipient pairs is feasible
2. Each patient is indifferent between compatible kidneys (Gjertson David and Cecka., 2000; Delmonico Francis, 2004). Technically, each agent has 0-1 cardinal preferences between compatible and incompatible kidneys.

In addition to the above mentioned constraints, the problem includes an additional constraint.

1. Each patient donor pair includes weight as an additional parameter. Here weight captures the strength of claim of patient donor pair.

Incentive compatibility is restricted to possible Pareto efficient matches which is later shown to be equivalent to matches of maximum cardinality. Under such circumstances, the first part of the problem reduces to cardinality matching problem(see for example, Korte Bernhard et al. (2011)). Each donor patient pair can be considered as a vertex of an undirected graph and each edge represents a feasible match between donor recipient pairs.Finding a maximum cardinality match between incompatible donor recipient pair, reduces to finding a maximum cardinality match in the undirected graph, which is well analyzed in literature. We will use the results of (Tibor, 1963, 1964; Edmonds, 1965) Decomposition Lemma (GED) in finding maximum cardinality matches. Our analysis will rely heavily on the results offered by the GED lemma.

We will also use existence results on feasible efficient matches offered by Anna and Moulin. (2004, 2002); Hervé and Moulin. (2001) to construct lotteries over efficient matches. The solution concept offered by the egalitarian mechanism of Roth Alvin et al. (2005) can be considered in some aspects derivative of Bhaskar and Ray. (1989). Our weighted egalitarian solution can also be considered as a Weighted Dutta Ray solution, and we introduce the idea of Weighted Lorenz Dominance.

In providing a normative critique of the allocation mechanism we will draw from the extensive works on distributive justice. We will be relying on characterizations of solutions to the bargaining problem for insights. Specifically, we will be using the work of Border and Segal(2000) in providing a normative critique of the mechanisms.

2.3 Generalized Pairwise Kidney Exchange [3]

Let **N** = {1, 2, ...,n} be the set of incompatible patient donor pairs (patient from now). Each patient is indifferent between compatible donors and each patient strictly prefers a compatible donor to an incompatible donor. Each patient strictly prefers her own incompatible donor to other incompatible donors. The last condition ensures that in any prospective match satisfying individual rationality, patient prefers to remain unmatched rather than be matched with an incompatible donor. The above assumptions on individual preferences can be summarized as follows.

1. For any patient i, and any compatible donor j we have j $\succ_i$ i

[3] For continuity, We will be borrowing notations and use preliminary analysis from Roth Alvin et al. (2005)

2. For any patient i, and any incompatible donor j we have i $\succ_i$ j

3. For any patient i, and any compatible donors j and h we have j $\sim_i$ i

4. For any patient i, and any incompatible donors j and h we have j $\sim_i$ h

$\succeq_i$ captures the preference relation of i over the set N, and $\succeq = (\succeq_\mathrm{i})_{\mathrm{i} \in \mathrm{N}}$ is a list containing the preference of all patients in set N.

A matching is a function μ: $\mathbf{N} \rightarrow \mathbf{N}$ such that, $\mu(\mathbf{i})$ = j if and only if $\mu(\mathbf{j})$ = i. A matching is individually rational if $\mu(\mathbf{i}) \neq$ i implies $\mu(\mathbf{i}) \succ_i$ i, $\forall$ i. Throughout this paper we will focus solely on matchings which are individually rational. Note that, given our preference structure for agents restricting attention to individually rational matchings ensures that we focus on matches where exchange occurs between mutually compatible pairs only.

Let $\mathbf{w}' = (\mathbf{w}'_\mathbf{i})_{i \in N}$ be a list containing the weights of individual agents, where $\mathbf{w}'_\mathbf{i}$ captures the weight of patient i . Since only relative weights are of significance, we will normalize $\mathbf{w}'$ so that the least weight patient has a weight 1. Let $\mathbf{w} = (\mathbf{w}_\mathbf{i})_{i \in N}$ be the normalized list of weights, where $\mathbf{w}_\mathbf{i}$ captures the normalized weight of patient i obtained by setting the weight of least weight patient to 1. The generalized pairwise kidney exchange problem is given by the triplet $(\mathbf{N}, \succeq, \mathbf{w})$.

Since for a given $(\mathbf{N}, \succeq, \mathbf{w})$ we will be focusing on matchings with exchanges between mutually compatible pairs, it is sufficient to use a symmetric $|\mathbf{N}|$ by $|\mathbf{N}|$ mutual compatibility matrix $\mathbf{R}$ defined by

$$[\mathbf{r_{i,j}}]_{\mathbf{i}\in\mathbf{N},\mathbf{j}\in\mathbf{N}} = \begin{cases} 1, & \mathbf{j} \succ_{\mathbf{i}} \mathbf{i}\, \mathbf{and}\, \mathbf{i} \succ_{\mathbf{j}} \mathbf{j} \\ 0, & \mathbf{otherwise} \end{cases} \tag{2.1}$$

The generalized kidney problem can be rewritten as a triplet $(\mathbf{N}, \mathbf{R}, \mathbf{w})$. A generalized kidney problem can be considered as an undirected graph $((\mathbf{N}, \mathbf{w}), \mathbf{R})$ whose vertices are the $\mathbf{N}$ patients and whose edges represent a connection between mutually compatible pairs (i.e) there exists an edge between patient $(\mathbf{i}, \mathbf{w_i})$ and $(\mathbf{j}, \mathbf{w_j})$ iff $\mathbf{r_{i,j}} = 1$. A matching can be thought of as a subset of edges in which each patient appears at most once. If $(\mathbf{i}, \mathbf{w_i})$ and $(\mathbf{j}, \mathbf{w_j})$ is an edge in the matching μ, then $(\mathbf{i}, \mathbf{w_i})$ and $(\mathbf{j}, \mathbf{w_j})$ are matched in the matching μ. A **mechanism** is a function/procedure which chooses a matching or a lottery over matchings for a given problem.

Let $\mathcal{M}$ denote the set of individually rational matchings for a given problem $(\mathbf{N}, \succeq, \mathbf{w})$.

Since patient's preferences are formed independently of weights the set of individual matchings should be the same given, for a given set of patients and a list of preferences. This can be summarized by the following lemma.

Lemma 3. *For a given set of patients* $\mathbf{N}$ *and a list of preferences* $\succeq$ *and for any list of weights* $\mathbf{w}$, $\mathbf{w}'$, *the set of individually rational matchings for* $(\mathbf{N}, \succeq, \mathbf{w})$ *and* $(\mathbf{N}, \succeq, \mathbf{w}')$ *are equal. Equivalently, the set of individually rational matchings for* $(\mathbf{N}, \succeq, \mathbf{w})$ *and* $(\mathbf{N}, \succeq, \mathbf{w}')$ *are equal, where* $\mathbf{R}$ *is the symmetric mutual compatibility matrix for a given list of preference relation* $\succeq$

Efficiency: A matching $\mu \in \mathcal{M}$ is efficient, if there exists no matching $\eta \in \mathcal{M}$ such that $\eta(\mathbf{i}) \succeq_i \mu(\mathbf{i})$ for all i $\in \mathbf{N}$ and $\eta(\mathbf{i}) \succ_i \mu(\mathbf{i})$ for some i $\in \mathbf{N}$. The following lemma will be very helpful in the characterization of efficient matchings

Lemma 4. Roth Alvin et al. (2005)*:Let* $\mathcal{I}$ *be the set of simultaneously matchable patients, i.e.* $\mathcal{I} = \{I \subseteq N : \exists \mu \in \mathcal{M}$ *such that patient in* I *are matched with another patient in* $\mu\}$. *Then* $(\mathbf{N}, \mathcal{I})$ *is a matroid.*

Lemma 5. *(*Roth Alvin et al. (2005)*): For any pair of Pareto efficient matchings* μ, η

$,|\mu| = |\eta|$

The above lemma gives an equivalence between Pareto efficient and maximum cardinal matches. Since Pareto efficiency is a desirable property for the mechanism, we will restrict our attention to matchings of maximum cardinality . Finding maximum cardinal matchings are well analyzed in combinatorial optimization. Let $\mathcal{E}$ represent the set of matchings of maximum cardinality.It follows that $\mathcal{E} \subseteq \mathcal{M}$. We will use the Gallai- Edmonds Decomposition(GED) lemma in characterizing maximum cardinal matchings.

2.3.1 Gallai-Edmonds Decomposition

Let $\{\mathbf{N^U}, \mathbf{N^O}, \mathbf{N^P}\}$ be the partition of set of patients $\mathbf{N}$ where,

$$\mathbf{N^U} = \{i \in \mathbf{N}, \exists \mu \in \mathcal{E} \text{ s.t. } \mu(i) = \text{i}\}$$

$$\mathbf{N^O} = \{i \in \mathbf{N} \setminus \mathbf{N^U}, \exists j \in \mathbf{N^U} \text{ s.t. } r_{i,j} = 1\}$$

$\mathbf{N^P} = \mathbf{N} \setminus (\mathbf{N^U} \cup \mathbf{N^O})$

$\mathbf{N^U}$ represents patients in $\mathbf{N}$ who are not matched with another patient in at least one matching in $\mathcal{E}$. $\mathbf{N^O}$ represents patients who share at least one edge with a patient in $\mathbf{N^U}$. $\mathbf{N^P}$ represents patients who are always matched with another patient in every matching belonging to $\mathcal{E}$ and who do not share an edge with a patient in $\mathbf{N^U}$.

Let $\mathbf{I} \subseteq \mathbf{N}$. Let $\mathbf{R_I} = [r_{i,j}]_{i\in\mathbf{I},j\in\mathbf{I}}$ be the reduced symmetric mutually compatible matrix for patients in $\mathbf{I}$. Let $\mathbf{w_I}$ be the list of weights for patients in $\mathbf{I}$. Then $(\mathbf{I}, \mathbf{R_I}, \mathbf{w_I})$ is the reduced sub-problem of the original problem $(\mathbf{N}, \mathbf{R}, \mathbf{w})$ restricted to $\mathbf{I}$. A reduced sub-problem $(\mathbf{I}, \mathbf{R_I}, \mathbf{w_I})$ is **connected** if there exist a sequence of patients $i_1 i_2 i_3 ... i_m$, possibly with repetition such that $r_{i_k, i_{k+1}} = 1$, for all k $\in \{1, 2, ..., m-1\}$ where $\mathbf{I} = \{i_1, i_2, i_3, ..., i_m\}$. A connected reduced sub-problem $(\mathbf{I}, \mathbf{R_I}, \mathbf{w_I})$ is a component of $(\mathbf{N}, \mathbf{R}, \mathbf{w})$ if $r_{i,j} = 0$, for $i \in \mathbf{I}, j \in \mathbf{N} \setminus \mathbf{I}$. $(\mathbf{I}, \mathbf{R_I}, \mathbf{w_I})$ is an odd component if $|\mathbf{I}|$ is odd and an even component if $|\mathbf{I}|$ is even.

Lemma 6. *(Tibor, 1963, 1964; Edmonds, 1965): Let* $(\mathbf{I}, \mathbf{R_I}, \mathbf{w_I})$ *be a reduced sub-problem with* $\mathbf{I} = \mathbf{N} \setminus \mathbf{N^O}$*. Let* μ *be a matching of maximum caridinality for the original* *problem* $(\mathbf{N},\mathbf{R},\mathbf{w})$*. Then*

1. *For any patient* $i \in \mathbf{N^O}$*,* $\mu(i) \in \mathbf{N^U}$
2. *For any even component* $(\mathbf{J},\mathbf{R_J},\mathbf{w_J})$ *of* $(\mathbf{I},\mathbf{R_I},\mathbf{w_I})$*,* $\mathbf{J} \subseteq \mathbf{N^P}$ *and for any patient* $i \in \mathbf{J}, \mu(i) \in \mathbf{J} \setminus \{i\}$
3. *For any odd component* $(\mathbf{J},\mathbf{R_J},\mathbf{w_J})$ *of* $(\mathbf{I},\mathbf{R_I},\mathbf{w_I})$*,* $\mathbf{J} \subseteq \mathbf{N^U}$ *and for any patient* $i \in \mathbf{J}$*, it is possible to match all remaining patients in* $\mathbf{J}$ *with each other (so that any patient* $j \in \mathbf{J}\setminus\{i\}$ *can be matched with another patient in* $\mathbf{J}\setminus\{i,j\}$*). Moreover for any odd component* $(\mathbf{J},\mathbf{R_J},\mathbf{w_J})$*, either,*
   a) *One and only one patient* $i \in \mathbf{J}$ *is matched with a patient in* $\mathbf{N^O}$ *and all the remaining patients in* $\mathbf{J}\setminus\{i\}$ *are matched with each other so that* $\mu(i) \in \mathbf{J}\setminus\{i,j\}$*, or*

   b) *One patient* $i \in \mathbf{J}$ *remains unmatched, while all the remaining patients in* $\mathbf{J}\setminus\{i\}$ *are matched with each other so that* $\mu(i) \in \mathbf{J}\setminus\{i,j\}$*.*

Based on the GED lemma, $\mathbf{N^U}$ can be described as the set of *underdemanded* patients, $\mathbf{N^O}$ can be described as the set of *overdemanded* patients, $\mathbf{N^P}$ can be described as the set of perfectly matched patients. Let$\mathcal{D} = \{\mathbf{D_1},\mathbf{D_2},\mathbf{D_3},,...,\mathbf{D_k},\}$ be the partition of set of *underdemanded* patients $\mathbf{N^U}$, such that $(\mathbf{D_k},\mathbf{R_{D_k}},\mathbf{w_{Dk}}$ is an odd component of $(\mathbf{N} \setminus \mathbf{N^O}, \mathbf{R_{N \setminus N^O}}, \mathbf{w_{N \setminus N^O}})$. Following (Roth, Sönmez and Ünver(2004)[5] we will also slightly abuse the notation, and call each $(\mathbf{D_k}, \mathbf{R_{D_k}}, \mathbf{w_{Dk}}$ as an odd component.

2.3.2 Induced two-sided matching market

Every patient in $\mathbf{N^P}$ are matched in every matching belonging to $\mathcal{E}$. Following GED lemma, in every matching of $\mathcal{E}$, patients in $\mathbf{N^O}$ are matched to one of the odd components in $\mathcal{D}$.Hence we,we will restrict our attention to the induced two sided matching market, comprising of the *overdemanded* patients on one side and odd components of *underdemanded* patients on other side.

For every odd component $\mathbf{J} \in \mathcal{D}$ and for every *overdemanded* patient i $\in \mathbf{N^O}$ let

$$[\tilde{\mathbf{r}}_{\mathbf{i},\mathbf{J}}]_{\mathbf{i} \in \mathbf{N^O}, \mathbf{J} \in \mathcal{D}} = \begin{cases} 1, & \exists j \in \mathbf{J}\, s.t.\, r_{i,j} = 1 \\ 0, & \textbf{otherwise} \end{cases} \tag{2.2}$$

and let $\tilde{\mathbf{R}} = [\tilde{\mathbf{r}}_{\mathbf{i},\mathbf{J}}]_{\mathbf{i} \in \mathbf{N^O}, \mathbf{J} \in \mathcal{D}}$. For a given problem with its equivalent symmetric mutual compatibility matrix $(\mathbf{N}, \mathbf{R}, \mathbf{w})$, the induced two-sided matching market is given by the quartet $(\mathbf{N^O}, \mathcal{D}, \tilde{\mathbf{R}}, \mathbf{w_{N \setminus N^P}})$. The pre-matching $\tilde{mu}$ for the induced two-sided matching is a function $\tilde{\mu} : \mathbf{N^O} \cup \mathcal{D} \rightarrow \mathbf{N^O} \cup \mathcal{D} \cup \{\phi\}$ given by,

1. $\tilde{\mu}(i) \in \mathcal{D} \cup \{\phi\}$, for all $i \in \mathbf{N^O}$

2. $\tilde{\mu}(\mathbf{J}) \in \mathbf{N^O} \cup \{\phi\}$, for all $\mathbf{J} \in \mathcal{D}$

3. $\tilde{\mu}(i) = \mathbf{J} \Leftrightarrow \tilde{\mu}(\mathbf{J}) = i$, for any $i \in \mathbf{N^O}$ and $\mathbf{J} \in \mathcal{D}$

4. $\tilde{\mu}(i) = \mathbf{J} \Rightarrow \tilde{r}_{i,\mathbf{J}} = 1$, for any $i \in \mathbf{N^O}$ and $\mathbf{J} \in \mathcal{D}$

Let $\tilde{\mathcal{M}}$ deno..te the set of individually rational pre-matchings in the induced two sided matching market. Let $\tilde{\mathcal{E}}$ denote the set of pre-matchings of maximum cardinality. From the above lemmas the pre-matchings of maximum cardinality are efficient. Also the set of efficient pre-matchings is non empty by the GED lemma.

2.3.3 Stochastic Exchange

Based on GED lemma, we can see that an *underdemanded* patient competes for an indivisible good(kidney) with other *underdemanded* patients in the odd component, and odd components of *underdemanded* patients compete with each other for *overdemanded* patients. Since in maximum cardinal matches not all patients are matched and kidneys are indivisible, we will use a stochastic mechanism. A stochastic mechanism will choose a lottery $\lambda = \{\lambda_\mu\}_{\mu \in \mathcal{M}}$, a probability distribution over the set of matchings $\mathcal{M}$. Let $\mathcal{L}$ be the set of lotteries associated with $(\mathbf{N},\mathbf{R},\mathbf{w})$. The definition can be extended to the induced two sided matching market. Let the set of *pre-lotteries* $\tilde{\mathcal{L}}$ be the set of lotteries associated with $(\mathbf{N^O},\mathcal{D},\tilde{\mathbf{R}},\mathbf{w}_{\mathbf{N}\setminus\mathbf{N^P}})$. A *pre-lottery* $\tilde{\lambda}=\{\tilde{\lambda}_\mu\}_{\mu\in\tilde{\mathcal{M}}}$, a probability distribution over matchings in $\tilde{\mathcal{M}}$.

Allocation Matrix

Given a lottery $\lambda\in\mathcal{L}$ for a given problem $(\mathbf{N},\mathbf{R},\mathbf{w})$, an allocation matrix is a symmetric matrix given by, $\mathbf{A}(\lambda)=[a_{i,j}]_{i\in\mathbf{N},j\in\mathbf{N}}$ where $\{a_{i,j}\}_{i\in\mathbf{N},j\in\mathbf{N}}$ gives the probability with which patient i is matched with patient j under the lottery λ. An allocation matrix is **feasible** if,

1. For all patient i, $\sum_{j\in\mathbf{N}\setminus\{i\}}a_{i,j}\le 1$ and $\sum_{j\in\mathbf{N}}a_{i,j}=1$.

2. For all patient i, $a_{i,j}>0\Rightarrow r_{i,j}=1$.

Let $\mathcal{A}=\{\mathbf{A}(\lambda)\}_{\lambda\in\mathcal{L}}$ be the set of allocation matrices. Similarly, for an induced two-sided matching market we can extend the definition of symmetric allocation matrix. Given a *pre-lottery* $\tilde{\lambda}\in\tilde{\mathcal{L}}$ for the associated two-sided induced matching market $(\mathbf{N^O},\mathcal{D},\tilde{\mathbf{R}},\mathbf{w}_{\mathbf{N}\setminus\mathbf{N^P}})$, the *pre-allocation* matrix $\tilde{\mathbf{A}}(\lambda)=[a_{i,j}]_{i\in\mathbf{N^O},j\in\mathcal{D}}$ where $\{a_{i,j}\}_{i\in\mathbf{N^O},j\in\mathcal{D}}$ gives the probability with which an *overdemanded* patient i is matched with an odd component $\mathbf{J}$. The pre-allocation matrix is feasible if,

1. For all *overdemanded* patient i, $\sum_{\mathbf{J} \in \mathcal{D}} a_{i,\mathbf{J}} \leq 1$.

2. For all odd components $\mathbf{J} \in \mathcal{D}$, $\sum_{i \in \mathbf{N^O}} a_{i,\mathbf{J}} \leq 1$.

3. For any *overdemanded* patient i and odd component $\mathbf{J}$, $a_{i,\mathbf{J}} > 0 \Rightarrow \tilde{r}_{i,\mathbf{J}} = 1$.

Induced utility profile

Given $\lambda \in \mathcal{L}$, let $\mathbf{A}(\lambda)$ be the associated allocation matrix. Given λ the induced utility profile is a non-negative vector $u(\lambda) = (u(\lambda)_i)_{i \in \mathbf{N}}$ where, $u(\lambda_i) = \sum_{j \in \mathbf{N} \setminus \{\mathbf{i}\}} a_{i,j}$, $a_{i,j} \in \mathbf{A}(\lambda)$ Utility feasible profile $\mathcal{U} = (u^{\lambda})_{\lambda \in \mathcal{L}}$ is the set of all possible induced utility profile

Ex-Ante Efficiency

An allocation matrix $\mathbf{A} \in \mathcal{A}$ is ex-ante efficient if there exists no allocation matrix $\mathbf{B} \in \mathcal{A}$ such that $\sum_{j \in \mathbf{N}} b_{i,j} \geq \sum_{j \in \mathbf{N}} a_{i,j}$ $i \in \mathbf{N}$, and $\sum_{j \in \mathbf{N}} b_{i,j} > \sum_{j \in \mathbf{N}} a_{i,j}$ for some $i \in \mathbf{N}$.

A *pre-allocation* $\tilde{\mathbf{A}} \in \tilde{\mathcal{A}}$ matrix is ex-ante efficient if there exists no *pre-allocation* $\tilde{\mathbf{B}} \in \tilde{\mathcal{A}}$ matrix such that $\sum_{\mathbf{J} \in \mathcal{D}} b_{i,\mathbf{J}} \geq \sum_{\mathbf{J} \in \mathcal{D}} a_{i,\mathbf{J}}$ $i \in \mathbf{N^O}$ and $\sum_{i \in \mathbf{N^O}} b_{i,\mathbf{J}} \geq \sum_{i \in \mathbf{N^O}} a_{i,\mathbf{J}}$ $\mathbf{J} \in \mathcal{D}$ for all $i \in \mathbf{N^O}, \mathbf{J} \in \mathcal{D}$, and strictly greater for some $\mathbf{J} \in \mathcal{D}$ for all $i \in \mathbf{N^O}, \mathbf{J} \in \mathcal{D}$

Ex-Post Efficiency

A lottery λ over matchings $\mathcal{M}$ is ex-post efficient if only efficient matches are in the support of λ. Since the set of efficient matchings is equivalent to set of maximum cardinal matchings in our setting, λ is ex-post efficient if only matches from $\mathcal{E}$ are in the support of λ.

A *pre-allocation* lottery $\tilde{\lambda}$ over matchings $\tilde{\mathcal{M}}$ is ex-post efficient if only efficient matches of the two-sided induced matching market are in the support of $\tilde{\lambda}$. Since the set of efficient matchings is equivalent to set of maximum cardinal matchings of two-sided induced matching market, $\tilde{\lambda}$ is ex-post efficient if only matches from $\tilde{\mathcal{E}}$ are in the support of $\tilde{\lambda}$.The following lemma will be very useful in constructing ex-post efficient lotteries.

Lemma 7. *(Anna and Moulin. (2004)) An allocation matrix $\mathbf{A} \in \mathcal{A}$ is ex-ante efficient iff there exists an ex-post efficient lottery $\lambda \in \mathcal{L}$ such that $\mathbf{A}(\lambda) = \mathbf{A}$*

2.4 Weighted Egalitaraian Mechanism

2.4.1 Preliminaries

A weighted egalitarian mechanism for a given $(\mathbf{N}, \mathbf{R}, \mathbf{w})$, is a stochastic mechanism which chooses a probability distribution over efficient matches, satisfying incentives with respect to weight. By GED lemma, patients in $\mathbf{N^P}$ are matched in all efficient matchings. Therefore we will focus on the induced two sided matching market $(\mathbf{N^O}, \mathcal{D}, \tilde{\mathbf{R}}, \mathbf{w_{N \setminus N^P}})$. WLOG, we will rearrange any coalition $\mathbf{S}$ of patients so that,

$$w_1^{\mathbf{S}} \geq w_2^{\mathbf{S}} \geq w_3^{\mathbf{S}} \geq \ldots \geq w_{|\mathbf{S}|}^{\mathbf{S}}$$

the first index is provided to the patient with maximum weight and the last index is provided to the patient with minimum weight. Note that, $w_{|\mathbf{S}|}^{S} \geq 1$

2.4.2 Allocation rule

By GED lemma, patients in $\mathbf{N^P}$ and $\mathbf{N^O}$ are perfectly matched in any Pareto efficient matching. Hence we will restrict our attention to *underdemanded* patients. Consider any coalition $\mathcal{J} \subseteq \mathcal{D}$ of odd components of *underdemanded* patients. For some $I \subseteq \mathbf{N^O}$, and $q = |\cup_{J\in\mathcal{J}} J|$,, the allocation rule is defined as follows.

Step 1:

$$f^w(\mathcal{J},I) = \frac{|\bigcup_{J\in\mathcal{J}} J| - (|\mathscr{J}| - |C(\mathscr{J},I)|)}{\sum\limits_{i\in\bigcup_{J\in\mathcal{J}} J} w_i}$$

If $w_1 f^w(\mathcal{J},I) < 1$ the procedure terminates. Otherwise, proceed to Step 2.

Step 2:

$$f^w(\mathcal{J},I) = \frac{|\bigcup_{J\in\mathcal{J}} J \setminus \{1\}| - (|\mathscr{J}| - |C(\mathscr{J},I)|)}{\sum\limits_{i\in\bigcup_{J\in\mathcal{J}} J} w_i - w_1}$$

$$\psi_i^w(\mathcal{J},I) = w_i f^w(\mathcal{J},I)$$

If $w_2 f^w(\mathcal{J},I) < 1$ the procedure terminates. Otherwise, proceed to Step 3.

In general,

Step k:

$$f^w(\mathcal{J},I) = \frac{|\bigcup_{J\in\mathcal{J}} J \setminus \{1,\dots,k-1\}| - (|\mathscr{J}| - |C(\mathscr{J},I)|)}{\sum\limits_{i\in\bigcup_{J\in\mathcal{J}} J} w_i - \sum\limits_{i=1}^{k-1} w_i}$$

$$\psi_i^w(\mathcal{J},I) = w_i f^w(\mathcal{J},I)$$

If $w_k f^w(\mathcal{J},I) < 1$ the procedure terminates. Otherwise, proceed Step k+1. If at the Step q, $w_q f^w(\mathscr{J},I) \geq 1$, then we set $f^w(\mathscr{J},I) = 1$.

Let us define $N_{\mathscr{J}} = \bigcup_{J \in \mathcal{J}} J$, $N^m_{\mathscr{J}} = \{i \in N_{\mathscr{J}} : \psi^w_i(\mathcal{J}, I) \geq 1\}$, and $N^u_{\mathscr{J}} = \{i \in N_{\mathscr{J}} : \psi^w_i(\mathcal{J}, I) < 1\}$. We rewrite $f^w(\mathcal{J}, I)$ using these notation:

$$f^w(\mathcal{J}, I) = \frac{|N^u_{\mathscr{J}}| - (|\mathscr{J}| - |C(\mathscr{J}, I)|)}{\sum_{i \in N^u_{\mathscr{J}}} w_i}$$

The allocation chosen by the allocation rule for *underdemanded* patients in $\mathcal{J} \subseteq \mathcal{D}$ given $I \subseteq \mathbf{N^O}$ is then,

$$u_i = \psi^w_i(\mathcal{J}, I) = w_i f^w(\mathcal{J}, I), i \in N^u_{\mathscr{J}}$$
$$u_i = 1, i \in N^m_{\mathscr{J}}$$

2.4.3 Mechanism

Consider the induced two sided matching market $(\mathbf{N^O}, \mathcal{D}, \tilde{\mathbf{R}}, \mathbf{w}_{\mathbf{N} \setminus \mathbf{N^P}})$, for a given problem $(\mathbf{N}, \mathbf{R}, \mathbf{w})$. This can be done since all patients in $\mathbf{N^P}$ are perfectly matched in all efficient matching. The following procedure is suggested as weighted egalitarian mechanism.

Step 1:

Let $\mathcal{D}_1 \subseteq \mathcal{D}$ be such that,

$$\mathcal{D}_1(\mathcal{D}, \mathbf{N^O}) = \underset{\mathcal{J} \subseteq \mathcal{D}}{\arg\min} f^w(\mathcal{J}, \mathbf{N^O})$$

where the associated f^w value of any coalition is obtained by the allocation rule.

Here f^w captures the value of per weight utility of patients in a coalition, who are not matched in at least one Pareto efficient matching given the allocation of *overdemanded* neighbors. In the first step, a coalition of odd components is selected from $\mathcal{D}$ such that, using the allocation rule suggested in the previous subsection, the attained f^w value for that coalition is minimum among all possible coalition of odd components. Intuitively, this captures a coalition where per weight allocation of utility of patients who are not matched in at least one Pareto efficient matching given the allocated *overdemanded* neighbors to the coalition, is minimum. Under such a scenario, the mechanism compensates the coalition by providing all *overdemanded* patients who are neighbors to odd components of the coalition. The total number of possible matches of *underdemanded* patients is then allocated among members of the coalition as probability of getting a match using the allocation rule. Let $\mathcal{C}(\mathbf{N^O}, \mathcal{D}_1) \equiv \mathbf{N^O_1}$

Since the patients in $\mathcal{D}_1$ are matched with $\mathbf{N_1^O}$, $\mathbf{D}_k \in \mathcal{D} \setminus \mathcal{D}_1$, $(\mathbf{D}_k, \mathbf{R}_{D_k})$ is an odd-component for the reduced problem. The induced two-sided matching market for the reduced problem consists of odd components which weren't part of $\mathcal{D}_1$ and *overdemanded* patients who weren't part of $\mathbf{N_1^O}$. Hence the induced two-sided matching market for the reduced problem is given by $(\mathbf{N^O} \setminus \mathbf{N_1^O}, \mathcal{D} \setminus \mathcal{D}_1, \tilde{\mathbf{R}}_{\mathbf{N^O} \setminus \mathbf{N_1^O}}, \mathbf{w}_{\mathbf{N} \setminus \mathbf{N^P} \cup \mathbf{N_1^O} \cup_{\mathbf{D_i} \in \mathcal{D}_1} \mathbf{D_i}})$.

Let $\mathcal{D}_2$ be such that,

$$\mathcal{D}_2(\mathcal{D} \setminus \mathcal{D}_1, \mathbf{N} \setminus \mathbf{N_1^O}) = \underset{\mathcal{J} \subseteq \mathcal{D} \setminus \mathcal{D}_1}{\arg\min} f^w(\mathcal{J}, \mathbf{N} \setminus \mathbf{N_1^O})$$

where the associated f value of the coalition is given by the allocation rule.

Here $\mathcal{D}_2$ is the set of odd components where per weight utility of a patient unmatched in at least one Pareto efficient matching is minimum. *Underdemanded* patients in coalition $\mathcal{D}_2$ of odd components along with their *overdemanded* neighbours $\mathbf{N_2^O} \equiv \mathcal{C}(\mathbf{N} \setminus \mathbf{N_1^O}, \mathcal{D}_2)$ are matched with probability provided by the allocation rule.

This motivates the recursive construction of partition of $\mathbf{N^O}$ of *overdemanded* patients as $\{\mathbf{N_1^O}, \mathbf{N_2^O}, ..., \mathbf{N_q^O}\}$ and $\mathcal{D}$ of odd components as $\{\mathcal{D}_1, \mathcal{D}_2, ..., \mathcal{D}_q\}$. Each set $\mathbf{N_i^O}$ of *overdemanded* patients are then matched with $\mathcal{D}_i$ of odd components, and the utilities of *underdemanded* patients in odd components are determined by the allocation rule.The utility profile such obtained is the weighted egalitarian induced utility profile u^{WE} for the given problem $(\mathbf{N}, \mathbf{R}, \mathbf{w})$.

Theorem 6. *: The weighted egalitarian utility profile u^{WE} is feasible.*

For any utility profile $u \in R^n$, take a permutation σ_u to re-order ratios of individual utility and weight in an increasing order: $\frac{u_{\sigma_u(1)}}{w_{\sigma_u(1)}} \leq \frac{u_{\sigma_u(2)}}{w_{\sigma_u(2)}} \leq ... \leq \frac{u_{\sigma_u(n)}}{w_{\sigma_u(n)}}$.

Weighted Lorenz Curve: The *weighted Lorenz curve* for a utility profile u is a piece-wise linear function $L_u^w : [0, \sum_{i \in N} w_i] \to \mathscr{R}$ defined by

$$L_u^w(p) = \begin{cases} p \cdot \frac{u_{\sigma_u(1)}}{w_{\sigma_u(1)}} & (p \in [0, w_{\sigma_u(1)}]) \\ \sum_{j=1}^{i} u_{\sigma_u(j)} + (p - \sum_{j=1}^{i} w_{\sigma_u(j)}) \cdot \frac{u_{\sigma_u(i+1)}}{w_{\sigma_u(i+1)}} & (p \in [\sum_{j=1}^{i} w_{\sigma_u(j)}, \sum_{j=1}^{i+1} w_{\sigma_u(j)}], i \in \{1, 2, ..., n-1\}) \end{cases}$$

Weighted Lorenz Dominance: Let $u, v \in \mathscr{R}^n$. A utility profile u *weighted Lorenz dominates* v if $L^w_u(p) \geq L^w_v(p)$ for all $p \in [0.\sum_{i\in N} w_i]$, and $L^w_u(p) > L^w_v(p)$ for some $p \in [0.\sum_{i\in N} w_i]$.

An utility profile u *weighted Lorenz dominates* another utility profile v if the *Weighted Lorenz curve* of u lies above the *Weighted Lorenz curve* of v.

Theorem 7. *The weighted egalitarian utility profile u^{WE} weighted Lorenz dominates all feasible utility profiles.*

The above theorem suggests that, per weight allocation of the Weighted Egalitarian mechanism is more egalitarian than the per weight allocation of any feasible utility profile. An important concern for any mechanism design problem is *i*mplementability (i.e.) the mechanism provides right incentives for the participants. In this regard the following theorem holds.

Theorem 8. *: The weighted egalitarian mechanism makes it dominant strategy for a patient to reveal both, (a) the available set of donors, (b) her full set of acceptable kidneys*

The above theorem states that, it is incentive compatible for a patient to reveal more of their incompatible donors since it increases probability of match and the patient doesn't suffer from revealing her full set of acceptable kidneys. It is important to note that, addition of weight as parameter doesn't provide any incentives for agents to reject any acceptable kidney. The following corollary is immediate.

Corollary 1. *The weighted egalitarian mechanism is donor monotonic*

2.5 Priority and Weighted Egalitarian mechanism

In Priority mechanism [5] patients are offered priority rankings instead of weights. The priority mechanism chooses an allocation by matching agents based on priority order. Priority mechanisms are widely used in practice. It is important to study the Weighted Egalitarian mechanism in contrast with the Priority mechanism to understand the changing incentives faced by the patients. The following theorem will be helpful in this regard.

Theorem 9. *Consider a sequence of pairwise kidney exchange problems* $(\mathbf{N}, \mathbf{R}, \mathbf{w^{(n)}})$ *such that for all* $i > j$, $\dfrac{w_i^n}{w_j^n} = o(1)$. *Let* ϕ_n^{WE} *be the sequence of Weighted Egalitarian* *mechanism allocations. Let ψ be the allocation chosen by the Priority mechanism where patient i has higher priority than patient j, if $i < j$. Then*

$$\lim_{n\to\infty} \phi_n^{WE} = \psi$$

The Weighted Egalitarian mechanism chooses the allocation of Egalitarian mechanism when the weights of participating patients are equal. When the weights of patients of are increasing at an exponential rate relative to each other, then the above theorem implies that at the limit the allocation chosen by the Weighted Egalitarian mechanism is equivalent to the allocation chosen by the Priority mechanism. This characterizes the continuum, where symmetric weights constitute the egalitarian allocation at one end, and priority allocation at the other end where the priority allocation signifies total asymmetry between matched and relatively unmatched patient

2.6 Conclusion

Data available from United States suggests that, there is an ever growing demand for kidneys. Considering logistics and other constraints, Roth, Sönmez and Ünver(2004)[5] suggested Priority mechanism and Egalitarian mechanism for organizing the market for pairwise kidney exchange. Since patients are asymmetric in terms of their medical needs and demands, the current system provides people with priority score and chooses a matching which maximizes the priority score. Patients with low priority score can have relatively lower proportional needs but the mechanism which maximizes the priority score can fail to reflect the relative strength of needs in the final allocation. Through this paper we propose the Weighted Egalitarian mechanism, a stochastic mechanism which can address this deficit. We have provided with the normative characterization of Weighted Egalitarian mechanism, which shows how the mechanism better represents our intuitive notions of fairness.

Appendix: Proofs and Results

Theorem 1: The proof of theorem 3 can be broken down into following lemmas.

Lemma 8. *Fix* $\mathcal{G} \subseteq \mathscr{D}$ *and* $I \subseteq \mathbf{N^O}$. *Suppose* $\mathcal{G}_1, \mathcal{G}_2 \in \arg\min_{\mathcal{J} \subseteq \mathcal{G}} f(\mathcal{J}, I)$. *Then* $\mathcal{G}_1 \cup \mathcal{G}_2 \in \arg\min_{\mathcal{J} \subseteq \mathcal{G}} f(\mathcal{J}, I)$ *as well.*

Proof. We show following facts.

1. $N^m_{\mathcal{G}_1} \cup N^m_{\mathcal{G}_2} \subseteq N^m_{\mathcal{G}_1 \cup \mathcal{G}_2}$
2. $N^m_{\mathcal{G}_1} \cap N^m_{\mathcal{G}_2} \subseteq N^m_{\mathcal{G}_1 \cap \mathcal{G}_2}$

For the first, suppose not: there exist $i \in N^m_{\mathcal{G}_1} \cup N^m_{\mathcal{G}_2}$ such that $i \notin N^m_{\mathcal{G}_1 \cup \mathcal{G}_2}$.

$$\frac{\psi^w_i(\mathcal{G}_1 \cup \mathcal{G}_2, I)}{w_i} = f^w(\mathcal{G}_1 \cup \mathcal{G}_2, I) < \frac{\psi^w_i(\mathcal{G}_1, I)}{w_i} \leq f^w(\mathcal{G}_1, I) \tag{2.3}$$

This violates minimality of $\mathcal{G}_1$. For the second, we can apply a similar argument. By the definition of f,

$$\sum_{i\in N^u_{\mathcal{G}_1}} w_i f^w(\mathcal{G}_1,I)+|N^m_{\mathcal{G}_1}|=|N_{\mathcal{G}_1}|-(|\mathscr{G}_1|-|C(\mathscr{G}_1,I)|)$$

$$\sum_{i\in N^u_{\mathcal{G}_2}} w_i f^w(\mathcal{G}_2,I)+|N^m_{\mathcal{G}_2}|=|N_{\mathcal{G}_2}|-(|\mathscr{G}_2|-|C(\mathscr{G}_2,I)|)$$

$$\sum_{i\in N^u_{\mathcal{G}_1\cap\mathcal{G}_2}} w_i f^w(\mathcal{G}_1\cap\mathcal{G}_2,I)+|N^m_{\mathcal{G}_1\cap\mathcal{G}_2}|=|N_{\mathcal{G}_1\cap\mathcal{G}_2}|-(|\mathcal{G}_1\cap\mathcal{G}_2|-|C(\mathcal{G}_1\cap\mathcal{G}_2,I)|)$$

Since $|C(\mathscr{G}_1,I)|)+|C(\mathscr{G}_2,I)|)\geq|C(\mathcal{G}_1\cap\mathcal{G}_2,I)|+|C(\mathcal{G}_1\cup\mathcal{G}_2,I)|$, we have

$$\begin{aligned}
|N_{\mathcal{G}_1}|-(|\mathscr{G}_1|-|C(\mathscr{G}_1,I)|)+|N_{\mathcal{G}_2}|-(|\mathscr{G}_2|-|C(\mathscr{G}_2,I)|)&\\
-\big(|N_{\mathcal{G}_1\cap\mathcal{G}_2}|-(|\mathcal{G}_1\cap\mathcal{G}_2|-|C(\mathcal{G}_1\cap\mathcal{G}_2,I)|)\big)&\\
\geq|N_{\mathcal{G}_1\cup\mathcal{G}_2}|-(|\mathcal{G}_1\cup\mathcal{G}_2|-|C(\mathcal{G}_1\cup\mathcal{G}_2,I)|)&\\
=\sum_{i\in N^u_{\mathcal{G}_1\cup\mathcal{G}_2}} w_i f^w(\mathcal{G}_1\cup\mathcal{G}_2,I)+|N^m_{\mathcal{G}_1\cup\mathcal{G}_2}|&
\end{aligned}$$

Since $f^w(\mathcal{G}_1,I)=f^w(\mathcal{G}_2,I)\leq f^w(\mathcal{G}_1\cap\mathcal{G}_2,I)$, we have

$$\begin{aligned}
&\sum_{i\in N^u_{\mathcal{G}_1}} w_i f^w(\mathcal{G}_1,I)+|N^m_{\mathcal{G}_1}|+\sum_{i\in N^u_{\mathcal{G}_2}} w_i f^w(\mathcal{G}_1,I)+|N^m_{\mathcal{G}_2}|-\Big(\sum_{i\in N^u_{\mathcal{G}_1\cap\mathcal{G}_2}} w_i f^w(\mathcal{G}_1,I)+|N^m_{\mathcal{G}_1\cap\mathcal{G}_2}|\Big)\\
\geq&\sum_{i\in N^u_{\mathcal{G}_1}} w_i f^w(\mathcal{G}_1,I)+|N^m_{\mathcal{G}_1}|+\sum_{i\in N^u_{\mathcal{G}_2}} w_i f^w(\mathcal{G}_2,I)+|N^m_{\mathcal{G}_2}|-\Big(\sum_{i\in N^u_{\mathcal{G}_1\cap\mathcal{G}_2}} w_i f^w(\mathcal{G}_1\cap\mathcal{G}_2,I)+|N^m_{\mathcal{G}_1\cap\mathcal{G}_2}|\Big)
\end{aligned}$$

Since $N^m_{\mathcal{G}_1} \cap N^m_{\mathcal{G}_2} \subseteq N^m_{\mathcal{G}_1 \cap \mathcal{G}_2}$ and $1 \geq w_i f^w(\mathcal{G}_1)$ for any i, we have

$$\sum_{i \in N^u_{\mathcal{G}_1 \cap \mathcal{G}_2}} w_i f^w(\mathcal{G}_1, I) + |N^m_{\mathcal{G}_1 \cap \mathcal{G}_2}| \geq \sum_{i \in N^u_{\mathcal{G}_1} \cap N^u_{\mathcal{G}_2}} w_i f^w(\mathcal{G}_1, I) + |N^m_{\mathcal{G}_1} \cap N^m_{\mathcal{G}_2}|$$

Thus we also have,

$$\begin{aligned} &\sum_{i \in N^u_{\mathcal{G}_1} \cup N^u_{\mathcal{G}_2}} w_i f^w(\mathcal{G}_1, I) + |N^m_{\mathcal{G}_1} \cup N^m_{\mathcal{G}_2}| \\ = &\sum_{i \in N^u_{\mathcal{G}_1}} w_i f^w(\mathcal{G}_1, I) + |N^m_{\mathcal{G}_1}| + \sum_{i \in N^u_{\mathcal{G}_2}} w_i f^w(\mathcal{G}_1, I) + |N^m_{\mathcal{G}_2}| - \Big(\sum_{i \in N^u_{\mathcal{G}_1} \cap N^u_{\mathcal{G}_2}} w_i f^w(\mathcal{G}_1, I) + |N^m_{\mathcal{G}_1} \cap N^m_{\mathcal{G}_2}| \Big) \\ \geq &\sum_{i \in N^u_{\mathcal{G}_1}} w_i f^w(\mathcal{G}_1, I) + |N^m_{\mathcal{G}_1}| + \sum_{i \in N^u_{\mathcal{G}_2}} w_i f^w(\mathcal{G}_1, I) + |N^m_{\mathcal{G}_2}| - \Big(\sum_{i \in N^u_{\mathcal{G}_1 \cap \mathcal{G}_2}} w_i f^w(\mathcal{G}_1, I) + |N^m_{\mathcal{G}_1 \cap \mathcal{G}_2}| \Big) \end{aligned}$$

Since $N^m_{\mathcal{G}_1} \cup N^m_{\mathcal{G}_2} \subseteq N^m_{\mathcal{G}_1 \cup \mathcal{G}_2}$, we have

$$\sum_{i \in N^u_{\mathcal{G}_1 \cup \mathcal{G}_2}} w_i f^w(\mathcal{G}_1, I) + |N^m_{\mathcal{G}_1 \cup \mathcal{G}_2}| \geq \sum_{i \in N^u_{\mathcal{G}_1} \cup N^u_{\mathcal{G}_2}} w_i f^w(\mathcal{G}_1, I) + |N^m_{\mathcal{G}_1} \cup N^m_{\mathcal{G}_2}|$$

By these results, we have $f^w(\mathcal{G}_1, I) \geq f^w(\mathcal{G}_1 \cup \mathcal{G}_2, I)$. But since $\mathcal{G}_1$ minimize f, we have $f^w(\mathcal{G}_1, I) = f^w(\mathcal{G}_1 \cup \mathcal{G}_2, I)$ and hence $\mathcal{G}_1 \cup \mathcal{G}_2$ minimize f as well. ■

Lemma 9. *For each $k \in \{1, 2, ..., q\}$ and $i \in N^u_{\mathcal{D}_k}$, we have*

1. $C(\mathcal{D}_k, N^O_K) = C(\mathcal{D}_k, N^O \setminus \bigcup_{l=1}^{k-1} N^O_l)$ *and* $= f(\mathcal{D}_k, N^O \setminus \bigcup_{l=1}^{k-1} N^O_l)$.

2. $w_i f(\mathscr{D}_k, N_K^O) < 1$

Proof. ■

Let $k \in \{1,2,...,q\}$ and $J \in \mathcal{D}_k$. Let us define $J^u_{\mathcal{D}_k} = J \cap N^u_{\mathcal{D}_k}$ and $J^m_{\mathcal{D}_k} = J \cap N^m_{\mathcal{D}_k}$. $\sum_{i \in J^u_{\mathcal{D}_k}} w_i f^w(\mathcal{D}_k, N_k^O) + |J^m_{\mathcal{D}_k}|$ is the aggregate utility in set J under u^E. The set J, by itself, generates an aggregate utility $|J|-1$ without any help the overdemanded patients. Therefore, the set J need $\sum_{i \in J^u_{\mathcal{D}_k}} w_i f^w(\mathcal{D}_k, N_k^O) + |J^m_{\mathcal{D}_k}| - (|J|-1)$ from the overdemanded patients. Let

$$\begin{aligned}
\alpha_J &= \sum_{i \in J^u_{\mathcal{D}_k}} w_i f^w(\mathcal{D}_k, N_k^O) + |J^m_{\mathcal{D}_k}| - (|J|-1) \\
&= \sum_{i \in J^u_{\mathcal{D}_k}} w_i f^w(\mathcal{D}_k, N_k^O) - (|J^u_{\mathcal{D}_k}|-1) \\
&= \frac{\left(|N^u_{\mathcal{D}_k}| - (|\mathcal{D}_k| - |C(\mathcal{D}_k, N_k^O)|)\right)\sum_{i \in J^u_{\mathcal{D}_k}} w_i}{\sum\limits_{i \in N^u_{\mathcal{D}_k}} w_i} - (|J^u_{\mathcal{D}_k}|-1) \\
&= \frac{\left(|N^u_{\mathcal{D}_k}| - (|\mathcal{D}_k| - |C(\mathcal{D}_k, N_k^O)|)\right)\sum_{i \in J^u_{\mathcal{D}_k}} w_i - (|J^u_{\mathcal{D}_k}|-1)\sum_{i \in N^u_{\mathcal{D}_k}} w_i}{\sum\limits_{i \in N^u_{\mathcal{D}_k}} w_i}
\end{aligned}$$

Lemma 10. *There exists a pre-allocation matrix $\tilde{A} \in \mathscr{A}$ such that*

1. *For each $i \in N^O$, $\sum_{J \in \mathscr{D}} \tilde{a}_{i,J} = 1$, and*

2. *For each* $k \in \{1, 2, \dots, q\}$ *and* $J \in \mathcal{D}_k$

   (a) $\tilde{a}_{i,J} = 0$ *for all* $i \in N^O \setminus N^O_k$*, and*

   (b) $\sum_{i \in N^O_k} \tilde{a}_{i,J} = \alpha_J$

Proof. Let $k \in \{1, 2, \dots, q\}$. We will show that there exists a matrix $\tilde{A}^{k,k} = [\tilde{a}_{i,J}]_{i \in N^O_k, J \in \mathscr{D}_k}$ such that

1. $\sum_{J \in \mathscr{D}_k} \tilde{a}_{i,J} = 1$ for all $i \in N^O_k$,

2. $\sum_{J \in \mathscr{D}_k} \tilde{a}_{i,J} = \alpha_J$ for all $J \in \mathscr{D}_k$, and

3. $\tilde{a}_{i,J} > 0 \Rightarrow \tilde{r}_{i,J} = 1$ for any pair $i \in N^O_k, J \in \mathscr{D}_k$

We will show this by defining an auxiliary task assignment problem and applying Hall's Theorem to the auxiliary task assignment problem. Given N^O_k and $\mathscr{D}_k$, construct the task assignment problem $(X, \mathscr{T}, \Gamma)$ as follows:

- For each overdemanded patient $i \in N^O_k$, introduce $\sum_{i \in N^u_{\mathcal{D}_k}} w_i$ identical agents. Let X_i ne the set of the identical agents associated with patient i, and $X = \bigcup_{i \in N^O_k} X_i$.

- For each odd component $J \in \mathscr{D}_k$, introduce $\big(\mid N^u_{\mathcal{D}_k} \mid - (\mid \mathcal{D}_k \mid - \mid C(\mathcal{D}_k, N^O_k) \mid) \big) \sum_{i \in J^u_{\mathcal{D}_k}} w_i - (\mid J^u_{\mathcal{D}_k} \mid - 1) \sum_{i \in N^u_{\mathcal{D}_k}} w_i$ identical tasks. Let $\mathscr{T}_J$ be the set of identical tasks associated with set J, and $\mathscr{T} = \bigcup_{J\in\mathscr{D}_k} \mathscr{T}_J$.

- Finally, introduce a matrix $\Gamma = (\gamma_{x,T})_{x\in X, T\in\mathscr{T}}$ such that $\gamma_{x,T} = 1$ if $\tilde{r}_{i,J} = 1$ for $x \in X$ and $T \in \mathscr{T}$, and 0 otherwise.

Note that $(w_i)_{i\in N} \in \mathbb{N}^{|N|}$ and hence $\sum_{i\in N^u_{\mathcal{D}_k}} w_i \in \mathbb{N}$. Given N^O_k and $\mathscr{D}_k$ we refer to $(X, \mathscr{T}, \Gamma)$ as the auxiliary task assignment problem. Note that

$$
\begin{aligned}
|\mathscr{T}| &= \sum_{J\in\mathscr{D}_k} \Big(\big(|N^u_{\mathcal{D}_k}| - (|\mathcal{D}_k| - |C(\mathcal{D}_k, N^O_k)|) \big) \sum_{i\in J^u_{\mathcal{D}_k}} w_i - (|J^u_{\mathcal{D}_k}| - 1) \sum_{i\in N^u_{\mathcal{D}_k}} w_i \Big) \\
&= \big(|N^u_{\mathcal{D}_k}| - (|\mathcal{D}_k| - |C(\mathcal{D}_k, N^O_k)|) \big) \sum_{i\in N^u_{\mathcal{D}_k}} w_i - (|N^u_{\mathcal{D}_k}| - |\mathscr{D}_k|) \sum_{i\in N^u_{\mathcal{D}_k}} w_i \\
&= |C(\mathcal{D}_k, N^O_k)| \sum_{i\in N^u_{\mathcal{D}_k}} w_i \\
&= \sum_{j\in N^O_k} \sum_{i\in N^u_{\mathcal{D}_k}} w_i = |X|.
\end{aligned}
$$

■

An auxiliary task assignment is a bijection $\nu : X \longrightarrow \mathscr{T}$. An auxiliary task assignment ν is feasible if and only if $\nu(x) = T$ implies that $\gamma_{x,T} = 1$. We will show that there exists a feasible auxiliary task assignment ν for $(X, \mathscr{T}, \Gamma)$ Given $\tau \subseteq \mathscr{T}$ define

$$C(\tau, X) = \{x \in X : \exists T \in \tau \text{ with } \gamma_{x,T} = 1\}.$$

By Hall's Theorem there exists a feasible auxiliary task assignment if and only if

$$|\tau| \leq |C(\tau, X)| \quad \text{for every} \quad \tau \subseteq \mathscr{T}$$

We will prove this by contradiction. Suppose there exists a subset $\tau \subseteq \mathscr{T}$ of tasks such that $|\tau| > |C(\tau, X)|$. Next construct the following set of tasks $\tau^* \supseteq \tau$. Note that since $C(\tau^*, X) = C(\tau, X)$, we have

$$|\tau^*| \geq |\tau| > |C(\tau, X)| = |C(\tau^*, X)|.$$

Let $\mathscr{J}^*$ be the set of odd components each of which is associated with a task in τ^*. Note that $\bigcup_{J \in \mathscr{J}^*} \mathscr{T}_J = \tau^*$, and therefore

$$\begin{aligned}
|\tau^*| &= \sum_{J \in \mathscr{J}^*} \Big(\big(| N^u_{\mathcal{D}_k} | - (| \mathcal{D}_k | - | C(\mathcal{D}_k, N^O_k) |) \big) \sum_{i \in J^u_{\mathcal{D}_k}} w_i - (| J^u_{\mathcal{D}_k} | - 1) \sum_{i \in N^u_{\mathcal{D}_k}} w_i \Big) \\
&= \big(| N^u_{\mathcal{D}_k} | - (| \mathcal{D}_k | - | C(\mathcal{D}_k, N^O_k) |) \big) \sum_{J \in \mathscr{J}^*} \sum_{i \in J^u_{\mathcal{D}_k}} w_i - (| \bigcup_{J \in \mathscr{J}^*} J^u_{\mathcal{D}_k} | - | \mathscr{J}^* |) \sum_{i \in N^u_{\mathcal{D}_k}} w_i
\end{aligned}$$

Moreover,

$$|C(\tau^*, X)| = | C(\mathscr{J}^*, N^O_k) | \sum_{i \in N^u_{\mathcal{D}_k}} w_i$$

By these results,

$$\Big(|N^u_{\mathcal{D}_k}|-(|\mathcal{D}_k|-|C(\mathcal{D}_k,N^O_k)|)\Big)\sum_{J\in\mathscr{J}^*}\sum_{i\in J^u_{\mathcal{D}_k}} w_i-(|\bigcup_{J\in\mathscr{J}^*} J^u_{\mathcal{D}_k}|-|\mathscr{J}^*|)\sum_{i\in N^u_{\mathcal{D}_k}} w_i$$
$$>|C(\mathscr{J}^*,N^O_k)|\sum_{i\in N^u_{\mathcal{D}_k}} w_i;$$

rearranging the terms, we have

$$\frac{|N^u_{\mathcal{D}_k}|-(|\mathcal{D}_k|-|C(\mathcal{D}_k,N^O_k)|)}{\sum_{i\in N^u_{\mathcal{D}_k}} w_i}=f(\mathscr{D}_k,N^O_k)>\frac{|\bigcup_{J\in\mathscr{J}^*} J^u_{\mathcal{D}_k}|-(|\mathscr{J}^*|-|C(\mathscr{J}^*,N^O_k)|)}{\sum_{J\in\mathscr{J}^*}\sum_{i\in J^u_{\mathcal{D}_k}} w_i} \tag{2.4}$$

We show that $\bigcup_{J\in\mathscr{J}^*} J^u_{\mathcal{D}_k}=N^u_{\mathscr{J}^*}$, which is equivalent to $\bigcup_{J\in\mathscr{J}^*} J^m_{\mathcal{D}_k}=N^m_{\mathscr{J}^*}$. Suppose not. There are two cases to consider. Case 1: There exists an agent i such that $i\in\bigcup_{J\in\mathscr{J}^*} J^m_{\mathcal{D}_k}$ and $i\notin N^m_{\mathscr{J}^*}$. This implies

$$w_i f(\mathscr{D}_k,N^O_k)\geq 1>w_i f(\mathscr{J}^*,N^O_k)$$
$$\Leftrightarrow f(\mathscr{D}_k,N^O_k)>f(\mathscr{J}^*,N^O_k)$$

This violates $\mathscr{D}_k$ minimize f. Case 2: There exists an agent i such that $i\notin\bigcup_{J\in\mathscr{J}^*} J^m_{\mathcal{D}_k}$

and $i \in N^m_{\mathscr{J}^*}$. Notice that $i \in \bigcup_{J\in\mathscr{J}^*} J^u_{\mathcal{D}_k}$ and thus

$$1 > w_i f(\mathscr{D}_k, N^O_k)$$

Moreover, $\bigcup_{J\in\mathscr{J}^*} J^m_{\mathcal{D}_k} \subseteq N^m_{\mathscr{J}^*}$ by Case 1. Let $|\bigcup_{J\in\mathscr{J}^*} J^m_{\mathcal{D}_k}| = l$. Together with these facts, at the step l in procedure of $f(\mathscr{J}^*, N^O_k)$,

$$f^{(l)}(\mathscr{J}^*, N^O_k) = \frac{|\bigcup_{J\in\mathscr{J}^*} J^u_{\mathcal{D}_k}| - (|\mathscr{J}^*| - |C(\mathscr{J}^*, N^O_k)|)}{\sum_{J\in\mathscr{J}^*}\sum_{i\in J^u_{\mathcal{D}_k}} w_i}$$

Since $i \in N^m_{\mathscr{J}^*}$, we have $w_1 f^{(l)}(\mathscr{J}^*, N^O_k) \geq 1$. Thus, we get $w_1 f^{(l)}(\mathscr{J}^*, N^O_k) \geq 1 > w_i f(\mathscr{D}_k, N^O_k)$, or $f^{(l)}(\mathscr{J}^*, N^O_k) > f(\mathscr{D}_k, N^O_k)$, contradicting Eq. (2.4). Now we get $\bigcup_{J\in\mathscr{J}^*} J^u_{\mathcal{D}_k} = N^u_{\mathscr{J}^*}$ and hence Eq. (2.4) is

$$\begin{aligned}
f(\mathscr{D}_k, N^O_k) &> \frac{|\bigcup_{J\in\mathscr{J}^*} J^u_{\mathcal{D}_k}| - (|\mathscr{J}^*| - |C(\mathscr{J}^*, N^O_k)|)}{\sum_{J\in\mathscr{J}^*}\sum_{i\in J^u_{\mathcal{D}_k}} w_i} \\
&= \frac{|N^u_{\mathscr{J}^*}| - (|\mathscr{J}^*| - |C(\mathscr{J}^*, N^O_k)|)}{\sum_{i\in N^u_{\mathscr{J}^*}} w_i} = f(\mathscr{J}^*, N^O_k)
\end{aligned}$$

contradicting the definition of $\mathscr{D}_k$, and showing that for each $\tau \subseteq \mathscr{T}$, $|\tau| \leq |C(\tau, X)|$. Therefore, there exists a feasible auxiliary assignment ν.

We next construct matrix $\tilde{A}^{k,k} = [\tilde{a}_{i,J}]_{i\in N^O_k, J\in\mathscr{D}_k}$ using the auxiliary assignment ν.

For each $J \in \mathscr{D}_k$ and $i \in N_k^O$, define

$$\nu_{i,J} = \{x \in X_i : \nu(x) = T \text{ for some } T \in \mathscr{T}_J\}$$

For each $J \in \mathscr{D}_k$ and $i \in N_k^O$, let

$$\tilde{a}_{i,J} = \frac{|\nu_{i,J}|}{\sum_{i \in N^u_{\mathscr{D}_k}} w_i}$$

and let $\tilde{A}^{k,k} = [\tilde{a}_{i,J}]_{i \in N_k^O, J \in \mathscr{D}_k}$. For each $J \in \mathscr{D}_k$, we have

$$\begin{aligned}
\sum_{i \in N_k^O} \tilde{a}_{i,J} &= \frac{\sum_{i \in N_k^O} |\nu_{i,J}|}{\sum_{i \in N^u_{\mathscr{D}_k}} w_i} = \frac{|\mathscr{T}_J|}{\sum_{i \in N^u_{\mathscr{D}_k}} w_i} \\
&= \frac{\left(|N^u_{\mathcal{D}_k}| - (|\mathcal{D}_k| - |C(\mathcal{D}_k, N_k^O)|)\right) \sum_{i \in J^u_{\mathcal{D}_k}} w_i - (|J^u_{\mathcal{D}_k}| - 1) \sum_{i \in N^u_{\mathcal{D}_k}} w_i}{\sum_{i \in N^u_{\mathscr{D}_k}}} \\
&= \alpha_J,
\end{aligned}$$

where the last inequality hods by the definition of α_J. Moreover, for each $i \in N_k^O$, we have

$$\sum_{J \in \mathscr{D}_k} \tilde{a}_{i,J} = \frac{\sum_{J \in \mathscr{D}_k} |\nu_{i,J}|}{\sum_{i \in N^u_{\mathscr{D}_k}} w_i} = \frac{X_i}{\sum_{i \in N^u_{\mathscr{D}_k}} w_i} = \frac{\sum_{i \in N^u_{\mathscr{D}_k}} w_i}{\sum_{i \in N^u_{\mathscr{D}_k}} w_i} = 1$$

We conclude the proof by constructing a pre-allocation matrix $\tilde{A} \in \tilde{\mathscr{A}}$ using the matrices $\{\tilde{A}^{k,k}\}_{k\in\{1,\dots,q\}}$ constructed above. For each k,k' with $k\neq k'$, for each $i\in N_k^O$ and each $J\in\mathscr{D}_{k'}$, let $\tilde{a}_{i,J}=0$. Let $\tilde{A}^{k,k'}=[\tilde{a}_{i,J}]_{i\in N_k^O,J\in\mathscr{D}_k'}$. Let $\tilde{A}=\{\tilde{A}^{k,k}\}_{k\in\{1,\dots,q\},k'\in\{1,\dots,q\}}=[\tilde{a}_{i,J}]_{i\in N^O,J\in\mathscr{D}}$. Notice that for each $J\in\mathscr{D}_k$, we have $\sum_{i\in N^O}\tilde{a}_{i,J}=\sum_{i\in N_k^O}\tilde{a}_{i,J}=\alpha_J$ and for each $i\in N_k^O$, we have $\sum_{J\in\mathscr{D}}\tilde{a}_{i,J}=\sum_{J\in\mathscr{D}_k}\tilde{a}_{i,J}=1$ concludting the proof.

Lemma 11. *There exists an ex post efficient lottery $\lambda^{WE}\in\mathscr{L}$ such that $u(\lambda^{WE})=u^{WE}$*

Proof. By the result, there exists a pre-allocation matrix $\tilde{A}\in\mathscr{A}$ such that

1. For each $i\in N^O$, $\sum_{J\in\mathscr{D}}\tilde{a}_{i,J}=1$, and
2. For each $k\in\{1,2,\dots,q\}$ and $J\in\boldsymbol{\mathcal{D}}_k$
   (a) $\tilde{a}_{i,J}=0$ for all $i\in N^O\setminus N_k^O$, and
   (b) $\sum_{i\in N_k^O}\tilde{a}_{i,J}=\alpha_J$

For each $k\in\{1,2,\dots,q\}$ and $J\in\mathscr{D}_k$, we have $\sum_{i\in N_k^O}\tilde{a}_{i,J}=\alpha_J$ and $\tilde{a}_{i,J}=0$ for all $i\in N^O\setminus N_k^O$. By Lemma 2.1 in Bogomolnai and Moulin there exists an ex post efficient pre-lottery $\tilde{\lambda}\in\tilde{\mathscr{L}}$ that implements $\tilde{A}$. For each pre-matching $\tilde{\mu}\in\tilde{\mathscr{M}}$ in the support of $\tilde{\lambda}$, partition set $\mathscr{D}$ as $\{\mathscr{D}^m(\tilde{\mu}),\mathscr{D}^u(\tilde{\mu})\}$ where

- $\mathscr{D}^m(\tilde{\mu})=\{J\in\mathscr{D}:\tilde{\mu}\neq\emptyset\}$ and
- $\mathscr{D}^u(\tilde{\mu})=\mathscr{D}\setminus\mathscr{D}^m(\tilde{\mu})$.

Define $J^u(\tilde{\mu}) = J \cap N^u_{\mathscr{D}}$ for each $J \in \mathscr{D}^u(\tilde{\mu})$. Pick one patient $i \in J^u(\tilde{\mu})$ for each $J \in \mathscr{D}^u(\tilde{\mu})$. Note that there are $\Pi_{J\in\mathscr{D}^u(\tilde{\mu})}|J^u(\tilde{\mu})|$ possible combinations. For each combination construct a Pareto-efficient matching μ such that:

- each of the chosen patient i is matched to herself
- each remaining patient in $J \in \mathscr{D}^u(\tilde{\mu})$ is matched with another patient in the same odd component J, and
- one patient in each $J \in \mathscr{D}^m(\tilde{\mu})$ is matched with the an overdemanded patient $i \in N^O$ whereas all other patients in each such odd component J is matched with another patient in J.

By the GED Lemma, there exists at least one such matching. Pick one and only one such matching for each of the $\Pi_{J\in\mathscr{D}^u(\tilde{\mu})}|J^u(\tilde{\mu})|$ possible combinations. Let $\mathscr{M}(\tilde{\mu})$ be the resulting set of matchings. Clearly, $|\mathscr{M}(\tilde{\mu})| = \Pi_{J\in\mathscr{D}^u(\tilde{\mu})}|J^u(\tilde{\mu})|$

We are finally ready to construct a lottery λ^{WE}, which induces the utility profile u^{WE}. The lottery λ^{WE} is constructed from the pre-lottery $\tilde{\lambda}^{WE}$ by replacing each pre-matching $\tilde{\mu}$ in the support of $\tilde{\lambda}^{WE}$ with a lottery over $\mathscr{M}(\tilde{\mu})$ based on the weight profile w. That is:

$$\lambda^{WE}_{\mu} = \begin{cases} \left(\Pi_{J\in\mathscr{D}^u(\tilde{\mu}), i\in J^u(\tilde{\mu}) \text{with} \mu(i)=i} \dfrac{1-w_i f(\mathscr{D}_k, N^O_K)}{1-\alpha_J}\right)\tilde{\lambda}_{\tilde{\mu}} & \text{if } \mu\in\mathscr{M}(\tilde{\mu}) \text{ and } \tilde{\lambda}_{\tilde{\mu}}>0 \\ 0 & \text{otherwise} \end{cases}$$

Clearly, λ^{WE} is a lottery:

$$\begin{aligned} \sum_{\mu\in\mathscr{M}} \lambda^{WE}_{\mu} &= \sum_{\tilde{\mu}\in\tilde{\mathscr{M}}} \left(\sum_{\mu\in\mathscr{M}(\tilde{\mu})} \lambda^{WE}_{\mu}\right) \\ &= \sum_{\tilde{\mu}\in\tilde{\mathscr{M}}} \left(\sum_{\mu\in\mathscr{M}(\tilde{\mu})} \left(\Pi_{J\in\mathscr{D}^u(\tilde{\mu}), i\in J^u(\tilde{\mu}) \text{with} \mu(i)=i} \frac{1-w_i f(\mathscr{D}_k, N^O_K)}{1-\alpha_J}\right)\tilde{\lambda}_{\tilde{\mu}}\right) \\ &= \sum_{\tilde{\mu}\in\tilde{\mathscr{M}}} \left(\Pi_{J\in\mathscr{D}^u(\tilde{\mu})} \left(\sum_{i\in J^u(\tilde{\mu})} \frac{1-w_i f(\mathscr{D}_k, N^O_K)}{1-\alpha_J}\right)\tilde{\lambda}_{\tilde{\mu}}\right) \\ &= \sum_{\tilde{\mu}\in\tilde{\mathscr{M}}} \left(\Pi_{J\in\mathscr{D}^u(\tilde{\mu})} \left(\frac{1-\alpha_J}{1-\alpha_J}\right)\right)\tilde{\lambda}_{\tilde{\mu}} = 1. \end{aligned}$$

Moreover, by construction λ^E is an ex post efficient lottery.

We conclude the proof by showing that $u(\lambda^{WE}) = u^{WE}$. Each patient in $N\setminus N^u$ is matched with another patient in every efficient matching by the GED Lemma. Since λ^{WE} is an ex post efficient, for each patient $i\in N\setminus N^u$ we have $u_i(\lambda^{WE}) = u^{WE}_i = 1$.

Consider a patient $i\in N^u$. Let $i\in J\in\mathscr{D}_k$ for some k. Recall that J is partitioned by

$J^m_{\mathscr{D}_k}$ and $J^u_{\mathscr{D}_k}$. If $i \in J^m_{\mathscr{D}_k}$, by construction of λ^{WE}, is matched with another patient under every matching in the support of λ^{WE}. Thus, we have $u_i(\lambda^{WE}) = u_i^{WE} = 1$. Now, we suppose that $i \in J^u_{\mathscr{D}_k}$ and show that $u_i(\lambda^{WE}) = u_i^{WE} = w_i f(\mathscr{D}_k, N_k^O)$.

Let $\tilde{\mu} \in \tilde{M}$ be a pre-matching with $\tilde{\lambda}_{\tilde{\mu}} > 0$.

1. If $J \in \mathscr{D}^m(\tilde{\mu})$ then all patients in J are matched under every matching $\mu \in \mathscr{M}(\tilde{\mu})$.

2. If $J \in \mathscr{D}^u(\tilde{\mu})$, then the probability that i is unmatched conditional on $J \in \mathscr{D}^u(\tilde{\mu})$ is $\dfrac{1 - w_i f(\mathscr{D}_k, N_K^O)}{1 - \alpha_J}$. In other words, the probability that i is matched with another patients conditional on $J \in \mathscr{D}^u(\tilde{\mu})$ is $\dfrac{1 - \alpha_J - (1 - w_i f(\mathscr{D}_k, N_K^O))}{1 - \alpha_J}$.

Since $\sum_{i \in N_k^O} \tilde{a}_{i,J} = \alpha_J$ is the probability that the odd component J is assigned a patient in N_k^O under the pre-lottery $\tilde{\lambda}$, we have

$$\sum_{\tilde{\mu} \in \tilde{\mathscr{M}} \text{ s.t. } J \in \mathscr{D}^m(\tilde{\mu})} \tilde{\lambda}_{\tilde{\mu}} = \alpha_J \quad \text{and} \quad \sum_{\tilde{\mu} \in \tilde{\mathscr{M}} \text{ s.t. } J \in \mathscr{D}^u(\tilde{\mu})} \tilde{\lambda}_{\tilde{\mu}} = 1 - \alpha_J$$

Therefore

$$
\begin{aligned}
u_i(\lambda^{WE}) &= \sum_{\mu \in \mathcal{M} \text{ s.t. } \mu(i) \neq i} \lambda^{WE} \\
&= \sum_{\tilde{\mu} \in \tilde{\mathcal{M}} \text{ s.t. } J \in \mathcal{D}^m(\tilde{\mu})} \left(\sum_{\mu \in \mathcal{M}(\tilde{\mu})} \lambda_\mu^{WE} \right) + \sum_{\tilde{\mu} \in \tilde{\mathcal{M}} \text{ s.t. } J \in \mathcal{D}^u(\tilde{\mu})} \left(\sum_{\mu \in \mathcal{M}(\tilde{\mu}) \text{ s.t. } \mu(i) \neq i} \lambda_\mu^{WE} \right) \\
&= \sum_{\tilde{\mu} \in \tilde{\mathcal{M}} \text{ s.t. } J \in \mathcal{D}^m(\tilde{\mu})} \left(\Pi_{J' \in \mathcal{D}^u(\tilde{\mu})} \left(\frac{1-\alpha_{J'}}{1-\alpha_{J'}} \right) \right) \tilde{\lambda}_{\tilde{\mu}} \\
&+ \sum_{\tilde{\mu} \in \tilde{\mathcal{M}} \text{ s.t. } J \in \mathcal{D}^u(\tilde{\mu})} \left(\left\{ \Pi_{J' \in \mathcal{D}^u(\tilde{\mu}) \text{ s.t. } J' \neq J} \left(\frac{1-\alpha_{J'}}{1-\alpha_{J'}} \right) \right\} \left\{ \frac{1-\alpha_J-(1-w_i f(\mathcal{D}_k, N_K^O))}{1-\alpha_J} \right\} \right) \tilde{\lambda}_{\tilde{\mu}} \\
&= \sum_{\tilde{\mu} \in \tilde{\mathcal{M}} \text{ s.t. } J \in \mathcal{D}^m(\tilde{\mu})} \tilde{\lambda}_{\tilde{\mu}} + \left\{ \frac{1-\alpha_J-(1-w_i f(\mathcal{D}_k, N_K^O))}{1-\alpha_J} \right\} \sum_{\tilde{\mu} \in \tilde{\mathcal{M}} \text{ s.t. } J \in \mathcal{D}^u(\tilde{\mu})} \tilde{\lambda}_{\tilde{\mu}} \\
&= \alpha_J + \left\{ \frac{1-\alpha_J-(1-w_i f(\mathcal{D}_k, N_K^O))}{1-\alpha_J} \right\} (1-\alpha_J) \\
&= w_i f(\mathcal{D}_k, N_K^O) = u_i^{WE}.
\end{aligned}
$$

This completes the proof of Theorem 1. ■

Theorem 2: The weighted egalitarian utility profile u^{WE} *w*eighted Lorenz dominates all feasible utility profiles.

Proof. The following lemma will be useful in proving the theorem

Lemma 12. *$f^w(\mathscr{D}_k, \mathscr{N}_k^O) < f^w(\mathscr{D}_{k+1}, \mathscr{N}_{k+1}^O)$ for each $k \in \{1,2,...,q\}$.*

Proof. Suppose not. There exists k such that $f^w(\mathscr{D}_{k+1}, \mathscr{N}_{k+1}^O) \leq f^w(\mathscr{D}_k, \mathscr{N}_k^O)$. Let $I = N^O \setminus \cup_{l=1}^{k-1} N_l^O$. Consider construction of $\{\mathscr{D}_1, \mathscr{D}_2, ..., \mathscr{D}_q\}$. Note that $\mathscr{D}_k \cup \mathscr{D}_{k+1} \subseteq \mathscr{D} \setminus \cup_{l=1}^{k-1} \mathscr{D}_l$. Since

$$f^w(\mathscr{D}_k, I) = \min_{\mathscr{J} \subseteq \mathscr{D} \setminus \cup_{l=1}^{k-1} \mathscr{D}_l} f^w(\mathscr{J}, I)$$

and $\mathscr{D}_k$ is the largest subset in $\mathscr{J} \subseteq \mathscr{D} \setminus \cup_{l=1}^{k-1} \mathscr{D}_l$ satisfying equality, we have

$$f^w(\mathscr{D}_k, I) < f^w(\mathscr{D}_k \cup \mathscr{D}_{k+1}, I)$$

Let us define $N_{\mathscr{D}_k}^u = \{i \in J_k : w_i f^w(\mathscr{D}_k, I) < 1\}$, $N_{\mathscr{D}_{k+1}}^u = \{i \in J_{k+1} : w_i f^w(\mathscr{D}_{k+1}, I \setminus \mathscr{N}_k^O) < 1\}$, and $N_{\mathscr{D}_k \cup \mathscr{D}_{k+1}}^u = \{i \in J_k \cup J_{k+1} : w_i f^w(\mathscr{D}_k \cup \mathscr{D}_{k+1}, I) < 1\}$. Let us also define $N_{\mathscr{D}_k}^m = J_k \setminus N_{\mathscr{D}_k}^u$, $N_{\mathscr{D}_{k+1}}^m = J_{k+1} \setminus N_{\mathscr{D}_{k+1}}^u$, and $N_{\mathscr{D}_k \cup \mathscr{D}_{k+1}}^m = (J_k \cup J_{k+1}) \setminus N_{\mathscr{D}_k \cup \mathscr{D}_{k+1}}^u$. The first definitions means sets of unmatched agents, receiving less than one utility. The second definitions means the sets of matched agents, receiving one utility. Based on these definitions, the aggregate utility of each odds components is represented by

$$|N^m_{\mathscr{D}_k}| + f^w(\mathscr{D}_k, I) \sum_{i \in N^u_{\mathscr{D}_k}} w_i = |J_k| - (|\mathscr{D}_k| - |C(\mathscr{D}_k, I)|)$$

$$|N^m_{\mathscr{D}_{k+1}}| + f^w(\mathscr{D}_{k+1}, I \setminus \mathscr{N}^O_k) \sum_{i \in N^u_{\mathscr{D}_{k+1}}} w_i = |J_{k+1}| - (|\mathscr{D}_{k+1}| - |C(\mathscr{D}_{k+1}, I \setminus \mathscr{N}^O_k)|)$$

$$\begin{aligned} |N^m_{\mathscr{D}_k \cup \mathscr{D}_{k+1}}| + f^w(\mathscr{D}_k \cup \mathscr{D}_{k+1}, I) \sum_{i \in N^u_{\mathscr{D}_k \cup \mathscr{D}_{k+1}}} w_i &= |J_k \cup J_{k+1}| - (|\mathscr{D}_k \cup \mathscr{D}_{k+1}| - |C(\mathscr{D}_k \cup \mathscr{D}_{k+1}, I)|) \\ &= |J_k| + |J_{k+1}| - (|\mathscr{D}_k| + |\mathscr{D}_{k+1}| - |C(\mathscr{D}_k, I)| - |C(\mathscr{D}_{k+1}, I \setminus \mathscr{N}^O_k)|) \end{aligned}$$

The last equality is held by $C(\mathscr{D}_k \cup \mathscr{D}_{k+1}, I) = C(\mathscr{D}_k, I) \cup C(\mathscr{D}_{k+1}, I \setminus \mathscr{N}^O_k)$ and $C(\mathscr{D}_k, I) \cap C(\mathscr{D}_{k+1}, I \setminus \mathscr{N}^O_k) = \emptyset$. We get the fact that the aggregate utility of J_k and J_{k+1} is equal to the aggregate utility of $J_k \cup J_{k+1}$:

$$\begin{aligned} |N^m_{\mathscr{D}_k}| + f^w(\mathscr{D}_k, I) \sum_{i \in N^u_{\mathscr{D}_k}} w_i + |N^m_{\mathscr{D}_{k+1}}| + f^w(\mathscr{D}_{k+1}, I \setminus \mathscr{N}^O_k) \sum_{i \in N^u_{\mathscr{D}_{k+1}}} w_i \\ = |N^m_{\mathscr{D}_k \cup \mathscr{D}_{k+1}}| + f^w(\mathscr{D}_k \cup \mathscr{D}_{k+1}, I) \sum_{i \in N^u_{\mathscr{D}_k \cup \mathscr{D}_{k+1}}} w_i \end{aligned}$$

Notice that $N^m_{\mathscr{D}_k}, N^m_{\mathscr{D}_{k+1}} \subseteq N^m_{\mathscr{D}_k \cup \mathscr{D}_{k+1}}$ since the definition of f^w and $f^w(\mathscr{D}_{k+1}, \mathscr{N}^O_{k+1}) \leq f^w(\mathscr{D}_k, \mathscr{N}^O_k) < f^w(\mathscr{D}_k \cup \mathscr{D}_{k+1}, I)$. Thus, we have $N^m_{\mathscr{D}_k} \cup N^m_{\mathscr{D}_{k+1}} \subseteq N^m_{\mathscr{D}_k \cup \mathscr{D}_{k+1}}$ and $N^u_{\mathscr{D}_k \cup \mathscr{D}_{k+1}} \subseteq N^u_{\mathscr{D}_k} \cup N^u_{\mathscr{D}_{k+1}}$. Together these facts and $w_i f^w(\mathscr{D}_k, I) < 1$ for each $i \in N^u_{\mathscr{D}_k}$ and $w_i f^w(\mathscr{D}_{k+1}, I \setminus$

$\mathcal{N}_k^O) < 1$ for each $i \in N^u_{\mathscr{D}_{k+1}}$, we have

$$
\begin{aligned}
&|N^m_{\mathscr{D}_k}| + f^w(\mathscr{D}_k, I) \sum_{i \in N^u_{\mathscr{D}_k}} w_i + |N^m_{\mathscr{D}_{k+1}}| + f^w(\mathscr{D}_{k+1}, I \setminus \mathcal{N}_k^O) \sum_{i \in N^u_{\mathscr{D}_{k+1}}} w_i \\
\leq\ & |N^m_{\mathscr{D}_k \cup \mathscr{D}_{k+1}}| + f^w(\mathscr{D}_k, I) \sum_{i \in N^u_{\mathscr{D}_k} \cap (N^u_{\mathscr{D}_k \cup \mathscr{D}_{k+1}})} w_i + f^w(\mathscr{D}_{k+1}, I \setminus \mathcal{N}_k^O) \sum_{i \in N^u_{\mathscr{D}_{k+1}} \cap (N^u_{\mathscr{D}_k \cup \mathscr{D}_{k+1}})} w_i \\
&< |N^m_{\mathscr{D}_k \cup \mathscr{D}_{k+1}}| + f^w(\mathscr{D}_k \cup \mathscr{D}_{k+1}, I) \sum_{i \in N^u_{\mathscr{D}_k \cup \mathscr{D}_{k+1}}} w_i
\end{aligned}
$$

which contradicts the previous equality.

■

Lemma 13. *Let $\boldsymbol{\mathcal{U}}$ denote the set of feasible utility profiles. Let $u, v \in \boldsymbol{\mathcal{U}}$ such that $u \neq v$. If $L^w_u(p) = L^w_v(p), \forall p \in [0, \sum_{i \in \mathbf{N}} w_i]$ then $\alpha u + (1-\alpha)v, \alpha \in (0,1)$ Weighted Lorenz dominates u and v*

Proof. Let Σ be the permutation of patients such that for any $\sigma \in \Sigma$, $\frac{u_{\sigma_u(1)}}{w_{\sigma_u(1)}} \leq \frac{u_{\sigma_u(2)}}{w_{\sigma_u(2)}} \leq \ldots \leq \frac{u_{\sigma_u(n)}}{w_{\sigma_u(n)}}$ and $\frac{v_{\sigma_v(1)}}{w_{\sigma_v(1)}} \leq \frac{v_{\sigma_v(2)}}{w_{\sigma_v(2)}} \leq \ldots \leq \frac{v_{\sigma_v(n)}}{w_{\sigma_v(n)}}$. Note that, the Weighted Lorenz curve is independent of such permutations. Also

$$L^w_u(p) = L^w_v(p)$$

$$\sum_{j=1}^{i} u_{\sigma_u(j)} + (p - \sum_{j=1}^{i} w_{\sigma_u(j)}) \cdot \frac{u_{\sigma_u(i+1)}}{w_{\sigma_u(i+1)}} = \sum_{j=1}^{k} v_{\sigma_v(j)} + (p - \sum_{j=1}^{k} w_{\sigma_v(j)}) \cdot \frac{v_{\sigma_v(k+1)}}{w_{\sigma_v(k+1)}}$$

where $p \in [\sum_{j=1}^{i} w_{\sigma_u(j)}, \sum_{j=1}^{i+1} w_{\sigma_u(j)}]$ and $p \in [\sum_{j=1}^{k} w_{\sigma_v(j)}, \sum_{j=1}^{k+1} w_{\sigma_v(j)}]$. For any subset of patients $\{1,2,3,..,m+1\}$ such that $p \in [\sum_{j=1}^{m} w_j, \sum_{j=1}^{m+1} w_j]$ we have

$$\sum_{j=1}^{m} u_j + (p - \sum_{j=1}^{m} w_j) \cdot \frac{u_{m+1}}{w_{m+1}} \geq \sum_{j=1}^{i} u_{\sigma_u(j)} + (p - \sum_{j=1}^{i} w_{\sigma_u(j)}) \cdot \frac{u_{\sigma_u(i+1)}}{w_{\sigma_u(i+1)}}$$

$$\sum_{j=1}^{m} v_j + (p - \sum_{j=1}^{m} w_j) \cdot \frac{v_{m+1}}{w_{m+1}} \geq \sum_{j=1}^{i} v_{\sigma_v(j)} + (p - \sum_{j=1}^{i} w_{\sigma_v(j)}) \cdot \frac{v_{\sigma_v(i+1)}}{w_{\sigma_v(i+1)}}$$

Consider any set of patients $\{1,2,3,..,r+1\}$ such that $p \in [\sum_{j=1}^{r} w_j, \sum_{j=1}^{r+1} w_j]$, where

$$\frac{(\alpha u + (1-\alpha)v)_1}{w_1} \leq \frac{(\alpha u + (1-\alpha)v)_2}{w_2} \leq ... \leq \frac{(\alpha u + (1-\alpha)v)_{r+1}}{w_{r+1}}$$

Applying the previous results for the set $\{1,2,3,..,r+1\}$ we have

$$\sum_{j=1}^{r} u_j + (p - \sum_{j=1}^{r} w_j) \cdot \frac{u_{r+1}}{w_{r+1}} \geq \sum_{j=1}^{i} u_{\sigma_u(j)} + (p - \sum_{j=1}^{i} w_{\sigma_u(j)}) \cdot \frac{u_{\sigma_u(i+1)}}{w_{\sigma_u(i+1)}}$$

$$\sum_{j=1}^{r} v_j + (p - \sum_{j=1}^{r} w_j) \cdot \frac{v_{r+1}}{w_{r+1}} \geq \sum_{j=1}^{i} v_{\sigma_v(j)} + (p - \sum_{j=1}^{i} w_{\sigma_v(j)}) \cdot \frac{v_{\sigma_v(i+1)}}{w_{\sigma_v(i+1)}}$$

Pre-multiplying the equations by α and $(1-\alpha)$ implies that,

$$\sum_{j=1}^{r}(\alpha u+(1-\alpha)v)_j+(p-\sum_{j=1}^{r}w_j)\cdot\frac{(\alpha u+(1-\alpha)v)_{r+1}}{w_{r+1}}\geq\sum_{j=1}^{i}v_{\sigma_v(j)}+(p-\sum_{j=1}^{i}w_{\sigma_v(j)})\cdot\frac{v_{\sigma_v(i+1)}}{w_{\sigma_v(i+1)}}$$

$$\Rightarrow L^w_{\alpha u+(1-\alpha)v}(p)\geq L^w_v(p)$$

Let $\sigma^{'}\in\Sigma$ such that, there exists $i\in\{1,2,...,n\}$, $\forall j<i,\sigma^{'}_u(j)=\sigma^{'}_v(j)$ and $\sigma^{'}_u(i)\neq\sigma^{'}_v(i)$, $u_{\sigma^{'}_u(i)}\neq v_{\sigma^{'}_u(i)},u_{\sigma^{'}_v(i)}\neq v_{\sigma^{'}_v(i)}$ and there exists no $j>i$ such that, $u_{\sigma^{'}_u(j)}=v_{\sigma^{'}_u(j)}$ and $\frac{u_{\sigma^{'}_u(j)}}{w_{\sigma^{'}_u(j)}}=\frac{u_{\sigma^{'}_u(i)}}{w_{\sigma^{'}_u(i)}}$. The i'th patient under permutation $\sigma^{'}$ in utility profiles u and v are different and their utility levels in the two profiles are different and no patient higher than them who have similar utility to weight in both the profiles and utility to weight is equal to that of i'th patient in either of the profile. Such an i and $\sigma^{'}$ exists since $u\neq v$ and $L^w_u(p)=L^w_v(p)$.

Let $\alpha\in(0,1)$. Observe that, for $j<i$, and any permutation $\sigma\in\Sigma$

$$\frac{u_{\sigma^{'}_u(j)}}{w_{\sigma^{'}_u(j)}}=\frac{v_{\sigma^{'}_v(j)}}{w_{\sigma^{'}_v(j)}}=\frac{\alpha u+(1-\alpha)v_{\sigma_{\alpha u+(1-\alpha)v}(j)}}{w_{\sigma_{\alpha u+(1-\alpha)v}(j)}}$$

Since for $j > i-1$,

$$\frac{u_{\sigma'_u(j)}}{w_{\sigma'_u(j)}} \geq \frac{u_{\sigma'_u(i-1)}}{w_{\sigma'_u(i-1)}}, \frac{v_{\sigma'_v(j)}}{w_{\sigma'_v(j)}} \geq \frac{v_{\sigma'_v(i-1)}}{w_{\sigma'_v(i-1)}}$$

Given i, $\sigma'_u(i) \neq \sigma'_v(i)$, $u_{\sigma'_u(i)} \neq v_{\sigma'_u(i)}, u_{\sigma'_v(i)} \neq v_{\sigma'_v(i)}$ and there exists no $j > i$ such that, $u_{\sigma'_u(j)} = v_{\sigma'_u(j)}$ and $\frac{u_{\sigma'_u(j)}}{w_{\sigma'_u(j)}} = \frac{u_{\sigma'_u(i)}}{w_{\sigma'_u(i)}}$.

$$\Rightarrow j = i, \alpha u_{\sigma'_u(j)} + (1-\alpha) v_{\sigma'_u(j)} > u_{\sigma'_u(i)}$$

$$\alpha u_{\sigma'_v(j)} + (1-\alpha) v_{\sigma'_v(j)} > v_{\sigma'_v(i)}$$

and

$$\forall j > i, \frac{u_{\sigma'_u(j)}}{w_{\sigma'_u(j)}} > \frac{u_{\sigma'_u(i)}}{w_{\sigma'_u(i)}}$$

$$\frac{v_{\sigma'_v(j)}}{w_{\sigma'_v(j)}} > \frac{v_{\sigma'_v(i)}}{w_{\sigma'_v(i)}}$$

$$\Rightarrow \frac{\alpha u + (1-\alpha) v_{\sigma_{\alpha u + (1-\alpha)v}(i)}}{w_{\sigma_{\alpha u + (1-\alpha)v}(i)}} > \frac{u_{\sigma'_u(i)}}{w_{\sigma'_u(i)}} = \frac{v_{\sigma'_v(i)}}{w_{\sigma'_v(i)}}$$

Let $w_i = min\{w_{\sigma'_u(i)}, w_{\sigma'_v(i)}\}$.Let $p = \sum_{j \leq \mathbf{i-1}} w_{\sigma'_u(j)} + w_i$, then

$$L_u^w(p) = L_v^w(p) < L_{\alpha u+(1-\alpha)v}^w(p)$$

$\Rightarrow \alpha u + (1-\alpha)v$ Lorenz dominates utility profiles u, v

■

We will show by construction, the profile of utilities obtained by Weighted Egalitarian mechanism, weighted Lorenz dominates all feasible utility profiles. For any given $(\mathbf{N}, \mathbf{R}, \mathbf{w})$, let $f_1^w, f_2^w, f_3^w, \ldots, f_n^w$ be the induced sequence of f values produced by the Weighted Egalitarian mechanism. Consider the set of patients w^1 such that

$$w^1 = \left\{ i : w_i.f_1^w > 1 \right\}$$

Let w^2 be the set of patients in step 1 of Weighted Egalitarian mechanism that doesn't belong to w^1.For any patient i in w^2,

$$\frac{u_i}{w_i} = f_1^w < f_2^w < \ldots < f_n^w$$

The patients in w^1 and w^2 include all the patients in Step 1 of Weighted Egalitarian mechanism. Therefore patients in w^1 and w^2 can be broken down into two sets,

$$w^1 \cup w^2 = w_{WE}^{Step1} \cup w_{WE}^{Step1'}$$

patient who belong to Step 1 of Weighted Egalitarian mechanism and patients who don't. Note that, patient i who belong to $w_{WE}^{Step1'}$ and not to step 1 of Weighted Egalitarian mechanism has an utility of 1 in Weighted Egalitarian mechanism. By previous lemma f's of a Weighted Egalitarian mechanism are an increasing sequence. Therefore patients in $w_{WE}^{Step1'}$ are assigned an utility of 1 by Weighted Egalitarian mechanism. Also

$$i \in w_{WE}^{Step1'} \Rightarrow f_1^w > \frac{1}{w_i} = \frac{u_i}{w_i}$$

Therefore, the total utility of patients in $w^1 \cup w^2$ is

$$\sum_{i \in w^1 \cup w^2} u_i = \sum_{j \in w_{WE}^{Step1}} u_j + \sum_{k \in w_{WE}^{Step1'}} u_k$$

maximum in Weighted Egalitarian mechanism since all feasible neighbors are allocated to patients in Step 1 of Weighted Egalitarian mechanism. This implies that for any utility profile the total sum of utilities in weighted Lorenz curve, from 0 upto weight $w_s^1 + w_s^2$ is always lesser than or equal to $\sum_{i \in w^1 \cup w^2} u_i$.

WLOG, let $w_1^1 \geq w_2^1 \geq w_3^1 \geq \ldots \geq w_n^1$ be the set of patients in w^1. Observe that,

$$\frac{u_1^1}{w_1^1} \leq \frac{u_2^1}{w_2^1} \leq \frac{u_3^1}{w_3^1} \leq \ldots \leq \frac{u_n^1}{w_n^1}$$

Let,

$$w_s^1 = \sum_{w_i^1 \in w^1} w_i^1, w_s^2 = \sum_{w_i^2 \in w^2} w_i^2$$

Observe that

$$\frac{(u_i^1)^{WE}}{w_i} < \frac{(u_j)}{w_j}, j \in w^2$$

Let u' be any other feasible utility profile. Note that, for any patient i in w^1,

$$\frac{(u_i^1)^{WE}}{w_i} \geq \frac{u_i'}{w_i}$$

since patient i in w^1 is allocated an utility 1 by the Weighted Egalitarian mechanism. This implies that, the weighted Lorenz curve of Weighted Egalitarian mechanism dominates the weighted Lorenz curve of all feasible utility profiles in the domain $[0, w_s^1]$.

Since for any utility profile, the weighted Lorenz curve at $w_s^1 + w_s^2$ is always lesser than or equal to $\sum_{i \in w^1 \cup w^2} u_i$. This implies the allocation Weighted Egalitarian mechanism, Weighted Lorenz dominates all feasible utility profiles in the domain of $[0, w_s^1 + w_s^2]$. (No profile can *u*ndominate, since such a profile won't be feasible).

Extending this argument inductively, it can be established that the allocation of Weighted Egalitarian mechanism, Weighted Lorenz dominates all feasible utility profiles.

■

Theorem 3: The weighted egalitarian mechanism makes it dominant strategy for a patient to reveal both, (a) the available set of donors, (b) her full set of acceptable kidneys

Proof. Let $\mathcal{U}$ denote the set of feasible utility profiles. The following lemmas will be helpful.

Lemma 14. *Let* $\mathbf{u}, \mathbf{v} \in \mathcal{U}$ *be such that* $\mathbf{u}$ *Weighted Lorenz dominates* $\mathbf{v}$. *Then for any*

$\alpha \in (0,1)$, the utility profile $\alpha\mathbf{u} + (1-\alpha)\mathbf{v}$ Weighted Lorenz dominates $\mathbf{v}$.

Proof. Let $\mathbf{z} = \alpha.\mathbf{u} + (1-\alpha).\mathbf{v}$. Since $\mathbf{u}, \mathbf{v} \in \mathcal{U}$ there exist lotteries $\lambda, \gamma \in \mathcal{L}$ that induce $\mathbf{u}, \mathbf{v}$. Let $\phi = \alpha\lambda + (1-\alpha)\gamma$. For any patient i,

$$u_i(\phi) = \alpha.u_i(\lambda) + (1-\alpha)u_i(\gamma) = \alpha.\mathbf{u}_i + (1-\alpha).\mathbf{v}_i = z_i$$

This implies that, there exists a probability distribution over Pareto efficient matchings which will give the utility profile generated by $\alpha\mathbf{u} + (1-\alpha)\mathbf{v}$.

Consider any arbitrary weight $\mathbf{p} \in [0, \sum_{i \in \mathbf{N}} w_i]$. Let σ_v, σ_u be the permutation of agents such that, $\frac{u_{\sigma_u(1)}}{w_{\sigma_u(1)}} \leq \frac{u_{\sigma_u(2)}}{w_{\sigma_u(2)}} \leq \dots \leq \frac{u_{\sigma_u(n)}}{w_{\sigma_u(n)}}$. and $\frac{v_{\sigma_v(1)}}{w_{\sigma_v(1)}} \leq \frac{v_{\sigma_v(2)}}{w_{\sigma_v(2)}} \leq \dots \leq \frac{v_{\sigma_v(n)}}{w_{\sigma_v(n)}}$.

Since $\mathbf{u}$ Weighted-Lorenz dominates $\mathbf{v}$, we know that

$$L_u^w(p) \geq L_v^w(p)$$

$$\sum_{j=1}^{i} u_{\sigma_u(j)} + (p - \sum_{j=1}^{i} w_{\sigma_u(j)}) \cdot \frac{u_{\sigma_u(i+1)}}{w_{\sigma_u(i+1)}} \geq \sum_{j=1}^{k} v_{\sigma_v(j)} + (p - \sum_{j=1}^{k} w_{\sigma_v(j)}) \cdot \frac{v_{\sigma_v(k+1)}}{w_{\sigma_v(k+1)}}$$

where $p \in [\sum_{j=1}^{i} w_{\sigma_u(j)}, \sum_{j=1}^{i+1} w_{\sigma_u(j)}]$ and $p \in [\sum_{j=1}^{k} w_{\sigma_v(j)}, \sum_{j=1}^{k+1} w_{\sigma_v(j)}]$. For any subset of patients $\{1,2,3,..,m+1\}$ such that $p \in [\sum_{j=1}^{m} w_j, \sum_{j=1}^{m+1} w_j]$ we have

$$\sum_{j=1}^{m} u_j + (p - \sum_{j=1}^{m} w_j) \cdot \frac{u_{m+1}}{w_{m+1}} \geq \sum_{j=1}^{i} u_{\sigma_u(j)} + (p - \sum_{j=1}^{i} w_{\sigma_u(j)}) \cdot \frac{u_{\sigma_u(i+1)}}{w_{\sigma_u(i+1)}}$$

$$\sum_{j=1}^{m} v_j + (p - \sum_{j=1}^{m} w_j) \cdot \frac{v_{m+1}}{w_{m+1}} \geq \sum_{j=1}^{i} v_{\sigma_v(j)} + (p - \sum_{j=1}^{i} w_{\sigma_v(j)}) \cdot \frac{v_{\sigma_v(i+1)}}{w_{\sigma_v(i+1)}}$$

Consider any set of patients $\{1,2,3,..,r+1\}$ such that $p \in [\sum_{j=1}^{r} w_j, \sum_{j=1}^{r+1} w_j]$, where

$$\frac{(\alpha u + (1-\alpha)v)_1}{w_1} \leq \frac{(\alpha u + (1-\alpha)v)_2}{w_2} \leq ... \leq \frac{(\alpha u + (1-\alpha)v)_{r+1}}{w_{r+1}}$$

Applying the previous results for the set $\{1,2,3,..,r+1\}$ we have

$$\sum_{j=1}^{r} u_j + (p - \sum_{j=1}^{r} w_j) \cdot \frac{u_{r+1}}{w_{r+1}} \geq \sum_{j=1}^{i} u_{\sigma_u(j)} + (p - \sum_{j=1}^{i} w_{\sigma_u(j)}) \cdot \frac{u_{\sigma_u(i+1)}}{w_{\sigma_u(i+1)}}$$

$$\sum_{j=1}^{r} v_j + (p - \sum_{j=1}^{r} w_j) \cdot \frac{v_{r+1}}{w_{r+1}} \geq \sum_{j=1}^{i} v_{\sigma_v(j)} + (p - \sum_{j=1}^{i} w_{\sigma_v(j)}) \cdot \frac{v_{\sigma_v(i+1)}}{w_{\sigma_v(i+1)}}$$

Pre-multiplying the equations by α and $(1-\alpha)$ implies that,

$$\sum_{j=1}^{r}(\alpha u+(1-\alpha)v)_j+(p-\sum_{j=1}^{r}w_j)\cdot\frac{(\alpha u+(1-\alpha)v)_{r+1}}{w_{r+1}}\geq\sum_{j=1}^{i}v_{\sigma_v(j)}+(p-\sum_{j=1}^{i}w_{\sigma_v(j)})\cdot\frac{v_{\sigma_v(i+1)}}{w_{\sigma_v(i+1)}}$$

$$\Rightarrow L^w_{\alpha u+(1-\alpha)v}(p)\geq L^w_v(p)$$

Similarly it can be shown that there exists p such that, $L^w_{\alpha u+(1-\alpha)v}(p)>L^w_v(p)$

■

We will introduce additional notations which will help in completing the proof. For any problem $(\mathbf{N},\mathbf{R},\mathbf{w})$ given a set of patients $\mathbf{N},\mathbf{w}$, let

$$\mathbf{J}_k(\mathbf{R})=\bigcup_{J\in\mathcal{D}_k}J\quad\text{and}\quad e(\mathbf{R})=\max_{\mu\in\mathcal{M}(\mathbf{R})}|\mu|$$

From lemma 3, if $\mu\in\mathcal{M}(\mathbf{R})$, then $\mu\in\mathcal{E}(\mathbf{R})\iff|\mu|=e(\mathbf{R})$. For a given compatibility matrix $\mathbf{R}$, and any two subsets of patients $I,J\in\mathbf{N}$ define, the neighbors of J among I as

$$\mathbf{C}(J, I, \mathbf{R}) = \{i \in I \setminus J : r_{i,j} = 1 \quad \text{for some} \quad j \in J\}$$

When J = $\{j\}$, for notational convenience we use $\mathbf{C}(j, I, \mathbf{R})$ instead of $\mathbf{C}(\{j\}, I, \mathbf{R})$.

For a Weighted Egalitarian mechanism ϕ^{WE} construct the Gallai-Edmonds decomposition $\{\mathbf{N^U}(\mathbf{R}), \mathbf{N^O}(\mathbf{R}), \mathbf{N^P}(\mathbf{R})\}$ for a given set of patients $\mathbf{N}$ and weights $\mathbf{w}$. Let $\mathcal{D}(\mathbf{R})$ represent the associated partition $\{\mathcal{D}_1(\mathbf{R}), \mathcal{D}_2(\mathbf{R}), ..., \mathcal{D}_q(\mathbf{R})\}$ of $\mathbf{N^U}(\mathbf{R})$ in the Weighted Egalitarian mechanism. Similarly, $\{\mathbf{N^O_1}(\mathbf{R}), \mathbf{N^O_2}(\mathbf{R}), ..., \mathbf{N^O_q}(\mathbf{R})\}$ is the associated partition of $\mathbf{N^O}(\mathbf{R})$. Let $u^{WE}(\mathbf{R})$ be the associated utility profile of Weighted Egalitarian mechanism.

Since any patient in $\mathbf{N^O}(\mathbf{R}) \cup \mathbf{N^P}(\mathbf{R})$ are always matched with probability 1, the don't have any incentive to deviate. Let $j \in \mathbf{N^U}(\mathbf{R})$ be such that $u_j^{WE}(\mathbf{R}) < 1$. We will show that patient j cannot increase her utility by declaring a mutually compatible patient to be incompatible. Then we can inductively extend the argument to show that patient j cannot increase her utility by declaring a subset of patients incompatible. Let J be the associated odd component of j.

Let $j^{'} \in \mathbf{C}(j, I, \mathbf{R})$. Either $j^{'} \in J$ or $j^{'} \in \mathbf{N^O}(\mathbf{R})$. Let $\mathbf{Q}$ be the reduced problem obtained by patient j declaring patient $j^{'}$ incompatible. Then $\mathbf{C}(j, \mathbf{N}, \mathbf{Q}) = \mathbf{C}(j, \mathbf{N}, \mathbf{R}) \setminus \{j^{'}\}, \mathbf{C}(j^{'}, \mathbf{N}, \mathbf{Q}) = \mathbf{C}(j^{'}, \mathbf{N}, \mathbf{R}) \setminus \{j\}, \mathbf{C}(i, \mathbf{N}, \mathbf{Q}) = \mathbf{C}(i, \mathbf{N}, \mathbf{R})$ for all $i \in \mathbf{N} \setminus \{j, j^{'}\}$ and $\mathcal{M}(\mathbf{Q}) = \{\mu \in \mathcal{M}(\mathbf{R}) : \mu(j) \neq j^{'}\}$. Construct the Gallai-Edmonds decomposition $\{\mathbf{N^U}(\mathbf{Q}), \mathbf{N^O}(\mathbf{Q}), \mathbf{N^P}(\mathbf{Q})\}$, for the modified problem where $\mathcal{D}(\mathbf{Q}) \equiv \{\mathcal{D}_1(\mathbf{Q}), \mathcal{D}_2(\mathbf{Q}), \ldots, \mathcal{D}_r(\mathbf{Q})\}$ is the Weighted Egalitarian partition of $\mathbf{N^U}(\mathbf{Q})$ and $\{\mathbf{N^O_1}(\mathbf{Q}),$
$\mathbf{N^O_2}(\mathbf{Q}), \ldots, \mathbf{N^O_r}(\mathbf{Q})\}$ is the associated partition of $\mathbf{N^O}(\mathbf{Q})$. The following Lemmas will be helpful in the proof.

Lemma 15. *(Roth, Sönmez and Ünver(2004)[5]): (i)* $e(\mathbf{Q}) = e(\mathbf{R})$

(ii) $\mathcal{E}(\mathbf{Q}) \subseteq \mathcal{E}(\mathbf{R})$ *and* $\mu \in \mathcal{E}(\mathbf{R}) \bigcap \mathcal{M}(\mathbf{Q}) \Rightarrow \mu \in \mathcal{E}(\mathbf{Q})$

Lemma 16. *(Roth, Sönmez and Ünver(2004)[5]):* $\mathbf{N^O}(\mathbf{R}) \subseteq \mathbf{N^O}(\mathbf{Q}) \bigcup \mathbf{N^P}(\mathbf{Q})$ *and* $\mathbf{N^U}(\mathbf{R}) \subseteq \mathbf{N^U}(\mathbf{Q})$

Lemma 17. *If* $u^{WE}_{j^{'}}(\mathbf{Q}) < 1$*, then* $\dfrac{u^{WE}_{j^{'}}(\mathbf{Q})}{w_{j^{'}}} \geq \dfrac{u^{WE}_{j}(\mathbf{Q})}{w_j}$

Proof. The lemma states that per weight allocation of utility for patient $j^{'}$ is at least as high as the per weight allocation of utility for patient j, when patient $j^{'}$'s utility is less than 1.

If $j' \in \mathbf{N^O}(\mathbf{R})$, then by previous lemma $j' \in \mathbf{N^O}(\mathbf{Q}) \bigcup \mathbf{N^P}(\mathbf{Q})$. Thus utility of patient j' is constant in both the cases being equal to 1.

Suppose $\frac{u_{j'}^{WE}(\mathbf{Q})}{w_{j'}} < \frac{u_{j}^{WE}(\mathbf{Q})}{w_{j}}$. Let $j' \in \mathbf{N^U}(\mathbf{Q})$. Since $u_{j'}^{WE}(\mathbf{Q}) < 1$, there exists a Pareto efficient matching $\mu \in \boldsymbol{\mathcal{E}}(\mathbf{Q})$ in the support of lottery chosen by Weighted Egalitarian mechanism, such that $\mu \in \phi^{WE}(Q), \mu(j') = j'$. By previous lemma, $\mu \in \boldsymbol{\mathcal{E}}(\mathbf{R})$. Let $j' \in \mathbf{J} \in \boldsymbol{\mathcal{D}}(\mathbf{Q})$. Since $u_{j'}^{WE}(\mathbf{Q}) < 1$, and by GED lemma at most one patient in $\mathbf{J}$ remains unmatched,$\mu(j) = \mathbf{J} \setminus \{j\}$. Again by GED lemma, there exists a Pareto efficient matching $\nu \in \boldsymbol{\mathcal{E}}(\mathbf{R})$, that matches the same patients and j' instead of j. $\nu(j) = j$ and hence $\nu \in \boldsymbol{\mathcal{M}}(\mathbf{Q})$. Let $0 < \epsilon \leq \min\left\{\phi_{\mu}^{WE}(\mathbf{Q}), \frac{\frac{u_{j}^{WE}(\mathbf{Q})}{w_{j}} - \frac{u_{j'}^{WE}(\mathbf{Q})}{w_{j'}}}{2}\right\}$. Construct a lottery λ from $\phi^{WE}(\mathbf{Q})$ by subtracting ϵ from μ and adding ϵ to ν

$$u_{h}^{WE}(\lambda) = \begin{cases} u_{h}^{WE}(\mathbf{Q}) - \epsilon, h = j \\ u_{h}^{WE}(\mathbf{Q}) + \epsilon, h = j' \\ u_{h}^{WE}(\mathbf{Q}), otherwise \end{cases}$$

Note that, $u^{WE}(\lambda)$ Weighted Lorenz dominates u^{WE} contradicting Theorem 2. Therefore if $u^{WE}_{j'}(\mathbf{Q}) < 1$, then $\dfrac{u^{WE}_{j'}(\mathbf{Q})}{w_{j'}} \geq \dfrac{u^{WE}_{j}(\mathbf{Q})}{w_j}$ ■

Suppose $u^{WE}_j(\mathbf{Q}) > u^{WE}_j(\mathbf{R})$. Let $\mathbf{J} \in \mathcal{D}_k(\mathbf{R})$. Since $\phi^{WE}(\mathbf{Q})$ is an ex post efficient lottery under $\mathbf{Q}$, $\phi^{WE}(\mathbf{R})$ is an ex post efficient lottery under $\mathbf{R}$, and $e(\mathbf{Q}) = e(\mathbf{R})$ we have,

$$\sum_{i \in \mathbf{N}} u^{WE}_i(\mathbf{Q}) = e(\mathbf{Q}) = e(\mathbf{R}) = \sum_{i \in \mathbf{N}} u^{WE}_i(\mathbf{R})$$

Since $u^{WE}_j(\mathbf{Q}) > u^{WE}_j(\mathbf{R})$, there exists an $h \in \mathbf{N^U}(\mathbf{Q})$ such that $u^{WE}_h(\mathbf{Q}) < u^{WE}_h(\mathbf{R})$. This implies, $u^{WE}_h(\mathbf{Q}) < 1$. Let $\left\{\bigcup_{i=1}^{k} \mathcal{D}_i(\mathbf{R}), \bigcup_{i=k+1}^{r} \mathcal{D}_i(\mathbf{R})\right\}$ be a partition of the *underdemanded* patients $\mathbf{N^U}(\mathbf{R})$.

Consider the *underdemanded* patients in $\bigcup_{i=k+1}^{r} \mathcal{D}_k(\mathbf{R})$. For any Pareto efficient matching in the support of Weighted Egalitarian mechanism, *underdemanded* patients in $\bigcup_{i=k+1}^{r} \mathcal{D}_i(\mathbf{R})$ are matched with *overdemanded* patients $\bigcup_{i=k+1}^{r} \mathbf{N^O}_i(\mathbf{R})$. *Underdemanded* patients in $\bigcup_{i=1}^{k} \mathcal{D}_i(\mathbf{R})$ are matched first in Weighted Egalitarian mechanism, they are incompatible with *overdemanded* patients $\bigcup_{i=k+1}^{r} \mathbf{N^O}_i(\mathbf{R})$ and they don't lie in the odd components belonging to $\bigcup_{i=k+1}^{r} \mathcal{D}_i(\mathbf{R})$. Therefore for any

$i \in \bigcup_{j=1}^{k} \mathcal{D}_j(\mathbf{R})$,

$$\mathbf{C}(i,\mathbf{N},\mathbf{R}) \bigcap \bigcup_{j=k+1}^{r} (\mathcal{D}_j(\mathbf{R}) \bigcup \mathbf{N_j^O}(\mathbf{R})) = \varnothing$$

Therefore, *underdemanded* patients in $\bigcup_{i=k+1}^{r} \mathcal{D}_i(\mathbf{R})$ cannot have an utility reduction under modified preferences $\mathbf{Q}$. This implies that, there exists an *underdemanded* patient $h \in \bigcup_{j=1}^{k} \mathcal{D}_j(\mathbf{R})$ such that, $u_h^{WE}(\mathbf{Q}) < u_h^{WE}(\mathbf{R})$.

Since $j \in \mathbf{J} \in \mathcal{D}_k(\mathbf{R})$, by the construction of utility profile in Weighted Egalitarian mechanism it should be that, $\dfrac{u_h^{WE}(\mathbf{R})}{w_h} \leq \dfrac{u_j^{WE}(\mathbf{R})}{w_j}$. By assumption $u_j^{WE}(\mathbf{Q}) > u_j^{WE}(\mathbf{R})$ and $u_h^{WE}(\mathbf{Q}) < 1$. Also, lets assume $u_{j'}^{WE} < 1$. By previous lemma, and from above conclusions we have,

$$\frac{u_h^{WE}(\mathbf{Q})}{w_h} < \frac{u_h^{WE}(\mathbf{R})}{w_h} \leq \frac{u_j^{WE}(\mathbf{R})}{w_j} < \frac{u_j^{WE}(\mathbf{Q})}{w_j} \leq \frac{u_{j'}^{WE}(\mathbf{Q})}{w_{j'}} \tag{2.5}$$

Let $\phi = \phi^{WE}(\mathbf{R})$ and $\psi = \phi^{WE}(\mathbf{Q})$. Let's construct the lottery over matchings λ given ϕ as follows. For any matching μ in the support of ϕ,

1. If $\mu(j) \neq j'$, then the matching μ has the same probability of being selected in

λ

2. If $\mu(j) = j^{'}$, then

   a) Construct the matching $\mu_{-j,j'}$ by breaking the match between j and $j^{'}$ and retaining all the other matches in μ.

   b) For each such matching $\mu \in \phi$, replace with $\mu_{-j,j'}$ in the lottery λ.

Observe that, λ is feasible under the modified preference $\mathbf{Q}$, and $u_i(\lambda) = u_i(\phi) = u_i^{WE}(\mathbf{R})$ for all $i \in \mathbf{N} \setminus \{j, j^{'}\}$.Given $\epsilon \in (0,1)$, let

$$\gamma^{\epsilon} = \epsilon\phi + (1-\epsilon)\psi \quad \text{and} \quad \lambda^{\epsilon} = \epsilon\lambda + (1-\epsilon)\psi$$

By construction of λ from $\phi, u_i(\lambda^{\epsilon}) = u_i(\gamma^{\epsilon})$ for all $i \in \mathbf{N} \setminus \{j, j^{'}\}$. The utility profile generated by ψ is feasible under original preference $\mathbf{R}$. By Theorem 2, the utility profile generated by ϕ Weighted Lorenz dominates the utility profile generated by lottery ψ and by Lemma 13, the utility profile generated by γ^{ϵ} Weighted Lorenz dominates ψ. Pick $\epsilon \in (0,1)$ small enough such that, $\frac{u_j(\lambda^{\epsilon})}{w_j} > \frac{u_h(\lambda^{\epsilon})}{w_h} \equiv \frac{u_h(\gamma^{\epsilon})}{w_h}$ and $\frac{u_{j'}(\lambda^{\epsilon})}{w_{j'}} > \frac{u_h(\lambda^{\epsilon})}{w_h} \equiv \frac{u_h(\gamma^{\epsilon})}{w_h}$. This is possible, given Equation(5).

Consider permutation of patients $\sigma_{\gamma^\epsilon}\sigma_{\lambda^\epsilon} \in \Sigma$ the set of possible of permutations of patients, such that $\frac{u_{\sigma_{\gamma^\epsilon}(1)}}{w_{\sigma_{\gamma^\epsilon}(1)}} \leq \frac{u_{\sigma_{\gamma^\epsilon}(2)}}{w_{\sigma_{\gamma^\epsilon}(2)}} \leq \ldots \leq \frac{u_{\sigma_{\gamma^\epsilon}(n)}}{w_{\sigma_{\gamma^\epsilon}(n)}}$. and $\frac{u_{\sigma_{\lambda^\epsilon}(1)}}{w_{\sigma_{\lambda^\epsilon}(1)}} \leq \frac{u_{\sigma_{\lambda^\epsilon}(2)}}{w_{\sigma_{\lambda^\epsilon}(2)}} \leq \ldots \leq \frac{u_{\sigma_{\lambda^\epsilon}(n)}}{w_{\sigma_{\lambda^\epsilon}(n)}}$. The Weighted Lorenz Curve is impervious to such permutations. Let m, n be such that the patient h is the m'th and n'th person under the permutation, (i.e.)

$$h = \sigma_{\gamma^\epsilon}(m) \quad \text{and} \quad h = \sigma_{\lambda^\epsilon}(n)$$

Since γ^ϵ Weighted Lorenz dominates ψ we have

$$L^w_{u(\gamma^\epsilon)}(p) \geq L^w_{u(\psi)}(p), \text{for all } p \in [0, \sum_{i \in \mathbf{N}} w_i]$$

Since only patients j, j' are affected between lotteries $\gamma^\epsilon, \lambda^\epsilon$. Also by Equation (5), the definition of permutations $\sigma_{\gamma^\epsilon}, \sigma_{\lambda^\epsilon}$ and the choice of ϵ implies,

$$j = \sigma_{\gamma^\epsilon}(k) \Rightarrow k > m \quad \text{and} \quad j' = \sigma_{\gamma^\epsilon}(k') \Rightarrow k' > m \quad \text{and}$$

$$j = \sigma_{\lambda^\epsilon}(l) \Rightarrow l > m \quad \text{and} \quad j' = \sigma_{\lambda^\epsilon}(l') \Rightarrow l' > m$$

$$\text{Let, } W = \left\{ i \in \mathbf{N} \quad | \quad \frac{u_{\lambda^\epsilon}(i)}{w_i} \leq \frac{u_{\lambda^\epsilon}(h)}{w_h} \right\}$$

$$w_* = \sum_{i \in W} w_i$$

Hence in the interval $[0, w_*]$,

$$L^w_{u(\gamma^\epsilon)}(p) = L^w_{u(\lambda^\epsilon)}(p) \geq L^w_{u(\psi)}(p)$$

Since λ^ϵ is feasible under modified preference $\mathbf{Q}$ and by Theorem 2, ψ Weighted Lorenz dominates all feasible utility profiles under modified preferences, it should be that

$$L^w_{u(\lambda^\epsilon)}(p) = L^w_{u(\psi)}(p), p \in [0, w_*]$$

Note that there exists at least one patient with different utilities in $u(\lambda^\epsilon)$ and $u(\psi)$ since patient h has different utilities in both the profiles, lying in $[0, w_*]$ of the Weighted Lorenz curve. Consider a convex combination such that $\alpha\lambda^\epsilon + (1-\alpha)\psi$. There exists patients i, i' such that $\frac{u^\psi_i}{w_i} = \frac{u^{\lambda^\epsilon}_{i'}}{w_{i'}}$ and their utilities under $\alpha\lambda^\epsilon + (1-\alpha)\psi$ and ψ are different . This along with above equation implies that there exist a weight $p \in [0, w_*]$ such that $L^w_{\alpha\lambda^\epsilon + (1-\alpha)\psi}(p) > L^w_\psi(p)$. This is a contradiction since ψ

Weighted Lorenz dominates $\alpha\lambda^{\epsilon}+(1-\alpha)\psi$.

The same argument holds with slight modification when utility of patient $j^{'}$, $u_{j^{'}}^{WE}(\mathbf{Q})=$ 1

■

Theorem 5: Consider a sequence of pairwise kidney exchange problems $(\mathbf{N},\mathbf{R},\mathbf{w^{(n)}})$ such that for all $i>j$, $\frac{w_i^n}{w_j^n}=o(1)$. Let ϕ_n^{WE} be the sequence of Weighted Egalitarian mechanism allocations. Let ψ be the allocation chosen by the Priority mechanism where patient i has higher priority than patient j, if $i<j$. Then

$$\lim_{n\to\infty}\phi_n^{WE}=\psi$$

Proof. Consider the induced two sided matching problem $(\mathbf{N^O},\mathcal{D},\tilde{\mathbf{R}},\mathbf{w^n_{N\setminus N^P}})$ for a given $(\mathbf{N},\mathbf{R},\mathbf{w^{(n)}})$. Let $\mathbf{N^U}$ be the set of underdemanded patients. For any coalition $\mathcal{J}\subseteq D$ of odd components of underdemanded patients and a subset $I\subseteq\mathbf{N^O}$, if $\mathcal{C}(\mathcal{J},I)\geq|\mathcal{J}|$, then $f^{w^n}(\mathcal{J},I)=1$. Suppose $\mathcal{C}(\mathcal{J},I)<|\mathcal{J}|$. Let patient i be such that

$$i\in\bigcup_{J\in\mathcal{J}}J\quad\text{and}\quad w_i^n<w_j^n, j\in\bigcup_{J\in\mathcal{J}}J$$

$$\text{For large n} \quad , \frac{u_i^{w^n}}{u_j^{w^n}} = \frac{w_i^n}{w_j^n} \quad \text{for} \quad u_j \neq 1$$

$$\text{As} \quad n \to \infty, \frac{w_i^n}{w_j^n} \to 0 \quad \text{if} \quad i > j$$

Let $\mathcal{V}_J^{n_i} = \{j \in \bigcup_{J \in \mathcal{J}} J \mid w_j^n > w_i^n\}$. $\mathcal{V}_J^{n_i}$ captures the set of underdemanded patients in with weights greater than weight of patient i in coalition $\mathcal{J}$. Let $\mathcal{W}_{\mathcal{J}}^n = \{i \in \bigcup_{J \in \mathcal{J}} J \mid \mathcal{V}_J^{n_i} \geq \mid j \in \bigcup_{J \in \mathcal{J}} J \mid -(\mid \mathcal{J} \mid - \mid \mathcal{C}(\mathcal{J}, I) \mid)\}$. Let $\mathcal{Q}_{\mathcal{J}}^n = \{j \mid j \in \bigcup_{J \in \mathcal{J}} J \setminus \mathcal{W}_{\mathcal{J}}^n\}$. $\mathcal{W}_{\mathcal{J}}^n$ captures the set of underdemanded patients excess to the possible number of matches with least weights, belonging to the coalition $\mathcal{J}$ of odd components. WLOG, we will refer $\mathcal{W}_{\mathcal{J}}^n$ as the set of *excess underdemanded patients* for coalition $\mathcal{J}$.

$$i \in \mathcal{W}_{\mathcal{J}}^n \Rightarrow \frac{u_i^{w^n}}{u_j^{w^n}} \to 0, j \in \mathcal{Q}_{\mathcal{J}}^n$$

Claim 1: There exists a large n such that for all $l \geq n$, $i \in \mathcal{Q}_{\mathcal{D}_k}^n, j \in \mathcal{W}_{\mathcal{D}_m}^n$, k, m, be the arbitrary steps of Weighted Egalitarian allocation such that

$$u_i^{WE(l)} > u_j^{WE(l)}$$

Proof. WLOG, take any arbitrary step k and m. WLOG, let $w(Q^n_{\mathcal{D}_k}(1)) \geq w(Q^n_{\mathcal{D}_k}(2)) \geq .. \geq w(Q^n_{\mathcal{D}_k}(r))$. Similarly, let $w(W^n_{\mathcal{D}_m}(1)) \geq w(W^n_{\mathcal{D}_m}(2)) \geq .. \geq w(Q^n_{\mathcal{D}_m}(s))$.

Note that,

$$u^{WE(n)}_{Q^n_{\mathcal{D}_k}(1)} = \frac{w_{Q^n_{\mathcal{D}_k}(1)}}{w(Q^n_{\mathcal{D}_k}(1)) + .. + w(Q^n_{\mathcal{D}_k}(r)) + ... + w(W^n_{\mathcal{D}_k}(t))}.(|\bigcup_{J \in \mathcal{D}_k} J| - (|\mathcal{D}_k| - \mathcal{C}(\mathcal{D}_k, \mathbf{N^O} \setminus \bigcup_{j=1}^{l} \mathbf{N^O_j}))$$

Fix any epsilon $\epsilon > 0$ such that $1 + |\mathbf{N}|\epsilon < 2$. Let n_1 be such that,

$$\frac{w_{Q^{n_1}_{\mathcal{D}_k}(1)}}{w(Q^{n_1}_{\mathcal{D}_k}(1)) + .. + w(Q^{n_1}_{\mathcal{D}_k}(r)) + ... + w(W^{n_1}_{\mathcal{D}_k}(t))} \geq \frac{1}{1 + |\mathbf{N}|\epsilon}$$

$$\Rightarrow u^{WE(n_1)}_{Q^{n_1}_{\mathcal{D}_k}(1)} = 1$$

Proceeding inductively with the argument we have

$$\Rightarrow u^{WE(n_1)}_{Q^{n_1}{}_{\mathcal{D}_k}(r-1)} = 1$$

Then by the allocation rule,

$$u^{WE(n_1)}_{Q^{n_1}_{\mathcal{D}_k}(r)} = \frac{w_{Q^{n_1}_{\mathcal{D}_k}(r)}}{w(Q^{n_1}_{\mathcal{D}_k}(r)) + w(W^{n_1}_{\mathcal{D}_k}(1))... + w(W^{n_1}_{\mathcal{D}_k}(t))} \geq \frac{1}{1 + |\mathbf{N}|\epsilon}$$

Similarly, For $n \geq n_1$, for any excess *overdemanded* patient $j \in \mathcal{W}^n_{\mathcal{D}_m}$ in step m

$$u^{WE(n)}_{W^{n_2}_{\mathcal{D}_m}(j)} = \frac{w_{W^n_{\mathcal{D}_m}(j)}}{w(Q^n_{\mathcal{D}_k}(u)) + w(W^n_{\mathcal{D}_k}(1))... + w(W^n_{\mathcal{D}_k}(s))}$$

There exists an $n_2 \geq n_1$ such that,

$$u^{WE(n_2)}_{W^{n_2}_{\mathcal{D}_m}(j)} < \frac{\epsilon}{1 + |\mathbf{N}|\epsilon}$$

Let $n^{k,l} = \max\{n_1, n_2\}$. The mechanism consists finite number of steps ,hence finite number of pairs. Let s be the total number of steps taken by the mechanism to arrive at the allocation. Then the maximum of all $n = max\{n^{i,j}\}$ where $i, j \in \{1, 2, 3..., s\}$ provides us with the required n. ■

Claim 2: There exists a large n such that, if patient i belongs to $\mathcal{W}^n_{\mathcal{D}_k}$, then for all $l \geq n$ patient i belongs to $\mathcal{W}^l_{\mathcal{D}_r}$. This establishes the continuity of the Weighted Egalitarian mechanism allocations for $(\mathbf{N}, \mathbf{R}, \mathbf{w^{(n)}})$

Proof. By previous claim, let n be such that $j \in \mathcal{Q}^n_{\mathcal{D}_g}, i \in \mathcal{W}^n_{\mathcal{D}_k}$ such that k, g be the arbitrary steps of Weighted Egalitarian mechanism, such that

$$u_j^{WE_n} > u_i^{WE_n}$$

Suppose, $i \in \mathcal{Q}^m_{\mathcal{D}_h}$ for some $m > n$. Consider $(\mathbf{N}, \mathbf{R}, \mathbf{w^n})$. Note that, patient i is not compatible with *overdemanded* neighbors of any step $p, p > k$. Since the total number of matches is a constant and given claim 1, it should be the case that there exists a patient $i^{'}$ belonging to a step $r, r \leq k$ such that $i^{'} \in \mathcal{Q}^n_{\mathcal{D}_r}$ and $i^{'} \in \mathcal{W}^m_{\mathcal{D}_t}$. Let i be the first patient so that there exists no patient $j \in \mathcal{W}^n_{\mathcal{D}_q}, q < k$ and $j \in \mathcal{W}^m_{\mathcal{D}_t}$. Patient i is chosen such that, there is no other patient in earlier steps of Weighted Egalitarian mechanism on $(\mathbf{N}, \mathbf{R}, \mathbf{w^n})$ that is similar to i between the two problems. If patient i switches from being an excess underdemanded patient for $(\mathbf{N}, \mathbf{R}, \mathbf{w^n})$ to not being one in $(\mathbf{N}, \mathbf{R}, \mathbf{w^m})$ then it should be the case that there exists a patient $i^{'}$ who is matched at an earlier or same step as patient i under Weighted Egalitarian mechanism in $(\mathbf{N}, \mathbf{R}, \mathbf{w^n})$ who is an excess underdemanded patient in $(\mathbf{N}, \mathbf{R}, \mathbf{w^m})$.

$$u_i^{WE_n} < u_{i'}^{WE_n}$$

Suppose $w_{i'} < w_i$

$$\Rightarrow f_k^{w^n} < f_r^{w^n}$$

which would be a contradiction to theorem 3. Hence it should be the case that $w_{i'} > w_i$. Since $i' \in \mathcal{W}^m_{\mathcal{D}_t}$ and $i \in \mathcal{Q}^m_{\mathcal{D}_h}$, then by theorem 3 it should be the case that $t < h$. Patient i' should be matched at a step earlier than patient i in $(\mathbf{N}, \mathbf{R}, \mathbf{w^m})$.

Consider patient i'. Since $i' \in \mathcal{W}^m_{\mathcal{D}_t}$, there exists an $i'' \in \mathcal{Q}^m_{\mathcal{D}_{t'}}, t' < t$ such that $i'' \in \mathcal{W}^n_{\mathcal{D}_{r'}}$. Drawing similarities from the analysis between patient i, i' we have that

$$w_{i''} > w_{i'}$$

If not,

$$\Rightarrow f_t^{w^m} < f_{t'}^{w^m}$$

which is a contradiction. But $i'' \in \mathcal{W}^n_{\mathcal{D}_{r'}}$ implies that, $r' < r$, since if $r' > r$ we have that

$$f_{r'}^{w^n} < f_r^{w^n}$$

But this implies that there exists a patient $i^{''}$ such that $i^{''} \in \mathcal{W}^n_{\mathcal{D}_{r'}}, r' < k$ and $i^{''} \in \mathcal{Q}^m_{\mathcal{D}_t}$. This is a contradiction to the selection of patient i.

■

We claim that the allocation chosen by the Weighted-Egalitarian mechanism is the allocation chosen by the priority mechanism. Suppose not, suppose there exists a patient i who is matched in Priority mechanism who isn't matched with probability 1 by the Weighted Egalitarian mechanism at the limit. This implies that, at the limit the Weighted Egalitarian mechanism matches with probability 1, a patient j with lower priority than patient i, who is matched instead of patient i. For large **n**, patient i is not compatible with the *underdemanded* and *overdemanded* patients of later steps.

This implies that, for large **n** patient j is matched at an earlier step than patient i in the Weighted Egalitarian mechanism. Let $f_i^{w^n}, f_j^{w^n}$ be the f values associated with the step at which patients i, j are matched in the Weighted Egalitarian mechanism.

The above assertion implies that,

$$f_i^{w^n} > f_j^{w^n} \quad \text{and}$$

$$\frac{w_j^n}{w_i^n} \to 0 \quad \text{also}$$

$$u_i^{w^n} \to 0 \quad \text{and} \quad u_j^{w^n} \to 1$$

Since patient j is matched with probability 1 and patient i is matched with probability 0 at the limit.

$$\Rightarrow f_i^{w^n} = \frac{u_i^{w^n}}{w_i^n} < \frac{u_j^{w^n}}{w_j^n} = f_j^{w^n}$$

which is a contradiction. Hence we have that, $\lim_{n\to\infty} \phi_n^{WE} = \psi$

■

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