GLOBAL, LOCAL ZETA FUNCTION AND P-ADIC INTEGRATION

ZI JIE ZHOU

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I have strong passion toward number theory when I was in high school, because prime numbers are interesting objects to investigate on. When I came to Boston College, Professor Dubi Kelmer, who taught me Introduction to Number Theory, provided more intriguing and deeper insights about this particular field. His teaching motivates me to study and think about number theory differently. It is my honor to have such great opportunity to do the honor thesis with Professor Dubi Kelmer. I cannot thank him enough for taking his extra times to teach me new things about math and most importantly, advising my thesis weekly.

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Abstract

In complex analysis, analytic continuation is a common and important tool to study the properties of complex functions. In this paper, we will introduce and define the global zeta function with an associated function f, where f is a polynomial with n variables with integer coefficients. With the usage of p-adic integration, we can conclude that the global zeta function is analytically continued to all $s \in \mathbb{C}$ when f is the sum of squares with n variables.

1. Introduction

Let $f \in \mathbb{Z}[x_1, x_2, \cdots, x_n]$ be a polynomial with integer coefficients, one natural question is to find out all of its zero solutions. In general, it is a difficult task to do, especially for a polynomial with several variable. However, it is easier to find the zero solutions locally. In other words, we can investigate the zero solutions of f in the finite group $(\mathbb{Z}/d\mathbb{Z})^n$, for any positive integer d.

Let $N_f(d)$ represents number of zero solutions for f in $(\mathbb{Z}/d\mathbb{Z})^n$ and this data can be placed into a global zeta function

$$Z_f(s) = \sum_{d=1}^{\infty} \frac{N_f(d)}{d^{s+n}}, s \in \mathbb{C} \text{ converging when Re(s) ia large} \tag{1}$$

One motivation to study this kind of global zeta function is that it appears in the constant term of Eisenstein series of certain arithmetic hyperbolic manifolds [6]. It turns out that for some special polynomials f, its associated global zeta function can be analytically continued to all $s \in \mathbb{C}$. One method to show that result is we first rewrite the global zeta function into a product of local zeta functions, then using stationary phase formula [1] and p-adic integration to show that each local zeta function can be written as a composition of local factors of Riemann zeta functions. In this paper, we will assume $f(x_1, \cdots, x_n) = \sum_{i=1}^n x_i^2$ to be the sum of squares with n variables and then its associated global zeta function can be evulated as follow:

Theorem 1. *For $f(x_1, \cdots, x_n) = \sum_{i=1}^n x_i^2$ and $Re(s) > 1$, we have:*

$$\sum_{d=1}^{\infty} \frac{N_f(d)}{d^{s+n}} = \begin{cases} \frac{(1-2^{-n-2s}+2^{-\frac{n}{2}-1-2s})\zeta(n+2s)\zeta(1+s)\zeta(\frac{n}{2}+s)}{(1-2^{-\frac{n}{2}-1-s})(1+2^{-\frac{n}{2}-s})\zeta(\frac{n}{2}+1+s)\zeta(n+2s)} & \text{where } n \equiv 4 \pmod 8 \\ \frac{(1-2^{-n-2s}-2^{-\frac{n}{2}-1-2s})\zeta(n+2s)\zeta(1+s)\zeta(\frac{n}{2}+s)}{(1-2^{-\frac{n}{2}-1-s})(1+2^{-\frac{n}{2}-s})\zeta(\frac{n}{2}+1+s)\zeta(n+2s)} & \text{where } n \equiv 0 \pmod 8 \\ \frac{(1-2^{-n-2s}-2^{-\frac{n}{2}-\frac{3}{2}-2s})\zeta(n+2s)\zeta(1+s)}{(1-2^{-n-1-2s})\zeta(n+1+2s)} & \text{where } n \equiv 5 \pmod 8 \\ \frac{(1-2^{-n-2s}+2^{-\frac{n}{2}-\frac{3}{2}-2s})\zeta(n+2s)\zeta(1+s)}{(1-2^{-n-1-2s})\zeta(n+1+2s)} & \text{where } n \equiv 1 \pmod 8 \\ \frac{(1-2^{-n-2s}+2^{-\frac{n}{2}-\frac{3}{2}-2s})\zeta(n+2s)\zeta(1+s)}{(1-2^{-n-1-2s})\zeta(n+1+2s)} & \text{where } n \equiv 7 \pmod 8 \\ \frac{(1-2^{-n-2s}-2^{-\frac{n}{2}-\frac{3}{2}-2s})\zeta(n+2s)\zeta(1+s)}{(1-2^{-n-1-2s})\zeta(n+1+2s)} & \text{where } n \equiv 8 \pmod 3 \end{cases}$$

where $\zeta(s) = \prod_p \frac{1}{1-p^{-s}}$. For the case when $n \equiv 2 \pmod 4$, we have

$$\sum_{d=1}^{\infty} \frac{N_f(d)}{d^{s+n}} = \frac{(1-2^{-n-2s})\zeta(1+s)\zeta(n+2s)}{L(\frac{n}{2}+1+s,\chi)L(\frac{n}{2}+s,\chi)}$$

where $L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \frac{1}{1-\chi(p)p^{-s}}$ is the Dirichlet L-function with χ, the Dirichlet character that is defined on primes

$$\chi(p) = \begin{cases} 1 & p \equiv 1 \pmod 4 \\ -1 & p \equiv 3 \pmod 4 \\ 0 & p = 2 \end{cases}$$

Since $\zeta(s)$ and $L(s, \chi)$ have analytical continuation to $\mathbb{C}$, this formula gives an analytical continuation to $Z_f(s)$.

2. p-adic integration

2.1. p-adic integers $\mathbb{Z}_p$

Before we proceed to the p-adic integration, it is convenient to know some basic properties of p-adic integers $\mathbb{Z}_p$. We will define $\mathbb{Z}_p$ as follow:

$$\mathbb{Z}_p = \{(a_1, a_2, \cdots) \in \prod_{i=1}^{\infty} \mathbb{Z}/p^i\mathbb{Z} \mid a_j \equiv a_k \pmod{p^j}, j < k\}$$

for any prime number p. Using this definition, there is a natural way to define a valuation for every element of $\mathbb{Z}_p$ [2]:

Defintion 1. For any nonzero $x \in \mathbb{Z}_p$, the p-adic valuation of x, denoted $v_p(x)$, is the index of the first nonzero entry in the p-adic expansion of x. For $x = 0$, we defined that $v_p(0) = \infty$.

Defintion 2. $\mathbb{Z}_p^\times$ is the multiplicative group of invertible elements in $\mathbb{Z}_p$.

Theorem 2. (1) $\mathbb{Z}_p^\times = \{a \in \mathbb{Z}_p : v_p(a) = 0\}$

(2) *Every nonzero $a \in \mathbb{Z}_p$ can be uniquely written as $p^n \cdot u$ with $n \in \mathbb{Z}_{\geq 0}$ and $u \in \mathbb{Z}_p^\times$.*

Proof. (1) Let $a \in \mathbb{Z}_p$ such that $v_p(a) = 0$. So we can write $a = (a_1, a_2, \cdots)$, where $a_i \nmid p$ for each a_i in the sequence. Therefore, each a_i has an unique inverse a_i^{-1} in $\mathbb{Z}/p^i\mathbb{Z}$. Therefore, the inverse for a is simply

$$a^{-1} = (a_1^{-1}, a_2^{-1}, \cdots)$$

Conversely, if $a \in \mathbb{Z}_p^\times$ and $v_p(a) = n > 0$ for some positive integer n. So we can represent

$$a = (0, \cdots, 0, a_{n+1}, \cdots)$$

Clearly a does not have a multiplicative inverse since every element multiply by 0 is not equal to 1.

(2) Let $a \neq 0 \in \mathbb{Z}_p$ and suppose $v_p(a) = m$. So we can write

$$a = (0, 0, \cdots, a_{n+1}, \cdots)$$

which means $a = p^m \cdot u$ for some $u \in \mathbb{Z}_p$. But $v_p(u) = 0 \implies u \in \mathbb{Z}_p^\times$. Moreover, u is unique and so is the representation $p^m \cdot u$.

□

2.2. **The field of p-adic integers $\mathbb{Q}_p$ and its topology**

We will start with a proposition:

Proposition 1. *$\mathbb{Z}_p$ is an integral domain.*

Proof. This is can be checked by the definition of p-adic valuation. That is, suppose there are two non-zero elements $x_1, x_2 \in \mathbb{Z}_p$ such that $x_1 \cdot x_2 = 0$, then we have

$$v_p(0) = v_p(x_1 \cdot x_2) = v_p(x_1) + v_p(x_2) = \infty$$

This is not possible since both $v_p(x_1), v_p(x_2)$ are finite when x_1, x_2 are non-zero. $\square$

Now since $\mathbb{Z}_p$ is an integral domain, we can denote its field of fraction $K(\mathbb{Z}_p) = \mathbb{Q}_p$, the field of p-adic integers. Using the valuation on $\mathbb{Z}_p$, we can define a metric d_p on $\mathbb{Q}_p$ [1]:

$$d_p(x,y) = \begin{cases} p^{-v_p(x-y)} & , \quad \text{if } x - y \neq 0 \\ 0 & , \quad \text{if } x - y = 0 \end{cases}$$

for any $x, y \in \mathbb{Q}_p$. Moreover, the associated metric topology is generated by the open balls

$$B_r(a) = \{x \in \mathbb{Q}_p : d_p(x,a) \leq p^r\}$$

where $r \in \mathbb{Z}$. In addition, we can explicitly describe the open ball

$$B_r(a) = a + p^r \mathbb{Z}_p$$

2.3. **p-adic integration on $\mathbb{Q}_p$**

Theorem 3. *Let $(G, \cdot)$ be a locally compact topological group. There exists a Borel measure dx, unique up to multiplication by a positive constant, such that $\int_U dx > 0$ for every non-empty Borel open set U, and $\int_{x \cdot E} dx = \int_E dx$, for every Borel set E.*

Proof. See [1, Thm 3.1]. $\square$

Since $(\mathbb{Q}_p, +)$ is a locally compact topological group, it has a Haar measure dx. Furthermore, we can normalize the measure by the condition

$$\int_{\mathbb{Z}_p} dx = 1$$

so that dx is unique. Inductively, we also have the condition

$$\int_{\mathbb{Z}_p^n} d^n x = 1$$

Although it is invariant under addition, which means

$$\int_{a+U} d^n x = \int_U d^n x$$

for any $a \in \mathbb{Q}_p$ and U a Borel set. But consider $a \in \mathbb{Q}_p^\times$, we then have this property

$$\int_{aU} d^n x = |a|_p^n \int_U d^n x \tag{2}$$

We can understand equation (2) as a way of changing variable, which is going to be useful for the computation later on.

2.4. Global and local zeta functions

Theorem 4.

$$Z_f(s) = \sum_{d=1}^{\infty} \frac{N_f(d)}{d^{s+n}} = \prod_{p \in \mathbb{P}} \left(\sum_{k=0}^{\infty} \frac{N_f(p^k)}{p^{k(s+n)}} \right)$$

where $\mathbb{P}$ represents the set of prime numbers and $f \in \mathbb{Z}[x_1, \cdots, x_n]$.

Before we proved the equality above, we need the following lemma:

Lemma 1. *Let $d \in \mathbb{Z}$ and according to the fundamental theorem of arithmetic, there exists a unique way to rewrite it as a product of prime numbers (up to signs and order):*

$$d = p_1^{e_1} \cdot p_2^{e_2} \cdots p_m^{e_m}$$

where each of p_i are distinct primes and e_i are positive integers. Then we have

$$N_f(d) = N_f(p_1^{e_1}) \cdot N_f(p_2^{e_2}) \cdots N_f(p_m^{e_m})$$

for any polynomial $f \in \mathbb{Z}[x_1, \cdots, x_n]$.

Proof. Let k_i be a zero solution of f in $\mathbb{Z}/p_i^{e_i}\mathbb{Z}$. According to the Chinese Reminder Theorem, there exists a unique $k \in \mathbb{Z}/(p_1^{e_1} \cdots p_m^{e_m})\mathbb{Z} = \mathbb{Z}/d\mathbb{Z}$ such that $k \equiv k_i \pmod{\mathbb{Z}/p_i^{e_i}\mathbb{Z}}$. There are a total of $\prod_{i=1}^{m} N_f(p_i^{e_i})$ choices and the uniqueness from the Chinese Reminder Theorem shows the desired equality. $\square$

Now we can prove theorem 3:

Proof. Let M be a fixed positive integer, then consider the difference

$$\prod_{\substack{p \in \mathbb{P} \\ p \leq M}} \left(\sum_{k=0}^{\infty} \frac{N_f(p^k)}{p^{k(s+n)}} \right) - \sum_{d \in A} \frac{N_f(d)}{d^{s+n}}$$

where A is the set of integers that has prime factors less than M. On the other hand, denote B to be the set of integers that at least has prime factor that is strictly greater than M. Then we have

$$\prod_{\substack{p\in\mathbb{P}\\ p\le M}}\left(\sum_{k=0}^{\infty}\frac{N_f(p^k)}{p^{k(s+n)}}\right)-\sum_{d\in A}\frac{N_f(d)}{d^{s+n}}=\sum_{d\in B}\frac{N_f(d)}{d^{s+n}}$$

$$\implies\left|\prod_{\substack{p\in\mathbb{P}\\ p\le M}}\left(\sum_{k=0}^{\infty}\frac{N_f(p^k)}{p^{k(s+n)}}\right)-\sum_{d\in A}\frac{N_f(d)}{d^{s+n}}\right|\le\sum_{d\in B}\left|\frac{N_f(d)}{d^{s+n}}\right|$$

$$\implies\left|\prod_{\substack{p\in\mathbb{P}\\ p\le M}}\left(\sum_{k=0}^{\infty}\frac{N_f(p^k)}{p^{k(s+n)}}\right)-\sum_{d\in A}\frac{N_f(d)}{d^{s+n}}\right|\le\sum_{d\in B}\left|\frac{N_f(d)}{d^{s+n}}\right|\le\sum_{d>M}\left|\frac{N_f(d)}{d^{s+n}}\right|$$

As $M\to\infty$, the last sum approaches to 0 since we assumed the global zeta function converges for some sufficiently large $\mathrm{Re}(s)$. So we finished the proof. □

From theorem 3, to calculate $Z_f(s)$, it suffices to calculate

$$Z_f^{(p)}(s)=\sum_{k=0}^{\infty}\frac{N_f(p^k)}{p^{k(s+n)}}$$

for each p.

Remark 1. For a general polynomial, it was proved by Igusa that $Z_f^{(p)}(s)$ is a rational function of p^{-s}, so can be analytically continued to all of $\mathbb{C}$. So what follows is that we calculate these functions explicitly for our choice of f.

Defintion 3. *Let $f(x)\in\mathbb{Z}_p[x_1,\cdots,x_n]\setminus\mathbb{Z}_p$, we set*

$$I_f^{(p)}(s):=\int_{\mathbb{Z}_p^n\setminus f^{-1}(0)}|f(x)|_p^s\,d^nx,\, s\in\mathbb{C}, Re(s)>0$$

Proposition 2 (1, Proposition 5.3)**.**

$$Z_f^{(p)}(s)=\frac{1-p^{-s}\cdot I_f^{(p)}(s)}{1-p^{-s}},\textit{for Re(s)>0}$$

Proof. Note that

$$Z_f^{(p)}(s)=\int_{\mathbb{Z}_p^n\setminus f^{-1}(0)}|f(x)|_p^s\,d^nx=\sum_{j=0}^{\infty}p^{-js}\int_{\{x\in\mathbb{Z}_p^n:|f(x)|=p^{-j}\}}d^nx$$

On the other hand,

$$\{x\in\mathbb{Z}_p^n:|f(x)|=p^{-j}\}=\{x\in\mathbb{Z}_p^n:\mathrm{ord}(f(x))=j\}$$
$$=\{x\in\mathbb{Z}_p^n:\mathrm{ord}(f(x))\ge j\}\setminus\{x\in\mathbb{Z}_p^n:\mathrm{ord}(f(x))\ge j+1\}$$

where $\mathrm{ord}_p(x)=v_p(x)$, the p-adic valuation of x. Now take $x_0\in\mathbb{Z}_p^n$ satisfying $\mathrm{ord}(f(x_0))\ge j$, then, by using Taylor expansion,

$$f(x_0+p^jz)=f(x_0)+p^j\sum_{j=1}^{n}\frac{\partial f}{\partial x_i}(x_0)(x_i-x_{0,i})+p^{2j}(\text{higher order terms})$$

we have $\operatorname{ord}(f(x_0 + p^j z)) \geq j$, for all $z \in \mathbb{Z}_p^n$, i.e. $\operatorname{ord}(f(x_0 + p^j \mathbb{Z}_p^n)) \geq j$. This fact implies:

(i) $x_0 \in (\mathbb{Z}_p/p^j\mathbb{Z}_p)^n$ satisfies $f(x_0) \equiv 0 \pmod{p^j}$

(ii) $A_j := \{x \in \mathbb{Z}_p^n : \operatorname{ord}(f(x)) \geq j\} = \displaystyle\bigsqcup_{f(x_0)\equiv 0 \pmod{p^j}} x_0 + p^j\mathbb{Z}_p^n$

(iii) $\displaystyle\int_{A_j} d^n x = N_f(j) \cdot p^{-jn}$

Therefore,

$$\begin{aligned}
I_f^{(p)}(s) &= \sum_{j=0}^{\infty} p^{-js}\left(N_f(j)\cdot p^{-jn} - N_f(j+1)\cdot p^{-(j+1)n}\right)\\
&= \sum_{j=0}^{\infty} N_f(j)\cdot p^{-js-jn} - \sum_{j=0}^{\infty} N_f(j+1)\cdot p^{-js-(j+1)n}\\
&= \sum_{j=0}^{\infty} N_f(j)\cdot (p^{-n}p^{-s})^j - p^s \sum_{j'=1}^{\infty} N_f(j')\cdot (p^{-n}p^{-s})^{j'}\\
&= Z_f^{(p)}(s) - p^s\cdot (Z_f^{(p)}(s) - 1)\\
\implies Z_f^{(p)}(s) &= \frac{1 - p^{-s}\cdot I_f^{(p)}(s)}{1 - p^{-s}}
\end{aligned}$$

□

2.5. **Stationary Phase Formula**

Here is another important proposition that we need to use in doing the computation:

Proposition 3 (Stationary Phase Formula). *Take $\overline{E} \subset \mathbb{F}_p^n$ and denote by $\overline{S}$ the subset consisting of all $\overline{a} \in \overline{E}$ such that $\overline{f}(\overline{a}) \equiv \frac{\partial \overline{f}}{\partial x_i}(\overline{a}) \equiv 0 \mod p$, for $1 \leq i \leq n$. Denote by E, S the preimages of $\overline{E}, \overline{S}$ under reduction $\mod p$ map $\mathbb{Z}_p^n \to \mathbb{F}_p^n$, and by N the number of zeros of $\overline{f}(x)$ in $\overline{E}$. Then*

$$\int_E |f(x)|_p^s\, d^n x = p^{-n}(\#\overline{E} - N) + \frac{p^{-n-s}(1-p^{-1})(N - \#\overline{S})}{1 - p^{-1-s}} + \int_S |f(x)|_p^s\, d^n x$$

Proof. See [1, Proposition 6.1]

□

The idea is to find out what is $I_f^{(p)}(s)$, where f is the sum of squares with n variables, via stationary phase formula, then using proposition 2 to evaluate each series $Z_f^{(p)}(s)$. After evaluating each $Z_f^{(p)}(s)$ for every $p \in \mathbb{P}$, we can rewrite the right hand side from theorem 3 as a composition of local factors of Riemann zeta functions.

3. Computation

3.1. $p = 2$

Here are some examples with $p = 2$:

Example 1. Suppose $f = x_1^2$, then $\overline{E} = \mathbb{F}_p^1$ and $\overline{S} = \{0\}$ since $\frac{\partial \overline{f}}{\partial x_1} \equiv 2x_1 \equiv 0 \pmod{p}$. Then $E = \mathbb{Z}_p$ and $S = \{p\mathbb{Z}_p\}$ and $N = |\{0\}| = 1$. In fact, when $p = 2$, we have $|\overline{S}| = N$, where $f(x) \in \mathbb{Z}_p[x_1, \ldots, x_n] \setminus \mathbb{Z}_p$ is the sum of square(s). So using stationary phase formula, we obtain:

$$\begin{aligned}
\int_{\mathbb{Z}_p} |f(x)|_p^s \, dx = p^{-1}(p-1) + \frac{p^{-1-s}(1-p^{-1})(1-1)}{1-p^{-1-s}} + \int_{p\mathbb{Z}_p} |f(x)|_p^s \, dx \\
&= 1 - p^{-1} + \int_{\mathbb{Z}_p} |(px_1)^2|_p^s \, d(px) \\
&= 1 - p^{-1} + p^{-1} \int_{\mathbb{Z}_p} |p^2 \cdot x_1^2|_p^s \, dx \\
&= 1 - p^{-1} + p^{-1-2s} \int_{\mathbb{Z}_p} |f(x)|_p^s \, dx \\
\implies I_f^{(2)}(s) &= \frac{1-p^{-1}}{1-p^{-1-2s}}
\end{aligned}$$

So we have

$$Z_f^{(2)}(s) = \frac{1 - p^{-s} \cdot I_f^{(2)}(s)}{1 - p^{-s}} = \frac{1 - p^{-s} \cdot \frac{1-p^{-1}}{1-p^{-1-2s}}}{1 - p^{-s}} = \frac{1 + p^{-1-s}}{1 - p^{-1-2s}}$$

Observation 1. *Let $p = 2$. If $f(x) \in \mathbb{Z}_p[x_1, \ldots, x_n] \setminus \mathbb{Z}_p$ is a sum of square(s), then*

$$\overline{S} = \{\textit{number of zeros of } \overline{f}(x) \textit{ in } \overline{E}\}$$

Proof. This is straightforward to show since every partial derivative $\frac{\partial \overline{f}}{\partial x_i} = 2x_i \equiv 0 \pmod{p}$ and every element in $\overline{S}$ is also a zero of the function $\overline{f}(x)$ in $\overline{E}$. So the equality holds. $\square$

Example 2. For this example, let $f = \sum_{i=1}^{5} x_i^2$. We first find that $\overline{E} = \mathbb{F}_p^5$,

$$\overline{S} = \{(0,0,0,0,0), (1,1,0,0,0), \ldots, (1,1,1,1,0)\}$$

Note that we can categorize the $\overline{S}$ into different forms. For example, there are a total of $\binom{5}{2} = 10$ of zero solutions of the form with two variables with value of 1; similarly, there are a total of $\binom{5}{4} = 5$ of zero solutions of the form with four variables with value of 1. As a result, we have preimages $E = \mathbb{Z}_p^5$ and

$$S = \{V_1 = (p\mathbb{Z}_p)^5, V_2 = (1 + p\mathbb{Z}_p)^2 \times (p\mathbb{Z}_p)^3, V_3 = (1 + p\mathbb{Z}_p)^4 \times (p\mathbb{Z}_p)\}$$

In addition,

$$N = \binom{5}{0} + \binom{5}{2} + \binom{5}{4} = 16 = p^4$$

Then stationary phase formula says that:

$$\begin{aligned}
\int_{\mathbb{Z}_p^5} |f(x)|_p^s \, d^n x &= p^{-5} \cdot (p^5 - p^4) + \int_{V_1 \cup V_2 \cup V_3} |f(x)|_p^s \, d^n x \\
&= 1 - p^{-1} + \binom{5}{0} \int_{V_1} |f(x)|_p^s \, d^n x + \binom{5}{2} \int_{V_2} |f(x)|_p^s \, d^n x + \binom{5}{4} \int_{V_3} |f(x)|_p^s \, d^n x
\end{aligned}$$

Note that we can split the integral into different parts is because V_i, V_j are pairwise disjoint when $i \neq j$. So we compute each integral separately:

$$\int_{V_1} |f(x)|_p^s \, d^n x = \int_{(p\mathbb{Z}_p)^5} |f(x)|_p^s \, d^5 x = p^{-5-2s} \int_{\mathbb{Z}_p^5} |f(x)|_p^s \, d^5 x$$

For V_2, we have

$$\begin{aligned}
\int_{V_2} |f(x)|_p^s \, d^n x &= \int_{(1+p\mathbb{Z}_p)^2 \times (p\mathbb{Z}_p)^3} |f(x)|_p^s \, d^5 x \\
&= p^{-5} \int_{\mathbb{Z}_p^5} |(1+px_1)^2 + (1+px_2)^2 + (px_3)^2 + (px_4)^2 + (px_5)^2|_p^s \, d^5 x \\
&= p^{-5-s} \int_{\mathbb{Z}_p^5} |1 + px_1 + px_1^2 + px_2 + px_2^2 + px_3^2 + px_4^2 + px_5^2|_p^s \, d^5 x \\
&= p^{-5-s}
\end{aligned}$$

Observation 2.

$$1 + px_1 + px_1^2 + px_2 + px_2^2 + px_3^2 + px_4^2 + px_5^p \equiv 1 \neq 0 \pmod p$$

Thus, we have $|1 + px_1 + px_1^2 + px_2 + px_2^2 + px_3^2 + px_4^2 + px_5^2|_p = 1.$

For V_3, we have

$$\begin{aligned}
\int_{V_3} |f(x)|_p^s \, d^n x &= \int_{(1+p\mathbb{Z}_p)^4 \times (p\mathbb{Z}_p)} |f(x)|_p^s \, d^5 x \\
&= p^{-5} \int_{\mathbb{Z}_p^5} |(1+px_1)^2 + (1+px_2)^2 + (1+px_3)^2 + (1+px_4)^2 + (px_5)^2|_p^s \, d^5 x \\
&= p^{-5-2s} \int_{\mathbb{Z}_p^5} |1 + x_1 + x_1^2 + x_2 + x_2^2 + x_3 + x_3^2 + x_4 + x_4^2 + x_5^2|_p^s \, d^5 x
\end{aligned}$$

Let $g(x) = 1 + x_1 + x_1^2 + x_2 + x_2^2 + x_3 + x_3^2 + x_4 + x_4^2 + x_5^2$, then observation 2 does not necessarily apply to $g(x)$. However, it is clear that $x_i + x_i^2 \equiv 0 \pmod p$ in the finite field $\mathbb{F}_p$, where $p = 2$. This forces $x_5 = 1$ so that $g(x) \equiv 0 \pmod p$.

Now use stationary phase formula for $g(x)$: $\overline{E} = \mathbb{F}_p^5$, note that not every partial derivative is equal to 0 in mod 2, such as $\frac{\partial}{\partial x_i} = 1 + 2x_i \not\equiv 0 \pmod p$, for $1 \leq i \leq 4$. As a result, $\overline{S} = \varnothing$. The preimages $E = \mathbb{Z}_p^5$ and $S = \varnothing$ and the zero of $g(x)$ is $\{(x_1, \dots, x_4) \in \mathbb{F}_p^4, x_5 = 1\} \implies N = p^4$.

$$\int_{\mathbb{Z}_p^5} |g(x)|_p^s \, d^5x = p^{-5} \cdot (p^5 - p^4) + \frac{p^{-5-s}(1-p^{-1})(p^4-0)}{1-p^{-1-s}} = \frac{1-p^{-1}}{1-p^{-1-s}}$$

Thus,

$$\int_{V_3} |f(x)|_p^s \, d^nx = p^{-5-2s} \cdot \frac{1-p^{-1}}{1-p^{-1-s}}$$

Combining these three terms all together, we obtain

$$\int_{\mathbb{Z}_p^5} |f(x)|_p^s \, d^nx = 1 - p^{-1} + p^{-5-2s} \int_{\mathbb{Z}_p^5} |f(x)|_p^s \, d^5x + 10 \cdot p^{-5-s} + 5 \cdot p^{-5-2s} \cdot \frac{1-p^{-1}}{1-p^{-1-s}}$$

$$I_f^{(2)}(s) = \int_{\mathbb{Z}_p^5} |f(x)|_p^s \, d^5x = \frac{1 - p^{-1-s} - p^{-1} + p^{-2-s} + 10 \cdot p^{-5-s} - 15 \cdot p^{-6-2s} + 5 \cdot p^{-5-2s}}{(1-p^{-5-2s})(1-p^{-1-s})}$$

So we have:

$$\begin{aligned} Z_f^{(2)}(s) &= \frac{1 - p^{-s} \cdot I_f^{(2)}(s)}{1-p^{-s}} \\ &= \frac{1 - p^{-s} + p^{-1-2s} - p^{-2-2s} - 11 \cdot p^{-5-2s} + 16 \cdot p^{-6-3s} - 5 \cdot p^{-5-3s}}{(1-p^{-5-2s})(1-p^{-1-s})(1-p^{-s})} \end{aligned}$$

Example 3 (general case)**.** Now let f be the sum of squares with n variables, we have $\overline{E} = \mathbb{F}_p^n$ and

$\overline{S} =$

$V_0 = \{\text{every variable is equal to } 0\}$	$\binom{n}{0}$
$V_2\{\text{only two of variables are equal to } 1\}$	$\binom{n}{2}$
$V_4 = \{\text{only four of variables are equal to } 1\}$	$\binom{n}{4}$
$\vdots$	$\vdots$
$V_m = \{\text{only } m \text{ of variables are equal to } 1\}$	$\binom{n}{m}$

where m is the largest even number that is smaller or equal to n. Then the preimages $\overline{E} = \mathbb{Z}_p^n$ and

$S =$

$\overline{V_0} = (p\mathbb{Z}_p)^n$	$\binom{n}{0}$
$\overline{V_2} = (1+p\mathbb{Z}_p)^2 \times (p\mathbb{Z}_p)^{n-2}$	$\binom{n}{2}$
$\overline{V_4} = (1+p\mathbb{Z}_p)^4 \times (p\mathbb{Z}_p)^{n-4}$	$\binom{n}{4}$
$\vdots$	$\vdots$
$\overline{V_m} = (1+p\mathbb{Z}_p)^m \times (p\mathbb{Z}_p)^{n-m}$	$\binom{n}{m}$

Theorem 5.

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

One can show the equality above is to use Pascal's identity $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$. More importantly, we have the following corollary:

Corollary 1.

$$\sum_{k \text{ odd}}^{n} \binom{n}{k} = \sum_{k \text{ even}}^{n} \binom{n}{k} = 2^{n-1}$$

Proof. There are two cases:

- Suppose n is odd, then we can write

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k} &= \binom{n}{0}+\cdots+\binom{n}{n}\\ &= \left(\binom{n}{0}+\binom{n}{n}\right)+\left(\binom{n}{1}+\binom{n}{n-1}\right)+\cdots\end{aligned}$$

  But notice that $\binom{n}{k}=\binom{n}{n-k}$, where if k is odd then $n-k$ is even, or k is even then $n-k$ is odd. In other words, for every $\binom{n}{k}$ and k is odd, we can find $\binom{n}{n-k}$ such that $\binom{n}{k}=\binom{n}{n-k}$. This splits the $\sum_{k=0}^{n}\binom{n}{k}$ into two components with the same sum, which is the desired equality.

- Suppose n is even, using Pascal's identity we have

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k} &= \sum_{k=0}^{n}\binom{n-1+1}{k}\\ &= \sum_{k=0}^{n}\binom{n}{k-1}+\binom{n}{k}\end{aligned}$$

  Similarly as above, for every $\binom{n}{k}$ with k odd, we can find a unique $\binom{n}{k-1}$, where k is even. So we obtain the desired equality.

□

From corollary 1, it is clear that $N=2^{n-1}$ when $f(x)=\sum_{i=1}^{n}x_i^2$. So using stationary phase formula:

$$\begin{aligned}\int_{\mathbb{Z}_p^n}|f(x)|_p^s\,d^nx &= p^{-n}\cdot(p^n-p^{n-1})+\int_{\overline{V_0}\cup\cdots\cup\overline{V_m}}|f(x)|_p^s\,d^nx\\ &= 1-p^{-1}+\sum_{i\text{ even}}^{n}\binom{n}{i}\int_{\overline{V_i}}|f(x)|_p^s\,d^nx\end{aligned}$$

If $i=0$, then we have:

$$\begin{aligned}\binom{n}{0}\int_{\overline{V_0}}|f(x)|_p^s\,d^nx &= \int_{(p\mathbb{Z}_p)^n}|f(x)|_p^s\,d^nx\\ &= \int_{\mathbb{Z}_p^n}|\sum_{i=1}^{n}(px_i)^2|_p^s\,d^n(px)\\ &= p^{-n-2s}\int_{\mathbb{Z}_p^n}|f(x)|_p^s\,d^nx\end{aligned}$$

If $i=2$, then we have:

$$
\begin{aligned}
\binom{n}{2}\int_{\overline{V_2}} |f(x)|_p^s \, d^n x &= \int_{(1+p\mathbb{Z}_p)^2\times(p\mathbb{Z}_p)^{n-2}} |f(x)|_p^s \, d^n x \\
&= p^{-n-s}\int_{\mathbb{Z}_p^n} |1+p\cdot h(x)|_p^s \, d^n x \\
&= p^{-n-s}
\end{aligned}
$$

More specifically, let $h(x) = (\sum_{i=1}^{2} x_i + x_i^2) + (\sum_{j=3}^{n} x_j^2)$.

If $i = 4$ for some $k \leq n$, then we have

$$
\begin{aligned}
\int_{\overline{V_4}} |f(x)|_p^s \, d^n x &= \int_{(1+p\mathbb{Z}_p)^4\times(p\mathbb{Z}_p)^{n-4}} |f(x)|_p^s \, d^n x \\
&= p^{-n}\int_{\mathbb{Z}_p^n} |p^2\cdot g(x)|_p^s \, d^n x \\
&= p^{-n-2s}\int_{\mathbb{Z}_p^n} |g(x)|_p^s \, d^n x
\end{aligned}
$$

where $g(x) = 1 + (\sum_{i=1}^{4} x_i + x_i^2) + (\sum_{j=5}^{n} x_j^2)$. But if we apply stationary phase formula for $g(x)$, we have $\overline{E} = \mathbb{F}_p^n$ and $\overline{S} = \varnothing$, since $\frac{\partial}{\partial x_i} \not\equiv 0 \pmod p$ for $1 \leq i \leq 4$. The preimages will be $E = \mathbb{Z}_p^n$ and $S = \varnothing$. Furthermore, the zeros of $g(x)$ in modular p is independent of choices of x_i, where $1 \leq i \leq 4$, because $x_i + x_i^2 \equiv 0 \pmod p$. In other words, the zero solutions of $g(x)$ consists of the form where in the sum $\sum_{j=5}^{n} x_j^2$, there has to be odd number of variables equal to 1 and there are a total number of

$$
\sum_{k \text{ odd}}^{n-5+1} \binom{n-5+1}{k}
$$

Since each form has p^4 distinct choices from $x_i + x_i^2$, thus

$$
N = p^4 \cdot \sum_{k \text{ odd}}^{n-5+1} \binom{n-5+1}{k} = p^4 \cdot p^{n-5} = p^{n-1}
$$

Thus, we have

$$
\int_{\mathbb{Z}_p^n} |g(x)|_p^s \, d^n x = p^{-n}\cdot(p^n - p^{n-1}) + \frac{p^{-n-s}(1-p^{-1})(p^{n-1})}{1-p^{-1-s}} = \frac{1-p^{-1}}{1-p^{-1-s}}
$$

So when $i = 4$, we have

$$
\binom{n}{4}\int_{\overline{V_4}} |f(x)|_p^s \, d^n x = \binom{n}{4}\cdot p^{-n-2s}\cdot\frac{1-p^{-1}}{1-p^{-1-s}}
$$

Notation 1. *In fact, the polynomial above $g(x)$ will appear frequently in the later computation, so we will denote $g_a(x) = (\sum_{i=1}^{a} x_i + x_i^2) + (\sum_{j=a+1}^{n} x_j^2)$, where $a < n$.*

Let $i > 4$ even such that $4 \nmid i$, then we can write $i = 2\cdot k = p\cdot k$ for some odd number k:

$$\begin{aligned}\int_{\overline{V_i}} |f(x)|_p^s \, d^n x &= \int_{(1+p\mathbb{Z}_p)^i \times (p\mathbb{Z}_p)^{n-i}} |f(x)|_p^s \, d^n x \\ &= p^{-n-s} \int_{\mathbb{Z}_p^n} |k + p \cdot g_i(x)|_p^s \, d^n x \\ &= p^{-n-s}\end{aligned}$$

On the other hand, if $i > 4$ and $4 \mid i$ but $8 \nmid i$, then we can write $i = p^2 \cdot k$ for some odd number k:

$$\begin{aligned}\int_{\overline{V_i}} |f(x)|_p^s \, d^n x &= \int_{(1+p\mathbb{Z}_p)^i \times (p\mathbb{Z}_p)^{n-i}} |f(x)|_p^s \, d^n x \\ &= p^{-n-2s} \int_{\mathbb{Z}_p^n} |k + g_i(x)|_p^s \, d^n x\end{aligned}$$

Apply stationary phase formula for the polynomial $h(x) = k + g_i(x)$, where $g_i(x)$ is defined above, we have: $\overline{E} = \mathbb{F}_p^n$ and $\overline{S} = \varnothing$, since $\frac{\partial}{\partial x_j} \not\equiv 0 \pmod{p}$, for $1 \leq j \leq i$. So the preimages are $E = \mathbb{Z}_p^n$ and $S = \varnothing$. Moreover, the zero solutions of $h(x)$ is dependent of number of variables whose value is 1. Since k is odd, we need to have odd numbers of variables in the second sum of $g_i(x)$ equal to 1. Thus,

$$N = p^i \cdot \sum_{k \text{ odd}}^{n-i} \binom{n-i}{k} = p^i \cdot p^{n-i-1} = p^{n-1}$$

So

$$\int_{\mathbb{Z}_p^n} |k + g_i(x)|_p^s \, d^n x = p^{-n} \cdot (p^n - p^{n-1}) + \frac{p^{-n-s}(1-p^{-1})(p^{n-1})}{1-p^{-1-s}} = \frac{1-p^{-1}}{1-p^{-1-s}}$$

Now suppose $8 \mid i$ and we can write $i = k \cdot p^m$, where $k \in \mathbb{Z}^+$ and $m \geq 3$, then we have

$$\begin{aligned}\int_{\overline{V_i}} |f(x)|_p^s \, d^n x &= \int_{(1+p\mathbb{Z}_p)^i \times (p\mathbb{Z}_p)^{n-i}} |f(x)|_p^s \, d^n x \\ &= p^{-n-2s} \int_{\mathbb{Z}_p^n} |p^{m'} + g_i(x)|_p^s \, d^n x \quad , \quad 1 \leq m' \leq \log_2(n)\end{aligned}$$

Apply stationary phase formula for the polynomial $h(x) = p^{m'} + g_i(x)$. We have: $\overline{E} = \mathbb{F}_p^n$ and $\overline{S} = \varnothing$, since $\frac{\partial}{\partial x_j} \not\equiv 0 \pmod{p}$, for $1 \leq j \leq i$. So the preimages are $E = \mathbb{Z}_p^n$ and $S = \varnothing$. Moreover, the zero solutions of $h(x)$ is dependent of number of variables whose value is 1. Since $p^{m'}$ is odd, we need to have even numbers of variables in the second sum of $g_i(x)$ equal to 1. Thus,

$$N = p^i \cdot \sum_{k \text{ even}}^{n-i} \binom{n-i}{k} = p^i \cdot p^{n-i-1} = p^{n-1}$$

So

$$\int_{\mathbb{Z}_p^n} |p^{m'} + g_i(x)|_p^s \, d^n x = p^{-n} \cdot (p^n - p^{n-1}) + \frac{p^{-n-s}(1-p^{-1})(p^{n-1})}{1-p^{-1-s}} = \frac{1-p^{-1}}{1-p^{-1-s}}$$

$$\begin{aligned}
I_f^{(2)}(s) &= \int_{\mathbb{Z}_p^n} |f(x)|_p^s \, d^n x \\
&= 1 - p^{-1} + \sum_{i \text{ even}}^{n} \binom{n}{i} \int_{\overline{V_i}} |f(x)|_p^s \, d^n x \\
&= 1 - p^{-1} + p^{-n-2s} \cdot I_f^{(2)}(s) + \sum_{4 \nmid i\ , i>0}^{n} \binom{n}{i} \cdot p^{-n-s} + \sum_{4 \mid i\ , i>0}^{n} \binom{n}{i} \cdot p^{-n-2s} \cdot \frac{1-p^{-1}}{1-p^{-1-s}}
\end{aligned}$$

$$I_f^{(2)}(s) = \frac{1 - p^{-1} + \sum_{4 \nmid i\ , i>0}^{n} \binom{n}{i} \cdot p^{-n-s} + \sum_{4 \mid i\ , i>0}^{n} \binom{n}{i} \cdot p^{-n-2s} \cdot \frac{1-p^{-1}}{1-p^{-1-s}}}{1 - p^{-n-2s}}$$

Denote $C = \sum_{4 \nmid i\ , i>0}^{n} \binom{n}{i}$ and $D = \sum_{4 \mid i\ , i>0}^{n} \binom{n}{i}$, then $Z_f^{(2)}(s)$ will be:

$$\begin{aligned}
Z_f^{(2)}(s) &= \frac{1 - p^{-s} \cdot I_f^{(2)}(s)}{1 - p^{-s}} \\
&= \frac{1 - p^{-n-2s} - p^{-s} + p^{-1-s} - C \cdot p^{-n-2s} - D \cdot p^{-n-3s} \cdot \frac{1-p^{-1}}{1-p^{-1-s}}}{(1 - p^{-n-2s})(1 - p^{-s})}
\end{aligned}$$

Now we want to find out the exact value of C and D. Consider the following equation:

$$\begin{aligned}
(1+i)^n + (1-i)^n &= \left(\sum_{k=0}^{n} \binom{n}{k} 1^{n-k} i^k \right) + \left(\sum_{j=0}^{n} \binom{n}{j} 1^{n-j} (-i)^j \right) \\
&= \left(\sum_{k=0}^{n} \binom{n}{k} i^k \right) + \left(\sum_{j=0}^{n} \binom{n}{j} (-i)^j \right) \\
&= \left(\sum_{k \equiv 0 \pmod 4}^{n} \binom{n}{k} (i)^k \right) + \left(\sum_{k \equiv 1 \pmod 4}^{n} \binom{n}{k} (i)^k \right) + \left(\sum_{k \equiv 2 \pmod 4}^{n} \binom{n}{k} (i)^k \right) \\
&\quad + \left(\sum_{k \equiv 3 \pmod 4}^{n} \binom{n}{k} (i)^k \right) + \left(\sum_{j \equiv 0 \pmod 4}^{n} \binom{n}{j} (-i)^j \right) + \left(\sum_{j \equiv 1 \pmod 4}^{n} \binom{n}{j} (-i)^j \right) + \\
&\quad \left(\sum_{j \equiv 2 \pmod 4}^{n} \binom{n}{j} (-i)^j \right) + \left(\sum_{j \equiv 3 \pmod 4}^{n} \binom{n}{j} (-i)^j \right) \\
&= 2 \cdot \left(\sum_{k \equiv 0 \pmod 4}^{n} \binom{n}{k} \right) - 2 \cdot \left(\sum_{k \equiv 2 \pmod 4}^{n} \binom{n}{k} \right) \\
&= 2 \cdot (D + 1 - C)
\end{aligned}$$

Moreover, we can also evaluate the expression on the left side in four different cases:

Observation 3.

$$\begin{cases} \text{If } n \equiv 0 \pmod 4, (1+i)^n + (1-i)^n = (-p^2)^k \cdot p & \text{where } n = 4k \\ \text{If } n \equiv 1 \pmod 4, (1+i)^n + (1-i)^n = (-p^2)^k \cdot p & \text{where } n = 4k+1 \\ \text{If } n \equiv 2 \pmod 4, (1+i)^n + (1-i)^n = 0 & \text{where } n = 4k+2 \\ \text{If } n \equiv 3 \pmod 4, (1+i)^n + (1-i)^n = (-p^2)^{k+1} & \text{where } n = 4k+3 \end{cases}$$

Proof. First note that

$$(1+i)^4 = (1-i)^4 = -4 = -p^2$$

So for any $n \equiv 0 \pmod 4$, we can write $n = 4k$ for some $k \in \mathbb{Z}^+$. Therefore we have

$$\begin{aligned} (1+i)^n + (1-i)^n &= (1+i)^{4k} + (1-i)^{4k} \\ &= (-p^2)^k + (-p^2)^k \\ &= (-p^2)^k \cdot p \end{aligned}$$

Then consider $(1+i)^1 + (1-i)^1 = p$, so for any $n \equiv 1 \pmod 4$, we can write $n = 4k+1$ for some $k \in \mathbb{Z}^+$, Therefore we have

$$\begin{aligned} (1+i)^n + (1-i)^n &= (1+i)^{4k} \cdot (1+i) + (1-i)^{4k} \cdot (1-i) \\ &= (1+i)^{4k} \cdot p \\ &= (-p^2)^k \cdot p \end{aligned}$$

Now consider $(1+i)^2 + (1-i)^2 = 0$, so for any $n \equiv 2 \pmod 4$, we have

$$\begin{aligned} (1+i)^n + (1-i)^n &= (1+i)^{4k} \cdot (1+i)^2 + (1-i)^{4k} \cdot (1-i)^2 \\ &= (1+i)^{4k} \cdot [(1+i)^2 + (1-i)^2] \\ &= 0 \end{aligned}$$

Since $(1+i)^3 + (1-i)^3 = -p^2$, so for any $n \equiv 3 \pmod 4$, we have

$$\begin{aligned} (1+i)^n + (1-i)^n &= (1+i)^{4k} \cdot (1+i)^3 + (1-i)^{4k} \cdot (1-i)^3 \\ &= (1+i)^{4k} \cdot [(1+i)^3 + (1-i)^3] \\ &= (-p^2)^k \cdot (-p^2) \\ &= (-p^2)^{k+1} \end{aligned}$$

$\square$

From Corollary 1, we obtained that $C + D + 1 = p^{n-1}$. Combining with observation 3, we then can find out the exact value of C and D:

(1) When $n \equiv 2 \pmod 4$, we have $p \cdot (D + 1 - C) = 0 \implies D + 1 = C$. So we have this system of equations:

$$\begin{aligned} D + 1 &= C \\ C + D + 1 &= p^{n-1} \end{aligned}$$

Solving this system of equations, we obtain that $C = p^{n-2}$ and $D = p^{n-2} - 1$. Substitute them into $Z_f^{(2)}(s)$, we have

$$\begin{aligned}
Z_f^{(2)}(s) &= \frac{1 - p^{-n-2s} - p^{-s} + p^{-1-s} - C \cdot p^{-n-2s} - D \cdot p^{-n-3s} \cdot \frac{1-p^{-1}}{1-p^{-1-s}}}{(1-p^{-n-2s})(1-p^{-s})} \\
&= \frac{1 - p^{-n-2s} - p^{-s} + p^{-1-s} - p^{n-2} \cdot p^{-n-2s} - (p^{n-1} - 1) \cdot p^{-n-3s} \cdot \frac{1-p^{-1}}{1-p^{-1-s}}}{(1-p^{-n-2s})(1-p^{-s})} \\
&= \frac{1 - p^{-n-2s} - p^{-s} + p^{-n-3s}}{(1-p^{-n-2s})(1-p^{-s})(1-p^{-1-s})} \\
&= \frac{1}{1-p^{-1-s}}
\end{aligned}$$

This makes sense since we obtained the same result when using stationary phase formula directly for $n = 2$.

$$Z_f^{(2)}(s) = \frac{1}{1-p^{-1-s}}$$

(2) When $n \equiv 3 \pmod 4$ and k is odd, we have the system of equations:

$$\begin{aligned}
p \cdot (D + 1 - C) &= p^{2k+2} \\
C + D + 1 &= p^{n-1}
\end{aligned}$$

We then have $C = p^{n-2} - p^{2k}$ and $D = p^{2k+1} + p^{n-2} - p^{2k} - 1$, thus:

$$\begin{aligned}
Z_f^{(2)}(s) &= \frac{1 - p^{-n-2s} - p^{-s} + p^{-1-s} - C \cdot p^{-n-2s} - D \cdot p^{-n-3s} \cdot \frac{1-p^{-1}}{1-p^{-1-s}}}{(1-p^{-n-2s})(1-p^{-s})} \\
&= \frac{1 - p^{-n-2s} - p^{-s} + p^{2k-n-2s} + p^{-n-3s} - p^{2k-n-3s}}{(1-p^{-n-2s})(1-p^{-s})(1-p^{-1-s})} \\
&= \frac{1 - p^{-n-2s} + p^{2k-n-2s}}{(1-p^{-n-2s})(1-p^{-1-s})}
\end{aligned}$$

(3) When $n \equiv 3 \pmod 4$ and k is even, we have the system of equations:

$$\begin{aligned}
p \cdot (D + 1 - C) &= -p^{2k+2} \\
C + D + 1 &= p^{n-1}
\end{aligned}$$

We then have $C = p^{n-2} + p^{2k}$ and $D = p^{n-1} - p^{n-2} - p^{2k} - 1$, thus:

$$\begin{aligned}
Z_f^{(2)}(s) &= \frac{1-p^{-n-2s}-p^{-s}+p^{-1-s}-C\cdot p^{-n-2s}-D\cdot p^{-n-3s}\cdot\frac{1-p^{-1}}{1-p^{-1-s}}}{(1-p^{-n-2s})(1-p^{-s})}\\
&= \frac{1-p^{-n-2s}-p^{-s}-p^{2k-n-2s}+p^{2k-n-3s}+p^{-n-3s}}{(1-p^{-n-2s})(1-p^{-s})(1-p^{-1-s})}\\
&= \frac{1-p^{-n-2s}-p^{2k-n-2s}}{(1-p^{-n-2s})(1-p^{-1-s})}
\end{aligned}$$

(4) When $n\equiv 0,1 \pmod 4$ and k is even, we have the system of equations:

$$\begin{aligned}
p\cdot(D+1-C) &= p^{2k+1}\\
C+D+1 &= p^{n-1}
\end{aligned}$$

We then have $C=p^{n-2}-p^{2k-1}$ and $D=p^{n-1}-p^{n-2}+p^{2k-1}-1$, thus:

$$\begin{aligned}
Z_f^{(2)}(s) &= \frac{1-p^{-n-2s}-p^{-s}+p^{-1-s}-C\cdot p^{-n-2s}-D\cdot p^{-n-3s}\cdot\frac{1-p^{-1}}{1-p^{-1-s}}}{(1-p^{-n-2s})(1-p^{-s})}\\
&= \frac{1-p^{-n-2s}-p^{-s}+p^{2k-n-1-2s}-p^{2k-n-1-3s}+p^{-n-3s}}{(1-p^{-n-2s})(1-p^{-s})(1-p^{-1-s})}\\
&= \frac{1-p^{-n-2s}+p^{2k-n-1-2s}}{(1-p^{-n-2s})(1-p^{-1-s})}
\end{aligned}$$

(5) When $n\equiv 0,1 \pmod 4$ and k is odd, we have the system of equations:

$$\begin{aligned}
p\cdot(D+1-C) &= -p^{2k+1}\\
C+D+1 &= p^{n-1}
\end{aligned}$$

We then have $C=p^{n-2}+p^{2k-1}$ and $D=p^{n-1}-p^{n-2}-p^{2k-1}-1$, thus:

$$\begin{aligned}
Z_f^{(2)}(s) &= \frac{1-p^{-n-2s}-p^{-s}+p^{-1-s}-C\cdot p^{-n-2s}-D\cdot p^{-n-3s}\cdot\frac{1-p^{-1}}{1-p^{-1-s}}}{(1-p^{-n-2s})(1-p^{-s})}\\
&= \frac{1-p^{-n-2s}-p^{-s}-p^{2k-n-1-2s}+p^{2k-n-1-3s}+p^{-n-3s}}{(1-p^{-n-2s})(1-p^{-s})(1-p^{-1-s})}\\
&= \frac{1-p^{-n-2s}-p^{2k-n-1-2s}}{(1-p^{-n-2s})(1-p^{-1-s})}
\end{aligned}$$

Therefore, we finished computing $Z_f^{(p)}(s)$ when $p=2$:

$$Z_f^{(2)}(s) = \begin{cases} \frac{1-2^{-n-2s}-2^{2k-n-1-2s}}{(1-2^{-n-2s})(1-2^{-1-s})} & n \equiv 0,1 \pmod 4 \text{, where } n = 4k \text{ or } n = 4k+1 \text{ and } k \text{ is odd} \\ \frac{1-2^{-n-2s}+2^{2k-n-1-2s}}{(1-2^{-n-2s})(1-2^{-1-s})} & n \equiv 0,1 \pmod 4 \text{, where } n = 4k \text{ or } n = 4k+1 \text{ and } k \text{ is even} \\ \frac{1}{1-2^{-1-s}} & n \equiv 2 \pmod 4 \\ \frac{1-2^{-n-2s}+2^{2k-n-2s}}{(1-2^{-n-2s})(1-2^{-1-s})} & n \equiv 3 \pmod 4 \text{, where } n = 4k+3 \text{ and } k \text{ is odd} \\ \frac{1-2^{-n-2s}-2^{2k-n-2s}}{(1-2^{-n-2s})(1-2^{-1-s})} & n \equiv 3 \pmod 4 \text{, where } n = 4k+3 \text{ and } k \text{ is even} \end{cases}$$

3.2. $p > 2$

Remark 2. *Recall the stationary phase formula and it is easy to check that $\overline{S} = \{0\}$ for any prime $p > 2$ for our choice of f. Therefore, it is expected that the computation in this section will be easier compared to the previous one.*

Now consider the case when $p > 2$ with the same polynomial:

$$f = x_1^2 + x_2^2 + \cdots + x_n^2$$

Then we have the following conditions:

$$\begin{aligned} \overline{E} = \mathbb{F}_p^n &\implies E = \mathbb{Z}_p^n \\ \overline{S} = \{0\} &\implies S = (p\mathbb{Z}_p)^n \end{aligned}$$

So the stationary phase formula says that

$$\begin{aligned} \int_{\mathbb{Z}_p^n} |f(x)|_p^s \, d^n x &= p^{-n} \cdot (p^n - N) + \frac{p^{-n-s}(1-p^{-1})(N-1)}{1-p^{-1-s}} + \int_{(p\mathbb{Z}_p)^n} |f(x)|_p^s \, d^n x \\ &= 1 - p^{-n} \cdot N + \frac{p^{-n-s}(1-p^{-1})(N-1)}{1-p^{-1-s}} + p^{-n-2s} \cdot \int_{\mathbb{Z}_p^n} |f(x)|_p^s \, d^n x \end{aligned}$$

We have to find N, which represents number of zeros of f in $\overline{E}$. Since we want to calculate:

$$\#\{(a_1, \cdots, a_n) \mid a_1^2 + a_2^2 + \cdots + a_n^n \equiv 0 \pmod p\}$$

which is equal to the following expression:

$$
\begin{aligned}
N &= \frac{1}{p} \cdot \left(\sum_{a \in \mathbb{Z}/p\mathbb{Z}} \sum_{(x_1, \cdots, x_n) \in \mathbb{Z}/p\mathbb{Z}} e^{\frac{2\pi i}{p} \cdot a(x_1^2 + \cdots + x_n^2)} \right) \\
&= \frac{1}{p} \cdot \sum_a \left(\sum_{x_1, \cdots, x_n} e^{\frac{2\pi i}{p}(ax_1^2)} \cdot e^{\frac{2\pi i}{p}(ax_2^2)} \cdots e^{\frac{2\pi i}{p}(ax_n^2)} \right) \\
&= \frac{1}{p} \cdot \sum_a \left(\sum_x (e^{\frac{2\pi i}{p} ax^2})^n \right) \\
&= \frac{1}{p} \cdot \sum_{a=0} \left(\sum_x (e^{\frac{2\pi i}{p} ax^2})^n \right) + \frac{1}{p} \sum_{a \neq 0} \left(\sum_x (e^{\frac{2\pi i}{p} ax^2})^n \right) \\
&= p^{n-1} + \frac{1}{p} \sum_{a \neq 0} (G_p(a))^n \\
&= p^{n-1} + \frac{1}{p} \sum_{a \neq 0} \left(\left(\frac{a}{p} \right) G_p(1) \right)^n
\end{aligned}
$$

where $\left(\frac{a}{p} \right)$ is the Legendre symbol and

$$
G_p(a) = \sum_{x \in \mathbb{Z}/p\mathbb{Z}} e^{\frac{2\pi i}{p} ax^2}
$$

Consider the following theorem:

Theorem 6. *Let $p > 2$ be a prime number, then*

$$
G_p(1) = \begin{cases} \sqrt{p} & p \equiv 1 \mod 4 \\ i\sqrt{p} & p \equiv 3 \mod 4 \end{cases}
$$

Proof. See [5].

□

In other words, the number N is determined by n, the number of variables, and the prime number p. Using the expression and the theorem above, we can obtain N in the following cases:

(1) Let $n \equiv 0 \pmod 4$ and $p \equiv 1 \pmod 4$, then we can write $n = 4k$ for some nonnegative integer k:

$$
\begin{aligned}
N &= p^{n-1} + \frac{1}{p} \sum_{a \neq 0} \left(\left(\frac{a}{p} \right) G_p(1) \right)^n \\
&= p^{n-1} + \frac{1}{p} \cdot (p-1) \cdot \left(\left(\frac{a}{p} \right) \sqrt{p} \right)^{4k} \\
&= p^{n-1} + \frac{1}{p} \cdot (p-1) \cdot p^{2k} \\
&= p^{n-1} + p^{2k} - p^{2k-1}
\end{aligned}
$$

(2) Let $n \equiv 0 \pmod 4$ and $p \equiv 3 \pmod 4$, then we can write $n = 4k$ and

$$\begin{aligned} N &= p^{n-1} + \frac{1}{p} \sum_{a \neq 0} \left(\left(\frac{a}{p} \right) G_p(1) \right)^n \\ &= p^{n-1} + \frac{1}{p} \cdot (p-1) \cdot \left(\left(\frac{a}{p} \right) i\sqrt{p} \right)^{4k} \\ &= p^{n-1} + \frac{1}{p} \cdot (p-1) \cdot p^{2k} \\ &= p^{n-1} + p^{2k} - p^{2k-1} \end{aligned}$$

(3) Let $n \equiv 1 \pmod 4$ and $p \equiv 1 \pmod 4$ and write $n = 4k+1$,

$$\begin{aligned} N &= p^{n-1} + \frac{1}{p} \sum_{a \neq 0} \left(\left(\frac{a}{p} \right) G_p(1) \right)^{4k+1} \\ &= p^{n-1} + \frac{1}{p} \sum_{a \neq 0} \cdot \left(\left(\frac{a}{p} \right) \sqrt{p} \right)^{4k} \cdot \left(\left(\frac{a}{p} \right) \sqrt{p} \right) \\ &= p^{n-1} \end{aligned}$$

(4) Let $n \equiv 1 \pmod 4$ and $p \equiv 3 \pmod 4$ and write $n = 4k+1$,

$$\begin{aligned} N &= p^{n-1} + \frac{1}{p} \sum_{a \neq 0} \left(\left(\frac{a}{p} \right) G_p(1) \right)^{4k+1} \\ &= p^{n-1} + \frac{1}{p} \sum_{a \neq 0} \cdot \left(\left(\frac{a}{p} \right) i\sqrt{p} \right)^{4k} \cdot \left(\left(\frac{a}{p} \right) i\sqrt{p} \right) \\ &= p^{n-1} \end{aligned}$$

(5) Let $n \equiv 2 \pmod 4$ and $p \equiv 1 \pmod 4$ and write $n = 4k+2$,

$$\begin{aligned} N &= p^{n-1} + \frac{1}{p} \sum_{a \neq 0} \left(\left(\frac{a}{p} \right) G_p(1) \right)^{4k+2} \\ &= p^{n-1} + \frac{1}{p} \sum_{a \neq 0} \cdot \left(\left(\frac{a}{p} \right) \sqrt{p} \right)^{4k} \cdot \left(\left(\frac{a}{p} \right) \sqrt{p} \right)^2 \\ &= p^{n-1} + \frac{1}{p} \cdot (p-1) \cdot p^{2k} \cdot p \\ &= p^{n-1} + p^{2k+1} - p^{2k} \end{aligned}$$

(6) Let $n \equiv 2 \pmod 4$ and $p \equiv 3 \pmod 4$ and write $n = 4k+2$,

$$\begin{aligned}
N &= p^{n-1} + \frac{1}{p}\sum_{a\neq 0}\left(\left(\frac{a}{p}\right)G_p(1)\right)^{4k+2} \\
&= p^{n-1} + \frac{1}{p}\sum_{a\neq 0}\cdot\left(\left(\frac{a}{p}\right)i\sqrt{p}\right)^{4k}\cdot\left(\left(\frac{a}{p}\right)i\sqrt{p}\right)^{2} \\
&= p^{n-1} + \frac{1}{p}\cdot(p-1)\cdot p^{2k}\cdot -p \\
&= p^{n-1} - p^{2k+1} + p^{2k}
\end{aligned}$$

(7) Let $n \equiv 3 \pmod 4$ and $p \equiv 1 \pmod 4$ and write $n = 4k+3$,

$$\begin{aligned}
N &= p^{n-1} + \frac{1}{p}\sum_{a\neq 0}\left(\left(\frac{a}{p}\right)G_p(1)\right)^{4k+3} \\
&= p^{n-1} + \frac{1}{p}\sum_{a\neq 0}\cdot\left(\left(\frac{a}{p}\right)\sqrt{p}\right)^{4k}\cdot\left(\left(\frac{a}{p}\right)\sqrt{p}\right)^{3} \\
&= p^{n-1}
\end{aligned}$$

(8) Let $n \equiv 3 \pmod 4$ and $p \equiv 3 \pmod 4$ and write $n = 4k+3$,

$$\begin{aligned}
N &= p^{n-1} + \frac{1}{p}\sum_{a\neq 0}\left(\left(\frac{a}{p}\right)G_p(1)\right)^{4k+3} \\
&= p^{n-1} + \frac{1}{p}\sum_{a\neq 0}\cdot\left(\left(\frac{a}{p}\right)i\sqrt{p}\right)^{4k}\cdot\left(\left(\frac{a}{p}\right)i\sqrt{p}\right)^{3} \\
&= p^{n-1}
\end{aligned}$$

In conclusion, we have

$$N = \begin{cases}
p^{n-1} + p^{2k} - p^{2k-1} & n \equiv 0 \pmod 4 \text{ and } p \equiv 1,3 \pmod 4 \\
p^{n-1} & n \equiv 1,3 \pmod 4 \text{ and } p \equiv 1,3 \pmod 4 \\
p^{n-1} + p^{2k+1} - p^{2k} & n \equiv 2 \pmod 4 \text{ and } p \equiv 1 \pmod 4 \\
p^{n-1} - p^{2k+1} + p^{2k} & n \equiv 2 \pmod 4 \text{ and } p \equiv 3 \pmod 4
\end{cases}$$

Once we find out the exact value for N and using stationary phase formula, we then can find $Z_f^{(p)}(s)$ when $p > 2$:

$$Z_f^{(p)}(s) = \begin{cases}
\frac{1-p^{-n-1-2s}+p^{2k-n-s}-p^{2k-n-1-s}}{(1-p^{-n-2s})(1-p^{-1-s})} & n \equiv 0 \pmod 4 \text{ and } p \equiv 1,3 \pmod 4 \\
\frac{1-p^{-n-1-2s}}{(1-p^{-n-2s})(1-p^{-1-s})} & n \equiv 1,3 \pmod 4 \text{ and } p \equiv 1,3 \pmod 4 \\
\frac{1-p^{-n-1-2s}+\chi(p)p^{2k+1-n-s}-p^{2k-n-s}}{(1-p^{-n-2s})(1-p^{-1-s})} & n \equiv 2 \pmod 4
\end{cases}$$

where the character χ is defined on prime numbers:

$$\chi(p) = \begin{cases}
1 & p \equiv 1 \pmod 4 \\
-1 & p \equiv 3 \pmod 4 \\
0 & p = 2
\end{cases}$$

Now recall theorem 3 and we can rewrite global zeta function $Z_f(s)$ as a Euler product, that is:

$$\begin{aligned}\sum_{d=1}^{\infty} \frac{N_f(d)}{d^{s+n}} &= \prod_{p\in\mathbb{P}} \left(\sum_{k=0}^{\infty} \frac{N_f(p^k)}{p^{k(s+n)}} \right) \\ &= \prod_{p=2} \left(\sum_{k=0}^{\infty} \frac{N_f(p^k)}{p^{k(s+n)}} \right) \cdot \prod_{p>2} \left(\sum_{k=0}^{\infty} \frac{N_f(p^k)}{p^{k(s+n)}} \right)\end{aligned}$$

For each prime number p, substitute our results for $Z_f^{(p)}(s)$. Then $Z_f(s)$ will look like:

$$Z_f(s) = \sum_{d=1}^{\infty} \frac{N_f(d)}{d^{s+n}} = \begin{cases} \frac{(1-2^{-n-2s}+2^{-\frac{n}{2}-1-2s})\zeta(n+2s)\zeta(1+s)\zeta(\frac{n}{2}+s)}{(1-2^{-\frac{n}{2}-1-s})(1+2^{-\frac{n}{2}-s})\zeta(\frac{n}{2}+1+s)\zeta(n+2s)} & \text{where } n \equiv 4 \pmod 8 \\ \frac{(1-2^{-n-2s}-2^{-\frac{n}{2}-1-2s})\zeta(n+2s)\zeta(1+s)\zeta(\frac{n}{2}+s)}{(1-2^{-\frac{n}{2}-1-s})(1+2^{-\frac{n}{2}-s})\zeta(\frac{n}{2}+1+s)\zeta(n+2s)} & \text{where } n \equiv 0 \pmod 8 \\ \frac{(1-2^{-n-2s}-2^{-\frac{n}{2}-\frac{3}{2}-2s})\zeta(n+2s)\zeta(1+s)}{(1-2^{-n-1-2s})\zeta(n+1+2s)} & \text{where } n \equiv 5 \pmod 8 \\ \frac{(1-2^{-n-2s}+2^{-\frac{n}{2}-\frac{3}{2}-2s})\zeta(n+2s)\zeta(1+s)}{(1-2^{-n-1-2s})\zeta(n+1+2s)} & \text{where } n \equiv 1 \pmod 8 \\ \frac{(1-2^{-n-2s}+2^{-\frac{n}{2}-\frac{3}{2}-2s})\zeta(n+2s)\zeta(1+s)}{(1-2^{-n-1-2s})\zeta(n+1+2s)} & \text{where } n \equiv 7 \pmod 8 \\ \frac{(1-2^{-n-2s}-2^{-\frac{n}{2}-\frac{3}{2}-2s})\zeta(n+2s)\zeta(1+s)}{(1-2^{-n-1-2s})\zeta(n+1+2s)} & \text{where } n \equiv 8 \pmod 3 \end{cases}$$

Recall that Riemann zeta function is defined as usual:

$$\zeta(s) = \prod_{p\in\mathbb{P}} \frac{1}{(1-p^{-s})}$$

In particular, when $n \equiv 2 \pmod 4$, we have that

$$\sum_{d=1}^{\infty} \frac{N_f(d)}{d^{s+n}} = \frac{(1-2^{-n-2s})\zeta(1+s)\zeta(n+2s)}{L(\frac{n}{2}+1+s,\chi)L(\frac{n}{2}+s,\chi)}$$

where $L(s,\chi) = \prod_p \frac{1}{1-\chi(p)p^{-s}}$ is the Dirichlet L-function [4] associated to the character χ that is defined above.

4. Conclusion

After the computations, we can explicitly write the global zeta function $Z_f(s)$ as a combination of both Dirichlet L-function and local factors of Riemann zeta functions, where f is sum of squares with n variables.

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