# ON APPLICATIONS OF KHOVANOV HOMOLOGY

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In 1999, Khovanov constructed a combinatorial categorification of the Jones polynomial. Since then there has been a question of to what extent the topology of a link is reflected in his homology theory and how Khovanov homology can be used for topological applications. This dissertation compiles some of the authors contributions to these avenues of mathematical inquiry.

In the first chapter, we prove that for a fixed braid index there are only finitely many possible shapes of the annular Rasmussen $d_t$ invariant of braid closures. Focusing on the case of 3-braids, we compute the Rasmussen $s$-invariant and the annular Rasmussen $d_t$ invariant of all 3-braid closures. As a corollary, we show that the vanishing/non-vanishing of the $\psi$ invariant is entirely determined by the $s$-invariant and the self-linking number for 3-braid closures.

In the second chapter, we show if $L$ is any link in $S^3$ whose Khovanov homology is isomorphic to the Khovanov homology of $T(2,6)$ then $L$ is isotopic to $T(2,6)$. We show this for unreduced Khovanov homology with $\mathbb{Z}$ coefficients.

Finally in the third chapter, we exhibit infinite families of annular links for which the maximum non-zero annular Khovanov grading grows infinitely large but the maximum non-zero annular Floer-theoretic gradings are bounded. We also show this phenomenon exists at the decategorified level for some of the infinite families. Our computations provide further evidence for the wrapping conjecture of Hoste-Przytycki and its categorified analogue. Additionally, we show that certain satellite operations cannot be used to construct counterexamples to the categorified wrapping conjecture.

# Contents

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# DEDICATION

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My life would not have been so bright or enjoyable without Karen in it. No matter how hard grad school may have felt she was always there to support me and I can't find words to express how important she is.

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# Chapter 1

# Introduction

Low-dimensional topology is the study of manifolds in dimensions of 4 and lower. One of the ways to study manifolds is by studying the knotting of embedded submanifolds, this perspective on manifolds, often referred to as knot theory, has a long history especially in 3-dimensions.

While the field of knot theory has existed for over a century there have been shifts in perspective as the field has development. A recent perspective has been to take a knot or link and associate graded vector spaces, coming from the homology of a chain complex. Often these algebraic invariants are drawing from either gauge theory or representation theory.

Khovanov homology is a specific knot homology theory coming from ideas in higher representation theory and also has a straightforward combinatorial definition. The combinatorial definition of Khovanov homology makes it relatively easy to compute but it is often unclear how much topological information is contained within Khovanov homology. Trying to determine the extent to which topological properties of a link are reflected in Khovanov homology has been an ongoing theme in recent research.

In spite of the difficulties, there have been applications of Khovanov homology to questions about knots and other objects in low-dimensional topology. Sometimes the applications of Khovanov homology have used analogies or specific algebraic relationships to Floer homology theories. The some of the first topological applications of Khovanov homology were when J. Rasmussen's gauge theory free proof of the Milnor conjecture [52] and Kronheimer-Mrowka's proof using a spectral sequence to a gauge theoretic invariant that Khovanov homology detects the unknot [36]. More recently, Khovanov homology was used by Piccirillo in her celebrated proof that the Conway knot does not bound a smooth disk in the 4-ball [49].

This dissertation is a compilation of some of the author's work on applying tools from Khovanov homology to study low-dimensional topology. Chapter 2 contains work understanding and computing invariants of braid closures defined with Khovanov homology as well as relating these invariants to dynamical properties of the braids and comparing them with invariants defined with Floer homology. In Chapter 3 gives an application of Khovanov homology to the question of link detection. Finally, Chapter 4 investigates a long standing conjecture on the relationship between Khovanov homology and the annular wrapping number and highlights a stark contrast between Khovanov homology and Floer theories.

In addition to the research presented in the following chapters, the author has also collaborated on other applications of Khovanov homology and knot Floer homology. Questions about link detection and knot Floer homology was explored in joint work with Binns and applications of Khovanov homology to detecting braid conjugacy classes were also given [13]. Work with Chernov and Petkova illuminated some applications of knot homology theories to mathematical physics and causality [18].

# Chapter 2

# Annular Rasmussen invariants: Properties and 3-braid classification

## 2.1 Introduction

In [35], Khovanov defined a bigraded homology theory $\mathrm{Kh}^{i,j}(L)$ associated to an oriented link $L \subseteq S^3$ [1]. Later in [37], Lee defined a homology theory $\mathrm{Lee}(L)$ by adding additional differentials to the Khovanov chain complex. Lee also showed that the total rank of $\mathrm{Lee}(L)$ is $2^{|L|}$ where $|L|$ is the number of components of $L$.

For a knot $K$, J. Rasmussen used a $\mathbb{Z}$ filtration on Lee homology to define an invariant $s(K)$ [52]. His invariant gives a lower bound on the smooth 4-ball genus of a knot $K$ and is strong enough to give a combinatorial proof of the Milnor conjecture about the smooth 4-ball genus of torus knots [43]. The definition of the $s$-invariant was later extended to oriented links by Beliakova and Wehrli [12]. Pardon gives a slightly different extension of the Rasmussen invariant to oriented links in [48] but in

[1] This article will appear in a forthcoming issue of Michigan Mathematical Journal.

the present paper we use the extension by Beliakova and Wehrli.

In a slightly different direction, Asaeda, Przytycki, and Sikora [1] and L. Roberts [53] define a triply-graded version of Khovanov homology called annular Khovanov homology for oriented links $L$ embedded in a thickened annulus $A \times I$. Additionally if the thickened annulus is embedded in $S^3$ so that is it unknotted, then the additional grading on annular Khovanov homology of $L$ induces a $\mathbb{Z}$ filtration on the standard Khovanov homology of $L$. Annular Khovanov homology detects the trivial braid closure [5] and detects some non-conjugate braids related by exchange moves [28].

Combining these two directions, Grigsby, A. Licata, and Wehrli [21] show that the Lee complex of an oriented annular link $L \subset A \times I \subset S^3$ is $\mathbb{Z} \oplus \mathbb{Z}$ filtered. From this data, using ideas of Ozsváth, Stipsicz, and Szabó [47], as reinterpreted by Livingston [40], they construct a piecewise linear function $d_t(L)$ called the annular Rasmussen invariant. Grigsby, A. Licata, and Wehrli show that for braid closures $d_t(\widehat{\beta})$ can be used to show that $\widehat{\beta}$ is right-veering and also to show that $\widehat{\beta}$ is not quasipositive. At $t = 0$, the $d_t$ invariant recovers the $s$-invariant of $L$ by $s(L) - 1 = d_0(L)$.

In the present paper we investigate the $\mathbb{Z} \oplus \mathbb{Z}$ filtered Khovanov-Lee complex of braid closures and use its algebraic structure to obtain strong restrictions on the annular Rasmussen invariant.

**Theorem 2.3.4.** For a fixed $n$, there is a finite set of piecewise linear functions $f_i : [0, 1] \to \mathbb{R}$ so that for any $n$-braid $\beta$ there is some $j$ so that $d_t(\widehat{\beta}) = f_j + w(\widehat{\beta})$. Furthermore there is an explicit method for enumerating all the functions $f_i$.

**Remark 2.1.1.** For 3-braids, the number of possible shapes of $d_t(\widehat{\beta})$ is three. For 4-braids, the number is seven and for 5-braids there are 18 possible shapes.

Because the value of $d_t(\widehat{\beta})$ at $t = 0$ is $s(\widehat{\beta}) - 1$, one may hope that Theorem 2.3.4 provides a new upper bound on the $s$-invariant of braid closures. Unfortunately, the upper bound provided by Theorem 2.3.4 for a braid $\beta$ with braid index $n$, i.e. an $n$-braid, is $s(\widehat{\beta}) \leq w(\widehat{\beta})+n-2$ which is never better than the bounds coming from [42] when Lobb's upper bound $U(D)$ is computed for the diagram $D = \widehat{\beta}$.

Applying this perspective on $d_t(\widehat{\beta})$ to 3-braids we get that the $d_t$ invariant of any 3-braid closure $\widehat{\beta}$ depends only on the writhe $w(\widehat{\beta})$ of the braid closure and the Rasmussen invariant $s(\widehat{\beta})$ of the closure.

**Theorem 2.3.7.** When $\beta$ is a 3-braid, then for $t$ between 0 and 1 one of the following holds: $d_t(\widehat{\beta}) = w(\beta) - 3 + 3t$, $d_t(\widehat{\beta}) = w(\widehat{\beta}) - 1 + t$ or $d_t(\widehat{\beta}) = w(\widehat{\beta}) + 1 - t$.

In other words, if $\beta$ is a 3-braid then $d_t(\widehat{\beta})$ is entirely determined by the $s$-invariant and the writhe.

With Theorem 2.3.7 in mind, all that is needed to compute the $d_t$ invariants of 3-braid closures is to compute the $s$-invariant of all 3-braid closures. Focusing on 3-braid closures allows us to use Murasugi's classification of 3-braids up to conjugacy. A discussion of why Murasugi's classification is used here instead of a more recent approach to the conjugacy problem for 3-braids can be found in Remark 2.2.2. By understanding how the $s$-invariant changes under adding crossings we are able to compute the $s$-invariant of all 3-braid closures.

**Theorem 2.4.1.** A 3-braid $\beta$ has $s(\widehat{\beta}) = w(\widehat{\beta}) - 2$ if and only if $\beta$ is conjugate to a braid of the form:

1. $h^d\sigma_1\sigma_2^{-a_1}\cdots\sigma_1\sigma_2^{-a_n}$ with $a_i \geq 0$ and some $a_i > 0$ and $d > 0$.
2. $h^d\sigma_2^m$ with $m \in \mathbb{Z}$ and either $d = 0$ $m \geq 0$, $d = 1$ $m \geq -4$, or $d > 1$.
3. $h^d\sigma_1^m\sigma_2^{-1}$ with $m \in \{-1, -2, -3\}$ and $d > 0$.

**Theorem 2.4.2.** A 3-braid $\beta$ has $s(\widehat{\beta}) = w(\widehat{\beta}) + 2$ if and only if the mirror of $\beta$ is conjugate to a braid of the form:

1. $h^d\sigma_1\sigma_2^{-a_1}\cdots\sigma_1\sigma_2^{-a_n}$ with $a_i \geq 0$ and some $a_i > 0$ and $d > 0$.

2. $h^d\sigma_2^m$ with $m \in \mathbb{Z}$ and either $d = 1$ $m \geq -3$, or $d > 1$.

3. $h^d\sigma_1^m\sigma_2^{-1}$ with $m \in \{-1, -2, -3\}$ and $d > 0$.

**Theorem 2.4.3.** All other 3-braids have $s(\widehat{\beta}) = w(\widehat{\beta})$.

It is a natural question to ask if this computation gives new obstructions to the sliceness of 3-braid closures. However the only 3-braids where it is not know if they are slice are known to have finite concordance order and so their $s$-invariants are necessarily zero [39].

Using braid foliations, Birman and Menasco completely classify which links are closures of 3-braids and in particular they find two infinite families of non-conjugate 3-braids whose closures are isotopic as links [15]. Knowing that the $d_t$ invariants of a braid closure are invariants of the conjugacy class of of the braid, one may ask if the $d_t$ invariants of 3-braid closures can detect these families of non-conjugate 3-braids. However, the $d_t$ invariants of 3-braid closures only detect the writhe of the braid closure and an invariant of the isotopy class of the braid closure, the $s$-invariant, so they can not distinguish the non-conjugate 3-braids.

In [50], Plamenevskaya introduced an invariant of transverse isotopy $\psi$ using Khovanov homology. This invariant can be thought of as an invariant of braid closures $\widehat{\beta}$ up to positive stabilization. An open question is if this invariant contains more information than the self-linking number of $\widehat{\beta}$ and the $s$-invariant.

By comparing the value of the $s$-invariant of 3-braid closures with the vanishing/non-vanishing of $\psi$ we obtain the following corollary of Theorem 2.4.1.

**Corollary 2.5.3.** The invariant $\psi$ is not effective for 3-braid closures. In particular, the vanishing/non-vanishing of $\psi$ for 3-braids is determined by the $s$-invariant and the self-linking number.

It is currently an open question if $\psi$ is an effective transverse invariant and relatedly it is also unknown if the vanishing/non-vanishing of $\psi$ is always determined only

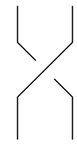


Figure 2.1: A local diagram for the generator $\sigma_i$

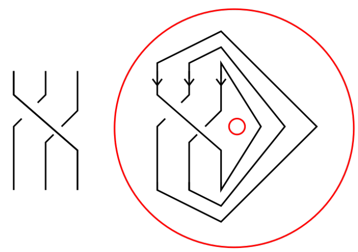

Figure 2.2: A braid and the annular closure of the braid equipped with the braid orientation

by the $s$-invariant and the self-linking number.

The organization of the paper is as follows. In Section 2.2, we review some basic facts about 3-braids and braid closure invariants from Khovanov homology. In Section 2.3, we investigate new properties of the annular Rasmussen $d_t$ invariants and prove Theorem 2.3.4. In Section 2.4, we compute the Rasmussen $s$-invariant for 3-braid closures and prove Theorems 2.4.1, 2.4.2, and 2.4.3. In Section 2.5, we compare our computations of the $s$-invariant of 3-braid closures to the computations of other categorified invariants of 3-braid closures. Section 2.6 contains an explicit computation of the $s$-invariant of a single 3-braid closure that was not computable via the methods used in Section 2.4.

## 2.2 Background

### 2.2.1 3-braids

A braid on $n$ strands is a proper embedding of $n$ disjoint copies of the interval $I = [0, 1]$ into $D^2 \times [0, 1]$ where each interval intersects every slice $D^2 \times t, t \in [0, 1]$ in exactly one point up to isotopy through such embeddings. The collection of all braids on $n$

strands form a group $B_n$ with multiplication given by stacking braids. The group $B_n$ has the Artin presentation with generators $\sigma_i$ for $1 \leq i \leq n-1$ and the relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| \geq 2$ and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$. Diagrammatically, the generator $\sigma_i$ is the crossing in Figure 2.1 between the $i$-th and $i+1$-st strands. The group $B_n$ can be identified with $\mathrm{Mod}(D_n)$ the mapping class group of the disk with $n$ punctures. See [14] for more about braids.

Given a braid $\beta$, there is a link $\widehat{\beta}$ in the thickened annulus associated to $\beta$ called the annular closure of $\beta$. The braid closure $\widehat{\beta}$ has an orientation called the braid orientation induced on it by the braid $\beta$ where all the strands of $\beta$ are oriented to flow from the top of the braid to the bottom. See Figure 2.2 for a diagram of the annular closure with the braid orientation. Two oriented annular braid closures $\hat{\beta}_1$ and $\hat{\beta}_2$ represent the same oriented annular link if and only if $\beta_1$ and $\beta_2$ are conjugate braids. All braid closures are assumed to be equipped with the braid orientation.

A braid $\beta$ is said to be quasipositive if it can be expressed as a product of conjugates of positive generators, that is if $\beta = \prod w_i\sigma_{j_i}w_i^{-1}$ for some words $w_i \in B_n$. A braid $\beta$ is said to be right-veering if it sends every arc $\gamma$ running from $\partial D$ to one of the marked points to the right. See Section 2.2 of [51] for more details and a precise definition. All quasipositive braids are right-veering, but right-veering braids are often no quasipositive, see for instances examples found in [51].

For each braid group $B_n$ there is a group homomorphism $w : B_n \to \mathbb{Z}$ given by $\sigma_i \to 1$ for every $i$. The image $w(\beta)$ is an invariant of the annular closure $\widehat{\beta}$ called the writhe $w(\widehat{\beta})$. The mirror of a braid $\beta$ is the image of $\beta$ under the homomorphism $m : B_n \to B_n$ given by $\sigma_i \to \sigma_i^{-1}$ for every $i$. The annular closure $\widehat{m(\beta)}$ is the topological mirror of the annular closure $\widehat{\beta}$. Note that $w(\widehat{m(\beta)}) = -w(\widehat{\beta})$.

Murasugi classified all 3-braids up to conjugation as having exactly one of the following three forms.

**Lemma 2.2.1** (Murasugi's classification of 3-braids [44])**.** *Every* 3*-braid is conjugate* *to one of the following braids:*

1. $h^d \sigma_1 \sigma_2^{-a_1} \cdots \sigma_1 \sigma_2^{-a_n}$ *with* $d \in \mathbb{Z}$, $a_i \geq 0$ *and some* $a_i > 0$.

2. $h^d \sigma_2^m$ *with* $d \in \mathbb{Z}$ *and* $m \in \mathbb{Z}$.

3. $h^d \sigma_1^m \sigma_2^{-1}$ *with* $d \in \mathbb{Z}$ *and* $m \in \{-1, -2, -3\}$.

*where* $h = (\sigma_1 \sigma_2)^3$.

In what follows, we will refer to the three different forms for a 3-braid listed above as families 1, 2, and 3 respectively.

**Remark 2.2.2.** Since the time since Murasugi published his classification of 3-braids, much more has been written about the structure of 3-braids. Specifically, work of Xu provided a new solution to the conjugacy problem for 3-braids and a solution to the shortest word problem for 3-braids [60]. Xu's approach also generalizes to solving the conjugacy problem for $n$-braids. The reason why we use Murasugi's approach in what follows is because Murasugi's classification is connected to properties of the braid $\beta$ in the mapping class group of the disk with 3 punctures. It was previously known that the $d_t$ invariants of braid closures are connected to certain properties of the conjugacy class of $\beta$ viewed as a mapping class, for example if the conjugacy class is right-veering. A summary of Xu's solution to the conjugacy problem for 3-braids along with a survey of recent work on 3-braids can be found in [17].

### 2.2.2 Braid closure invariants from Khovanov homology

Given a diagram of an annular braid closure $\widehat{\beta}$, we will build two $\mathbb{Z} \oplus \mathbb{Z}$ filtered chain complexes using constructions of Khovanov [35] and Lee [37] along with ideas from J. Rasmussen [52], Asaeda-Przytycki-Sikora [1], and L. Roberts [53]. Given a braid closure $\widehat{\beta}$, we can form the cube of resolutions of $\widehat{\beta}$ as in Section 4.2 of [35]. This cube of resolutions is then assigned a triply graded vector space $C^{i,j,k}(\widehat{\beta})$. The $i, j$, and

$k$ gradings are the homological, quantum, and annular gradings respectively. This triply graded vector space can be equipped with differentials $\partial$ and $\Phi$ so that the homologies with respect to $\partial$ and with respect to $\partial + \Phi$ are called the Khovanov homology $\mathrm{Kh}^{i,j}(\widehat{\beta})$ and the Lee homology $\mathrm{Lee}^{i}(\widehat{\beta})$ respectively. Both homologies are invariants of the oriented link $L \subseteq S^3$ represented by $\widehat{\beta}$. Because $\partial$ preserves the $j$ grading, Khovanov homology is bi-graded with $i$ and $j$ gradings and filtered by $j-2k$. Lee homology is $i$ graded and $\mathbb{Z} \oplus \mathbb{Z}$ filtered by $j$ and $j-2k$. The $\mathbb{Z} \oplus \mathbb{Z}$ filtered chain homotopy type of Lee homology is an invariant of the annular closure of $\widehat{\beta}$. The chain complex for Lee homology decomposes into two subcomplexes $L_1$ and $L_2$ which represent the two non-zero quantum gradings of vectors mod 4 because $\partial$ preserves the $j$ grading and $\Phi$ increases it by 4.

The vector space $C^{i,j,k}(\widehat{\beta})$ has a basis of distinguished generators that are all homogeneous with respect to each grading. A complete resolution of $\widehat{\beta}$ is a choice of smoothing for each crossing of $\widehat{\beta}$. A complete resolution consists of a collection of circles in the annulus. Some of the cirlces wrap around the puncture of the annulus while others will not. Associated to each vertex of the cube of resolutions of $\widehat{\beta}$ is a complete resolution. Generators of $C^{i,j,k}(\widehat{\beta})$ are labellings of each circle of this complete resolution with either a $+$ sign or a $-$ sign. Each circle with a $+$ increases the quantum grading by one and each circle with a $-$ decreases the quantum grading by one. The quantum grading also increases by one for each 1 resolution used in making the complete resolution. Only the labels of circles that wrap around the annulus are counted for the annular grading. Those that are labeled with a $+$ increase the annular grading by one and the ones labeled with a $-$ decrease the annular grading by one. For an $n$ braid, the annular grading of any generator is in the set $\{-n, -n+2, \dots, n-2, n\}$. The homological grading tracks the number of 1 resolutions used and is shifted so that the oriented resolution coming from the braid orientation $o$ then is in homological grading 0. The quantum grading is given an overall shift so that the generator in the oriented resolution with all the circles labelled with $-$ sits in quantum grading $sl(\widehat{\beta})$.

Lee showed that if $\widehat{\beta}$ is a link with $m$ components then $\mathrm{Lee}(\widehat{\beta})$ has dimension $2^m$ and for each possible orientation $o$ of $\widehat{\beta}$ constructed an explicit cycle $\mathfrak{s}_o$ so that the homology classes of these cycles generate $\mathrm{Lee}(\widehat{\beta})$ [37]. The class $\mathfrak{s}_0$ is a linear combination of distinguished generators in the oriented resolution coming from $o$. Each possible labelling of the resolution appears in the linear combination with coefficient $\pm 1$ depending on the orientation and which circles are labeled with a $+$ and which are labeled with $-$. The reader can refer to [52] for a detailed description of how to construct the cycle $\mathfrak{s}_o$.

Using Lee homology, J. Rasmussen defined the $s$-invariant for knots in $S^3$ and his definition was extended to oriented links in $S^3$ by Beliakova and Wehrli [52] [12]. We recall some basic properties of this invariant. For a braid closure $\widehat{\beta}$ with its braid orientation $o$ there is a non-zero homology class $[\mathfrak{s}_o]$ in $\mathrm{Lee}^0(\widehat{\beta})$. It follows immediately from Definition 3.4 of [52] that $s(\widehat{\beta}) = \max\{gr_j(x) \mid [x] = [\mathfrak{s}_o]\} + 1$ where $gr_j$ is the filtered $j$ degree. The $s$-invariant gives bounds on the four ball genus of knots and links and is a group homomorphism from the smooth knot concordance group to $\mathbb{Z}$ [52] [12].

**Proposition 2.2.3** (Proposition 3.9 of [52])**.** *If $K$ is a knot then $s(m(K)) = -s(K)$.*

The following proposition follows directly from Lemma 3.5 of [52].

**Proposition 2.2.4.** *For an oriented link $L$, the homology class $[\mathfrak{s}_o]$ is non-zero in both subcomplexes of* $\mathrm{Lee}(L)$*.*

*Proof from Lemma 3.5 of [52].* Lemma 3.5 of [52] gives that $[\mathfrak{s}_o]+[\mathfrak{s}_{\bar{o}}]$ and $[\mathfrak{s}_o]-[\mathfrak{s}_{\bar{o}}]$ are contained in separate subcomplexes of $\mathrm{Lee}(L)$. The proof of the Lemma shows that $\pm[\mathfrak{s}_{\bar{o}}]$ is the image of $[\mathfrak{s}_o]$ under some automorphism of $\mathrm{Lee}(L)$ that preserves the two subcomplexes. Therefore both $[\mathfrak{s}_o]$ and $[\mathfrak{s}_{\bar{o}}]$ must be non-zero in each subcomplex. $\square$

By considering the additional $j-2k$ filtration on Lee homology, Grigsby, A. Licata, and Wehrli extended the $s$-invariant to a family of annular braid closure invariants $d_t$ one for each $t$ in the interval $[0,2]$ [21]. This family of invariants is also called the annular Rasmussen invariants. For each $t \in [0,2]$ the Lee homology has a filtration $j_t = j - tk$ so for each $t$ it is possible to define $d_t(\widehat{\beta}) = \max\{j_t(x) \mid [x] = [\mathfrak{s}_o]\}$. If a generator $x$ with $[x] = [\mathfrak{s}_o]$ satisfies $j_{t_0}(x) = d_{t_0}(\widehat{\beta})$ then $x$ is said to determine $d_t(\widehat{\beta})$ at $t_0$. From the definition it is immediate that $d_0(\widehat{\beta}) = s(\widehat{\beta}) - 1$. After defining the invariant, they show that it has a symmetry $d_{1-t}(\widehat{\beta}) = d_{1+t}(\widehat{\beta})$ for $t \in [0,1]$ [21]. Because of the symmetry, in the rest of this paper we restrict to the interval $[0,1]$ without losing information about the invariant $d_t$. We recall key properties of the $d_t$ invariant needed for the present work.

**Proposition 2.2.5** (Theorem 1 of [21] and its proof)**.** *The $d_t$ invariant is a piecewise linear function on $[0,1]$ and the right-hand slope of $d_t(\widehat{\beta})$, called $m_t(\widehat{\beta})$, at any $t \in [0,1]$ is the negative $k$ grading of the cycle $x$ with $[x] = [\mathfrak{s}_o]$ and $j_t(x) = d_t(\widehat{\beta})$.*

**Proposition 2.2.6** (Theorem 3 of [21])**.** *Let $\widehat{\beta}$ be a braid closure with writhe $w(\widehat{\beta})$, then $d_1(\widehat{\beta}) = w(\widehat{\beta})$.*

**Proposition 2.2.7** (Theorem 1 of [21])**.** *Let $S$ be an oriented cobordism with $n$ odd index critical points and no even index critical points between two braid closures $\widehat{\beta}$ and $\widehat{\beta'}$ so that every component of $S$ has boundary components in both $\widehat{\beta}$ and $\widehat{\beta'}$. Then the cobordism $S$ gives a bound on the difference of their $d_t$ invariants $|d_t(\widehat{\beta}) - d_t(\widehat{\beta'})| \leq n$.*

**Proposition 2.2.8** (Proposition 4 of [21])**.** *If $\beta_0$ and $\beta_1$ are braids and $\beta_2$ is the braid that is the disjoint union of $\beta_0$ and $\beta_1$ then $d_t(\widehat{\beta_2}) = d_t(\widehat{\beta_1}) + d_t(\widehat{\beta_2})$.*

Given a braid closure $\widehat{\beta}$, Plamenevskaya showed how to associate to $\widehat{\beta}$ a homology class $\psi(\widehat{\beta})$ in Khovanov homology which is well-defined up to sign [50]. The homology class $\psi(\widehat{\beta})$ is invariant not only under braid conjugation but also under positive stabilization and destabilization and so it is an invariant of the transverse isotopy class of $\widehat{\beta}$. We recall the definition and a few properties of this invariant. The invariant is defined to be the homology class of the distinguished generator $v_-$ where all the circles of the oriented resolution are labeled with a $-$ sign in the Khovanov homology of $\widehat{\beta}$.

**Proposition 2.2.9** (Proposition 3.1 of [5]). *If $\psi(\widehat{\beta}) \neq 0$ then $\beta$ is right-veering.*

Note that there are examples of right-veering braids whose transverse invariants $\psi$ vanish [51].

**Proposition 2.2.10** ([50]). *For any $n$-braid, if $s(\widehat{\beta}) - 1 = w(\widehat{\beta}) - n$ then $\psi(\widehat{\beta}) \neq 0$.*

This proposition includes all quasipositive braids as examples. More generally for any $n$-braid, if $d_t(\widehat{\beta})$ has slope $n$ at any point in the interval $[0, 1)$ then $\psi(\widehat{\beta}) \neq 0$ [21].

The transverse invariant $\psi$ is also functorial in the the following sense.

**Proposition 2.2.11** (Theorem 4 of [50]). *If $\beta'$ is a braid obtained from $\beta$ by adding a positive crossing then the crossing change induces a map $f : \mathrm{Kh}(\widehat{\beta'}) \to \mathrm{Kh}(\widehat{\beta})$ and $f(\psi(\widehat{\beta'})) = \pm\psi(\widehat{\beta})$*

In particular if $\psi(\widehat{\beta}) \neq 0$ then $\psi(\widehat{\beta'}) \neq 0$; equivalently if $\psi(\widehat{\beta'}) = 0$ then $\psi(\widehat{\beta}) = 0$.

For 3-braids, it is known exactly when the invariant $\psi$ is non-zero. This result was well known to experts but the author is not aware of a complete proof in the literature. The case when $d > 1$ is also treated in [29]. The strategy of proof in [29] is essentially the same as in the proof produced here.

**Lemma 2.2.12.** *A 3-braid $\beta$ has $\psi(\widehat{\beta}) \neq 0$ if and only if $\beta$ is conjugate to one of the following braids.*

1. *$h^d \sigma_1 \sigma_2^{-a_1} \cdots \sigma_1 \sigma_2^{-a_n}$ with all $a_i \geq 0$ and some $a_i > 0$ and $d > 0$.*
2. *$h^d \sigma_2^m$ with $m \in \mathbb{Z}$ and either $d = 0$ and $m \geq 0$, $d = 1$ and $m \geq -4$, or $d > 1$.*
3. *$h^d \sigma_1^m \sigma_2^{-1}$ with $m \in \{-1, -2, -3\}$ and $d > 0$.*

*Proof.* The Murasugi form of $\beta$ is in one of the three families. For the first family, when $d < 1$ the braids are not right-veering and so their $\psi$ invariants vanish by Proposition 2.2.9. When $d = 1$ the argument that these braids are quasi-alternating gives you that the $\psi$ invariants are non-vanishing (see Remark 7.6 of [7]). For $d > 1$ these braids can be obtained from braids in this family with $d = 1$ by adding positive crossings and so their $\psi$ invariants are also non-vanishing by Proposition 2.2.11.

For the second family, when $d < 0$ or $d = 0$ and $m < 0$ the braids are all not right-veering so their $\psi$ invariants vanish by Proposition 2.2.9 [51]. When $d = 1$ and $m = -5$ a calculation shows that the $\psi$ invariant vanishes and so it also vanishes when $m < -5$ by Proposition 2.2.11 [7]. When $d = 1$ and $m = -4$, this braid is quasipositive so the $\psi$ invariant is non-zero by Proposition 2.2.10 and then so are the $\psi$ invariants of all braids with $d = 1$ and $m > -4$ by Proposition 2.2.11. Finally, if $d > 1$ then the braids can be obtained by adding positive crossings to braids in the first family with non-vanishing $\psi$ invariants and so the $\psi$ invariant is non-zero for these braids as well by Proposition 2.2.11.

For the third family, note that when $d \leq 0$ the braids are not right-veering and so their $\psi$ invariants vanish by Proposition 2.2.9 and when $d > 0$ the braids are quasipositive so their $\psi$ invariants are all non-zero by Proposition 2.2.10 [51].

□

## 2.3 Additional properties of the annular Rasmussen invariant

In this section we aim to place constraints on the $d_t$ invariant of a braid. These constraints will limit the possible shapes of $d_t$. For a 3-braid, these constraints will imply that $d_t$ is determined entirely by the $s$-invariant and the writhe of the braid.

Throughout the remaining sections of the paper, at times we use a graphical perspective to describe $d_t(\widehat{\beta})$ and the $j_t$ gradings of generators. If a generator $x$ has grading $(j, k)$ then the $j_t$ gradings of the generator $x$ can be plotted as the line from $t = 0$ to $t = 2$ starting at $(0, j)$ of slope $-k$. Then the graph for $d_t(\widehat{\beta})$ can be drawn by following the plots for the generators determining $d_t(\widehat{\beta})$ for different values of $t$.

**Proposition 2.3.1.** *If $\beta$ is a negative $n$ braid, then the slope $m_t(\widehat{\beta})$ is in the set $\{n, n-2, \ldots, -n+2\}$ for $t \in (0, 1)$.*

In [21] it was proved that $m_t(\widehat{\beta})$ is in the set $\{n, n-2, \ldots, -n+2, -n\}$ so it remains to rule out the possibility of $m_t(\widehat{\beta}) = -n$ for $t \in (0, 1)$.

*Proof.* For a negative braid closure $\widehat{\beta}$, the oriented resolution is at the far right of the cube of resolutions, so the distinguished generators from the oriented resolution can only be homologous to linear combinations of other generators from the oriented resolution. The only generator $x$ with $k$-grading $n$ is the labelling of every circle with a $+$. The generator $x$ is contained in exactly one of the subcomplexes $L_1, L_2$ of the Lee chain complex, assume that it is in $L_1$. The homology class $[\mathfrak{s}_0]$ is non-zero in both $L_1$ and $L_2$ by Proposition 2.2.4, and so any representative of the homology class $[\mathfrak{s}_0]$ has elements in both $L_1$ and $L_2$. Every distinguished generator $y$ in $L_2$ living in homological grading 0 has $j_t = j - tk$ grading less than $x$ for any $t \in [0, 1)$ which means that $x$ is never the generator that determines $d_t(\widehat{\beta})$ for $t \in [0, 1)$. The slope at $t$ is the negative $k$ grading of the generator determining $d_t(\widehat{\beta})$ so the slope $m_t(\widehat{\beta})$ is never $-n$ for $t \in [0, 1)$.

□

The condition that the $d_t$ invariant of negative $n$-braids doesn't have slope $-n$ is enough to show that the $d_t$ invariant of all $n$-braids can't have slope $-n$.

**Proposition 2.3.2.** *If $\beta$ is an $n$-braid, then the slope $m_t(\widehat{\beta})$ is in the set $\{n, n-2, \ldots, -n+2\}$ for $t \in (0, 1)$.*

In [21] it was proved that $m_t(\widehat{\beta})$ is in the set $\{n, n-2, \ldots, -n+2, -n\}$ so we rule out the possibility of $m_t(\widehat{\beta}) = -n$ for $t \in (0,1)$.

*Proof.* The only generator with $k$ grading $n$ is the labelling of all circles in the oriented resolution with a $+$ and this generator lies on the line $w(\widehat{\beta}) - n(t-1)$ so it is enough to show that $d_t(\widehat{\beta})$ does not lie on this line for $t \in [0,1)$ because that place where it is possible to have slope $-n$.

Let $\beta$ have $n_+$ positive crossings and $n_-$ negative crossings and $\beta'$ be the negative $n$ braid that is the result of removing all positive crossings from $\beta$. There is a cobordism from $\beta$ to $\beta'$ with $n_+$ odd-index annular critical points, one for each positive crossing removed from $\beta$. By Proposition 2.2.7, the cobordism gives the bound $|d_t(\widehat{\beta}) - d_t(\widehat{\beta'})| \leq n_+$. For any $t \in [0,1)$ the previous proposition implies that $d_t(\widehat{\beta'}) < -n_- - n(t-1)$ so then $d_t(\widehat{\beta}) < n_+ - n_- - n(t-1) = w - n(t-1)$ and $d_t(\widehat{\beta})$ doesn't lie on this line for $t \in [0,1)$.

□

The following technical proposition will be used to provide strong restrictions on the shape of the $d_t$ invariant.

**Proposition 2.3.3.** *Suppose $x, y$ are homogenous elements in the two different subcomplexes $L_1$ and $L_2$ of the Lee complex and there are numbers $t_0, t_1$, and $t_2$ so the following holds. In some neighborhood $(t_0, t_2)$ that $j_t(y) < j_t(x)$ on $(t_0, t_1)$ and $j_t(x) < j_t(y)$ on $(t_1, t_2)$. Additionally if $a \in L_1$ with $x$ and $j_t(a) > j_t(x)$ on $(t_1, t_2)$ then $j_t(a) > j_t(x)$ on $(t_0, t_2)$ and similarly if $b \in L_2$ with $y$ and $j_t(b) > j_t(y)$ on $(t_0, t_1)$ then $j_t(b) > j_t(y)$ on $(t_1, t_2)$. Then it is not possible to have $x$ determine $d_t$ on $(t_0, t_1)$ and $y$ determine $d_t$ on $(t_1, t_2)$.*

Note that a sufficient condition on when it is possible to find numbers $t_0, t_1$, and $t_2$ so the assumptions of the theorem are met is when no other generator's line passes through the point of intersection of the lines of $x$ and $y$.

*Proof.* Suppose that $x$ determines $d_t$ on $(t_0, t_1)$ and $y$ determines $d_t$ on $(t_1, t_2)$. Then there is a cycle $c_1$ representing $[\mathfrak{s}_0]$ with $c_1 = x+\sum a_i+\sum b_j$ with $a_i \in L_1$ and $b_j \in L_2$ and $j_t(x) < j_t(b_j)$ for $t \in (t_0, t_1)$. It is not possible to have $j_t(x)$ equal to $j_t(b_j)$ for $t \in (t_0, t_1)$ because if generators agree at more than one point then they agree for all values of $t$ and so must lie in the same subcomplex because they have the same $j_0$ grading and if they agree for only a single value of $t$ then $b_j$ would have a smaller $j_t$ value than $x$ for some values of $(t_1, t_2)$ which contradicts the assumption that $x$ determines $d_t$ on this interval.

Similarly there is a cycle $c_2$ representing $[\mathfrak{s}_0]$ with $c_2 = y+\sum a'_i+\sum b'_j$ with $a'_i \in L_1$ and $b'_j \in L_2$ and $j_t(y) < j_t(a'_i)$ for $t \in (t_1, t_2)$. Then $j_t(x) < j_t(a'_i)$ for $t \in (t_0, t_2)$ and $j_t(y) < j_t(b_j)$ for $t \in (t_0, t_2)$.

Notice that $c_3 = \sum a'_i + \sum b_j$ also represents $[\mathfrak{s}_0]$ because $[\mathfrak{s}_0] = [A] + [B]$ where $[A]$ is supported only in $L_1$ and $[B]$ is supported only in $L_2$ so $[\sum a'_i] = [A]$ because $[y + \sum a'_i + \sum b'_j] = [A] + [B]$ and a similar argument shows $[\sum b_j] = [B]$. Then $c_3$ is a representative of $[\mathfrak{s}_0]$ with $j_t(x) < j_t(c_3)$ for $t \in (t_0, t_1)$ and $j_t(y) < j_t(c_3)$ for $t \in (t_1, t_2)$. Constructing this representative of $[\mathfrak{s}_0]$ with higher $j_t$ grading contradicts that $x$ and $y$ determined $d_t$ on these intervals. □

Informally, Proposition 2.3.3 means that you can't see a change from being determined by an element in one subcomplex to being determined by an element in the other subcomplex with lower $k$ grading/higher $j_t$ slope under certain additional assumptions.

The $d_t$ invariant of a braid is determined by the $j_t$ grading of some generator of the braid's Lee chain complex. Noticing this and a few properties of the $d_t$ invariant, it is possible to restrict the shape of the $d_t$ invariant of a braid.

**Theorem 2.3.4.** *For a fixed $n$, there is a finite set of piecewise linear functions $f_i : [0, 1] \to \mathbb{R}$ so that for any $n$-braid $\beta$ there is some $j$ so that $d_t(\widehat{\beta}) = f_j + w(\widehat{\beta})$. Furthermore there is an explicit method for enumerating all the functions $f_i$.*

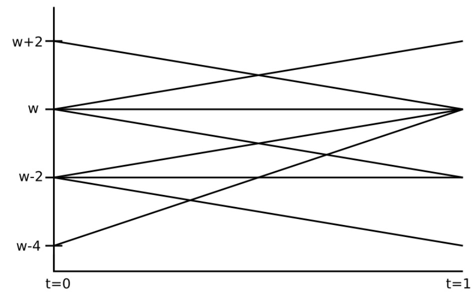


Figure 2.3: Possible generators for a 4-braid

*Proof.* At $t = 1$ the value of $d_t(\widehat{\beta})$ is known, $d_1(\widehat{\beta}) = w(\widehat{\beta})$ and in the interval $[0, 1)$ the slope $m_t(\widehat{\beta})$ is bounded between $n$ and $-n+2$. So in this interval, $d_t(\widehat{\beta})$ lies in the triangular region bounded by the lines passing through $(1, w(\widehat{\beta}))$ with slopes $n$ and $-n+2$. A start to understanding the possible shapes of $d_t$ is listing all possible $j_t$ gradings of generators in this region for some $t \in (0, 1)$.

There are generators with $j_t$ gradings on the line of slope $-n+2$ from $(0, w(\widehat{\beta})+n-2)$ to $(1, w(\widehat{\beta}))$ and a single generator with $j_t$ grading on the line from $(0, w(\widehat{\beta}) - n)$ to $(1, w(\beta))$. Additionally there could be generators with $j_0$ grading from the set $\{w(\beta)-n+2, w(\beta)-n+4, \ldots, w(\widehat{\beta})+n-4\}$ with any slope from the set $\{n-2, n-4, \ldots, -n+2\}$. This is an exhaustive list of generators which can determine $d_t(\widehat{\beta})$ in $[0, 1)$ because of restrictions on the parity of the $j_0$ grading in the Lee complex and restrictions on the annular gradings of generators in the complex. Now every possible shape of $d_t(\beta)$ must be some path along the lines coming from these generators which stays in the triangular region, never has $t$ decrease, and ends at the point $(1, w(\widehat{\beta}))$.

□

Using the restriction from Proposition 2.3.3, some of these paths can be ruled out as possibilities for the shape of $d_t(\widehat{\beta})$.

**Example 2.3.5.** As an example of Theorem 2.3.4, we can enumerate all possible shapes of $d_t(\widehat{\beta})$ when $\beta$ is a 4-braid. For a 4-braid, the shape of $d_t$ is restricted to paths to $(1, w(\widehat{\beta}))$ in Figure 2.3. Examining the figure, there are ten possible paths;

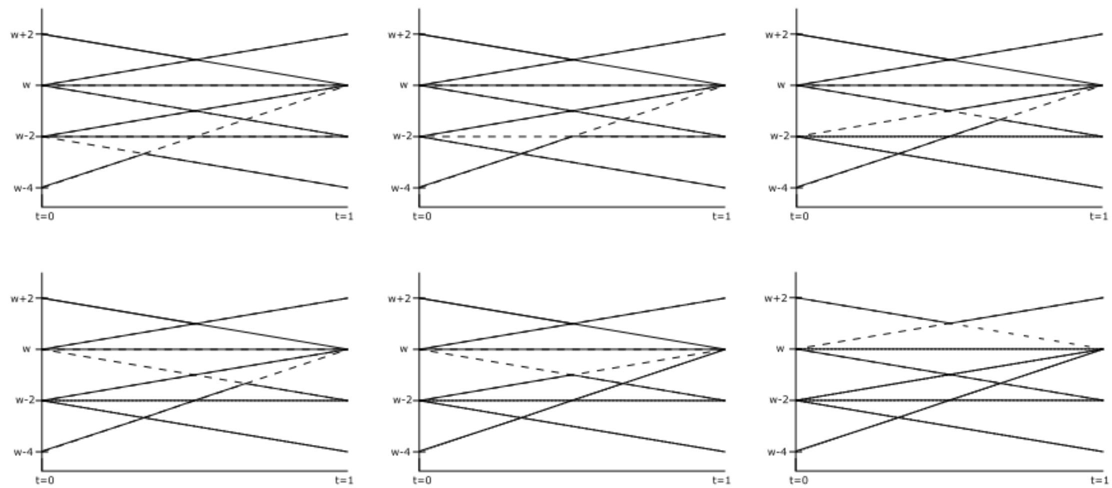


Table 2.1: The six possible shapes of $d_t$ of a 4-braid with non-constant slope

the six with non-constant slope are shown in Table 2.1. Some of these paths can not be the shape of $d_t$ however because of the restrictions on $d_t$ from Proposition 2.3.3. Specifically the first and second paths in the top row and the second path in the bottom row can not be the shape of $d_t$ because of Proposition 2.3.3.

**Remark 2.3.6.** The application of Theorem 2.3.4 in the case of 4-braids verifies that the example of a $d_t$ invariant given in Section 7.4 of [21] does not contain more points of non-differentiability than those shown. The authors of [21] stated that they did not expect more discontinuities but were unable to rule out their existence. More generally, Theorem 2.3.4 provides an approach to answering the question raised in Section 7.4 of [21] on the values of $t$ where it is possible that $d_t(\widehat{\beta})$ is not differentiable.

Replicating the same process for 3-braids provides even stronger restrictions on the shape of the $d_t$ invariant.

**Theorem 2.3.7.** *When $\beta$ is a 3-braid, then for $t$ between $0$ and $1$ one of the following holds: $d_t(\widehat{\beta}) = w(\beta) - 3 + 3t$, $d_t(\widehat{\beta}) = w(\widehat{\beta}) - 1 + t$ or $d_t(\widehat{\beta}) = w(\widehat{\beta}) + 1 - t$.*

*Proof.* Following the process described in Theorem 2.3.4 for a 3-braid shows that the shape $d_t$ is restricted by the paths to $(1, w(\widehat{\beta}))$ in Figure 2.4. Examining the figure,

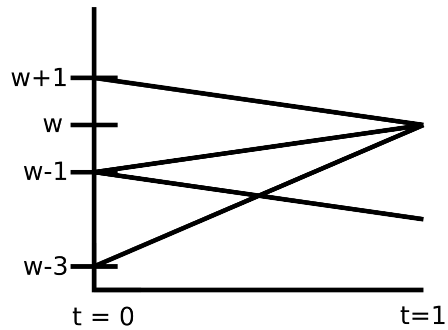


Figure 2.4: Possible generators for a 3-braid

there are exactly four paths, three with constant slope and a single path that starts at $w-1$ with slope $-1$ until it intersects the line with slope 3 and then follows the line with slope 3 to the endpoint $(1, w(\widehat{\beta}))$. Note that the generators on these two lines are in different subcomplexes. So then this single path with non-constant slope can not be the shape of $d_t$ because it is ruled out by Proposition 2.3.3.

□

For the $d_t$ invariant of 3-braids, these three possibilities can be enumerated by comparing the $s$-invariant and the writhe. The possibilities are $s(\widehat{\beta}) = w(\widehat{\beta}) - 2$, $s(\widehat{\beta}) = w(\widehat{\beta})$, or $s(\widehat{\beta}) = w(\widehat{\beta}) + 2$.

**Remark 2.3.8.** A similar perspective can be applied to study the possible forms of other algebraically analogous invariants. For example, in the case of the concordance invariant $\Upsilon_K(t)$ an application can show that there are 13 possibilities for $\Upsilon_K(t)$ if $K$ is concordant to a genus 2 knot and gives an explicit enumeration of these possibilities. This could previously be obtained through an application of Corollary 7.2, Theorems 7.1, 8.1, and 8.2 of [40].

## 2.4 The annular Rasmussen invariants of 3-braids

For a 3-braid $\beta$, the $s$-invariant of $\widehat{\beta}$ is either $w(\widehat{\beta}) - 2$, $w(\widehat{\beta})$, or $w(\widehat{\beta}) + 2$ depending on if the slope of $d_t$ in $(0, 1)$ is $3, 1$, or $-1$. The following theorems completely classify when each of these possibilities occur.

**Theorem 2.4.1.** *A 3-braid $\beta$ has $s(\widehat{\beta}) = w(\widehat{\beta}) - 2$ if and only if $\beta$ is conjugate to a braid of the form:*

1. $h^d\sigma_1\sigma_2^{-a_1}\cdots\sigma_1\sigma_2^{-a_n}$ *with* $a_i \geq 0$ *and some* $a_i > 0$ *and* $d > 0$.

2. $h^d\sigma_2^m$ *with* $m \in \mathbb{Z}$ *and either* $d = 0$ $m \geq 0$, $d = 1$ $m \geq -4$, *or* $d > 1$.

3. $h^d\sigma_1^m\sigma_2^{-1}$ *with* $m \in \{-1,-2,-3\}$ *and* $d > 0$.

**Theorem 2.4.2.** *A 3-braid $\beta$ has $s(\widehat{\beta}) = w(\widehat{\beta}) + 2$ if and only if the mirror of $\beta$ is conjugate to a braid of the form:*

1. $h^d\sigma_1\sigma_2^{-a_1}\cdots\sigma_1\sigma_2^{-a_n}$ *with* $a_i \geq 0$ *and some* $a_i > 0$ *and* $d > 0$.

2. $h^d\sigma_2^m$ *with* $m \in \mathbb{Z}$ *and either* $d = 1$ $m \geq -3$, *or* $d > 1$.

3. $h^d\sigma_1^m\sigma_2^{-1}$ *with* $m \in \{-1,-2,-3\}$ *and* $d > 0$.

**Theorem 2.4.3.** *All other 3-braids have $s(\widehat{\beta}) = w(\widehat{\beta})$.*

**Remark 2.4.4.** Almost all non-split 3 braids satisfy the equation $d_t(\widehat{m(\beta)}) = -d_t(\beta) - 2 + 2t$. However, the example of the braids $h^1\sigma_2^{-4}$ and $h^{-1}\sigma_2^4$ does not satisfy this formula and shows that there is no simple formula for the behavior of $d_t$ under mirroring for links. It is not known if there is a simple formula for the behavior of $d_t$ under mirroring for knots.

The following property of the $d_t$ invariant of 3-braids is useful for the computation of the $s$-invariant of 3-braid closures.

**Lemma 2.4.5.** *Adding positive crossings to a 3-braid can only preserve or increase the slope of $d_t(\widehat{\beta})$ in the interval $(0,1)$. Similarly adding negative crossings to a 3-braid can only preserve or decrease the slope of $d_t(\widehat{\beta})$ in the interval $(0,1)$.*

**Remark 2.4.6.** This fact about adding crossings and how that changes the slope of the $d_t$ invariant is a specific property of 3-braids and not true in general. For example, for 4-braids it should be possible to begin with a braid whose shape is shown in the top right of Table 2.1 add a positive crossing and get a braid whose shape has constant slope of 2. This would decrease the slope for some values of $t$.

*Proof.* Let $\beta$ be a 3-braid and consider $\beta\sigma_i$ the result of adding a positive crossing. Notice that there is a cobordism from $\beta$ to $\beta\sigma_i$ with a single index one critical point. Now by Proposition 2.2.6, $d_1(\widehat{\beta}) + 1 = w(\widehat{\beta}) + 1 = d_1(\widehat{\beta\sigma_i})$ which means that if $d_0(\widehat{\beta\sigma_i}) = d_0(\widehat{\beta}) + 1$ then the two braids have the same $d_t$ slope. The bounds on $d_t$ from cobordisms in Proposition 2.2.7 give that $d_0(\widehat{\beta\sigma_i}) \leq d_0(\widehat{\beta}) + 1$. If equality holds then $d_t$ of $\widehat{\beta\sigma_i}$ has the same slope as that of $\widehat{\beta}$ and if equality does not hold then $d_t$ of $\widehat{\beta\sigma_i}$ has a more positive slope than that of $\widehat{\beta}$. The same argument proves the case of adding a negative crossing. □

With this lemma, we are now ready to prove Theorems 2.4.1, 2.4.2, and 2.4.3. Throughout the proof of Theorems 2.4.1, 2.4.2, and 2.4.3 braids are considered up to conjugacy.

*Proof of Theorem 2.4.1.* If a 3-braid $\beta$ has $s(\widehat{\beta}) = w(\widehat{\beta}) - 2$ then $\psi(\widehat{\beta}) \neq 0$ by Proposition 2.2.10 so the braids listed are the only possible 3-braids with $s(\widehat{\beta}) = w(\widehat{\beta}) - 2$ by Lemma 2.2.12.

The braids in the family 1 with $d = 1$ are all quasi-alternating and have $\psi(\widehat{\beta}) \neq 0$ so they have $s(\widehat{\beta}) = w(\widehat{\beta}) - 2$. For $d > 1$ then the braids can be obtained from a braid with $d = 1$ by adding positive crossings and so they also have $s(\widehat{\beta}) = w(\widehat{\beta}) - 2$.

In family 2, when $d > 1$ then the braids can be obtained from the braids in the first family by adding positive crossings and so these braids have maximal $d_t$ slope on (0,1) and $s(\widehat{\beta}) = w(\widehat{\beta}) - 2$ by Lemma 2.4.5. If $d = 1$ and $m = -4$ then the braid is quasipositive and so $s(\widehat{\beta}) = w(\widehat{\beta}) - 2$ by Proposition 2.2.10. Adding positive crossings shows that the $s(\widehat{\beta}) = w(\widehat{\beta}) - 2$ also holds when $m > -4$ by Lemma 2.4.5. Finally if $d = 0$ and $m \geq 0$ then the braids are quasipositive and so have $s(\widehat{\beta}) = w(\widehat{\beta}) - 2$.

The braids in family 3 are all quasipositive and so they have $s(\widehat{\beta}) = w(\widehat{\beta}) - 2$ by Proposition 2.2.10. □

*Proof of Theorem 2.4.2.* First, notice that if $\beta$ is a 3-braid whose closure is a knot and $m(\beta)$ has $\psi(m(\widehat{\beta})) \neq 0$ then $s(\widehat{\beta}) = -s(m(\widehat{\beta})) = -(w(m(\widehat{\beta})) - 2) = w(\widehat{\beta}) + 2$ by Proposition 2.2.3. More generally if $\beta'$ is any 3-braid where it is possible to obtain $\beta$ as above by adding positive generators to $\beta'$ then $d_0(\widehat{\beta}') = w+1$ because of Lemma 2.4.5. Also, adding positive generators to $\beta'$ is equivalent to adding negative generators to $m(\beta')$ and arriving at $m(\beta)$.

If $m(\beta')$ is conjugate to a braid in family 1 and has $d > 0$ for $m(\beta')$ then you can add negative crossings to $m(\beta')$ to get to the braid $m(\beta) = h^d \sigma_1 \sigma_2^{-n}$ with $d > 0$ and then after maybe adding an additional negative crossing $m(\beta)$ closes up to a knot and $m(\widehat{\beta})$ also has $\psi$ non-vanishing. Then $\beta$ has $s(\widehat{\beta}) = w(\widehat{\beta}) + 2$ and so then the same holds for $\widehat{\beta}'$.

If $m(\beta')$ is conjugate to a braid in family 2 with $\psi(m(\widehat{\beta}')) \neq 0$ and $d > 1$ then you can add negative crossings to the mirror to get to a braid $m(\beta)$ in family 1 with $d \geq 1$ by canceling out all of a full twist except a single $\sigma_1$. Then $\beta$ has $s(\widehat{\beta}) = w(\widehat{\beta}) + 2$ and so then the same holds for $\widehat{\beta}'$.

For $\beta$ with $d = 1$ and $m = -3$ then the Khovanov homology of $\widehat{\beta}$ in homological grading 0 has dimension 2 and is supported in quantum gradings -2 and 0. This implies that $s(\widehat{\beta}) = -1 = w(\widehat{\beta}) + 2$. For $\beta'$ with $d = 1$ and $m > -3$, it is possible to add negative crossings to $m(\beta')$ to arrive at $m(\beta)$ and so $s(\widehat{\beta}') = w(\widehat{\beta}') + 2$ as well.

If $m(\beta')$ is conjugate to a braid in family 3 and has $\psi(m(\widehat{\beta}')) \neq 0$ then you can add negative crossings to $m(\beta')$ to get to $m(\beta) = h^d \sigma_1^{-3} \sigma_2^{-1}$ which is a knot with $\psi(m(\widehat{\beta})) \neq 0$. So then $\beta$ has $s(\widehat{\beta}) = w(\widehat{\beta}) + 2$ and so then the same holds for $\widehat{\beta}'$.

To complete the proof that these are the only 3-braids with $s(\widehat{\beta}) = w(\widehat{\beta}) + 2$ we will show that all other 3 braids have $s(\widehat{\beta}) = w(\widehat{\beta})$. $\square$

*Proof of Theorem 2.4.3.* If $\beta$ is conjugate to a braid in family 2 with $d = 0$ and $m < 0$ then $\beta$ is conjugate to a split braid. Specifically it is the union of a trivial braid on a single strand $\mathbb{1}_1$ and a braid $\alpha$ with $|m|$ half-twists on two strands. Proposition 2.2.8 implies that $d_0(\widehat{\beta}) = d_0(\widehat{\mathbb{1}_1}) + d_0(\widehat{\alpha}) = -1 + (w(\widehat{\alpha})) = w(\widehat{\beta}) - 1$ and so then $s(\widehat{\beta}) = w(\widehat{\beta})$.

If $\beta$ is conjugate to the braid $h^{-1}\sigma_2^4$ in family 2 then an explicit computation shows that $s(\widehat{\beta}) = -2 = w(\widehat{\beta})$. This computation is included in Section 2.6.

It remains to compute $d_t(\widehat{\beta})$ for braids $\beta$ where $\psi(\widehat{\beta}) = 0 = \psi(m(\widehat{\beta}))$.

First notice that if $\beta$ is a 3-braid whose closure is a knot and $\psi(\widehat{\beta}) = 0 = \psi(m(\widehat{\beta}))$ then $s(\widehat{\beta}) = w(\widehat{\beta})$ by Proposition 2.2.3. More generally if $\beta'$ is a 3-braid with $\psi(\widehat{\beta'}) = 0$ and it is possible to add negative crossings to $\beta'$ to arrive at $\beta$ with $s(\widehat{\beta}) = w(\widehat{\beta})$ then $s(\widehat{\beta'}) = w(\widehat{\beta'})$ as well. This is because by Lemma 2.4.5 we know that $s(\widehat{\beta'}) = w(\widehat{\beta'})$ or $w(\widehat{\beta'}) - 2$ but the fact that $\psi(\widehat{\beta'}) = 0$ rules out the second possibility by Proposition 2.2.10. Finally, adding negative crossings to $\beta'$ is the same as adding positive crossings to $m(\beta')$.

Notice that if we add negative crossings to $\widehat{\beta'}$ to arrive at $\widehat{\beta}$ and $\psi(\widehat{\beta'}) = 0 = \psi(m(\widehat{\beta'}))$ then the functoriality of $\psi$ implies that $\psi(\widehat{\beta}) = \pm f(\psi(\widehat{\beta'})) = 0$ so we only need to check that $\psi(m(\widehat{\beta})) = 0$ as well if $\beta$ closes up to a knot.

For $m(\beta')$ conjugate to a braid in family 1 with $\psi(\widehat{\beta'}) = 0 = \psi(m(\widehat{\beta'}))$, you can add positive crossings tFor $m(\beta')$ conjugateo $m(\beta')$ to get to $h^d\sigma_1^k\sigma_2^{-1}\sigma_1^j$ with $d \leq 0$ and after possibly adding another positive crossing this is a braid $m(\beta)$ in family 1 whose closure is a knot and has $\psi(\widehat{\beta}) = 0 = \psi(m(\widehat{\beta}))$. So then $s(\widehat{\beta'}) = w(\widehat{\beta'})$. Note that we only require that $d \leq 0$ and don't identify specific values of $k, j$, and $d$ which satisfy $\psi(\widehat{\beta}) = 0 = \psi(m(\widehat{\beta}))$ but as long as the braid $\widehat{\beta'}$ we started with satisfies $\psi(\widehat{\beta'}) = 0 = \psi(m(\widehat{\beta'}))$ then we know that the braid $\widehat{\beta}$ that we obtain will as well.

For $m(\beta')$ conjugate to a braid in family 2 with $\psi(\widehat{\beta'}) = 0 = \psi(m(\widehat{\beta'}))$, we have that either $\beta' = h\sigma_2^k$ for $k \leq -5$ or $\beta' = h^{-1}\sigma_2^k$ for $k \geq 5$. If $\beta'$ is of the form $\beta' = h^{-1}\sigma_2^k$ for $k \geq 5$ then it is possible to add negative crossings to $\beta'$ and arrive at $\beta = h^{-1}\sigma_2^5$. Examining the Khovanov homology of $\widehat{\beta}$ in homological grading 0, it has dimension 1 in quantum grading $-2$ and dimension 2 in quantum grading 0 and is zero in all other quantum gradings. This means that $s(\widehat{\beta}) = -1 = w(\widehat{\beta})$ and so then $s(\widehat{\beta'}) = w(\widehat{\beta'})$ as well for all $\beta' = h^{-1}\sigma_2^k$ with $k \geq 5$.

If $\beta' = h\sigma_2^k$ for $k \leq -5$ then you can rewrite $\beta'$ as the word $\sigma_1\sigma_2^2\sigma_1\sigma_2^{k+2}$ and then you can add negative crossings to $\beta'$ to arrive at the braid $\beta = \sigma_2^{k+2}$ which has $s(\widehat{\beta}) = w(\widehat{\beta})$ and so then $s(\widehat{\beta'}) = w(\widehat{\beta'})$.

For $m(\beta')$ conjugate to a braid in family 3 with $\psi(\widehat{\beta'}) = \psi(m(\widehat{\beta'})) = 0$, you can add positive crossings to $m(\beta')$ and get to the braid $m(\beta) = h^d\sigma_1^{-1}\sigma_2^{-1}$ with $d \leq 0$ which is a knot with $\psi(m(\widehat{\beta})) = 0 = \psi(\widehat{\beta})$, so then $s(\widehat{\beta'}) = w(\widehat{\beta'})$.

□

## 2.5 Comparisons with other invariants of 3-braids

The results of Theorems 2.2.12, 2.4.1 and 2.4.2 show a close connection between $d_t$ and $\psi$ for 3-braids.

**Corollary 2.5.1.** *A 3-braid $\beta$ has $\psi(\widehat{\beta}) \neq 0$ if and only if $d_t(\widehat{\beta})$ has constant maximal slope, i.e. $s(\widehat{\beta}) = w(\widehat{\beta}) - 2$.*

**Corollary 2.5.2.** *If $\beta$ is a non-split 3-braid with $\psi(m(\widehat{\beta})) \neq 0$ then $\beta$ is conjugate to $h^{-1}\sigma_2^4$ or $d_t(\widehat{\beta})$ has constant minimal slope, i.e. $s(\widehat{\beta}) = w + 2$.*

An open problem is if $\psi$ is an "effective" transverse invariant, that is if it contains more information about transverse links than the self-linking number and the smooth link type. A weaker question which is also unknown is if vanishing/non-vanishing of $\psi$ contains more information than the $s$-invariant and the self-linking number. Birman and Menasco showed that there are transversaly non-isotopic 3-braid closures with the same underlying smooth link type and the same self linking number [16] so it is meaningful to ask if a transverse invariant is effective for 3-braid closures.

**Corollary 2.5.3.** *The invariant $\psi$ is not effective for 3-braid closures. In particular, the vanishing/non-vanishing of $\psi$ for 3-braids is determined by the s-invariant and the self-linking number.*

Along with the $d_t$ invariant and $\psi$ invariant, there are other invariants that have been previously computed for 3-braids. Two examples are the transverse invariant from knot Floer homology $\widehat{\Theta}$ and the contact invariant of double branched covers of 3-braids $c(T,\phi)$.

**Theorem 2.5.4** (Theorem 4.1 of [51] )**.** *For a 3-braid $\beta$, the invariant $\widehat{\Theta}(\widehat{\beta})$ is non-zero if and only if $\beta$ is conjugate to a braid of the following form:*

1. $h^d\sigma_1\sigma_2^{-a_1}\cdots\sigma_1\sigma_2^{-a_n}$ *with* $a_i \geq 0$ *and some* $a_i > 0$ *and* $d > 0$.
2. $h^d\sigma_2^m$ *with* $m \in \mathbb{Z}$ *and either* $d = 0$ $m \geq 0$ *or* $d \geq 1$.
3. $h^d\sigma_1^m\sigma_2^{-1}$ *with* $m \in \{-1,-2,-3\}$ *and* $d > 0$.

**Theorem 2.5.5** (Theorem 4.2 of [4])**.** *For a 3-braid $\beta$, the contact invariant $c(T,\phi)$ of $\Sigma(\widehat{\beta})$ is non-vanishing if and only if $\widehat{\Theta}(\widehat{\beta})$ is non-vanishing.*

The following statements summarize how the four invariants, $d_t$, $\psi$, $\widehat{\Theta}$, and $c(T,\phi)$, compare for 3-braids.

**Corollary 2.5.6.** *The following 3-braids have $s(\widehat{\beta}) = w(\widehat{\beta}) - 2$, $\psi(\widehat{\beta}) \neq 0$, $\widehat{\Theta}(\widehat{\beta}) \neq 0$ and $c(T,\phi) \neq 0$:*

1. $h^d\sigma_1\sigma_2^{-a_1}\cdots\sigma_1\sigma_2^{-a_n}$ *with* $a_i \geq 0$ *and some* $a_i > 0$ *and* $d > 0$.
2. $h^d\sigma_2^m$ *with* $m \in \mathbb{Z}$ *and either* $d = 0$ $m \geq 0$, $d = 1$ $m \geq -4$, *or* $d > 1$.
3. $h^d\sigma_1^m\sigma_2^{-1}$ *with* $m \in \{-1,-2,-3\}$ *and* $d > 0$.

**Corollary 2.5.7.** *The 3-braids $h\sigma_2^m$ with $m \leq -5$ have $\widehat{\Theta}(\widehat{\beta}) \neq 0$ and $c(T,\phi) \neq 0$ but $\psi(\widehat{\beta}) = 0$ and $s(\widehat{\beta}) = w(\widehat{\beta})$.*

For the case of 3-braids the Heegaard Floer invariants completely classify the braids as right-veering and non right-veering depending on if the invariants vanish or not. However the comparison shows that the invariants defined from Khovanov homology may detect slightly more subtle information about the conjugacy class of $\beta$ as an element of $\mathrm{Mod}(D_3)$ because the invariants vanish/have non-maximal slope on the closures of $h\sigma_2^m$ with $m \leq -5$ and these braids are right-veering but contain large amounts of negative twisting inside a fixed circle in $D_3$ fixed by $\beta$.

## 2.6 Computation of $s(\widehat{h^{-1}\sigma_2^4})$

The computation of the $s$-invariant of the braid closure of $h^{-1}\sigma_2^4$ makes use of Bar-Natan's cobordism category, which allows for a "divide and conquer" approach to calculations. Recently Schuetz wrote a paper [54] where he described using a "divide and conquer" approach to compute the $s$-invariant of knots. While the specific ideas in his paper are not used in our calculation, his paper did prompt the author to consider using a "divide and conquer" approach to compute $s(\widehat{h^{-1}\sigma_2^4})$.

The computation makes repeated use of the ideas in [10], which allow us to build up the formal Bar-Natan complex crossing by crossing. We use Bar-Natan's delooping and cancellation lemmas (Lemmas 4.1 and 4.2 of [10]) to simplify the complex as much as possible.

For this braid $s(\widehat{h^{-1}\sigma_2^4})$, computing the $s$-invariant is equivalent to determining if a specific cycle is non-zero in the Khovanov homology of the braid. The following two paragraphs justify this assertion.

A computer computation shows that the Khovanov homology of $\widehat{h^{-1}\sigma_2^4}$ with coefficients in $\mathbb{Q}$ in homological grading zero has dimension 6. This is exactly the same as the dimension of the Lee homology of $\widehat{h^{-1}\sigma_2^4}$ in homological grading zero. So starting with the original Lee complex and using filtered chain homotopies to simplify it to a complex whose underlying generators are the generators of Khovanov homology, we find that there are no induced differentials entering or leaving homological grading 0. So in homological grading 0, the filtration on Lee homology is determined entirely by the quantum grading on Khovanov homology.

The Khovanov homology of $\widehat{h^{-1}\sigma_2^4}$ in homological grading zero has dimension one in quantum grading $-3$, dimension three in quantum grading $-1$, and dimension two in quantum grading 1. When we say "the part of the distinguished generator $\mathfrak{s}_o$ in quantum grading $-3$" what we mean is if $\mathfrak{s}_o$ is written as a linear combination of distinguished generators restricting to just the linear combination of those generators in quantum grading $-3$. The part of the distinguished generator $\mathfrak{s}_o$ in quantum grading $-3$ is a cycle in the Khovanov homology of $\widehat{h^{-1}\sigma_2^4}$. It is a cycle in Khovanov homology because $\mathfrak{s}_o$ is a cycle in Lee homology and the additional differentials in the Lee complex raise the quantum grading by 4. There is no part of $\mathfrak{s}_o$ in quantum grading $-7$ to cancel the Khovanov differential of the part of $\mathfrak{s}_o$ in quantum grading $-3$ so it must be that it is a cycle in Khovanov homology. So if it is non-zero in Khovanov homology then $s(\widehat{h^{-1}\sigma_2^4}) = -2$.

To compute that the part of the distinguished generator $\mathfrak{s}_o$ in quantum grading $-3$ is non-zero in the Khovanov homology of the closure of $h^{-1}\sigma_2^4 = \sigma_1^{-1}\sigma_2^{-2}\sigma_1^{-1}\sigma_2^2$, first we split the braid into the words $\sigma_1^{-1}\sigma_2^{-2}\sigma_1^{-1}$ and $\sigma_2^2$ and compute a tangle complex for each. While computing and simplifying each tangle complex, one can keep track of which are in the image of the oriented resolution under chain homotopy. In this example, from the choice of where to cut the braid, the image is only the oriented resolution. In all of the tangle complexes, the homological and quantum grading of the objects are shown below the object as a pair $(i, j)$. Throughout the computation, dots on arrows represent dotted cobordisms.

**Lemma 2.6.1.** *The tangle complex associated to $\sigma_2^2$ with the global grading shifts needed for the braid $\sigma_1^{-1}\sigma_2^{-2}\sigma_1^{-1}\sigma_2^2$ is chain homotopy equivalent to the complex $B$ in Figure 2.5.*

*Proof.* A straightforward computation using the delooping and cancelation lemmas shows that the tangle complex for $\sigma_2^2$ with the appropriate grading shifts is equivalent to the complex $B$ in Figure 2.5.

$\square$

**Lemma 2.6.2.** *The tangle complex associated to $\sigma_1^{-1}\sigma_2^{-2}\sigma_1^{-1}$ is chain homotopy equivalent to the complex $E'$ whose objects are $\{E_1, E_2, E_3, E_5, E_6, E_7, E_9, E_{10}, E_{12}\}$ in Figure 2.6 and whose differentials are shown in Figure 2.8.*

*Proof.* A straightforward computation using the delooping and cancelation lemmas shows that the tangle complex for $\sigma_1^{-1}\sigma_2^{-2}\sigma_1^{-1}$ is equivalent to the complex $E$ in Figures 2.6 and 2.7. Two more applications of the cancelation lemma gives the complex $E'$ whose objects are $\{E_1, E_2, E_3, E_5, E_6, E_7, E_9, E_{10}, E_{12}\}$ in Figure 2.6 and whose differentials are shown in Figure 2.8.

$\square$

We are now ready to show the part of the distinguished generator $\mathfrak{s}_o$ in quantum grading $-3$ is non-zero in the Khovanov homology of $\widehat{h^{-1}\sigma_2^4}$.

The tensor complex $E' \otimes B$ is a tangle complex for the braid $\sigma_1^{-1}\sigma_2^{-2}\sigma_1^{-1}\sigma_2^2$. Closing the tangles off gives a complex for the braid closure $\widehat{\sigma_1^{-1}\sigma_2^{-2}\sigma_1^{-1}\sigma_2^2}$. A straightforward computation with the closed off complex verifies that the part of the distinguished generator $\mathfrak{s}_o$ in quantum grading $-3$ is non-zero in the Khovanov homology of $\widehat{h^{-1}\sigma_2^4}$.

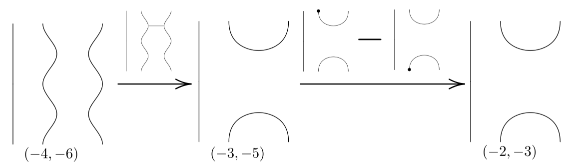


Figure 2.5: The complex $B$

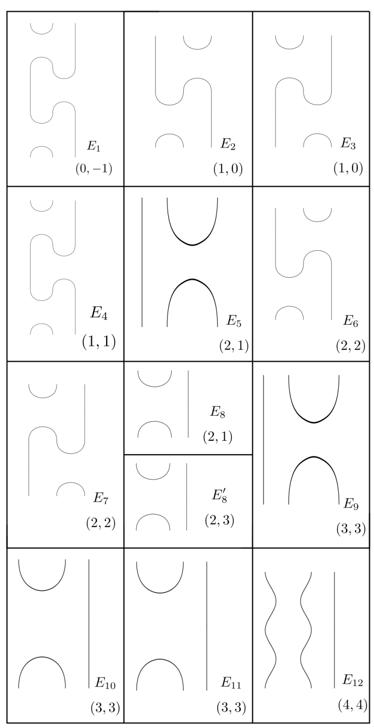


Figure 2.6: The generators of the complex $E$

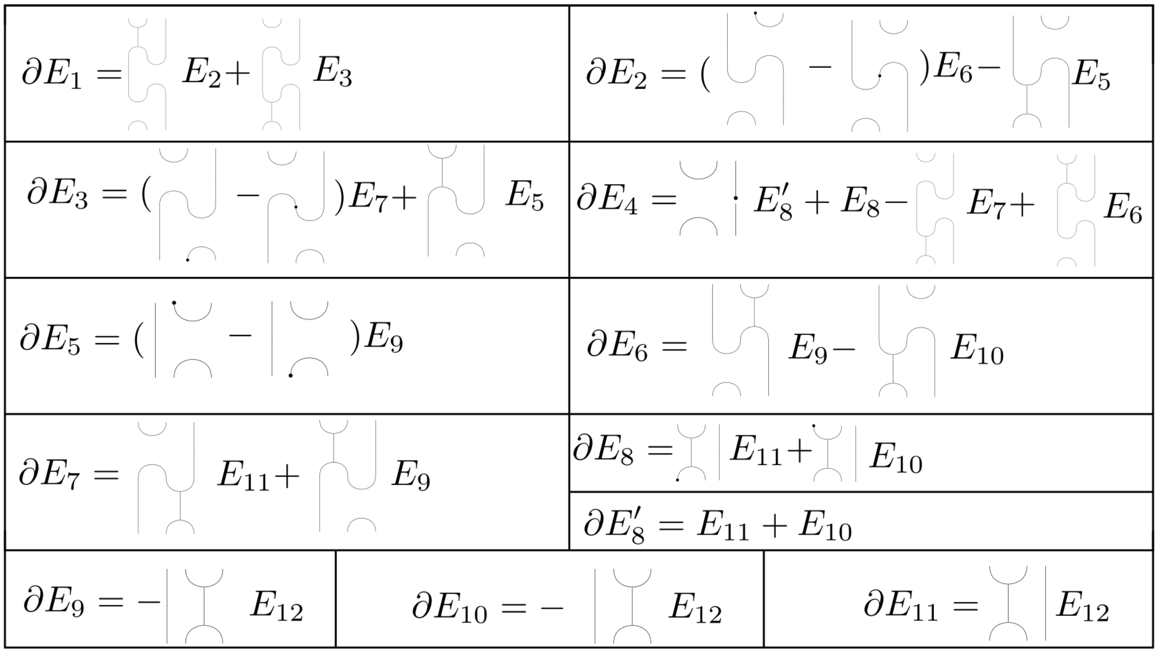


Figure 2.7: The differentials of the complex $E$

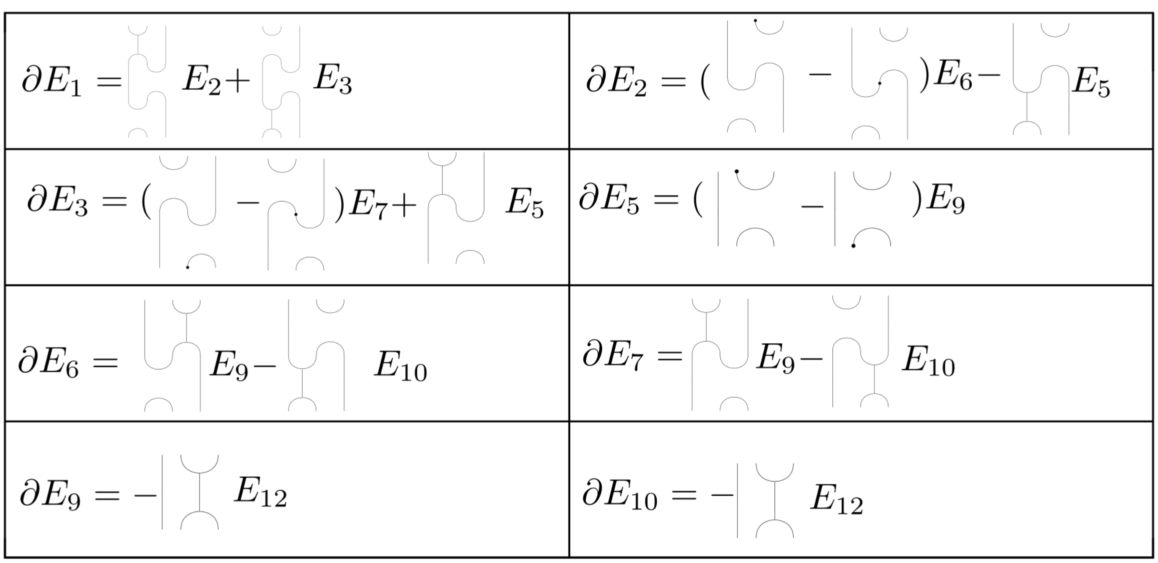


Figure 2.8: The differentials of the complex $E'$ obtained by eliminations of $E$

# Chapter 3

# Khovanov homology detects T(2,6)

## 3.1 Introduction

Khovanov homology is a combinatorially defined, bi-graded $R$ module $\mathrm{Kh}^{i,j}(L,R)$ which is associated to an oriented link $L \subseteq S^3$ [35] [1]. The graded Euler characteristic of Khovanov homology is the Jones polynomial. Many of the topological applications of Khovanov homology come from algebraic relationships to Floer homologies either implicitly through an analogy (e.g. the definition of the $s$-invariant) or explicitly through a spectral sequence.

In the spirit of finding connections between topological information and $\mathrm{Kh}^{i,j}(L,R)$ is the question of detection. Specifically, Khovanov homology is said to detect a link $L_0$ if given any link $L$ then $L$ is isotopic to $L_0$ if and only if $\mathrm{Kh}^{i,j}(L,R) \cong \mathrm{Kh}^{i,j}(L_0,R)$. Kronheimer and Mrowka showed that Khovanov homology detects the unknot [36]. Khovanov homology is also known to detect the unlink [25] [11], the Hopf link [9], the trefoil [8], the connected sum of two Hopf links [58], the torus link $T(2,4)$ [58], and split links [38].

In this paper we prove an additional detection result for Khovanov homology

[1]This article will appear in a forthcoming issue of Mathematical Research Letters published by International Press.

**Theorem 3.3.1.** Let $L$ be a link with $\mathrm{Kh}(L,\mathbb{Z}) \cong \mathrm{Kh}(T(2,6),\mathbb{Z})$. Then $L$ is isotopic to $T(2,6)$.

The proof of Theorem 3.3.1 is similar in spirt to the proof in [58] that Khovanov homology detects $T(2,4)$ but uses Dowlin's spectral sequence to knot Floer homology, rather than the Kronheimer-Mrowka spectral sequence to singular instanton Floer homology.

## 3.2 Background

### 3.2.1 Khovanov homology

Khovanov homology is a combinatorially defined invariant that assigns to an oriented link $L \subseteq S^3$ a bi-graded $R$-module $\mathrm{Kh}^{i,j}(L,R)$ which is the homology of a chain complex $\mathrm{CKh}(D)$ associated to a diagram $D$ for $L$ [35]. The $i$ grading is the homological grading and the $j$ grading is the quantum grading.

A choice of a basepoint $p \in L$ defines an action of $R[X]/X^2 = 0$ on $\mathrm{CKh}(D)$. Quotienting by the image of this action and then taking homology gives rise to reduced Khovanov homology $\widetilde{\mathrm{Kh}}^{i,j}(L,R)$. The rank of $\widetilde{\mathrm{Kh}}^{i,j}(L,\mathbb{F}_2)$ is exactly half the rank of $\mathrm{Kh}^{i,j}(L,\mathbb{F}_2)$ [55, Corollaries 3.2.B-C].

Multiple spectral sequences starting at Khovanov homology and converging to other homology theories have been constructed. We briefly recall the spectral sequences that will be needed in the proof of Theorem 3.3.1.

Using a similar construction to Khovanov homology, Lee defined an invariant of an oriented link $L \subseteq S^3$ called Lee homology $\mathrm{Lee}^i(L,\mathbb{Q})$, and from the construction there is a spectral sequence from $\mathrm{Kh}^{i,j}(L,\mathbb{Q})$ to $\mathrm{Lee}^i(L,\mathbb{Q})$. Lee showed that the total rank of $\mathrm{Lee}^i(L,\mathbb{Q})$ for an $n$-component link $L$ is $2^n$ and showed an explicit bijection between generators of $\mathrm{Lee}^i(L,\mathbb{Q})$ and choices of orientations of $L$. The homological gradings in which $\mathrm{Lee}^i(L,\mathbb{Q})$ is non-zero are determined by the pairwise linking numbers of the different components of $L$ [37].

Batson-Seed constructed a link splitting spectral sequence from Khovanov homology. To simplify the exposition, we will restrict to the case that $L = K_1 \cup K_2$ is a 2-component link which is what is relevant to the proof of Theorem 3.3.1. We also define an additional grading $\ell$ on Khovanov homology given by $\ell = i - j$. If $L$ has two components, then the Batson-Seed construction gives a spectral sequence from $\mathrm{Kh}^{i,j}(L,\mathbb{F}_2)$ to a homology theory $\mathrm{BS}^{\ell}(L,\mathbb{F}_2)$ where $\mathrm{BS}^{\ell+A}(L,\mathbb{F}_2) \cong \mathrm{Kh}^{\ell}(K_1 \sqcup K_2) \cong \oplus_{\ell_1+\ell_2=\ell} \mathrm{Kh}^{\ell_1}(K_1) \otimes \mathrm{Kh}^{\ell_2}(K_2)$ where $A$ is some overall shift determined by $L$ [11].

Pointed Khovanov homology is a generalization of reduced Khovanov homology to a link $L$ with a set of base points $p_1, \ldots, p_m \in L$ and a corresponding action of $R[X_1, \ldots, X_m]/X_1^2 = \cdots = X_m^2 = 0$ on $\mathrm{CKh}(L)$ for each base point [19]. Dowlin constructed a spectral sequence from relatively $\delta' = j - 2i$ graded pointed Khovanov homology to relatively $\delta' = 2M - 2A$ graded knot Floer homology [19]. The version of pointed Khovanov homology Dowlin constructs is similar but slightly different than the earlier version defined by Baldwin-Levine-Sarkar [6]. As an example, when applied to a link with a single basepoint, the Dowlin construction gives the reduced Khovanov homology reduced at that point while the Baldwin-Levine-Sarkar construction does not. However, many formal properties of the theories are analogous, compare the following lemma with Lemma 2.11 of [6].

**Lemma 3.2.1.** *Let $(L,\boldsymbol{p})$ be a pointed link diagram, and suppose that $\boldsymbol{p}$ contains some point $p_0$, and $\boldsymbol{p}' = \boldsymbol{p} \setminus p_0$. Then there is a short exact sequence*

$$0 \to \Sigma^{0,1}\,\mathrm{CKh}(L,\boldsymbol{p}') \to \mathrm{CKh}(L,\boldsymbol{p}) \to \Sigma^{-1,-1}\,\mathrm{CKh}(L,\boldsymbol{p}') \to 0$$

*where $\Sigma^{i,j}$ denotes shifts in the homological and quantum gradings. In particular*

$$\operatorname{rank} \mathrm{Kh}^{\delta'}(L, \boldsymbol{p}) \leq 2 \operatorname{rank}^{\delta'+1} \mathrm{Kh}(L, \boldsymbol{p}')$$

*where $\delta'$ is a single $\mathbb{Z}$ grading given by $j - 2i$.*

*Proof.* The chain level exact sequence follows immediately from the construction of $\mathrm{CKh}(L, \mathbf{p})$ using an iterated tensor product of basepoint maps and keeping track of the bi-gradings. □

### 3.2.2 Knot Floer homology, link Floer homology, and sutured Floer homology

Knot Floer homology is an invariant that assigns to an oriented link $L \subseteq S^3$ a bi-graded $R$-module $\widehat{\mathrm{HFK}}(L)$. The two gradings are the Maslov grading $M$ and the Alexander grading $A$ [45].

Link Floer homology $\widehat{\mathrm{HFL}}(L)$ is a generalization of knot Floer homology which is graded by an Alexander grading $a_i$ for each component of $L$ in addition to the Maslov grading $M$ [46].

To recover knot Floer homology from link Floer homology for an $n$-component link $L$, start with $\widehat{\mathrm{HFL}}(L)$ and define a single Alexander grading as the sum over the Alexander gradings of all components, $A = \sum a_i$. Then $\widehat{\mathrm{HFL}}(L)$ graded by $A$ and $M + \frac{n-1}{2}$ is isomorphic to $\widehat{\mathrm{HFK}}(L)$ [46].

Sutured Floer homology $\mathrm{SFH}(M)$ is a version of Heegaard Floer homology defined for a balanced sutured manifold $M$. The homology $\mathrm{SFH}(M)$ splits over relative $\mathrm{Spin}^C$ structures on $M$ [32].

The complement of a link $S^3 \setminus L$ is naturally a balanced sutured manifold. The sutures are pairs of two oppositely oriented meridional sutures on each component of the boundary. The sutured Floer homology $\mathrm{SFH}(S^3 \setminus L)$ is isomorphic to $\widehat{\mathrm{HFL}}(L)$

with the relative $\mathrm{Spin}^C$ structures corresponding to the multi-Alexander gradings.

Given a properly embedded oriented surface with boundary $S$ in a sutured manifold $M$, which satisfies some technical conditions about how $\partial S$ intersects the the sutures of $M$, $S$ defines a sutured manifold decomposition from $M$ to $M' = M \setminus \mathrm{Int}(N(S))$. Juhász showed that $\mathrm{SFH}(M')$ is isomorphic to the direct summands of $\mathrm{SFH}(M)$ corresponding to "outer" $\mathrm{Spin}^C$ structures [33].

### 3.2.3 Link Floer homology detects braids

An argument similar in spirit to arguments in [33, Theorem 1.5] and [22] shows that link Floer homology detects braids in the complement of a fibered component. This braid detection result is known to some experts but the author is unaware of a proof in the literature so one is produced here. We provide a proof a more general statement is needed in the proof of Theorem 3.3.1. A version of the following argument was communicated to the author by John Baldwin [3]. For a definition of a braid in the complement of a fibered knot refer to [34, Definition 1.2].

**Proposition 3.2.2.** *Suppose $L \subseteq M$ is a link with $l$ components with a fibered component $K$ and $M \setminus L$ is irreducible. Then $L \setminus K$ is a braid in the complement of $K$ if and only if $\widehat{\mathrm{HFL}}(L)$ has rank $2^{l-1}$ in the highest (and lowest) non-zero Alexander grading associated to $K$.*

*Proof.* Consider a fiber surface $S$ bounded by $K$ which intersects $L \setminus K$ minimally. Cutting open the sutured manifold $M \setminus L$ along $S \setminus L$ gives a new sutured manifold $N$. The sutured Floer homology of $N$ is isomorphic to the link Floer homology of $L$ supported in a constant $a_K$ grading of $\frac{1}{2}c(S,t) = \chi(S)+I(S)-r(S,t)$, where $a_K$ is the Alexander grading associated to $K$ [33, Theorem 3.11]. For a definition of $\frac{1}{2}c(S,t)$, see [33, Definition 3.8].

The sutured manifold $N$ contains $l-1$ pairs of parallel sutures corresponding to the base points on the components of $L \setminus K$. Removing these superfluous pairs of sutures gives a new sutured manifold $N'$ and $\mathrm{rk}(\mathrm{SFH}(N)) = 2^{l-1}\,\mathrm{rk}(\mathrm{SFH}(N'))$. Finally, $\mathrm{rk}(\mathrm{SFH}(N')) = 1$ if and only if $N'$ is a product sutured manifold [32, Prop 9.4] [33, Theorem 1.4]. The manifold $N'$ is a product sutured manifold exactly when $L \setminus K$ is a braid in the complement of $K$.

To see that $c(S,t)$ is the lowest non-zero $a_K$ grading, consider increasing the genus of the Seifert surface for $K$ by adding $h$ handles to the genus $g$ surface $S$ in the complement of $L$ to obtain a new surface $S'$ of genus $g+h$. Then the sutured manifold obtained by cutting open along $S'$ is not taut if $h \geq 1$ so the link Floer homology in $a_U$ grading $\frac{1}{2}c(S',t)$ is zero [32, Prop 9.8] and one can compute that $c(S',t) = c(S,t) - 2h$.

The rank in the lowest non-zero Alexander grading associated to $K$ is the same as the rank in the highest non-zero Alexander grading associated to $K$ because of the symmetry of Link Floer homology. $\square$

Taking $L \subseteq S^3$ and the fibered knot to be the unknot gives the following corollary.

**Corollary 3.2.3.** *Suppose $L \subseteq S^3$ is a link with $l$ components with an unknotted component $U$ and each component of $L \setminus U$ has non-zero geometric linking with $U$. Then $L \setminus U$ is a braid in the complement of $U$ if and only if $\widehat{\mathrm{HFL}}(L)$ has rank $2^{l-1}$ in the highest (and lowest) non-zero Alexander grading associated to $U$.*

*Proof.* The condition that each component of $L \setminus U$ has non-zero geometric linking with $U$ ensures that $S^3 \setminus L$ is irreducible and so the result follows from Proposition 3.2.2. $\square$

**Remark 3.2.4.** For the case with the unknot, if $D$ intersects $L$ in $n$ points then a simple computation of $c(D,t)$ shows that the highest non-zero Alexander grading will be $n/2$.

| | $i = 0$ | $i = 1$ | $i = 2$ | $i = 3$ | $i = 4$ | $i = 5$ | $i = 6$ |
|---|---|---|---|---|---|---|---|
| $j = 18$ | | | | | | | $\mathbb{Z}$ |
| $j = 16$ | | | | | | $\mathbb{Z}$ | $\mathbb{Z}$ |
| $j = 14$ | | | | | | $\mathbb{Z}/2\mathbb{Z}$ | |
| $j = 12$ | | | | $\mathbb{Z}$ | $\mathbb{Z}$ | | |
| $j = 10$ | | | | $\mathbb{Z}/2\mathbb{Z}$ | | | |
| $j = 8$ | | | $\mathbb{Z}$ | | | | |
| $j = 6$ | $\mathbb{Z}$ | | | | | | |
| $j = 4$ | $\mathbb{Z}$ | | | | | | |

Table 3.1: The Khovanov homology of the torus link T(2,6) computed using SageMath [56]

## 3.3 Khovanov Homology detects T(2,6)

In this section, we show that Khovanov homology detects the torus link T(2,6). For reference, the Khovanov homology is shown in Table 3.1.

**Theorem 3.3.1.** *Let $L$ be a link with $\mathrm{Kh}(L, \mathbb{Z}) \cong \mathrm{Kh}(T(2,6), \mathbb{Z})$, then $L$ is isotopic to $T(2,6)$.*

Theorem 3.3.1 follows from the two propositions below.

**Proposition 3.3.2.** *If $\mathrm{Kh}(L, \mathbb{Z}) \cong \mathrm{Kh}(T(2,6), \mathbb{Z})$, then $L$ is a 2-component link with linking number 3 and each of the components is an unknot.*

**Proposition 3.3.3.** *If $\mathrm{Kh}(L, \mathbb{Z}) \cong \mathrm{Kh}(T(2,6), \mathbb{Z})$, then one component of $L$ is a braid in the complement of the other component.*

*Proof of Theorem 3.3.1 from Propositions 3.3.2 and 3.3.3.* From Propositions 3.3.2 and 3.3.3, $L$ must be $\widehat{\beta} \cup U$ where $\beta$ is a 3-braid whose closure is an unknot and $U$ is the braid axis.

Up to isotopy in the complement of the braid axis, there are only three possible 3-braids whose closures are the unknot, $\sigma_1\sigma_2$, $\sigma_1^{-1}\sigma_2^{-1}$ and $\sigma_1\sigma_2^{-1}$ so $L$ must be one of these braids together with its braid axis [44, Theorem 12.1]. The first two possibilities both represent $T(2,6)$. The final possibility using the braid $\sigma_1\sigma_2^{-1}$ gives the link L6a2 and $\mathrm{Kh}(\mathrm{L6a2}, \mathbb{Z}) \ncong \mathrm{Kh}(T(2,6), \mathbb{Z})$ because they have different ranks [41]. □

*Proof of Proposition 3.3.2.* The fact that $\mathrm{Kh}(L, \mathbb{Z}) \cong \mathrm{Kh}(T(2,6), \mathbb{Z})$ means that $\mathrm{Kh}(L, \mathbb{Z})$ is supported in even quantum gradings and so $L$ has an even number of components because the non-zero quantum gradings of $\mathrm{Kh}(L, R)$ agrees mod 2 with the number of components of $L$.

The Lee homology of $L$ has even rank in each homological grading and has total rank $2^n$ where $n$ is the number of components of $L$. So then rank inequalities from the spectral sequence between Khovanov homology and Lee homology show that $L$ has exactly two components because there are only two homological gradings where the rank of Khovanov homology is more than 1 and in each of these gradings the rank is exactly 2. Furthermore, these homological gradings are $i = 0$ and $i = 6$ so the linking number of the two components is $6/2 = 3$ [37, Proposition 4.3].

Considering the Batson-Seed spectral sequence over $\mathbb{F}_2$ from $\mathrm{Kh}(L)$ to $\mathrm{Kh}(L')$ where $L'$ is the split link comprised of the two components of $L$ [11, Theorem 1.1]. The total rank of $\mathrm{Kh}(L, \mathbb{F}_2)$ is 12 and the total rank of $\mathrm{Kh}(L')$ is the product of the ranks of the Khovanov homology of the two components. Additionally, over $\mathbb{F}_2$, the rank of Khovanov homology of a knot over $\mathbb{F}_2$ must be twice an odd number because it is twice the rank of reduced Khovanov homology over $\mathbb{F}_2$ which always has odd rank for a knot.

Then the only possible ranks for the Khovanov homologies of the components of $L$ are 2 and 6. Then the only possibilities for the components of $L$ are either two unknots or an unknot and a trefoil because the unknot is the only knot whose Khovanov homology has rank two over $\mathbb{F}_2$ [36, Theorem 1.1] and the trefoil is the only knot whose Khovanov homology has rank 6 over $\mathbb{F}_2$ [8, Theorem 1.4]. Examining the rank of Khovanov homology of $L$ over $\mathbb{Q}$ in each $i - j$ grading, which is preserved by the Batson-Seed spectral sequence up to an overall shift, rules out the possibility that one of the components of $L$ is a trefoil because there is no overall shift possible to make the ranks agree with the ranks in $i - j$ gradings of the tensor product $\mathrm{Kh}(U) \otimes \mathrm{Kh}(T)$

where $U$ and $T$ are the unknot and trefoil respectively. $\square$

*Proof of Proposition 3.3.3.* To show that one component of $L$ is braided with respect to the other, we will use the spectral sequence from the pointed Khovanov homology of $L$ to a singly graded version of knot Floer homology constructed by Dowlin [19]. In this proof, we will use $\delta'$ to refer to both grading Khovanov homology by $j - 2i$ and grading knot Floer homology by $2M - 2A$. We will use $\delta$ to refer to grading knot Floer homology by $A - M$.

From knowing $\mathrm{Kh}(L, \mathbb{Z})$ we can see that the reduced Khovanov homology of $L$ over $\mathbb{F}_2$ is rank 6. So then the reduced Khovanov homology of $L$ over $\mathbb{Q}$ has rank no greater than 6 for any choice of basepoint. Let $(L, \mathbf{p})$ be the pointed link $L$ with a single basepoint on each component of $L$. Because $L$ is a 2 component link, the pointed Khovanov homology $\mathrm{Kh}(L, \mathbf{p})$ over $\mathbb{Q}$ has rank no greater than 12. The fact that $L$ is Khovanov thin means that the reduced Khovanov homology of $L$ over $\mathbb{F}_2$ is supported in a single $\delta' = j - 2i$ grading and then this is also true over $\mathbb{Q}$ for either choice of basepoint. This implies that $\mathrm{Kh}(L, \mathbf{p})$ is supported in a single $\delta' = j - 2i$ grading by Lemma 3.2.1.

The Dowlin spectral sequence preserves the relative $\delta'$ grading so $\widehat{\mathrm{HFK}}(L)$ is supported in a single $\delta = -1/2\delta'$ grading.

Now we consider $\widehat{\mathrm{HFL}}(L)$ in order to show that one component is a braid in the complement of the other component. By Corollary 3.2.3, we want to show that in the top non-zero grading of either $a_1$ or $a_2$ the rank of $\widehat{\mathrm{HFL}}(L)$ is exactly two.

Link Floer homology of a 2-component link $L = K_1 \cup K_2$ admits a spectral sequence from $\widehat{\mathrm{HFL}}(L = K_1 \cup K_2)$ to $\widehat{\mathrm{HFL}}(K_1) \otimes V$. The grading $a_1$ is corresponds to the Alexander grading on $\widehat{\mathrm{HFL}}(K_1) \otimes V$ up to an overall shift by half the linking number of $L$ [5, Lemma 2.4]. The differentials of the spectral sequence lower the $a_2$ grading. There is a similar spectral sequence from $\widehat{\mathrm{HFL}}(L = K_1 \cup K_2)$ to $\widehat{\mathrm{HFL}}(K_2) \otimes V$.

The fact that each of the two components of $L$ is an unknot and the existence of the spectral sequence from $\widehat{\mathrm{HFL}}(L = K_1 \cup K_2)$ to $\widehat{\mathrm{HFL}}(K_1) \otimes V$ implies that $\widehat{\mathrm{HFL}}(L)$ has rank at least 1 in the gradings $M = 0, a_1 = 3/2$ and $M = -1, a_1 = 3/2$ where each generator sits in some unknown $a_2$ grading [5, Lemma 2.4]. Similarly $\widehat{\mathrm{HFL}}(L)$ has rank at least 1 in the gradings $M = 0, a_2 = 3/2$ and $M = -1, a_2 = 3/2$ where each generator sits in some unknown $a_1$ grading. The fact that $\widehat{\mathrm{HFL}}(L)$ is supported in a single $\delta = a_1 + a_2 - M - 1/2$ grading allows us to write the unknown gradings in terms of a variable $x$. There is at least one generator in $(M, a_1, a_2)$ gradings $(0, 3/2, x)$ and $(-1, 3/2, x-1)$, where these generators survive in the spectral sequence induced by $a_2$. Also there is at least one generator in $(0, x, 3/2)$ and $(-1, x-1, 3/2)$, where these generators survive in the spectral sequence induced by $a_1$.

The symmetry of $\widehat{\mathrm{HFL}}(L)$ [46, Proposition 8.2] then tells us that $\widehat{\mathrm{HFL}}(L)$ also has rank at least 1 in the following four gradings $(-3-2x, -3/2, -x)$, $(-3-2x+1, -3/2, -x+1)$, $(-3-2x, -x, -3/2)$, $(-3-2x+1, -x+1, -3/2)$.

From here the proof breaks into seven cases depending on the value of $x$. There is a case where $x > 5/2$, a case where $x < -3/2$ and a case for each of the following values $5/2, 3/2, 1/2, -1/2, -3/2$. For each case we deduce that one component of $L$ is a braid in the complement of the other component.

We first address the case $x = 3/2$ which is the case that occurs if $L = T(2,6)$. If the Dowlin spectral sequence was known to preserve the absolute $\delta'$ grading then this would be the only case that needed to be considered.

*The case* $x = 3/2$

Setting $x = 3/2$ we have that $\widehat{\mathrm{HFL}}(L)$ has rank at least 1 in the tri-gradings $(0, 3/2, 3/2)$, $(-1, 1/2, 3/2)$ and $(-1, 3/2, 1/2)$ and these generators survive in one or both of the spectral sequences induced by $a_i$. Additionally, $\widehat{\mathrm{HFL}}(L)$ also has rank at least 1 in gradings $(-6, -3/2, -3/2)$, $(-5, -3/2, -1/2)$, and $(-5, -1/2, -3/2)$ and these generators do not survive in either spectral sequence. The partial complex with these six generators is shown in Figure 3.1.

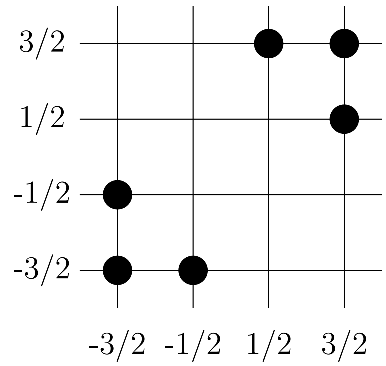


Figure 3.1: A partial $\widehat{\mathrm{HFL}}(L)$ complex when $x = 3/2$ with 6 generators. The dots represent bi-degrees where the partial complex has rank one.

At this point, there are at most 6 more generators we can add to construct a possible $\widehat{\mathrm{HFL}}(L)$. When we have finished adding generators to a possible $\widehat{\mathrm{HFL}}(L)$, the end result must have even rank in every $a_1$ grading and every $a_2$ grading, otherwise it is impossible to have spectral sequences to $\widehat{\mathrm{HFL}}(K_1) \otimes V$ and $\widehat{\mathrm{HFL}}(K_2) \otimes V$. Additionally, it must have the symmetry that $\widehat{\mathrm{HFL}}_M(L, a_1, a_2) \cong \widehat{\mathrm{HFL}}_{M-a_1-a_2}(L, -a_1, -a_2)$ [46, Proposition 8.2], and it must be possible to add differentials changing the Maslov index by 1 so that there are spectral sequences induced by each $a_i$ grading with the $E_\infty$ pages mentioned in the previous paragraph.

If we add generators with $a_i$ grading larger than 3/2 then the above requirements about even parity in every grading and symmetry mean that we must add exactly two generators in that grading (which is now the top grading) and so one of the components is a braid in the complement of the other. So now we can assume that we only add generators whose $a_i$ gradings are less than or equal to 3/2 in absolute value.

Having the appropriate $E_\infty$ pages of the two spectral sequences now means that there must be a generator in grading $M = -4, a_1 = -1/2, a_2 = -1/2$ and one in grading $M = -2, a_1 = 1/2, a_2 = 1/2$. The partial complex with these generators is shown in Figure 3.2.

Now there are at most 4 more generators to add. If the end result will have

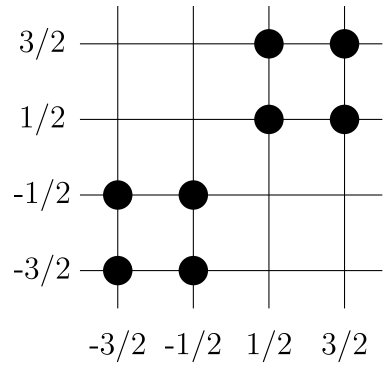


Figure 3.2: A partial $\widehat{\mathrm{HFL}}(L)$ complex when $x = 3/2$ with 8 generators if every generator has $a_i$ grading no greater than 3/2 in absolute value. The dots represent bi-degrees where the partial complex has rank one.

that neither component is a braid in the complement of the other then two those generators must be added at either $a_1 = 3/2, a_2 = -3/2$ or $a_1 = 3/2, a_2 = 3/2$ and the other two are then added in the appropriate place for symmetry. There are three possible ways to add the two generators, either both in $a_1 = 3/2, a_2 = -3/2$, both in $a_1 = 3/2, a_2 = 3/2$, or one in $a_1 = 3/2, a_2 = -3/2$ and the other in $a_1 = 3/2, a_2 = 3/2$. For each of these ways of adding generators, it is impossible to add differentials that give the desired $E_\infty$ pages of both spectral sequences. So then the link $L$ must have that one of its components is a braid in the complement of the other.

*The case $x = 5/2$*

Setting $x = 5/2$ we have that $\widehat{\mathrm{HFL}}(L)$ has rank at least 1 in the tri-gradings $(0, 5/2, 3/2)$, $(0, 3/2, 5/2)$ and $(-1, 3/2, 3/2)$ and these generators survive in one or both of the spectral sequences induced by $a_i$. Additionally, $\widehat{\mathrm{HFL}}(L)$ also has rank at least 1 in gradings $(-6, -5/2, -3/2)$, $(-6, -3/2, -5/2)$, and $(-5, -3/2, -3/2)$ and these generators do not survive in either spectral sequence. The partial complex with these six generators is shown in Figure 3.3.

Generators must be added to the gradings $a_i = \pm 5/2$ to allow for the desired spectral sequences to exist. There are two possible ways to do this while preserving the symmetry, the first is adding generators at the $(a_1, a_2)$ gradings $(5/2, 1/2)$ and

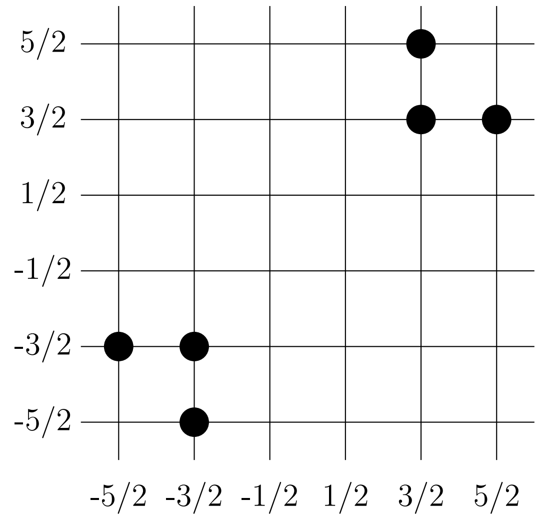


Figure 3.3: A partial $\widehat{\mathrm{HFL}}(L)$ complex when $x = 5/2$ with 6 generators. The dots represent bi-degrees where the partial complex has rank one. The vertical columns represent $a_1$ gradings with grading increasing by 1 from $-5/2$ to $5/2$.

$(1/2, 5/2)$. Then two more generators must be added to preserve symmetry leaving a complex with 10 generators and only being able to add up to two additional generators. The partial complex has exactly two generators in the maximal $a_i$ grading for $i = 1, 2$. There is no way to add the two additional generators in a way that increases the rank in these maximal gradings while maintaining the needed spectral sequences and symmetry.

The second possibility is adding a generator in the $(a_1, a_2)$ grading $(5/2, 5/2)$ and adding on in the grading $(-5/2, -5/2)$ to preserve the symmetry. The partial complex with these generators is shown in Figure 3.4. At this point there are eight generators in the complex and up to four more that can be placed.

If we add generators with $a_i$ grading larger than $5/2$ then the requirements about even parity in every grading and symmetry mean that we must add exactly two generators in that grading (which is now the top grading) and so one of the components is a braid in the complement of the other. So now we can assume that we only add generators whose $a_i$ gradings are less than or equal to $5/2$ in absolute value.

Now there are at most 4 more generators to add. If the end result will have that neither component is a braid in the complement of the other then two those generators must be added at either $a_1 = 5/2, a_2 = -5/2$ or $a_1 = 5/2, a_2 = 5/2$ and the other two are then added in the appropriate place for symmetry. There are three possible ways to add the two generators, either both in $a_1 = 5/2, a_2 = -5/2$, both in $a_1 = 5/2, a_2 = 5/2$, or one in $a_1 = 5/2, a_2 = -5/2$ and the other in $a_1 = 5/2, a_2 = 5/2$. For each of these ways of adding generators, it is impossible to add differentials that give the desired $E_\infty$ pages of both spectral sequences. So then the link $L$ must have that one of its components is a braid in the complement of the other.

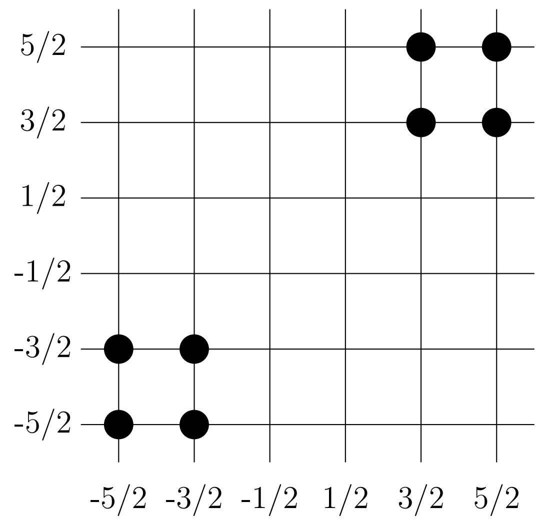


Figure 3.4: A partial $\widehat{\mathrm{HFL}}(L)$ complex when $x = 5/2$ with 8 generators. The dots represent bi-degrees where the partial complex has rank one. The vertical columns represent $a_1$ gradings with grading increasing by 1 from $-5/2$ to $5/2$.

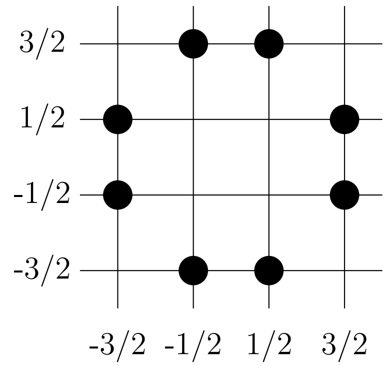


Figure 3.5: A partial $\widehat{\mathrm{HFL}}(L)$ complex when $x = 1/2$ with 8 generators. The dots represent bi-degrees where the partial complex has rank one.

*The case* $x = 1/2$

Setting $x = 1/2$ we have that $\widehat{\mathrm{HFL}}(L)$ has rank at least 1 in the tri-gradings $(0, 3/2, 1/2)$, $(0, 1/2, 3/2)$, $(-1, -1/2, 3/2)$ and $(-1, 3/2, -1/2)$ and these generators survive in one of the spectral sequences induced by $a_i$. Additionally, $\widehat{\mathrm{HFL}}(L)$ also has rank at least 1 in gradings $(-4, -3/2, -1/2)$, $(-4, -1/2, -3/2)$, $(-3, 1/2, -3/2)$ and $(-3, -3/2, 1/2)$ and these generators do not survive in either spectral sequence. The partial complex with these six generators is shown in Figure 3.5.

If we add generators with $a_i$ grading larger than $3/2$ then the requirements about even parity in every grading and symmetry mean that we must add exactly two generators in that grading (which is now the top grading) and so one of the components is a braid in the complement of the other. So now we can assume that we only add generators whose $a_i$ gradings are less than or equal to $3/2$ in absolute value.

Having the appropriate $E_\infty$ pages of the two spectral sequences now means that there must be a generator in each of the following four gradings $(-3, -1/2, -1/2)$, $(-1, 1/2, 1/2)$, $(-2, 1/2, -1/2)$, and $(-2, -1/2, 1/2)$. After adding these four generators the rank of the complex is 12 and there are no more generators to add. There are exactly two generators in the top $a_1$ grading and so one of the components is a braid in the complement of the other.

*The case* $x > 5/2$

When $x > 5/2$ the tri-gradings $(0, 3/2, x)$, $(-1, 3/2, x-1)$, $(0, x, 3/2)$, $(-1, x-1, 3/2)$, $(-3-2x, -3/2, -x)$, $(-3-2x+1, -3/2, -x+1)$, $(-3-2x, -x, -3/2)$, and $(-3-2x+1, -x+1, -3/2)$ are all distinct and so represent eight different generators of $\widehat{\mathrm{HFL}}(L)$. The only way to add four generators so that all the $a_i$ gradings have even rank are by adding two at $(a_1, a_2)$ gradings $(x, x)$ and $(x-1, x-1)$ or at $(x, x-1)$ $(x-1, x)$. The Maslov gradings are determined by $\widehat{\mathrm{HFL}}(L)$ being supported in a single $\delta$ grading. The other two generators are then added to the appropriate gradings to maintain symmetry. After adding these four generators the rank of the complex is 12 and there are no more generators to add. There are exactly two generators in the top $a_1$ grading and so one of the components is a braid in the complement of the other.

*All other cases*

The arguments to show braidedness in the remaining cases are almost identical to the cases shown and are not repeated. The argument for the case $x < -3/2$ is similar to the case $x > 5/2$. The argument for the case $x = -3/2$ is similar to the case $x = 5/2$. Finally the argument for the case $x = -1/2$ is similar to the case $x = 3/2$. $\square$

# Chapter 4

# Annular Khovanov homology and meridional disks

## 4.1 Introduction

Jones brought new ideas to the field of low-dimensional topology with his construction of the Jones polynomial for links in $S^3$ [31]. A decade and a half later, Khovanov categorified the Jones polynomial with Khovanov homology, a bi-graded abelian group whose graded Euler characteristic recovers the Jones polynomial [35].

Annular Khovanov homology was defined by Asaeda-Przytycki-Sikora [1], who introduced a version of Khovanov homology for links in thickened surfaces. They also showed that the graded Euler characteristic of annular Khovanov homology is the Kauffman skein bracket of annular links defined by Hoste-Przytycki [26].

Since the introduction of these invariants, some natural questions have arisen about what, if any, relationship exists between topological properties of links and their Jones polynomial or Khovanov homology. Conjecturally there is a relationship between the Kauffman skein bracket of annular links and certain embedded disks in the annular complement of the link.

**Conjecture 4.1.1** (The wrapping conjecture [27]). *Let $L$ be an annular link in $A \times I$ and let $n$ be the minimal intersection of $L$ with a meridional disk. Then the maximal non-zero annular degree of the Kauffman skein bracket of $L$ is $n$.*

When the conjecture was stated, Hoste-Przytycki give an argument that the wrapping conjecture holds for any annular link with a $\pm$-adequately wrapped diagram [27].

Since the graded Euler characteristic of annular Khovanov homology is the Kauffman skein bracket, an immediate consequence of Conjecture 4.1.1 would be:

**Conjecture 4.1.2** (The categorified wrapping conjecture). *Let $L$ be an annular link in $A \times I$ and let $n$ be the minimal intersection of $L$ with a meridional disk. Then the maximal non-zero annular grading of the annular Khovanov homology of $L$ is $n$.*

The statement of the categorified wrapping conjecture first appeared in a talk by Eli Grigsby at the MSRI semester-long program *Homology theories of knots and links* in spring 2010, where she claimed a proof of the conjecture relying on the spectral sequence to the Floer homology of the double-branched cover [53] [23] and Juhasz's Thurston norm detection results [33]. In the week after the talk, Matt Hedden and Stephan Wehrli independently found examples that are a subset of the family that appears here in Theorems 4.4.2 and 4.5.1, which proved that the argument she suggested did not work. These examples were not pursued further until recent work of Xie [57] relating annular Khovanov homology to instanton Floer homology sparked new interest in the topic.

Some results about annular gradings where $\mathrm{AKh}(L)$ is non-zero were already known. Grigsby-Ni showed that the categorified wrapping conjecture holds for string links [22]. If we allow surfaces of higher genus, work of Xie [57] and Xie-Zhang [59] shows that $\mathrm{AKh}(L)$ is non-zero in the annular grading of the generalized Thurston norm for all meridional surfaces. However, we will see examples of annular links where the generalized Thurston norm is much smaller than the minimal intersection number with a meridional disk.

In this paper we verify that categorified wrapping conjecture holds for some new families of annular links. For a subset of these links we also show that the wrapping conjecture holds on the decategorified level.

**Theorem 4.4.1.** Let $L$ be an $n$-component annular link for which the categorified wrapping conjecture holds and let $L_{m,\mathbf{s}}$ be the link where the $i$-th component of $L$ is replaced with a link $s_i$ obtained as the closure of a tangle $T_i$ consisting of an $m$-string link and possibly additional closed components. Then the categorified wrapping conjecture also holds for the cable $L_{m,\mathbf{s}}$.

**Theorem 4.4.2.** Let $L^n$ be the annular link built by vertically stacking $n$ copies of tangle $J$ from Figure 4.1 and then taking the annular closure. Also, let $L^n_{\mathbf{m},\mathbf{s}}$ be the link built by replacing the $i$-th component of $L^n$ with a link $s_i$ obtained as the closure of a tangle $T_i$ consisting of an $m_i$-string link and possibly additional closed components. Then the categorified wrapping conjecture holds for the cable $L^n_{\mathbf{m},\mathbf{s}}$.

**Theorem 4.4.4.** Let $L^n$ be the annular link built by vertically stacking $n$ copies of tangle $J$ from Figure 4.1 and then taking the annular closure. Also, let $L^n_{m,\beta_1,\ldots,\beta_n}$ be the link built by replacing the $i$-th component of $L^n$ with the closure of the $m$-braid $\beta_i$. Then the categorified wrapping conjecture holds for the cable $L^n_{m,\beta_1,\ldots,\beta_n}$.

**Theorem 4.4.5.** Let $K^n$ be the iterated Whitehead double of the annular link $L^1$ obtained as the annular closure of the tangle $J$ from Figure 4.1 and let $K^n_{m,\beta}$ be the annular knot obtained by replacing $K^n$ with the closure of the $m$-braid $\beta$. Then the categorified wrapping conjecture holds for $K^n_{m,\beta}$.

**Theorem 4.5.1.** Let $L^n$ be the annular link built by vertically stacking $n$ copies of tangle $J$ from Figure 4.1 and then taking the annular closure. Also, let $L^n_{m,\beta_1,\ldots,\beta_n}$ be the link built by replacing the $i$-th component of $L^n$ with the closure of the $m$-braid $\beta_i$. Then the wrapping conjecture holds for the cable $L^n_{m,\beta_1,\ldots,\beta_n}$.

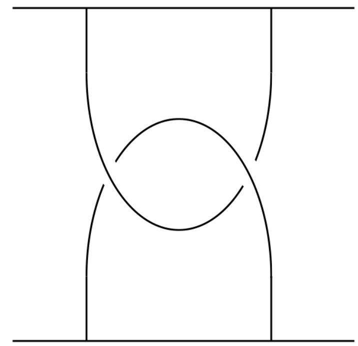

Figure 4.1: The tangle $J$ we use to construct some annular links.

**Theorem 4.5.2.** Let $K^n$ be the iterated Whitehead double of the annular link $L^1$ obtained as the annular closure of the tangle $J$ from Figure 4.1 and let $K^n_{m,\beta}$ be the annular knot obtained by replacing $K^n$ with the closure of the $m$-braid $\beta$. Then the wrapping conjecture holds for $K^n_{m,\beta}$.

The links considered in Theorems 4.4.2, 4.4.4, 4.4.5, 4.5.1, and 4.5.2 are examples of links where the generalized Thurston norm is much smaller than the minimal intersection of $L$ with a meridional disk. The generalized Thurston norm is no larger than two for any of these links because there is an embedded meridional torus which does not intersect the link. However, for these examples, the minimal intersection with a meridional disk can be made arbitrarily large. This gives rise to a difference between the Kauffman bracket and annular Khovanov homology on the one hand and the multi-variable Alexander polynomial and various annular Floer theories on the other hand.

**Corollary 4.1.3.** *There is an infinite family of links $L_i$ such that the maximal non-zero annular grading for the annular Khovanov homology of the links $L_i$ grows infinitely large but the maximal non-zero annular grading for annular instanton homology and annular link Floer homology of the links $L_i$ is bounded.*

**Corollary 4.1.4.** *There is an infinite family of links $L_i$ such that the maximal non-zero annular grading for the Kauffman bracket of the links $L_i$ grows infinitely large but* *the maximal non-zero annular grading for the multi-variable Alexander polynomial of the links $L_i$ is bounded.*

Recalling the relationship between annular Khovanov homology and knot Floer homology using double branched covers [53, 23], we also have the following differences.

**Corollary 4.1.5.** *There is an infinite family of links $L_i$ such that the maximal non-zero annular grading for the annular Khovanov homology of the links $L_i$ grows infinitely large but the maximal non-zero Alexander grading for an associated link in $\Sigma(L_i)$ is bounded.*

**Corollary 4.1.6.** *There is an infinite family of links $L_i$ such that the maximal non-zero annular grading for the Kauffman bracket of the links $L_i$ grows infinitely large but the maximal non-zero degree of the Alexander polynomial for an associated link in $\Sigma(L_i)$ is bounded.*

The proofs of Theorems 4.5.1 and 4.5.2 are entirely diagrammatic and involve understanding a specific resolution of the links in question. To prove Theorems 4.4.1, 4.4.2, 4.4.4, and 4.4.5 we extend the Batson-Seed link splitting spectral sequence [11] to the annular setting.

**Theorem 4.3.1.** Let $L$ be an annular link and $R$ a ring. Choose weights $w_c \in R$ for each component $c$ of $L$. Then there is a spectral sequence with pages $E_k(L, w)$, and

$$E_1(L, w) \cong \mathrm{AKh}(L; R)$$

If the difference $w_c - w_d$ is invertible in $R$ for each pair of components $c$ and $d$ with distinct weight, then the spectral sequence converges to

$$\mathrm{AKh}\left(\coprod_{r \in R} L^{(r)}; R\right)$$

where $L^{(r)}$ denotes the sub-link of $L$ consisting of those components with weight $r$.

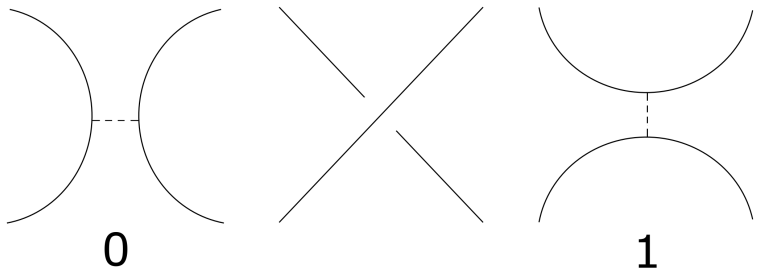


Figure 4.2: The 0-resolution and 1-resolutions of a crossing. The dotted lines indicate where to attach bands to change between the 0-resolution and the 1-resolution.

The organization of the paper is as follows. In Section 4.2 we give relevant background on annular links, the Kauffman bracket, annular Khovanov homology, and the Batson-Seed link splitting spectral sequence in $S^3$. In Section 4.3 we extend the Batson-Seed construction to the annular setting and prove Theorem 4.3.1. In Section 4.4 we apply the annular link splitting spectral sequence to prove Theorems 4.4.1, 4.4.2, 4.4.4, and 4.4.5. Finally in Section 4.5 we turn our attention to the Kauffman bracket and prove Theorems 4.5.1 and 4.5.2.

## 4.2 Background

An $n$-component annular link $L$ is an embedding of $n$ circles into the thickened annulus $A \times I$ considered up to ambient isotopy. Alternatively, an $n$-component annular link is an $n+1$-component link in $S^3$ with a distinguished unknotted component. A diagram of an annular link $L$ is a choice of a generic projection of $L$ onto $A \times \{0\}$ which records crossing information.

Given a crossing of an annular link, there are two possible resolutions of the crossing. These are referred to as the 0-resolution and the 1-resolution and are shown in Figure 4.2. Notice that it is possible to add a band to transform the 0-resolution into the 1-resolution or to transform the 1-resolution back into the 0-resolution. The locations for attaching the bands is indicated by dashed lines in Figure 4.2.

Asaeda-Przytycki-Sikora constructed a version of Khovanov homology for annular links which is now referred to as annular Khovanov homology. Annular Khovanov homology categorifies the Kauffman bracket of annular links [1]. Since the introduction of the theory, the main application of annular Khovanov homology to low-dimensional topology is the study of braid closures [2, 13, 22, 28, 30]. Additionally there have been spectral sequences constructed relating annular Khovanov homology to various Floer theories [53, 23, 57].

The annular Khovanov homology of an annular link $L$ is constructed by taking a diagram for $L$ in the annulus and constructing a cube of resolutions from the diagram, assigning a triply graded vector space to each complete resolution, and assigning linear maps to the edges of the cube. For our applications, the details of the vectors spaces and maps are not needed but a full definition of this invariant appears in [1] where it was introduced.

### 4.2.1 Batson-Seed link splitting spectral sequence

In [11], Batson-Seed constructed a link-splitting spectral sequence for Khovanov homology of links in $S^3$. In Section 4.3 we will verify that their arguments also work for constructing a similar spectral sequence for annular Khovanov homology. Here we will briefly recall some details of their construction relevant to Section 4.3.

Batson-Seed construct their spectral sequence by taking the Khovanov chain complex and perturbing it by adding an additional differential $\partial^{BS}$. This differential does not respect the $i$ or $j$-gradings on the chain complex but does respect an $\ell$-grading defined as the difference $i-j$. Additionally, the entire chain complex with the perturbation is $g$-filtered where for an $n$-component link, the $g$-filtration is defined as $\frac{j-n}{2}$. This filtration is what induces the spectral sequence from the Khovanov homology of a link to the Khovanov homology of a splitting of the link.

The construction of the perturbed differential requires a choice of sign assignment $s$ on a diagram of the link but Batson-Seed show that the filtered chain homotopy type of the construction does not depend on these choices. Additionally the construction requires that every component of $L$ be given a weight $w_i$, when the difference $w_i - w_j$ between any two non-equal weights is a multiplicative unit then the spectral sequence converges to the Khovanov homology of a link built by taking disjoint union of the sublinks $L_i$ consisting of all the components weighted by $w_i$. For the applications in this paper we will be working over the field $\mathbb{C}$ so the unit condition is always satisfied.

### 4.2.2 Sutured Khovanov homology of balanced tangles in $D \times I$

The computations in Section 4.5 require working with Khovanov-type invariants of balanced tangles. We briefly recall some information about these invariants and their relationship with annular Khovanov homology here.

A tangle $T$ is an embedding of some number of circles and intervals into $D^2 \times I$ so that the boundary of $T$ is a subset of $D^2 \times \{0\} \cup D^2 \times \{1\}$. A tangle is balanced if the intersection number of $T$ with $D^2 \times \{0\}$ is the same as the intersection number of $T$ with $D^2 \times \{1\}$. We can assume that the intersections of $T$ with the two disks happen in the same points.

As with the other Khovanov-type invariants, the sutured Khovanov homology of a balanced tangle $T$ is constructed by taking a diagram for $T$ and constructing a cube of resolution made up of flat tangles, tangle diagrams with no crossings. Each balanced tangle is then replaced with a graded vector space and the edges of the cube are replaced with linear maps. For our applications, the details of the vectors spaces and maps are not needed but a full definition of this invariant appears in [24] where it was introduced.

Given a balanced tangle $T$, we can construct an annular link by identifying $D^2 \times \{0\}$ with $D^2 \times \{1\}$ via the identity map which we will call the annular closure of the tangle. Reversing this process, it is possible to construct a balanced tangle from an annular link $L$ by decomposing or "cutting open" $A \times I$ along a meridional disk that $L$ intersects transversely.

There is a relationship between the annular Khovanov homology of $L$ and the sutured Khovanov homology of a tangle $T$ obtained by decomposing $A \times I$ along a meridional disk $D$.

**Theorem 4.2.1** ([23])**.** *Let $L$ be an annular link and let $T$ be the tangle obtained from $L$ by decomposing along a meridional disk $D$. Then the annular Khovanov homology of $L$ in k-grading $w$ is isomorphic to the sutured Khovanov homology of $T$ where $w$ is the number of intersection points of $L$ with $D$.*

Because the wrapping conjecture relates to a specific annular grading, we will use ideas from the sutured Khovanov homology of balanced tangles in parts of Section 4.5 to assist with computations.

## 4.3 Annular link splitting spectral sequence

In this section we replicate many of the arguments from [11] to verify the existence of an annular link splitting spectral sequence and show that it has similar properties to the non-annular version.

**Theorem 4.3.1.** *Let $L$ be an annular link and $R$ a ring. Choose weights $w_c \in R$ for each component $c$ of $L$. Then there is a spectral sequence with pages $E_k(L, w)$, and*

$$E_1(L, w) \cong \mathrm{AKh}(L; R)$$

*If the difference $w_c - w_d$ is invertible in $R$ for each pair of components $c$ and $d$ with distinct weight, then the spectral sequence converges to*

$$\mathrm{AKh}\left(\coprod_{r \in R} L^{(r)}; R\right)$$

*where $L^{(r)}$ denotes the sub-link of $L$ consisting of those components with weight $r$.*

*Proof.* Considering the annular link $L$ as a link in $S^3$ and choosing a diagram $D$ we can associate to it an $\ell$-graded, $g$-filtered, and $k$-filtered chain complex $\mathrm{CKh}(D; R)$ with the Khovanov differential $\partial$ and the Batson-Seed perturbation $\partial^{BS}$. Both of these differentials decompose into a portion that preserves the $k$-filtration, which we call $\partial_0$ and $\partial_0^{BS}$, and a portion that lowers the $k$-filtration by 2, which we call $\partial_-$ and $\partial_-^{BS}$. Decomposing the relations $\partial^2 = 0$, $(\partial^{BS})^2 = 0$, and $\partial\partial^{BS} + \partial^{BS}\partial = 0$ into their $k$-homogeneous components immediately gives that $\partial_0^2 = 0$, $\partial_0^{BS} = 0$ and $\partial_0\partial_0^{BS} + \partial_0^{BS}\partial_0 = 0$. This shows that equipping the chain complex for $L$ only with these differentials would give a bi-graded chain complex by gradings $\ell$ and $k$ which is also filtered by the $g$-filtration. Following the notation from [11], we refer to this chain complex as $AC(D, w, s)$ where $w$ represents the weighting of the components and $s$ represents a choice of sign assignment.

In Section 2.3 of [11], to show that the filtered chain homotopy type of their construction did not depend on the choice of sign assignment, Batson-Seed construct an explicit chain map giving the equivalence. Their map preserves the $k$-grading so it also shows that the filtered chain homotopy type of $AC(D, w, s)$ does not depend on the sign assignment. Similarly the chain maps in Proposition 2.3 of [11] used to show that the relatively $\ell$-graded total homology is unchanged by crossing changes also preserve the $k$-grading so their argument also applies in the annular setting.

The arguments to show that the filtered chain homotopy type does not depend on the choice of diagram in Section 2.3 of [11] work in the annular setting as well. The arguments in [11] consider local diagrams for the Reidemeister moves, resolve the local diagrams, construct some local cancelations and then produce an isomorphism on the level of a diagrammatic chain complex. The local nature of the arguments ensures that they also will work in the annular setting to show invariance under the Reidemeister moves.

$\square$

The existence of this annular link splitting spectral sequence gives the following rank inequality as an immediate consequence. The proof follows exactly as in the proof of Corollary 3.4 in [11].

**Corollary 4.3.2.** *Let $\mathbb{F}$ be any field, and let $L$ be an annular link with components $K_1, \ldots, K_m$. Then*

$$\operatorname{rank}^{\ell,k} \mathrm{AKh}^*(L, \mathbb{F}) \geq \operatorname{rank}^{\ell+t,k} \otimes_{c=1}^{m} \mathrm{AKh}^*(K_c, \mathbb{F})$$

*where each side is bi-graded by $\ell, k$ and the shift $t$ is given by*

$$t = \sum_{c \leq d} 2 \cdot \mathrm{lk}(L_c, L_d)$$

*where $L_c$ and $L_d$ are components of the link $L$.*

As in the non-annular version, this annular link splitting spectral sequence can provide lower bounds on the splitting number of a link. We state the bound here but we will not use it in the rest of this paper. The proof of the bound is the same as the proof of Theorem 1.2 in [11].

**Definition 4.3.3.** We say that an $n$-component link $L$ is an annular split link if after isotopy of the link in $A \times I$ it is possible to find numbers $t_1, \ldots, t_{n-1} \in I$ such that the surfaces $(S^1 \times t_i) \times I$ in $A \times I$ separate the components of $L$.

**Definition 4.3.4.** The annular splitting number of an annular link $L$ is the minimum number of times different components of the link must be passed through one another to obtain an annular split link.

**Theorem 4.3.5.** *Let $L$ be an annular link and let $w_c \in R$ be a set of component weights such that $w_c - w_d$ is invertible for each pair of components $c$ and $d$. Let*

*$b(L, w)$ be the largest integer $k$ such that $E_k \neq E_\infty(L, w)$. Then $b(L, w)$ is less than or equal to the annular splitting number of $L$.*

## 4.4 Link splitting and the maximal annular grading

Now we look at some applications of the spectral sequence from Theorem 4.3.1 to verifying the categorified wrapping conjecture for some families of examples. The idea of the applications is to consider a specific splitting of the link so that it is easier to show the annular Khovanov homology of the resulting splitting is non-zero in the desired annular grading.

**Theorem 4.4.1.** *Let $L$ be an $n$-component annular link for which the categorified wrapping conjecture holds and let $L_{m,\boldsymbol{s}}$ be the link where the $i$-th component of $L$ is replaced with a link $s_i$ obtained as the closure of a tangle $T_i$ consisting of an $m$-string link and possibly additional closed components. Then the categorified wrapping conjecture also holds for the cable $L_{m,\boldsymbol{s}}$.*

*Proof.* Notice that if $D$ is a disk which intersects $L$ in $\mathrm{wrap}(L)$ points, then $D$ intersects $L_{m,\mathbf{s}}$ in $m \cdot \mathrm{wrap}(L)$ points. So we know $\mathrm{wrap}(L_{m,\mathbf{f}}) \leq m \cdot \mathrm{wrap}(L)$. Now we will show that $\mathrm{AKh}(L_{m,\mathbf{f}})$ is non-zero in $k$-grading $m \cdot \mathrm{wrap}(L)$. Together this will show that the categorified wrapping conjecture holds for $L_{m,\mathbf{s}}$.

Consider the link $L_{m,\mathbf{s}}$ and choose $m+1$ distinct weights $w_1, \ldots, w_{m+1} \in \mathbb{C}$ and weight the link $L_{m,\mathbf{s}}$ so that for each individual satellite, the $m$ components of the string link closure are weighted differently using weights $w_1, \ldots, w_m$ and then all the other components are weighted with $w_{m+1}$. Using these weights, the link splitting spectral sequence converges to the $k, \ell$ bi-graded annular Khovanov homology of a link built from the disjoint union of $m$ copies of $L$ by taking additional disjoint unions and connected sums with links in $S^3$.

The annular Khovanov homology of the disjoint union of $m$ copies of $L$ is non-zero in the $k$-grading $m \cdot \mathrm{wrap}(L)$ and taking disjoint unions and connected sums with links in $S^3$ does not change the maximal non-zero annular grading. For disjoint unions this is immediate from the decomposition as a tensor product and for connected sums this is the content of [22, Lemma 3.5]. Then rank inequality from the annular link splitting spectral sequence ensures that $\mathrm{AKh}(L_{m,\mathbf{s}})$ is non-zero in $k$-grading $m \cdot \mathrm{wrap}(L)$ as well. $\square$

**Theorem 4.4.2.** *Let $L^n$ be the annular link built by vertically stacking $n$ copies of tangle $J$ from Figure 4.1 and then taking the annular closure. Also, let $L^n_{\boldsymbol{m},\boldsymbol{s}}$ be the link built by replacing the $i$-th component of $L^n$ with a link $s_i$ obtained as the closure of a tangle $T_i$ consisting of an $m_i$-string link and possibly additional closed components. Then the categorified wrapping conjecture holds for the cable $L^n_{\boldsymbol{m},\boldsymbol{s}}$.*

**Remark 4.4.3.** Notice that Theorem 4.4.2 does not follow immediately from Theorem 4.4.1 because here we are allowing the different components of the link to be replaced by links built from string links of varying numbers of strands.

*Proof.* First notice the fact that the annular Khovanov homology of $L^n$ is non-zero in $k$-grading 2 follows immediately from the existence of a spectral sequence to annular instanton Floer homology [57] and that this theory is known to detect the Thurston norm of meridional surfaces [59].

Let $p = m_j$ be the minimum over all the $m_i$, then there is a disk which intersects $L^n_{\mathbf{m},\mathbf{f}}$ in $2p$ points. Now we choose weights $w_1, \ldots, w_p, w_{p+1} \in \mathbb{C}$ and weight the components of $L^n_{\mathbf{m},\mathbf{s}}$ so that the first $p$ components of each string link are weighted by $w_1, \ldots, w_p$ and all remaining components are weighted by the final weight $w_{p+1}$.

Using these weights, the link splitting spectral sequence converges to the $k, \ell$ bigraded annular Khovanov homology of a link built from the disjoint union of $p$ copies of $L^n$ by taking additional disjoint unions and connected sums with links in $S^3$.

Because the annular Khovanov homology of $L^n$ is non-zero in $k$-grading 2, arguments from the proof of Theorem 4.4.1 show the annular Khovanov homology of a link built from the disjoint union of $p$ copies of $L^n$ by taking additional disjoint unions and connected sums with links in $S^3$ is non-zero in grading $2p$. Then rank inequality from the annular link splitting spectral sequence ensures that $\mathrm{AKh}(L_{\mathbf{m},\mathbf{s}})$ is non-zero in $k$-grading $2p$ as well.

□

There are other families of annular links that we can also show satisfy the categorified wrapping conjecture. The arguments work on the decategorified level and so the proofs of these theorems are delayed until Section 4.5. For completeness, we state that the categorified wrapping conjecture holds for these families as well.

**Theorem 4.4.4.** *Let $L^n$ be the annular link built by vertically stacking $n$ copies of tangle $J$ from Figure 4.1 and then taking the annular closure. Also, let $L^n_{m,\beta_1,\ldots,\beta_n}$ be the link built by replacing the $i$-th component of $L^n$ with the closure of the $m$-braid $\beta_i$. Then the categorified wrapping conjecture holds for the cable $L^n_{m,\beta_1,\ldots,\beta_n}$.*

**Theorem 4.4.5.** *Let $K^n$ be the iterated Whitehead double of the annular link $L^1$ obtained as the annular closure of the tangle $J$ from Figure 4.1 and let $K^n_{m,\beta}$ be the annular knot obtained by replacing $K^n$ with the closure of the $m$-braid $\beta$. Then the categorified wrapping conjecture holds for $K^n_{m,\beta}$.*

## 4.5 The maximal annular degree of the Kauffman bracket

In this section we show that for a subset of the families of links we have considered previously that not only is the annular Khovanov homology non-zero in the top annular grading but the Kauffman bracket is as well. The main technique we use to show that the Kauffman bracket is non-zero for this subset is to work diagramatically and demonstrate that there is a quantum grading with exactly one generator in the annular Khovanov chain complex and that this generator is in the maximal annular grading. The generator in question will always be the generator obtained by taking a 1-resolution at each crossing and labeling every circle with a "+".

This generator is in the maximal quantum grading for the chain complex and it has been observed that any other generator in the maximal quantum grading must come from a resolution that be connected to the all 1's resolution by a path of changes of resolutions where any time a 0-resolution changes to a 1-resolution two distinct circles merge together. Alternatively working backwards from the all 1's resolution changing a 1-resolution back to a 0-resolution must result in the splitting of a circle [20, Proposition 1].

**Theorem 4.5.1.** *Let $L^n$ be the annular link built by vertically stacking $n$ copies of tangle $J$ from Figure 4.1 and then taking the annular closure. Also, let $L^n_{m,\beta_1,\ldots,\beta_n}$ be the link built by replacing the $i$-th component of $L^n$ with the closure of the $m$-braid $\beta_i$. Then the wrapping conjecture holds for the cable $L^n_{m,\beta_1,\ldots,\beta_n}$.*

The proof of Theorem 4.5.1 is broken up into two parts. In the first two part of the proof we show the wrapping conjecture holds for the blackboard framed $m$-cable of $L^n$ by identifying a specific tangle and computing its all 1's resolution. Then to finish the proof we apply a result of Hoste-Przytycki.

*Proof. Step 1:* We first compute the all 1's resolution for the sutured Khovanov homology of the balanced tangle whose annular closure is the annular link $L^1_{m,1,\ldots,1}$ where all the braids are the identity braid. For the 2-cable, the all 1's resolution is shown in Figure 4.3. The dashed lines represent the bands that would be added to change a single 1-resolution back to a 0-resolution.

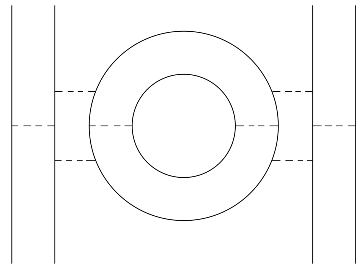

Figure 4.3: The resolution of the balanced tangle obtained from the 2-cable $L_2$. The dotted lines indicate where to attach bands to change back to a 0-resolution.

We will verify that for every $m$, the resolution in question is a tangle with $m$ strands running from top to bottom on the left followed by $m$ concentric circles followed by $m$ strands runnings from top to bottom on the right. We will also show that all the bands that should be attached to change a 1-resolution back to a 0-resolution are either between adjacent strands, between adjacent circles, or between the outermost circle and one of the strands adjacent to it.

Notice that the diagram for the sutured Khovanov homology of the $m$-cable can be viewed as a combination of the crossings involving either of the outer most strands and then the crossings that only involve the inner $m-1$ strands. The crossings involving only the inner $m-1$ cable correspond exactly to a diagram for the sutured Khovanov homology of the $m-1$-cable. We will consider resolving each of these halves individually and then place them together.

Considering the outer half and resolving all the crossings with the 1-resolution gives a tangle that is pictured for the $m$-cable in Figure 4.4. The dashed lines indicate where to attach a band to go back to a 0-resolution and the bold square represents where to place the inner half of the tangle. Notice that the bands all connect two different components of the tangle.

We can obtain the resolution of the $m$-cable by taking the resolved tangle from the previous paragraph and filling the middle hole with the resolution of the $m-1$-cable. Doing this gives a tangle with $m$ strands running from top to bottom on

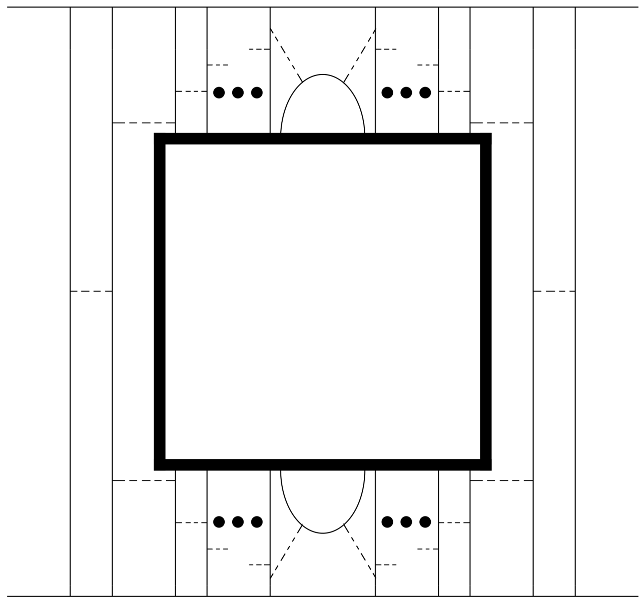

Figure 4.4: The all 1's resolution of the outer half of the tangle. The dashed lines indicate where to attach a band to go back to a 0-resolution and the bold square represents where to place the inner half of the tangle.

the left followed by $m$ concentric circles followed by $m$ strands runnings from top to bottom on the right. Furthermore, all the bands that should be attached to change a 1-resolution back to a 0-resolution are either between adjacent strands, between adjacent circles, or between the outermost circle and one of the two strands adjacent to the circle.

Now the all 1's resolution of the link $L^n_{m,1,\ldots,1}$ can be obtained by stacking $m$ copies of the resolution considered above and taking the annular closure. Doing this gives a resolution that consists of $2m$ circles running around the annular axis and $n$ groups of $m$ concentric homotopically trivial circles. Furthermore all of the bands we would attach to revert a 1-resolution back to a 0-resolution run between distinct circles. This guarantees that the generator where all of these circles are labeled with a "+" is the only generator of the chain complex in its quantum grading. It also sits in the maximal annular grading of $2m$ showing that the Kauffman bracket is non-zero in this annular grading and that the wrapping conjecture holds for the link $L^n_{m,1,\ldots,1}$. In the language of [27] we have shown that these links have minus-adequately wrapped diagrams.

*Step 2:* A diagram for the link $L^n_{m,\beta_1,\ldots,\beta_n}$ can be obtained from a diagram for the link $L^n_{m,1,\ldots,1}$ by the addition of the braids $\beta_i$ in the appropriate places in the diagram. Hoste-Przytycki give an argument showing that starting with a minus-adequately wrapped diagram and adding in a braid $\beta$ produces a new link for which the wrapping conjecture also holds [27, Lemma 10]. Repeated applications of this argument show that the wrapping conjecture holds for the link $L^n_{m,\beta_1,\ldots,\beta_n}$. $\square$

**Theorem 4.5.2.** *Let $K^n$ be the iterated Whitehead double of the annular link $L^1$ obtained as the annular closure of the tangle $J$ from Figure 4.1 and let $K^n_{m,\beta}$ be the annular knot obtained by replacing $K^n$ with the closure of the $m$-braid $\beta$. Then the wrapping conjecture holds for $K^n_{m,\beta}$.*

*Proof.* Notice that a diagram for $K^n_{m,1}$ can be built by stacking blackboard framed cables of the tangle $J$ from Figure 4.1 along with trivial braids to the left or right of these tangles and then taking an annular closure. This observation and the arguments from the proof of Theorem 4.5.1 show that the generator from all 1's resolution with every circle marked with a $+$ sign is the only generator in its quantum grading and it is in the appropriate $k$-degree so the wrapping conjecture holds for $K^n_{m,1}$. In other words, there is a minus-adequately wrapped diagram for $K^n_{m,1}$. Applying [27, Lemma 10] ensures that the wrapping conjecture also holds for $K^n_{m,\beta}$.

$\square$

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