On the Construction of Supercuspidal Representations: New Examples from Shallow Characters

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A dissertation

submitted to the Faculty of

the department of Mathematics

in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

Boston College

Morrissey College of Arts and Sciences

Graduate School

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Abstract

This thesis contributes to the construction of supercuspidal representations in small residual characteristics. Let $\mathbf{G}$ be a connected, quasi-split, semisimple reductive algebraic group defined and quasi-split over a non-archimedean local field k and splitting over a tamely, totally ramified extension of k. To each parahoric subgroup of $\mathbf{G}(k)$, Moy and Prasad have attached a natural filtration by compact open subgroups, the first of which is called the pro-unipotent radical of the parahoric subgroup. The first main result of this thesis is to characterize *shallow characters* of a pro-unipotent radical, those being complex characters that vanish on the smallest Moy-Prasad subgroup containing all commutators of linearly-dependent affine k-root groups. Through low-rank examples, we illustrate how this characterization can be used to explicitly construct all shallow characters. Next, we provide a natural sufficient condition under which a shallow character compactly induces as a direct sum of supercuspidal representations of $\mathbf{G}(k)$. Through examples, however, we show that this sufficient condition need not be necessary, all while constructing new supercuspidal representations of $\mathrm{Sp}_4(k)$ when $p = 2$ and the split form of G_2 over k when $p = 3$. This work extends the construction of the simple supercuspidal representations given by Gross and Reeder and the epipelagic supercuspidal representations given by Reeder and Yu.

In memory of Harrold.

All of my best ideas were his.

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Chapter 1

Introduction

The seed of representation theory, as a discipline of mathematical research, was planted by Gauss in the early 19th century with his study of characters of finite abelian groups. Gauss's seed would lay dormant through the century and eventually germinate in 1896, when Frobenius extended the notion of a character to non-abelian finite groups. In the 126 years since taking root, representation theory has grown and branched off, becoming one of the farthest reaching fields of active research in all of mathematics.

Broadly speaking, the goal of a representation theorist studying a general algebraic object is to "represent" it as a well-understood prototypical example. One then hopes to gain a better understanding of their object through investigating all the ways to "represent" it as the simpler prototype. Once a representation theorist has precisely defined what it means to "represent" their object, they can begin to classify all the possible representations. To Gauss, the first representation theorist, this meant representing an arbitrary finite abelian group A as $\mathbb{C}^\times$ via a group homomorphism $\chi : A \to \mathbb{C}^\times$, called a *character*. This is the first, and simplest, example of a (matrix) representation.

To a student taking a first course in representation theory, the goal is to understand an arbitrary finite group G by relating it to the finite-dimensional complex matrix group $\mathrm{GL}_n(\mathbb{C})$, via a group homomophism

$$\pi : G \to \mathrm{GL}_n(\mathbb{C})$$

called a n-dimensional complex representation. The simplest of these representations are 1-dimensional *characters* of G. The characters of a finite group are of particular note, since they cannot be decomposed into smaller sub-representations, and thus we call them *irreducible*. Any student learning about finite-dimensional complex representations of finite groups is sure to quickly discover that any arbitrary representation can always be decomposed into irreducible sub-representations; thus, in order to understand all finite-dimensional complex representations, one need only classify the irreducible ones.

For a representation theorist studying p-adic groups, such as myself, the goal is to understand a p-adic group G by relating it to the linear group $\mathrm{GL}(V)$ for an arbitrary complex vector space V, via an open group homomophism

$$\pi : G \to \mathrm{GL}(V)$$

called a smooth representation. Specifically, there is a strong focus on constructing *supercuspidal* representations, which act as the irreducible building blocks from which more general smooth representations can be constructed. Much progress has been made on this goal in recent years for the case in which the residual characteristic p is large; however, the progress has been relatively limited when p is small. In this thesis, we will use what we call *shallow characters* to provide a method for explicitly constructing *new* supercuspidal representations when p is small.

1.1 Constructing Supercuspidal Representations

Let $\mathbf{G}$ be a reductive algebraic group defined over a non-archimedean local field k. A *smooth representation* of $G = \mathbf{G}(k)$ is a group homomorphism

$$\pi : G \to \mathrm{GL}(V)$$

where V is a complex vector space such that for every $v \in V$ there is a compact open subgroup $H \subseteq G$ with $\pi(h)v = v$ for every $h \in H$. A smooth irreducible representation (π, V) is *supercuspidal* if every matrix coefficient is compactly supported modulo the center of G.

Much of the work on constructing supercuspidal representations is highly dependant on the residual characteristic p: The supercuspidal representations of SL_2 were first classified when $p > 2$ by Sally and Shalika in 1969 [28]. Similarly, a classification for PGL_2 when $p > 2$ was given by Silberger [29]. In 1977, Howe gave a construction for the "tame" supercuspidal representations of GL_n [13], which was proven to be exhaustive when p does not divide n [23]. In 1991, Bushnell and Kutzco were able to classify the supercuspidal representations of GL_n, independent of the prime p [5]. A method for constructing supercuspidal representation for general groups was given by Adler in 1998 [1] and generalized by Yu in 2001 [36]. Kim proved that Yu's method is exhaustive when p is large [18]. Finally, in 2008, Stevens gave a construction of all supercuspidal representations for a split connected classical group when $p > 2$ [33].

A common thread among many of these exhaustive methods of construction is that they rely on compact induction: If $H \subseteq \mathbf{G}(k)$ is a compact open subgroup modulo the center of $\mathbf{G}(k)$ and V is an irreducible smooth representation of H then the compactly-induced representation

$$\mathrm{ind}_G^N(V) = \left\{ f : G \to V \;\middle|\; \begin{array}{c} f(hx) = h \cdot f(x) \\ f \text{ is compactly supported} \end{array} \right\}.$$

is a supercuspidal representation of G whenever it is irreducible. In general, a compactly-induced representation is highly reducible. Therefore, when constructing supercuspidal representations, one must be careful in choosing a compact open subgroup and an irreducible representation thereof, and thus in order to be exhaustive, these constructions are relatively complicated and highly dependent on the residual characteristic.

There are very few constructions of supercuspidal representations that are independent of residual characteristic p and they are far from exhaustive. One notable example is the construction of the simple supercuspidal representation given by Gross and Reeder in 2010 [12]. This construction requires the input of an affine generic character of the pro-unipotent radical of the Iwahori subgroup, and the relatively simplicity of the resulting irreducible compactly-induced representation of G gives it its structure. This construction was then reformulated in terms of Geometric Invariant Theory (GIT) and generalized by Reeder and Yu to give the epipelagic[1] supercuspidal representation which has minimal non-zero depth. The input for this construction was a stable vector (in the sense of GIT) belonging to the finite abelian quotient of the first two piece of the Moy-Prasad filtration of a general parahoric subgroup [27]. The existence of these stable vectors was initially only known for large enough p and for small p only in the case of the Iwahori subgroup. The existence of stable vectors was eventually extended to all parahoric subgroups for small p and sufficiently large residue fields in the case that $\mathrm{char}(k) = 0$ and G is split [7].

We now give a more detailed explanation of the construction of [12]: Let f denote the residue field of k, and let I be an Iwahori subgroup of G. The natural filtration of the field k, Moy and Prasad proved, is mirrored in I which is filtered by open compact subgroups

$$I \geq I_{1/h} \geq I_{2/h} \geq \cdots$$

where h is the Coxeter number of G [24][25].[2] The first of these Moy-Prasad subgroups, often denoted by $I_+ := I_{1/h}$, is called the pro-unipotent radical of I, and the second of these Moy-Prasad subgroups, denoted by $I_{++} = I_{r_2}$, is a normal subgroup of I_+. The quotient I_+/I_{++} is an elementary abelian p-group isomorphic to $\mathsf{f}^{\ell+1}$. A group homomorphism

$$\chi : I_+/I_{++} \to \mathbb{C}^\times$$

is called an affine generic character if it is non-trivial on each factor of f. If, through abuse of notation, we also denote by $\chi : Z(G)I_+ \to \mathbb{C}^\times$ the lift and extension of an affine generic character χ to $Z(G)I_+$, Reeder and Gross proved that the compactly induced representation

$$\pi(\chi) := \mathrm{ind}_{Z(G)I_+}^G(\chi)$$

is an irreducible superecuspidal representation of G called the *simple supercuspidal repre-*

[1]The *epipelagic zone* is the shallowest layer of the ocean where photosynthesis can occur.

[2]This particular filtration of I is the one corresponding to the barycenter of the alcove attached to I. Choosing a different point in this alcove results in a different filtration.

sentation [12].

We now assume that $\mathbf{G}$ is connected, semisimple, quasi-simple, and split or quasi-split over k, splitting over a tamely purely ramified Galois extension of k. The main results of Chapter 3 are an attempt to generalize the work of Gross-Reeder on affine generic characters. In §3.1, we classify characters that are non-trivial on subgroups that are deeper, but not too deep, in the Moy-Prasad filtration. More specifically, in Theorem 3.1.4, given any point λ in the Bruhat-Tits building, we identify the minimal Moy-Prasad subgroup of the parahoric subgroup G_λ containing commutators of linearly dependent positive affine root groups. This parahoric subgroup, which we denote by $G_{\lambda,\mathsf{s}(\lambda)}$, is a normal subgroup of the pro-unipotent radical $G_{\lambda+}$ of G_λ, and while the quotient $G_{\lambda+}/G_{\lambda,\mathsf{s}(\lambda)}$ is not necessarily abelian, we see in Corollary 3.1.9 that its commutator subgroup is generated by the commutators of pairwise linearly independent positive affine root groups. These commutators have a relatively simple form which we reference in Proposition 2.2.4 and verify in Appendix A. This simple form of the commutator subgroup allows us to completely classify all λ-*shallow characters*

$$\chi : G_{\lambda+}/G_{\lambda,\mathsf{s}(\lambda)} \to \mathbb{C}^\times,$$

as we see through example in §3.1.3. With our ingredients identified, we use §3.2 to investigate which λ-shallow characters compactly induce to give supercuspidal representations of G. We provide a naive extension of Reeder-Yu's stability condition in Theorem 3.2.3, which we prove is sufficient for constructing supercuspidal representations. And finally, we show in §3.2.3 and §3.2.4, through examples, that this naive extension is not a necessary condition while simultaneously constructing *new* supercuspidal representations of $\mathrm{Sp}_4(k)$ when $p = 2$ and the split form of G_2 over k when $p = 3$.

The methods used in this thesis were first presented for split groups in a preprint on the arXiv [8]. Here we have extended the argument to construct λ-shallow characters for quasi-split G. I am of the belief that these arguments can be extended further to residually non-split quasi-split groups.

1.2 Structure of the Thesis

If our goal is to explicitly construct supercuspidal representations, we must have a complete understanding of our group. Chapter 2 is therefore devoted to providing the necessary background in quasi-split groups. The vast majority of this information is not new, but is rather a collection of results from various sources for easy reference.

In §2.1.1, we lay out how to fold a simple reduced root system along a symmetry of its Dynkin diagram. The orbits of this diagram symmetry form a root system, called a twisted root system, which need not be reduced. Along with the twisted root system, in §2.1.2 we define a companion affine twisted root system. which is an affine root system in the sense of [22]. These twisted root and affine root systems will respectively become the relative root and affine root systems of our connected, semisimple, quasi-simple algebraic group $\mathbf{G}$ defined and quasi-split over a non-archimedean local field k.

In section 2.2, we present the necessary background information found in [26] on Galois descent for a quasi-split group defined over a non-archimedean local field. As is the case for any quasi-split group (not necessarily over a non-archimedean local field), we see in §2.2.1 that the the Galois group acts on the absolute root system of our group via a Dynkin diagram folding, so that the k-root system of our group can be realized as a twisted root system. By fixing an épinglage on which the Galois group acts, we are able to give a Chevalley-Steinberg system in §2.2.2 for our group $G = \mathbf{G}(k)$. This subsection closes with a description of commutators of k-root groups, coming from [26]. Then in §2.2.3 we see that when the natural filtration of k induces a natural filtration on the k-root groups in G, and the resulting affine k-root system is then easily identifiable as the affine twisted root system constructed in §2.1.2. The culmination of this chapter is the definition of the Moy-Prasad filtration of a parahoric subgroup of G by compact open subgroups, which will be necessary for constructing supercuspidal representations in the next chapter.

Chapter 3 is where the bulk of the results of this thesis can be found. In §3.1.2, we use the Moy-Prasad filtration from the previous chapter to define a finite p-group, not necessarily abelian. We call the characters of this group *shallow*, and using the commutator formulas calculated previously we are able to characterize them. Finally, in §3.1.3 we explicitly illustrate how to classify the shallow characters for both split and non-split quasi-split examples.

Section 3.2 is devoted to a discussion on constructing supercuspidal representations from the shallow characters classified in the previous section. In §3.2.1, we give a brief overview of compact induction, and in §3.2.2, §3.2.3, and §**??** we show that compact induction and shallow characters can yield supercuspidal representations under appropriate conditions.

We then close with two appendices: In Appendix A we provide computations that justify the commutator formulas given in §2.2.2, and in Appendix B we have the various tables that are referenced throughout the thesis.

1.3 Acknowledgements

This thesis is based on the research I conducted as a graduate student at Boston College from 2016 to present. I am grateful for my friends and colleagues whose encouragement sustained me while I persevered through these difficult years.

First and foremost, I would like to thank my advisor Mark Reeder for his mentorship and friendship throughout my time in graduate school. His work was instrumental for guiding my own research, and without his patience and many suggestions, this thesis would not have been possible. It was also through him that I discovered a love for math history, and this has played a significant role in shaping my teaching style as well.

I am grateful to the staff and faculty of the Department of Mathematics at Boston College for providing me with the environment in which my research could thrive. I would like to give special thanks to my defense committee, Mark Reeder, Benjamin Howard, and David Truman, for their insights and careful considerations. I would also like to thank my dear friend Eli Grigsby, who was a constant source of support during many of the challenges that arose during my time in Boston.

Many thanks to the friends who have provided me with emotional and mathematical support over the last six years. Thank you to all of my friends who have helped in discussions relating to this thesis; namely, thank you to Ethan Farber, Maria Fox, Siddharth Mahendraker, Gwyneth Moreland, and Laura Seaberg.

I would also like to acknowledged my parents, Charles and Suzanne Gastineau, and my brothers, Robert and Philip Gastineau. I am very fortunate to have had the love and support of my family to guide me on my journey.

Finally, above all else, I would like to express my most heartfelt gratitude for my wife Martha McAlister-Raeburn-Gastineau and our late cat Harrold. The last six years presented me with numerous trials, and it was only with their strength was I able to endure. Without their love and understanding, my work would not be possible; this thesis is as much theirs as it is mine.

Chapter 2

Quasi-split Groups over Filtered Fields

2.1 Twisted Root Systems

In this section we develop the notions of a *twisted root system* and an *affine twisted root system* which will be vital for understanding the structure of quasi-split groups in the sequel. We show how one can, starting with a simple reduced root system, construct a twisted root system, not necessarily reduced, by *folding* its Dynkin diagram. We then construct an affine root system by taking affine translations of the twisted roots constructed previously.

Our discussion on affine root systems will involve affine linear functionals on a real vector space. Those more familiar with these objects may recognize that it is possible to work over affine spaces instead of vector spaces. Once an origin is chosen in the affine space, however, these two notions coincide. Our future discussions of quasi-split groups over a local field will assume an origin has been chosen; thus, we find it appropriate to assume an origin has already been chosen in what follows.

A working knowledge of root and affine root systems is assumed, more-so for the former. For anyone unfamiliar with these topics, I strongly recommend any of the widely-available classical texts [2][16][22].

Notation 2.1.1. Let $\mathbf{E}$ be a real vector space, and let $\mathbf{R}$ be a root system of linear functionals on $\mathbf{E}$; in other words, there exists a set of coroots $\mathbf{R}^\vee = \{\mathbf{a}^\vee \mid \mathbf{a} \in \mathbf{R}\} \subseteq \mathbf{E}$ and a finite group of automorphisms $\mathbf{W}_0 \subseteq \mathrm{GL}(\mathbf{E}^*)$ generated by reflections such that the following hold:

(1) $\mathbf{R}$ spans the dual space $\mathbf{E}^*$.

(2) $w(\mathbf{R}) = \mathbf{R}$ for all $w \in \mathbf{W}_0$.

(3) $\langle \mathbf{a}, \mathbf{b}^\vee \rangle := \mathbf{a}(\mathbf{b}^\vee) \in \mathbb{Z}$ and $\langle \mathbf{a}, \mathbf{a}^\vee \rangle = 2$ for all $\mathbf{a}, \mathbf{b} \in \mathbf{R}$.

(4) $\mathbf{R}$ is finite.

Fix a subset $\mathbf{R}^+ \subseteq \mathbf{R}$, called a *positive system* of $\mathbf{R}$, satisfying

(5) For each $\mathbf{a} \in \mathbf{R}$, exactly one of $\mathbf{a}, -\mathbf{a}$ belongs to $\mathbf{R}^+$.

(6) For each $\mathbf{a}, \mathbf{b} \in \mathbf{R}^+$, if $\mathbf{a} + \mathbf{b} \in \mathbf{R}$ then $\mathbf{a} + \mathbf{b} \in \mathbf{R}^+$.

The indecomposable elements in $\mathbf{R}^+$, denoted by $\mathbf{D}$, form a *base* of $\mathbf{R}$ satisfying

(7) $\mathbf{a} \in \mathbf{D}$ if and only if $\mathbf{a}$ cannot be written as the sum of roots in $\mathbf{R}^+$.

We assume that $\mathbf{R}$ is *reduced*, meaning that the only scalar multiples of each $\mathbf{a} \in \mathbf{R}$ belonging to $\mathbf{R}$ are $\mathbf{a}$ and $-\mathbf{a}$. We also assume that $\mathbf{R}$ is *simple*, meaning that it cannot be decomposed as the direct product of two subroot systems. //

2.1.1 Twisted roots

We denote by $\mathrm{Aut}(\mathbf{R})$ the set of automorphisms in $\mathrm{GL}(\mathbf{E}^*)$ which preserve $\mathbf{R}$. We say that an automorphism $\sigma \in \mathrm{Aut}(\mathbf{R})$ is **based** whenever σ preserves a base of $\mathbf{R}$. In particular, for our fixed base $\mathbf{D}$ of $\mathbf{R}$, we denote by $\mathrm{Aut}(\mathbf{R}, \mathbf{D})$ the subgroup of $\mathrm{Aut}(\mathbf{R})$ of based automorphisms preserving $\mathbf{D}$. We have the splitting

$$\mathrm{Aut}(\mathbf{R}) = \mathbf{W}_0 \rtimes \mathrm{Aut}(\mathbf{R}, \mathbf{D}).$$

The non-trivial based automorphisms of $\mathbf{R}$ arise as symmetries of the Dynkin diagram of $\mathbf{R}$, and therefore must have order $e \in \{2, 3\}$, as seen in Table B.1 in Appendix B. In particular, once a base $\mathbf{D}$ of $\mathbf{R}$ has been chosen, a based automorphism of $\mathbf{R}$ preserving $\mathbf{D}$ is uniquely determined by its order e except when $\mathbf{R}$ is type D_4 and $e = 3$; in this case there are two order 3 symmetries of the Dynkin diagram, but they are inverses of each other.

Definition 2.1.1. Let $\sigma \in \mathrm{Aut}(\mathbf{R}, \mathbf{D})$ be an order e based automorphism of $\mathbf{R}$ preserving a fixed base $\mathbf{D}$. The action of σ on $\mathbf{R} \subseteq \mathbf{E}^*$ naturally induces an automorphism of $\mathbf{E}$ which we also denote by σ. We denote by the unbolded $E = \mathbf{E}^\sigma$ the set of points in $\mathbf{E}$ which are fixed by σ, and we denote by the unbolded $R = \mathbf{R}_\sigma$ the set of restrictions to E of roots in $\mathbf{R}$:

$$R = \mathbf{R}_\sigma = \{a = \bar{\mathbf{a}} \mid \mathbf{a} \in \mathbf{R}\}$$

where $\bar{\mathbf{a}}$ is the restriction to E of a root $\mathbf{a} \in \mathbf{R}$. The set R is called the **twisted root system** of linear functionals on E, and the elements therein are called **twisted roots**.

Since σ leaves stable the base $\mathbf{D}$ and the positive system $\mathbf{R}^+$ associated to it, we say that a twisted root is **simple** (resp. **positive**) provided that it is the restriction to E of a simple (resp. positive) root in $\mathbf{R}$. We denote by

$$D = \mathbf{D}_\sigma = \{a = \bar{\mathbf{a}} \mid \mathbf{a} \in \mathbf{D}\}$$

the set of simple twisted roots and by

$$R^+ = \mathbf{R}^+_\sigma = \{a = \bar{\mathbf{a}} \mid \mathbf{a} \in \mathbf{R}^+\}$$

the set of positive twisted roots.

Given any twisted root $a \in R$, we denote by (a) the set of all roots in $\mathbf{R}$ whose restriction to E is a. Two roots in $\mathbf{R}$ have the same restriction if and only if they belong to the same $\langle\sigma\rangle$-orbit in $\mathbf{R}$; thus, we can write

$$(a) = \{\mathbf{a}, \sigma\mathbf{a}, \sigma^2\mathbf{a}, \dots\}$$

for any $\mathbf{a} \in (a)$. If for any $a \in R$ we let e_a denote the cardinality of (a), then we have the following dichotomy:

(*i*) (a) generates a type $(A_1)^{e_a}$ root subsystem of $\mathbf{R}$.

(*ii*) (a) generates a type A_2 root subsystem of $\mathbf{R}$.

Cycles with type (*ii*) imply that $2a$ is a twisted root in R and can only occur when σ is an involution on a root system of type A_{2n}.

Below we will see that R forms a root system of linear functionals over E. But first we must construct the coroot system $R^\vee$ lying inside E and the reflection group W_0 contained in $\mathrm{GL}(E^*)$. For each twisted root $a \in R$, the definition of the coroot $a^\vee$ and the reflection w_a depends on the cycle type of (a) in the following way:

(*i*) If (a) generates a type $(A_1)^{e_a}$ root subsystem of $\mathbf{R}$, then we define

$$a^\vee = \sum_{\mathbf{a}\in(a)} \mathbf{a}^\vee$$

and

$$w_a = \prod_{\mathbf{a}\in(a)} w_{\mathbf{a}}.$$

Note that since all roots in (a) are pairwise orthogonal, both the sum and product above are independent of order.

(*ii*) If $(a) = \{\mathbf{a}, \mathbf{b}\}$ generates a type A_2 root subsystem of $\mathbf{R}$, then we define $a^\vee = 2\mathbf{a}^\vee + 2\mathbf{b}^\vee$ and $w_a = w_{\mathbf{a}+\mathbf{b}}$.

In both cases, we see that both $a^\vee$ and w_a are invariant under the respective actions of σ. Therefore, we denote by $R^\vee \subseteq E$ the set of all $a^\vee$, and by $W_0 \subseteq \mathrm{GL}(E^*)$ the subgroup generated by all w_a. The perfect pairing between $\mathbf{R}$ and $\mathbf{R}^\vee$ restricts to a σ-invariant pairing between R and $R^\vee$ so that

$$w_a(b) = b - \langle b, a^\vee \rangle\, a$$

for all $a, b \in R$.

If one wishes to identify the vector spaces E and E^* by using the pairing $\langle \cdot, \cdot \rangle$ between R and $R^\vee$, then we this vector space can be equiped with a inner-product thus allowing us to talk about the relative length of twisted roots. However, it is also possible to define the relative length of twisted roots without making this identification: we say that two twisted root $a, b \in R$ have distinct relative length if $\langle a, b^\vee \rangle$ and $\langle b, a^\vee \rangle$ do not equal. In this case, we say that a is longer (resp. shorter) than b if $\langle a, b^\vee \rangle$ is greater (resp. less) than $\langle b, a^\vee \rangle$. One can check that that the roots of R are partitioned by their relative lengths [15, Section 10.4]. There are exactly two relative lengths, except when σ is an involution of a root system of type A_{2n}, in which case there are three relative lengths. We say that a twisted root is **short** if it has the shortest relative length and **long** if it has the second-to-shortest relative length. If σ is an involution of a root system of type A_{2n}, then the twisted roots that are of the longest relative length are precisely the divisible roots of form $2a$ for a twisted root $a \in R$.

We are now ready to prove that R forms a root system, not necessarily reduced, of linear functionals on E. The proof is straightforward and is included for the sake of completeness:

Proposition 2.1.1. *Let $\sigma, \mathbf{E}, \mathbf{R}, \mathbf{D}, \mathbf{R}^+, \mathbf{R}^\vee, \mathbf{W}_0$ be as above. Then $R = \mathbf{R}_\sigma$ is a root system of linear functionals on $E = \mathbf{E}^\sigma$ with coroot system $R^\vee = \mathbf{R}_\sigma^\vee$ and reflection group $W_0 = \mathbf{W}_0^\sigma$ in the sense that the following hold:*

(1) *The roots in R span E^*.*

(2) *$w_a(R) = R$ for all $a \in R$.*

(3) *$\langle a, b^\vee \rangle \in \mathbb{Z}$ and $\langle a, a^\vee \rangle = 2$ for all $a, b \in R$.*

(4) *R is finite.*

Moreover, $R^+ = \mathbf{R}_\sigma^+$ forms a positive system of R in the sense that the following hold:

(5) *For each $a \in R$, exactly one of $a, -a$ belongs to R^+.*

(6) *For each $a, b \in R^+$, if $a + b \in R$ then $a + b \in R^+$.*

Finally, $D = \mathbf{D}_\sigma$ forms a base of R in the following sense:

(7) *$a \in D$ if and only if a cannot be written as the sum of roots in R^+.*

Proof. This proposition vacuously holds if σ is trivial; therefore we will assume that σ has order either 2 or 3. These results will be proven one at a time, but most follow directly from the appropriate structures of $\mathbf{E}$, $\mathbf{R}$, $\mathbf{D}$, $\mathbf{R}^+$, $\mathbf{R}^\vee$, and $\mathbf{W}_0$ laid out in Notation 2.1.1.

(1): Recall that $\mathbf{R}$ spans $\mathbf{E}^*$. For each $a \in R$, the subspace of $\mathbf{E}^*$ spanned by (a) has a 1-dimensional σ-invariant subspace spanned by a. Therefore, R must span $E^* = (\mathbf{E}^*)^\sigma$.

(2): Fix any $a \in R$. From the definitions above, we see that $w_a \in W_0$ is the restriction to E^* of some element in $\mathbf{W}_0$. Since elements in $\mathbf{W}_0$ leave stable $\mathbf{R}$, their restrictions to E^* leave stable R whose elements are restrictions to E of roots in $\mathbf{R}$.

(3): Fix $a, b \in R$, and let $\mathbf{a} \in (a)$ and $\mathbf{b} \in (b)$ so that

$$\langle a, b^\vee \rangle = x \cdot \sum_{i=1}^{e_a} \langle \mathbf{a}, (\sigma^i \mathbf{b})^\vee \rangle,$$

where x is 2 or 1 respectively depending on whether $2b$ is a root in R or not. In either case, $\langle a, b^\vee \rangle$ is an integer since each $\langle \mathbf{a}, (\sigma^i \mathbf{b})^\vee \rangle$ is one. Furthermore, we can see that $\langle a, a^\vee \rangle = 2$. Indeed, if (a) generates a type $(A_1)^{e_a}$ root subsystem of $\mathbf{R}$, then

$$\langle a, a^\vee \rangle = \langle \mathbf{a}, \mathbf{a}^\vee \rangle = 2$$

since $\mathbf{a}$ commutes with each $\sigma^i \mathbf{a}$, $i = 1, \ldots, e_a - 1$; if (a) generates a type A_2 root subsystem of $\mathbf{R}$, then

$$\langle a, a^\vee \rangle = 2\langle \mathbf{a}, \mathbf{a}^\vee + \sigma \mathbf{a}^\vee \rangle = 2(1) = 2.$$

(4): R must be finite, since $\mathbf{R}$ is assumed to be so.

(5): Given any $a \in R$, either (a) or $(-a)$ must intersect R^+ since it forms a positive system on $\mathbf{R}$. In fact, exactly one of these intersects $\mathbf{R}^+$, since σ preserves $\mathbf{R}^+$.

(6): Given $a, b \in R^+$, if $a + b$ is a root in R then there must exist some $\mathbf{a} \in (a)$ and $\mathbf{b} \in (b)$ such that $\mathbf{a} + \mathbf{b} \in \mathbf{R}$. Since $\mathbf{a}, \mathbf{b} \in \mathbf{R}^+$, this means that $\mathbf{a} + \mathbf{b} \in \mathbf{R}^+$ so that $a + b \in R^+$.

(7): Let $a \in R^+$ and fix any $\mathbf{a} \in (a)$. If $\mathbf{a} = \mathbf{b} + \mathbf{c}$ for $\mathbf{b}, \mathbf{c} \in \mathbf{R}^+$, then $a = b + c$ where $\mathbf{b} \in (b)$ and $\mathbf{c} \in (c)$. On the other hand, if a is decomposable in R^+ then, since $\mathbf{D}$ is a base of $\mathbf{R}$, $\mathbf{a}$ is decomposable in $\mathbf{R}^+$ so that $\mathbf{a} \notin \mathbf{D}$ and $a \notin D$. $\square$

Remark. A reader familiar with root systems may notice the omission of the condition in the definition of a root system that often appears in the literature; namely, here we do not require that R satisfy the condition of a **reduced root system** that the only scalar multiples of a twisted root a that belong to R are a and $-a$. In this sense, we say that R need not necessarily be reduced. In fact, R is non-reduced if and only if σ is an involution of a root system of type A_{2n}.

For any root system R, not necessarily reduced, with base D, its **Dynkin diagram** is the graph with vertex set D and $\langle a, b^\vee \rangle \langle b, a^\vee \rangle$ edges between the vertices $a, b \in D$. If there is more than 1 edge between a, b then we label them with an arrow pointing towards the shorter root. Since R need not be reduced, we will also shade in a vertex $a \in D$ whenever $2a \in R$.

Let R be any twisted root system constructed above. When σ is trivial, so that $R = \mathbf{R}$, then the Dynkin diagram of R is identical to that of $\mathbf{R}$. When σ is non-trivial and R is reduced, then its Dynkin diagram is one of a simple reduced root system. When σ is an involution of a root system of type A_{2n}, then R is non-reduced and its Dynkin diagram is one of type BC_n. This is summarized in Table B.2 of Appendix B.

2.1.2 Affine twisted roots

Given real vector spaces V, U, a function $\psi : V \to U$ is called an **affine linear map** if there exists a linear map $\dot{\psi} : V \to U$ such that

$$\psi(x + y) = \psi(x) + \dot{\psi}(y)$$

for all $x, y \in V$. The linear function $\dot{\psi}$ is called the **linear part** (or **gradient**) of ψ. The **constant part** (or **intercept**) of ψ is the vector $\psi(0) \in U$. An affine linear map ψ can then be recovered from its linear and constant parts via the formula

$$\psi(x) = \dot{\psi}(x) + \psi(0).$$

In the specific case that the codomain equals $\mathbb{R}$, we call an affine linear map an **affine linear functional**, and through an abuse notation, we write $a + r : V \to \mathbb{R}$ to be the affine linear functional with linear part $a \in V^*$ and constant part $r \in \mathbb{R}$.

Definition 2.1.2. For each root $a \in R$ we denote by $\Psi(a)$ a set of affine linear functionals on E whose gradients are all a. The intercepts of the affine functionals in $\Psi(a)$ depend on the cycle type of (a) and whether $a/2$ is a k-root according to the following trichotomy:

(i_1) If (a) generates a type $(A_1)^{e_a}$ root subsystem in $\mathbf{R}$ and $a/2 \notin R$, then let

$$\Psi(a) := \{a + n/e_a \mid n \in \mathbb{Z}\}$$

be the set of all affine linear functionals on E with gradient a and intercept an integer multiple of $1/e_a$.

(i_2) If (a) generates a type A_1 root subsystem of $\mathbf{R}$ and $a/2 \in R$, then let

$$\Psi(a) := \{a + (2n+1)/2 \mid n \in \mathbb{Z}\}$$

be the set of all affine linear functions on E with gradient a and intercept an odd-integer multiple of $1/2$ $(\neq 1/e_a)$.

(ii) If (a) generates a type A_2 subsystem in $\mathbf{R}$, then set

$$\Psi(a) := \{a + n/2 \mid n \in \mathbb{Z}\}$$

to be the set of all affine linear functionals on E with gradient a and intercept an integer multiple of $1/2 = 1/e_a$.

The affine linear functionals appearing above, called **affine twisted roots**, form

$$\Psi = \Psi(\mathbf{R}, \sigma) := \bigsqcup_{a \in R} \Psi(a),$$

the **affine twisted root system** of affine linear functionals on E.

For each $a_i \in D = \{a_1, \dots, a_\ell\}$, we denote by

$$\alpha_i := a_i + 0$$

the affine twisted root with gradient a_i and intercept 0. If σ is trivial, set a_0 to be the lowest root in $R = \mathbf{R}$; if σ is non-trivial, then let a_0 be the lowest short twisted root in R except when σ is an involution and of a root system of type A_{2n}, in which case we let a_0 be twice the lowest short twisted root in R. In each case, we denote by

$$\alpha_0 := a_0 + 1/e$$

the affine twisted root with gradient a_0 and intercept $1/e$. The set

$$\Delta = \{\alpha_0, \alpha_1, \ldots, \alpha_\ell\}$$

will be called the **base** of Ψ, with the affine twisted roots therein called **simple**.

To see that Ψ forms an affine root system, it will be necessary that we construct a perfect pairing. If one wishes to identify E and its dual E^* using the perfect pairing $\langle \cdot, \cdot \rangle$ between R and $R^\vee$, then it is possible to define a twisted affine coroot and thus a pairing. However, it is also possible to define the pairing without making this identification: for each affine twisted root $\alpha, \beta \in \Psi$ we abuse notation to write $\langle \alpha, \beta^\vee \rangle$ in place of the pairing between their gradients

$$\langle \alpha, \beta^\vee \rangle := \langle \dot{\alpha}, \dot{\beta}^\vee \rangle.$$

In a similar vein, it makes sense to talk about the relative length of an affine twisted root without ever defining an inner-product; that is, we say that an affine twisted root is **short** (resp. **long**) if its gradient is short (resp. long).

Definition 2.1.3. Let α be an affine twisted root in Ψ. The **vanishing hyperplane** $\ker(\alpha)$ is the affine hyperplane in E consisting of points at which α vanishes. We denote by w_α the affine linear involution of E given by reflection along $\ker(\alpha)$ via

$$w_\alpha(\lambda) = \lambda - \alpha(\lambda)\dot{\alpha}^\vee$$

for all $\lambda \in E$. Through the pairing defined above, w_α also acts on Ψ according to the formula

$$w_\alpha(\beta) = \beta - \langle \beta, \alpha^\vee \rangle \alpha$$

for all $\beta \in \Psi$. The group W generated by each w_α, $\alpha \in \Psi$ is the **affine reflection group** of Ψ.

Let $\mathcal{H}$ denote the union of all vanishing hyperplanes of affine twisted roots in Ψ. The open connected components of the complement $E \setminus \mathcal{H}$ are called **alcoves** in E. One of these alcoves, called the **fundamental alcove**, is bounded by the vanishing hyperplanes of the simple affine twisted roots in Δ, and it will be denoted by $\mathcal{C} \subseteq E$. (c.f., Example 4.3 in [22] or Example 4.7 in [16])

An affine twisted root in Ψ is said to be **positive** if it takes positive values on $\mathcal{C}$. The set of positive affine twisted roots in Ψ will be denoted by Ψ^+.

We are now ready to see that Ψ forms an **affine root system** in the sense of [22], satisfying the following:

(1) Ψ spans the space of affine linear functionals on E.

(2) $w_\alpha(\Psi) = \Psi$ for all $\alpha \in \Psi$.

(3) $\langle \alpha, \beta^\vee \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in \Psi$.

(4) W (as a discrete group) acts properly on E.

The construction of these affine root systems has appeared in the literature (for example see [26]). We will, however, include the proof for a number of them, as they are fairly straightforward:

Proposition 2.1.2. *Let $E, \Psi, \Delta, \Psi^+, W$ be as above. Then Ψ is a affine root system of affine linear functionals on E with reflection group W in the sense that the following hold:*

(1) *Ψ spans the space of affine linear functionals on E.*

(2) *$w_\alpha(\Psi) = \Psi$ for all $\alpha \in \Psi$.*

(3) *$\langle \alpha, \beta^\vee \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in \Psi$.*

(4) *W (as a discrete group) acts properly on E.*

Moreover, Ψ^+ forms a positive system of Ψ in the sense that the following hold:

(5) *For each $\alpha \in \Psi$, exactly one of $\alpha, -\alpha$ belongs to Ψ^+.*

(6) *For each $\alpha, \beta \in \Psi^+$, if $\alpha + \beta \in \Psi$ then $\alpha + \beta \in \Psi^+$.*

Finally, Δ forms a base of Ψ in the following sense:

(7) *$\alpha \in \Delta$ if and only if α cannot be written as the sum of roots in Ψ^+.*

Proof. If σ is either trivial or an involution of a root system of type A_{2n} then, up to normalizing the constant parts, Ψ is one of the affine root systems described [22, Proposition 2.1]. These prototypical affine root systems are well-understood. We therefore will focus on when σ is non-trivial and R is reduced.

(1): This follows from R spanning the space of linear functionals on E.

(2): Let $\alpha = a + m/e_a$ and $\beta = b + n/e_b$ be two affine roots in Ψ. Then

$$w_\alpha(\beta) = (b + n/e_b) + \langle b, a^\vee \rangle (a + m/e_a) = w_a(b) + n/e_b + \langle b, a^\vee \rangle m/e_a.$$

Since W_0 is length-preserving on gradients, we know that $e_{w_a(b)} = e_b$. We also know that $\langle b, a^\vee \rangle$ is an integer. Thus,

$$n/e_b + \langle b, a^\vee \rangle m/e_a = k/e_b \tag{2.1.1}$$

for some integer k. Indeed, if $e_a = e_b$, then certainly (2.1.1) holds for $k = n + \langle b, a^\vee \rangle m$; if $e_a = e \neq 1$ and $e_b = 1$, so that a is short and b is long, then $\langle b, a^\vee \rangle m/e_a = \pm m$ and (2.1.1) holds for $k = n \pm m$; and if $e_a = 1$ and $e_b = e \neq 1$, so that a is long and b is short, then $\langle b, a^\vee \rangle = \pm 1$ and (2.1.1) holds for $k = n \pm me$.

(3): This is a restatement of the fact that $\langle a, b^\vee \rangle \in \mathbb{Z}$ for all $a, b \in R$.

(5): No affine twisted root vanishes on the fundamental alcove $\mathcal{C}$, and so either α or $-\alpha$ evaluates positively thereon.

(6): If $\alpha(\lambda), \beta(\lambda) > 0$ then $\alpha(\lambda) + \beta(\lambda) > 0$, for all $\lambda \in \mathcal{C}$. □

Remark. There are a couple things to note about the definition of an affine root system appearing in MacDonald's influential paper [22]. First is that it is not an entirely correct definition, as independently noted in 2011 by Mark Reeder and Jasper Stokman. Missing from this definition is the condition that the affine root system be *non-finite*, meaning that for each gradient $a \in R$, there are at least two (and thus infinitely many) affine roots whose gradient is a. Our twisted root systems Ψ certainly satisfy this non-finiteness condition, and therefore the condition was omitted from the above proposition.

With the addition of the non-finiteness condition to the definition of affine root systems, Macdonald's definition of affine root systems is equivalent to that of a *echellonage* appearing in Bruhat-Tits [3].

Definition 2.1.4. The closure of the fundamental alcove is characterized as the set of points in E on which each simple affine twisted root in Δ takes non-negative values. A **facet** of $\mathcal{C}$ is a equivalence class of this closure under the relation $\sim$, where $\lambda \sim \mu$ when $\alpha(\lambda) > 0$ if and only if $\alpha(\mu) > 0$ for all $\alpha \in \Delta$. A facet of $\mathcal{C}$ is then uniquely determined by the vanishing of a subset of simple affine twisted roots.

A facet corresponding to the vanishing of all but one simple affine root in Δ contains exactly one point, which we call a **vertex** of $\mathcal{C}$. We say that a vertex λ of $\mathcal{C}$ is **strongly-special** whenever there is a simple affine twisted root $\alpha_i \in \Delta$ such that $\alpha_i(\lambda) = 1/e$ (and thus vanishes on all other simple affine twisted roots in Δ). Not to be confused with the notion of a *hyperspecial vertex*, the relationship between strongly-special vertices and *special vertices*, found elsewhere in the literature [34], is as follows: if R is reduced, then a vertex is special if and only if it is strongly-special, but if R has type C-BC_ℓ then a vertex is strongly-special if and only if it is the special vertex corresponding to the vanishing of α_0. Alternatively, strongly-special vertices can be understood as coming from weight-1 vertices in the weighted Dynkin diagram of Ψ as defined below.

Recall that $-a_0$ is a positive twisted root in R. While we have not proven it here, it is well known that any element of R^+ can be written as the the integral sum of the simple twisted roots in D with all positive coefficients (c.f., [2, Chapter 6 Theorem 3]). We define the **minimal constant relation** among the simple affine twisted roots in Δ to be

$$1/e = m_0\alpha_0 + m_1\alpha_1 + \cdots + m_\ell\alpha_\ell \tag{2.1.2}$$

where $m_0 = 1$ and $m_1, \dots, m_\ell$ are all positive integers coming from the decomposition of $-a_0$ into simple twisted roots.

The **Dynkin diagram** of Ψ is the graph with vertex set Δ and $\langle \alpha_i, \alpha_j^\vee \rangle \langle \alpha_j, \alpha_i^\vee \rangle$ edges between vertices α_i, α_j except when the vanishing hyperplanes of α_i and α_j are parallel in which case we put a single bold edge. If there are more than 1 edges between α_i, α_j then we label them with an arrow pointing towards the shorter affine twisted root. To each vertex $\alpha_i \in \Delta$ in the Dynkin diagram of Ψ, we assign the positive integer m_i appearing in the minimal constant relation above. The Dynkin diagram with vertices adorned with these weights will be called the **weighted Dynkin diagram** of Ψ. Table B.3 in Appendix B shows the weighted weighted Dynkin diagrams for all affine twisted root systems.

Technical lemmas

What follows are some technical results on affine twisted root systems. They are tools that will help us later in Chapter 3:

Lemma 2.1.3. *If α is a non-long affine twisted root in Ψ, then $1/e - \alpha$ and $1/e + \alpha$ are also.*

Proof. Since α is non-long, it is of the form $\alpha = a + n/e$. Then indeed $1/e - \alpha = -a + (1-n)/e$ and $1/e + \alpha = a + (1+n)/e$ are both non-long affine roots in Ψ. $\square$

Lemma 2.1.4. *Suppose that R is not simply-laced, and let α_{i_1} be any long, simple affine root whose gradient is not twice the lowest short root when R is of type C-BC_ℓ. Then there exists a non-repeating sequence of vertices $(\alpha_{i_1}, \dots, \alpha_{i_n})$ in the Dynkin diagram of Ψ such that the following hold:*

(1) *$\{\alpha_{i_j}, \alpha_{i_{j+1}}\}$ is an edge for each $j = 1, \dots, n-1$.*

(2) *$\alpha_{i_1}, \dots, \alpha_{i_{n-1}}$ are all long affine k-roots.*

(3) *α_{i_n} is short.*

Moreover, in this situation, the affine functional $\alpha = \alpha_{i_1} + \cdots + \alpha_{i_n}$ is a short affine twisted root in Ψ.

Proof. This existence claim can be individually checked for each Dynkin diagram given in Table B.3 in Appendix B. Graphically this sequence can occur as one of three classes of sub-diagrams: First, if Ψ has either type G_2 or G_2^I, then our sequence can appear as the sub-diagram

$$\alpha_{i_1} \Rrightarrow \alpha_{i_2};$$

next, if Ψ_σ has type G_2, then our sequence can appear as the sub-diagram

$$\alpha_{i_1} \,\text{——}\, \alpha_{i_2} \Rrightarrow \alpha_{i_3};$$

and finally, if Ψ has any type other than G_2 or G_2^I, then our sequence appears as a sub-diagram of the form

$$\alpha_{i_1} \,\text{——} \cdots \text{——}\, \alpha_{i_{n-1}} \Rightarrow \alpha_{i_n}.$$

In all three cases, $\alpha = \alpha_{i_1} + \cdots + \alpha_{i_n}$ is a short affine twisted root. $\square$

2.2 Quasi-split Groups

A reductive group is said to be **quasi-split** over a field k provided that it has a Borel subgroup defined over k. We say that a quasi-split group is **split** provided that it has a k-split maximal torus. In this section, we will develop the theory of quasi-split groups defined over local fields. For a general, in-depth treatment of split reductive groups, there are many classical sources that one can turn to, such as [9][14][32][30]. I personally recommend [6], as it most naturally lends itself to the computations performed throughout. Literature on the structure of non-split quasi-split groups is slightly less abundant, but I have modeled my treatment in this section based of that of [26].

We will begin by defining relative root system of any quasi-split reductive group over any field of definition, not necessarily local. As with the the the absolute root system of a split group, the relative root system of a quasi-split group contains combinatoric information that controls the structure of the group via the Chevalley-Steinberg system. Like the previous subsection, this will merely be an overview of well-established ideas found in the above references.

In §2.2.3, we will focus on a structure of reductive groups over filtered fields, like the ones considered in this thesis. Namely, we will define the notion of an affine root system which melds the root data and the filtration of the field of definition. This structure was first studied by Iwahori and Mastumoto [17] and then further developed by many mathematicians in the late 20th century. Notably, the work on Bruhat and Tits [3][34] act as comprehensive references for anyone looking to learn more.

Notation 2.2.1. Let k be a non-archimedean local field with ring of integers A_k, a local ring with unique maximal ideal P_k. We denote by $\mathrm{val} : k^\times \to \mathbb{Z}$ a surjective integral valuation on k, so that $\mathrm{val}(k^\times) = \mathbb{Z}$. The residue field of k is denoted by $\mathsf{f} = A_k/P_k$. The residual characteristic of k is the positive characteristic of its residue field, denoted by p.

Let K denote a degree $e \in \{1, 2, 3\}$ totally ramified Galois extension of k, and assume that $e \neq p$ so that K/k is tamely ramified. We fix an order e cyclic generator σ of the Galois group $\mathrm{Gal}(K/k)$, and write $\sigma(x) = \bar{x}$ when $e = 2$. We denote by A_K the ring of integers of K, a local ring with unique maximal ideal denoted P_K. Through an abuse of notation, we also denote by $\mathrm{val} : K^\times \to \mathbb{Z}e^{-1}$ the valuation of K extending that on k. Since K/k is totally ramified, the residue field of K is isomorphic to that of k, and thus we also denote it by $\mathsf{f} = A_K/P_K$.

Let $\mathbf{G}$ be a connected, quasi-simple, semisimple reductive algebraic group defined over k and splitting over K. Over k, we assume that $\mathbf{G}$ is either split (in the case that $K = k$ and $e = 1$) or non-split quasi-split (in the case that $e \in \{2, 3\}$).Let $\mathbf{S}$ be a maximal k-split torus, and let $\mathbf{Z}$ the centralizer of $\mathbf{S}$ in $\mathbf{G}$, a maximal torus of $\mathbf{G}$ defined over k. Let $\mathbf{B}$ be a Borel subgroup of $\mathbf{G}$ defined over k and containing $\mathbf{Z}$.

As a convention, we use unbolded letters G, B, Z, S to respectively denote the groups k-rational points in $\mathbf{G}, \mathbf{B}, \mathbf{Z}, \mathbf{S}$. More generally, if a subgroup of $\mathbf{G}$, represented with a bolded letter, is defined over k then the group of k-rational points is denoted by the corresponding unbolded letter. //

2.2.1 Root systems

Let $\mathbf{R} = R(\mathbf{G}, \mathbf{Z}, K)$ denote the K**-root system** of $\mathbf{G}$ relative to the K-split maximal torus $\mathbf{Z}$ consisting of all K-characters of $\mathbf{Z}$ appearing in the adjoint representation of $\mathbf{Z}(K)$ on the Lie K-algebra Lie($\mathbf{G}(K)$). Through the natural pairing between K-characters and K-cocharacters of $\mathbf{Z}$, the K-root system forms a simple reduced system of roots acting as linear functionals on the real vector space $X_*(\mathbf{Z}, K) \otimes_{\mathbb{Z}} \mathbb{R}$, where $X_*(\mathbf{Z}, K)$ is the K-cocharacter lattice of $\mathbf{Z}$. The set of the K-roots appearing in the adjoint representation of $\mathbf{Z}(K)$ on the Borel K-subalgebra Lie($\mathbf{B}(K)$) form a positive system $\mathbf{R}^+ \subseteq \mathbf{R}$, and therein lies the set of simple K-roots of $\mathbf{D} = D(\mathbf{G}, \mathbf{B}, \mathbf{Z}, K)$ of $\mathbf{G}$ relative to the pair $(\mathbf{B}, \mathbf{Z})$.

The Galois group Gal(K/k) has a right action of $\mathbf{G}(K)$ which we denote by $g \mapsto g^\gamma$ for all $g \in \mathbf{G}(K)$ and $\gamma \in \text{Gal}(K/k)$. This action preserves $\mathbf{B}(K), \mathbf{Z}(K), \mathbf{S}(K)$. The action on $\mathbf{Z}(K)$ induces a natural left action of Gal(K/k) on the K-character lattice of $\mathbf{Z}$ defined as follows: given any K-character $\chi : \mathbf{Z}(K) \to K^\times$, the K-character $\sigma\chi$ is defined via

$$[\sigma\chi](z) := \sigma(\chi(z^\sigma))$$

for all $z \in \mathbf{Z}(K)$. This action leaves stable the K-roots of $\mathbf{G}$ relative to $\mathbf{Z}$ as well as the base therein corresponding to $\mathbf{B}$. Thus, σ acts as a based automorphism in Aut$(\mathbf{R}, \mathbf{D})$.

The natural pairing between K-characters and K-cochcaracters of $\mathbf{Z}$ induces a Galois group Gal(K/k) acts on the $X_*(\mathbf{Z}, K)$ according to

$$[\sigma^{-1}\lambda](t) := \lambda(\sigma(t))^\sigma$$

for all $\lambda \in X_*(\mathbf{Z}, K)$ and $t \in K^\times$. We denote by $X_*(\mathbf{S}, k)$ the lattice of k-cocharacters of $\mathbf{S}$ consisting of the restriction to $k^\times$ of σ-invariant K-cocharacters of $\mathbf{Z}$. The images of the elements in $X_*(\mathbf{S}, k)$ generate

$$S = \mathbf{S}(k) = \left\langle \lambda(t) \mid \lambda \in X_*(\mathbf{S}, k), t \in k^\times \right\rangle,$$

the maximal split k-tori in $G = \mathbf{G}(k)$.

Let $R = R(\mathbf{G}, \mathbf{S}, k)$ denote the **k-root system** of $\mathbf{G}$ relative to the maximal k-split torus $\mathbf{S}$, consisting of all k-characters of $\mathbf{S}$ appearing in the adjoint representation of S on the Lie k-algebra $\mathrm{Lie}(G)$. The set of k-roots appearing in the adjoint representation of S on the Lie k-algebra of the Borel subgroup $B = \mathbf{B}(k)$ will form a positive system $R^+ \subseteq R$, and therein will lie the simple k-roots $D = D(\mathbf{G}, \mathbf{B}, \mathbf{S}, k)$ of $\mathbf{G}$ relative to $(\mathbf{B}, \mathbf{S})$.

Each k-root of $\mathbf{G}$ relative to $\mathbf{S}$, when viewed as a linear functional on $X_*(\mathbf{S}, k) \otimes_{\mathbb{Z}} \mathbb{R}$, is the restriction of a K-root in $\mathbf{R}$ acting as a linear functional on $X_*(\mathbf{Z}, K) \otimes_{\mathbb{Z}} \mathbb{R}$. Two K-roots restrict to the same k-root if and only if they belong to the same orbit under the Galois group $\mathrm{Gal}(K/k)$ acting as a based automorphisms. Thus, we have shown that one can realize the k-root system R as one of the twisted root systems constructed in Definition 2.1.1:

Proposition 2.2.1. *Let $\mathbf{G}, \mathbf{Z}, \mathbf{S}, K, k$ be as in Notation 2.2.1. If σ is a cyclic generator of the Galois group $\mathrm{Gal}(K/k)$, acting as a based automorphism of the K-root system $R(\mathbf{G}, \mathbf{Z}, K)$, then the k-root system $R(\mathbf{G}, \mathbf{S}, k)$ can be realized as the twisted root system $R(\mathbf{G}, \mathbf{Z}, K)_\sigma$ of linear functionals on the real vector space $X_*(\mathbf{S}, k) \otimes_{\mathbb{Z}} \mathbb{R}$ defined in Definition 2.1.1.*

2.2.2 Chevalley-Steinberg systems and root groups

For each positive K-root $\mathbf{a} \in \mathbf{R}^+$, we denote by $\mathbf{U_a}$ and $\mathbf{U_{-a}}$ the **K-root groups** respectively corresponding to $\mathbf{a}$ and $-\mathbf{a}$, uniquely characterized as being the non-trivial connected unipotent subgroups of $\mathbf{G}$, defined over K, on which $\mathbf{Z}$ respectively acts via $\mathbf{a}$ and $-\mathbf{a}$. The group $\mathbf{G_a} = \langle \mathbf{U_a}, \mathbf{U_{-a}} \rangle$ is a connected, quasi-simple, semisimple reductive algebraic group, defined over K, with absolute-rank 1 maximal torus $\mathbf{G_a} \cap \mathbf{Z}$; therefore, there must exist a (non-unique) central K-isogeny

$$\varphi_{\mathbf{a}} : \mathrm{SL}_2(K) \to \mathbf{G_a}(K)$$

characterized as isomorphically mapping the upper and lower-triangular unipotent subgroups onto $\mathbf{U_a}(K)$ and $\mathbf{U_{-a}}(K)$ respectively. We then consider the (non-unique) **K-root**

morphisms, which are defined to be the restriction of $\varphi_{\mathbf{a}}$ to these unipotent subgroups:

$$u_{\mathbf{a}}(x) := \varphi_{\mathbf{a}} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \qquad \text{and} \qquad u_{-\mathbf{a}}(x) := \varphi_{\mathbf{a}} \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}$$

for all $x \in K$.

The non-uniqueness of K-root morphisms is addressed by fixing a K**-pinning** (or an **épinglage**) of $\mathbf{G}$ relative to $(\mathbf{B}, \mathbf{Z})$ consisting of a choice of central K-isogenies $\varphi_{\mathbf{a}}$ for the simple roots $\mathbf{a} \in \mathbf{D}$. Once a pinning has been chosen, the central K-isogenies for the remaining positive k-roots $\mathbf{a} \in \mathbf{R}^+$ are chosen so that vectors $du_{\pm\mathbf{a}}(1)$ collectively form a **Chevalley basis** of the Lie K-algebra Lie$(\mathbf{G}(K))$, meaning that

$$[du_{\mathbf{a}}(1), du_{\mathbf{b}}(1)] = \begin{cases} 0 & \text{if } \mathbf{a}+\mathbf{b} \notin \mathbf{R} \\ \pm(n+1)\, du_{\mathbf{a}+\mathbf{b}}(1) & \text{if } \mathbf{a}+\mathbf{b} \in \mathbf{R}, \end{cases}$$

for all $\mathbf{a}, \mathbf{b} \in \mathbf{R}$, where n is the greatest positive integer for which $\mathbf{b} - n\mathbf{a}$ is a root in $\mathbf{R}$. For more detail on this construction, please refer to [6, Section 4.2].

Notation 2.2.2. Throughout the remainder of the thesis, we will assume that a K-pinning of $\mathbf{G}$ relative to $(\mathbf{B}, \mathbf{Z})$ has been chosen so that for $\mathbf{a} \in \mathbf{R}$, we have

$$u_{\mathbf{a}}(x)^{\sigma} = u_{\sigma\mathbf{a}}(\epsilon_{\mathbf{a}}\, \sigma(x)) \tag{2.2.1}$$

for all $x \in K$, where $\epsilon_{\mathbf{a}} \in \{1, -1\}$ such that $\epsilon_{\mathbf{a}} = 1$ whenever $\mathbf{a} \in \mathbf{D}$. Such an assumption is reasonable due to the classical work of Steinberg which says that any automorphism of a split reductive group preserving a Borel and maximal torus therein must preserve some pinning [31, proof of Theorem 8.2]. The remaining $\epsilon_{\mathbf{a}}$ are computed using the relations

$$[u_{\mathbf{a}}(x), u_{\mathbf{b}}(y)]^{\sigma} = [u_{\mathbf{a}}(x)^{\sigma}, u_{\mathbf{b}}(y)^{\sigma}].$$

for all $\mathbf{a}, \mathbf{b} \in \mathbf{R}$ and $x, y \in K$. In fact, without loss of generality, we can assume that the K-root morphisms are chosen so that $\epsilon_{\mathbf{a}} = -1$ if and only if $a/2 \in R$ [6, Lemma 13.6.2]. //

Definition 2.2.1. Let a be any k-root in R. So as to align with the definition of twisted root systems given in Definition 2.1.1, we let (a) be the $\mathrm{Gal}(K/k)$-orbit of all K-roots in $\mathbf{R}$ whose restriction to $\mathbf{S}$ is a. The order of this orbit, equal to the index of the stabilizer in $\mathrm{Gal}(K/k)$ of $\mathbf{a}$, is denoted by e_a. We then define K_a to be the subset of K or K^2 according to the following trichotomy:

(i_1) If (a) generates a type $(A_1)^{e_a}$ root subsystem of type $\mathbf{R}$ and $a/2 \notin R$, then let K_a denote the fixed field in K of $\langle \sigma^{e_a} \rangle$, a degree e_a field extension of k with Galois group $\mathrm{Gal}(K_a/k) = \langle \sigma^{e/e_a} \rangle$.

(i_2) If (a) generates a type A_1 root subsystem of type $\mathbf{R}$ and $a/2 \in R$, then let K_a denote the set of all $y \in K$ such that $y + \bar{y} = 0$.

(ii) If (a) generates a type A_2 root system of type $\mathbf{R}$, then let K_a denote the subset of K^2 consisting of all pairs (x, y) such that $x\bar{x} + y + \bar{y} = 0$.

Throughout, it will often be convenient to refer to the union of cases (i_1) and (i_2), which we denote by (i) and corresponds with case (i) in Definition 2.1.1.

Corresponding to a, we define the k**-root group** $\mathbf{U}_a$ to be the subgroups of $\mathbf{G}$ generated by all K-root groups $\mathbf{U}_{\mathbf{a}}$ for $\mathbf{a} \in (a)$. The structure of $\mathbf{U}_a$ can be described in terms of the cycle type of (a) as follows:

(i) If (a) generates a type $(A_1)^{e_a}$ root subsystem of $\mathbf{R}$, then $\mathbf{U}_a$ is the internal direct product

$$\mathbf{U}_a = \prod_{\mathbf{a} \in (a)} \mathbf{U}_{\mathbf{a}}.$$

This product is independent of the order of $\mathbf{a} \in (a)$, since the $\mathbf{U}_{\mathbf{a}}$ commute with one another.

(ii) If $(a) = \{\mathbf{a}, \sigma\mathbf{a}\}$ generates a type A_2 root subsystem of $\mathbf{R}$, so that $\mathbf{a} + \sigma\mathbf{a} \in \mathbf{R}$, then $\mathbf{U}_a$ is not abelian. Rather, it is the internal semidirect product

$$\mathbf{U}_a = \mathbf{U}_{\mathbf{a}} \mathbf{U}_{\sigma\mathbf{a}} \mathbf{U}_{\mathbf{a}+\sigma\mathbf{a}}.$$

Here $\mathbf{U}_{\mathbf{a}+\sigma\mathbf{a}}$ is central, and thus is normalized by both $\mathbf{U}_{\mathbf{a}}$ and $\mathbf{U}_{\sigma\mathbf{a}}$.

Since each $\mathbf{U_a}$ is defined over K, the group $\mathbf{U}_a$ must be also. Furthermore, since (a) is a single Galois orbit, $\mathbf{U}_a$ is defined over k. We denote by

$$U_a = \mathbf{U}_a(k) = \{u \in \mathbf{U}_a(K) \mid u^\sigma = u\}$$

the group of k-rational points in $\mathbf{U}_a$.

Proposition 2.2.2. *Let a be any k-root in R. The σ-invariant elements of $\mathbf{U}_a(K)$ can be described as follows:*

(i) Suppose that (a) generates a type $(A_1)^{e_a}$ root subsystem of $\mathbf{R}$. Fixing $\mathbf{a} \in (a)$, an element in $\mathbf{U}_a(K)$ is σ-invariant if and only if it is of the form

$$u_a(x) := \prod_{i=0}^{e_a-1} u_{\mathbf{a}}(x)^{\sigma^i} \tag{2.2.2}$$

for $x \in K_a$.

(ii) Suppose that (a) generates a type A_2 root subsystem of $\mathbf{R}$. Fixing $\mathbf{a} \in (a)$, an element in $\mathbf{U}_a(K)$ is σ-invariant if and only if it is of the form

$$u_a(x, y) := u_{\mathbf{a}}(x) \cdot u_{\sigma\mathbf{a}}(\bar{x}) \cdot u_{\mathbf{a}+\sigma\mathbf{a}}(y) \tag{2.2.3}$$

for $(x, y) \in K_a$.

Proof. (i): Suppose that $(a) = \{\mathbf{a}, \sigma\mathbf{a} \ldots, \sigma^{e_a-1}\mathbf{a}\}$ generates a type $(A_1)^{e_a}$ root subsystem of $\mathbf{R}$. Fix any element $u \in \mathbf{U}_a(K)$ and write u as the unique product

$$u = \prod_{i=0}^{e_a-1} u_{\sigma^i\mathbf{a}}(x_{\sigma^i\mathbf{a}})$$

for $x_{\sigma^i\mathbf{a}} \in K$, independent of order of the product. Applying σ to this product yields

$$u^\sigma = \prod_{i=0}^{e_a-1} u_{\sigma^i\mathbf{a}}(x_{\sigma^i\mathbf{a}})^\sigma.$$

If u is σ-invariant then each $u_{\sigma^i\mathbf{a}}(x_{\sigma^i\mathbf{a}})^\sigma = u_{\sigma^{i+1}\mathbf{a}}(x_{\sigma^{i+1}\mathbf{a}})$. Thus U_a consists precisely of all elements of the form

$$u_a(x) = \prod_{i=0}^{e_a-1} u_{\mathbf{a}}(x)^{\sigma^i}$$

for $x \in K$ such that $\sigma^{e_a}(x) = \epsilon_a\, x$ where

$$\epsilon_a = \prod_{i=1}^{e_a} \epsilon_{\sigma^i \mathbf{a}} \in \{1, -1\}.$$

But according to the assumptions made in Notation 2.2.2, $e_a = -1$ if and only if $a/2 \in R$, and so $\sigma^{e_a}(x) = \epsilon_a\, x$ if and only if $x \in K_a$.

(ii): Suppose that $(a) = \{\mathbf{a}, \sigma\mathbf{a}\}$ generates a type A_2 root subsystem of $\mathbf{R}$. Without loss of generality, we can assume that $u_{\mathbf{a}+\sigma\mathbf{a}}$ is scaled so that

$$[u_{\mathbf{a}}(x_1), u_{\sigma\mathbf{a}}(x_2)] := u_{\mathbf{a}+\sigma\mathbf{a}}(x_1 x_2).$$

Note that this assumption is consistent with every assumption made in Notation 2.2.2. Let u be any element of U_a, and so an element of $\mathbf{U}_a(K)$ that is σ-invariant. We can write u as the product

$$u = u_{\mathbf{a}}(x_1) \cdot u_{\sigma\mathbf{a}}(x_2) \cdot u_{\mathbf{a}+\sigma\mathbf{a}}(y)$$

for $x_1, x_2, y \in K$. Applying σ to this product yields

$$\begin{aligned} u^\sigma &= u_{\sigma\mathbf{a}}(\bar{x}_1) \cdot u_{\mathbf{a}}(\bar{x}_2) \cdot u_{\mathbf{a}+\sigma\mathbf{a}}(-\bar{y}) \\ &= u_{\mathbf{a}}(\bar{x}_2) \cdot u_{\sigma\mathbf{a}}(\bar{x}_1) \cdot [u_{\sigma\mathbf{a}}(\bar{x}_1), u_{\mathbf{a}}(\bar{x}_2)] u_{\mathbf{a}+\sigma\mathbf{a}}(-\bar{y}) \\ &= u_{\mathbf{a}}(\bar{x}_2) \cdot u_{\sigma\mathbf{a}}(\bar{x}_1) \cdot u_{\mathbf{a}+\sigma\mathbf{a}}(-\bar{x}_1\bar{x}_2 - \bar{y}) \end{aligned}$$

Since u is assumed to be fixed by σ, this means that $x_1 = \bar{x}_2$ and $y = -\bar{y} - \bar{x}_1\bar{x}_2$. Putting these all together, we see that U_a consists of all elements of the form

$$u_a(x, y) = u_{\mathbf{a}}(x) \cdot u_{\sigma\mathbf{a}}(\bar{x}) \cdot u_{\mathbf{a}+\mathbf{b}}(y)$$

for all $x, y \in K$ such that $x\bar{x} + y + \bar{y} = 0$, or equivalently $(x, y) \in K_a$. $\square$

Definition 2.2.2. For each k-root $a \in R$, the function $u_a : K_a \to U_a$ defined by either (2.2.2) or (2.2.3) is a group isomorphism, which we logically call a **k-root morphism**. The non-uniqueness seen in K-root morphisms is inherited here, but more-so since (2.2.2) and (2.2.3) required an additional choice of a K-root $\mathbf{a} \in (a)$. This will not be a problem for the discussions that follow, since a different choice of K-root in (a) amounts to pre-composing u_a with an element of $\mathrm{Gal}(K/k)$.

Commutator formulas

Let a, b be linearly independent k-roots. For all $u \in U_a$ and $v \in U_b$, we consider the **commutator**

$$[u, v] := u^{-1}v^{-1}uv.$$

We will denote by $[U_a, U_b]$ the subgroup of G generated by all commutators $[u, v]$ with $u \in U_a$ and $v \in U_b$. If $a+b$ is not a k-root, then U_a and U_b must commute with one another and thus $[U_a, U_b] = 1$. We will now investigate the structure of $[U_a, U_b]$ in the case that U_a and U_b do not commute.

When $a + b$ is a k-root, let $\mathbf{G}_{a,b}$ be the group generated by k-root groups $\mathbf{U}_{\pm a}$ and $\mathbf{U}_{\pm b}$, defined over k with k-rank 2. In the case that $\mathbf{G}_{a,b}$ splits over k, the structure of the commutators of elements in U_a and U_b is given by the following classical result:

Proposition 2.2.3 (Chevalley)**.** *Let a, b be linearly independent k-roots such that $a + b$ is a k-root, and suppose that $\mathbf{G}_{a,b}$ splits over k. Then*

$$[U_a, U_b] \subseteq \prod_{i,j} U_{ia+jb}$$

where the product is increasing over all positive integers i, j such that $ia + jb$ is a k-root. In particular,

$$[u_a(x), u_b(y)] = \prod_{i,j} u_{ia+jb}(\pm C_{abij}\, x^i y^j)$$

where the product is over all positive integers i, j such that $ia + jb$ is a k-root with the constant $C_{abij} \in \{1, 2, 3\}$.

Proof. The proof of this claim, along with a more in-depth description of the constants C_{abij} can be found in [6, Theorem 5.2.2]. □

Now we suppose that $\mathbf{G}_{a,b}$ does not split over k. As in this split case, the commutators of elements in U_a and U_b can be written in terms of the groups U_{ia+jb} for positive integers $i, j > 0$ such that $ia + jb$ is a k-root. Unlike the split case, there is not a uniform way to describe what the factor in U_{ia+jb} is, and they must instead be studied in a case-by-case basis.

The following results can be found in [26, Section 1] without accompanying computations. In its statement, we will be continuing the notation established in (2.2.2) and (2.2.3) for the Galois-invariant elements in the k-root groups:

Proposition 2.2.4 (Prasad-Raghunathan)**.** *Let a, b be linearly independent k-roots such that $a + b$ is also a k-root, and suppose that $\mathbf{G}_{a,b}$ does not split over k. Then*

$$[U_a, U_b] \subseteq \prod_{i,j} U_{ia+jb}$$

where the product is increasing over all positive integers i, j such that $ia + jb$ is a k-root. In particular, we have the following:

(1) *If the k-root system of $\mathbf{G}_{a,b}$ has type A_2 then there exists $\rho, \tau \in \mathrm{Gal}(K/k)$ such that*

$$[u_a(x), u_b(y)] = u_{a+b}(\pm\rho(x)\tau(y))$$

for all $x, y \in K$.

(2) *If the k-root system of $\mathbf{G}_{a,b}$ has type B_2, then*

a. *If a, b are short roots and $a + b$ is long, then there exists a $\rho \in \mathrm{Gal}(K/k)$ such that*

$$[u_a(x), u_b(y)] = u_{a+b}(\pm\,\mathrm{Trace}_{K/k}(x\rho(y))$$

for all $x, y \in K$.

b. *If a is short and b is long, there there exists a $\rho \in \mathrm{Gal}(K/k)$ such that*

$$[u_a(x), u_b(y)] = u_{a+b}(\pm\rho(x)y) \cdot u_{2a+b}(\pm x\bar{x}y),$$

for all $x \in K$ and $y \in k$.

(3) *If the k-root system of $\mathbf{G}_{a,b}$ has type G_2, then the following hold:*

a. *If a, b, and $a+b$ are all short roots, then there are k-embeddings ρ, τ of K into K different than the natural one such that*

$$\begin{aligned}[u_a(x), u_b(y)] = u_{a+b}&(\pm(\rho(x)\tau(y) + \tau(x)\rho(y)) \\ &\cdot u_{2a+b}(\pm(\rho(x)\tau(x)y + \tau(x)x\rho(y) + x\rho(x)\tau(y)) \\ &\qquad \cdot u_{a+2b}(\pm(x\rho(y)\tau(y) + \rho(x)\tau(y)y + \tau(x)y\rho(y)))\end{aligned}$$

for all $x, y \in K$.

b. *If a, b are short roots and $a+b$ is long, then there exists $\rho \in \mathrm{Gal}(K/k)$ such that*

$$[u_a(x), u_b(y)] = u_{a+b}(\pm \mathrm{Trace}_{K/k}(x\rho(y)))$$

for all $x, y \in K$.

c. *If a is short and b is long, then there exists $\gamma, \rho, \tau \in \mathrm{Gal}(K/k)$ such that*

$$\begin{aligned}[u_a(x), u_b(y)] = u_{b+c}&(\pm\gamma(x)y) \cdot u_{2a+b}(\pm\rho(x)\tau(x)y) \\ &\cdot u_{3a+b}(\pm y\,\mathrm{Norm}_{K/k}(x)) \cdot u_{3a+2b}(\pm 2y^2\,\mathrm{Norm}_{K/k}(x))\end{aligned}$$

for all $x \in K$ and $y \in k$.

(4) *If the k-root system of $\mathbf{G}_{a,b}$ has type BC_2, then*

a. *If a, b are short roots and $a+b$ is long, then there exists a $\rho \in \mathrm{Gal}(K/k)$ such that*

$$[u_a(x), u_b(y)] = u_{a+b}(\pm(x\rho(y) - \bar{x}\rho(\bar{y}))$$

for all $x, y \in K$*.*

b. *If* a *is short and* b *is long, then there there exists a* $\rho \in \mathrm{Gal}(K/k)$ *such that*

$$[u_a(x), u_b(y)] = u_{a+b}(\pm\rho(x)y) \cdot u_{2a+b}(\pm x\bar{x}y),$$

for all $x, y \in K$ *with* $y + \bar{y} = 0$*.*

c. *If* $2a, 2b$ *are* k*-roots, then there there exist* $\rho, \tau \in \mathrm{Gal}(K/k)$ *such that*

$$[u_a(x, y), u_b(z, w)] = u_{a+b}(\pm\rho(x)\tau(z))$$

for all $x, y, z, w \in K$ *with* $x\bar{x} + y + \bar{y} = 0$ *and* $z\bar{z} + w + \bar{w} = 0$*.*

d. *If* $2a$ *is a* k*-root but* $2b$ *is not, then there exist* $\gamma, \delta, \rho, \tau \in \mathrm{Gal}(K/k)$ *such that*

$$[u_a(x, y), u_b(z)] = u_{b+c}(\pm\rho(x)\tau(z), \pm\gamma(y)z\bar{z}) \cdot u_{2a+b}(\pm\delta(y)\tau(z))$$

for all $x, y, z \in K$ *with* $x\bar{x} + y + \bar{y} = 0$*.*

Proof. The proof of this proposition can be found in Appendix A, where we fix a group with the indicated k-root data and perform the commutator computation for arbitrary Galois-invariant elements for each pair of k-root groups. □

Remark. The commutator formulas in both Proposition 2.2.3 and Proposition 2.2.4 have a layer of ambiguity coming from the $\pm$ signs, another consequence of the persistent fact that K-root morphisms are not unique. Moreover, the choice of a K-pinning is not enough to determine these signs. Rather, these signs follow from a choice of *structure constants* when fixing a Chevalley basis of $\mathrm{Lie}(\mathbf{G}(K))$ as in Notation 2.2.2. A detailed discussion of how to choose these constants can be found in [6][10][11]

Apart from the non-uniqueness of signs appearing above, Proposition 2.2.4 has an additional layer of ambiguity coming from the elements $\gamma, \delta, \rho, \tau \in \mathrm{Gal}(K/k)$. This is a consequence of the choice of $\mathbf{a} \in (a)$ when defining the k-root morphism u_a in Proposition 2.2.2. Indeed, a different choice of $\mathbf{a} \in (a)$ amounts to a pre-composition of u_a by an element in $\mathrm{Gal}(K/k)$ and this is being reflected above.

2.2.3 Affine k-roots and parahoric subgroups

Definition 2.2.3. Let a be any k-root in R. Proposition 2.2.2 establishes a natural isomorphism between a set K_a and the group U_a via the k-root morphism. If (a) generates a type $(A_1)^{e_a}$ root subsystem of $\mathbf{R}$, then K_a is a subset of K and thus there is a natural valuation on U_α derived from that on K_a given by setting

$$\mathrm{val}_a(u_a(x)) := \mathrm{val}(x)$$

for all $x \in K_a$. On the other hand, when (a) generates a type A_2 root subsystem of $\mathbf{R}$, we set

$$\mathrm{val}_a(u_a(x,y)) := \tfrac{1}{2}\,\mathrm{val}(y) \leq \mathrm{val}(x)$$

for all $(x,y) \in K_a$. In both cases, the valuation is well-defined and independent of the choice of K-root $\mathbf{a} \in (a)$ used to define the k-root morphism, since a different choice amounts to pre-composing u_a in Proposition 2.2.2 with an element of $\mathrm{Gal}(K/k)$.

For each real number $r \in \mathbb{R}$, we consider the following subgroups:

$$\begin{aligned} U_{a,r} &:= \{u \in U_a \mid \mathrm{val}_a(u) \geq r\} \\ U_{a,r+} &:= \{u \in U_a \mid \mathrm{val}_a(u) > r\} \leq U_{a,r}. \end{aligned} \tag{2.2.4}$$

For each r, the quotient group $\mathsf{U}_{a,r} := U_{a,r}/U_{a,r+}$ is a finite-dimensional f-vector space. Furthermore, if $2a$ is a k-root, then $\mathsf{U}_{2a,2r} := U_{2a,2r}/U_{2a,2r+}$ is naturally a f-vector subspace. We will denote by $d_a(r)$ the following f-dimensions:

$$d_a(r) := \begin{cases} \dim_{\mathsf{f}} \mathsf{U}_{a,r} & \text{if } 2a \text{ is not a } k\text{-root}, \\ \dim_{\mathsf{f}} \mathsf{U}_{a,r}/\mathsf{U}_{2a,2r} & \text{if } 2a \text{ is a } k\text{-root}. \end{cases}$$

If $d_a(r)$ is non-zero then we say that $U_{a,r}$ is an **affine k-root group**.

Denote by $\Psi(a)$ the set of all pairs (a,r) for which $d_a(r)$ is non-zero. The union

$$\Psi = \Psi(\mathbf{G},\mathbf{S},k) := \bigsqcup_{a \in R} \Psi(a)$$

of all sets $\Psi(a)$ as a ranges over all k-roots in R forms the **affine k-root system** of $\mathbf{G}$ relative to $\mathbf{S}$. An **affine k-root** is an element $(a, r) \in \Psi$, consisting of its **gradient** (or **vector part**) $a \in R$ and its **intercept** (or **constant part**) $r \in \mathbb{R}$.

We will now show that the affine k-root system Ψ can be identified with a affine twisted root system of linear functions on the real vector space $X_*(\mathbf{S}, k) \otimes_{\mathbb{Z}} \mathbb{R}$. Therefore, throughout the remainder of the thesis, we will adopt the definitions laid out in Definition 2.1.2. For example, we will write $a + r$ to be the affine k-root (a, r) and treat it as a affine linear functional given by

$$[a + r](\lambda) := \langle a, \lambda \rangle + r$$

for all $\lambda \in X_*(\mathbf{S}, k) \otimes_{\mathbb{Z}} \mathbb{R}$.

Proposition 2.2.5. *Let $\mathbf{G}, \mathbf{Z}, \mathbf{S}$ be as in Notation 2.2.1. If σ is a cyclic generator of the Galois group $\mathrm{Gal}(K/k)$ acting as a based automorphism of the K-root system $R(\mathbf{G}, \mathbf{Z}, K)$, then the affine k-root system $\Psi(\mathbf{G}, \mathbf{S}, k)$ can be realized as the affine twisted root system $\Psi(R(\mathbf{G}, \mathbf{Z}, k), \sigma)$ of affine linear functions on the real vector space $X_*(\mathbf{S}, k) \otimes_{\mathbb{Z}} \mathbb{R}$ defined in Definition 2.1.2.*

Proof. Let a be any k-root. We will show that the set of real numbers r such that $d_a(r) \neq 0$ is precisely equal to the constant parts of the affine roots in $\Psi(a)$ defined in Definition 2.1.2. This will be split into cases according to the trichotomy given in Definition 2.2.1. In the first two cases, (i_1) and (i_2), $2a$ is not a k-root and so $d_a(r)$ is non-zero if and only if r is the valuation of an element in K_a. In the final two cases, (i_2) and (ii), K must be a totally, tamely ramified quadratic Galois extension of k, and so we can write $K = k(\sqrt{u})$ for some unit $u \in k$ with $\mathrm{val}(u) = 1$.

(i_1): Suppose that (a) generates a type $(A_1)^{e_a}$ root subsystem of $\mathbf{R}$ and $a/2 \notin R$. Here K_a is a totally ramified degree e_a extension of k with value group

$$\mathrm{val}(K_a^\times) = \{n/e_a \mid n \in \mathbb{Z}\}.$$

Thus, $d_a(r)$ is non-zero and (a, r) is an affine k-root if and only if $r = n/e_a$ for some integer n.

(i_2): Suppose that (a) generates a type A_1 root subsystem of $\mathbf{R}$ and $a/2 \in R$. An element $x \in K$ satisfies $x = -\bar{x}$ if and only if $x = y\sqrt{u}$ for some $y \in k$. The valuation of such an element $x = y\sqrt{u}$ is

$$\mathrm{val}(x\sqrt{u}) = \mathrm{val}(y) + \mathrm{val}(\sqrt{u}) = \mathrm{val}(y) + \tfrac{1}{2} \in \mathbb{Z} + \tfrac{1}{2}.$$

for $y \in k$. Thus, $d_a(r)$ is non-zero and (a, r) is a affine k-root if and only if $r = (2n + 1)/2$ for some $n \in \mathbb{Z}$.

(ii): For this case, we rely on an argument similar to the one found in [26]. First consider the value

$$\nu = \max\{\mathrm{val}(w) \in \mathbb{Z} \mid 1 + w + \bar{w} = 0 \text{ for } w \in K\} \leq 0.$$

Let $w \in K$ such that $1 + w + \bar{w}$ and $\mathrm{val}(w) = \nu$. We now show that $\nu \in \mathbb{Z}$: By way of contradiction, suppose that ν belongs to

$$\mathbb{Z} + \tfrac{1}{2} = \{\mathrm{val}(z) \mid z + \bar{z} = 0 \text{ for } z \in K\},$$

where the equality follows from the computations done in case (i_2) above. Let $z \in K$ be any element such that $\mathrm{val}(z) = \nu \leq 0$ and such that $1+w+\bar{w} = 0 = z+\bar{z}$. Since K/k is ramified, there exists a unit $v \in k^\times$ such that $z/w \in v + P_K$. By replacing w by $-vw$ if necessary, we may assume that $z/w \in -1 + P_K$ so that $w + z \in wP_K$. But then $1 + (w + z) + (\bar{w} + \bar{z}) = 0$ with $\mathrm{val}(w + z) > \mathrm{val}(w) = \nu$, contradicting the maximality of ν.

Suppose that (a) generates a type A_2 root subsystem of $\mathbf{R}$. If $d_a(r)$ is non-zero for a real number r, then there must exist an element $u_a(x, y)$ with $r = \mathrm{val}(y)/2$ belonging to $U_{a,r} - U_{a,r+}U_{2a,2r}$, so that y is "maximal" in the sense that

$$\mathrm{val}(y) = \max\{\mathrm{val}(w) = \nu + 2\,\mathrm{val}(z) \mid z\bar{z} + w + \bar{w} = 0 \text{ for } z, w \in K\}.$$

For each $x \in K$, such a maximal y exists. Thus, $d_a(r)$ is non-zero and (a, r) is an affine k-root if and only if $r = n/2$ for some integer n. □

Definition 2.2.4. Given a point $\lambda \in X_*(\mathbf{S}, k) \otimes_{\mathbb{Z}} \mathbb{R}$, we define the **parahoric subgroup** of

G attached to λ, denoted by G_λ, to be the subgroup of G generated by the compact torus

$$Z_0 := \{z \in Z \mid \mathrm{val}(\chi(z)) \geq 0 \text{ for all } \chi \in X^*(\mathbf{Z}, k)\}$$

and the affine k-root groups U_α for affine k-roots α that evaluate non-negatively at λ:

$$G_\lambda = \langle Z_0, U_\alpha \mid \alpha(\lambda) \geq 0 \text{ for affine } k\text{-root } \alpha \in \Psi\rangle.$$

This group has a natural filtration by open compact subgroups, called the **Moy-Prasad filtration**, induced from the natural filtration of k which we now describe.

For each positive real number $r > 0$, we define the **Moy-Prasad subgroup** of G_λ, denoted by $G_{\lambda,r}$, generated by the compact subgroup

$$Z_r := \{z \in Z \mid \mathrm{val}(1 - \chi(z)) \geq r \text{ for all } \chi \in X^*(\mathbf{Z}, k)\} \leq Z_0$$

and the affine k-root groups U_α for affine k-roots α such that $\alpha(\lambda) \geq r$:

$$G_{\lambda,r} := \langle Z_r, U_\alpha \mid \alpha \text{ is an affine } k\text{-root with } \alpha(\lambda) \geq r\rangle \leq G_\lambda.$$

If $s > r$ are positive real numbers, then $G_{\lambda,s}$ is a normal subgroup of $G_{\lambda,r}$. The positive real numbers for which there is a non-trivial quotient

$$\mathsf{G}_{\lambda,r} := G_{\lambda,r}/G_{\lambda,r+}, \qquad \text{where } G_{\lambda,r+} := \bigcup_{s>r} G_{\lambda,s}$$

forms a discrete, well-ordered set.

The first Moy-Prasad subgroup, denoted by $G_{\lambda+} := G_{\lambda,\mathsf{r}_1(\lambda)}$ where $\mathsf{r}_1(\lambda)$ is the smallest positive real number r such that $\mathsf{G}_{\lambda,r}$ is non-trivial, is called the **pro-unipotent radical** of G_λ. Alternatively, the pro-unipotent radical of G_λ is characterized as being the subgroup thereof generated by the compact torus

$$Z_+ := \{z \in Z \mid \mathrm{val}(1 - \chi(z)) > 0 \text{ for all } \chi \in X_*(\mathbf{Z}, k)\}$$

and the affine k-root groups U_α for affine k-root groups α that evaluate positively at λ:

$$G_{\lambda+} := \langle Z_+, U_\alpha \mid \alpha \text{ is an affine } k\text{-root with } \alpha(\lambda) > 0\rangle \leq G_\lambda.$$

The pro-unipotent radical and its complex representations will be the main object of study throughout the remainder of this thesis.

We denote the second Moy-Prasad subgroup by $G_{\lambda++} := G_{\lambda,\mathsf{r}_2(\lambda)}$ where $\mathsf{r}_2(\lambda)$ is the second-to-smallest positive real number r such that $\mathsf{G}_{\lambda,r}$ is non-trivial. Although $G_{\lambda++}$ does not have a distinguished name, the quotient

$$\mathsf{G}_{\lambda+} = G_{\lambda+}/G_{\lambda++}$$

has been thoroughly studied in [12] and [27] and will serve as a useful point of reference for what follows.

Chapter 3

Supercuspidal Representations

3.1 Shallow Characters

In 2010, Gross and Reeder studied characters

$$\chi : G_{\lambda+} \to \mathbb{C}^{\times}$$

that vanish on $G_{\lambda++}$ in the case that λ was the barycenter of the fundemental alcove [12, Section 9.2]. A few years later, in 2014, Reeder and Yu extended their methods to study these characters for more general λ [27]. In this section we will dive slightly deeper down the Moy-Prasad filtration and consider λ-*shallow characters* of $G_{\lambda+}$ which vanish on the Moy-Prasad subgroup $G_{\lambda,\mathsf{s}(\lambda)} \subseteq G_{\lambda++}$, defined below. Unlike $\mathsf{G}_{\lambda+} = G_{\lambda+}/G_{\lambda++}$, the quotient

$$\mathsf{H}_{\lambda} := G_{\lambda+}/G_{\lambda,\mathsf{s}(\lambda)}$$

is not necessarily abelian, and therefore its commutator subgroup need not be trivial. We show in Theorem 3.1.4, however, that $G_{\lambda,\mathsf{s}(\lambda)}$ is precisely defined so it is the minimal Moy-Prasad subgroup of G_{λ} containing commutators of linearly dependent positive affine k-root groups belonging to $G_{\lambda+}$. Thus, we see in Corollary 3.1.9 that the commutator subgroup of H_{λ} is generated by the commutators of the linearly independent λ-shallow affine k-root groups. In §3.1.3 we then show how to use these commutators to explicitly classify λ-shallow characters, with some low-rank examples.

Notation 3.1.1. We assume the notation set out in Notation 2.2.1. Additionally, we fix a prime element ϖ of $A_k \subseteq k$, called a *uniformizer*, generating the unique prime ideal $P_k = A_k\varpi$. The extension K/k is purely ramified, and therefore we can fix a uniformizer of $A_K \subseteq K$ whose e-th power is ϖ. For convenience of notation, this element will be denoted $\varpi^{1/e}$.

As before, we will let σ denote a fixed order e cyclic generator of $\mathrm{Gal}(K/k)$, and if $e = 2$, we will write $\sigma(x) = \bar{x}$ for all $x \in K$. Additionally, if $e = 3$, we may also write $\sigma(x) = x'$ and $\sigma^2(x) = x''$ for all $x \in K$.

Let $\mathbf{R}$ denote the K root system of $\mathbf{G}$ relative to $\mathbf{Z}$, acting as a set of linear functionals on the real vector space $X_*(\mathbf{Z}, K) \otimes_{\mathbb{Z}} \mathbb{R}$. A K-pinning of $\mathbf{G}$ relative to the pair $(\mathbf{B}, \mathbf{Z})$ is chosen so that it satisfies the assumptions made in Notation 2.2.2.

Let R denote the k-root system of $\mathbf{G}$ relative to $\mathbf{S}$, acting as a set of linear functionals on the real vector space $X_*(\mathbf{S}, k) \otimes_{\mathbb{Z}} \mathbb{R}$. From the K-pinning of $\mathbf{G}$ chosen above, we will fix a set of k-root morphisms as defined in (2.2.2) and (2.2.3).

Let Ψ denote the affine k-root system of $\mathbf{G}$ relative to $\mathbf{S}$, acting as a set of affine linear functionals on the real vector space $X_*(\mathbf{S}, k) \otimes_{\mathbb{Z}} \mathbb{R}$. The valuation on the k-root groups and the corresponding affine k-root groups are as defined in Definition 2.2.3.

We denote by ℓ the rank of the k-cocharacter lattice $X_*(\mathbf{S}, k)$. The k-roots in the base D of R corresponding to the Borel subgroup $\mathbf{B}$ will be enumerated $a_1, \ldots, a_\ell$. The simple affine k-roots $\alpha_0, \alpha_1, \ldots, \alpha_\ell$, defined according to Definition 2.1.2, then form a base of Ψ denoted by Δ. We will let $\mathcal{C}$ denote the fundamental alcove of $X_*(\mathbf{S}, k) \otimes_{\mathbb{Z}} \mathbb{R}$ bounded by the vanishing hyperplanes of the simple affine k-roots in Δ.

Throughout this section, we fix a point λ belonging to the closure of the the fundamental alcove $\mathcal{C}$. //

3.1.1 Shallow affine k-roots

Definition 3.1.1. We denote by $\mathsf{s}(\lambda)$ the largest positive real number such that $\alpha(\lambda) + \beta(\lambda) \geq \mathsf{s}(\lambda)$ whenever α, β satisfy the following:

(s1) $\alpha(\lambda) > 0$ and $\beta(\lambda) > 0$, and (s2) their gradients a, b are linearly dependent and either $a+b=0$ or $a+b$ is a k-root.

A *maximal* value $\mathsf{s}(\lambda)$ exists and is equal to the *minimal* value of $\alpha(\lambda)+\beta(\lambda)$ for α, β satisfying (s1) and (s2). An affine k-root α is called **λ-shallow** provided that $0 < \alpha(\lambda) < \mathsf{s}(\lambda)$.

Lemma 3.1.1. *The following hold:*

a. If λ is not a strongly-special vertex, then there exists a non-long affine k-root α such that $0 < \alpha(\lambda) < 1/e$.

b. If λ is a strongly-special vertex, then there exists a non-long affine k-root α such that $\alpha(\lambda) = 0$.

Proof. For this proof, recall the minimal constant relationship (2.1.2) which says that

$$1/e = m_0\alpha_0 + m_1\alpha_1 + \cdots + m_\ell\alpha_\ell,$$

where $m_0, m_1, \ldots, m_\ell$ are the weights of the weighted Dynkin diagram of Ψ. Then λ is a strongly-special vertex if and only if there exists a simple affine k-root $\alpha_i \in \Delta$ such that $m_i = 1$ and $\alpha_j(\lambda) \neq 0$ if and only if $j = i$.

(a): Suppose that λ is not a strongly-special vertex. When R is simply-laced and all roots are non-long, α can be chosen to be any simple affine k-root not vanishing at λ. When R is not simply-laced, α can be chosen to be any short, simple affine k-root not vanishing at λ if one exists. Therefore, it will be assume that R is not simply-laced and that all short, simple affine k-roots vanish at λ.

Let α_{i_1} be any long, simple affine k-root which does not vanish at λ. Since λ is not strongly-special, we can assume without loss of generality that the gradient of α_{i_1} is not twice the lowest short k-root if Ψ is C-BC_ℓ. Lemma 2.1.4 says that there exists a non-repeating sequence of vertices $(i_1, \ldots, i_n)$ in the Dynkin diagram of Ψ such that the following hold:

(1) $\{\alpha_{i_j}, \alpha_{i_{j+1}}\}$ is an edge for each $j = 1, \ldots, n-1$.

(2) $\alpha_{i_1}, \ldots, \alpha_{i_{n-1}}$ are all long affine k-roots.

(3) α_{i_n} is short.

I claim that we can assume, without loss of generality, that $(i_1, \ldots, i_n)$ also satisfies the following:

(4) $\alpha_{i_j}(\lambda) \neq 0$ if and only if $j = 1$.

Indeed, if $(i_1, \ldots, i_n)$ does not satisfy condition (4) then it can be replaced with the subsequence $(i_j, \ldots, i_n)$ where $1 \leq j < n$ is maximal such that α_{i_j} is non-vanishing at λ. Lemma 2.1.4 also says that

$$\alpha = \alpha_{i_1} + \cdots + \alpha_{i_n}$$

is a short affine k-root, and condition (4) implies that $\alpha(\lambda) = \alpha_{i_1}(\lambda) > 0$. But since λ is not a strongly-special vertex, $\alpha(\lambda) = \alpha_{i_1}(\lambda) < 1/e$.

(b): The following fact can be checked individually for each affine root system with weighted Dynkin diagram given in Table B.3 in Appendix B: there either exists a non-long vertex with weight greater than 1 or at least two non-long vertices with weight equal to 1. Thus, one can always choose α to be a non-long, simple affine k-root that vanishes at a given strongly-special vertex. □

Proposition 3.1.2. *The following hold:*

a. If λ is not a strongly-special vertex, then $\mathsf{s}(\lambda) \leq 1/e$, with equality when R is reduced.

b. If λ is a strongly-special vertex, then $\mathsf{s}(\lambda) = \mathsf{r}_2(\lambda) = 2/e$, where $\mathsf{r}_2(\lambda)$ is defined as in Definition 2.2.4.

Proof. (a): Suppose that λ is not a strongly-special vertex. Let α be as in Lemma 3.1.1(a), and let $\beta = 1/e - \alpha$, which Lemma 2.1.3 says is an affine k-root since α is non-long; therefore, α, β satisfy $(\mathsf{s}1)$ and $(\mathsf{s}2)$ while

$$\alpha(\lambda) + \beta(\lambda) = \alpha(\lambda) + [1/e - \beta(\lambda)] = 1/e,$$

proving that $\mathsf{s}(\lambda) \leq 1/e$.

Now assume that R is reduced, and let α, β be any affine k-roots satisfying $(\mathsf{s}1)$ and $(\mathsf{s}2)$. This can only be true if their gradients sum to 0. In this case, $\alpha(\lambda) + \beta(\lambda)$ must be a positive scalar multiple of the minimal constant $1/e$ and so at least $1/e$. Since we already showed that $\mathsf{s}(\lambda) \leq 1/e$, this yields equality.

(b): Let λ be a strongly-special vertex so that all simple affine k-roots vanish at λ except one. Let $\alpha_i \in \Delta$ be this simple affine k-root, with $\alpha_i(\lambda) = 1/e$. Every positive affine k-root is an integral combination of simple affine k-roots with positive coefficients [22, Proposition 4.6]. Therefore, if $\alpha(\lambda) > 0$ for a positive affine k-root α, then the minimal possible value $\alpha(\lambda)$ can take is $1/e$, and thus, if α, β are any affine k-roots satisfying ($\mathsf{s}1$) and ($\mathsf{s}2$), then the minimal possible value that $\alpha(\lambda) + \beta(\lambda)$ can take is $2/e$. We now see that this minimum can be achieved: if γ is any non-long affine k-root as in Lemma 3.1.1(b), then consider the affine k-roots $\alpha = 1/e - \gamma$ and $\beta = 1/e + \gamma$ as in Lemma 2.1.3. These α, β satisfy ($\mathsf{s}1$) and ($\mathsf{s}2$) and $\alpha(\lambda) + \beta(\lambda) = 2/e$. □

Remark. When G splits over k and λ is a strongly-special vertex, Proposition 3.1.2 and Definition 3.1.1 say that an affine root $\alpha \in \Psi$ is λ-shallow provided that $0 < \alpha(\lambda) < \mathsf{s}(\lambda) = 2$. An observant reader may note that this is different than the definition of a shallow affine root given in [8], where I universally defined a shallow affine root to be $\alpha \in \Psi$ such that $0 < \alpha(\lambda) < 1$. This slight change in definition strengthens the results of the next section since, as we see in Corollary 3.1.3 below, the set of affine k-roots that are shallow under the old definition would be empty in this case.

Corollary 3.1.3. *If α, β are distinct λ-shallow affine k-roots, then their gradients a, b are distinct. Moreover, the following hold:*

a. *If λ is not a strongly-special vertex, then every λ-shallow affine k-root has minimal positive height among all affine k-roots with the same gradient.*

b. *If λ is a strongly-special vertex, then every λ-shallow affine k-root takes value $1/e$ at λ and has either minimal positive or second-to-minimal positive height among all affine k-roots with the same gradient, with the second-to-minimal positive height affine k-root being λ-shallow only if the minimal positive height affine k-root with the same gradient vanishes at λ.*

Proof. This follows immediately from Proposition 3.1.2 since $|\alpha - \beta| \geq 1/e$ for all distinct affine k-roots α, β with equal gradients. □

Remark. When the k-root system is reduced, the value of $\mathsf{s}(\lambda)$ and consequently the set of λ-shallow affine k-roots depends only on the facet containing λ and not precisely on λ itself. When the k-root system is non-reduced, both are dependent on the precise choice of λ. For example, consider the affine k-root system of type C-BC_2 with base $\{\alpha, \beta, \gamma\}$ satisfying

$$1/2 = 2\alpha + 2\beta + \gamma.$$

If λ is the barycenter of the alcove corresponding to this base, then $\mathsf{s}(\lambda) = 1/10$ and $\{\alpha, \beta, \gamma\}$ forms the set of λ-shallow affine k-roots. If λ is not the barycenter, with $\alpha(\lambda) = 3/20$ and $\beta(\lambda) = 1/20$, then $\mathsf{s}(\lambda) = 3/10$ and $\{\alpha, \beta, \gamma, \alpha + \beta, \beta + \gamma, 2\beta + \gamma\}$ is the set of λ-shallow affine k-roots.

3.1.2 Shallow characters

Notation 3.1.2. Throughout this subsection, we let λ denote a fixed point in the closure of $\mathcal{C}$, the fundamental alcove in $X_*(\mathbf{S}, k) \otimes_{\mathbb{Z}} \mathbb{R}$. We also fix an enumeration on λ-shallow affine k-roots

$$\psi_1, \psi_2, \ldots, \psi_n$$

so that if $i < j$ then $\psi_i(\lambda) < \psi_j(\lambda)$. //

Recall from Definition 2.2.4 that $G_{\lambda+}$ is the pro-unipotent radical of the parahoric subgroup G_λ, generated by the compact torus Z_+ and the affine k-root groups U_α for all affine k-roots α whose value at λ is positive:

$$G_{\lambda+} := \big\langle Z_+, U_\alpha \mid \alpha \text{ is an affine } k\text{-root with } \alpha(\lambda) > 0 \big\rangle.$$

Contained within $G_{\lambda+}$ is $G_{\lambda,\mathsf{s}(\lambda)}$, the normal Moy-Prasad subgroup of G_λ generated by the compact torus $Z_{\mathsf{s}(\lambda)}$ and the affine k-root groups U_α for all affine k-roots α whose value at

λ is at least $\mathsf{s}(\lambda)$:

$$G_{\lambda,\mathsf{s}(\lambda)} := \left\langle Z_{\mathsf{s}(\lambda)}, U_\alpha \mid \alpha \text{ is an affine } k\text{-root with } \alpha(\lambda) \geq \mathsf{s}(\lambda) \right\rangle.$$

The definition of $\mathsf{s}(\lambda)$ allows us to characterize the group $G_{\lambda,\mathsf{s}(\lambda)}$ as follows:

Theorem 3.1.4. *$G_{\lambda,\mathsf{s}(\lambda)}$ is the smallest Moy-Prasad subgroup of G_λ containing subgroups $[U_\alpha, U_\beta]$ whenever α, β are affine k-roots satisfying the following:*

(1) *$U_\alpha, U_\beta \subseteq G_{\lambda+}$, and*

(2) *the respective gradients a, b are linearly dependent.*

Proof. Suppose that α, β are affine k-roots satisfying conditions (1) and (2) of the theorem. Exactly one of the following must hold:

- $a + b = 0$,
- $a + b$ is a k-root, or
- $a + b$ is non-zero and not a k-root.

If $a + b$ is non-zero and not a k-root, then U_α and U_β commute so that $[U_\alpha, U_\beta] = \{1\} \subseteq G_{\lambda,\mathsf{s}(\lambda)}$. If either $a + b = 0$ or $a + b$ is a k-root, then α, β satisfy ($\mathsf{s}1$) and ($\mathsf{s}2$) so that $\alpha(\lambda) + \beta(\lambda) \geq \mathsf{s}(\lambda)$ and $[U_\alpha, U_\beta] \subseteq G_{\lambda,\mathsf{s}(\lambda)}$. The minimality of $G_{\lambda,\mathsf{s}(\lambda)}$ is equivalent to the maximality of $\mathsf{s}(\lambda)$. □

Definition 3.1.2. Denote by

$$\mathsf{H}_\lambda := G_{\lambda+}/G_{\lambda,\mathsf{s}(\lambda)},$$

a finite group generated by abelian subgroups of the form

$$\mathsf{U}_0 := Z_+ G_{\lambda,\mathsf{s}(\lambda)}/G_{\lambda,\mathsf{s}(\lambda)} \cong Z_+/Z_{\mathsf{s}(\lambda)},$$

and

$$\mathsf{U}_i := U_{\psi_i} G_{\lambda,\mathsf{s}(\lambda)}/G_{\lambda,\mathsf{s}(\lambda)} \cong U_{\psi_i}/U_{\psi_i} \cap G_{\lambda,\mathsf{s}(\lambda)}$$

for λ-shallow affine k-roots $\psi_1, \ldots, \psi_n$.

Proposition 3.1.5. *The following hold:*

a. *If λ is not a strongly-special vertex, then U_0 is trivial and each $\mathsf{U}_1, \ldots, \mathsf{U}_n$ is isomorphic to the additive group f.*

b. *If λ is a strongly-special vertex, then H_λ is the abelian group $\mathsf{G}_{\lambda+} = G_{\lambda+}/G_{\lambda++}$ defined in Definition 2.2.4.*

Proof. (a): When λ is not a strongly-special vertex, Proposition 3.1.2(a) tells us that $\mathsf{s}(\lambda) \leq 1/e$. But the minimal positive real value taken by the valuation on K is $1/e$; thus, by Definition 2.2.4, $Z_+ = Z_{1/e} \subseteq G_{r,\mathsf{s}(\lambda)}$. Hence, we have our claim that $\mathsf{U}_0 = Z_+/Z_{\mathsf{s}(\lambda)}$ is trivial.

Let $\psi_i = (a, r)$ be a λ-shallow affine k-root. Corollary 3.1.3 says that gradients are unique among λ-shallow affine k-roots. Therefore,

$$U_{\psi_i} \cap G_{\lambda,\mathsf{s}(\lambda)} = U_{\psi+}$$

so that U_i is equal to the U_{ψ_i} found in Definition 2.2.3. There it was noted that U_i is an f-vector space whose dimension is equal to $d_a(r)$, except when $2a \in R$ where its dimension is equal to $d_a(r) + d_{2a}(2r)$. But $2\psi_i \notin \Psi$, and so $d_{2a}(2r) = 0$ whenever $2a \in R$. It can be found in the literature (for example in [34, §1.8.1]) that $d_a(r) = 1$ since $\mathbf{G}$ is *residually split* over k, but this terminology is outside the scope of this thesis. Therefore, we will briefly discuss the proofs in a case-by-case manner according to the trichotomy given in Definition 2.2.1. In cases (i_2) and (ii), K is a tamely, totally ramified quadratic Galois extension of k, and so we can write $K = k(\sqrt{u})$ for some unit $u \in k$ with $\mathrm{val}(u) = 1$.

(i_1): Suppose that (a) generates a type $(A_1)^{e_a}$ root subsystem of $\mathbf{R}$ and $a/2 \notin R$. For each affine k-root $(a, n/e_a) \in \Psi(a)$, $n \in \mathbb{Z}$, the group $U_{a,n/e_a}$ contains all elements of the form $u_a(x\varpi^{n/e_a})$ for $x \in K_a$ with $\mathrm{val}(x) \geq 0$. Therefore

$$\mathsf{U}_i = U_{\psi_i}/U_{\psi_i+1/e_a} \cong \{x \bmod \varpi^{1/e_a} \mid x \in K_a \text{ and } \mathrm{val}(x) \geq 0\} \cong \mathsf{f},$$

where the second isomorphism follows from K_a/k being purely ramified.

(i_2): Suppose that (a) generates a type A_1 root subsystem of $\mathbf{R}$ and $a/2 \in R$. For each affine k-root $(a, n+1/2) \in \Psi(a)$, $n \in \mathbb{Z}$, the group $U_{a,n+1/2}$ contains all elements of the form $u_a(x\varpi^n\sqrt{u})$ for $x \in k$ with $\mathrm{val}(x) \geq 0$. Therefore

$$\mathsf{U}_i = U_{\psi_i}/U_{\psi_i+1} \cong \{x \bmod \varpi \mid x \in k \text{ and } \mathrm{val}(x) \geq 0\} \cong \mathsf{f}.$$

(ii): Suppose that (a) generates a type A_2 root subsystem of $\mathbf{R}$. Consider the homomorphism $\pi : \mathsf{U}_i \to \mathsf{f}$ given by

$$\pi(u_a(x,y)U_{\psi_i+}) = x \bmod P_K$$

for all $(x,y) \in K_a$. As we saw in the proof of Proposition 2.2.5, the elements in U_{ψ_i} with non-trivial projections into U_i are precisely $u_a(x,y)$ with $r = \mathrm{val}(y)/2$ and "maximal" y in the sense that

$$\mathrm{val}(y) = \max\{\mathrm{val}(w) \in \mathbb{Z} \mid x\bar{x} + w + \bar{w} = 0 \text{ for } w \in K\}.$$

For each $x \in K$, a "maximal" y uniquely exists, and so π is an isomorphism.

(b): Recall Proposition 3.1.2(b) where we showed that $\mathsf{s}(\lambda) = \mathsf{r}_2(\lambda)$ whenever λ is a strongly-special vertex. But $G_{\lambda++} := G_{\lambda,\mathsf{r}_2(\lambda)}$ in Definition 2.2.4, and thus H_λ is precisely the quotient $\mathsf{G}_{\lambda+} = G_{\lambda+}/G_{\lambda++}$. □

The group H_λ is a finite p-group, not necessarily abelian. Its **commutator subgroup** $[\mathsf{H}_\lambda, \mathsf{H}_\lambda]$ is the normal subgroup generated by all commutators

$$[\mathsf{h}, \mathsf{g}] := \mathsf{h}^{-1}\mathsf{g}^{-1}\mathsf{h}\mathsf{g}$$

for $\mathsf{h}, \mathsf{g} \in \mathsf{H}_\lambda$. The quotient $\mathsf{H}_\lambda/[\mathsf{H}_\lambda, \mathsf{H}_\lambda]$ is called the **abelianization** of H_λ, and it has the universal property that any group homomorphism from H_λ into an abelian group must factor through it.

Lemma 3.1.6. *Suppose that $\psi_i = a + r$ and $\psi_j = b + s$ are λ-shallow affine k-roots with linearly independent gradients. If x, y are positive integers such that $xa + yb$ is a k-root but*

$x\psi_i + y\psi_j$ is not a λ-shallow affine k-root, then $U_{xa+yb,yr+xs} \subseteq G_{\lambda,\mathsf{s}(\lambda)}$.

Proof. Certainly, if $x\psi_i + y\psi_j$ is not λ-shallow, then either (1) it is an affine k-root but not λ-shallow or (2) it is not an affine k-root. We will now show that in both cases, $U_{xa+yb,yr+xs} \subseteq G_{\lambda,\mathsf{s}(\lambda)}$.

(1): Suppose that $\alpha := x\psi_i + y\psi_j$ is an affine k-root so that $U_{xa+yb,yr+xs} = U_\alpha$. Suppose further that α is not λ-shallow. Since ψ_i is λ-shallow and x, y are positive,

$$\alpha(\lambda) = x\psi_i(\lambda) + y\psi_j(\lambda) > \psi_i(\lambda) > 0.$$

Since α is not λ-shallow, $\alpha(\lambda)$ must be greater than $\mathsf{s}(\lambda)$, and thus $U_\alpha \subseteq G_{\lambda,\mathsf{s}(\lambda)}$.

(2): Suppose that $x\psi_i + y\psi_j$ is not an affine k-root. If λ is a strongly-special vertex, then both $\psi_i(\lambda) = 1/e$ and $\psi_j(\lambda) = 1/e$, and thus $x\psi_i(\lambda) + y\psi_j(\lambda) \geq 2/e = \mathsf{s}(\lambda)$. Suppose instead that λ is not a strongly-special vertex. If $x\psi_i + y\psi_j$ is not an affine root, then let t be the minimal positive real number such that

$$\alpha := x\psi_i + y\psi_j + t = (xa + yb) + (xr + ys + t)$$

is an affine k-root, so that $U_{xa+yb,xr+ys} = U_\alpha$. Since ψ_i, ψ_j and α are all affine k-roots, we know that their intercepts r, s, and $xr + ys + t$ are all integer multiples of the minimal constant $1/e$, and thus so is t. Since t is positive, it must be at least $1/e$. Thus,

$$\alpha(\lambda) = x\psi_i(\lambda) + y\psi_j(\lambda) + t \geq 0 + 1/e \geq \mathsf{s}(\lambda)$$

and so $U_{xa+yb,xr+ys} = U_\alpha \subseteq G_{\lambda,\mathsf{s}(\lambda)}$ □

Theorem 3.1.7. *Let $\mathsf{H}_\lambda, \mathsf{U}_0, \mathsf{U}_1, \ldots, \mathsf{U}_n$ be as above. The natural map $\mathsf{U}_0 \times \mathsf{U}_1 \times \cdots \times \mathsf{U}_n \to \mathsf{H}_\lambda$ given by $(\mathsf{u}_0, \mathsf{u}_1, \ldots, \mathsf{u}_n) \mapsto \mathsf{u}_0\mathsf{u}_1 \cdots \mathsf{u}_n$ is a bijection, meaning that for all $\mathsf{h} \in \mathsf{H}_\lambda$ there exists a unique decomposition*

$$\mathsf{h} = \mathsf{u}_0(\mathsf{h})\mathsf{u}_1(\mathsf{h}) \cdots \mathsf{u}_n(\mathsf{h}) \tag{3.1.1}$$

with $\mathsf{u}_i(\mathsf{h}) \in \mathsf{U}_i$. Moreover, $\mathsf{u}_l([\mathsf{u}, \mathsf{v}]) = 1$ for all $\mathsf{u} \in \mathsf{U}_i$ and $\mathsf{v} \in \mathsf{U}_j$ with $0 \leq l \leq \max(i, j)$.

Proof. Since the U_i, $i = 0, \dots, n$ generate H_λ, each element $\mathsf{h} \in \mathsf{H}_\lambda$ has a decomposition as in (3.1.1); the uniqueness of this decomposition is a consequence of [34, §3.1.1], which says that once an order on k-roots has been chosen, each element in $G_{\lambda+}$ has a unique decomposition in $Z \prod_{a \in R} U_a$. Since there are no repeated gradients among λ-shallow affine k-roots (as shown in Corollary 3.1.3), each U_a contains at most one U_{ψ_i}, and thus the decomposition of h in (3.1.1) must be the unique decomposition coming from [34].

We now turn to proving that $\mathsf{u}_l([\mathsf{u}, \mathsf{v}]) = 1$ for all $\mathsf{u} \in \mathsf{U}_i$ and $\mathsf{v} \in \mathsf{U}_j$ with $0 \leq l \leq \max(i, j) \leq n$. If ψ_i, ψ_j are λ-shallow affine k-roots with linearly dependent gradients, then Theorem 3.1.4 says that

$$[U_{\psi_i}, U_{\psi_j}] \subseteq G_{\lambda, \mathsf{s}(\lambda)};$$

thus, in H_λ, the commutator subgroup $[\mathsf{U}_i, \mathsf{U}_j]$ is trivial. Therefore, let $\psi_i = a + r$ and $\psi_j = b + s$ with a, b linearly independent k-roots. In this case, the commutator formulas in Propositions 2.2.3 and 2.2.4 say that

$$[U_{\psi_i}, U_{\psi_j}] = [U_{a,r}, U_{b,s}] =\subseteq \prod_{x,y} U_{xa+yb, yr+xs}$$

where the product is in increasing order over all positive integers x, y such that $xa + yb$ is a k-root. Lemma 3.1.6 says that if x, y are positive integers such that $xa + yb$ is a k-root but $x\psi_i + y\psi_j$ is not a λ-shallow affine k-root, then $U_{xa+yb, yr+xs} \subseteq G_{\lambda, \mathsf{s}(\lambda)}$. On the other hand, since

$$x\psi_i(\lambda) + y\psi_j(\lambda) > \psi_i(\lambda), \psi_j(\lambda),$$

the enumeration fixed in Notation 3.1.2 says that if $x\psi_i + y\psi_j$ is a λ-shallow affine k-root ψ_l, then $\max(i, j) < l \leq n$. Hence in H_λ, the commutator subgroup $[\mathsf{U}_i, \mathsf{U}_j]$ is contained in the subgroup generated by U_l with $\max(i, j) < l \leq n$. □

Definition 3.1.3. We define a **λ-shallow character** to be any group homomorphism

$$\chi : \mathsf{H}_\lambda \to \mathbb{C}^\times.$$

For each λ-shallow affine k-root ψ_i, the restriction to U_i of a λ-shallow character will be denoted by χ_i. Using the unique decomposition given in (3.1.1), a shallow character χ can be recovered from the restrictions χ_i via the formula

$$\chi(\mathsf{h}) = \prod_{i=1}^{n} \chi_i(\mathsf{u}_i(\mathsf{h}))$$

for all $\mathsf{h} \in \mathsf{H}_\lambda$.

Theorem 3.1.8. *Let $\mathsf{H}_\lambda, \mathsf{U}_0, \mathsf{U}_1, \ldots, \mathsf{U}_n$ be as above. Suppose we are given group homomorphisms*

$$\chi_i : \mathsf{U}_i \to \mathbb{C}^\times$$

for $i = 0, 1, \ldots, n$. The function $\chi : \mathsf{H}_\lambda \to \mathbb{C}^\times$ given by

$$\chi(\mathsf{h}) := \prod_{i=0}^{n} \chi_i(\mathsf{u}_i(\mathsf{h})) \tag{3.1.2}$$

is a group homomorphism, and thus a λ-shallow character, if and only if $\chi([\mathsf{u}, \mathsf{v}]) = 1$ for all $\mathsf{u} \in \mathsf{U}_i$ and $\mathsf{v} \in \mathsf{U}_j$ with $1 \leq i, j \leq n$.

Proof. First note that χ given in (3.1.2) is well-defined, since the decomposition given in (3.1.1) is unique. If χ were a group homomorphism, then it would evaluate trivially on all commutators, since the codomain $\mathbb{C}^\times$ is abelian. Therefore, for the remainder of the proof, we will conversely assume that

$$1 = \chi([\mathsf{u}, \mathsf{v}]) \tag{3.1.3}$$

for all $\mathsf{u} \in \mathsf{U}_i$ and $\mathsf{v} \in \mathsf{U}_j$ with $1 \leq i, j \leq n$, and we will show that χ defines a group homomorphism.

In order to show that the well-defined χ defines a group homomorphism, it will be sufficient to show that

$$\chi(\mathsf{hv}) = \chi(\mathsf{h}) \cdot \chi_j(\mathsf{v}) \tag{3.1.4}$$

for all $\mathsf{h} \in \mathsf{H}_\lambda$ and $\mathsf{v} \in \mathsf{U}_j$ for $j = 0, 1, \ldots, n$. In the case, $j = 0$ we see that Proposition 3.1.5

implies that U_0 is central, and so (3.1.2) yields

$$\begin{aligned}
\chi(\mathsf{hv}) &= \chi\left(\left[\prod_{i=0}^{n}\mathsf{u}_i(\mathsf{h})\right]\mathsf{v}\right) \\
&= \chi\left(\mathsf{v}\left[\prod_{i=0}^{n}\mathsf{u}_i(\mathsf{h})\right]\right) \\
&= \chi\left(\mathsf{vu}_0(\mathsf{h})\left[\prod_{i=1}^{n}\mathsf{u}_i(\mathsf{h})\right]\right) \\
&= \chi_0(\mathsf{vu}_0(\mathsf{h}))\left[\prod_{i=1}^{n}\chi_i(\mathsf{u}_i(\mathsf{h}))\right] \\
&= \chi_0(\mathsf{v})\left[\prod_{i=0}^{n}\chi_i(\mathsf{u}_i(\mathsf{h}))\right] \\
&= \chi(\mathsf{h})\cdot\chi_0(\mathsf{v})
\end{aligned}$$

for all $\mathsf{h} \in \mathsf{H}_\lambda$ and $\mathsf{v} \in \mathsf{U}_0$.

We will now proceed to use reverse strong induction on j to show that (3.1.4) holds for $j = 1, \ldots, n$. For the base case, we let $j = n$ so that (3.1.2) directly yields

$$\begin{aligned}
\chi(\mathsf{hv}) &= \chi\left(\left[\prod_{i=0}^{n}\mathsf{u}_i(\mathsf{h})\right]\mathsf{v}\right) \\
&= \chi\left(\left[\prod_{i=0}^{n-1}\mathsf{u}_i(\mathsf{h})\right]\mathsf{u}_n(\mathsf{h})\mathsf{v}\right) \\
&= \left[\prod_{i=0}^{n-1}\chi_i(\mathsf{u}_i(\mathsf{h}))\right]\chi_n(\mathsf{u}_n(\mathsf{h})\mathsf{v}) \\
&= \left[\prod_{i=0}^{n-1}\chi_i(\mathsf{u}_i(\mathsf{h}))\right]\chi_n(\mathsf{u}_n(\mathsf{h}))\cdot\chi_n(\mathsf{v}) \\
&= \left[\prod_{i=0}^{n}\chi_i(\mathsf{u}_i(\mathsf{h}))\right]\chi_n(\mathsf{v}) \\
&= \chi(\mathsf{h})\cdot\chi_n(\mathsf{v})
\end{aligned}$$

for all $\mathsf{h} \in \mathsf{H}_\lambda$ and $\mathsf{v} \in \mathsf{U}_n$.

Next, for the induction step, we assume that

$$\chi(\mathsf{hu}) = \chi(\mathsf{h})\cdot\chi_l(\mathsf{u}) \tag{3.1.5}$$

for all $\mathsf{h} \in \mathsf{H}_\lambda$ and $\mathsf{u} \in \mathsf{U}_l$ with $j < l \leq n$. In this case, we look at products of the form

$$\begin{aligned}
\mathsf{hv} &= \left[\prod_{i=0}^{n} \mathsf{u}_i(\mathsf{h})\right] \mathsf{v} \\
&= \left[\prod_{i=0}^{j-1} \mathsf{u}_i(\mathsf{h})\right] \mathsf{u}_j(\mathsf{h})\mathsf{v} \left[\prod_{i=j+1}^{n} \mathsf{u}_i(\mathsf{h})[\mathsf{u}_i(\mathsf{h}), \mathsf{v}]\right] \\
&= \left[\prod_{i=0}^{j-1} \mathsf{u}_i(\mathsf{h})\right] \mathsf{u}_j(\mathsf{h})\mathsf{v} \left[\prod_{i=j+1}^{n} \mathsf{u}_i(\mathsf{h}) \left(\prod_{l=0}^{n} \mathsf{u}_l([\mathsf{u}_i(\mathsf{h}), \mathsf{v}])\right)\right]
\end{aligned}$$

for $\mathsf{h} \in \mathsf{H}_\lambda$ and $\mathsf{v} \in \mathsf{U}_j$. Theorem 3.1.7 tells us that $\mathsf{u}_l([\mathsf{u}_i(\mathsf{h}), \mathsf{v}]) = 1$ whenever $0 \leq l \leq j$, so that

$$\mathsf{hv} = \left[\prod_{i=0}^{j-1} \mathsf{u}_i(\mathsf{h})\right] \mathsf{u}_j(\mathsf{h})\mathsf{v} \left[\prod_{i=j+1}^{n} \mathsf{u}_i(\mathsf{h}) \left(\prod_{l=j+1}^{n} \mathsf{u}_l([\mathsf{u}_i(\mathsf{h}), \mathsf{v}])\right)\right] \tag{3.1.6}$$

for all $\mathsf{h} \in \mathsf{H}_\lambda$ and $\mathsf{v} \in \mathsf{U}_j$. Since every factor to the right of $\mathsf{u}_j(\mathsf{h})\mathsf{v}$ in (3.1.6) is contained in some U_l with $j < l \leq n$, repeated use of induction hypothesis (3.1.5) gives us

$$\begin{aligned}
\chi(\mathsf{hv}) &= \chi\left(\left[\prod_{i=0}^{j-1} \mathsf{u}_i(\mathsf{h})\right] \mathsf{u}_j(\mathsf{h})\mathsf{v} \left[\prod_{i=j+1}^{n} \mathsf{u}_i(\mathsf{h}) \left(\prod_{l=j+1}^{n} \mathsf{u}_l([\mathsf{u}_i(\mathsf{h}), \mathsf{v}])\right)\right]\right) \\
&= \chi\left(\left[\prod_{i=0}^{j-1} \mathsf{u}_i(\mathsf{h})\right] \mathsf{u}_j(\mathsf{h})\mathsf{v}\right) \left[\prod_{i=j+1}^{n} \chi_i(\mathsf{u}_i(\mathsf{h})) \left(\prod_{l=j+1}^{n} \chi_l\big(\mathsf{u}_l([\mathsf{u}_i(\mathsf{h}), \mathsf{v}])\big)\right)\right] \qquad (3.1.7)
\end{aligned}$$

for every $\mathsf{h} \in \mathsf{H}_\lambda$ and $\mathsf{v} \in \mathsf{U}_j$. Next, use (3.1.2) to rewrite

$$\begin{aligned}
\chi\left(\left[\prod_{i=0}^{j-1} \mathsf{u}_i(\mathsf{h})\right] \mathsf{u}_j(\mathsf{h})\mathsf{v}\right) &= \left[\prod_{i=0}^{j-1} \chi(\mathsf{u}_i(\mathsf{h}))\right] \chi_j(\mathsf{u}_j(\mathsf{h})\mathsf{v}) \\
&= \left[\prod_{i=0}^{j} \chi(\mathsf{u}_i(\mathsf{h}))\right] \chi_j(\mathsf{v})
\end{aligned}$$

for all $\mathsf{h} \in \mathsf{H}_\lambda$ and $\mathsf{v} \in \mathsf{U}_j$. Similarly, use (3.1.2) and assumption (3.1.3) to rewrite

$$\begin{aligned}
\prod_{i=j+1}^{n} \chi_i(\mathsf{u}_i(\mathsf{h})) \left(\prod_{l=j+1}^{n} \chi_l\big(\mathsf{u}_l([\mathsf{u}_i(\mathsf{h}), \mathsf{v}])\big)\right) &= \prod_{i=j+1}^{n} \chi_i(\mathsf{u}_i(\mathsf{h}))\chi([\mathsf{u}_i, \mathsf{v}]) \\
&= \prod_{i=j+1}^{n} \chi_i(\mathsf{u}_i(\mathsf{h}))
\end{aligned}$$

for all $\mathsf{h} \in \mathsf{H}_\lambda$ and $\mathsf{v} \in \mathsf{U}_j$. Thus, (3.1.7) can be reduced to

$$\begin{aligned}
\chi(\mathsf{hv}) &= \left[\prod_{i=0}^{j} \chi(\mathsf{u}_i(\mathsf{h}))\right] \chi_j(\mathsf{v}) \left[\prod_{i=j+1}^{n} \chi_i(\mathsf{u}_i(\mathsf{h}))\right] \\
&= \left[\prod_{i=0}^{n} \chi(\mathsf{u}_i(\mathsf{h}))\right] \chi_j(\mathsf{v}) \\
&= \chi(\mathsf{h}) \cdot \chi_j(\mathsf{v})
\end{aligned}$$

for all $\mathsf{h} \in \mathsf{H}_\lambda$ and $\mathsf{v} \in \mathsf{U}_j$, completing the induction step. Hence, we have shown that χ is indeed a group homomorphism. □

Corollary 3.1.9. *The commutator subgroup of H_λ is generated by commutators of the form $[\mathsf{u}, \mathsf{v}]$ for $\mathsf{u} \in \mathsf{U}_i$ and $\mathsf{v} \in \mathsf{U}_j$ with $1 \leq i, j \leq n$.*

Proof. This is an immediate consequence of Theorem 3.1.8 since the abelianization of H_λ, a finite group, is isomorphic to the dual group $\check{\mathsf{H}}_\lambda$ consisting of all group homomorphisms $\mathsf{H}_\lambda \to \mathbb{C}^\times$ (i.e., λ-shallow characters). □

3.1.3 Rank-2 examples

For the rank-2 affine root system of type C_2, G_2, and G_2^I, we will now give a connected, quasi-simple, semisimple reductive algebraic group $\mathbf{G}$ defined and quasi-split over k having affine k-root system Ψ of the given types. For this $\mathbf{G}$, an explicit Chevalley-Steinberg system will be provided; if $\mathbf{G}$ is non-split quasi-split over k, this will require that we first describe the K-structure of $\mathbf{G}$, a K-pinning, and a $\mathrm{Gal}(K/k)$-action preserving this pinning.

Next, we will describe the affine k-root system Ψ in more detail, giving a base and the minimal-height positive affine k-roots, one for each gradient. The commutators of the affine k-root groups for these minimal-height positive affine k-roots are then computed using either Appendix A or Propositions 2.2.3 and 2.2.4.

Finally, we will describe the abelianization of H_λ whenever λ is the barycenter of our fundamental alcove. The method provided can be easily generalized to compute the abelianization of H_λ when λ is the barycenter of any facet of our fundamental alcove and not equal to a strongly-special vertex. These results are summarized in Tables B.4 and B.6 in Appendix B.

For this final step, we will need the following three technical lemmas. The first appears in [26], where it is used to compute the commutator subgroups for Iwahori subgroups. The second and third are natural generalizations that will be necessary for computing commutators in more general groups.

Lemma 3.1.10 (Prasad-Raghunathan)**.**

$$\langle (xy, x^2y) \in \mathsf{f}^2 \mid x, y \in \mathsf{f} \rangle \cong \begin{cases} \{(x,y) \in \mathsf{f}^2 \mid \ x, y \in \mathsf{f} \textit{ such that } x = y\} & \textit{if } \#\mathsf{f} = 2, \\ \{(x,y) \in \mathsf{f}^2 \mid \ x, y \in \mathsf{f}\} = \mathsf{f}^2 & \textit{if else.} \end{cases}$$

Proof. Denote by N the subgroup of f^2 generated by all pairs (xy, x^2y) for $x, y \in \mathsf{f}$. Note that when $\#\mathsf{f} = 2$, we can individually check the 4 combinations of $x, y \in \mathsf{f}$ and see that N is the subgroup $\{(0,0), (1,1)\}$. For the remainder of the proof we will therefore assume that $\#\mathsf{f} \neq 2$.

Let $z \in \mathsf{f}^\times$ and note that

$$(0, [1-z]y) = (y, y) - (z[y/z], z^2[y/z])$$

belongs to N for all $y \in \mathsf{f}$. Since $\#\mathsf{f} \neq 2$, we can choose z so that $1 - z$ is invertible in f, and therefore the group N contains every element of the form $(0, y)$ for $y \in \mathsf{f}$. For any $x \in \mathsf{f}$, we also have that $(x, 0) = (x, x^2) - (0, x^2)$ belongs to N. Hence, we have shown that N contains any element of the form (x, y) for $x, y \in \mathsf{f}$. □

Lemma 3.1.11.

$$\langle (xy, -x^3y) \in \mathsf{f}^2 \mid x, y \in \mathsf{f} \rangle \cong \begin{cases} \{(x,y) \mid x, y \in \mathsf{f} \textit{ such that } x = -y\} & \textit{if } \#\mathsf{f} \in \{2,3\} \\ \{(x,y) \mid x, y \in \mathsf{f}\} = \mathsf{f}^2 & \textit{if else} \end{cases}$$

Proof. The proof given below is essentially identical to that given for Lemma 3.1.10: Denote by N the subgroup of f^2 generated by all pairs $(xy, -x^3y)$ for $x, y \in \mathsf{f}$. Note that when $\#\mathsf{f} \in \{2, 3\}$, we can individually check the small number of combinations of $x, y \in \mathsf{f}$ and see that N equals the subgroup $\{(0,0),(1,-1),(-1,1)\}$, with $1=-1$ when $\#\mathsf{f}=2$. For the remainder of the proof we will therefore assume that $\#\mathsf{f}\notin\{2,3\}$.

Let $z\in\mathsf{f}^\times$ and note that

$$(0,[1-z^2]y)=(z[y/z],-z^3[y/z])-(y,-y)$$

belongs to N for all $y\in\mathsf{f}$. Since $\#\mathsf{f}\notin\{2,3\}$, we can choose z so that $1-z^2$ is invertible in f, and therefore the group N contains every element of the form $(0,y)$ for $y\in\mathsf{f}$. For any $x\in\mathsf{f}$, we also have that $(x,0)=(x,-x^3)-(0,-x^3)$ belongs to N. Hence, we have shown that N contains any element of the form (x,y) for $x,y\in\mathsf{f}$. $\square$

Lemma 3.1.12.

$$\langle(0,0,0,xy),(0,2xy,3x^2y,3y^2x),(xy,x^2y,-x^3y,2x^3y^2)\in\mathsf{f}^4\mid x,y\in\mathsf{f}\rangle$$
$$\cong\begin{cases}\{(x,y,z,w)\in\mathsf{f}^4\mid x,y,z,w\in\mathsf{f}\textit{ such that }x=y\} & \textit{if }\#\mathsf{f}=2,\\ \{(x,y,z,w)\in\mathsf{f}^4\mid x,y,z,w\in\mathsf{f}\textit{ such that }x=-z\} & \textit{if }\#\mathsf{f}=3,\\ \{(x,y,z,w)\in\mathsf{f}^4\mid x,y,z,w\in\mathsf{f}\}=\mathsf{f}^4 & \textit{if else.}\end{cases}$$

Proof. Denote by N the subgroup of f^4 generated by all quadruples of the form $(0,0,0,xy)$, $(0,2xy,3x^2y,3xy^2)$, and $(xy,x^2y,-x^3y,2x^3y^2)$ for any $x,y\in\mathsf{f}$. First note that for all possible f, N contains any element of the form $(0,0,0,w)$ for $w\in\mathsf{f}$. To understand the remaining elements of N, we will need to consider the various primes $p=\operatorname{char}\mathsf{f}$.

($p=2$): In the case that 3 is invertible and $2=0$ in f, the generators of N are $(0,0,0,xy)$, $(0,0,3x^2y,3xy^2)$, and $(xy,x^2y,x^3y,0)$ for $x\in\mathsf{f}$. Note that when $\#\mathsf{f}=2$, it can be individually checked that

$$N=\{(x,y,z,w)\in\mathsf{f}^4\mid x,y,z,w\in\mathsf{f}\text{ such that }x=y\}.$$

Assuming that $\#\mathsf{f}\neq 2$, N contains all elements of the form

$$(xy,x^2y,0,0)=(xy,x^2y,x^3y,0)-(0,0,3x^2[xy/3],3x[xy/3]^2)+(0,0,0,x^3y^2/3)$$

with $x, y \in \mathsf{f}$. Lemma 3.1.10 therefore implies that N contains every element of the form $(x, y, 0, 0)$ for $x, y \in \mathsf{f}$ with $x = y$ when $\#\mathsf{f} = 2$. We also see that N contains every element of the form

$$(0, 0, z, 0) = (z, z, z, 0) - (z, z, 0, 0)$$

with $z \in \mathsf{f}$. Hence, we have shown that if $\operatorname{char} \mathsf{f} = 2$ then N contains all quadruples (x, y, z, w) for $x, y, z, w \in \mathsf{f}$.

$(p = 3)$: In the case that 2 is invertible and $3 = 0$ in f, the generators of N are $(0, 0, 0, xy)$, $(0, 2xy, 0, 0)$, and $(xy, x^2y, -x^3y, 2x^3y^2)$ for $x \in \mathsf{f}$. Note that when $\#\mathsf{f} = 3$, it can be individually checked that

$$N = \{(x, y, z, w) \in \mathsf{f}^4 \mid x, y, z, w \in \mathsf{f} \text{ such that } x = -y\}.$$

Assuming that $\mathsf{f} \neq 3$, N contains all elements of the form

$$(xz, 0, -x^2z, 0) = (xz, x^2z, -x^3z, 2x^3z^2) - (0, 2x[xz/2], 0, 0) - (0, 0, 0, 2x^3z^2)$$

with $x, z \in \mathsf{f}$. Lemma 3.1.11 therefore implies that N contains every element of the form $(x, 0, z, 0)$ for $x, z \in \mathsf{f}$ with $x = -z$ when $\#\mathsf{f} = 3$. We also see that N contains every element of the form

$$(0, y, 0, 0) = (y, y, -y, 2y^2) - (y, 0, -y, 0) - (0, 0, 0, 2y^2)$$

for $y \in \mathsf{f}$. Hence, we have shown that if $\operatorname{char} \mathsf{f} = 3$ then N contains all quadruples (x, y, z, w) for $x, y, z, w \in \mathsf{f}$ with $x = z$ when $\#\mathsf{f} = 3$.

$(p > 3)$: In the case that both 3 and 2 are invertible in f, we see that N contains all elements of the form

$$(0, yz, y^2z, 0) = \bigl(0, 2[2y/3][3z/4], 3[2y/3]^2[3z/4], 3[2y/3][3z/4]^2\bigr) - \bigl(0, 0, 0, 9yz^2/8\bigr)$$

with $y, z \in \mathsf{f}$. Lemma 3.1.10 therefore implies that N contains every element of the form

$(0, y, z, 0)$. Finally, we conclude that N contains every element of the form

$$(x, 0, 0, 0) = (x, x, -x, 2x^2) - (0, x, -x, 0) - (0, 0, 0, 2x^2)$$

with $x \in \mathsf{f}$. Hence, we have shown that $N = \{(x, y, z, w) \in \mathsf{f}^4 \mid x, y, z, w \in \mathsf{f}\} = \mathsf{f}^4$. □

Type C_2

Let $G = \mathrm{Sp}_4(k)$ be the split group of 4×4 k-matrices g fixed under the endomorphism

$$g \mapsto Q^{-1}(g^{\mathrm{tr}})^{-1}Q \qquad \text{with } Q = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix},$$

where g^{tr} is the transpose of $g \in \mathrm{Sp}_4(k)$. Therein lies the diagonal, maximal k-torus

$$S = \left\{ s = \begin{pmatrix} s_1 & & & \\ & s_2 & & \\ & & s_3 & \\ & & & s_4 \end{pmatrix} \;\middle|\; \begin{array}{c} s_1, s_2, s_3, s_4 \in k^\times \text{ with} \\ s_1 s_4 = 1 \text{ and } s_2 s_3 = 1 \end{array} \right\}$$

A base D of the k-root system $R = R(G, S)$ consists of a short root $a(s) = s_1/s_2$ and a long root $b(s) = s_2/s_3$. The Borel subgroup corresponding to this simple system consists of all upper-triangular matrices in G. We give the standard pinning of G with respect to the diagonal torus and upper-triangular Borel:

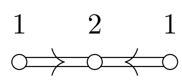
$$u_a(x) = \begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{pmatrix} \qquad \text{and} \qquad u_b(x) = \begin{pmatrix} 1 & & & \\ & 1 & x & \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

for $x \in k$. The remaining positive k-root morphisms are

$$u_{a+b}(x) = \begin{pmatrix} 1 & 0 & x & \\ & 1 & 0 & x \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \quad \text{and} \quad u_{2a+b}(x) = \begin{pmatrix} 1 & 0 & 0 & x \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix},$$

for $x \in k$.

The affine k-root system Ψ has type C_2, with weighted Dynkin diagram



and a base Δ of Ψ consists of simple affine k-roots

$$\begin{aligned} \alpha &= a + 0, \\ \beta &= b + 0, \\ \gamma &= c + 1, \end{aligned} \tag{3.1.8}$$

where $c = -2a - b$ is the lowest long k-root in R. An enumeration of the minimal-height positive affine k-roots $\psi_1, \ldots, \psi_8$ for each gradient in R is given below:

i	gradient $\dot{\psi}_i$	intercept $\psi_i(0)$	simple affine k-root decomposition ψ_i
1	c	1	γ
2	a	0	α
3	b	0	β
4	$a+c$	1	$\alpha+\gamma$
5	$a+b$	0	$\alpha+\beta$
6	$2a+c$	1	$2\alpha+\gamma$
7	$2a+b$	0	$2\alpha+\beta$
8	$a+b+c$	1	$\alpha+\beta+\gamma$

(3.1.9)

Through direct computation, one can produce the commutators of affine k-root groups for all pairs ψ_i, ψ_j of minimal-height positive affine k-roots with non-linearly dependent gradients $\dot{\psi}_i, \dot{\psi}_j$ such that $\dot{\psi}_i + \dot{\psi}_j$ is a k-root which we now give:

$$\begin{aligned}
[u_a(x\,\varpi^0), u_{a+c}(y\,\varpi^1)] &= u_{2a+c}(-2xy\,\varpi^1)\\
[u_a(x\,\varpi^0), u_{a+b}(y\,\varpi^0)] &= u_{2a+b}(+2xy\,\varpi^0)\\
[u_{a+c}(x\,\varpi^1), u_{a+b+c}(y\,\varpi^1)] &= u_c(+2xy\,\varpi^2)\\
[u_{a+b}(x\,\varpi^0), u_{a+b+c}(y\,\varpi^1)] &= u_b(-2xy\,\varpi^1)
\end{aligned}$$

for $x, y \in A_k$ and

$$\begin{aligned}
[u_a(x\,\varpi^0), u_c(y\,\varpi^1)] &= u_{a+c}(-xy\,\varpi^1)u_{2a+c}(-x^2y\,\varpi^1)\\
[u_a(x\,\varpi^0), u_b(y\,\varpi^0)] &= u_{a+b}(+xy\,\varpi^0)u_{2a+b}(-x^2y\,\varpi^0)\\
[u_{a+c}(x\,\varpi^1), u_b(y\,\varpi^0)] &= u_{a+b+c}(-xy\,\varpi^1)u_c(+x^2y\,\varpi^2)\\
[u_{a+b}(x\,\varpi^0), u_c(y\,\varpi^1)] &= u_{a+b+c}(+xy\,\varpi^1)u_b(+x^2y\,\varpi^1)\\
[u_{a+c}(x\,\varpi^1), u_{2a+b}(y\,\varpi^0)] &= u_a(-xy\,\varpi^1)u_{2a+c}(+x^2y\,\varpi^2)\\
[u_{a+b}(x\,\varpi^0), u_{2a+c}(y\,\varpi^1)] &= u_a(+xy\,\varpi^1)u_{2a+b}(+x^2y\,\varpi^1)\\
[u_{a+b+c}(x\,\varpi^1), u_{2a+c}(y\,\varpi^1)] &= u_{a+c}(-xy\,\varpi^2)u_c(-x^2y\,\varpi^3)\\
[u_{a+b+c}(x\,\varpi^1), u_{2a+b}(y\,\varpi^0)] &= u_{a+b}(-xy\,\varpi^1)u_b(+x^2y\,\varpi^2)
\end{aligned}$$

for $x, y \in A_k$.

For each λ that is the barycenter of a facet of the fundamental alcove and not a strongly-special vertex, the group

$$\mathsf{H}_\lambda := G_{\lambda+}/G_{\lambda,1}$$

is generated by subgroups

$$\mathsf{U}_i := U_{\psi_i} G_{\lambda,1}/G_{\lambda,1} \cong \mathsf{f}$$

for $i = 1, \ldots, 8$ such that ψ_i is λ-shallow. Corollary 3.1.9 says that the commutator subgroup of H_λ is generated by the commutators $[\mathsf{u}, \mathsf{v}]$ with $\mathsf{u} \in \mathsf{U}_i$ and $\mathsf{v} \in \mathsf{U}_j$ for λ-shallow ψ_i, ψ_j; these commutators are computed from the commutators above. For example, when λ is the barycenter of the fundamental alcove, each ψ_i, $i = 1, \ldots, 8$ is a λ-shallow affine k-root. The commutator subgroup of H_λ is then generated by commutators of the form

$$\begin{aligned}
[u_a(x\varpi^0), u_{a+c}(y\varpi^1)]G_{\lambda,1} &= u_{2a+c}(-2xy\varpi^1)G_{\lambda,1}\\
[u_a(x\varpi^0), u_{a+b}(y\varpi^0)]G_{\lambda,1} &= u_{2a+b}(+2xy\varpi^0)G_{\lambda,1}\\
[u_{a+c}(x\varpi^1), u_{a+b+c}(y\varpi^1)]G_{\lambda,1} &= G_{\lambda,1}\\
[u_{a+b}(x\varpi^0), u_{a+b+c}(y\varpi^1)]G_{\lambda,1} &= G_{\lambda,1}
\end{aligned}$$

for $x, y \in A_k$ and

$$\begin{aligned}
[u_a(x\varpi^0), u_c(y\varpi^1)]G_{\lambda,1} &= u_{a+c}(-xy\varpi^1)u_{2a+c}(-x^2y\varpi^1)G_{\lambda,1}\\
[u_a(x\varpi^0), u_b(y\varpi^0)]G_{\lambda,1} &= u_{a+b}(+xy\varpi^0)u_{2a+b}(-x^2y\varpi^0)G_{\lambda,1}\\
[u_{a+c}(x\varpi^1), u_b(y\varpi^0)]G_{\lambda,1} &= u_{a+b+c}(-xy\varpi^1)G_{\lambda,1}\\
[u_{a+b}(x\varpi^0), u_c(y\varpi^1)]G_{\lambda,1} &= u_{a+b+c}(+xy\varpi^1)G_{\lambda,1}\\
[u_{a+c}(x\varpi^1), u_{2a+b}(y\varpi^0)]G_{\lambda,1} &= G_{\lambda,1}\\
[u_{a+b}(x\varpi^0), u_{2a+c}(y\varpi^1)]G_{\lambda,1} &= G_{\lambda,1}\\
[u_{a+b+c}(x\varpi^1), u_{2a+c}(y\varpi^1)]G_{\lambda,1} &= G_{\lambda,1}\\
[u_{a+b+c}(x\varpi^1), u_{2a+b}(y\varpi^0)]G_{\lambda,1} &= G_{\lambda,1}
\end{aligned}$$

for $x, y \in A_k$, and the abelianization of H_λ is isomorphic to

$$\begin{aligned}
\mathsf{H}_\lambda/[\mathsf{H}_\lambda, \mathsf{H}_\lambda] = \mathsf{f}_1 \oplus \mathsf{f}_2 \oplus \mathsf{f}_3 \oplus \frac{\mathsf{f}_4 \oplus \mathsf{f}_6}{\langle (0, -2xy), (-xy, -x^2y) \mid x, y \in \mathsf{f}\rangle}\\
\oplus \frac{\mathsf{f}_5 \oplus \mathsf{f}_7}{\langle (0, 2xy), (xy, -x^2y) \mid x, y \in \mathsf{f}\rangle} \oplus \frac{\mathsf{f}_8}{\langle \pm xy \mid x, y \in \mathsf{f}\rangle}
\end{aligned}$$

where $\mathsf{f}_i = \mathsf{f}$ is the abelian group isomorphic to $U_{\psi_i}/U_{\psi_i} \cap G_{\lambda,1}$ for $i = 1, \ldots, 8$. Lemma 3.1.10 allows us to simply this expression, so that

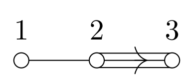
$$\mathsf{H}_\lambda/[\mathsf{H}_\lambda, \mathsf{H}_\lambda] \cong \begin{cases} \mathsf{f}^5 & \text{if } \#\mathsf{f} = 2, \\ \mathsf{f}^3 & \text{if else.} \end{cases} \tag{3.1.10}$$

Similar computations can be done for each barycenter not equal to a strongly-special vertex, only using the commutators $[\mathsf{U}_i, \mathsf{U}_j]$ for affine k-roots $\psi_1, \ldots, \psi_8$ that do not vanish at λ. The results of these computations are summarized in Table B.4 in Appendix B.

Remark. In the case that λ is the barycenter of the fundamental alcove, the abelianization of H_λ computed above is isomorphic to the abelianization of the pro-unipotent radical $G_{\lambda+}$ of the Iwahori subgroup G_λ whose commutator subgroup is computed by Prasad and Raghunathan in [26, Theorem 6.6]. But it should be noted that the above computation is considerably simpler than the one in [26].

Type G_2

Let $\mathbf{G}$ be a connected, quasi-simple, semisimple reductive group defined and split over k with a type G_2 k-root system. Then $G = \mathbf{G}(k)$ is the split Chevalley group of type G_2 over k, the description of which can be found in [6]. The affine k-root system Ψ has type G_2, with weighted Dynkin diagram



and a base Δ consisting of simple affine k-roots

$$\begin{aligned} \alpha &= a + 0, \\ \beta &= b + 0, \\ \gamma &= c + 1, \end{aligned} \tag{3.1.11}$$

where $c = -3a - 2b$ is the lowest long k-root in R. An enumeration of the minimal-height positive affine k-roots $\psi_1, \ldots, \psi_{12}$ for each gradient in R is given below:

index i	gradient $\dot{\psi}_i$	intercept $\psi_i(0)$	simple affine k-root decomposition ψ_i
1	c	1	γ
2	b	0	α
3	a	0	β
4	$b+c$	1	$\beta+\gamma$
5	$a+b$	0	$\alpha+\beta$
6	$a+b+c$	1	$\alpha+\beta+\gamma$
7	$2a+b$	0	$2\alpha+\beta$
8	$2a+b+c$	1	$2\alpha+\beta+\gamma$
9	$3a+b$	0	$3\alpha+\beta$
10	$3a+b+c$	1	$3\alpha+\beta+\gamma$
11	$3a+2b$	0	$3\alpha+2\beta$
12	$2a+2b+c$	1	$2\alpha+2\beta+\gamma$

(3.1.12)

From Proposition 2.2.3 and the methods of [6, Section 5.2], one can recover the commutators of affine k-root groups for all pairs ψ_i, ψ_j of minimal-height positive affine k-roots with non-linearly dependent gradients $\dot{\psi}_i, \dot{\psi}_j$ such that $\dot{\psi}_i + \dot{\psi}_j$ is a k-root. We give these commutators below:

$$[u_c(x\,\varpi^1), u_b(y\,\varpi^0)] = u_{b+c}(+xy\,\varpi^1)$$
$$[u_c(x\,\varpi^1), u_{3a+b}(y\,\varpi^0)] = u_{3a+b+c}(-xy\,\varpi^1)$$
$$[u_b(x\,\varpi^0), u_{3a+b}(y\,\varpi^0)] = u_{3a+2b}(+xy\,\varpi^0)$$
$$[u_{b+c}(x\,\varpi^1), u_{3a+2b}(y\,\varpi^0)] = u_b(-xy\,\varpi^1)$$
$$[u_{b+c}(x\,\varpi^1), u_{3a+b+c}(y\,\varpi^1)] = u_c(+xy\,\varpi^2)$$
$$[u_{3a+2b}(x\,\varpi^0), u_{3a+b+c}(y\,\varpi^1)] = u_{3a+b}(-xy\,\varpi^1)$$

for all $x, y \in A_k$;

$$
\begin{aligned}
[u_a(x\,\varpi^0), u_{a+b}(y\,\varpi^0)] &= u_{2a+b}(-2xy\,\varpi^0)u_{3a+b}(+3x^2y\,\varpi^0)u_{3a+2b}(-3xy^2\,\varpi^0)\\
[u_a(x\,\varpi^0), u_{a+b+c}(y\,\varpi^1)] &= u_{2a+b+c}(+2xy\,\varpi^1)u_{3a+b+c}(+3x^2y\,\varpi^1)u_c(+3xy^2\,\varpi^2)\\
[u_{a+b}(x\,\varpi^0), u_{a+b+c}(y\,\varpi^1)] &= u_{2a+2b+c}(-2xy\,\varpi^1)u_b(-3x^2y\,\varpi^1)u_{b+c}(+3xy^2\,\varpi^2)\\
[u_{2a+b}(x\,\varpi^0), u_{2a+b+c}(y\,\varpi^1)] &= u_a(-2xy\,\varpi^1)u_{3a+b}(-3x^2y\,\varpi^1)u_{3a+b+c}(+3xy^2\,\varpi^2)\\
[u_{2a+b}(x\,\varpi^0), u_{2a+2b+c}(y\,\varpi^1)] &= u_{a+b}(+2xy\,\varpi^1)u_{3a+2b}(-3x^2y\,\varpi^1)u_b(-3xy^2\,\varpi^2)\\
[u_{2a+b+c}(x\,\varpi^1), u_{2a+2b+c}(y\,\varpi^1)] &= u_{a+b+c}(-2xy\,\varpi^2)u_c(+3x^2y\,\varpi^3)u_{b+c}(-3xy^2\,\varpi^3)
\end{aligned}
$$

for all $x, y \in A_k$;

$$
\begin{aligned}
[u_a(x\,\varpi^0), u_{2a+b}(y\,\varpi^0)] &= u_{3a+b}(+3xy\,\varpi^0)\\
[u_a(x\,\varpi^0), u_{2a+b+c}(y\,\varpi^1)] &= u_{3a+b+c}(-3xy\,\varpi^1)\\
[u_{a+b}(x\,\varpi^0), u_{2a+b}(y\,\varpi^0)] &= u_{3a+2b}(-3xy\,\varpi^0)\\
[u_{a+b}(x\,\varpi^0), u_{2a+b+c}(y\,\varpi^1)] &= u_b(-3xy\,\varpi^1)\\
[u_{a+b+c}(x\,\varpi^1), u_{2a+b+c}(y\,\varpi^1)] &= u_c(-3xy\,\varpi^2)\\
[u_{a+b+c}(x\,\varpi^1), u_{2a+2b+c}(y\,\varpi^1)] &= u_{b+c}(+3xy\,\varpi^2)
\end{aligned}
$$

for all $x, y \in A_k$;

$$
\begin{aligned}
[u_{a+b}(x\,\varpi^0), u_c(y\,\varpi^1)] &= u_{a+b+c}(+xy\,\varpi^1)u_{2a+2b+c}(+x^2y\,\varpi^1)\\
&\quad\cdot u_b(+x^3y\,\varpi^1)u_{b+c}(-2x^3y^2\,\varpi^2)\\
[u_{2a+b}(x\,\varpi^0), u_c(y\,\varpi^1)] &= u_{2a+b+c}(-xy\,\varpi^1)u_a(-x^2y\,\varpi^1)\\
&\quad\cdot u_{3a+b}(-x^3y\,\varpi^1)u_{3a+b+c}(-2x^3y^2\,\varpi^2)\\
[u_a(x\,\varpi^0), u_b(y\,\varpi^0)] &= u_{a+b}(+xy\,\varpi^0)u_{2a+b}(+x^2y\,\varpi^0)\\
&\quad\cdot u_{3a+b}(-x^3y\,\varpi^0)u_{3a+2b}(+2x^3y^2\,\varpi^0)\\
[u_a(x\,\varpi^0), u_{b+c}(y\,\varpi^1)] &= u_{a+b+c}(-xy\,\varpi^1)u_{2a+b+c}(+x^2y\,\varpi^1)\\
&\quad\cdot u_{3a+b+c}(+x^3y\,\varpi^1)u_c(-2x^3y^2\,\varpi^2)
\end{aligned}
$$

$$\begin{aligned}
[u_{2a+b+c}(x\,\varpi^1), u_b(y\,\varpi^0)] &= u_{2a+2b+c}(-xy\,\varpi^1)u_{a+b+c}(-x^2y\,\varpi^2) \\
&\qquad \cdot u_c(+x^3y\,\varpi^3)u_{b+c}(+2x^3y^2\,\varpi^3) \\
[u_{2a+b}(x\,\varpi^0), u_{b+c}(y\,\varpi^1)] &= u_{2a+2b+c}(+xy\,\varpi^1)u_{a+b}(-x^2y\,\varpi^1) \\
&\qquad \cdot u_{3a+2b}(+x^3y\,\varpi^1)u_b(+2x^3y^2\,\varpi^2) \\
[u_{a+b}(x\,\varpi^0), u_{3a+b+c}(y\,\varpi^1)] &= u_a(+xy\,\varpi^1)u_{2a+b}(-x^2y\,\varpi^1) \\
&\qquad \cdot u_{3a+2b}(-x^3y\,\varpi^1)u_{3a+b}(+2x^3y^2\,\varpi^2) \\
[u_{a+b+c}(x\,\varpi^1), u_{3a+b}(y\,\varpi^0)] &= u_a(-xy\,\varpi^1)u_{2a+b+c}(-x^2y\,\varpi^2) \\
&\qquad \cdot u_c(-x^3y\,\varpi^3)u_{3a+b+c}(+2x^3y^2\,\varpi^3) \\
[u_{a+b+c}(x\,\varpi^1), u_{3a+2b}(y\,\varpi^0)] &= u_{a+b}(+xy\,\varpi^1)u_{2a+2b+c}(-x^2y\,\varpi^2) \\
&\qquad \cdot u_{b+c}(+x^3y\,\varpi^3)u_b(-2x^3y^2\,\varpi^3) \\
[u_{2a+b+c}(x\,\varpi^1), u_{3a+2b}(y\,\varpi^0)] &= u_{2a+b}(-xy\,\varpi^1)u_a(+x^2y\,\varpi^2) \\
&\qquad \cdot u_{3a+b+c}(-x^3y\,\varpi^3)u_{3a+b}(-2x^3y^2\,\varpi^3) \\
[u_{2a+2b+c}(x\,\varpi^1), u_{3a+b}(y\,\varpi^0)] &= u_{2a+b}(+xy\,\varpi^1)u_{a+b}(+x^2y\,\varpi^2) \\
&\qquad \cdot u_b(-x^3y\,\varpi^3)u_{3a+2b}(-2x^3y^2\,\varpi^3) \\
[u_{2a+2b+c}(x\,\varpi^1), u_{3a+b+c}(y\,\varpi^1)] &= u_{2a+b+c}(-xy\,\varpi^2)u_{a+b+c}(+x^2y\,\varpi^3) \\
&\qquad \cdot u_{b+c}(+x^3y\,\varpi^4)u_c(+2x^3y^2\,\varpi^5)
\end{aligned}$$

for all $x, y \in A_k$.

For each λ that is the barycenter of a facet of the fundamental alcove and not a strongly-special vertex, the group

$$\mathsf{H}_\lambda := G_{\lambda+}/G_{\lambda,1}$$

is generated by subgroups

$$\mathsf{U}_i := U_{\psi_i}G_{\lambda,1}/G_{\lambda,1} \cong \mathsf{f}$$

for $i = 1, \ldots, 12$ such that ψ_i is λ-shallow. Corollary 3.1.9 says that the commutator subgroup of H_λ is generated by the the commutators $[\mathsf{u}, \mathsf{v}]$ with $\mathsf{u} \in \mathsf{U}_i$ and $\mathsf{v} \in \mathsf{U}_j$ for

λ-shallow ψ_i, ψ_j, which can be found among the following:

$$
\begin{aligned}
[u_c(x\,\varpi^1), u_b(y\,\varpi^0)]G_{\lambda,1} &= u_{b+c}(+xy\,\varpi^1)G_{\lambda,1} \\
[u_c(x\,\varpi^1), u_{3a+b}(y\,\varpi^0)]G_{\lambda,1} &= u_{3a+b+c}(-xy\,\varpi^1)G_{\lambda,1} \\
[u_b(x\,\varpi^0), u_{3a+b}(y\,\varpi^0)]G_{\lambda,1} &= u_{3a+2b}(+xy\,\varpi^0)G_{\lambda,1} \\
[u_{b+c}(x\,\varpi^1), u_{3a+2b}(y\,\varpi^0)]G_{\lambda,1} &= G_{\lambda,1} \\
[u_{b+c}(x\,\varpi^1), u_{3a+b+c}(y\,\varpi^1)]G_{\lambda,1} &= G_{\lambda,1} \\
[u_{3a+2b}(x\,\varpi^0), u_{3a+b+c}(y\,\varpi^1)]G_{\lambda,1} &= G_{\lambda,1}
\end{aligned}
$$

for all $x, y \in A_k$;

$$
\begin{aligned}
[u_a(x\,\varpi^0), u_{a+b}(y\,\varpi^0)]G_{\lambda,1} &= u_{2a+b}(-2xy\,\varpi^0)u_{3a+b}(+3x^2y\,\varpi^0)u_{3a+2b}(-3xy^2\,\varpi^0)G_{\lambda,1} \\
[u_a(x\,\varpi^0), u_{a+b+c}(y\,\varpi^1)]G_{\lambda,1} &= u_{2a+b+c}(+2xy\,\varpi^1)u_{3a+b+c}(+3x^2y\,\varpi^1)G_{\lambda,1} \\
[u_{a+b}(x\,\varpi^0), u_{a+b+c}(y\,\varpi^1)]G_{\lambda,1} &= u_{2a+2b+c}(-2xy\,\varpi^1)G_{\lambda,1} \\
[u_{2a+b}(x\,\varpi^0), u_{2a+b+c}(y\,\varpi^1)]G_{\lambda,1} &= G_{\lambda,1} \\
[u_{2a+b}(x\,\varpi^0), u_{2a+2b+c}(y\,\varpi^1)]G_{\lambda,1} &= G_{\lambda,1} \\
[u_{2a+b+c}(x\,\varpi^1), u_{2a+2b+c}(y\,\varpi^1)]G_{\lambda,1} &= G_{\lambda,1}
\end{aligned}
$$

for all $x, y \in A_k$;

$$
\begin{aligned}
[u_a(x\,\varpi^0), u_{2a+b}(y\,\varpi^0)]G_{\lambda,1} &= u_{3a+b}(+3xy\,\varpi^0)G_{\lambda,1} \\
[u_a(x\,\varpi^0), u_{2a+b+c}(y\,\varpi^1)]G_{\lambda,1} &= u_{3a+b+c}(-3xy\,\varpi^1)G_{\lambda,1} \\
[u_{a+b}(x\,\varpi^0), u_{2a+b}(y\,\varpi^0)]G_{\lambda,1} &= u_{3a+2b}(-3xy\,\varpi^0)G_{\lambda,1} \\
[u_{a+b}(x\,\varpi^0), u_{2a+b+c}(y\,\varpi^1)]G_{\lambda,1} &= G_{\lambda,1} \\
[u_{a+b+c}(x\,\varpi^1), u_{2a+b+c}(y\,\varpi^1)]G_{\lambda,1} &= G_{\lambda,1} \\
[u_{a+b+c}(x\,\varpi^1), u_{2a+2b+c}(y\,\varpi^1)]G_{\lambda,1} &= G_{\lambda,1}
\end{aligned}
$$

for all $x, y \in A_k$;

$$\begin{aligned}
[u_{a+b}(x\,\varpi^0), u_c(y\,\varpi^1)]G_{\lambda,1} &= u_{a+b+c}(+xy\,\varpi^1)u_{2a+2b+c}(+x^2y\,\varpi^1)G_{\lambda,1}\\
[u_{2a+b}(x\,\varpi^0), u_c(y\,\varpi^1)]G_{\lambda,1} &= u_{2a+b+c}(-xy\,\varpi^1)G_{\lambda,1}\\
[u_a(x\,\varpi^0), u_b(y\,\varpi^0)]G_{\lambda,1} &= u_{a+b}(+xy\,\varpi^0)u_{2a+b}(+x^2y\,\varpi^0)\\
&\qquad \cdot u_{3a+b}(-x^3y\,\varpi^0)u_{3a+2b}(+2x^3y^2\,\varpi^0)G_{\lambda,1}\\
[u_a(x\,\varpi^0), u_{b+c}(y\,\varpi^1)]G_{\lambda,1} &= u_{a+b+c}(-xy\,\varpi^1)u_{2a+b+c}(+x^2y\,\varpi^1)\\
&\qquad \cdot u_{3a+b+c}(+x^3y\,\varpi^1)G_{\lambda,1}\\
[u_{2a+b+c}(x\,\varpi^1), u_b(y\,\varpi^0)]G_{\lambda,1} &= u_{2a+2b+c}(-xy\,\varpi^1)G_{\lambda,1}\\
[u_{2a+b}(x\,\varpi^0), u_{b+c}(y\,\varpi^1)]G_{\lambda,1} &= u_{2a+2b+c}(+xy\,\varpi^1)G_{\lambda,1}\\
[u_{a+b}(x\,\varpi^0), u_{3a+b+c}(y\,\varpi^1)]G_{\lambda,1} &= G_{\lambda,1}\\
[u_{a+b+c}(x\,\varpi^1), u_{3a+b}(y\,\varpi^0)]G_{\lambda,1} &= G_{\lambda,1}\\
[u_{a+b+c}(x\,\varpi^1), u_{3a+2b}(y\,\varpi^0)]G_{\lambda,1} &= G_{\lambda,1}\\
[u_{2a+b+c}(x\,\varpi^1), u_{3a+2b}(y\,\varpi^0)]G_{\lambda,1} &= G_{\lambda,1}\\
[u_{2a+2b+c}(x\,\varpi^1), u_{3a+b}(y\,\varpi^0)]G_{\lambda,1} &= G_{\lambda,1}\\
[u_{2a+2b+c}(x\,\varpi^1), u_{3a+b+c}(y\,\varpi^1)]G_{\lambda,1} &= G_{\lambda,1}
\end{aligned}$$

for all $x, y \in A_k$. For example, when λ is the barycenter of the facet of $\mathcal{C}$ corresponding to the vanishing of β and the non-vanishing of α and γ, the commutator subgroup of H_λ is generated by all the commutators $[\mathsf{U}_{\psi_i}, \mathsf{U}_{\psi_j}]$ appearing above with λ-shallow affine k-roots $\psi_i, \psi_j \notin \{\beta\}$. In particular, the abelianization of H_λ is isomorphic to

$$\begin{aligned}
&\mathsf{f}_1 \oplus \mathsf{f}_3 \oplus \mathsf{f}_4 \oplus \mathsf{f}_5 \oplus \frac{\mathsf{f}_7 \oplus \mathsf{f}_9 \oplus \mathsf{f}_{11}}{\langle (-2xy, 3x^2y, -3xy), (0, 3xy, 0), (0, 0, -3xy) \mid x, y \in \mathsf{f}\rangle}\\
&\quad \oplus \frac{\mathsf{f}_6 \oplus \mathsf{f}_8 \oplus \mathsf{f}_{10} \oplus \mathsf{f}_{12}}{\left\langle \begin{matrix} (0,0,-xy,0), (0,2xy,3x^2y,0), (0,0,0,-2xy), (0,0,-3xy,0), \\ (xy,0,0,x^2y), (0,-xy,0,0), (-xy,x^2y,x^3y,0), (0,0,0,\pm xy) \end{matrix} \;\middle|\; x, y \in \mathsf{f} \right\rangle}
\end{aligned} \tag{3.1.13}$$

where $\mathsf{f}_i = \mathsf{f}$ is the abelian group isomorphic to $U_{\psi_i}/U_{\psi_i} \cap G_{\lambda,1}$ for $i = 1, \ldots, 12$. Lemma 3.1.12

and some simple reductions allow us to simplify this expression, so that

$$\mathsf{H}_\lambda/[\mathsf{H}_\lambda,\mathsf{H}_\lambda] \cong \begin{cases} \mathsf{f}^6 & \text{if } \mathrm{char}(\mathsf{f}) = 3, \\ \mathsf{f}^5 & \text{if } \mathrm{char}(\mathsf{f}) = 2, \\ \mathsf{f}^4 & \text{if else.} \end{cases}$$

Similar computations can be done for each barycenter not equal to a strongly-special vertex, and the results are summarized in Table B.5 in Appendix B.

Type G_2^I

Let K be a tamely, totally ramified Galois extension of k of degree 3 with ring of integers A_K having maximal ideal P_K generated by a uniformizer which we denote by $\varpi^{1/3}$. Fix a cyclic generator σ of $\mathrm{Gal}(K/k)$ and denote by $x' = \sigma(x)$ and $x'' = \sigma^2(x)$ for all $x \in K$. It will also be convenient to denote by $\zeta_1 = \sigma(\varpi^{1/3})/\varpi^{1/3}$ and $\zeta_2 = \sigma^2(\varpi^{1/3})/\varpi^{1/3} = \zeta_1\zeta_1'$, both of which are units in A_K. We will let ζ denote the image of ζ_1 under the natural projection into $\mathsf{f} = A_K/P_K$. Since σ acts trivially on f, the natural projection of $\zeta_2 = \zeta_1\zeta_1'$ into $\mathsf{f} = A_K/P_K$ is equal to ζ^2.

Let $\mathbf{G}$ be the connected, quasi-simple, semisimple, adjoint-type reductive group defined and non-split quasi-split over k with type G_2 k-root system, and let $\mathrm{SO}_8(K)$ be the split group of 8×8 K-matrices g fixed under the endomorphism

$$g \mapsto Q^{-1}(g^{\mathrm{tr}})^{-1}Q \qquad \text{with } Q = \begin{pmatrix} &&&&&&&1\\ &&&&&&1&\\ &&&&&1&&\\ &&&&1&&&\\ &&&1&&&&\\ &&1&&&&&\\ &1&&&&&&\\ 1&&&&&&& \end{pmatrix}$$

where g^{tr} denotes the transpose of $g \in \mathrm{SO}_8(K)$. The adjoint isogeny $\mathrm{SO}_8 \to \mathrm{SO}_8/\{\pm 1\}$ is not surjective on K-rational points, but the K-roots of $\mathbf{G}$ lift to the diagonal maximal torus of $\mathrm{SO}_8(K)$:

$$\left\{ z = \begin{pmatrix} z_1 &&&&&&& \\ & z_2 &&&&&& \\ && z_3 &&&&& \\ &&& z_4 &&&& \\ &&&& z_5 &&& \\ &&&&& z_6 && \\ &&&&&& z_7 & \\ &&&&&&& z_8 \end{pmatrix} \;\middle|\; \begin{array}{c} z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8 \in K^\times, \\ z_1 z_8 = 1,\ z_2 z_7 = 1, \\ z_3 z_6 = 1,\ z_4 z_5 = 1 \end{array} \right\} \subseteq \mathrm{SO}_8(K).$$

A base $\mathbf{D}$ of the root system $\mathbf{R} = R(\mathbf{G}, \mathbf{Z}, K)$ consists of simple K-roots $\mathbf{a}_1(z) = z_1/z_2$, $\mathbf{b}(z) = z_2/z_3$, $\mathbf{a}_2(z) = z_3/z_4$, and $\mathbf{a}_3(z) = z_4/z_5$. The Borel K-subgroup corresponding to this simple system lifts to the upper-triangular subgroup of $\mathrm{SO}_8(K)$. Through an abuse of notation, the K-pinning of $\mathbf{G}(K)$ will be identified with its lifts in $\mathrm{SO}_8(K)$; namely, we denote by

$$u_{\mathbf{a}_1}(x) = \begin{pmatrix} 1 & x &&&&&& \\ & 1 &&&&&& \\ && 1 &&&&& \\ &&& 1 &&&& \\ &&&& 1 &&& \\ &&&&& 1 && \\ &&&&&& 1 & -x \\ &&&&&&& 1 \end{pmatrix}, \qquad u_{\mathbf{b}}(x) = \begin{pmatrix} 1 &&&&&&& \\ & 1 & x &&&&& \\ && 1 &&&&& \\ &&& 1 &&&& \\ &&&& 1 &&& \\ &&&&& 1 & -x & \\ &&&&&& 1 & \\ &&&&&&& 1 \end{pmatrix},$$

$$u_{\mathbf{a}_2}(x) = \begin{pmatrix} 1 &&&&&&& \\ & 1 &&&&&& \\ && 1 & x &&&& \\ &&& 1 &&&& \\ &&&& 1 & -x && \\ &&&&& 1 && \\ &&&&&& 1 & \\ &&&&&&& 1 \end{pmatrix}, \qquad u_{\mathbf{a}_3}(x) = \begin{pmatrix} 1 &&&&&&& \\ & 1 &&&&&& \\ && 1 & 0 & x &&& \\ &&& 1 & 0 & -x && \\ &&&& 1 & 0 && \\ &&&&& 1 && \\ &&&&&& 1 & \\ &&&&&&& 1 \end{pmatrix}$$

for $x \in K$. The remaining positive K-root morphisms are then denoted by

$$u_{\mathbf{a}_1+\mathbf{b}}(x) = \begin{pmatrix} 1 & 0 & x & & & & & \\ & 1 & 0 & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & 0 & -x \\ & & & & & & 1 & 0 \\ & & & & & & & 1 \end{pmatrix},$$

$$u_{\mathbf{a}_2+\mathbf{b}}(x) = \begin{pmatrix} 1 & & & & & & & \\ & 1 & 0 & x & & & & \\ & & 1 & 0 & & & & \\ & & & 1 & & & & \\ & & & & 1 & 0 & -x & \\ & & & & & 1 & 0 & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{pmatrix},$$

$$u_{\mathbf{a}_3+\mathbf{b}}(x) = \begin{pmatrix} 1 & & & & & & & \\ & 1 & 0 & 0 & x & & & \\ & & 1 & 0 & 0 & & & \\ & & & 1 & 0 & 0 & -x & \\ & & & & 1 & 0 & 0 & \\ & & & & & 1 & 0 & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{pmatrix},$$

$$u_{\mathbf{a}_1+\mathbf{a}_2+\mathbf{b}}(x) = \begin{pmatrix} 1 & 0 & 0 & x & & & & \\ & 1 & 0 & 0 & & & & \\ & & 1 & 0 & & & & \\ & & & 1 & & & & \\ & & & & 1 & 0 & 0 & -x \\ & & & & & 1 & 0 & 0 \\ & & & & & & 1 & 0 \\ & & & & & & & 1 \end{pmatrix},$$

$$u_{\mathbf{a}_1+\mathbf{a}_3+\mathbf{b}}(x) = \begin{pmatrix} 1 & 0 & 0 & 0 & x & & & \\ & 1 & 0 & 0 & 0 & & & \\ & & 1 & 0 & 0 & & & \\ & & & 1 & 0 & 0 & 0 & -x \\ & & & & 1 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & 1 & 0 \\ & & & & & & & 1 \end{pmatrix},$$

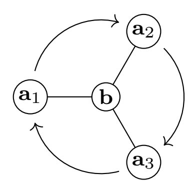
$$u_{\mathbf{a}_2+\mathbf{a}_3+\mathbf{b}}(x) = \begin{pmatrix} 1 & & & & & & & \\ & 1 & 0 & 0 & 0 & x & & \\ & & 1 & 0 & 0 & 0 & -x & \\ & & & 1 & 0 & 0 & 0 & \\ & & & & 1 & 0 & 0 & \\ & & & & & 1 & 0 & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{pmatrix},$$

$$u_{\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3+\mathbf{b}}(x) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & x & & \\ & 1 & 0 & 0 & 0 & 0 & & \\ & & 1 & 0 & 0 & 0 & 0 & -x \\ & & & 1 & 0 & 0 & 0 & 0 \\ & & & & 1 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & 1 & 0 \\ & & & & & & & 1 \end{pmatrix},$$

$$u_{\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3+2\mathbf{b}}(x) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & x & \\ & 1 & 0 & 0 & 0 & 0 & 0 & -x \\ & & 1 & 0 & 0 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 & 0 & 0 \\ & & & & 1 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & 1 & 0 \\ & & & & & & & 1 \end{pmatrix}$$

for $x \in K$.

We define a left action of the Galois group $\mathrm{Gal}(K/k)$ on $\mathbf{R}$ via based automorphisms so that σ acts according to the following triality of the Dynkin diagram



The right Galois action on $\mathbf{G}(K)$ is then given so that $u_{\mathbf{c}}(x)^\sigma = u_{\sigma\mathbf{c}}(\epsilon_{\mathbf{c}}\, x')$ for all $\mathbf{c} \in \mathbf{R}$ and $x \in K$, with $\epsilon_{\mathbf{c}} \in \{1, -1\}$. Keeping with the convention set out in Notation 2.2.2, we will be assuming that $\epsilon_{\mathbf{c}} = 1$ for all simple roots $\mathbf{c} \in \mathbf{D}$. The remaining signs $\epsilon_{\mathbf{c}}$ are recovered from the commutators $[u_{\mathbf{c}}(x), u_{\mathbf{d}}(y)]^\sigma = [u_{\mathbf{c}}(x)^\sigma, u_{\mathbf{d}}(y)^\sigma]$ for all $\mathbf{c}, \mathbf{d} \in \mathbf{R}$ and $x, y \in K$; in particular, for positive roots we have

$$\epsilon_{\mathbf{a}_1+\mathbf{b}} = -1, \qquad \epsilon_{\mathbf{a}_2+\mathbf{b}} = +1, \qquad \epsilon_{\mathbf{a}_3+\mathbf{b}} = -1,$$
$$\epsilon_{\mathbf{a}_1+\mathbf{a}_2+\mathbf{b}} = +1, \qquad \epsilon_{\mathbf{a}_1+\mathbf{a}_3+\mathbf{b}} = +1, \qquad \epsilon_{\mathbf{a}_2+\mathbf{a}_3+\mathbf{b}} = +1,$$
$$\epsilon_{\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3+\mathbf{b}} = +1,$$
$$\epsilon_{\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3+2\mathbf{b}} = +1.$$

For negative roots $\mathbf{c} \in \mathbf{R}^-$, the signs are determined by $\epsilon_{\mathbf{c}} = \epsilon_{-\mathbf{c}}$.

We now consider the k-group $G = \mathbf{G}(k)$ consisting of all σ-fixed matrices in $\mathrm{PSO}_8(K)$. The Galois action preserves the diagonal torus $\mathbf{Z}(K)$, permuting the fundamental coweights dual to the base $\mathbf{D}$. The k-root system $R = R(\mathbf{G}, \mathbf{S}, k)$ is identified with the twisted root system $R = \mathbf{R}_\sigma$. A base D of R consists of a short k-root $a \in R$, the restriction to $\mathbf{S}$ of $\mathbf{a}_i \in \mathbf{D}$, and $b \in R$, the restriction to $\mathbf{S}$ of $\mathbf{b} \in \mathbf{D}$. The simple k-root morphisms are defined as in Proposition 2.2.2:

$$u_a(x) = \begin{pmatrix} 1 & x & & & & & & \\ & 1 & & & & & & \\ & & 1 & x' & x'' & -x'x'' & & \\ & & & 1 & 0 & -x'' & & \\ & & & & 1 & -x' & & \\ & & & & & 1 & & \\ & & & & & & 1 & -x \\ & & & & & & & 1 \end{pmatrix}, \qquad u_b(y) = \begin{pmatrix} 1 & & & & & & & \\ & 1 & y & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & -y & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{pmatrix}$$

for $x \in k$ and $y \in K$. The remaining positive k-root morphisms are defined similarly:

$$u_{a+b}(x) = \begin{pmatrix} 1 & 0 & x & & & & & \\ & 1 & 0 & -x' & -x'' & 0 & -x'x'' & \\ & & 1 & 0 & 0 & 0 & 0 & \\ & & & 1 & 0 & 0 & x'' & \\ & & & & 1 & 0 & x' & \\ & & & & & 1 & 0 & -x \\ & & & & & & 1 & 0 \\ & & & & & & & 1 \end{pmatrix}$$

$$u_{2a+b}(x) = \begin{pmatrix} 1 & 0 & 0 & x & x'' & 0 & 0 & -xx'' \\ & 1 & 0 & 0 & 0 & x' & 0 & 0 \\ & & 1 & 0 & 0 & 0 & -x' & 0 \\ & & & 1 & 0 & 0 & 0 & -x'' \\ & & & & 1 & 0 & 0 & -x \\ & & & & & 1 & 0 & 0 \\ & & & & & & 1 & 0 \\ & & & & & & & 1 \end{pmatrix}$$

$$
u_{3a+b}(y) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & y & & \\
 & 1 & 0 & 0 & 0 & 0 & & \\
 & & 1 & 0 & 0 & 0 & 0 & -y \\
 & & & 1 & 0 & 0 & 0 & 0 \\
 & & & & 1 & 0 & 0 & 0 \\
 & & & & & 1 & 0 & 0 \\
 & & & & & & 1 & 0 \\
 & & & & & & & 1
\end{pmatrix}
$$

$$
u_{3a+2b}(y) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & y & \\
 & 1 & 0 & 0 & 0 & 0 & 0 & -y \\
 & & 1 & 0 & 0 & 0 & 0 & 0 \\
 & & & 1 & 0 & 0 & 0 & 0 \\
 & & & & 1 & 0 & 0 & 0 \\
 & & & & & 1 & 0 & 0 \\
 & & & & & & 1 & 0 \\
 & & & & & & & 1
\end{pmatrix}
$$

for $x \in k$ and $y \in K$.

The affine k-root system Ψ has type G_2^I, with a base Δ consisting of simple affine k-roots

$$
\begin{aligned}
\alpha &= a + 0 \\
\beta &= b + 0 \\
\gamma &= c + 1/3,
\end{aligned}
\tag{3.1.14}
$$

where $c = -2a - b$ is the lowest short root in R. We now give an enumeration of the minimal-height positive affine k-roots $\psi_1, \ldots, \psi_{12}$ for each gradient in R. The enumeration given to these affine k-roots will be height preserving so that $i > j$ whenever ψ_i has a greater height than ψ_j:

index i	gradient $\dot{\psi}_i$	intercept $\psi_i(0)$	simple affine k-root decomposition ψ_i
1	c	$1/3$	γ
2	a	0	α
3	b	0	β
4	$a+c$	$1/3$	$\alpha+\gamma$
5	$a+b$	0	$\alpha+\beta$
6	$a+b+c$	$1/3$	$\alpha+\beta+\gamma$
7	$2a+b$	0	$2\alpha+\beta$
8	$3a+b$	0	$3\alpha+\beta$
9	$3a+2b$	0	$3\alpha+2\beta$
10	$a+2c$	1	$3\alpha+1\beta+3\gamma$
11	$a+b+2c$	1	$3\alpha+2\beta+3\gamma$
12	$2a+c$	1	$6\alpha+2\beta+3\gamma$

(3.1.15)

Since $1/3 = 2\alpha+\beta+\gamma$, we note that $\psi_{10}, \psi_{11}, \psi_{12}$ are never λ-shallow for λ belonging to the closure of the fundamental alcove, and therefore in what follows we will only consider ψ_i for $i = 1, \ldots, 9$: from Appendix A, we recover the commutators of affine k-root groups for all pairs ψ_i, ψ_j of minimal-height positive affine k-roots with non-linearly dependent gradients $\dot{\psi}_i, \dot{\psi}_j$ such that $\dot{\psi}_i + \dot{\psi}_j$ is a k-root, but we reproduce them here for convenience:

$$[u_b(x\,\varpi^0), u_{3a+b}(y\,\varpi^0)] = u_{3a+2b}(xy\,\varpi^0)$$

for all $x, y \in A_k$;

$$\begin{aligned}[u_a(x\,\varpi^0), u_{a+b}(y\,\varpi^0)] = u_{2a+b}&((-xy' - x'y)\,\varpi^0)\\ &\cdot u_{3a+b}((x'x''y + xx''y' + xx'y'')\,\varpi^0)\\ &\cdot u_{3a+2b}((-xy'y'' - x'yy'' - x''yy')\,\varpi^0)\end{aligned}$$

$$
\begin{aligned}
[u_a(x\,\varpi^0), u_c(y\,\varpi^{1/3})] &= u_{a+c}((x'y + \zeta_2 x''y'')\,\varpi^{1/3}) \\
&\qquad \cdot u_{2a+c}((xx'y + \zeta_1 x'x''y' + \zeta_2 xx''y'')\,\varpi^{1/3}) \\
&\qquad \cdot u_{a+2c}((\zeta_2 xyy'' + \zeta_1 x'yy' + \zeta_1\zeta_2 x''y'y'')\,\varpi^{2/3}) \\
[u_{a+b}(x\,\varpi^0), u_c(y\,\varpi^{1/3})] &= u_{a+b+c}((-x'y - \zeta_2 x''y'')\,\varpi^{1/3}) \\
&\qquad \cdot u_{a+b+2c}((\zeta_2 xyy'' + \zeta_1 x'yy' + \zeta_1\zeta_2 x''y'y'')\,\varpi^{2/3}) \\
&\qquad \cdot u_b((-xx'y - \zeta_1 x'x''y' - \zeta_2 xx''y'')\,\varpi^{1/3}) \\
[u_{2a+b}(x\,\varpi^0), u_{a+b+c}(y\,\varpi^{1/3})] &= u_{a+b}((\zeta_1 xy' + \zeta_2 x''y'')\,\varpi^{1/3}) \\
&\qquad \cdot u_{3a+2b}((-xx''y - \zeta_1 xx'y' - \zeta_2 x'x''y'')\,\varpi^{1/3}) \\
&\qquad \cdot u_b((-\zeta_1 xyy' - \zeta_1\zeta_2 x'y'y'' - \zeta_2 x''yy'')\,\varpi^{2/3}) \\
[u_{2a+b}(x\,\varpi^0), u_{a+c}(y\,\varpi^{1/3})] &= u_a((-\zeta_1 xy' - \zeta_2 x''y'')\,\varpi^{1/3}) \\
&\qquad \cdot u_{3a+b}((-xx''y - \zeta_1 xx'y' - \zeta_2 x'x''y'')\,\varpi^{1/3}) \\
&\qquad \cdot u_{2a+c}((\zeta_1 xyy' + \zeta_1\zeta_2 x'y'y'' + \zeta_2 x''yy'')\,\varpi^{2/3}) \\
[u_{a+b+c}(x\,\varpi^{1/3}), u_{a+c}(y\,\varpi^{1/3})] &= u_c((\zeta_1 xy' + \zeta_1 x'y)\,\varpi^{2/3}) \\
&\qquad \cdot u_{a+b+2c}((\zeta_1\zeta_2 x'x''y + \zeta_1\zeta_2 xx''y' + \zeta_1\zeta_2 xx'y'')\,\varpi^{1}) \\
&\qquad \cdot u_{a+2c}((-\zeta_1\zeta_2 xy'y'' - \zeta_1\zeta_2 x'yy'' - \zeta_1\zeta_2 x''yy')\,\varpi^{1})
\end{aligned}
$$

for all $x, y \in A_K$, where $\zeta_1 = \sigma(\varpi^{1/3})/\varpi^{1/3}$ and $\zeta_2 = \sigma^2(\varpi^{1/3})/\varpi^{1/3}$, as defined above;

$$
\begin{aligned}
[u_a(x\,\varpi^0), u_{2a+b}(y\,\varpi^0)] &= u_{3a+b}((xy' + x'y'' + x''y)\,\varpi^0) \\
[u_a(x\,\varpi^0), u_{a+c}(y\,\varpi^{1/3})] &= u_{2a+c}((-xy - \zeta_1 x'y' - \zeta_2 x''y'')\,\varpi^{1/3}) \\
[u_{a+b}(x\,\varpi^0), u_{2a+b}(y\,\varpi^0)] &= u_{3a+2b}((-xy' - x'y'' - x''y)\,\varpi^0) \\
[u_{a+b}(x\,\varpi^0), u_{a+b+c}(y\,\varpi^{1/3})] &= u_b((-xy - \zeta_1 x'y' - \zeta_2 x''y'')\,\varpi^{1/3}) \\
[u_{a+b+c}(x\,\varpi^{1/3}), u_c(y\,\varpi^{1/3})] &= u_{a+b+2c}((-\zeta_1 xy' - \zeta_1\zeta_2 x'y'' - \zeta_2 x''y)\,\varpi^{2/3}) \\
[u_{a+c}(x\,\varpi^{1/3}), u_c(y\,\varpi^{1/3})] &= u_{a+2c}((\zeta_1 xy' + \zeta_1\zeta_2 x'y'' + \zeta_2 x''y)\,\varpi^{2/3})
\end{aligned}
$$

for all $x, y \in A_K$; and

$$\begin{aligned}
[u_a(x\,\varpi^0), u_b(y\,\varpi^0)] &= u_{a+b}(xy\,\varpi^0)u_{2a+b}(xx'y\,\varpi^0) \\
&\quad\cdot u_{3a+b}(-xx'x''y\,\varpi^0)u_{3a+2b}(2xx'x''y^2\,\varpi^0) \\
[u_{a+b+c}(x\,\varpi^{1/3}), u_{3a+b}(y\,\varpi^0)] &= u_{2a+b}(\zeta_2 x''y\,\varpi^{1/3})u_{a+b}(\zeta_1\zeta_2 x'x''y\,\varpi^{2/3}) \\
&\quad\cdot u_b(-\zeta_1\zeta_2 xx'x''y\,\varpi^1)u_{3a+2b}(-2\zeta_1\zeta_2 xx'x''y^2\,\varpi^1) \\
[u_{a+c}(x\,\varpi^{1/3}), u_b(y\,\varpi^0)] &= u_{a+b+c}(-xy\,\varpi^{1/3})u_c(-\zeta_1 xx'y\,\varpi^{2/3}) \\
&\quad\cdot u_{a+2c}(\zeta_1\zeta_2 xx'x''y\,\varpi^1)u_{a+b+2c}(2\zeta_1\zeta_2 xx'x''y^2\,\varpi^1) \\
[u_{a+c}(x\,\varpi^{1/3}), u_{3a+2b}(y\,\varpi^0)] &= u_{2a+b}(-\zeta_2 x''y\,\varpi^{1/3})u_a(\zeta_1\zeta_2 x'x''y\,\varpi^{2/3}) \\
&\quad\cdot u_{2a+c}(-\zeta_1\zeta_2 xx'x''y\,\varpi^1)u_{3a+2b}(-2\zeta_1\zeta_2 xx'x''y^2\,\varpi^1) \\
[u_c(x\,\varpi^{1/3}), u_{3a+b}(y\,\varpi^0)] &= u_a(-\zeta_1 x'y\,\varpi^{1/3})u_{a+c}(-\zeta_2 xx''y\,\varpi^{2/3}) \\
&\quad\cdot u_{a+2c}(-\zeta_1\zeta_2 xx'x''y\,\varpi^1)u_{2a+c}(2\zeta_1\zeta_2 xx'x''y^2\,\varpi^1) \\
[u_c(x\,\varpi^{1/3}), u_{3a+2b}(y\,\varpi^0)] &= u_{a+b}(\zeta_1 x'y\,\varpi^{1/3})u_{a+b+c}(-\zeta_2 xx''y\,\varpi^{2/3}) \\
&\quad\cdot u_{a+b+2c}(\zeta_1\zeta_2 xx'x''y\,\varpi^1)u_b(-2\zeta_1\zeta_2 xx'x''y^2\,\varpi^1)
\end{aligned}$$

for all $x \in A_K$ and $y \in A_k$.

For each λ that is the barycenter of a facet of the fundamental alcove and not a strongly-special vertex, the group

$$\mathsf{H}_\lambda := G_{\lambda+}/G_{\lambda,1/3}$$

is generated by subgroups

$$\mathsf{U}_i := U_{\psi_i}G_{\lambda,1/3}/G_{\lambda,1/3} \cong \mathsf{f}$$

for $i = 1, \ldots, 9$ such that ψ_i is λ-shallow. Corollary 3.1.9 says that the commutator subgroup of H_λ is generated by the commutators $[\mathsf{u}, \mathsf{v}]$ with $\mathsf{u} \in \mathsf{U}_i$ and $\mathsf{v} \in \mathsf{U}_j$ for λ-shallow ψ_i, ψ_j; these commutators are computed from the above commutators. For example, when λ is the barycenter of the fundamental alcove, each ψ_i, $i = 1, \ldots, 9$ is a λ-shallow affine k-root and the commutator subgroup of H_λ is generated by commutators of the following forms:

$$[u_b(x\,\varpi^0), u_{3a+b}(y\,\varpi^0)]G_{\lambda,1/3} = u_{3a+2b}(xy\,\varpi^0)G_{\lambda,1/3}$$

for all $x, y \in A_k$;

$$
\begin{aligned}
[u_a(x\,\varpi^0), u_{a+b}(y\,\varpi^0)]G_{\lambda,1/3} &= u_{2a+b}((-xy' - x'y)\,\varpi^0) \\
&\quad \cdot u_{3a+b}((x'x''y + xx''y' + xx'y'')\,\varpi^0) \\
&\quad \cdot u_{3a+2b}((-xy'y'' - x'yy'' - x''yy')\,\varpi^0)G_{\lambda,1/3} \\
[u_a(x\,\varpi^0), u_c(y\,\varpi^{1/3})]G_{\lambda,1/3} &= u_{a+c}((x'y + \zeta_2 x''y'')\,\varpi^{1/3})G_{\lambda,1/3} \\
[u_{a+b}(x\,\varpi^0), u_c(y\,\varpi^{1/3})]G_{\lambda,1/3} &= u_{a+b+c}((-x'y - \zeta_2 x''y'')\,\varpi^{1/3})G_{\lambda,1/3} \\
[u_{2a+b}(x\,\varpi^0), u_{a+b+c}(y\,\varpi^{1/3})]G_{\lambda,1/3} &= G_{\lambda,1/3} \\
[u_{2a+b}(x\,\varpi^0), u_{a+c}(y\,\varpi^{1/3})]G_{\lambda,1/3} &= G_{\lambda,1/3} \\
[u_{a+b+c}(x\,\varpi^{1/3}), u_{a+c}(y\,\varpi^{1/3})]G_{\lambda,1/3} &= G_{\lambda,1/3}
\end{aligned}
$$

for all $x, y \in A_K$ where ζ is the natural projection of $\zeta_1 \in A_K^\times$ into f, as defined above;

$$
\begin{aligned}
[u_a(x\,\varpi^0), u_{2a+b}(y\,\varpi^0)]G_{\lambda,1/3} &= u_{3a+b}((xy' + x'y'' + x''y)\,\varpi^0)G_{\lambda,1/3} \\
[u_a(x\,\varpi^0), u_{a+c}(y\,\varpi^{1/3})]G_{\lambda,1/3} &= G_{\lambda,1/3} \\
[u_{a+b}(x\,\varpi^0), u_{2a+b}(y\,\varpi^0)]G_{\lambda,1/3} &= u_{3a+2b}((-xy' - x'y'' - x''y)\,\varpi^0)G_{\lambda,1/3} \\
[u_{a+b}(x\,\varpi^0), u_{a+b+c}(y\,\varpi^{1/3})]G_{\lambda,1/3} &= G_{\lambda,1/3} \\
[u_{a+b+c}(x\,\varpi^{1/3}), u_c(y\,\varpi^{1/3})]G_{\lambda,1/3} &= G_{\lambda,1/3} \\
[u_{a+c}(x\,\varpi^{1/3}), u_c(y\,\varpi^{1/3})]G_{\lambda,1/3} &= G_{\lambda,1/3}
\end{aligned}
$$

for all $x, y \in A_K$; and for all $x \in A_K$ and $y \in A_k$ we have

$$
\begin{aligned}
[u_a(x\,\varpi^0), u_b(y\,\varpi^0)]G_{\lambda,1/3} &= u_{a+b}(xy\,\varpi^0)u_{2a+b}(xx'y\,\varpi^0) \\
&\quad \cdot u_{3a+b}(-xx'x''y\,\varpi^0)u_{3a+2b}(2xx'x''y^2\,\varpi^0)G_{\lambda,1/3}
\end{aligned}
$$

$$[u_{a+b+c}(x\,\varpi^{1/3}), u_{3a+b}(y\,\varpi^{0})]G_{\lambda,1/3} = G_{\lambda,1/3}$$
$$[u_{a+c}(x\,\varpi^{1/3}), u_{b}(y\,\varpi^{0})]G_{\lambda,1/3} = u_{a+b+c}(-xy\,\varpi^{1/3})G_{\lambda,1/3}$$
$$[u_{a+c}(x\,\varpi^{1/3}), u_{3a+2b}(y\,\varpi^{0})]G_{\lambda,1/3} = G_{\lambda,1/3}$$
$$[u_{c}(x\,\varpi^{1/3}), u_{3a+b}(y\,\varpi^{0})]G_{\lambda,1/3} = G_{\lambda,1/3}$$
$$[u_{c}(x\,\varpi^{1/3}), u_{3a+2b}(y\,\varpi^{0})]G_{\lambda,1/3} = G_{\lambda,1/3}.$$

The abelianization of H_λ is isomorphic to

$$\begin{aligned}\mathsf{H}_\lambda/[\mathsf{H}_\lambda,\mathsf{H}_\lambda] \cong \mathsf{f}_1 \oplus \mathsf{f}_2 \oplus \mathsf{f}_3 \oplus \frac{\mathsf{f}_4}{\langle (1+\zeta^2)xy \mid x,y\in\mathsf{f}\rangle} \oplus \frac{\mathsf{f}_6}{\langle -(1+\zeta^2)xy, -xy \mid x,y\in\mathsf{f}\rangle} \\ \oplus \frac{\mathsf{f}_5\oplus\mathsf{f}_7\oplus\mathsf{f}_8\oplus\mathsf{f}_9}{\left\langle \begin{matrix}(0,0,0,xy),(0,-2xy,3x^2y,-3xy^2),(0,0,3xy,0),\\ (0,0,0,-3xy),(xy,x^2y,-x^3y,2x^3y^2)\end{matrix} \;\middle|\; x,y\in\mathsf{f} \right\rangle}\end{aligned}$$

where $\mathsf{f}_i = \mathsf{f}$ is the abelian group isomorphic to $U_{\psi_i}/U_{\psi_i}\cap G_{\lambda,1/3}$ for $i=1,\ldots,9$. Lemma 3.1.12 allows us to simplify the above expression so that

$$\mathsf{H}_\lambda/[\mathsf{H}_\lambda,\mathsf{H}_\lambda] \cong \begin{cases} \mathsf{f}^5 & \text{if } \#\mathsf{f}\in\{2,3\} \text{ and } 1+\zeta^2=0,\\ \mathsf{f}^4 & \text{if } \#\mathsf{f}\in\{2,3\} \text{ and } 1+\zeta^2\neq 0,\\ \mathsf{f}^4 & \text{if } \#\mathsf{f}\notin\{2,3\} \text{ and } 1+\zeta^2=0,\\ \mathsf{f}^3 & \text{if else.}\end{cases} \tag{3.1.16}$$

Similar computations can be done for each barycenter which is not strongly-special, and the results are summarized in Table B.6 in Appendix B.

Remark. There do not exist any tamely, totally ramified Galois extensions of degree 3 of a non-archimedean local field whose residue field has order 2 or 3, and so not all of the cases in (3.1.16) can occur.

3.2 Supercuspidal Representations

Recall that a **smooth representation** of G is a group homomorphism $\pi : G \to \mathrm{GL}(V)$ where V is a complex vector space, such that for every $v \in V$ there is a compact open subgroup $H \subseteq G$ with $\pi(h)v = v$ for every $h \in H$. A smooth irreducible representation π is **supercuspidal** if every matrix coefficient of G is compactly supported. This section is devoted to the construction of supercuspidal representations of G through the method of compact induction, which we now briefly review.

From here we turn to using the shallow characters constructed in the previous section to give one-dimensional smooth representations of the compact open pro-unipotent radical of the parahoric subgroup. In Proposition 3.2.2 we provide a blueprint for constructing supercuspidal representations of G from shallow characters and a sequence of compact inductions. This method was first used by Gross and Reeder to construct the simple supercuspidal representation from affine generic characters [12], and again by Reeder and Yu to construct epipelagic supercuspidal representations from *stable* λ-shallow characters of $G_{\lambda+}$ vanishing at the Moy-Prasad subgroup $G_{\lambda++}$ [27]. Here we are using stable in the sense of geometric invariant theory, as these shallow characters belong to a graded Lie algebra where one can apply the methods of Vinberg [35] and Levy [19][20][21].

The λ-shallow characters not vanishing on $G_{\lambda++}$ do not have a natural identification with elements of a single piece of a graded Lie algebra, and therefore the arguments of [27] do not immediately extend to our situation. Instead, in Theorem 3.2.3 we provide a naive extension of [27, Lemma 2.3 and Proposition 2.4] sufficient for constructing supercuspidal representations. We then show in §3.2.3 and §3.2.4, by way of example, that this naive condition is not necessary. Indeed, when $\#\mathsf{f} = 2$, we give four λ-shallow characters that yield new supercuspidal representations of $G = \mathrm{Sp}_4(k)$; and when $\mathrm{char}(\mathsf{f}) = 3$, we give a class of $(\#\mathsf{f})^4$ λ-shallow characters that yield new supercuspidal representations of the split form of G_2 over k.

Notation 3.2.1. In this section, we will be continuing the notation set out in Notation 2.2.1 and Notation 3.1.1. In addition, we will let $\mathbf{N}$ denote the normalizer in $\mathbf{G}$ of the maximal k-split torus $\mathbf{S}$. This algebraic group is defined over k, and thus we denote by the unbolded

N its group of k-rational points. //

3.2.1 Compact induction

Let $H \subseteq M$ be subgroups of G and suppose that H is compact open containing the center of G. Let (ϕ, V) be an irreducible smooth representation of H. Denote by $\mathrm{ind}_H^M(\phi)$ the **compactly induced representation** of M consisting of all compactly supported functions $f : M \to V$ that commute with H:

$$\mathrm{ind}_H^M(\phi) = \left\{ f : M \to V \;\middle|\; \begin{array}{c} f(hx) = \phi(h) \cdot f(x) \\ f \text{ is compactly supported} \end{array} \right\}.$$

The left M-action on $\mathrm{ind}_H^M(\phi)$ is given by right translation so that $[m \cdot f](x) = f(xm)$ for all $m, x \in M$.

For each $m \in M$, we denote by $({}^m\phi, V)$ the **conjugate representation** of ${}^mH := mHm^{-1}$ given by

$${}^m\phi(mhm^{-1}) = \phi(h)$$

for all $h \in H$. The **intertwining set** $\mathscr{I}(M, H, \phi)$ is then the set of all elements $m \in M$ such that ${}^m\phi$ and ϕ are isomorphic on the intersection ${}^mH \cap H$:

$$\mathscr{I}(M, H, \phi) = \{m \in M \mid {}^m\phi \cong \phi \text{ on } {}^mH \cap H\}.$$

Lemma 3.2.1. *Let M, H, ϕ be as above. Then the following are equivalent:*

(1) $\mathscr{I}(M, H, \phi) = H$.

(2) $\mathrm{ind}_H^M(\phi)$ *is irreducible.*

If $M = G$ and the above conditions hold, then $\mathrm{ind}_H^G(\phi)$ is supercuspidal.

Proof. A proof of this basic result can be found in [4, 3.11.4]. $\square$

3.2.2 Supercuspidal representations and shallow characters

Let $\chi : G_{\lambda+} \to \mathbb{C}^\times$ be any group homomorphism, and thus a one-dimensional smooth representation of $G_{\lambda+}$. The parahoric subgroup G_λ normalizes its pro-unipotent radical, and so we consider the **stabilizer** in G_λ of χ denoted by

$$G_\lambda(\chi) = \{n \in G_\lambda \mid \chi(nhn^{-1}) = \chi(h) \text{ for all } h \in G_{\lambda+}\} \subseteq \mathscr{I}(G, G_{\lambda+}, \chi)$$

The quotient $G_\lambda(\chi)/G_{\lambda+}$ is a finite group whose order is equal to the dimension of the semisimple **intertwining algebra**

$$\mathscr{A}_\chi = \mathrm{End}_{G_\lambda(\chi)}\left(\mathrm{ind}_{G_{\lambda+}}^{G_\lambda(\chi)}(\chi)\right)$$

There is a bijection $\rho \mapsto \chi_\rho$ between equivalence classes of irreducible $\mathscr{A}_\chi$-modules and the irreducible $G_\lambda(\chi)$-representations appearing in the isotypic decomposition

$$\mathrm{ind}_{G_{\lambda+}}^{G_\lambda(\chi)}(\chi) = \bigoplus_\rho \dim(\rho) \cdot \chi_\rho.$$

Then we have the following result from §2.1 of [27]:

Proposition 3.2.2 (Reeder-Yu)**.** *Let $\chi : G_{\lambda+} \to \mathbb{C}^\times$ be any group homomorphism. If $\mathscr{I}(G, G_{\lambda+}, \chi) = G_\lambda(\chi)$, then we have the following isotypic decomposition:*

$$\mathrm{ind}_{G_{\lambda+}}^{G}(\chi) = \bigoplus_\rho \dim(\rho) \cdot \mathrm{ind}_{G_\lambda(\chi)}^{G}(\chi_\rho),$$

where the direct sum is over all simple $\mathscr{A}_\chi$-modules ρ. Moreover, all compactly induced representations

$$\pi(\chi; \rho) := \mathrm{ind}_{G_\lambda(\chi)}^{G}(\chi_\rho) \tag{3.2.1}$$

are inequivalent irreducible supercuspidal representations of G.

Proof. The direct sum decomposition follows from the transitivity of compact induction:

$$\begin{aligned}\mathrm{ind}_{G_{\lambda+}}^{G}(\chi) &= \mathrm{ind}_{G_\lambda(\chi)}^{G}(\mathrm{ind}_{G_{\lambda+}}^{G_\lambda(\chi)}(\chi)) \\ &= \mathrm{ind}_{G_\lambda(\chi)}^{G}\left(\bigoplus_\rho \dim(\rho)\cdot\chi_\rho\right) \\ &= \bigoplus_\rho \dim(\rho)\cdot \mathrm{ind}_{G_\lambda(\chi)}^{G}(\chi_\rho).\end{aligned}$$

For proofs of the remaining claims, please refer to [27, Lemma 2.2]. □

The group $N = \mathbf{N}(k)$ acts on the k-root system via the finite reflection group W_0 of R so that given an element $n \in N$ we have that

$$nU_a n^{-1} = U_{na}$$

for all k-roots $a \in R$. This action does not leave invariant the valuation on k-root groups given in Definition 2.2.3. For example, given an element $n = w\lambda(t) \in N$ for $w \in W_0$, $\lambda \in X_*(\mathbf{Z}, K)$, and $t \in K^\times$, we have

$$\mathrm{val}_{wa}(nun^{-1}) = \langle a, \lambda\rangle \,\mathrm{val}(t) + \mathrm{val}_a(u).$$

for all $a \in R$ and $u \in U_a$ so that

$$nU_{a,r}n^{-1} = U_{wa,r+\langle a,\lambda\rangle\,\mathrm{val}(t)}.$$

In this case, we write $n(a,r) = (wa, r+\langle a,\lambda\rangle\,\mathrm{val}(t)) \in \Psi$ and see that N acts on $X_*(\mathbf{S},k)\otimes_{\mathbb{Z}}\mathbb{R}$ via affine linear transformations and the induced action on affine functions preserves Ψ. [17].

The group of affine linear transformations generated by the action of N on Ψ is a finite extension of the affine reflection group W of Ψ defined in Definition 2.1.3, dependent on the isogeny type of G [17]. Here we will call this group the **affine Weyl group** (in the literature it is sometimes called the **extended affine Weyl group** while the affine reflection group W is called the affine Weyl group). The general theory of the affine Weyl group is beyond the scope of this thesis, and so we will simply refer to N and its action on Ψ.

Theorem 3.2.3. *Let $\chi : G_\lambda \to \mathbb{C}^\times$ be any group homomorphism that is trivial on $G_{\lambda,\mathsf{s}(\lambda)}$, and let r be the minimal real number such that χ is trivial on U_α whenever $\alpha(\lambda) > r$. Suppose that the following holds:*

(∗) *If $n \in N$ and χ is trivial on U_α whenever α is a shallow affine k-root such that $[n^{-1}\alpha](\lambda) > r$, then $n \in G_\lambda$.*

Then $\mathscr{I}(G, G_{\lambda+}, \chi) = G_\lambda(\chi)$.

Proof. Recall the **affine Bruhat decomposition** [17][34, §3.3.1] for the group G, which says that

$$G = G_{\mathcal{C}} N G_{\mathcal{C}},$$

where $G_{\mathcal{C}}$ is the parahoric subgroup attached to any point in the fundamental alcove $\mathcal{C}$. The parahoric subgroup G_λ contains $G_{\mathcal{C}}$. Therefore, in order to show that $\mathscr{I}(G, G_{\lambda+}, \chi) = G_\lambda(\chi)$ it is only necessary to consider $n \in N$ and show that

$$^n\chi = \chi \text{ on } {}^nG_{\lambda+} \cap G_{\lambda+} \tag{3.2.2}$$

implies that $n \in G_\lambda$.

Let $n \in N$ be such that (3.2.2) holds. Then it is certainly true that

$$^n\chi = \chi \text{ on } {}^nG_{\lambda,r+} \cap G_{\lambda+} \tag{3.2.3}$$

for the Moy-Prasad subgroup $G_{\lambda,r} \subseteq G_\lambda$. Let $\alpha \in \Psi$ be any shallow affine k-root such that $[n^{-1}\alpha](\lambda) > r$. The affine k-root group $U_{n^{-1}\alpha}$ then belongs to $G_{\lambda,r+}$ and thus

$$nU_{n^{-1}\alpha}n^{-1} = U_\alpha \subseteq {}^nG_{\lambda,r+} \cap G_{\lambda+}.$$

Then (3.2.3) says that

$$\chi(u) = {}^n\chi(u) = \chi(n^{-1}un)$$

for all $u \in U_\alpha$. But the definition of r implies that $\chi(n^{-1}un) = 1$ for all $u \in U_\alpha$, since $[n^{-1}\alpha](\lambda) > r$. This must hold for every λ-shallow affine k-root α such that $[n^{-1}\alpha](\lambda) > r$, and thus condition $(*)$ of the theorem says that $n \in G_\lambda$, as desired. □

Remark. Reeder and Yu show that a λ-shallow character vanishing on any λ-shallow affine k-root group U_α such that $\alpha(\lambda) > \mathsf{r}_1(\lambda)$, as defined in Definition 2.2.4, satisfies condition $(*)$ of Theorem 3.2.3 whenever it is **stable** in the sense that its G_λ-orbit is closed with a finite stabilizer (as an algebraic group) [27, Proposition 2.4].

3.2.3 New supercuspidal representations of $\mathrm{Sp}_4(k)$ when $\#\mathsf{f} = 2$

In this subsection, we will continue the notation laid out in §3.1.3 for the group split group $\mathrm{Sp}_4(k)$ with type C_2 k-root system, including the enumeration $\psi_1, \ldots, \psi_8$ of positive affine k-roots having minimal height given in (3.1.9). Since $G = \mathrm{Sp}_4(k)$ is simply connected, its affine Weyl group, by which N acts on Ψ, is isomorphic to the affine reflection group W. Here, any element of W acts as an affine linear transformation on E with gradient w in the finite reflection group

$$W_0 = \langle w_a, w_b \mid w_a^2 = w_b^2 = (w_a w_b)^4 = 1 \rangle$$

and intercept μ in the coroot lattice $\mathbb{Z}R^\vee = \mathbb{Z}a^\vee \oplus \mathbb{Z}b^\vee$ [17]. In particular, given $w \in W_0$ and $\mu \in \mathbb{Z}R^\vee$, the element $\mu w \in W$ acts on any affine k-root via

$$[\mu w]\psi = w\dot{\psi} + \langle w\dot{\psi}, \mu \rangle + \psi(0) \in \Psi$$

for all $\psi \in \Psi$.

We now make the additional assumption that $\#\mathsf{f} = 2$, and let λ be the barycenter of the fundamental alcove $\mathcal{C}$. In (3.1.10) we saw that the abelianization of $\mathsf{H}_\lambda = G_{\lambda+}/G_{\lambda,1}$ is isomorphic to

$$\begin{aligned} \mathsf{f}^5 \cong \mathsf{f}_1 \oplus \mathsf{f}_2 \oplus \mathsf{f}_3 \oplus \frac{\mathsf{f}_4 \oplus \mathsf{f}_6}{\langle (xy, x^2y) \mid x, y \in \mathsf{f} \rangle} \oplus \frac{\mathsf{f}_5 \oplus \mathsf{f}_7}{\langle (xy, x^2y) \mid x, y \in \mathsf{f} \rangle} \\ \oplus \frac{\mathsf{f}_8}{\langle xy \mid x, y \in \mathsf{f} \rangle} \end{aligned} \tag{3.2.4}$$

where $\mathsf{f}_i = \mathsf{f}$ is the additive group isomorphic to $\mathsf{U}_i = U_{\psi_i}/U_{\psi_i} \cap G_{\lambda,1}$. Let $\chi_i : \mathsf{f}_i \to \mathbb{C}^\times$ be the restriction to f_i of χ for each $i = 1, \ldots, 8$. From (3.2.4), we can conclude the following relations between the χ_i:

$$\begin{cases} 1 = \chi_4(xy) \cdot \chi_6(xy^2) \\ 1 = \chi_5(xy) \cdot \chi_7(xy^2) \\ 1 = \chi_8(xy) \end{cases} \tag{3.2.5}$$

for all $x, y \in \mathsf{f}$. Applying Lemma 3.1.10 to the first two of these relations implies that $\chi_6(1) = \chi_4(1)$ and $\chi_7(1) = \chi_5(1)$. The third relation of (3.2.5) implies that $\chi_8(1) = 1$. Thus, each of the 32 λ-shallow characters of H_λ is uniquely determined by a 5-tuple

$$(\chi_1(1), \chi_2(1), \chi_3(1), \chi_4(1), \chi_5(1)) \in \{\pm 1\}^5$$

In what follows, we will say that χ is *given by* its corresponding 5-tuple.

Each λ-shallow character $\chi : \mathsf{H}_\lambda \to \mathbb{C}^\times$ lifts to a group homomorphism $G_{\lambda+} \to \mathbb{C}^\times$, which through an abuse of notation we will also denote by χ. Of these 32 λ-shallow characters, exactly 5 satisfy

$$\mathscr{I}(G, G_{\lambda+}, \chi) = G_\lambda(\chi). \tag{3.2.6}$$

For the 27 λ-shallow characters for which (3.2.6) does not hold, there exists at least one $n \in N$ not in G_λ such that ${}^n\chi = \chi$ on ${}^nG_\lambda \cap G_\lambda$; this is summarized in Table B.7 in Appendix B.

There is exactly one λ-shallow character of H_λ that satisfies condition $(*)$ in Theorem 3.2.3, and it is given by the 5-tuple

$$(\chi_1(1), \chi_2(1), \chi_3(1), \chi_4(1), \chi_5(1)) = (-1, -1, -1, +1, +1).$$

The corresponding one-dimensional representation of $G_{\lambda+}$ is an **affine generic character** discussed first by Gross and Reeder in [12] and again by Reeder and Yu in [27]. The irreducible supercuspidal representations $\pi(\chi; \rho)$ in (3.2.1) are called the **simple supercuspidal representations** of G.

The remaining four λ-shallow characters for which (3.2.6) holds but not condition $(*)$

of Theorem 3.2.3 are given by 5-tuples

$$(\chi_1(1), \chi_2(1), \chi_3(1), \chi_4(1), \chi_5(1)) \in \begin{Bmatrix} (-1, +1, +1, -1, -1), \\ (+1, +1, -1, -1, -1), \\ (-1, -1, +1, -1, -1), \\ (+1, -1, -1, -1, -1) \end{Bmatrix}. \tag{3.2.7}$$

The justification that each of these λ-shallow characters satisfies (3.2.6) is essentially the same, and thus we illustrate it for only a single example:

Example 3.2.1. Let χ be the λ-shallow character of H_λ given by the 5-tuple

$$(\chi_1(1), \chi_2(1), \chi_3(1), \chi_4(1), \chi_5(1)) = (-1, -1, +1, -1, -1),$$

and note that the following facts hold:

- If α is a short affine k-root then $n\alpha$ is also short for all $n \in N$.
- The only short affine k-roots ψ for which the restriction to U_ψ of χ is non-trivial are ψ_2, ψ_4, and ψ_5. The only other λ-shallow affine k-roots for which the restriction to U_ψ of χ is non-trivial are the long ψ_1, ψ_6, and ψ_7, and the restriction to U_ψ of χ is trivial for all affine k-roots ψ that are not λ-shallow.
- For any $n \in N$, either $n\psi_4$ or $n\psi_5$ is a positive affine k-root. Indeed, $\psi_4 = -(a+b)+1$ and $\psi_5 = a + b + 0$, and so one of

  $$\begin{cases} [\mu w]\psi_4 = -w(a+b) - \langle w(a+b), \mu\rangle + 1 \\ [\mu w]\psi_5 = +w(a+b) + \langle w(a+b), \mu\rangle \end{cases}$$

  must be positive for any $\mu w \in W$. Alternatively, one can note that the vanishing hyperplanes of ψ_4 and ψ_5 are parallel with the fundamental alcove $\mathcal{C}$ between them; therefore, there does not exist an alcove in $X_*(\mathbf{S}, k) \otimes_{\mathbb{Z}} \mathbb{R}$ that is separated from $\mathcal{C}$ by both hyperplanes.

Consequently, for any $n \in N$, ${}^n\chi$ and χ can only agree on ${}^nG_{\lambda+} \cap G_{\lambda+}$ if either n fixes both ψ_4 and ψ_5, n permutes ψ_4 and ψ_5, or either $n\psi_4 = \psi_2$ or $n\psi_5 = \psi_2$. If $n \in N$ fixes both ψ_4 and ψ_5 then either

$$\begin{cases} n\psi_1 = \psi_1 - 2m \\ n\psi_7 = \psi_7 + 2m \end{cases} \quad \text{or} \quad \begin{cases} n\psi_1 = \psi_6 - 2m \\ n\psi_7 = \psi_3 + 2m \end{cases}$$

holds for some $m \in \mathbb{Z}$. If $n \in N$ permutes ψ_4 and ψ_5 then either

$$\begin{cases} n\psi_1 = \psi_7 - 2m + 1 \\ n\psi_6 = \psi_3 + 2m + 1 \end{cases} \quad \text{or} \quad \begin{cases} n\psi_1 = \psi_3 - 2m + 1 \\ n\psi_6 = \psi_7 + 2m + 1 \end{cases}$$

holds for some $m \in \mathbb{Z}$. If $n\psi_4 = \psi_2$ then $n\psi_6 = \psi_8$, and if $n\psi_5 = \psi_2$ then $n\psi_4 = \psi_8$. For all n acting non-trivially on Ψ, we have given a λ-shallow affine root ψ_i such that $\psi_i(1) = -1$ while χ is trivial on $U_{n\psi_i} \subseteq G_{\lambda+}$. Thus,

$${}^n\chi \neq \chi \text{ on } U_{n\psi_i} \subseteq {}^nG_{\lambda+} \cap G_{\lambda+}$$

for some λ-shallow ψ_i whenever $n \notin G_\lambda$. Hence $\mathscr{I}(G, G_{\lambda+}, \chi) = G_\lambda(\chi)$.

Remark. The supercuspidal representations $\pi(\chi;\rho)$ for λ-shallow characters given by 5-tuples in (3.2.7) are *new* in the sense that they cannot be constructed from the methods of Reeder and Yu [27] because the minimal Moy-Prasad subgroup of G_λ on which χ is non-vanishing is $G_{\lambda,3/4} \subsetneq G_{\lambda++}$. Nor can they be constructed using the exhaustive methods of Stevens [33] or Yu [36], as the residual characteristic is 2.

3.2.4 New supercuspidal representations of G_2 when $\mathrm{char}(\mathsf{f}) = 3$

In this subsection, we will continue the notation laid out in §3.1.3 for G the split Chevalley group of type G_2 over k, including simple affine k-roots given in (3.1.11) and the enumeration $\psi_1, \ldots, \psi_{12}$ of positive affine k-roots given in (3.1.12). Since G is simply connected, its affine Weyl group by which N acts on Ψ is isomorphic to the affine reflection group W. Here, any element of W acts as an affine linear transformation on E with gradient w in the finite reflection group

$$W_0 = \langle w_a, w_b \mid w_a^2 = w_b^2 = (w_a w_b)^6 = 1\rangle$$

and intercept μ in the coroot lattice $\mathbb{Z}R^\vee = \mathbb{Z}a^\vee \oplus \mathbb{Z}b^\vee$ [17]. In particular, given $w \in W_0$ and $\mu \in \mathbb{Z}R^\vee$, the element $\mu w \in W$ acts on any affine k-root via

$$[\mu w]\psi = w\dot{\psi} + \langle w\dot{\psi}, \mu\rangle + \psi(0) \in \Psi$$

for all $\psi \in \Psi$.

We now make the additional assumption that $\mathrm{char}(\mathsf{f}) = 3$, and let λ be the barycenter of the facet of the fundamental alcove corresponding to the vanishing of β and non-vanishing of α and γ. In (3.1.13) we saw that the abelianization of $\mathsf{H}_\lambda = G_{\lambda+}/G_{\lambda,1}$ is isomorphic to

$$\mathsf{f}_1 \oplus \mathsf{f}_3 \oplus \mathsf{f}_4 \oplus \mathsf{f}_5 \oplus \frac{\mathsf{f}_7 \oplus \mathsf{f}_9 \oplus \mathsf{f}_{11}}{\langle(-2xy, 3x^2y, -3xy), (0, 3xy, 0), (0, 0, -3xy) \mid x, y \in \mathsf{f}\rangle}$$

$$\oplus \frac{\mathsf{f}_6 \oplus \mathsf{f}_8 \oplus \mathsf{f}_{10} \oplus \mathsf{f}_{12}}{\left\langle \begin{array}{c} (0,0,-xy,0), (0,2xy,3x^2y,0), (0,0,0,-2xy), (0,0,-3xy,0), \\ (xy,0,0,x^2y), (0,-xy,0,0), (-xy,x^2y,x^3y,0), (0,0,0,\pm xy) \end{array} \middle| \; x, y \in \mathsf{f} \right\rangle}$$

where $\mathsf{f}_i = \mathsf{f}$ is the additive group isomorphic to $\mathsf{U}_i = U_{\psi_i}/U_{\psi_i} \cap G_{\lambda,1}$. After some simple reductions, this can be rewritten as

$$\mathsf{f}^6 \cong \mathsf{f}_1 \oplus \mathsf{f}_3 \oplus \mathsf{f}_4 \oplus \mathsf{f}_5 \oplus \mathsf{f}_9 \oplus \mathsf{f}_{11} \tag{3.2.8}$$

Let $\chi_i : \mathsf{f}_i \to \mathbb{C}^\times$ be the restriction to f_i of χ for each $i = 1, \ldots, 8$. From (3.2.8), we see that there are $(\#\mathsf{f})^6$ λ-shallow characters of H_λ, each uniquely determined by 6 characters

$$\chi_1, \chi_3, \chi_4, \chi_5, \chi_9, \chi_{11} : \mathsf{f} \to \mathbb{C}^\times$$

on which there are no restrictions.

As we saw in the previous subsection, each λ-shallow character $\chi : \mathsf{H}_\lambda \to \mathbb{C}^\times$ lifts to a group homomorphism $G_{\lambda+} \to \mathbb{C}^\times$, which through an abuse of notation we will also denote by χ. In order to completely classify which λ-shallow characters satisfy

$$\mathscr{I}(G, G_{\lambda+}, \chi) = G_\lambda(\chi) \tag{3.2.9}$$

a case-by-case approach may be necessary, but we now give a class of $(\#\mathsf{f})^4$ λ-shallow characters satisfying (3.2.9).

Example 3.2.2. Consider any shallow character $\chi : \mathsf{H}_\lambda \to \mathbb{C}^\times$ where the restrictions χ_1, χ_{11} are trivial and $\chi_3, \chi_4, \chi_5, \chi_9$ are non-trivial, and note that the following facts hold:

- If α is a long affine k-root, then $n\alpha$ is also long for all $n \in N$.
- The only long affine k-roots ψ for which the restriction to U_ψ of χ is non-trivial are ψ_4 and ψ_9. The only other λ-shallow affine k-roots for which the restriction to U_ψ of χ is non-trivial are the short ψ_3 and ψ_5, and the restriction to U_ψ of χ is trivial for all affine k-roots that are not λ-shallow.
- For any $n \in N$, either $n\psi_4$ or $n\psi_9$ is positive. Indeed, $\psi_4 = -(3a+b)+1$ and $\psi_9 = 3a+b+0$, and so one of

  $$\begin{cases} [\mu w]\psi_4 = -w(3a+b) - \langle w(3a+b), \mu\rangle + 1 \\ [\mu w]\psi_9 = +w(3a+b) + \langle w(3a+b), \mu\rangle \end{cases}$$

  must be positive for any $\mu w \in W$. Alternatively, one can note that the vanishing hyperplanes of ψ_4 and ψ_9 are parallel with the fundamental alcove $\mathcal{C}$ between them; therefore, there does not exist an alcove in $X_*(\mathbf{S}, k) \otimes_{\mathbb{Z}} \mathbb{R}$ that is separated from $\mathcal{C}$ by both hyperplanes.

Consequently, for any $n \in N$, ${}^n\chi$ and χ can only agree on ${}^nG_{\lambda+} \cap G_{\lambda+}$ if either n fixes both ψ_4 and ψ_9 or n permutes ψ_4 and ψ_9. If $n \in N$ fixes both ψ_4 and ψ_9, then either

$$\begin{cases} n\psi_3 = \psi_3 - m \\ n\psi_5 = \psi_5 + 2m \end{cases} \quad \text{or} \quad \begin{cases} n\psi_3 = \psi_7 - m \\ n\psi_5 = \psi_8 + 2m - 1 \end{cases}$$

holds for some $m \in \mathbb{Z}$. If $n \in N$ permutes ψ_4 and ψ_9, then either

$$\begin{cases} n\psi_3 = \psi_6 - m \\ n\psi_5 = \psi_5 + 2m - 1 \end{cases} \quad \text{or} \quad \begin{cases} n\psi_3 = \psi_{12} - m \\ n\psi_5 = \psi_8 + 2m - 2 \end{cases}$$

for some $m \in \mathbb{Z}$. For all n acting non-trivially on Ψ, we have given a λ-shallow affine root ψ_i such that ψ_i is non-trivial but χ is trivial on $U_{n\psi_i} \subseteq G_{\lambda+}$. Thus,

$${}^n\chi \neq \chi \text{ on } U_{n\psi_i} \subseteq {}^nG_{\lambda+} \cap G_{\lambda+}$$

for some λ-shallow ψ_i whenever $n \notin G_\lambda$. Hence $\mathscr{I}(G, G_{\lambda+}, \chi) = G_\lambda(\chi)$.

Remark. The supercuspidal representations $\pi(\chi;\rho)$ for λ-shallow characters given in Example 3.2.2 are *new* in the sense that they cannot be constructed from the methods of Reeder and Yu [27] because the minimal Moy-Prasad subgroup of G_λ on which χ is non-vanishing is $G_{\lambda,3/4} \subsetneq G_{\lambda++}$. Nor can they be constructed using the exhaustive methods of Stevens [33] and Yu [36], as G is not a classical group and the residual characteristic is 3.

Appendix A

Commutator Computations

In this appendix we will be adopting the notation of Section 2.2 in order to verify the formulas given in Proposition 2.2.4. For fixed k-roots a, b we will look at the group $\mathbf{G}_{a,b}$ generated by $\mathbf{U}_{\pm a}$ and $\mathbf{U}_{\pm b}$, defined and quasi-split over k with k-rank 2. Keeping with Proposition 2.2.4, we assume that $\mathbf{G}_{a,b}$ is non-split over k.

In each of the following subsections, we will either prove the corresponding case in Proposition 2.2.4 directly or by fixing a group isogenous to $\mathbf{G}_{a,b}$ and compute all possibilities for commutators in $[U_a, U_b]$.

A.1 Non-split group with k-root system of type A_2

If $\mathbf{G}_{a,b}$ is non-split with a k-root system of type A_2, then $a + b$ is a k-root and both (a) and (b) have cycle type (i) as in Definition 2.2.1 with $e_a = e_b = e$. In this case, there exist K-roots $\mathbf{a} \in (a)$ and $\mathbf{b} \in (b)$ such that $\sigma^i\mathbf{a} + \sigma^j\mathbf{b}$ is a K-root if and only if $i = j$, and so

$$[u_{\sigma^i\mathbf{a}}(x), u_{\sigma^j\mathbf{b}}(y)] = \begin{cases} u_{\sigma^i\mathbf{a}+\sigma^j\mathbf{b}}(\pm xy) & \text{if } i = j, \\ 0 & \text{if else.} \end{cases}$$

for all $x, y \in K$. We now directly compute

$$\begin{aligned}\left[\prod_{i=1}^{e} u_{\mathbf{a}}(x)^{\sigma^i}, \prod_{j=1}^{e} u_{\mathbf{b}}(y)^{\sigma^j}\right] &= \prod_{i=1}^{e}[u_{\mathbf{a}}(x)^{\sigma^i}, u_{\mathbf{b}}(y)^{\sigma^i}] \\ &= \prod_{i=1}^{e}[u_{\mathbf{a}}(x), u_{\mathbf{b}}(y)]^{\sigma^i} \\ &= \prod_{i=1}^{e} u_{\mathbf{a}+\mathbf{b}}(\pm xy)^{\sigma^i}\end{aligned}$$

for all $x \in K_a$ and $y \in K_b$. Up to a choice of k-root morphisms, this is what Proposition 2.2.4(1) claims.

A.2 Non-split group with k-root system of type $B_2 = C_2$

Let K/k be a tamely, purely ramified quadratic Galois extension of k, and $x \mapsto \bar{x}$ is a cyclic generator of the Galois group $\mathrm{Gal}(K/k)$. Suppose that $\mathbf{G}_{a,b}$ has K-structure isogenous to $\mathrm{SL}_4(K)$ with Galois action given by a non-trivial involution of the Dynkin diagram of type A_3.

(a) For all $x, y \in K$ we have the following commutators:

$$\left[\begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \bar{x} \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & y & \\ & 1 & 0 & -\bar{y} \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 & 0 & -\bar{x}y - x\bar{y} \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}$$

$$\left[\begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \bar{x} \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ y & 0 & 1 & \\ & -\bar{y} & 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & -xy - \bar{x}\bar{y} & 1 & \\ & & & 1 \end{pmatrix}$$

$$\left[\begin{pmatrix} 1 & & & \\ x & 1 & & \\ & & 1 & \\ & & \bar{x} & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & y & \\ & 1 & 0 & -\bar{y} \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & & & \\ & 1 & xy + \bar{x}\bar{y} & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$\left[\begin{pmatrix} 1 & & & \\ x & 1 & & \\ & & 1 & \\ & & \bar{x} & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ y & 0 & 1 & \\ & -\bar{y} & 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ \bar{x}y + x\bar{y} & 0 & 0 & 1 \end{pmatrix}$$

(b) For all $x \in K$ and $y \in k$ we have the following commutators:

$$\left[\begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \bar{x} \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & 1 & y & \\ & & 1 & \\ & & & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 & xy & x\bar{x}y \\ & 1 & 0 & -\bar{x}y \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}$$

$$\left[\begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \bar{x} \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ y & 0 & 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ \bar{x}y & x\bar{x}y & 1 & \\ & -xy & 0 & 1 \end{pmatrix}$$

$$\left[\begin{pmatrix} 1 & 0 & x & \\ & 1 & 0 & -\bar{x} \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ y & 0 & 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & & & \\ -\bar{x}y & 1 & -x\bar{x}y & \\ & & 1 & \\ & & -xy & 1 \end{pmatrix}$$

$$\left[\begin{pmatrix} 1 & 0 & x & \\ & 1 & 0 & -\bar{x} \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & 1 & & \\ & y & 1 & \\ & & & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & xy & 0 & -x\bar{x}y \\ & 1 & 0 & 0 \\ & & 1 & \bar{x}y \\ & & & 1 \end{pmatrix}$$

$$\left[\begin{pmatrix} 1 & & & \\ x & 1 & & \\ & & 1 & \\ & & \bar{x} & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & y \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 & -\bar{x}y & \\ & 1 & x\bar{x}y & xy \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}$$

$$\left[\begin{pmatrix} 1 & & & \\ x & 1 & & \\ & & 1 & \\ & & x & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & 1 & & \\ & y & 1 & \\ & & & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ -xy & 0 & 1 & \\ x\bar{x}y & \bar{x}y & 0 & 1 \end{pmatrix}$$

$$\left[\begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ x & 0 & 1 & \\ & -\bar{x} & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & y \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & \bar{x}y & & \\ & 1 & & \\ & -x\bar{x}y & 1 & xy \\ & & & 1 \end{pmatrix}$$

$$\left[\begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ x & 0 & 1 & \\ & -\bar{x} & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & 1 & y & \\ & & 1 & \\ & & & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & & & \\ -xy & 1 & & \\ 0 & 0 & 1 & \\ -x\bar{x}y & 0 & -\bar{x}y & 1 \end{pmatrix}$$

A.3 Non-split group with k-root system of type G_2

Let K/k be a tamely, purely ramified degree-3 Galois extension of k, and $x \mapsto x'$ is a cyclic generator of the Galois group $\mathrm{Gal}(K/k)$. Suppose that $\mathbf{G}_{a,b}$ has K-structure isogenous to $\mathrm{SO}_8(K)$ with Galois action given by a non-trivial triality of the Dynkin diagram of type D_4. Note that an in-depth discussion on this group and its structure can be found in §3.1.3.

(a) For all $x, y \in K$ we have the following commutators:

$$
\left[\begin{bmatrix}
1 & x & & & & & & \\
 & 1 & & & & & & \\
 & & 1 & x' & x'' & -x'x'' & & \\
 & & & 1 & 0 & -x'' & & \\
 & & & & 1 & -x' & & \\
 & & & & & 1 & & \\
 & & & & & & 1 & -x \\
 & & & & & & & 1
\end{bmatrix},
\begin{bmatrix}
1 & 0 & y & & & & & \\
 & 1 & 0 & -y' & -y'' & 0 & -y'y'' & \\
 & & 1 & 0 & 0 & 0 & 0 & \\
 & & & 1 & 0 & 0 & y'' & \\
 & & & & 1 & 0 & y' & \\
 & & & & & 1 & 0 & -y \\
 & & & & & & 1 & 0 \\
 & & & & & & & 1
\end{bmatrix}\right]
$$

$$
= \begin{bmatrix}
1 & 0 & 0 & -xy' - x'y & -xy'' - x''y & x'x''y + xx''y' + xx'y'' & -xy'y'' - x'yy'' - x''yy' & -(xy' + x'y)(xy'' + x''y) \\
 & 1 & 0 & 0 & 0 & -x'y'' - x''y' & 0 & xy'y'' + x'yy'' + x''yy' \\
 & & 1 & 0 & 0 & 0 & x'y'' + x''y' & -x'x''y - xx''y' - xx'y'' \\
 & & & 1 & 0 & 0 & 0 & xy'' + x''y \\
 & & & & 1 & 0 & 0 & xy' + x'y \\
 & & & & & 1 & 0 & 0 \\
 & & & & & & 1 & 0 \\
 & & & & & & & 1
\end{bmatrix}
$$

$$\left[\begin{bmatrix} 1 & x & & & & & & \\ & 1 & & & & & & \\ & & 1 & x' & x'' & -x'x'' & & \\ & & & 1 & 0 & -x'' & & \\ & & & & 1 & -x' & & \\ & & & & & 1 & & \\ & & & & & & 1 & -x \\ & & & & & & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ 0 & 0 & 1 & & & & & \\ y & 0 & 0 & 1 & & & & \\ y'' & 0 & 0 & 0 & 1 & & & \\ 0 & y' & 0 & 0 & 0 & 1 & & \\ 0 & 0 & -y' & 0 & 0 & 0 & 1 & \\ -yy'' & 0 & 0 & -y'' & y & 0 & 0 & 1 \end{bmatrix}\right]$$

$$= \begin{bmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ x'y + x''y'' & xx'y + x'x''y' + xx''y'' & 1 & & & & & \\ 0 & -xy - x''y' & 0 & 1 & & & & \\ 0 & -xy'' - x'y' & 0 & 0 & 1 & & & \\ 0 & 0 & 0 & 0 & 0 & 1 & & \\ xyy'' + x'yy' + x''y'y'' & -(xy + x''y')(xy'' + x'y') & 0 & xy'' + x'y' & xy + x''y' & -xx'y - x'x''y' - xx''y'' & 1 & \\ 0 & -xyy'' - x'yy' - x''y'y'' & 0 & 0 & 0 & -x'y - x''y'' & 0 & 1 \end{bmatrix}$$

$$
\left[\begin{bmatrix}
1 & 0 & x & & & & & \\
 & 1 & 0 & -x' & -x'' & 0 & -x'x'' & \\
 & & 1 & 0 & 0 & 0 & 0 & \\
 & & & 1 & 0 & 0 & x'' & \\
 & & & & 1 & 0 & x' & \\
 & & & & & 1 & 0 & x \\
 & & & & & & 1 & 0 \\
 & & & & & & & 1
\end{bmatrix},
\begin{bmatrix}
1 & & & & & & & \\
0 & 1 & & & & & & \\
0 & 0 & 1 & & & & & \\
y & 0 & 0 & 1 & & & & \\
y'' & 0 & 0 & 0 & 1 & & & \\
0 & y' & 0 & 0 & 0 & 1 & & \\
0 & 0 & -y' & 0 & 0 & 0 & 1 & \\
-yy'' & 0 & 0 & -y'' & y & 0 & 0 & 1
\end{bmatrix}\right]
$$

$$
= \begin{bmatrix}
1 & & & & & & & \\
-x'y - x''y'' & 1 & -xx'y - x'x''y' - xx''y'' & & & & & \\
0 & 0 & 1 & & & & & \\
0 & 0 & -xy - y'x'' & 1 & & & & \\
0 & 0 & -xy'' - x'y' & 0 & 1 & & & \\
xyy'' + x'yy' + x''y'y'' & 0 & -(xy + x''y')(xy'' + x'y') & xy'' + x'y' & xy + y'x'' & 1 & xx'y + x'x''y' + xx''y'' & \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & \\
0 & 0 & -xyy'' - x'yy' - x''y'y'' & 0 & 0 & 0 & x'y + x''y'' & 1
\end{bmatrix}
$$

$$\left[\begin{bmatrix} 1 & 0 & 0 & x & x'' & 0 & 0 & -xx'' \\ & 1 & 0 & 0 & 0 & x' & 0 & 0 \\ & & 1 & 0 & 0 & 0 & -x' & 0 \\ & & & 1 & 0 & 0 & 0 & -x'' \\ & & & & 1 & 0 & 0 & -x \\ & & & & & 1 & 0 & 0 \\ & & & & & & 1 & 0 \\ & & & & & & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & & & & & & \\ y & 1 & & & & & & \\ & & 1 & & & & & \\ & & y' & 1 & & & & \\ & & y'' & 0 & 1 & & & \\ & & -y'y'' & -y'' & -y' & 1 & & \\ & & & & & & 1 & \\ & & & & & & -y & 1 \end{bmatrix}\right]$$

$$= \begin{bmatrix} 1 & 0 & xy'+x''y'' & 0 & 0 & 0 & -xx''y-xx'y'-x'x''y'' & 0 \\ & 1 & -xyy'-x'y'y''-x''yy'' & -xy-x'y'' & -x'y'-x''y & 0 & -(x'y'+x''y)(xy+x'y'') & xx''y+xx'y'+x'x''y'' \\ & & 1 & 0 & 0 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 & x'y'+yx'' & 0 \\ & & & & 1 & 0 & xy+x'y'' & 0 \\ & & & & & 1 & xyy'+x'y'y''+x''yy'' & -xy'-x''y'' \\ & & & & & & 1 & 0 \\ & & & & & & & 1 \end{bmatrix}$$

$$\left[\begin{bmatrix}1&0&0&x&x''&0&0&-xx''\\ &1&0&0&0&x'&0&0\\ &&1&0&0&0&-x'&0\\ &&&1&0&0&0&-x''\\ &&&&1&0&0&-x\\ &&&&&1&0&0\\ &&&&&&1&0\\ &&&&&&&1\end{bmatrix},\begin{bmatrix}1&&&&&&&\\ 0&1&&&&&&\\ y&0&1&&&&&\\ &-y'&0&1&&&&\\ &-y''&0&0&1&&&\\ &0&0&0&0&1&&\\ &-y'y''&0&y''&y'&0&1&\\ &&&&&-y&0&1\end{bmatrix}\right]$$

$$=\begin{bmatrix}1&-xy'-x''y''&0&0&0&-xx''y-xx'y'-x'x''y''&0&0\\ &1&0&0&0&0&0&0\\ &xyy'+x'y'y''+x''yy''&1&-xy-x'y''&-x'y'-x''y&-(x'y'+x''y)(xy+x'y'')&0&xx''y+xx'y'+x'x''y''\\ &&&1&0&x'y'+x''y&0&0\\ &&&&1&xy+x'y''&0&0\\ &&&&&1&0&0\\ &&&&&-xyy'-x'y'y''-x''yy''&1&xy'+x''y''\\ &&&&&&&1\end{bmatrix}$$

$$\left[\begin{bmatrix} 1 &&&&&&& \\ x & 1 &&&&&& \\ && 1 &&&&& \\ && x' & 1 &&&& \\ && x'' & 0 & 1 &&& \\ && -x'x'' & -x'' & -x' & 1 && \\ &&&&&& 1 & \\ &&&&&& -x & 1 \end{bmatrix}, \begin{bmatrix} 1 &&&&&&& \\ 0 & 1 &&&&&& \\ y & 0 & 1 &&&&& \\ & -y' & 0 & 1 &&&& \\ & -y'' & 0 & 0 & 1 &&& \\ & 0 & 0 & 0 & 0 & 1 && \\ & -y'y'' & 0 & y'' & y' & 0 & 1 & \\ &&&&& -y & 0 & 1 \end{bmatrix}\right]$$

$$= \begin{bmatrix} 1 &&&&&&& \\ 0 & 1 &&&&&& \\ 0 & 0 & 1 &&&&& \\ xy'+yx' & 0 & 0 & 1 &&&& \\ xy''+x''y & 0 & 0 & 0 & 1 &&& \\ x'x''y+xx''y'+xx'y'' & x'y''+x''y' & 0 & 0 & 0 & 1 && \\ -xy'y''-x'yy''-x''yy' & 0 & -x'y''-x''y' & 0 & 0 & 0 & 1 & \\ -(xy'+x'y)(xy''+x''y) & xy'y''+x'yy''+x''yy' & -x'x''y-xx''y'-xx'y'' & -xy''-x''y & -xy'-x'y & 0 & 0 & 1 \end{bmatrix}$$

(b) For $x, y \in K$ we have the following commutators:

$$\left[\begin{bmatrix} 1 & x & & & & & & \\ & 1 & & & & & & \\ & & 1 & x' & x'' & -x'x'' & & \\ & & & 1 & 0 & -x'' & & \\ & & & & 1 & -x' & & \\ & & & & & 1 & & \\ & & & & & & 1 & -x \\ & & & & & & & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & y & y'' & 0 & 0 & -yy'' \\ & 1 & 0 & 0 & 0 & y' & 0 & 0 \\ & & 1 & 0 & 0 & 0 & -y' & 0 \\ & & & 1 & 0 & 0 & 0 & -y'' \\ & & & & 1 & 0 & 0 & -y \\ & & & & & 1 & 0 & 0 \\ & & & & & & 1 & 0 \\ & & & & & & & 1 \end{bmatrix}\right]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & xy' + x'y'' + x''y & & \\ & 1 & 0 & 0 & 0 & 0 & & \\ & & 1 & 0 & 0 & 0 & 0 & -xy' - x'y'' - x''y \\ & & & 1 & 0 & 0 & 0 & 0 \\ & & & & 1 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & 1 & 0 \\ & & & & & & & 1 \end{bmatrix}$$

$$
\left[\begin{bmatrix}
1 & x & & & & & & \\
 & 1 & & & & & & \\
 & & 1 & x' & x'' & -x'x'' & & \\
 & & & 1 & 0 & -x'' & & \\
 & & & & 1 & -x' & & \\
 & & & & & 1 & & \\
 & & & & & & 1 & -x \\
 & & & & & & & 1
\end{bmatrix},
\begin{bmatrix}
1 & & & & & & & \\
0 & 1 & & & & & & \\
y & 0 & 1 & & & & & \\
 & -y' & 0 & 1 & & & & \\
 & -y'' & 0 & 0 & 1 & & & \\
 & 0 & 0 & 0 & 0 & 1 & & \\
 & -y'y'' & 0 & y'' & y' & 0 & 1 & \\
 & & & & & -y & 0 & 1
\end{bmatrix}\right]
$$

$$
= \begin{bmatrix}
1 & & & & & & & \\
 & 1 & & & & & & \\
 & -xy - x'y' - x''y'' & 1 & & & & & \\
 & & & 1 & & & & \\
 & & & & 1 & & & \\
 & & & & & 1 & & \\
 & & & & & xy + x'y' + x''y'' & 1 & \\
 & & & & & & & 1
\end{bmatrix}
$$

$$\left[\begin{bmatrix} 1 & 0 & x & & & & & \\ & 1 & 0 & -x' & -x'' & 0 & -x'x'' & \\ & & 1 & 0 & 0 & 0 & 0 & \\ & & & 1 & 0 & 0 & x'' & \\ & & & & 1 & 0 & x' & \\ & & & & & 1 & 0 & x \\ & & & & & & 1 & 0 \\ & & & & & & & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & y & y'' & 0 & 0 & -yy'' \\ & 1 & 0 & 0 & 0 & y' & 0 & 0 \\ & & 1 & 0 & 0 & 0 & -y' & 0 \\ & & & 1 & 0 & 0 & 0 & -y'' \\ & & & & 1 & 0 & 0 & -y \\ & & & & & 1 & 0 & 0 \\ & & & & & & 1 & 0 \\ & & & & & & & 1 \end{bmatrix}\right]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -xy' - x'y'' - x''y & \\ & 1 & 0 & 0 & 0 & 0 & 0 & xy' + x'y'' + x''y \\ & & 1 & 0 & 0 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 & 0 & 0 \\ & & & & 1 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & 1 & 0 \\ & & & & & & & 1 \end{bmatrix}$$

$$
\left[\begin{bmatrix}
1 & 0 & x & & & & & \\
 & 1 & 0 & -x' & -x'' & 0 & -x'x'' & \\
 & & 1 & 0 & 0 & 0 & 0 & \\
 & & & 1 & 0 & 0 & x'' & \\
 & & & & 1 & 0 & x' & \\
 & & & & & 1 & 0 & x \\
 & & & & & & 1 & 0 \\
 & & & & & & & 1
\end{bmatrix},
\begin{bmatrix}
1 & & & & & & & \\
y & 1 & & & & & & \\
 & & 1 & & & & & \\
 & & y' & 1 & & & & \\
 & & y'' & 0 & 1 & & & \\
 & & -y'y'' & -y'' & -y' & 1 & & \\
 & & & & & & 1 & \\
 & & & & & & -y & 1
\end{bmatrix}\right]
$$

$$
= \begin{bmatrix}
1 & & & & & & & \\
 & 1 & -xy - x'y' - x''y'' & & & & & \\
 & & 1 & & & & & \\
 & & & 1 & & & & \\
 & & & & 1 & & & \\
 & & & & & 1 & xy + x'y' + x''y'' & \\
 & & & & & & 1 & \\
 & & & & & & & 1
\end{bmatrix}
$$

$$\left[\begin{bmatrix} 1 & & & & & & & \\ x & 1 & & & & & & \\ & & 1 & & & & & \\ & & x' & 1 & & & & \\ & & x'' & 0 & 1 & & & \\ & & -x'x'' & -x'' & -x' & 1 & & \\ & & & & & & 1 & \\ & & & & & & -x & 1 \end{bmatrix}, \begin{bmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ 0 & 0 & 1 & & & & & \\ y & 0 & 0 & 1 & & & & \\ y'' & 0 & 0 & 0 & 1 & & & \\ 0 & y' & 0 & 0 & 0 & 1 & & \\ 0 & 0 & -y' & 0 & 0 & 0 & 1 & \\ -yy'' & 0 & 0 & -y'' & y & 0 & 0 & 1 \end{bmatrix}\right]$$

$$= \begin{bmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ 0 & 0 & 1 & & & & & \\ 0 & 0 & 0 & 1 & & & & \\ 0 & 0 & 0 & 0 & 1 & & & \\ -xy' - x'y'' - x''y & 0 & 0 & 0 & 0 & 1 & & \\ & & 0 & 0 & 0 & 0 & 1 & \\ & & xy' + x'y'' + x''y & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{bmatrix} 1 &&&&&&& \\ 0 & 1 &&&&&& \\ x & 0 & 1 &&&&& \\ & -x' & 0 & 1 &&&& \\ & -x'' & 0 & 0 & 1 &&& \\ & 0 & 0 & 0 & 0 & 1 && \\ & -x'x'' & 0 & x'' & x' & 0 & 1 & \\ &&&&& -x & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 &&&&&&& \\ 0 & 1 &&&&&& \\ 0 & 0 & 1 &&&&& \\ y & 0 & 0 & 1 &&&& \\ y'' & 0 & 0 & 0 & 1 &&& \\ 0 & y' & 0 & 0 & 0 & 1 && \\ 0 & 0 & -y' & 0 & 0 & 0 & 1 & \\ -yy'' & 0 & 0 & -y'' & y & 0 & 0 & 1 \end{bmatrix}\right]$$

$$= \begin{bmatrix} 1 &&&&&&& \\ 0 & 1 &&&&&& \\ 0 & 0 & 1 &&&&& \\ 0 & 0 & 0 & 1 &&&& \\ 0 & 0 & 0 & 0 & 1 &&& \\ 0 & 0 & 0 & 0 & 0 & 1 && \\ xy' + x'y'' + x''y & 0 & 0 & 0 & 0 & 0 & 1 & \\ & -xy' - x'y'' - x''y & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) For all $x \in K$ and $y \in k$ we have the following commutators:

$$\left[\begin{bmatrix} 1 & x & & & & & & \\ & 1 & & & & & & \\ & & 1 & x' & x'' & -x'x'' & & \\ & & & 1 & 0 & -x'' & & \\ & & & & 1 & -x' & & \\ & & & & & 1 & & \\ & & & & & & 1 & x \\ & & & & & & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & & & & & & \\ & 1 & y & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & -y & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix}\right] = \begin{bmatrix} 1 & 0 & xy & xx'y & xx''y & -xx'x''y & xx'x''y^2 & 0 \\ & 1 & 0 & -x'y & -x''y & x'x''y & -x'x''y^2 & 0 \\ & & 1 & 0 & 0 & 0 & -x'x''y & xx'x''y \\ & & & 1 & 0 & 0 & x''y & -xx''y \\ & & & & 1 & 0 & x'y & -xx'y \\ & & & & & 1 & 0 & -xy \\ & & & & & & 1 & 0 \\ & & & & & & & 1 \end{bmatrix}$$

$$\left[\begin{bmatrix} 1 & x & & & & & & \\ & 1 & & & & & & \\ & & 1 & x' & x'' & -x'x'' & & \\ & & & 1 & 0 & -x'' & & \\ & & & & 1 & -x' & & \\ & & & & & 1 & & \\ & & & & & & 1 & x \\ & & & & & & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ 0 & 0 & 1 & & & & & \\ 0 & 0 & 0 & 1 & & & & \\ 0 & 0 & 0 & 0 & 1 & & & \\ y & 0 & 0 & 0 & 0 & 1 & & \\ & & 0 & 0 & 0 & 0 & 1 & \\ & & -y & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right] = \begin{bmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ x'x''y & xx'x''y & 1 & & & & & \\ -x''y & -xx''y & 0 & 1 & & & & \\ -x'y & -xx'y & 0 & 0 & 1 & & & \\ 0 & -xy & 0 & 0 & 0 & 1 & & \\ -xx'x''y^2 & 0 & xy & xx'y & xx''y & -xx'x''y & 1 & \\ -x'x''y^2 & 0 & 0 & x'y & x''y & -x'x''y & 0 & 1 \end{bmatrix}$$

$$\left[\begin{bmatrix} 1 & 0 & x & & & & & \\ & 1 & 0 & -x' & -x'' & 0 & -x'x'' & \\ & & 1 & 0 & 0 & 0 & 0 & \\ & & & 1 & 0 & 0 & x'' & \\ & & & & 1 & 0 & x' & \\ & & & & & 1 & 0 & x \\ & & & & & & 1 & 0 \\ & & & & & & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & y & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & -y & 1 & \\ & & & & & & & 1 \end{bmatrix}\right] = \begin{bmatrix} 1 & xy & 0 & -xx'y & -xx''y & xx'x''y^2 & -xx'x''y & 0 \\ & 1 & 0 & 0 & 0 & -x'x''y & 0 & xx'x''y \\ & & 1 & x'y & x''y & -x'x''y^2 & x'x''y & 0 \\ & & & 1 & 0 & -x''y & 0 & xx''y \\ & & & & 1 & -x'y & 0 & xx'y \\ & & & & & 1 & 0 & 0 \\ & & & & & & 1 & -xy \\ & & & & & & & 1 \end{bmatrix}$$

$$\left[\begin{bmatrix} 1 & 0 & x & & & & & \\ & 1 & 0 & -x' & -x'' & 0 & -x'x'' & \\ & & 1 & 0 & 0 & 0 & 0 & \\ & & & 1 & 0 & 0 & x'' & \\ & & & & 1 & 0 & x' & \\ & & & & & 1 & 0 & x \\ & & & & & & 1 & 0 \\ & & & & & & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ 0 & 0 & 1 & & & & & \\ 0 & 0 & 0 & 1 & & & & \\ 0 & 0 & 0 & 0 & 1 & & & \\ 0 & 0 & 0 & 0 & 0 & 1 & & \\ y & 0 & 0 & 0 & 0 & 0 & 1 & \\ & -y & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right] = \begin{bmatrix} 1 & & & & & & & \\ x'x''y & 1 & xx'x''y & & & & & \\ 0 & 0 & 1 & & & & & \\ x''y & 0 & xx''y & 1 & & & & \\ x'y & 0 & xx'y & 0 & 1 & & & \\ -xx'x''y^2 & xy & 0 & -xx'y & -xx''y & 1 & -xx'x''y & \\ 0 & 0 & -xy & 0 & 0 & 0 & 1 & \\ -x'x''y^2 & 0 & 0 & -x'y & -x''y & 0 & -x'x''y & 1 \end{bmatrix}$$

$$\left[\begin{bmatrix} 1 & 0 & 0 & x & x'' & 0 & 0 & -xx'' \\ & 1 & 0 & 0 & 0 & x' & 0 & 0 \\ & & 1 & 0 & 0 & 0 & -x' & 0 \\ & & & 1 & 0 & 0 & 0 & -x'' \\ & & & & 1 & 0 & 0 & -x \\ & & & & & 1 & 0 & 0 \\ & & & & & & 1 & 0 \\ & & & & & & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ 0 & 0 & 1 & & & & & \\ 0 & 0 & 0 & 1 & & & & \\ 0 & 0 & 0 & 0 & 1 & & & \\ y & 0 & 0 & 0 & 0 & 1 & & \\ & & 0 & 0 & 0 & 0 & 1 & \\ & & -y & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right] = \begin{bmatrix} 1 & 0 & -xx''y & 0 & 0 & 0 & xx'x''y & 0 \\ x'y & 1 & xx'x''y^2 & xx'y & x'x''y & 0 & 0 & -xx'x''y \\ & & 1 & 0 & 0 & 0 & 0 & 0 \\ & & x''y & 1 & 0 & 0 & -x'x''y & 0 \\ & & xy & 0 & 1 & 0 & -xx'y & 0 \\ & & -xx''y^2 & -xy & -x''y & 1 & 0 & xx''y \\ & & & & & & 1 & 0 \\ & & & & & & -x'y & 1 \end{bmatrix}$$

$$\left[\begin{bmatrix} 1 & 0 & 0 & x & x'' & 0 & 0 & -xx'' \\ & 1 & 0 & 0 & 0 & x' & 0 & 0 \\ & & 1 & 0 & 0 & 0 & -x' & 0 \\ & & & 1 & 0 & 0 & 0 & -x'' \\ & & & & 1 & 0 & 0 & -x \\ & & & & & 1 & 0 & 0 \\ & & & & & & 1 & 0 \\ & & & & & & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ 0 & 0 & 1 & & & & & \\ 0 & 0 & 0 & 1 & & & & \\ 0 & 0 & 0 & 0 & 1 & & & \\ 0 & 0 & 0 & 0 & 0 & 1 & & \\ y & 0 & 0 & 0 & 0 & 0 & 1 & \\ & -y & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right] = \begin{bmatrix} 1 & -xx''y & 0 & 0 & 0 & -xx'x''y & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -x'y & -xx'x''y^2 & 1 & -xx'y & -x'x''y & 0 & 0 & xx'x''y \\ & x''y & 0 & 1 & 0 & x'x''y & 0 & 0 \\ & xy & 0 & 0 & 1 & xx'y & 0 & 0 \\ & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ & -xx''y^2 & 0 & -xy & -x''y & 0 & 1 & xx''y \\ & & & & & x'y & 0 & 1 \end{bmatrix}$$

$$\left[\begin{bmatrix} 1 &&&&&&& \\ x & 1 &&&&&& \\ && 1 &&&&& \\ && x' & 1 &&&& \\ && x'' & 0 & 1 &&& \\ && -x'x'' & -x'' & -x' & 1 && \\ &&&&&& 1 & \\ &&&&&& -x & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & y && \\ & 1 & 0 & 0 & 0 & 0 && \\ && 1 & 0 & 0 & 0 & 0 & -y \\ &&& 1 & 0 & 0 & 0 & 0 \\ &&&& 1 & 0 & 0 & 0 \\ &&&&& 1 & 0 & 0 \\ &&&&&& 1 & 0 \\ &&&&&&& 1 \end{bmatrix}\right] = \begin{bmatrix} 1 & 0 & x'x''y & x''y & x'y & 0 & 0 & -x'x''y^2 \\ & 1 & -xx'x''y & -xx''y & -xx'y & xy & 0 & xx'x''y^2 \\ && 1 & 0 & 0 & 0 & -xy & 0 \\ &&& 1 & 0 & 0 & xx'y & -x'y \\ &&&& 1 & 0 & xx''y & -x''y \\ &&&&& 1 & xx'x''y & -x'x''y \\ &&&&&& 1 & 0 \\ &&&&&&& 1 \end{bmatrix}$$

$$\left[\begin{bmatrix} 1 &&&&&&& \\ x & 1 &&&&&& \\ && 1 &&&&& \\ && x' & 1 &&&& \\ && x'' & 0 & 1 &&& \\ && -x'x'' & -x'' & -x' & 1 && \\ &&&&&& 1 & \\ &&&&&& -x & 1 \end{bmatrix}, \begin{bmatrix} 1 &&&&&&& \\ & 1 &&&&&& \\ & y & 1 &&&&& \\ &&& 1 &&&& \\ &&&& 1 &&& \\ &&&&& 1 && \\ &&&&& -y & 1 & \\ &&&&&&& 1 \end{bmatrix}\right] = \begin{bmatrix} 1 &&&&&&& \\ 0 & 1 &&&&&& \\ -xy & 0 & 1 &&&&& \\ xx'y & x'y & 0 & 1 &&&& \\ xx''y & x''y & 0 & 0 & 1 &&& \\ xx'x''y & x'x''y & 0 & 0 & 0 & 1 && \\ 0 & -x'x''y^2 & -x'x''y & -x''y & -x'y & 0 & 1 & \\ 0 & -xx'x''y^2 & -xx'x''y & -xx''y & -xx'y & xy & 0 & 1 \end{bmatrix}$$

$$\left[\begin{bmatrix} 1 &&&&&&& \\ 0 & 1 &&&&&& \\ x & 0 & 1 &&&&& \\ & -x' & 0 & 1 &&&& \\ & -x'' & 0 & 0 & 1 &&& \\ & 0 & 0 & 0 & 0 & 1 && \\ & -x'x'' & 0 & x'' & x' & 0 & 1 & \\ &&&&& -x & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 &&&&&&& \\ & 1 & y &&&&& \\ && 1 &&&&& \\ &&& 1 &&&& \\ &&&& 1 &&& \\ &&&&& 1 & -y & \\ &&&&&& 1 & \\ &&&&&&& 1 \end{bmatrix}\right] = \begin{bmatrix} 1 &&&&&&& \\ -xy & 1 &&&&&& \\ 0 & 0 & 1 &&&&& \\ -xx'y & 0 & -x'y & 1 &&&& \\ -xx''y & 0 & -x''y & 0 & 1 &&& \\ 0 & -x'x''y & -x'x''y^2 & x''y & yx' & 1 && \\ xx'x''y & 0 & x'x''y & 0 & 0 & 0 & 1 & \\ 0 & -xx'x''y & -xx'x''y^2 & xx''y & xx'y & 0 & xy & 1 \end{bmatrix}$$

$$\left[\begin{bmatrix} 1 &&&&&&& \\ 0 & 1 &&&&&& \\ x & 0 & 1 &&&&& \\ & -x' & 0 & 1 &&&& \\ & -x'' & 0 & 0 & 1 &&& \\ & 0 & 0 & 0 & 0 & 1 && \\ & -x'x'' & 0 & x'' & x' & 0 & 1 & \\ &&&&& -x & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & y & \\ & 1 & 0 & 0 & 0 & 0 & 0 & -y \\ && 1 & 0 & 0 & 0 & 0 & 0 \\ &&& 1 & 0 & 0 & 0 & 0 \\ &&&& 1 & 0 & 0 & 0 \\ &&&&& 1 & 0 & 0 \\ &&&&&& 1 & 0 \\ &&&&&&& 1 \end{bmatrix}\right] = \begin{bmatrix} 1 & x'x''y & 0 & -x''y & -x'y & 0 & 0 & -x'x''y^2 \\ & 1 & 0 & 0 & 0 & -xy & 0 & 0 \\ & -xx'x''y & 1 & xx''y & xx'y & 0 & xy & xx'x''y^2 \\ &&& 1 & 0 & -xx'y & 0 & x'y \\ &&&& 1 & -xx''y & & x''y \\ &&&&& 1 & 0 & 0 \\ &&&&& xx'x''y & 1 & -x'x''y \\ &&&&&&& 1 \end{bmatrix}$$

$$\left[\begin{bmatrix} 1 &&&&&&& \\ 0 & 1 &&&&&& \\ 0 & 0 & 1 &&&&& \\ x & 0 & 0 & 1 &&&& \\ x'' & 0 & 0 & 0 & 1 &&& \\ 0 & x' & 0 & 0 & 0 & 1 && \\ 0 & 0 & -x' & 0 & 0 & 0 & 1 & \\ -x'' & 0 & 0 & -x'' & y & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & y && \\ & 1 & 0 & 0 & 0 & 0 && \\ && 1 & 0 & 0 & 0 & 0 & -y \\ &&& 1 & 0 & 0 & 0 & 0 \\ &&&& 1 & 0 & 0 & 0 \\ &&&&& 1 & 0 & 0 \\ &&&&&& 1 & 0 \\ &&&&&&& 1 \end{bmatrix}\right] = \begin{bmatrix} 1 & -x'y &&&&&& \\ 0 & 1 &&&&&& \\ -xx''y & 0 & 1 & -x''y & -xy & -xx''y^2 && \\ 0 & xx'y & 0 & 1 & 0 & xy && \\ 0 & x'x''y & 0 & 0 & 1 & x''y && \\ 0 & 0 & 0 & 0 & 0 & 1 && \\ -xx'x''y & 0 & 0 & -x'x''y & -xx'y & -xx'x''y^2 & 1 & x'y \\ 0 & xx'x''y & 0 & 0 & 0 & xx''y & 0 & 1 \end{bmatrix}$$

$$\left[\begin{bmatrix} 1 &&&&&&& \\ 0 & 1 &&&&&& \\ 0 & 0 & 1 &&&&& \\ x & 0 & 0 & 1 &&&& \\ x'' & 0 & 0 & 0 & 1 &&& \\ 0 & x' & 0 & 0 & 0 & 1 && \\ 0 & 0 & -x' & 0 & 0 & 0 & 1 & \\ -x'' & 0 & 0 & -x'' & y & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & y & \\ & 1 & 0 & 0 & 0 & 0 & 0 & -y \\ && 1 & 0 & 0 & 0 & 0 & 0 \\ &&& 1 & 0 & 0 & 0 & 0 \\ &&&& 1 & 0 & 0 & 0 \\ &&&&& 1 & 0 & 0 \\ &&&&&& 1 & 0 \\ &&&&&&& 1 \end{bmatrix}\right] = \begin{bmatrix} 1 & 0 & x'y &&&&& \\ -xx''y & 1 & 0 & -x''y & -xy & 0 & -xx''y^2 & \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \\ 0 & 0 & -xx'y & 1 & 0 & 0 & xy & \\ 0 & 0 & -x'x''y & 0 & 1 & 0 & x''y & \\ xx'x''y & 0 & 0 & x'x''y & xx'y & 1 & xx'x''y^2 & -x'y \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -xx'x''y & 0 & 0 & 0 & xx''y & 1 \end{bmatrix}$$

A.4 Non-split group with k-root system of type BC_2

Let K/k be a tamely, purely ramified quadratic Galois extension of k, and $x \mapsto \bar{x}$ is a cyclic generator of the Galois group $\mathrm{Gal}(K/k)$. Suppose that $\mathbf{G}_{a,b}$ has K-structure isogenous to $\mathrm{SL}_5(K)$ with Galois action given by non-trivial involution of the Dynkin diagram of type A_4.

(a) For all $x, y \in K$, we have the following commutators:

$$\left[\begin{pmatrix} 1 & x & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \bar{x} \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & y & \\ & 1 & 0 & 0 & \bar{y} \\ & & 1 & 0 & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 & 0 & 0 & x\bar{y} - \bar{x}y \\ & 1 & 0 & 0 & 0 \\ & & 1 & 0 & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix}$$

$$\left[\begin{pmatrix} 1 & x & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \bar{x} \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & 0 & 1 & & \\ y & 0 & 0 & 1 & \\ & \bar{y} & 0 & 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & 0 & 1 & & \\ & \bar{x}\bar{y} - xy & 0 & 1 & \\ & & & & 1 \end{pmatrix}$$

$$\left[\begin{pmatrix} 1 & & & & \\ x & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & \bar{x} & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & y & \\ & 1 & 0 & 0 & \bar{y} \\ & & 1 & 0 & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & & & & \\ & 1 & 0 & xy - \bar{x}\bar{y} & \\ & & 1 & 0 & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$$

$$\left[\begin{pmatrix} 1 & & & & \\ x & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & \bar{x} & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & 0 & 1 & & \\ y & 0 & 0 & 1 & \\ & \bar{y} & 0 & 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & 1 & \\ \bar{x}y - x\bar{y} & 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) For all $x, y \in K$ with $y + \bar{y} = 0$, we have the following commutators:

$$\left[\begin{pmatrix} 1 & x & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \bar{x} \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & & \\ & 1 & 0 & -y & \\ & & 1 & 0 & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 & 0 & xy & x\bar{x}y \\ & 1 & 0 & 0 & -\bar{x}y \\ & & 1 & 0 & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix}$$

$$\left[\begin{pmatrix} 1 & x & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \bar{x} \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & 1 & \\ y & 0 & 0 & 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & 0 & 1 & & \\ \bar{x}y & x\bar{x}y & 0 & 1 & \\ & -xy & 0 & 0 & 1 \end{pmatrix}$$

$$\left[\begin{pmatrix} 1 & & & & \\ x & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & \bar{x} & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & y \\ & 1 & 0 & 0 & 0 \\ & & 1 & 0 & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 & 0 & -\bar{x}y & \\ & 1 & 0 & x\bar{x}y & xy \\ & & 1 & 0 & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix}$$

$$\left[\begin{pmatrix} 1 & & & & \\ x & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & \bar{x} & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & 0 & 1 & & \\ & -y & 0 & 1 & \\ & & & & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & 0 & 1 & & \\ -xy & 0 & 0 & 1 & \\ x\bar{x}y & \bar{x}y & 0 & 0 & 1 \end{pmatrix}$$

(c) For all $x, y, z, w \in K$ with $x\bar{x} + y + \bar{y} = 0$ and $z\bar{z} + w + \bar{w} = 0$, we have the following commutators:

$$\left[\begin{pmatrix} 1 & & & & \\ & 1 & x & -y & \\ & & 1 & \bar{x} & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & z & 0 & w \\ & 1 & 0 & 0 & 0 \\ & & 1 & 0 & -\bar{z} \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 & 0 & -\bar{x}z & \\ & 1 & 0 & 0 & -x\bar{z} \\ & & 1 & 0 & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix}$$

$$\left[\begin{pmatrix} 1 & & & & \\ & 1 & x & -y & \\ & & 1 & \bar{x} & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & & \\ 0 & 1 & & & \\ z & 0 & 1 & & \\ 0 & 0 & 0 & 1 & \\ w & 0 & -\bar{z} & 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & & & & \\ xz & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & \bar{x}\bar{z} & 1 \end{pmatrix}$$

$$\left[\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & x & 1 & & \\ & -y & \bar{x} & 1 & \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & z & 0 & w \\ & 1 & 0 & 0 & 0 \\ & & 1 & 0 & -\bar{z} \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & -xz & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & -\bar{x}\bar{z} \\ & & & & 1 \end{pmatrix}$$

$$\left[\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & x & 1 & & \\ & -y & \bar{x} & 1 & \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & & \\ 0 & 1 & & & \\ z & 0 & 1 & & \\ 0 & 0 & 0 & 1 & \\ w & 0 & -\bar{z} & 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & 0 & 1 & & \\ \bar{x}z & 0 & 0 & 1 & \\ & x\bar{z} & 0 & 0 & 1 \end{pmatrix}$$

(d) For all $x, y, z \in K$ with $x\bar{x} + y + \bar{y} = 0$, we have the following commutators:

$$\left[\begin{pmatrix} 1 & & & & \\ & 1 & x & -y & \\ & & 1 & \bar{x} & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & z & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \bar{z} \\ & & & & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 & -xz & yz & yz\bar{z} \\ & 1 & 0 & 0 & \bar{y}\bar{z} \\ & & 1 & 0 & \bar{x}\bar{z} \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix}$$

$$\left[\begin{pmatrix} 1 & 0 & x & 0 & y \\ & 1 & 0 & 0 & 0 \\ & & 1 & 0 & -\bar{x} \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & 0 & 1 & & \\ z & 0 & 0 & 1 & \\ & \bar{z} & 0 & 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & -\bar{y}\bar{z} & & & \\ & 1 & & & \\ & -\bar{x}\bar{z} & 1 & & \\ & -yz\bar{z} & -xz & 1 & -yz \\ & & & & 1 \end{pmatrix}$$

$$\left[\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & x & 1 & & \\ & -y & \bar{x} & 1 & \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & & \\ z & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & \bar{z} & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & & & & \\ 0 & 1 & & & \\ xz & 0 & 1 & & \\ \bar{y}z & 0 & 0 & 1 & \\ yz\bar{z} & y\bar{z} & -\bar{x}\bar{z} & 0 & 1 \end{pmatrix}$$

$$\left[\begin{pmatrix} 1 & & & & \\ 0 & 1 & & & \\ x & 0 & 1 & & \\ 0 & 0 & 0 & 1 & \\ y & 0 & -\bar{x} & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & z & \\ & 1 & 0 & 0 & \bar{z} \\ & & 1 & 0 & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & & & & \\ -y\bar{z} & 1 & \bar{x}\bar{z} & -yz\bar{z} & \\ & & 1 & xz & \\ & & & 1 & \\ & & & -\bar{y}z & 1 \end{pmatrix}$$

Appendix B

Tables

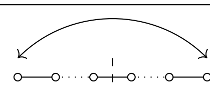
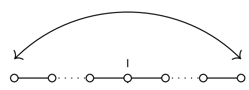
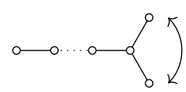
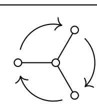
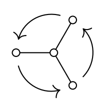
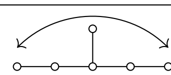
Type	**Diagram Symmetry**	**Order**
A_{2n}		2
A_{2n-1}		2
D_{n+1}		2
D_4	 or 	3
E_6		2

Table B.1: Non-trivial based automorphisms of a simple reduced root system arising from symmetries of its Dynkin diagram.

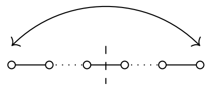
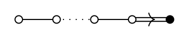
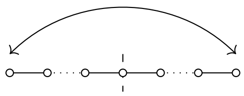
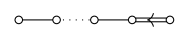
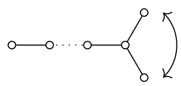
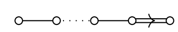
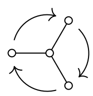
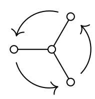
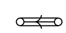
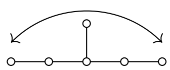
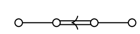
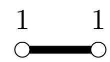
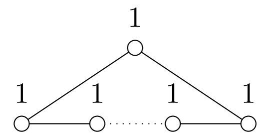
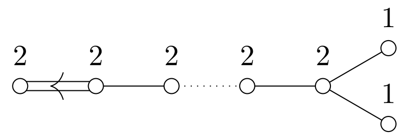
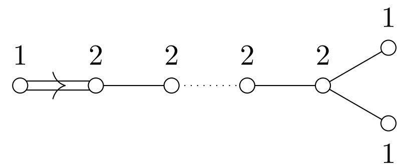
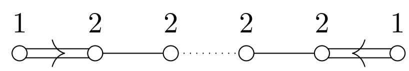
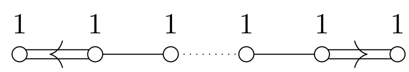
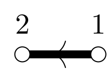
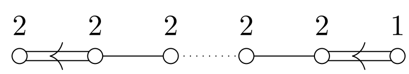
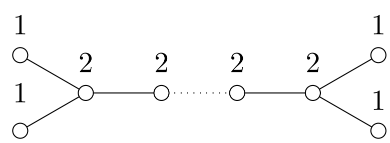
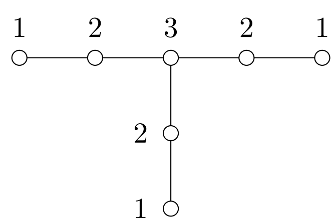
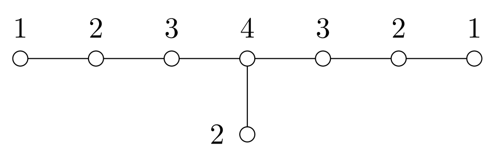
R Type	**Diagram Symmetry**	σ **Order**	R **Type**	**Twisted Diagram**
A_{2n}		2	BC_n	
A_{2n-1}		2	C_n	
D_{n+1}		2	B_n	
D_4	 or 	3	G_2	
E_6		2	F_4	

Table B.2: Twisted root systems R arising from non-trivial based automorphism of a simple reduced root system $\mathbf{R}$. Note that for the non-reduced twisted root system of type BC_n, the black vertex indicates a non-reduced twisted root where twice it is also a twisted root.

Ψ Name	Weighted Dynkin Diagram of Ψ
A_1	
A_ℓ $(\ell \geq 2)$	
B_ℓ $(\ell \geq 3)$	
$B\text{-}C_\ell$ $(\ell \geq 3)$	
C_ℓ $(\ell \geq 2)$	
$C\text{-}B_\ell$ $(\ell \geq 2)$	
$C\text{-}BC_1$	
$C\text{-}BC_\ell$ $(\ell \geq 2)$	
D_ℓ $(\ell \geq 4)$	
E_6	
E_7	

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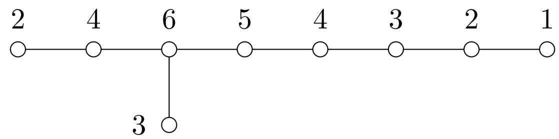
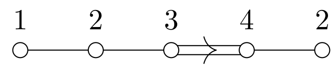
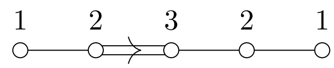
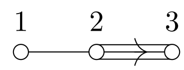
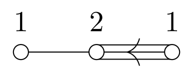
Ψ Name	Weighted Dynkin Diagram of Ψ
E_8	
F_4	
F_4^I	
G_2	
G_2^I	

Table B.3: Affine twisted root systems and their weighted Dynkin diagrams.

$[\gamma(\lambda),\ \alpha(\lambda),\ \beta(\lambda)]$	abelianization of H_λ	$\cong \mathsf{f}^n$
$\left[\frac{1}{4},\ \frac{1}{4},\ \frac{1}{4}\right]$	$\mathsf{f}_1 \oplus \mathsf{f}_2 \oplus \mathsf{f}_3 \oplus \dfrac{\mathsf{f}_4 \oplus \mathsf{f}_6}{\langle (0,-2xy),(-xy,-x^2y) \mid x,y \in \mathsf{f}\rangle} \oplus \dfrac{\mathsf{f}_5 \oplus \mathsf{f}_7}{\langle (0,2xy),(xy,-x^2y) \mid x,y \in \mathsf{f}\rangle} \oplus \dfrac{\mathsf{f}_8}{\langle \pm xy \mid x,y \in \mathsf{f}\rangle}$	$\cong \begin{cases} \mathsf{f}^5 & \text{if } \#\mathsf{f} = 2, \\ \mathsf{f}^3 & \text{if else.} \end{cases}$
$\left[\frac{1}{3},\ \frac{1}{3},\ 0\right]$	$\mathsf{f}_1 \oplus \mathsf{f}_2 \oplus \dfrac{\mathsf{f}_4 \oplus \mathsf{f}_6}{\langle (0,-2xy),(-xy,-x^2y) \mid x,y \in \mathsf{f}\rangle} \oplus \mathsf{f}_5 \oplus \dfrac{\mathsf{f}_7}{\langle 2xy \mid x,y \in \mathsf{f}\rangle} \oplus \dfrac{\mathsf{f}_8}{\langle xy \mid x,y \in \mathsf{f}\rangle}$	$\cong \begin{cases} \mathsf{f}^5 & \text{if } \#\mathsf{f} = 2, \\ \mathsf{f}^4 & \text{if } \mathrm{char}(\mathsf{f}) = 2 \text{ and } \#\mathsf{f} \neq 2, \\ \mathsf{f}^3 & \text{if else.} \end{cases}$
$\left[\frac{1}{2},\ 0,\ \frac{1}{2}\right]$	$\mathsf{f}_1 \oplus \mathsf{f}_3 \oplus \mathsf{f}_4 \oplus \mathsf{f}_5 \oplus \mathsf{f}_6 \oplus \mathsf{f}_7 \oplus \dfrac{\mathsf{f}_8}{\langle \pm xy \mid x,y \in \mathsf{f}\rangle}$	$\cong \mathsf{f}^6$
$\left[0,\ \frac{1}{3},\ \frac{1}{3}\right]$	$\mathsf{f}_2 \oplus \mathsf{f}_3 \oplus \mathsf{f}_4 \oplus \dfrac{\mathsf{f}_5 \oplus \mathsf{f}_7}{\langle (0,2xy),(xy,-x^2y) \mid x,y \in \mathsf{f}\rangle} \oplus \dfrac{\mathsf{f}_6}{\langle -2xy \mid x,y \in \mathsf{f}\rangle} \oplus \dfrac{\mathsf{f}_8}{\langle -xy \mid x,y \in \mathsf{f}\rangle}$	$\cong \begin{cases} \mathsf{f}^5 & \text{if } \#\mathsf{f} = 2, \\ \mathsf{f}^4 & \text{if } \mathrm{char}(\mathsf{f}) = 2 \text{ and } \#\mathsf{f} \neq 2, \\ \mathsf{f}^3 & \text{if else.} \end{cases}$

(continued on the next page)

$[\gamma(\lambda),\ \alpha(\lambda),\ \beta(\lambda)]$	abelianization of H_λ	$\cong \mathsf{f}^n$
$\left[0,\ \frac{1}{2},\ 0\right]$	$\mathsf{f}_2 \oplus \mathsf{f}_4 \oplus \mathsf{f}_5 \oplus \dfrac{\mathsf{f}_6}{\langle -2xy \mid x, y \in \mathsf{f}\rangle} \oplus \dfrac{\mathsf{f}_7}{\langle 2xy \mid x, y \in \mathsf{f}\rangle} \oplus \mathsf{f}_8$	$\cong \begin{cases} \mathsf{f}^6 & \text{if } \mathrm{char}(\mathsf{f}) = 2, \\ \mathsf{f}^4 & \text{if else.} \end{cases}$

Table B.4: Here $\mathbf{G}$ is a connected, quasi-simple, semisimple reductive algebraic group defined and splitting over a non-archimedean field k with residue field f with a type C_2 affine k-root system. The simple affine k-roots are as in (3.1.8), and the enumeration for the minimal-height positive affine k-roots is as in (3.1.9). The table shows the abelianization of the group H_λ for the barycenter λ of each facet not a strongly-special vertex. We denote by $\mathsf{f}_i = \mathsf{f}$ the abelian group isomorphic to $\mathsf{U}_i = U_{\psi_i}/U_{\psi_i} \cap G_{\lambda,1}$ for $i = 1, \ldots, 8$.

$[\gamma(\lambda),\ \beta(\lambda),\ \alpha(\lambda)]$	abelianization of H_λ	$\cong \mathsf{f}^n$
$\left[\frac{1}{6},\ \frac{1}{6},\ \frac{1}{6}\right]$	$\mathsf{f}_1 \oplus \mathsf{f}_2 \oplus \mathsf{f}_3 \oplus \dfrac{\mathsf{f}_4}{\langle xy \mid x, y \in \mathsf{f}\rangle}$ $\oplus \dfrac{\mathsf{f}_6 \oplus \mathsf{f}_8 \oplus \mathsf{f}_{10} \oplus \mathsf{f}_{12}}{\left\langle \begin{matrix} (0,0,-xy,0), (0,2xy,3x^2y,0), (0,0,0,-2xy), (0,0,3xy,0), \\ (xy,0,0,0,x^2y), (0,-xy,0,0), (-xy,x^2y,x^3y,0), (0,0,0,\pm xy) \end{matrix} \,\middle\|\, x, y \in \mathsf{f} \right\rangle}$ $\oplus \dfrac{\mathsf{f}_5 \oplus \mathsf{f}_7 \oplus \mathsf{f}_9 \oplus \mathsf{f}_{11}}{\left\langle \begin{matrix} (0,0,0,xy), (0,-2xy,3x^2y,-3xy^2) \\ (0,0,0,\pm 3xy), (xy,x^2y,-x^3y,2x^3y^2) \end{matrix} \,\middle\|\, x, y \in \mathsf{f} \right\rangle}$	$\cong \begin{cases} \mathsf{f}^4 & \text{if } \#\mathsf{f} \in \{2,3\}, \\ \mathsf{f}^3 & \text{if else.} \end{cases}$
$\left[\frac{1}{3},\ \frac{1}{3},\ 0\right]$	$\mathsf{f}_1 \oplus \mathsf{f}_2 \oplus \dfrac{\mathsf{f}_4}{\langle xy \mid x, y \in \mathsf{f}\rangle} \oplus \mathsf{f}_5 \oplus \dfrac{\mathsf{f}_6 \oplus \mathsf{f}_{12}}{\langle (0,-2xy), (xy,x^2y), (0,\pm xy) \mid x, y \in \mathsf{f}\rangle}$ $\oplus \mathsf{f}_7 \oplus \dfrac{\mathsf{f}_8}{\langle -xy \mid x, y \in \mathsf{f}\rangle} \oplus \mathsf{f}_9 \oplus \dfrac{\mathsf{f}_{10}}{\langle -xy \mid x, y \in \mathsf{f}\rangle} \oplus \dfrac{\mathsf{f}_{11}}{\langle xy, -3xy \mid x, y \in \mathsf{f}\rangle}$	$\cong \mathsf{f}^5$
$\left[0,\ \frac{1}{2},\ 0\right]$	$\mathsf{f}_2 \oplus \mathsf{f}_4 \oplus \mathsf{f}_5 \oplus \mathsf{f}_6 \oplus \mathsf{f}_7 \oplus \mathsf{f}_8 \oplus \mathsf{f}_9 \oplus \mathsf{f}_{10} \oplus \dfrac{\mathsf{f}_{11}}{\langle xy, -3xy \mid x, y \in \mathsf{f}\rangle} \oplus \dfrac{\mathsf{f}_{12}}{\langle -2xy, \pm xy \mid x, y \in \mathsf{f}\rangle}$	$\cong \mathsf{f}^8$

(continued on the next page)

$[\gamma(\lambda),\ \alpha(\lambda),\ \beta(\lambda)]$	abelianization of H_λ	$\cong \mathsf{f}^n$
$\left[\frac{1}{4},\ 0,\ \frac{1}{4}\right]$	$\mathsf{f}_1 \oplus \mathsf{f}_3 \oplus \mathsf{f}_4 \oplus \mathsf{f}_5 \oplus \dfrac{\mathsf{f}_7 \oplus \mathsf{f}_9 \oplus \mathsf{f}_{11}}{\langle (-2xy, 3x^2y, -3xy), (0, 3xy, 0), (0, 0, -3xy) \mid x, y \in \mathsf{f} \rangle}$ $\oplus \dfrac{\mathsf{f}_6 \oplus \mathsf{f}_8 \oplus \mathsf{f}_{10} \oplus \mathsf{f}_{12}}{\left\langle \begin{array}{c} (0,0,-xy,0), (0,2xy,3x^2y,0), (0,0,0,-2xy), (0,0,-3xy,0), \\ (xy,0,0,x^2y), (0,-xy,0,0), (-xy,x^2y,x^3y,0), (0,0,0,\pm xy) \end{array} \middle\vert\ x, y \in \mathsf{f} \right\rangle}$	$\cong \begin{cases} \mathsf{f}^6 & \text{if } \operatorname{char}(\mathsf{f}) = 3, \\ \mathsf{f}^5 & \text{if } \operatorname{char}(\mathsf{f}) = 2, \\ \mathsf{f}^4 & \text{if else.} \end{cases}$
$\left[0,\ 0,\ \frac{1}{3}\right]$	$\mathsf{f}_3 \oplus \mathsf{f}_5 \oplus \mathsf{f}_6 \oplus \dfrac{\mathsf{f}_8 \oplus \mathsf{f}_{10}}{\langle (2xy, 3x^2y), (0, -3xy) \mid x, y \in \mathsf{f} \rangle} \oplus \dfrac{\mathsf{f}_{12}}{\langle -2xy \mid x, y \in \mathsf{f} \rangle}$ $\oplus \dfrac{\mathsf{f}_7 \oplus \mathsf{f}_9 \oplus \mathsf{f}_{11}}{\langle (-2xy, 3x^3y, -3xy^2), (0, 3xy, 0), (0, 0, -3xy) \mid x, y \in \mathsf{f} \rangle}$	$\cong \begin{cases} \mathsf{f}^6 & \text{if } \operatorname{char}(\mathsf{f}) \in \{2, 3\}, \\ \mathsf{f}^3 & \text{if else.} \end{cases}$

(continued on the next page)

$[\gamma(\lambda),\ \alpha(\lambda),\ \beta(\lambda)]$	abelianization of H_λ	$\cong \mathsf{f}^n$
$\left[0,\ \frac{1}{5},\ \frac{1}{5}\right]$	$\mathsf{f}_2 \oplus \mathsf{f}_3 \oplus \mathsf{f}_4 \oplus \dfrac{\mathsf{f}_{12}}{\langle -2xy, \pm xy \mid x, y \in \mathsf{f}\rangle}$ $\oplus \dfrac{\mathsf{f}_6 \oplus \mathsf{f}_8 \oplus \mathsf{f}_{10}}{\langle (0, 2xy, 3x^2y), (0, 0, -3xy), (-xy, x^2y, x^3y) \mid x, y \in \mathsf{f}\rangle}$ $\oplus \dfrac{\mathsf{f}_5 \oplus \mathsf{f}_7 \oplus \mathsf{f}_9 \oplus \mathsf{f}_{11}}{\left\langle \begin{matrix} (0,0,0,xy), (0,-2xy,3x^2y,-3xy^2), \\ (0,0,0,\pm 3xy), (xy, x^2y, -x^3y, 2x^3y^2) \end{matrix} \,\middle\vert\, x, y \in \mathsf{f} \right\rangle}$	$\cong \begin{cases} \mathsf{f}^5 & \text{if } \#\mathsf{f} \in \{2,3\}, \\ \mathsf{f}^4 & \text{if } \mathrm{char}(\mathsf{f}) \in \{2,3\} \text{ and } \#\mathsf{f} \notin \{2,3\}, \\ \mathsf{f}^3 & \text{if else.} \end{cases}$

Table B.5: Here $\mathbf{G}$ is a connected, quasi-simple, semisimple reductive algebraic group defined and splitting over a non-archimedean field k with residue field f and with a type G_2 affine k-root system. The simple affine k-roots are as in (3.1.11), and the enumeration of the minimal-height positive affine k-roots is as in (3.1.12). The table shows the abelianization of the group H_λ for the barycenter λ of each facet not a strongly-special vertex. We denote by f_i the abelian group isomorphic to $\mathsf{U}_i = U_{\psi_i}/U_{\psi_i} \cap G_{\lambda,1}$ for $i = 1, \ldots, 12$.

$[\gamma(\lambda),\ \alpha(\lambda),\ \beta(\lambda)]$	abelianization of H_λ	$\cong \mathsf{f}^n$
$\left[\frac{1}{12},\ \frac{1}{12},\ \frac{1}{12}\right]$	$\mathsf{f}_1 \oplus \mathsf{f}_2 \oplus \mathsf{f}_3 \oplus \frac{\mathsf{f}_4}{\langle (1+\zeta^2)xy \mid x,y \in \mathsf{f}\rangle} \oplus \frac{\mathsf{f}_6}{\langle -(1+\zeta^2)xy, -xy \mid x,y \in \mathsf{f}\rangle} \oplus \frac{\mathsf{f}_5 \oplus \mathsf{f}_7 \oplus \mathsf{f}_8 \oplus \mathsf{f}_9}{\left\langle \begin{matrix}(0,0,0,xy),(0,-2xy,3x^2y,-3xy^2),(0,0,3xy,0),\\ (0,0,0,-3xy),(xy,x^2y,-x^3y,2x^3y^2)\end{matrix} \;\middle\vert\; x,y \in \mathsf{f} \right\rangle}$	$\cong \begin{cases} \mathsf{f}^5 & \text{if } \#\mathsf{f} \in \{2,3\} \text{ and } 1+\zeta^2 = 0, \\ \mathsf{f}^4 & \text{if } \#\mathsf{f} \in \{2,3\} \text{ and } 1+\zeta^2 \neq 0, \\ \mathsf{f}^4 & \text{if } \#\mathsf{f} \notin \{2,3\} \text{ and } 1+\zeta^2 = 0, \\ \mathsf{f}^3 & \text{if else.} \end{cases}$
$\left[\frac{1}{9},\ \frac{1}{9},\ 0\right]$	$\mathsf{f}_1 \oplus \mathsf{f}_2 \oplus \frac{\mathsf{f}_4}{\langle (1+\zeta^2)xy \mid x,y \in \mathsf{f}\rangle} \oplus \mathsf{f}_5 \oplus \frac{\mathsf{f}_6}{\langle -(1+\zeta^2)xy \mid x,y \in \mathsf{f}\rangle} \oplus \frac{\mathsf{f}_7 \oplus \mathsf{f}_8 \oplus \mathsf{f}_9}{\langle (-2xy,3x^2y,-3xy^2),(0,3xy,0),(0,0,-3xy) \mid x,y \in \mathsf{f}\rangle}$	$\cong \begin{cases} \mathsf{f}^7 & \text{if } \operatorname{char}\mathsf{f} = 3 \text{ and } 1+\zeta^2 = 0, \\ \mathsf{f}^6 & \text{if } \operatorname{char}\mathsf{f} = 2 \text{ and } 1+\zeta^2 = 0, \\ \mathsf{f}^5 & \text{if } \operatorname{char}\mathsf{f} > 3 \text{ and } 1+\zeta^2 = 0, \\ \mathsf{f}^5 & \text{if } \operatorname{char}\mathsf{f} = 3 \text{ and } 1+\zeta^2 \neq 0, \\ \mathsf{f}^4 & \text{if } \operatorname{char}\mathsf{f} = 2 \text{ and } 1+\zeta^2 \neq 0, \\ \mathsf{f}^3 & \text{if else.} \end{cases}$
$\left[\frac{1}{6},\ 0,\ \frac{1}{6}\right]$	$\mathsf{f}_1 \oplus \mathsf{f}_3 \oplus \mathsf{f}_4 \oplus \mathsf{f}_5 \oplus \frac{\mathsf{f}_6}{\langle -(1+\zeta^2)xy, -xy \mid x,y \in \mathsf{f}\rangle} \oplus \mathsf{f}_7 \oplus \mathsf{f}_8 \oplus \frac{\mathsf{f}_9}{\langle xy, -3xy \mid x,y \in \mathsf{f}\rangle}$	$\cong \mathsf{f}^6$

(continued on the next page)

$[\gamma(\lambda),\ \alpha(\lambda),\ \beta(\lambda)]$	abelianization of H_λ	$\cong \mathsf{f}^n$
$\left[0,\ \frac{1}{9},\ \frac{1}{9}\right]$	$\mathsf{f}_2 \oplus \mathsf{f}_3 \oplus \mathsf{f}_4 \oplus \dfrac{\mathsf{f}_6}{\langle -xy \mid x, y \in \mathsf{f} \rangle} \oplus \dfrac{\mathsf{f}_5 \oplus \mathsf{f}_7 \oplus \mathsf{f}_8 \oplus \mathsf{f}_9}{\left\langle \begin{matrix} (0,0,0,xy), (0,-2xy,3x^2y,-3xy^2), (0,0,3xy,0), \\ (0,0,0,-3xy), (xy, x^2y, -x^3y, 2x^3y^2) \end{matrix} \;\middle\vert\; x, y \in \mathsf{f} \right\rangle}$	$\cong \begin{cases} \mathsf{f}^4 & \text{if } \#\mathsf{f} \in \{2,3\}, \\ \mathsf{f}^3 & \text{if else.} \end{cases}$
$\left[0,\ \frac{1}{6},\ 0\right]$	$\mathsf{f}_2 \oplus \mathsf{f}_4 \oplus \mathsf{f}_5 \oplus \mathsf{f}_6 \oplus \dfrac{\mathsf{f}_7 \oplus \mathsf{f}_8 \oplus \mathsf{f}_9}{\langle (-2xy, 3x^2y, -3xy^2), (0, 3xy, 0), (0, 0, -3xy) \mid x, y \in \mathsf{f} \rangle}$	$\cong \begin{cases} \mathsf{f}^6 & \text{if } \operatorname{char} \mathsf{f} = 2, \\ \mathsf{f}^5 & \text{if } \operatorname{char} \mathsf{f} = 3, \\ \mathsf{f}^4 & \text{if else.} \end{cases}$

Table B.6: Here $\mathbf{G}$ is a connected, quasi-simple, semisimple reductive algebraic group defined and non-split quasi-split over a non-archimedean field k with residue field f with a type G_2^I affine k-root system. We assume that $\mathbf{G}$ splits over K, a degree 3 tamely, totally ramified extension of k having ring of integers A_K. The simple affine k-roots are as in (3.1.14), and the enumeration for the minimal-height positive affine k-roots is as in (3.1.15). The table shows the abelianization of the group H_λ for the barycenter λ of each facet not a strongly-special vertex. We denote by $\mathsf{f}_i = \mathsf{f}$ the abelian group isomorphic to $\mathsf{U}_i = U_{\psi_i}/U_{\psi_i} \cap G_{\lambda,1/3}$ for $i = 1, \ldots, 9$. We also denote by ζ the natural projection into f of a the unit $\sigma(\varpi^{1/3})/\varpi^{1/3} \in A_K^\times$ for a fixed uniformizer $\varpi^{1/3}$ of K and a fixed cyclic generator σ of the Galois group $\mathrm{Gal}(K/k)$.

$(\chi_1(1), \chi_2(1), \chi_3(1), \chi_4(1), \chi_5(1))$	non-trivial $w \in W$ such that ${}^w\chi = \chi$ on ${}^wG_{\lambda+} \cap G_{\lambda+}$
$(+1, +1, +1, +1, +1)$	w_a
$(-1, +1, +1, +1, +1)$	$(a^\vee + b^\vee)(w_a w_b) w_a$
$(+1, -1, +1, +1, +1)$	w_a
$(+1, +1, -1, +1, +1)$	$(w_a w_b)^3 w_a$
$(-1, -1, +1, +1, +1)$	$(a^\vee + 2b^\vee)(w_a w_b)^2$
$(-1, +1, -1, +1, +1)$	$(a^\vee + b^\vee)(w_a w_b)^2$
$(+1, -1, -1, +1, +1)$	$(w_a w_b)^2$
$(+1, +1, +1, -1, +1)$	$(a^\vee + 2b^\vee)(w_a w_b)^2$
$(-1, +1, +1, -1, +1)$	$(a^\vee + 2b^\vee)(w_a w_b)^2$
$(+1, -1, +1, -1, +1)$	$(a^\vee + 2b^\vee)(w_a w_b)^2$
$(+1, +1, -1, -1, +1)$	$(2a^\vee + 2b^\vee)(w_a w_b) w_a$
$(-1, -1, +1, -1, +1)$	$(a^\vee + 2b^\vee)(w_a w_b)^2$
$(-1, +1, -1, -1, +1)$	$(2a^\vee + 2b^\vee)(w_a w_b)^2 w_a$
$(+1, -1, -1, -1, +1)$	$(a^\vee + b^\vee)(w_a w_b) w_a$
$(-1, -1, -1, -1, +1)$	$(a^\vee + b^\vee)(w_a w_b) w_a$
$(+1, +1, +1, +1, -1)$	$(w_a w_b)^2$
$(-1, +1, +1, +1, -1)$	$-b^\vee (w_a w_b)^3 w_a$
$(+1, -1, +1, +1, -1)$	$(w_a w_b)^2$
$(+1, +1, -1, +1, -1)$	$(w_a w_b)^2$

(continued on the next page)

$(\chi_1(1), \chi_2(1), \chi_3(1), \chi_4(1), \chi_5(1))$	non-trivial $w \in W$
$(-1, -1, +1, +1, -1)$	$(w_a w_b)^3 w_a$
$(-1, +1, -1, +1, -1)$	$-b^\vee (w_a w_b)^3 w_a$
$(+1, -1, -1, +1, -1)$	$(w_a w_b)^2$
$(-1, -1, -1, +1, -1)$	$(w_a w_b)^3 w_a$
$(+1, +1, +1, -1, -1)$	$-a^\vee w_a$
$(+1, -1, +1, -1, -1)$	$-a^\vee w_a$
$(-1, +1, -1, -1, -1)$	w_a
$(-1, -1, -1, -1, -1)$	w_a

Table B.7: Here $G = \mathrm{Sp}_4(k)$, the residue field of k has order $\#\mathsf{f} = 2$, and λ is the barycenter of the fundamental alcove. Each λ-shallow χ is determined by a 5-tuple of integers. In the left-hand column of the above table, we have given the 27 λ-shallow characters χ such that $\mathscr{I}(G, G_\lambda, \chi) \neq G_\lambda(\chi)$. In the right-hand column, we have provided a non-trivial element μw in the reflection group $W = (\mathbb{Z}R^\vee)W_0$ of the affine root system of type C_2, chosen so that some lift $n \in N$ of w is contained in $\mathscr{I}(G, G_{\lambda+}, \chi)$ and not $G_\lambda(\chi)$.

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Index of Notation

§2.1. Twisted Root Systems

$\mathbf{E}$	a real vector space
$\mathbf{R}$	a simple reduced root system of linear functionals on $\mathbf{E}$
$\mathbf{R}^\vee$	the coroot system contained in $\mathbf{E}$
$\mathbf{W}_0$	the finite reflection group of $\mathbf{R}$
$\langle\ ,\ \rangle$	the perfect pairing between $\mathbf{R}$ and $\mathbf{R}^\vee$
$\mathbf{R}^+$	a positive system in $\mathbf{R}$
$\mathbf{D}$	the simple system in $\mathbf{R}$ inside of $\mathbf{R}^+$
σ	an order $e \in \{1,2,3\}$ automorphism of $\mathbf{R}$ preserving $\mathbf{D}$
$E = \mathbf{E}^\sigma$	the vectors in $\mathbf{E}$ fixed by σ
$R = \mathbf{R}_\sigma$	the twisted root system of restrictions to E of roots in $\mathbf{R}$
$D = \mathbf{D}_\sigma$	the simple system in R restricting from $\mathbf{D}$
$R^+ = \mathbf{R}_\sigma^+$	the positive system in R restricting from $\mathbf{R}^+$
(a)	the set of roots in $\mathbf{R}$ whose restriction to E is $a \in R$
e_a	the order of (a)
$R^\vee$	the coroot system of R contained in E
$W_0 = \mathbf{W}_0^\sigma$	the reflection group of R
$\Psi = \Psi(\mathbf{R}, \sigma)$	the twisted affine root system with gradients in R
$\Psi(a)$	the set of affine twisted roots with gradient $a \in R$

$\alpha_0, \ldots, \alpha_\ell$	the simple affine twisted root corresponding to the twisted root $a_i \in R$
Δ	the simple system in Ψ
$\mathcal{C}$	the alcove in E corresponding to Δ
W	the affine reflection group of Ψ with gradients in W_0

§2.2. Quasi-split Groups

k	a non-archimedean local field with surjective valuation $\mathrm{val} : k^\times \to \mathbb{Z}$
A_k	the ring of integers of k
P_k	the unique maximal ideal in A_k
ϖ	a prime element in A_k generating P_k
f	the residue field of k
p	the residual characteristic of k and the characteristic of f
K	a degree $e \in \{1, 2, 3\}$ tamely, totally ramified Galois extension of k
A_K	the ring of integers in K
P_K	the unique maximal ideal in A_K
$\varpi^{1/e}$	a prime element in A_K generating P_K whose e-th power is ϖ
f	the residue field of K, isomorphic to that of k
σ	a cyclic generator of the Galois group $\mathrm{Gal}(K/k)$
$\sigma(x) = \bar{x}$	the image under σ of any $x \in K$ when $e = 2$
$\mathbf{G}$	a connected, quasi-simple, semisimple reductive algebraic group defined and quasi-split over k and splitting over K, with $G = \mathbf{G}(k)$
$\mathbf{S}$	a maximal k-split torus in $\mathbf{G}$, with $S = \mathbf{S}(k)$
$\mathbf{Z}$	the maximal torus of $\mathbf{G}$ defined over k and centralizing $\mathbf{S}$, with $Z = \mathbf{Z}(k)$
$\mathbf{B}$	a Borel subgroup of $\mathbf{G}$ defined over k and containing $\mathbf{Z}$, with $B = \mathbf{B}(k)$

$\mathbf{R}$	the K-root system of $\mathbf{G}$ relative to $\mathbf{Z}$, acted on by σ
$\mathbf{D}$	the set of simple K-roots of $\mathbf{G}$ relative to $(\mathbf{B}, \mathbf{Z})$, preserved under σ
$R = \mathbf{R}_\sigma$	the k-root system of $\mathbf{G}$ relative to $\mathbf{S}$
$D = \mathbf{D}_\sigma$	the set of simple k-roots of $\mathbf{G}$ relative to $(\mathbf{B}, \mathbf{S})$

$\mathbf{U_a}$	the K-root group of on which $\mathbf{Z}$ acts via $\mathbf{a} \in \mathbf{R}$
$u_{\mathbf{a}}$	a K-root morphism isomorphically mapping K onto $\mathbf{U_a}(K)$
(a)	the set of K-roots in $\mathbf{R}$ whose restriction to $\mathbf{S}$ is $a \in R$
e_a	the order of (a)
K_a	a subset of either K or K^2 defined in Definition 2.2.1
$\mathbf{U}_a$	the k-root group generated by all K-root groups $\mathbf{U_a}$ for $\mathbf{a} \in (a)$
U_a	the group of σ-fixed elements in $\mathbf{U}_a(K)$
u_a	a k-root morphism isomorphically mapping K_a onto U_a
val_a	the valuation on U_a inherited from val
$U_{a,r}$	$= \{u \in U_a \mid \mathrm{val}_a(u) \geq r\}$ for $r \in \mathbb{R}$
$U_{a,r+}$	$= \{u \in U_a \mid \mathrm{val}_a(u) > r\}$ for $r \in \mathbb{R}$
$\mathsf{U}_{a,r}$	$= U_{a,r}/U_{a,r+}$
$d_a(r)$	the f-dimension of $\mathsf{U}_{a,r}/\mathsf{U}_{2a,2r}$
$\Psi = \Psi(\mathbf{R}, \sigma)$	the affine k-root system of $\mathbf{G}$ relative to $\mathbf{S}$ whose elements are pairs (a, r) for which $d_a(r) \neq 0$
Δ	the simple affine k-roots of $\mathbf{G}$ corresponding to $(\mathbf{B}, \mathbf{S})$
$\mathcal{C}$	the alcove of $X_*(\mathbf{S}, k) \otimes_{\mathbb{Z}} \mathbb{R}$ corresponding to Δ
G_λ	a parahoric subgroup of G attached to $\lambda \in X_*(\mathbf{S}, k) \otimes_{\mathbb{Z}} \mathbb{R}$
Z_0	$= Z \cap G_\lambda$
$G_{\lambda,r}$	the Moy-Prasad subgroup of G_λ for $r \in \mathbb{R}$
Z_r	$= Z \cap G_{\lambda,r}$
$G_{\lambda+}$	the pro-unipotent radical and first Moy-Prasad subgroup of of G_λ
$G_{\lambda++}$	the second Moy-Prasad subgroup of G_λ

§3.1. Shallow Characters

λ	a fixed point in the closure of the fundamental alcove $\mathcal{C}$
$\mathsf{s}(\lambda)$	the minimal value of $\alpha(\lambda) + \beta(\lambda)$ when α, β are affine k-roots satisfying $(\mathsf{s}1)$ and $(\mathsf{s}2)$ defined in Definition 3.1.1

$\psi_1, \ldots, \psi_n$	the λ-shallow affine k-roots that take value at λ between 0 and $\mathsf{s}(\lambda)$
H_λ	$= G_{\lambda+}/G_{\lambda,\mathsf{s}(\lambda)}$
U_0	$= Z_+/Z_{\mathsf{s}(\lambda)}$
U_i	$= U_{\psi_i} G_{\lambda,\mathsf{s}(\lambda)}/G_{\lambda,\mathsf{s}(\lambda)} \cong U_{\psi_i}/U_{\psi_i} \cap G_{\lambda,\mathsf{s}(\lambda)}$
χ	a homomorphism $\mathsf{H}_\lambda \to \mathbb{C}^\times$
χ_i	the restriction of χ to U_i

$\sigma(x) = \bar{x}$	the image under σ of any $x \in K$ when $e = 2$
$\sigma(x) = x'$	the image under σ of any $x \in K$ when $e = 3$
f_i	the abelian group f, isomorphic to $\mathsf{U}_i \cong U_{\psi_i}/U_{\psi_i} \cap G_{\lambda,\mathsf{s}(\lambda)}$

§3.2. Supercuspidal Representations

$\mathbf{N}$	the normalizer in $\mathbf{G}$ of $\mathbf{S}$, defined over k, with $N = \mathbf{N}(k)$

H	a compact open subgroup of G containing the center of G
M	a subgroup of G containing H
ϕ	an irreducible smooth representation of H
$\operatorname{ind}_H^M(\phi)$	the compactly induced representation of M
${}^mH$	$= mHm^{-1}$ for $m \in M$
${}^m\phi$	the conjugate representation of ${}^mH$
$\mathscr{I}(M, H, \phi)$	the intertwining set in M of χ

χ	any group homomorphism $G_{\lambda+} \to \mathbb{C}^\times$
$G_\lambda(\chi)$	the stabilizer in G_λ of χ
$\mathscr{A}_\chi$	the intertwining algebra of χ
ρ	a simple $\mathscr{A}_\chi$-module
χ_ρ	irreducible constituent of $\operatorname{ind}_{G_{\lambda+}}^{G_\lambda(\chi)}(\chi)$ corresponding to ρ
$\pi(\chi;\rho)$	$= \operatorname{ind}_{G_\lambda(\chi)}^{G}(\chi_\rho)$, an irreducible supercuspidal representation of G

Bibliography

[1] Jeffrey D. Adler. Refined anisotropic K-types and supercuspidal representations. *Pacific J. Math.*, 185(1):1–32, 1998. doi:10.2140/pjm.1998.185.1.

[2] Nicolas Bourbaki. *Lie groups and Lie algebras. Chapters 4–6*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2002. Translated from the 1968 French original by Andrew Pressley. doi:10.1007/978-3-540-89394-3.

[3] F. Bruhat and J. Tits. Groupes réductifs sur un corps local. *Inst. Hautes Études Sci. Publ. Math.*, (41):5–251, 1972. URL: http://www.numdam.org/item?id=PMIHES_1972__41__5_0.

[4] Colin J. Bushnell and Guy Henniart. *The local Langlands conjecture for* GL(2), volume 335 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006. doi:10.1007/3-540-31511-X.

[5] Colin J. Bushnell and Philip C. Kutzko. The admissible dual of GL_N via restriction to compact open subgroups. In *Harmonic analysis on reductive groups (Brunswick, ME, 1989)*, volume 101 of *Progr. Math.*, pages 89–99. Birkhäuser Boston, Boston, MA, 1991.

[6] Roger W. Carter. *Simple groups of Lie type*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1989. Reprint of the 1972 original, A Wiley-Interscience Publication.

[7] Jessica Fintzen and Beth Romano. Stable vectors in Moy-Prasad filtrations. *Compos. Math.*, 153(2):358–372, 2017. doi:10.1112/S0010437X16008228.

[8] Stella Sue Gastineau. Shallow characters and supercuspidal representations. 2020. arXiv:2011.00049.

[9] Meinolf Geck. *An introduction to algebraic geometry and algebraic groups*, volume 20 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2013. First paperback reprinting of the 2003 original.

[10] Meinolf Geck. Minuscule weights and Chevalley groups. In *Finite simple groups: thirty years of the atlas and beyond*, volume 694 of *Contemp. Math.*, pages 159–176. Amer. Math. Soc., Providence, RI, 2017.

[11] Meinolf Geck. On the construction of semisimple Lie algebras and Chevalley groups. *Proc. Amer. Math. Soc.*, 145(8):3233–3247, 2017. doi:10.1090/proc/13600.

[12] Benedict H. Gross and Mark Reeder. Arithmetic invariants of discrete Langlands parameters. *Duke Math. J.*, 154(3):431–508, 2010. doi:10.1215/00127094-2010-043.

[13] Roger E. Howe. Tamely ramified supercuspidal representations of Gl_n. *Pacific J. Math.*, 73(2):437–460, 1977. URL: http://projecteuclid.org/euclid.pjm/1102810618.

[14] James E. Humphreys. *Linear algebraic groups*. Graduate Texts in Mathematics, No. 21. Springer-Verlag, New York-Heidelberg, 1975.

[15] James E. Humphreys. *Introduction to Lie algebras and representation theory*, volume 9 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1978. Second printing, revised.

[16] James E. Humphreys. *Reflection groups and Coxeter groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990. doi:10.1017/CBO9780511623646.

[17] N. Iwahori and H. Matsumoto. On some Bruhat decomposition and the structure of the Hecke rings of p-adic Chevalley groups. *Inst. Hautes Études Sci. Publ. Math.*, (25):5–48, 1965. URL: http://www.numdam.org/item?id=PMIHES_1965__25__5_0.

[18] Ju-Lee Kim. Supercuspidal representations: an exhaustion theorem. *J. Amer. Math. Soc.*, 20(2):273–320, 2007. doi:10.1090/S0894-0347-06-00544-3.

[19] Paul Levy. Involutions of reductive Lie algebras in positive characteristic. *Adv. Math.*, 210(2):505–559, 2007. doi:10.1016/j.aim.2006.07.002.

[20] Paul Levy. Vinberg's θ-groups in positive characteristic and Kostant-Weierstrass slices. *Transform. Groups*, 14(2):417–461, 2009. doi:10.1007/s00031-009-9056-y.

[21] Paul Levy. KW-sections for Vinberg's θ-groups of exceptional type. *J. Algebra*, 389:78–97, 2013. doi:10.1016/j.jalgebra.2013.04.035.

[22] I. G. Macdonald. Affine root systems and Dedekind's η-function. *Invent. Math.*, 15:91–143, 1972. doi:10.1007/BF01418931.

[23] Allen Moy. Local constants and the tame Langlands correspondence. *Amer. J. Math.*, 108(4):863–930, 1986. doi:10.2307/2374518.

[24] Allen Moy and Gopal Prasad. Unrefined minimal K-types for p-adic groups. *Invent. Math.*, 116(1-3):393–408, 1994. doi:10.1007/BF01231566.

[25] Allen Moy and Gopal Prasad. Jacquet functors and unrefined minimal K-types. *Comment. Math. Helv.*, 71(1):98–121, 1996. doi:10.1007/BF02566411.

[26] Gopal Prasad and M. S. Raghunathan. Topological central extensions of semisimple groups over local fields. II. *Ann. of Math. (2)*, 119(2):203–268, 1984. doi:10.2307/2007040.

[27] Mark Reeder and Jiu-Kang Yu. Epipelagic representations and invariant theory. *J. Amer. Math. Soc.*, 27(2):437–477, 2014. doi:10.1090/S0894-0347-2013-00780-8.

[28] P. J. Sally, Jr. and J. A. Shalika. The Plancherel formula for SL(2) over a local field. *Proc. Nat. Acad. Sci. U.S.A.*, 63:661–667, 1969. doi:10.1073/pnas.63.3.661.

[29] Allan J. Silberger. PGL_2 *over the p-adics: its representations, spherical functions, and Fourier analysis.* Lecture Notes in Mathematics, Vol. 166. Springer-Verlag, Berlin-New York, 1970.

[30] T. A. Springer. *Linear algebraic groups.* Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, second edition, 2009.

[31] Robert Steinberg. *Endomorphisms of linear algebraic groups.* Memoirs of the American Mathematical Society, No. 80. American Mathematical Society, Providence, R.I., 1968.

[32] Robert Steinberg. *Lectures on Chevalley groups*, volume 66 of *University Lecture Series.* American Mathematical Society, Providence, RI, 2016. Notes prepared by John Faulkner and Robert Wilson, Revised and corrected edition of the 1968 original [MR0466335], With a foreword by Robert R. Snapp. doi:10.1090/ulect/066.

[33] Shaun Stevens. The supercuspidal representations of p-adic classical groups. *Invent. Math.*, 172(2):289–352, 2008. doi:10.1007/s00222-007-0099-1.

[34] J. Tits. Reductive groups over local fields. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 29–69. Amer. Math. Soc., Providence, R.I., 1979.

[35] È. B. Vinberg. The Weyl group of a graded Lie algebra. *Izv. Akad. Nauk SSSR Ser. Mat.*, 40(3):488–526, 709, 1976.

[36] Jiu-Kang Yu. Construction of tame supercuspidal representations. *J. Amer. Math. Soc.*, 14(3):579–622, 2001. doi:10.1090/S0894-0347-01-00363-0.