Essays on Behavioral Matching and Apportionment Methods for Affirmative Action

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A dissertation

submitted to the Faculty of

the Department of Economics

in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

Boston College
Morrissey College of Arts and Sciences
Graduate School

February 2022

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This thesis is a collection of three essays in market design concerning designs of matching markets, affirmative action schemes, and COVID-19 testing policies.

In Chapter 1, we explore the possibility of designing matching mechanisms that can accommodate non-standard choice behavior. In the standard model of matching markets, preferences over potential assignments encode participants' choice behavior. Our contribution to this literature is introducing behavioral participants to matching theory's setup. We pin down the necessary and sufficient conditions on participants' choice behavior for the existence of stable and incentive compatible matching mechanisms. Our results imply that well-functioning matching markets can be designed to adequately accommodate a plethora of non-standard (and standard) choice behaviors. We illustrate the applicability of our results by demonstrating that a simple modification in a commonly used matching mechanism enables it to accommodate non-standard choice behavior.

In Chapter 2, we show that commonly used methods in reserving positions for beneficiaries of affirmative action are often inadequate in settings where affirmative action policies apply at two levels simultaneously, for instance, at university and its departments. We present a comprehensive evaluation of existing procedures and formally and empirically document their shortcomings. We propose a new solution with appealing theoretical properties and quantify the benefits of adopting it using recruitment advertisement data from India. Our theoretical analysis hints at new possibilities for future work in the literature on the theory of apportionment (of parliamentary seats).

Chapter 3 delves into the designs of the commonly used and advocated COVID-19 testing policies to resolve a conflict between their allocative efficiency and the ability to identify the infection rates. We present a novel comparison of various COVID-19 testing policies that allows us to pin down ordinally efficient testing policies that generate reliable estimates of infection rates while prioritizing testing of persons suspected of having the disease.

Acknowledgments

I owe an immeasurable intellectual debt to my advisers, Tayfun Sönmez, M. Utku Ünver, and M. Bumin Yenmez, whose research and lectures inspired me to follow them into the field of market design. Their brilliance is contagious, and their influence is visible in almost every aspect of my research output and habits. I am enormously grateful to them for providing me with ample time and attention. The most valuable lesson I learned from them is "Loose ideas have no value."

I also had the good fortune to have been mentored by Olivier Bochet and Christopher P. Chambers. Their intellectual influence and advice have played a defining role in my development as a scholar over the last seven years. I have also learnt an enormous amount from Arthur Lewbel, Yusuke Narita, Uzi Segal, and the several teachers and colleagues that I met at Delhi School of Economics, New York University Abu Dhabi, Paris School of Economics, Boston College, and Massachusetts Institute of Technology.

The inspiration and support of my coauthors, Gian Caspari, Haydar Evren, and Simon Siegenthaler, has been invaluable. Gian Caspari and Haydar Evren deserve credit for the work jointly produced and presented in Chapters 1 and 2, respectively. I wish to thank Lucas Coffman, Umut Dur, Mehmet Ekmekci, Michael Grubb, Sarthak Joshi, Hideo Konishi, Thayer Morrill, Parag Pathak, William Phan, and participants of the North Carolina State Theory Seminar and the Boston College Dissertation Workshops for helpful comments and discussions on earlier drafts of the chapters.

I thank my family, Ritu, Neelam, Jeewan, Bharat, and Kriti, for their continuous encouragement and care. A special thanks to Richa Vijay for her companionship since 2014 and continual expression of faith in me during my volatile years of adapting to new academic environments. Last but most, my graduate school experience would not have been the same without my lovely friends and flatmates: Marco Errico, Marius Huber, Luigi Pollio, Simone Pesce, Paul Sarkis, and Pietro Visaggio, among many others.

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Chapter 1

Non-Standard Choice in Matching Markets

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1.1 Introduction

The market design approach of using microeconomic theory to solve real-life resource allocation problems has helped transport important economic insights from theory to practice. Research, triggered by exchanges between researchers and practitioners, has generated several mechanisms tailored for real markets — prominent examples include entry-level labor markets, school choice, refugee resettlement, spectrum auctions, organ transplantation, and internet advertising.[1] Vital to this approach's success has been its fastidious attention to contextual details of allocation problems, details ranging from laws and regulatory constraints to aspects of participants' strategic behavior. In this spirit, this paper presents an

[1]The initial leading applications of matching were school choice and kidney exchange (Abdulkadiroglu and Sönmez (2003a), Abdulkadiroğlu et al. (2005b), Abdulkadiroğlu et al. (2005a), Roth et al. (2005)). Auction applications include radio spectrum, electricity, and internet advertising (see McMillan (1994), Milgrom (2000), Wilson (2002), Edelman et al. (2007) and Milgrom and Segal (2020). Market design has since developed in various directions, for recent surveys see Sönmez and Ünver (2011), Sönmez and Ünver (2017), Kominers et al. (2017), Roth (2018) and Milgrom and Tadelis (2018).

analysis of matching markets with participants' choice behavior in focus.

Preferences over potential assignments encode participants' choice behavior in the standard model of matching theory.[2] However, there is plentiful evidence in marketing, psychology, and economics suggesting that participants' choices need not be consistent with the maximization of a preference relation. Participants may exhibit non-standard choice behavior due to behavioral biases and mistakes, among other possibilities. Established phenomenons include choice overload, framing and attraction effects, temptation and self-control, and status-quo biases.[3] This paper extends matching theory to the case where participants may exhibit such non-standard choice behavior. The following examples illustrate the significance of such an exercise in the context of matching markets.

(i) (Choice Complexity and Overload) Take the case of kidney exchange programs. The complexity of the choice problem is apparent given the amount of information needed to decide whether a kidney is a good match. A practical difficulty with the procedures that match donors to recipients is that doctors hesitate to state preferences over kidneys. However, they do not struggle to select the "best" kidney for a particular patient from a given "menu".[4] Another example is that of the US Army's branching system, where assignments have two attributes — branch assignment and length of service commitment. Ranking both branch assignment and length of service commitment jointly is considered too complex (see

[2]This assumption not only facilitates direct use of elegant algorithms in practice (e.g., Gale and Shapley (1962)'s Deferred Acceptance algorithm and Gale's Top Trading Cycles algorithm by Shapley and Scarf (1974)), but by limiting participants' strategic considerations to preference manipulations it also makes analysis of several commonly used mechanisms tractable (see, e.g., Roth (1982), Roth (1984) and Sönmez (1997)).

[3]The literature on non-standard choice is too large to summarize here. Instead, we mention a few papers that can help interested readers find the many strands of this literature. Non-standard choice behavior can be seen resulting from status-quo bias (Masatlioglu and Ok (2005), Masatlioglu and Ok (2014)), multiple conflicted selves (Kalai et al. (2002), Xu and Zhou (2007), Ambrus and Rozen (2015)), framing and order effects (Rubinstein and Salant (2006), Rubinstein and Salant (2008), Bernheim and Rangel (2009)), sequential procedures such as shortlisting (Manzini and Mariotti (2007), Horan (2016)), limited attention (Lleras et al. (2017), Manzini and Mariotti (2012), Masatlioglu et al. (2012), Cherepanov et al. (2013)) and lastly temptation and self-control (Lipman et al. (2013)).

[4]See Bade (2016) for a conversation between Sophie Bade and Utku Ünver regarding this hesitancy. Utku Ünver has been actively involved in the design and practical implementation of several kidney exchange mechanisms. To learn more about his contributions, see — `https://sites.bc.edu/utku-unver/policy-impact/`.

Greenberg et al. (2021)). In general, when potential assignments have multiple attributes considerations about the complexity of choice cannot be kept aside.[5] In such cases, eliciting a ranking over all alternatives from participants will likely inaccurately reflect their actual choice behavior.[6] Thus, analysis of such instances falls beyond the scope of standard matching theory.

(ii) (Groups as Participants) Consider the case of school admissions, where parents report a ranking over schools to a centralized authority. Preferences of parents and the various persons they consult to make this decision need not be perfectly aligned. They may therefore reach decisions by aggregating several preferences in some fashion. As seen in social choice theory, such decisions need not be consistent with maximization of a single preference. Thus participants may exhibit non-standard choice behavior even without behavioral biases and mistakes.

(iii) (Hiring with Attraction Effect) Consider a hypothetical choice situation where a manager is choosing among three job candidates: $\{a, b, c\}$. Candidate a and b are similar, but a is better. The manager's choice of candidate maybe influenced by the availability of a similar inferior alternative due to the attraction effect.[7] For example, choosing c out of $\{a, c\}$, but choosing a out of $\{a, b, c\}$. Thus exhibiting choices that cannot be rationalized by a single preference relation.

In this paper, we incorporate more general choice behavior into the classical theory of stable matchings (Gale and Shapley, 1962). We consider an admissions problem that

[5] Multi-attribute assignments are commonplace in real-life matching problems. For example, assignments in the US Army's branching system consist of branch assignment and length of service (Sönmez and Switzer (2013), Sönmez (2013), Greenberg et al. (2021)). Assignments in centralized college admissions markets (e.g., that of the University of Delhi) consist of college-course pairs.

[6] There is evidence suggesting that in choice situations involving alternatives with multiple attributes, participants make use of operational procedures and consequently exhibit non-standard choice behavior. See, e.g., Apesteguia and Ballester (2013).

[7] Identified by Huber et al. (1982), the attraction effect has been observed in job candidate evaluation (Highhouse (1996), Slaughter (2007), Slaughter et al. (1999)) among various other settings. See footnote 3 in Ok et al. (2015) for other settings and references.

consists of individuals and institutions. Institutions are non-strategic agents, equipped with exogenously determined capacities and priority orderings over individuals.[8] In contrast to the standard setup, where individuals have preferences over potential assignments, we equip the individuals in our model with choice functions that determine choice (singleton or empty) from any non-empty subset of assignments.[9] The advantage of having choice functions is that with varying restrictions on choice functions, we can contrast the results we will obtain in this perturbed setup with known results from the classical setup.

A matching is a solution to the admissions problem. It matches individuals and institutions with each other. A matching is *(pairwise) stable* if no individual is assigned an unacceptable institution,[10] no institution is assigned an unacceptable individual, and no individual-institution pair (who are originally not matched with each other) prefer being matched with each other, possibly instead of some of their current assignments. We import this notion of stability into our setup with choice functions. Section 1.2 formally introduces the model and relevant definitions. Our objective is to analyze the existence of stable matchings when individuals exhibit non-standard choice behavior.[11]

Without any sophistication in choice behavior, stable matchings may not exist. Section 1.3 provides two necessary and sufficient conditions on individuals' choice behavior for the existence of stable matchings (Theorem 1). The first condition, *weak acyclicity*,

[8]Priority orderings are typically determined by exam scores, neighborhood proximity, interviews, or affirmative action considerations. For example, centralized admissions are based entirely on priority orderings determined by exam scores in Turkey and China. It is a generally held belief that these institutions do not have an incentive to alter or manipulate their priority orderings strategically. In matching theory, this is a defining feature of the student placement model of Balinski and Sönmez (1999) and the school choice model of Abdulkadiroglu and Sönmez (2003a).

[9]In many real-life matching mechanisms, individuals report choices, not preferences. See, e.g., the college admissions procedures used in Brazil (Bo and Hakimov (2019)) and Inner Mongolia, China (Gong and Liang (2020)).

[10]An assignment is *unacceptable* for an individual if it is not chosen from the singleton set containing it.

[11]There are two motivations for studying stable matchings. First, there is a strong correlation between the success of a matching exchange and its capability of delivering stable matchings (see Roth (2002)). Matching markets in the UK provide field evidence that supports this finding. Moreover, Kagel and Roth (2000) confirm this hypothesis in a controlled lab environment. The second motivation comes from a mathematically equivalent fairness notion introduced for priority-based allocation mechanisms, where failure to respect priorities can have legal implications. This fairness notion, known as *elimination of justified envy*, was introduced by Balinski and Sönmez (1999) in the context of centralized school admissions and since then has appeared in several other real-world matching market proposals to emphasize the requirement of respecting priorities.

rules out the presence of strict cycles in choices for any sequence of binary menus. The second condition, *acceptable-consistency*, requires that an unacceptable assignment is not chosen over an acceptable assignment when offered as a pair. In Section 1.3.2 we illustrate how these conditions differ from the ones that standard choice behavior requires.

In Theorem 2, we present a novel characterization of individuals' ability to strictly order (rank) alternatives in terms of Plott (1973)'s *path independence*.[12] Path independence requires that if a menu of institutions is segmented arbitrarily, choice from the menu consisting of only the chosen assignments from each segment, must be the same as the choice made from the unsegmented menu. The purpose of Theorem 2 is twofold. First, it connects our model to the standard model in matching theory with (strict) preferences over potential assignments. Second, it helps contrast the requirements of standard choice behavior with the two conditions we have identified in Theorem 1. In Proposition 1 we show that path independence demands more sophistication in choice behavior than required for the existence of stable matchings.

In Section 1.4 we discuss whether there is a way to reconcile our perturbed setup with the standard one. In particular, we ask the following questions. For, given choice functions of individuals, is it possible to construct an associated market with proxy preferences that induce the corresponding choice functions? The answer is yes (Lemma 2). Second, do the two markets yield the same set of stable matchings? The answer is yes if the choice functions are path independent (Proposition 3). However, if the choice functions are weakly acyclic and acceptable-consistent, there are stable matchings that do not belong to the set of stable matchings of any associated proxy market (Proposition 2). In other words, allowing for more general choice behavior not only affects how one finds a stable matching but also the structure of stable matchings. The idiosyncrasies of our setup also imply that the lattice structure of stable matchings is absent, and there are no side-optimal matchings.[13] In fact,

[12]Path independent choice rules have been studied in the context of matching markets with contracts in Chambers and Yenmez (2017).

[13]See Roth and Sotomayor (1990) for the lattice property and side-optimality results of stable matchings in Gale and Shapley (1962)'s college admissions model.

we show an even stronger result that there are markets where every stable matching is Pareto dominated (for individuals) by another stable matching (Proposition 4).

Section 1.5 is dedicated to studying the incentives of individuals. A mechanism is said to be *incentive compatible (for individuals)* if for any admission problem with individual i's choice function, denoted C_i, there does not exist another choice function C_i' such that the assignment of individual i under C_i' is better than that under C_i (when analyzed with respect to the original choice function C_i). In Theorem 3, we show that weakly acyclic and acceptable-consistent choice functions are sufficient for stable and incentive compatible mechanisms to exist. Furthermore, these two conditions are necessary for the existence of a wider class of individually rational, weakly non-wasteful, and incentive compatible mechanisms containing the class of stable and incentive compatible mechanisms. Individually rational mechanisms require that no individual is assigned an unacceptable institution and vice versa, thus ruling out trivial incentive compatible mechanisms that assign every individual the same institution regardless of choices reported. Weakly non-wasteful mechanisms ensure that no unassigned individual prefers an institution with one or more empty slots where she is acceptable, thus ruling out trivial incentive compatible mechanisms that leave every individual unassigned. In other words, weakly acyclic and acceptable-consistent choice functions are not only necessary and sufficient for the existence of a stable mechanism but also — under two mild requirements — for the existence of an incentive compatible mechanism.

Section 1.6 presents an application that shows how a commonly used procedure for assigning undergraduates to university programs can be modified to accommodate non-standard choice behavior.[14] Publicly announced cut-offs demarcate each step of the procedure and circumvent the requirement of eliciting a rank-ordered list of programs from students. At each step, the cut-offs reveal the minimum score required for acceptance at each program, thus offering students a menu of programs to choose from (or switch to). Proposition 5 shows that even though the procedure offers choice menus, when programs

[14] Such procedures are used in many countries, e.g., Brazil and China. See Bó and Hakimov (2020b) for details of the mechanisms used in Brazil and China.

announce cut-offs simultaneously, it takes path independent choice functions for the procedure to yield stable outcomes. Proposition 6 shows that requiring programs to announce their cut-offs sequentially instead of simultaneously enables the mechanism to accommodate non-standard choice behavior. The intuition underlying these results is rather immediate. Announcing cut-offs sequentially ensures that individuals decide between at most two alternatives at a time, so that no irrelevant options can distort their choices. The message here is that reducing the menu size in matching contexts facilitates choice and therefore benefits individuals, especially in admissions problems where comparing institutions is likely complex and non-standard choice behavior is expected.

Contributions with respect to the Related Literature

The interest in designing mechanisms that accommodate non-standard choice behavior has spurred a rich academic literature. Bounded rationality in strategic play and choice biases have featured in game theory, mechanism design, implementation theory and industrial organization among others (see, e.g., Compte and Postlewaite (2019), Jehiel (2020), De Clippel et al. (2019), De Clippel (2014), Bochet and Tumennasan (2019), Grubb (2015)). In matching theory, departures from standard preferences have been motivated by bounded rationality, mistakes, indifferences, complementarities, externalities, and peer preferences.[15] Zhang (2021) and Bade (2016) are studies motivated by bounded rationality and therefore are most relevant to our study. Zhang (2021) studies implications of heterogeneous strategic sophistication of individuals under the Boston mechanism and the deferred acceptance mechanism.[16] Bade (2016)'s analysis of boundedly rational individuals focuses on Pareto optimality of matching mechanisms in housing markets when Pápai (2000)'s hierarchical

[15] See, e.g., Bade (2016) and Zhang (2021) for bounded rationality, see Echenique et al. (2016) for mistakes incorporated in individuals' behavior, see Erdil and Ergin (2008) and Erdil and Ergin (2017) for indifferences, see Hatfield and Kojima (2010), Pycia (2012) and Hatfield and Kominers (2015) for preferences exhibiting complementarity, see Sasaki and Toda (1996) and Pycia and Yenmez (2021) for analysis of matching problems with externalities, see Leshno (2021) and Cox et al. (2021) for peer-dependent preferences.

[16] First identified by Pathak and Sönmez (2008), the Boston mechanism has been shown to favor strategically sophisticated parents (Dur et al. (2018a)).

exchange mechanisms are used. By contrast, we are the first to extend matching theory of admissions markets to problems where individuals may exhibit non-standard choice behavior due to choice biases among other possibilities.

Features affecting real-world performance of mechanisms have gained considerable interest in economic theory, and complexity considerations are at the forefront (see, e.g., Oprea (2020)). Today's market designer strives to design cognitively simple mechanisms by primarily easing the complexity of strategic considerations (see, e.g., Li (2017), Börgers and Li (2019), Bochet and Tumennasan (2018)). Another source of complexity concerns choice situations faced by participants when interacting with the mechanism (see Salant and Spenkuch (2021)). These take the form of shortlisting and ranking schools in school choice,[17] or choosing an assignment after higher priority individuals have made their pick in a serial dictatorship procedure. There is growing evidence on preference-reporting errors and their detrimental effects on mechanism's performance (see, e.g., Rees-Jones (2018), Rees-Jones and Skowronek (2018), Hassidim et al. (2021)). Choice complexity leading to non-standard choice behavior analyzed in this paper poses a possible explanation for these occurrences.

There are multiple reasons to believe that choice complexity could be a cause for real-world underperformance of matching mechanisms. For instance, in school admissions, parents report challenges in navigating choice.[18] In the US Army's branching system, ranking both branch assignment and length of service commitment jointly is considered complex (see Greenberg et al. (2021)). Moreover, in practice, individuals have often been found to make mistakes that strategic considerations cannot explain (see, e.g., Narita (2018) and Shorrer and Sóvágó (2018)).

One approach to mitigating choice complexity is simply reducing the number of alterna-

[17] See Calsamiglia et al. (2010) and Haeringer and Klijn (2009) for complications surrounding this research.

[18] Jochim et al. (2014) reports "Parents with less education, minority parents, and parents of children with special needs are more likely to report challenges navigating choice," and recommends investing heavily in information systems as "Parents in high-choice cities are seeking information on their options, but sorting through it all can be overwhelming."

tives in the various choice situations that may arise. Recent laboratory experiments suggest that this approach could be advantageous. Sequential (step-by-step) implementation of the Deferred Acceptance algorithm, in which participants choose from relevant menus that occur at each step of the algorithm, is shown to outperform its static counterpart in Bó and Hakimov (2020a), Klijn et al. (2019) and Grenet et al. (2019). Bó and Hakimov (2020b) and Mackenzie and Zhou (2020) have theoretically investigated the advantages of sequential implementation, yet only strategic considerations have been analyzed. The analysis presented in our paper hints that sequential mechanisms may also be better at accommodating non-standard choice behavior.

Another approach to mitigate choice complexity is designing better preference-reporting language (as discussed in Milgrom (2009) and Milgrom (2011)). Experimental findings of Budish and Kessler (2021) show that this is a promising direction to explore. Budish and Kessler (2021) show that Budish (2011)'s mechanisms for combinatorial assignments can be successfully implemented with a limited set of preference data on binary choices. Therefore, if eliciting entire choice functions seems impractical, one way to accommodate non-standard choice would be tailoring messages. We take this approach in Section 1.6 when discussing a particular application of university admissions.

1.2 Model

We start by introducing a model for two-sided matching markets that consists of institutions and individuals. Examples include assigning students to schools, children to daycare centers, asylum seekers to member states, refugees to localities, or undergraduates to university programs. Institutions in our model are not strategic agents, while individuals potentially are. Institutions have limited seats available for individuals represented by their capacities. Moreover, institutions have priority orderings over individuals that, depending on the context, are based on exam scores, interviews, or other criteria such as geographic proximity to the institution and affirmative action considerations. We deviate from the standard matching models in the way we model individuals' preferences. In order to allow for more general choice behavior, we equip the individuals in our model with choice functions instead of preference relations. Let us formally define the model — referred to as an admissions problem.

An **admissions problem** $\gamma \in \Gamma$ is a five-tuple $\langle I, S, q, C, \pi \rangle$ that consists of:

(i) a non-empty finite set of individuals I,

(ii) a non-empty finite set of institutions S,

(iii) a list of capacities of institutions $q = (q_s)_{s \in S}$,

(iv) a list of priority orders of institutions $\pi = (\pi_s)_{s \in S}$ over $I \cup \{\varnothing\}$, and

(v) a list of choice functions of individuals $C = (C_i)_{i \in I}$ over 2^S.

Each institution s has a capacity of q_s seats that represents the maximum number of individuals it can accept. Priority order π_s represents the way institution s ranks individuals. Formally, a **priority order** π_s is a strict simple order over $I \cup \{\varnothing\}$. Let Π denote the of all possible lists of priority orders. We assume that, from an institutional viewpoint there are no complementarities between individuals, so the priority order π_s and capacity q_s of an institution s translate into a (partial order) preference over sets of individuals in a straightforward way.[19]

[19]In a nutshell, an institution chooses the q_s highest priority individuals from any set of acceptable individuals. Formally, let $\succ_s$ be an partial order over 2^I. We assume that $\succ_s$ is responsive (Roth (1985)), that is,

(i) for any $I' \subset I$ with $|I'| < q_s$ and any $i \in I \setminus I'$,

$$(I' \cup \{i\}) \succ_s I' \iff \{i\} \; \pi_s \; \varnothing,$$

(ii) for any $I' \subset I$ with $|I'| < q_s$ and any $i, i' \in I \setminus I'$,

$$(I' \cup \{i\}) \succ_s (I' \cup \{i'\}) \iff \{i\} \; \pi_s \; \{i'\}.$$

Each individual i is equipped with a choice function C_i that represents her choice from any menu of institutions. Formally, a (unit demand) **choice function** C_i is a mapping $C_i : 2^S \to 2^S$ such that for every $S' \subseteq S$ we have $C_i(S') \subseteq S'$ and $|C_i(S')| \leq 1$.

Let us define a few basic terms. An institution s is **acceptable to individual** i if $C_i(\{s\}) = \{s\}$ and unacceptable if $C_i(\{s\}) = \varnothing$. Similarly, an individual i is **acceptable to institution** s if $i \; \pi_s \; \varnothing$ and unacceptable otherwise.

We are seeking matchings such that each individual is assigned a seat at only one institution and no institution exceeds its capacity. Formally, a (feasible) **matching** is a correspondence $\mu : I \cup S \mapsto I \cup S \cup \{\varnothing\}$ that satisfies:

(i) $\mu(i) \subseteq S$ such that $|\mu(i)| \leq 1$ for all $i \in I$,

(ii) $\mu(s) \subseteq I$ such that $|\mu(s)| \leq q_s$ for all $s \in S$, and

(iii) $i \in \mu(s)$ if and only if $s \in \mu(i)$ for all $i \in I$ and $s \in S$.

Let $\mathcal{M}$ denote the set of all (feasible) matchings.

We next define an analog of the standard notion of (pairwise) stability in our setup.[20] A matching μ is individually rational if no individual is assigned an unacceptable institution and no institution is assigned an unacceptable individual. A matching μ has no blocking pair if no individual-institution pair (who are originally not matched with each other) prefer being matched with each other, possibly instead of some of their current assignments. A matching that is individually rational and has no blocking pair is said to be stable. Formally, a matching μ is **(pairwise) stable** if

(i) it is **individually rational**, that is, there is no individual i such that $C_i(\mu(i)) = \varnothing$ and no institution s such that $\varnothing \; \pi_s \; i$ for some $i \in \mu(s)$, and

(ii) there is no **blocking pair**, that is, there is no pair $(i, s) \in I \times S$ such that

[20]Pairwise stability is the standard (but not the only) notion of stability in two-sided matching markets. *Group stability* is another prominent stability concept (see Konishi and Ünver (2006) and Pycia (2012)). Pairwise stability is equivalent to group stability in the standard setup of many-to-one matching markets (see Roth and Sotomayor (1990)). The same result holds in our setup (see Proposition 9).

(a) $\mu(i) \neq s$,

(b) $C_i(\mu(i) \cup \{s\}) = \{s\}$, and

(c) (1) either $i \ \pi_s \ i'$ for some $i' \in \mu(s)$, or

(2) $|\mu(s)| < q_s$ and $i \ \pi_s \ \varnothing$.

Finally, a **mechanism** is a function $\psi : \Gamma \rightarrow \mathcal{M}$ that assigns a matching $\psi[\gamma] \in \mathcal{M}$ to each admission problem $\gamma \in \Gamma$. A **mechanism is stable** if $\psi[\gamma]$ is stable for any admission problem $\gamma \in \Gamma$.

1.3 Stable Matchings

In our setup, the existence of stable matchings is not guaranteed. Therefore, in the first step, in Section 1.3.1 we establish necessary and sufficient conditions — weak acyclicity and acceptable-consistency — on individuals' choice functions for the existence of stable matchings. In Section 1.3.2 we show that choice behavior consistent with the maximization of a single preference relation is equivalent to having a (unit demand) choice function that satisfy a well-known condition called path independence. Lastly, in Proposition 1 and Example 1 we illustrate the contrast between our necessary and sufficient conditions with path independence (that the standard setup assumes), which is a stronger condition. Analysis of the structure of the set of stable matchings is presented in Section 1.4.

1.3.1 Stable Matchings under Non-Standard Choice

Without any sophistication in choice behavior, stable matchings may not exist. For instance, a choice function that selects unacceptable alternatives over acceptable ones would certainly lead to violations of individual rationality. This subsection describes the weakest requirements from individual choice behavior for the existence of stable matchings.

The first condition, weak acyclicity, rules out the possibility that an individual, regardless of the institution assigned to it, can always find another institution to block with. Formally, choice function C_i is **weakly acyclic (over acceptable institutions)** if for all positive integer $t \geq 3$ and t distinct and acceptable institutions $s^1, s^2, \ldots, s^t \in S$,[21]

$$C_i(\{s^1, s^2\}) = \{s^1\}, \ldots, C_i(\{s^{t-1}, s^t\}) = \{s^{t-1}\} \text{ implies } C_i(\{s^1, s^t\}) \neq \{s^t\}.$$

The second condition, acceptable-consistency, ensures that an individual does not choose an unacceptable institution over an acceptable one in pairwise comparisons. Formally, choice function C_i is **acceptable-consistent** if for all distinct institutions $s, s' \in S$,

$$C_i(\{s\}) = \{s\} \text{ and } C_i(\{s'\}) = \varnothing \text{ implies } C_i(\{s, s'\}) \neq \{s'\}.$$

The following result shows that, for an individual with a weakly acyclic and acceptable-consistent choice function, any set of institutions with at least one acceptable institution, contains at least one such institution that no other institution in the set is chosen over in pairwise comparisons. Formally, let the set of **C-maximal institutions** for individual i in subset $S' \subseteq S$ be denoted by $U_i(S') \equiv \Big\{s \in S' : \{C_i(\{s, s'\}) = \{s'\} \text{ for some } s' \in S \setminus \{s\}\} = \varnothing\Big\}$. The following lemma holds.

Lemma 1. *For a weakly acyclic and acceptable-consistent choice function C_i and a subset of institutions $S' \subseteq S$ containing at least one acceptable institution, the set of C-maximal institutions $U_i(S')$ is non-empty.*

An analogous but much stronger version of this result trivially holds for the case of strict preferences. That is, every set of institutions, with at least one acceptable institution, must contain an institution preferred to any other institution in the set. Our next result shows that

[21] Weak acyclicity resembles *Strong Axiom of Revealed Preference (SARP)*, however there are important distinctions. Unlike SARP, weak acylicty does not imply *independence of irrelevance alternatives (IIA)* because it places no restriction on choices from menus of size larger than two. This is shown in Example 1. For definitions and an excellent exposition of SARP and IIA, see Bossert et al. (2010).

the weaker version presented in Lemma 1 is enough to construct a mechanism that always leads to a stable outcome. Moreover, if either weak acyclicity or acceptable-consistency are violated, the existence of a stable matching is no longer guaranteed.

Theorem 1. *Fix I, S, q, C. There exists a stable matching for every priority order profile $\pi \in \Pi$ if and only if the choice functions are weakly acyclic and acceptable-consistent.*

The proof of Theorem 1, given in Section 1.8, describes an algorithm that always yields a stable matching for weakly acyclic and acceptable-consistent choice functions. This result highlights that matching markets can be designed to accommodate a plethora of choice behaviors that are not allowed under the standard setup consisting of individuals with preference relations. However, the exact connection between preference relations and weak acyclic and acceptable-consistent choice functions remains to be established. This is the point of the following subsection.

1.3.2 Standard Assumptions on Choice Behavior

The two identified conditions are new to the literature on stable matching theory. The standard assumption in the literature is that individuals can rank the institutions (together with the option of remaining unassigned) in a single order. We next show that the ability to rank institutions corresponds to a well-known condition on choice functions in our setup called path independence.

Let us first define path independence formally. A choice function is **path independent** if for all $S', S'' \subseteq S$ we have

$$C_i(S' \cup S'') = C_i(C_i(S') \cup S'').$$

Path independence requires that if a set is segmented arbitrarily, choice from the menu consisting of only the chosen assignments from each segment, must be the same as the choice made from the unsegmented set. We next show that a path independent choice function reflects choice behavior that a strict order over institutions can rationalize. For a path independent choice function, there is a unique strict order over (acceptable) institutions such that the institution chosen from each menu of institutions is simply the best in that menu with respect to the strict order. This benchmark result will facilitate a direct comparison of the perturbed setup consisting of the two conditions identified in Theorem 1 with the standard setup of stable matching theory.

Let us proceed to make the idea of choice behavior consistent with the ability to rank institutions formal. Let R_i be a binary relation over $S \cup \{\varnothing\}$. A binary relation R_i over $S \cup \{\varnothing\}$ is (strongly) **complete over acceptable alternatives** if

(i) for all $s \in S$ either $s \; R_i \; \varnothing$ or $\varnothing \; R_i \; s$, and

(ii) for all $s, s' \in \{s \in S : s \; R_i \; \varnothing\}$ either $s \; R_i \; s'$ or $s' \; R_i \; s$.

A binary relation is **transitive over acceptable choices** if for all $s, s', s'' \in \{s \in S : s \; R_i \; \varnothing\}$ we have that $s \; R_i \; s'$ and $s' \; R_i \; s''$ implies $s \; R_i \; s''$. Finally, a binary relation is **anti-symmetric over acceptable choices** if for all $s, s' \in \{s \in S : s \; R_i \; \varnothing\}$, $s \; R_i \; s'$ and $s' \; R_i \; s$ implies $s = s'$. We say R_i is a **simple order over acceptable choices** in $S \cup \{\varnothing\}$ if it is (strongly) complete, transitive, and anti-symmetric over acceptable choices.

A choice function C_i can be **rationalized by a simple order over acceptable choices** R_i if and only if for all subsets $S' \subseteq S$, we have

(i) $C_i(S') = \varnothing$ if $\varnothing \; R_i \; s$ for all $s \in S'$, and

(ii) $C_i(S') = \big\{s \in S' : s \; R_i \; s' \text{ for all } s' \in \{s \in S : s \; R_i \; \varnothing\}\big\}$ otherwise.

Notice that our choice functions allow for the possibility of choosing nothing (the empty set). This possibility is not present in Plott (1973)'s original analysis of path independent choice functions where individuals' choice from a non-empty menu is not allowed to be empty. The possibility of empty choices from non-empty menus requires some additional careful considerations that lead to the following novel result.[22]

[22] It is worth noting that a similar result has been proved in Plott (1973) but for non-empty choices from non-empty menus.

Theorem 2. *A (unit demand) choice function C_i can be rationalized by a simple order over acceptable choices if and only if it is path independent.*

Theorem 2 shows that the choice behavior of an individual consistent with maximization of a single preference relation over the set of acceptable institutions can be represented with a path independent choice function. It is instructive to understand that path independence is a stronger requirement on choice sophistication than weak acyclicity and acceptable-consistency combined. In Proposition 1 we present a simple observation that path independence implies weak acyclicity and acceptable-consistency.

Proposition 1. *If a choice function is path independent, then it is weakly acyclic and acceptable-consistent. The converse statement may not hold.*

Corollary 1. *Fix I, S, q, C. There exists a stable matching for every priority order profile $\pi \in \Pi$ if the choice functions are path independent.*

Therefore, our identified conditions allow for more general choice behavior than the standard setup that assumes path independent choice behavior. Let us revisit the attraction effect (Huber et al. (1982)) example from the introduction to illustrate choice behavior that satisfies weak acyclicity and acceptable-consistency but not path independence.

Example 1 (Hiring with Attraction Effect)**.** Consider a hypothetical choice situation where a manager is choosing among three job candidates: $\{a, b, c\}$. Candidate a and b are similar, but a is better. The manager's choice of candidate may be influenced by the availability of a similar inferior alternative due to the attraction effect. For example, choosing c out of $\{a, c\}$, but choosing a out of $\{a, b, c\}$. Thus exhibiting choices that a single preference relation cannot rationalize.

Consider an admissions problem $\gamma = \langle I, S, q, C, \pi \rangle$ where the manager takes the role of an individual looking to match with a job candidate, and the job candidates take the role of institutions, each with capacity one (assuming that a job candidate cannot work in two firms).

(i) $I = \{i\}$,

(ii) $S = \{a, b, c\}$,

(iii) $q_s = 1$ for all $s \in S$,

(iv) $C_i(\{s\}) = \{s\}$ for all $s \in S$,

$C_i(\{a, b\}) = \{a\}$,

$C_i(\{a, c\}) = \{c\}$,

$C_i(\{b, c\}) = \{c\}$,

$C_i(\{a, b, c\}) = \{a\}$, and

(v) $i \; \pi_s \; \varnothing$ for all $s \in S$.

Three simple observations follow. First, C_i is not path independent because $C_i(\{a, b, c\}) = \{a\} \neq C_i\big(C_i(\{a, c\}) \cup \{b\}\big) = \{c\}$, that is, the choices cannot be rationalized by a single preference relation. Second, C_i is weakly acyclic and acceptable-consistent. Third, there exists a stable matching for this market, in particular, $\mu(i) = \{c\}$ and $\mu(a) = \mu(b) = \varnothing$.

Similar examples can be constructed for other choice behaviors that exhibits context effects related to a variety of psychological, social, or environmental factors such as status-quo bias (Masatlioglu and Ok (2005), Masatlioglu and Ok (2014)), framing and order effects (Rubinstein and Salant (2006), Rubinstein and Salant (2008), Bernheim and Rangel (2009)) and limited attention (Lleras et al. (2017), Manzini and Mariotti (2012), Masatlioglu et al. (2012), Cherepanov et al. (2013)). It is worth noting that although weak acyclicity and acceptable-consistency allow for more general choice behavior than path independence, there are well-known choice biases that violate even these (weaker) conditions. For example, consider the behavior that involves sequential shortlisting of alternatives or behavior of an individual who is maximizing a preference relation but may overlook some alternatives when making choices.[23] For instance, parents may restrict attention to schools in a five mile

[23] Such behavior was first highlighted in marketing literature, where the set of alternatives describes the

radius from their place of residence when there are numerous options but would drop that restriction if only a few schools are capable of nurturing their child's unique talents. Such behavior can lead to a pairwise cycle of choices and thus violate weak-acyclicity (see, e.g., the choice behavior presented in Manzini and Mariotti (2007)).

1.4 Richness

Shedding light on implications of having weakly acyclic and acceptable-consistent choice functions rather than standard preferences (represented by path independent choice functions) requires more than stating the relationship between the two. One must also understand what they imply for stable matchings. This section shows that the set of stable matchings under weak acyclic and acceptable-consistent choice functions is richer than the set of stable matchings in the standard setup. Moreover, well-known results regarding the lattice structure and side-optimality of stable matchings under the standard setup do not hold in our setup.

We start by constructing an associated proxy admission problem that differs only in that individuals have strict preferences as opposed to choice functions, and the preferences of individuals are in line with their choices from binary menus. Lemma 2 shows that for any admissions problem with weakly acyclic and acceptable-consistent choice functions, there always exists at least one associated proxy admissions problem. Interestingly, each stable matching in the proxy admissions problem is also stable in the original admissions problem. However, there are stable matchings of some admissions problems that are not stable in any associated admissions problem (Proposition 2). On the other hand, for admissions markets with path independent choice functions, the set of stable matchings for any admission problem is identical to the set of stable matchings for its associated proxy admission problems

set of all options available to choose from, while the consideration set is a subset of those options that the individual actively consider when making a choice (see, e.g., Masatlioglu et al. (2012) for a recent discussion of this idea in economics).

(Proposition 3).

Let us proceed to the definitions. Let P_i be a binary relation over $S \cup \{\varnothing\}$. A binary relation P_i over $S \cup \{\varnothing\}$ is **complete over acceptable alternatives** if

(i) for all $s \in S$ either $s \; P_i \; \varnothing$ or $\varnothing \; P_i \; s$, and

(ii) for all $s, s' \in \{s \in S : s \; P_i \; \varnothing\}$ such that $s \neq s'$ either $s \; P_i \; s'$ or $s' \; P_i \; s$.

A binary relation is **transitive over acceptable choices** if for all $s, s', s'' \in \{s \in S : s \; P_i \; \varnothing\}$ we have that $s \; P_i \; s'$ and $s' \; P_i \; s''$ implies $s \; P_i \; s''$. Finally, a binary relation is **asymmetric over acceptable choices** if for all $s, s' \in \{s \in S : s \; P_i \; \varnothing\}$, $s \; P_i \; s'$ implies $\neg(s' \; P_i \; s)$. We say P_i is a **strict simple order over acceptable choices** in $S \cup \{\varnothing\}$ if it is complete, transitive, and asymmetric over acceptable choices.

Let $\gamma_P = \langle I, S, q, P, \pi \rangle$ be a **proxy admissions problem** for admission problem $\gamma = \langle I, S, q, C, \pi \rangle$, where P is a profile of strict simple orders over acceptable choices satisfying the following conditions:

(i) If $C_i(\{s, s'\}) = \{s\}$ then $s \; P_i \; s'$,

(ii) If $C_i(\{s\}) = \{s\}$ then $s \; P_i \; \varnothing$, and

(iii) If $C_i(\{s\}) = \varnothing$ then $\varnothing \; P_i \; s$.

Let Γ_γ denote the set of all proxy admissions problems for admission problem γ. In order to make sure that an associated market constructed in such a way always exists, we need to check that Γ_γ is non-empty. For admissions problems with weakly acyclic and acceptable-consistent choice functions, the answer is affirmative.

Lemma 2. *For an admissions problem γ with choice functions that are weakly acyclic and acceptable-consistent, the set of proxy admissions problems Γ_γ is non-empty.*

Our following result shows that weakly acyclicity and acceptable-consistency are not only weaker than path independence but that they also yield a larger set of stable matchings. Let us first define stability for proxy problems.

A matching μ is **stable for proxy admissions problem** γ_P if

(1) it is **individually rational**, that is, there is no individual i such that $\varnothing\ P_i\ \mu(i)$ and no institution s such that $\varnothing\ \pi_s\ i$ for some $i \in \mu(s)$, and

(2) there is no **blocking pair**, that is, there is no pair $(i, s) \in I \times S$ such that

   (a) $s\ P_i\ \mu(i)$, and

   (b) (i) either $i\ \pi_s\ i'$ for some $i' \in \mu(s)$, or

   (ii) $|\mu(s)| < q_s$ and $i\ \pi_s\ \varnothing$.

Proposition 2. *Fix an admissions problem γ with choice functions that are weakly acyclic and acceptable-consistent.*

1. *If a matching is stable for some associated proxy admissions problem, then it is also stable for the admissions problem.*

2. *The converse statement may not hold.*

We wish to emphasize that this result stems from the possibility that individuals may exhibit more general (non-standard) choice behavior than previously studied. The following example illustrates one such possibility.

Example 2. Consider an admissions problem $\gamma = \langle I, S, q, C, \pi \rangle$ where

(i) $I = \{i\}$,

(ii) $S = \{s_1, s_2, s_3\}$,

(iii) $q_s = 1$ for all $s \in S$,

(iv) $C_i(\{s\}) = \{s\}$ for all $s \in S$,

$C_i(\{s_1, s_2\}) = \{s_1\}$,

$C_i(\{s_2, s_3\}) = \{s_2\}$,

$C_i(\{s_1, s_3\}) = \varnothing$,

(v) $i \; \pi_{s_1} \; \varnothing$,

$\varnothing \; \pi_{s_2} \; i$, and

$i \; \pi_{s_3} \; \varnothing$.

A few simple observations follow. First, under the original admissions problem both the matchings $\mu(i) = s_1$ and $\mu(i) = s_3$ are stable. Second, notice that there is a unique associated proxy admissions problem for this problem. The only preference consistent with these choices is P_i such that $s_1 \; P_i \; s_2 \; P_i \; s_3$. Finally, for the associated proxy problem $\mu(i) = s_1$ is the only stable matching. $\mu(i) = s_3$ is not stable as it is blocked by s_1 and i in the proxy problem.

This is not the case for path independent choice behavior. Our next result shows that for problems with path independent choice functions, each stable matching of the admissions problem is also stable for some associated proxy admissions problem, and each stable matching of proxy admissions problem is also stable in the corresponding admissions problem.

Proposition 3. *Fix an admissions problem γ with choice functions that are path independent.*

1. *If a matching is stable for some associated proxy admissions problem, then it is also stable for the admissions problem.*

2. *The converse statement holds.*

In the standard setup consisting of individuals with preference relations without indifferences, just like in our proxy admissions problem, there exists a unique *individual-optimal stable matching* (Gale and Shapley, 1962) that every individual (weakly) prefers to any other stable matching. For matching markets consisting of individuals that are allowed to have indifferences, a weakening of the notion of individual-optimality, called *constrained efficiency*, is seen in Erdil and Ergin (2008). A constrained efficient stable matching corresponds to a matching that is not Pareto dominated by any other stable matching. We next define this property formally and then show that for some problems in our setup, even a constrained efficient matching might not exist (Proposition 4).

A stable matching μ is **constrained efficient (for individuals)** if it is not Pareto dominated by any other stable matching μ'. That is, μ' **Pareto dominates (for individuals)** μ if $C_i(\mu'(i) \cup \mu(i)) \neq \mu(i)$ for every $i \in I$ and $C_j(\mu'(j) \cup \mu(j)) = \mu'(j)$ for some $j \in I$. The following result shows that even a constrained efficient stable matching is not guaranteed in our setup.

Proposition 4. *There exists an admissions problem $\gamma \in \Gamma$ with choice functions that are weakly acyclic and acceptable-consistent, that does not have a constrained efficient stable matching.*

The result again stems from the possibility that individuals may exhibit seemingly non-standard choices. The case is rather easy to understand by means of an example where individual choices are weakly acyclic and acceptable-consistent but not path independent.

Example 3. Consider an admissions problem $\gamma = \langle I, S, q, C, \pi \rangle$ where

(i) $I = \{i_1, i_2, i_3\}$,

(ii) $S = \{s_1, s_2, s_3\}$,

(iii) $q_s = 1$ for all $s \in S$,

(iv) $C_i(\{s\}) = \{s\}$ for all $i \in I$ and $s \in S$. Moreover,

Choice Menu	Choices: C_{i_1}	C_{i_2}	C_{i_3}
$\{s_1, s_2\}$	s_1	$\varnothing$	$\varnothing$
$\{s_2, s_3\}$	s_2	$\varnothing$	s_2
$\{s_1, s_3\}$	$\varnothing$	$\varnothing$	s_3

(v) $i_2 \; \pi_s \; i_3 \; \pi_s \; i_1$ for all $s \in S$.

Note that i_2 would never block a matching where she is assigned some institution. Similarly, i_1 having the lowest priority, would never contest another individual's assignment. Moreover, i_3 can only block assignments of i_1. It follows that there are exactly three stable matchings for problem γ.

(i) $\mu(i_1) = \{s_2\}$, $\mu(i_2) = \{s_3\}$ and $\mu(i_3) = \{s_1\}$;

(ii) $\nu(i_1) = \{s_1\}$, $\nu(i_2) = \{s_2\}$ and $\nu(i_3) = \{s_3\}$; and

(iii) $\eta(i_1) = \{s_3\}$, $\eta(i_2) = \{s_1\}$ and $\eta(i_3) = \{s_2\}$.

Recall the definition of the Pareto domination relation – μ' Pareto dominates (for individuals) μ if $C_i(\mu'(i) \cup \mu(i)) \neq \mu(i)$ for every $i \in I$ and $C_j(\mu'(j) \cup \mu(j)) = \mu'(j)$ for some $j \in I$. In this case, ν Pareto dominates μ, μ Pareto dominates η and η Pareto dominates ν. Since every stable matching has a Pareto improvement which is still stable, a constrained efficient stable matching does not exist in this case.

In conclusion, the results presented in this section imply that more general choice behavior, taking the form of weakly acyclic and acceptable-consistent choice functions, leads to a richer set of stable matchings. The set is shown to be different from the stable set of Gale and Shapley (1962) that has a lattice structure and stable matchings that are preferred over other stable matchings by all the participants on one side of the market.

1.5 Incentives

We next turn to analyze the strategic incentives of individuals when choices are non-standard. We show that weakly acyclic and acceptable-consistent choice functions are both necessary and sufficient for a large class of incentive compatible mechanisms.

Let us now define incentive compatibility for our setup but first recall the definition of a mechanism. A **mechanism** is a function $\psi : \Gamma \to \mathcal{M}$ that assigns a matching $\psi[\gamma] \in$

$\mathcal{M}$ to each admission problem $\gamma \in \Gamma$. A **mechanism is stable** if $\psi[\gamma]$ is stable for any admission problem $\gamma \in \Gamma$. A mechanism is said to be incentive compatible (for individuals) if for any admission problem with individual i's choice function denoted C_i, there does not exist another choice function C_i' such that the assignment of individual i under C_i' is better than that under C_i. Formally, a mechanism ψ is said to be **incentive compatible (for individuals)** if for any admissions problem $\gamma = \langle I, S, q, C, \pi \rangle$ there does not exist $\hat{\gamma} = \langle I, S, q, (\hat{C}_i, C_{-i}), \pi \rangle$ such that

$$\psi[\gamma](i) \neq \psi[\hat{\gamma}](i) \quad \text{and} \quad C_i(\psi[\hat{\gamma}](i) \cup \psi[\gamma](i)) = \psi[\hat{\gamma}](i).$$

Theorem 3 shows that weakly acyclic and acceptable-consistent choice functions are sufficient for the existence of stable and incentive compatible mechanisms. However, the two conditions are necessary for a bigger class of mechanisms, a class that contains stable and incentive compatible mechanisms. To define this class, we need two additional definitions that pin down meaningful incentive compatible mechanisms.

First, we need individual rationality to make sure the mechanism assigns an acceptable institution to every individual. This rules out trivial incentive compatible mechanisms that assign every individual the same institution regardless of choices reported. Formally, a matching μ is **individually rational** if there is no individual i such that $C_i(\mu(i)) = \varnothing$ and no institution s such that $\varnothing \pi_s i$ for some $i \in \mu(s)$. A **mechanism is individually rational** if $\psi[\gamma]$ is individually rational for any admission problem $\gamma \in \Gamma$. It is worth noting that a stable matching is always individual rational. Therefore, stability implies individual rationality.

Second, we need to ensure that no unassigned individual prefers an institution with one or more empty slots and where she is acceptable. Thus rules out trivial incentive compatible and individual rational mechanisms that leave all individuals unassigned. Formally, a matching μ is **weakly non-wasteful** if there exists no individual i with $\mu(i) = \varnothing$ and s with $|\mu(s)| < q_s$ such that $C_i(\{s\}) = \{s\}$ and $s \pi_s \varnothing$. A **mechanism is weakly non-wasteful** if

$\psi[\gamma]$ is weakly non-wasteful for any admission problem $\gamma \in \Gamma$. Again it is worth noting that a stable matching is always weakly non-wasteful because weak non-wastefulness is a weaker requirement than having no blocking pairs. Under weak non-wastefulness, only unassigned individuals block with only those institutions that have empty seats available. Therefore, stability implies weak non-wastefulness.

We next show that weakly acyclic and acceptable-consistent choice functions are both necessary and sufficient for existence of individually rational, weakly non-wasteful, and incentive compatible mechanisms (Corollary 2). Moreover, they are sufficient for the existence of stable and incentive compatible mechanisms (Theorem 3).

Theorem 3. *Fix I, S, q.*

1. *If the choice functions are weakly acyclic and acceptable-consistent, then a stable and incentive compatible mechanism exists.*

2. *There exists an individually rational, weakly non-wasteful, and incentive compatible mechanism for every priority order profile $\pi \in \Pi$ only if the choice functions are weakly acyclic and acceptable-consistent.*

Corollary 2. *Fix I, S, q. There exists an individually rational, weakly non-wasteful, and incentive compatible mechanism for every priority order profile $\pi \in \Pi$ if and only if the choice functions are weakly acyclic and acceptable-consistent.*

1.6 Application

Many admissions procedures use mechanisms where individuals get multiple opportunities to report their choices. For instance, college admissions in Brazil allow students to revise choices over four consecutive days after knowing the cut-off score at each university (Bo and Hakimov (2019)). In Inner Mongolia, Chinese students choose one college at a time and are allowed to change their choices at any time before a pre-announced deadline

(Gong and Liang (2020)). Such procedures are used in many other settings that have been analyzed by market designers — Wake County in North Carolina, Tunisia, Germany and France.[24]

There is no set term for such mechanisms in the market design literature. Some researchers have referred to them as sequential (implementation of static) mechanisms (Bó and Hakimov (2020b)), and others have termed them as dynamic (implementation of static) mechanisms (Mackenzie and Zhou (2020)). Dynamic/Sequential mechanisms have been shown to outperform their static counterparts in lab and field experiments (Bó and Hakimov (2020a), Klijn et al. (2019) and Grenet et al. (2019)). Explanations range from the simplicity of strategic considerations to transparency and credibility. Our analysis shows that sequential mechanisms are a promising avenue for accommodating non-standard choice behavior, circumventing the daunting task of (ex-ante) collecting all relevant choice information for running a direct mechanism.

Intuitively, dynamic/sequential mechanisms can serve as compelling alternatives to direct mechanisms by limiting the number of options under consideration at each step, thereby mitigating the problem of choice overload (Grenet et al. (2019), Hakimov et al. (2021)).[25] This section shows that dynamic/sequential mechanisms can be tailored further to adequately accommodate non-standard choice behavior by reducing the size of the encountered choice sets.

We consider the University of Delhi's college admissions procedure, which resembles the steps of college-proposing Deferred Acceptance algorithm by Gale and Shapley (1962).[26] The admissions procedure begins with high-school graduates applying to college programs and reporting their national high school exam scores to the university. The university uses these scores to release a public list of cut-off marks for college programs

[24]See Dur et al. (2018a) for Wake County, North Carolina; Luflade (2018) for Tunisia; Grenet et al. (2019) for Germany; and Haeringer and Iehlé (2019) for France.

[25]Thus explaining the wide-spread use of such mechanisms in nationwide college admissions, that come with a very large number of options.

[26]See University of Delhi's undergraduate admissions procedure on page 18 of the following file: https://www.du.ac.in/adm2019/pdf/BulletinForUpload30May2019.pdf

that represent the lowest score necessary to be accepted at a college program. All college programs are required to admit all applicants who meet the announced cut-off criteria. On declaration of the list of cut-offs, applicants are required to choose a college program from the list of college programs that they are eligible for, that is, the programs where their score is higher than the cut-off. The applicants then take (provisional) admission in their choice of program by submitting required documents within the prescribed duration of the cut-off round. For college programs that could not fill all their seats, a new list of (weakly) lower cut-offs is released. If in the updated list, applicants find themselves eligible for admission to another college program that they prefer over their current assignment, they can cancel their previous admission and take provisional admission at this preferred program. Once an applicant cancels a provisional admission, the applicant cannot be re-admitted to the same program.

Notice that whenever a new list of cut-offs is released, an applicant faces a choice situation where only the relevant programs are under consideration. That is, the programs where the applicant satisfies the cut-off criteria. We will show that even though only relevant choice menus are offered during each step of the procedure, the mechanism demands great sophistication in choice behavior. So much so that one can claim that the outcomes are stable if and only if the applicants exhibit path independent choice behavior (Proposition 5).

Let us adopt the University of Delhi's admissions procedure to our model and give a schematic algorithm to compute the matching. The only difference between our model and the problem that the University of Delhi faces is that merit scores determine the priority orderings of the programs over applicants.[27] Let us establish this connection before moving on to the algorithm.

Let the **merit score** of individual i at institution s be defined as

[27]Since priorities are strict, our model corresponds to situations where there is a tie-break rule that determines how to rank applicants that got the same exam score.

$$m_s(i) = \begin{cases} |\{i' \in I : i \pi_s i'\}| + 1, & \text{if } i \; \pi_s \; \varnothing \\ 0, & \text{otherwise} \end{cases}$$

Let the lowest non-zero merit score at institution s be $\underline{m}_s$.

Admissions using Simultaneous Cut-offs Algorithm

- **Step 0:** Each individual i applies to all acceptable institutions s such that $C_i(\{s\}) = \{s\}$. Let $\psi[\gamma]^0(i) \equiv \varnothing$, $\psi[\gamma]^0(s) \equiv \varnothing$ and $c_s^0 \equiv n + 1$.

- **Step t (t $\geq$ 1) with Simultaneous Cut-offs:** Each institution s announces a cut-off $c_s^t \in \{\underline{m}_s, \ldots, n+1\}$ such that $|\{i \in I : C_i(\{s\}) = \{s\}$ and $c_s^t \leq m_s(i) < c_s^{t-1}\}| + |\psi[\gamma]^{t-1}(s)| \leq q_s$ and $c_s^t < c_s^{t-1}$. If there is no such cut-off, the institution announces $c_s^t = c_s^{t-1}$.

  Let $B_i^t \equiv \{s \in S : C_i(\{s\}) = \{s\}$ and $c_s^t \leq m_s(i) < c_s^{t-1}\} \cup \psi[\gamma]^{t-1}(i)$ denote the set of institutions that individual i can choose from. Individual i's assignment is updated as follows:

$$\psi[\gamma]^t(i) = \begin{cases} C_i(B_i^t), & \text{if } C_i(B_i^t) \neq \varnothing \\ \psi[\gamma]^{t-1}(i), & \text{otherwise} \end{cases}$$

  Institution s's assignment is $\psi[\gamma]^t(s) = \{i \in I : \psi[\gamma]^t(i) = \{s\}\}$. If each institution s has either filled its capacity or has announced the lowest cut-off possible ($c_s = \underline{m}_s$), then stop and return $\psi[\gamma]^t$ as the outcome.

The algorithm terminates in a finite number of steps because there is a new rejection in every step that is not terminal, and there are only a finite number of students (and therefore merit scores). Therefore, the outcome of the algorithm is well-defined.

Let ψ^{sim} be the mechanism based on the simultaneous cut-offs algorithm. The following result formalizes the requirements of this mechanism in terms of the choice sophistication of the applicants.

Proposition 5. *The simultaneous cut-offs mechanism ψ^{sim} leads to a stable outcome $\psi^{sim}[\gamma]$ for every admissions problem $\gamma \in \Gamma$ if and only if the choice functions are path independent.*

This result shows that not all dynamic/sequential mechanisms are proficient at handling choice complexity. This particular mechanism requires choices consistent with a ranking of programs. We next check whether there is a way to tailor this mechanism so that it is less demanding in terms of choice sophistication. The answer is affirmative.

Next, we modify the mechanism such that the cut-offs for programs are released not simultaneously but sequentially. The modified mechanism identifies and presents the individuals with a pair of programs at each step with the presumption that applicants have no difficulty discarding the program that they dislike when choosing from two options. Thus, the individual reaches the final choice by discarding programs in binary comparisons in a some order. Let us formally define this mechanism.

Admissions with Sequential Cut-offs Algorithm

- **Step 0:** Each individual i applies to all acceptable institutions s such that $C_i(\{s\}) = \{s\}$. Let $\psi[\gamma]^0(i) \equiv \varnothing$, $\psi[\gamma]^0(s) \equiv \varnothing$ and $c_s^0 \equiv n+1$.

- **Step t (t $\geq$ 1) with Sequential Cut-offs:** A **single** institution s announces a cut-off $c_s^t \in \{1, \ldots, n+1\}$ such that $|\{i \in I : c_s^t \leq \pi_s(i) < c_s^{t-1}\}| + |\psi[\gamma]^{t-1}(s)| \leq q_s$ and $c_s^t < c_s^{t-1}$. Other institutions $s' \neq s$ announce $c_{s'}^t = c_{s'}^{t-1}$.

  Let $B_i^t \equiv \{s \in S : C_i(\{s\}) = \{s\} \text{ and } c_s^t \leq m_s(i) < c_s^{t-1}\} \cup \psi[\gamma]^{t-1}(i)$ denote the set of institutions that individual i can choose from. Individual i's assignment is updated as follows:

$$\psi[\gamma]^t(i) = \begin{cases} C_i(B_i^t), & \text{if } C_i(B_i^t) \neq \varnothing \\ \psi[\gamma]^{t-1}(i), & \text{otherwise} \end{cases}$$

Institution s's assignment is $\psi[\gamma]^t(s) = \{i \in I : \psi[\gamma]^t(i) = \{s\}\}$. If each institution s has filled its capacity or has announced the lowest cut-off possible ($c_s = m_s$), then stop and return $\psi[\gamma]^t$ as the outcome.

The algorithm terminates in a finite number of steps for the same reason as the simultaneous cut-offs algorithm. Therefore, the outcome of the algorithm is well-defined.

Let ψ^{seq} be the mechanism based on the sequential cut-offs algorithm. Notice that the outcomes of this mechanism are order dependent. That is, the outcome depends on the order of cut-offs announcements from institutions (which is not fixed). Therefore, the mechanism is said to lead to stable outcomes if they lead to stable outcomes for any order of cut-offs announcements from institutions.

Restricting choice situations to binary menus allows the sequential cut-offs mechanism to accommodate more choice behaviors than the simultaneous cut-offs mechanism. Even so, the weakly acyclic choices are not accommodated. In addition to acceptable-consistency, this mechanism requires acyclic choices, that is, choices without any weak and strict cycles in binary menus. This requirement is stronger than weak acyclicity, which rules out only strict cycles.

Formally, a choice function C_i is **acyclic (over acceptable institutions)** if for all sequences of acceptable institutions $s^1, s^2, \ldots, s^t \in S$,

$$C_i(\{s^1, s^2\}) = \{s^1\}, \ldots, C_i(\{s^{t-1}, s^t\}) = \{s^{t-1}\} \text{ implies } C_i(\{s^1, s^t\}) = \{s^1\}.$$

Proposition 6. *The sequential cut-offs mechanism ψ^{seq} leads to a stable outcome $\psi^{seq}[\gamma]$ for every admissions problem $\gamma \in \Gamma$ if and only if the choice functions are acyclic and acceptable-consistent.*

Therefore, in line with the intuition that making consistent choices from binary sets is easier than choosing from potentially larger sets, the sequential cut-offs mechanism requires much less choice sophistication than its simultaneous counterpart. Note that both ψ^{sim} and the proposed modification ψ^{seq} are not incentive compatible.[28] That is, the two mechanisms yield stable outcomes for the observed choices, given that the choices satisfy the corresponding conditions.

1.7 Conclusion

We extend matching theory to problems where individuals may exhibit a plethora of non-standard choice behaviors. We show that weak acyclic and acceptable-consistent choice functions are both necessary and sufficient for the existence of stable matchings and a large class of incentive compatible mechanisms. Compared to the standard choice behavior, characterized by path independent choice functions, our identified conditions allow for more general choice behavior of individuals and lead to a larger set of stable matchings. In our setup, classical results, such as the existence of an individual-optimal matching or a lattice structure, cease to exist. We find a stronger implication of non-standard choice behavior on the set of stable matchings. We show that for some problems, with weak acyclic and acceptable-consistent choice functions, even a Pareto undominated stable matching may not exist. In other words, allowing for more general choice behavior not only affects how one finds a stable matching but also the structure of the set of stable matchings.

We investigate an application in the context of centralized university admissions. Building on insights from the literature on preference-reporting language (see, e.g., Milgrom (2009), Milgrom (2011), Budish and Kessler (2021)), we tweak a commonly used mech-

[28]In rare situations, misrepresenting choices by rejecting an acceptable institution (that is preferred over the current tentative assignment) can lead to better outcomes as follows. The rejected institution lowers its cut-off, attracting an individual from another institution, which in turn lowers its cut-off — opening up a slot for the original manipulator. Notice that the problem disappears if the relevant market is sufficiently large as a single individual can no longer affect an institution's cut-off by rejecting that institution.

anism to reduce the complexity of choice situations that individuals face when interacting with the mechanism to mere binary comparisons. In contrast to the original mechanism, the modified mechanism is shown to accommodate non-standard choice behavior adequately. Focusing on mechanisms where choices are made directly — as opposed to elicited prior to the match by a centralized authority — circumvent the need to choose a preference-reporting language that accurately reflects individuals' choice behaviors. An important dimension is identifying the minimum amount of choice information required to arrive at stable outcomes, knowledge of which could greatly facilitate the use of such mechanisms in practice. This will be the subject of future work.

1.8 Mathematical Appendix

Lemma 1

Proof. Consider a subset $S' \subseteq S$ and individual $i \in I$ with at least one acceptable institution, that is, $s \in S'$ such that $C_i(\{s\}) = \{s\}$. Moreover, suppose that $U_i(S') = \varnothing$. Define the set of acceptable institutions as $\overline{S'} = \{s \in S' : C_i(\{s\}) = \{s\}\} \neq \varnothing$ and the set of unacceptable institutions as $\underline{S'} = \{s \in S' : C_i(\{s\}) = \{s\}\}$.

First, suppose there exists an unacceptable institution $s' \in \underline{S'}$ that is chosen over an acceptable institution $s \in \overline{S'}$, i.e., $C_i(\{s, s'\}) = \{s'\}$. Since choice functions are acceptable-consistent we have that $C_i(\{s\}) = \{s\}$ and $C_i(\{s'\}) = \varnothing$ implies $C_i(\{s, s'\}) \neq \{s'\}$ — a contradiction. Therefore, only an acceptable institution can be chosen over another acceptable institution.

Second, suppose that $|\overline{S'}| \leq 2$. Then there trivially exists a C-maximal institution, as only an acceptable institution can be chosen over another acceptable institution.

Third, suppose that for every acceptable institution $s \in \overline{S'}$, there is another institution $s' \in S'$ that is chosen in pairwise comparison. By the first part, only an acceptable institution can be chosen over another acceptable institution. That is, for all $s \in \overline{S'}$ there exists $s' \in$

$\overline{S'} \setminus \{s\}$ such that $C_i(\{s, s'\}) = \{s'\}$. By the second part, $|\overline{S'}| \geq 3$. But this implies that there exists a positive integer $t \geq 3$ and t distinct and acceptable alternatives $s^1, s^2, \ldots, s^t$, such that $C_i(\{s^1, s^2\}) = \{s^2\}, \ldots C_i(\{s^{t-1}, s^t\}) = \{s^t\}$, and $C_i(\{s^t, s^1\}) = \{s^1\}$ — a contradiction to weak acyclicity. ■

Theorem 1

We construct an outcome for every admission problem $\gamma \in \Gamma$ using the algorithm described next.

Non-Block Algorithm

Step 0: For each $i \in I$ consider the set of mutually acceptable institutions $S_i^0 \equiv \{s \in S : C_i(s) = \{s\}$ and $s \pi_s \varnothing\}$. Let the set of individuals proposing to institution s be denoted by $I_s^0 \equiv \varnothing$. Finally, let $A_s(I') \equiv \{i \in I' : |\{i' \in I' : i' \pi_s i\}| < q_s\}$ denote the set of individuals in $I' \subseteq I$ tentatively assigned to institution s.

Step $k \geq 1$: If there are any individual currently not tentatively admitted, i.e., $i \notin \bigcup_{s \in S} A_s(I_s^{k-1})$, and that still has a mutually acceptable institution left to propose, i.e., $S_i^{k-1} \neq \varnothing$. Then let each such individual i propose to an C-maximal institution $s_i^k \in S_i^{k-1}$ (if there are multiple, take the lowest-subscript institution) — by Lemma 1 such an institution exists since $S_i^{k-1} \neq \varnothing$ and contains only acceptable institutions. Let P_s^k denote the set of individuals proposing to institution s at step k. Set $S_i^k = S_i^{k-1} \setminus \{s_i^k\}$ and $I_s^k = I_s^{k-1} \cup P_s^k$. Go to step $k+1$.

Otherwise, if no such individual exists, the algorithm stops. Then each institution is assigned $A_s(I_s^{k-1})$ while each individual i is assigned $s \in S$ such that $i \in A_s(I_s^{k-1})$ and $\varnothing$ otherwise.

Let the **non-block mechanism** ψ^{nb} be described by the function that associates outcome of the non-block algorithm to any admission problem $\gamma \in \Gamma$. We now prove Theorem 1.

Proof. **The "if" part:** If the choice functions $(C_i)_{i \in I}$ are weakly acyclic and acceptable-consistent, then the non-block mechanism ψ^{nb} yields a stable matching $\psi^{nb}[\gamma]$ for every admission problem $\gamma \in \Gamma$.

For any admission problem $\pi \in \Pi$, consider the outcome of the non-block mechanism $\psi^{nb}[\pi]$. Since individuals propose to mutually acceptable institutions only, the outcome $\psi^{nb}[\pi]$ is trivially individually rational for both institutions and individuals.

Next, suppose that there exists $s \in S \backslash \psi^{nb}[\pi](i)$ with $C_i(\{s\} \cup \psi[\pi](i)) = \{s\}$ and $s \pi_s \varnothing$. By acceptable-consistency, s must be acceptable. Together with Lemma 1, this implies that i must have proposed to s at some step k and subsequently been rejected. Therefore, for institution s it must be the case that $|\psi^{nb}[\pi](s)| = q_s$ and $i' \pi_s i$ for all $i' \in \mu[\pi](s)$.

The "only if" part: A stable matching μ exists for every admission problems $\gamma \in \Gamma$ only if the choice functions $(C_i)_{i \in I}$ are weakly acyclic and acceptable-consistent. The contrapositive statement is, if the choice functions $(C_i)_{i \in I}$ are not weakly acyclic or acceptable-consistent, then for some admissions problem $\gamma \in \Gamma$ a stable matching μ does not exist.

Part 1. Weak Acyclicity is necessary.

Suppose that for some C_i we have a cycle for $t \geq 3$ and distinct acceptable alternatives $s^1, s^2, \ldots, s^t$ in S such that $C_i(\{s^1, s^2\}) = \{s^2\}, \ldots C_i(\{s^{t-1}, s^t\}) = \{s^t\},$ and $C_i(\{s^t, s^1\}) = \{s^1\}$.

Consider an admissions problem $\tilde{\gamma} = \langle I, S, q, C, \tilde{\pi} \rangle$ such that $i \; \tilde{\pi}_s \; i'$ for all $i' \in I \setminus \{i\}$ and $s \in \{s^1, s^2, \ldots, s^t\}$ and $\varnothing \; \tilde{\pi}_s \; i$ for all $s \in S \setminus \{s^1, s^2, \ldots, s^t\}$. Consider a matching μ for problem $\tilde{\gamma}$. If $\mu(i) = \{s\}$ for some $s \in \{s^1, s^2, \ldots, s^t\}$ or $\mu(i) = \varnothing$, i forms a blocking pair with some $s' \in \{s^1, s^2, \ldots, s^t\} \setminus \mu(i)$. While if $\mu(i) = \{s\}$ for $s \in S \setminus \{s^1, s^2, \ldots, s^t\}$ we have a violation of individual rationality for the appropriate institution. Therefore, there is no stable matching for problem $\tilde{\gamma}$.

Part 2. Acceptable-consistency is necessary.

Suppose that for some C_i we have $C_i(\{s^1\}) = \{s^1\}$ and $C_i(\{s^2\}) = \varnothing$ but $C_i(\{s^1, s^2\}) =$

$\{s^2\}$.

Consider an admission problem $\tilde{\gamma} \in \Gamma$ such that $i \; \tilde{\pi}_s \; i'$ for all $i' \in I \setminus \{i\}$ and $s \in \{s^1, s^2\}$ and $\varnothing \; \tilde{\pi}_s \; i$ for all $s \in S \setminus \{s^1, s^2\}$. Consider a matching μ for problem $\tilde{\gamma}$. There are four possibilities. (i) If $\mu(i) = \{s^1\}$, i forms a blocking pair with s^2. (ii) If $\mu(i) = \{s^2\}$ we have a violation of individual rationality for individual i. (iii) If $\mu(i) = \varnothing$, i forms a blocking pair with s^1. (iv) While if $\mu(i) = \{s\}$ for $s \in S \setminus \{s^1, s^2\}$ we have a violation of individual rationality for the appropriate institution. Therefore, there is no stable matching for problem $\tilde{\gamma}$. ■

Theorem 2

We start by proving the following lemma.

Lemma 3. *Consider a path independent choice function C_i, then*

(i) an unacceptable alternative cannot be chosen, that is,

$$C_i(\{s\}) = \varnothing \implies C_i(\{s\} \cup S') \neq \{s\} \quad \textit{for any} \quad S' \subseteq S \setminus \{s\}.$$

(ii) choice from a set containing at least one acceptable alternative is non-empty, that is,

$$C_i(\{s\}) = \{s\} \implies C_i(\{s\} \cup S') \neq \varnothing \quad \textit{for any} \quad S' \subseteq S \setminus \{s\}.$$

(iii) choice from a set containing unacceptable alternatives is empty, that is,

$$C_i(\{s\}) = \varnothing \quad \textit{for all} \quad s \in S' \subseteq S \implies C_i(S') = \varnothing.$$

Proof. (i) Consider $s \in S$ such that $C_i(\{s\}) = \varnothing$ and any $S' \subseteq S \setminus \{s\}$. By path independence we have

$$
\begin{aligned}
C_i(\{s\} \cup S') &= C_i(C_i(\{s\}) \cup S') \\
&= C_i(S') \subseteq S' \\
&\neq \{s\}.
\end{aligned}
$$

(ii) Consider $s \in S$ such that $C_i(\{s\}) = \{s\}$ and any $S' \subseteq S \setminus \{s\}$. By path independence we have

$$
\begin{aligned}
C_i(\{s\} \cup S') &= C_i(\{s\} \cup (\{s\} \cup S')) \\
&= C_i(\{s\} \cup C_i(S' \cup \{s\})) \\
&= C_i(\{s\}) \\
&= \{s\} \\
&\neq \varnothing.
\end{aligned}
$$

(iii) Consider any $S' \subseteq S$ with $C_i(\{s\}) = \varnothing$ for all $s \in S'$. Let $S' = \{s_1, \ldots, s_k\}$. By path independence we can remove alternatives one by one to reach the desired conclusion, that is,

$$
\begin{aligned}
C_i(S') &= C_i(C_i(\{s_1\}) \cup (S' \setminus \{s_1\})) \\
&= C_i(S' \setminus \{s_1\}) \\
&= C_i(C_i(\{s_2\}) \cup (S' \setminus \{s_1, s_2\})) \\
&= \ldots \\
&= C_i(\{s_k\}) \\
&= \varnothing.
\end{aligned}
$$

■

We now prove Theorem 2.

Proof. **The "if" part:** If a unit demand choice function C_i is path independent then it is rationalizable by a simple order R_i over acceptable choices.

Consider C_i such that $C_i(\{s, s'\}) = \{s\}$ and $C_i(\{s', s''\}) = \{s'\}$. Note that path independence implies that

$$
\begin{aligned}
\{s\} &= C_i(\{s\} \cup \{s'\}) \\
&= C_i(\{s\} \cup C_i(\{s', s''\})) \\
&= C_i(\{s\} \cup \{s', s''\}) && \text{by path independence} \\
&= C_i(C_i(\{s, s'\}) \cup \{s''\}) && \text{by path independence} \\
&= C_i(\{s, s''\}).
\end{aligned}
$$

Therefore, C_i is transitive over binary menus.

Next, define binary relation P_i such that $s \ P_i \ s'$ for $s, s' \in S$ with $s \neq s'$ if $C_i(\{s, s'\}) = \{s\}$, $C_i(\{s\}) = \{s\}$, and $C_i(\{s'\}) = \{s'\}$. Moreover, $s \ P_i \ \varnothing$ if $C_i(\{s\}) = \{s\}$, and $\varnothing \ P_i \ s$ if $C_i(\{s\}) = \varnothing$. Similarly, define R_i such that sR_is' if and only if $[s \ P_i \ s'$ or $s = s']$, $sR_i\varnothing$ if and only if $s \ P_i \ \varnothing$, and $\varnothing R_i s$ if and only if $\varnothing \ P_i \ s$. Note that, R_i is anti-symmetric, and (strongly) complete over acceptable choices by construction. Moreover, R_i is transitive over acceptable choices as C_i is transitive over binary choices. For the constructed order R_i we next show that

(i) $C_i(S') = \varnothing$ if $\varnothing R_i s$ for all $s \in S'$, and

(ii) $C_i(S') = \{s \in S' : sR_is' \text{ for all } s' \in S'\}$ otherwise.

(i) Consider $S' \subseteq S$ such that $\varnothing R_i s$ for all $s \in S'$. For any $s \in S'$, by construction $\varnothing R_i s$ implies $\varnothing \ P_i \ s$ which implies $C_i(\{s\}) = \varnothing$. By Lemma 3, if $C_i(\{s\}) = \varnothing$ for all $s \in S'$ then $C_i(S') = \varnothing$.

(ii) Consider $S' \subseteq S$ with at least one $s \in S'$ such that $sR_i\varnothing$. By construction $sR_i\varnothing$ implies $s\ P_i\ \varnothing$ which implies $C_i(\{s\}) = \{s\}$. That is, S' contains at least one acceptable alternative. Suppose by contradiction that $C_i(S') \neq \{s \in S' : sR_is' \text{ for all } s' \in S'\}$. By Lemma 3, $C_i(S') \neq \varnothing$. Hence, let $C_i(S') = \{s'\}$ and $\{s \in S' : sR_is' \text{ for all } s' \in S'\} = \{s\}$. Since $s \neq s'$ and sR_is' we have $s\ P_i\ s'$ respectively $\{s\} = C_i(\{s, s'\})$. From this observation we reach a contradiction since

$$\begin{aligned}
\{s'\} &= C_i(\{s, s'\} \cup (S' \setminus \{s, s'\})) \\
&= C_i(\{s\} \cup (S' \setminus \{s, s'\}) && \text{by path independence} \\
&= C_i(S' \setminus \{s'\}) \\
&\neq \{s'\}.
\end{aligned}$$

The "only if" part: If a unit demand choice function C_i is rationalizable by a simple order R_i then it is path independent.

There are three cases to consider.

(i) Consider $S', S'' \subseteq S$ both containing at least one acceptable alternative, that is, s such that $sR_i\varnothing$. Let $\{s^*\} = \{s \in S' \cup S'' : sR_is' \text{ for all } s' \in S' \cup S''\}$, and $\{s^{*\prime}\} = \{s \in S' : sR_is' \text{ for all } s' \in S'\}$. Since s^*R_is' for all $s' \in S' \cup S''$ and $s^{*\prime} \in S'$ we have $\{s^*\} = \{s \in \{s^{*\prime}\} \cup S'' : sR_is' \text{ for all } s' \in \{s^{*\prime}\} \cup S''\}$. Rewriting this in terms of choice functions leads

$$\begin{aligned}
C_i(S' \cup S'') &= C_i(\{s^{*\prime}\} \cup S'') \\
&= C_i(C_i(S') \cup S'')
\end{aligned}$$

(ii) Consider $S', S'' \subseteq S$ both containing no acceptable alternatives. We have $C(S') = \varnothing$, $C(S'') = \varnothing$ and $C(S' \cup S'') = \varnothing$. It directly follows that $C_i(S' \cup S'') = C_i(S'') = C_i(C_i(S') \cup S'')$.

(iii) Consider $S', S'' \subseteq S$ where only S' contains at least one acceptable alternative. Let $\{s^*\} = \{s \in S' \cup S'' : sR_is' \text{ for all } s' \in S' \cup S''\}$. It follows that $\{s^*\} = \{s \in S' : sR_is' \text{ for all } s' \in S'\}$ as well as $\{s^*\} = \{s \in S'' \cup \{s^*\} : sR_is' \text{ for all } s' \in S'' \cup \{s^*\}\}$. Again, we get that $C_i(S' \cup S'') = C_i(C_i(S') \cup S'')$.

■

Proposition 1

Part 1. Path independent choice functions are weakly acyclic.

Consider a path independent choice function C_i, an integer $t \geq 3$ and t distinct and acceptable institutions $s^1, s^2, \ldots, s^t \in S$, $C_i(\{s^1, s^2\}) = \{s^1\}, \ldots, C_i(\{s^{t-1}, s^t\}) = \{s^{t-1}\}$.

First, notice that path independence implies that

$$\begin{aligned} C_i(\{s^1, s^2, \ldots, s^t\}) &= C_i(\{s^1, s^2, \ldots, s^{t-2}\} \cup C_i\{s^{t-1}, s^t\}) \\ &= C_i(\{s^1, s^2, \ldots, s^{t-1}\}) \\ &\neq \{s^t\}. \end{aligned}$$

Second, path independence implies that

$$\begin{aligned} C_i(\{s^1, s^2, \ldots, s^t\}) &= C_i(\{s^1, s^2, \ldots, s^{t-3}\} \cup C_i\{s^{t-2}, s^{t-1}\} \cup \{s^t\}) \\ &= C_i(\{s^1, s^2, \ldots, s^{t-4}\} \cup C_i\{s^{t-3}, s^{t-2}\} \cup \{s^t\}) \\ &= \ldots \\ &= C_i(\{s^1, s^t\}. \end{aligned}$$

Therefore, $C_i(\{s^1, s^t\} \neq \{s^t\}$, that is, C_i is weakly acyclic.

Part 2. Path independent choice functions are acceptable-consistent.

Consider a path independent choice function C_i and distinct institutions $s, s' \in S$ such that

$C_i(\{s\}) = \{s\}$ and $C_i(\{s'\}) = \varnothing$. Path independence implies that

$$\begin{aligned} C_i(\{s, s'\}) &= C_i(\{s\} \cup C_i\{s'\}) \\ &= C_i(\{s\}) \\ &\neq \{s'\}. \end{aligned}$$

Therefore, C_i is acceptable-consistent.

Lemma 2

Proof. Consider the following construction for each individual $i \in I$:

Step 0: First, consider the unacceptable assignments, that is, $\{s \in S : C_i(\{s\}) = \varnothing\}$ and order them as $\varnothing P_i s$. Second, consider the acceptable assignments $S^0 = S' \setminus \{s \in S : C_i(\{s\}) = \varnothing\}$. If $S^0 = \varnothing$ the construction stops. Otherwise consider the set of C-maximal institutions $U(S^0) = \{s \in S^0 : \{C_i(\{s, s'\}) = \{s'\}$ for some $s' \in S \setminus \{s\}\} = \varnothing\}$ — which is non-empty by Lemma 1. Let

(i) sPs' for all $s \in U(S^0)$ and $s' \in S^0$,

(ii) order $s, s' \in U(S^0)$ in any order, that is, sPs' or $s'Ps$.

Step $k \geq 1$: Consider $S^k = S^{k-1} \setminus U(S^{k-1})$. If $S^k = \varnothing$ the construction stops. Otherwise consider the set of C-maximal institutions $U(S^k)$ —- which is non-empty by Lemma 1. Let

(i) sPs' for all $s \in U(S^k)$ and $s' \in S^k$,

(ii) order $s, s' \in U(S^k)$ in any order, that is, sPs' or $s'Ps$.

This construction yields a simple order over acceptable choices satisfying the conditions that (1) if $C_i(\{s, s'\}) = \{s\}$ then $s \ P_i \ s'$; (2) if $C_i(\{s\}) = \{s\}$ then $s \ P_i \ \varnothing$; and, (3) if $C_i(\{s\}) = \varnothing$ then $\varnothing \ P_i \ s$.

■

Proposition 2

Proof. Fix an admissions problem Γ with choice functions that are weakly acyclic and acceptable-consistent.

The "if" part: If a matching is stable for some associated proxy admissions problem then it is also stable for the admissions problem.

Consider a one-to-one mapping $f : \Gamma \mapsto \Gamma_{\mathcal{P}}$ such that $f(\gamma) \in \Gamma_{\gamma}$ — which is non-empty by Lemma 2. Let $\hat{\psi}^s$ denote a stable mechanism for proxy admissions problems. Now consider the mechanism ψ^s for admissions problems such that $\psi^s[\gamma] \equiv \hat{\psi}^s[f(\gamma)]$ for all $\gamma \in \Gamma$.

(i) Suppose $\psi^s[\gamma]$ is not individually rational for some $\gamma \in \Gamma$, then there exists $i \in I$ with $\psi^s[\gamma](i) \neq \varnothing$ such that $C_i(\psi^s[\gamma](i)) = \varnothing$. This implies that in the associated proxy admissions problem $\varnothing\ P_i\ \hat{\psi}^s[f(\gamma)](i)$. Thus contradicting that $\hat{\psi}^s$ is stable for proxy admissions problems.

(ii) Suppose there exist $\gamma \in \Gamma$ such that $\psi^s[\gamma]$ has a blocking pair (i, s), i.e., $C_i(\psi^s[\gamma](i) \cup s) = s$ and $i \pi_s i'$ for some $i' \in \psi^s[\gamma](s)$, or $|\psi^s[\gamma](s)| < q_s$ with $i \pi_s \varnothing$. Note that, $C_i(\psi^s[\gamma](i) \cup s) = s$ implies that in the associated proxy admissions problem $s\ P_i\ \hat{\psi}^s[f(\gamma)]$. Thus contradicting that $\hat{\psi}^s$ is stable for proxy admissions problems.

The converse may not hold: A stable matching for the admissions problem may not be stable for any associated proxy admissions problem in Γ_{γ}.

See Example 2. ■

Proposition 3

Proof. Fix an admissions problem Γ with choice functions that are path independent.

The "if" part: If a matching is stable for some associated proxy admissions problem then it is also stable for the admissions problem.

Since path independent choice functions are weakly acyclic and acceptable-consistent (Proposition 1), this part is a corollary to Proposition 2.

The converse holds: A stable matching for the admissions problem is also stable for some associated proxy admissions problem in Γ_γ.

Notice that with path independent choice functions there is a unique proxy admissions problem for every admissions problem with the preferences constructed in the same way as in Theorem 2.

Consider a one-to-one mapping $f : \Gamma \mapsto \Gamma_{\mathcal{P}}$ such that $f(\gamma) \in \Gamma_\gamma$ — where P the simple order over acceptable choices is constructed in the same way as in Theorem 2. Let ψ^s denote a stable mechanism for admissions problems. Now consider the mechanism $\hat{\psi}^s$ for the proxy admissions problems such that $\hat{\psi}^s[f(\gamma)] \equiv \psi^s[\gamma]$ for all $\gamma \in \Gamma$.

(i) Suppose $\hat{\psi}^s[f(\gamma)]$ is not individually rational for some $\gamma \in \Gamma$, then there exists $i \in I$ with $\hat{\psi}^s[f(\gamma)](i) \neq \varnothing$ such that $\varnothing \; P_i \; \hat{\psi}^s[f(\gamma)](i)$. This implies that in the associated admissions problem $C_i(\psi^s[\gamma](i)) = \varnothing$. Thus contradicting that ψ^s is stable for proxy admissions problems.

(ii) Suppose there exist $\gamma \in \Gamma$ such that $\hat{\psi}^s[f(\gamma)]$ has a blocking pair (i, s), that is, $s \; P_i \; \hat{\psi}^s[f(\gamma)]$ and $i \pi_s i'$ for some $i' \in \hat{\psi}^s[f(\gamma)](s)$, or $|\hat{\psi}^s[f(\gamma)](s)| < q_s$ with $i \pi_s \varnothing$. Note that, $s \; P_i \; \hat{\psi}^s[f(\gamma)]$ implies that in the associated admissions problem $C_i(\psi^s[\gamma](i) \cup s) = s$. Thus contradicting that ψ^s is stable for proxy admissions problems.

■

Theorem 3

We start by defining a related proxy admission problem for any assignment problem, as well as the stability and strategy-proofness in the proxy admission problem. In a second step, we will use the connection between the two problems to prove our result.

Recall the **proxy admissions problem** $\gamma_P = \langle I, S, q, P, \pi \rangle$ for admissions problem $\gamma = \langle I, S, q, C, \pi \rangle$, where P is a profile of strict simple orders over acceptable choices satisfying the following conditions for each choice function:

(1) If $C_i(\{s, s'\}) = \{s\}$ then $s \; P_i \; s'$;

(2) If $C_i(\{s\}) = \{s\}$ then $s \; P_i \; \varnothing$; and,

(3) If $C_i(\{s\}) = \varnothing$ then $\varnothing \; P_i \; s$.

We denote the set of all proxy admission problems by $\Gamma_{\mathcal{P}}$.

A **mechanism for proxy admissions problems** is a function $\hat{\psi} : \Gamma_{\mathcal{P}} \to \mathcal{M}$ that assigns a matching $\hat{\psi}[\gamma_P] \in \mathcal{M}$ to each proxy admission problem $\gamma_P \in \Gamma_{\mathcal{P}}$. A mechanism $\hat{\psi}$ is said to be **incentive compatible (for individuals) for proxy admissions problems** if for any $\gamma_P = \langle I, S, q, P, \pi \rangle$ there does not exist $\gamma_{\hat{P}} = \langle I, S, q, (\hat{P}_i, P_{-i}), \pi \rangle$ such that

$$\hat{\psi}[\gamma_{\hat{P}}](i) \; P_i \; \hat{\psi}[\gamma_P](i).$$

A matching μ is **stable for proxy admissions problems** γ_P if

(1) it is **individually rational**, that is, there is no individual i such that $\varnothing \; P_i \; \mu(i)$ and no institution s such that $\varnothing \; \pi_s \; i$ for some $i \in \mu(s)$, and

(2) there is no **blocking pair**, that is, there is pair $(i, s) \in I \times S$ such that

   (a) $s \; P_i \; \mu(i)$, and

   (b) (i) either $i \; \pi_s \; i'$ for some $i' \in \mu(s)$, or

   (ii) $|\mu(s)| < q_s$ and $i \; \pi_s \; \varnothing$.

A mechanism $\hat{\psi}$ for proxy admissions problems is said to be **stable for proxy admissions problems** if it assigns a stable matching $\hat{\psi}[\gamma_P]$ to each proxy admission problem $\gamma_P \in \Gamma_{\mathcal{P}}$. Let $\hat{\psi}^{GS}$ denote the individual-proposing deferred acceptance algorithm defined in Gale and Shapley (1962). Recall that this mechanism is both stable and incentive compatible.

Proposition 7 (Gale and Shapley (1962))**.** *The individual-proposing deferred acceptance mechanism* $\hat{\psi}^{GS}$ *is stable for proxy admissions problems.*

Proposition 8 (Dubins and Freedman (1981), Roth (1982))**.** *The individual-proposing deferred acceptance mechanism* $\hat{\psi}^{GS}$ *is incentive compatible (for individuals) for proxy admissions problems.*

We now prove Theorem 3.

Proof. **The "if" part.** Consider a one-to-one mapping $f : \Gamma \mapsto \Gamma_{\mathcal{P}}$ such that $f(\gamma) \in \Gamma_\gamma$ — which is non-empty by Lemma 2. Moreover let f be such that for any $\gamma = \langle I, S, q, C, \pi \rangle$ and $\hat{\gamma} = \langle I, S, q, (\hat{C}_i, C_{-i}), \pi \rangle$ the proxy admissions problem only differ by the preference relation of individual i. Such a requirement can be easily accommodated following a construction akin to the one in Lemma 2.

Consider the mechanism ψ^{GS} such that $\psi^{GS}[\gamma] \equiv \hat{\psi}^{GS}[f(\gamma)]$ for all $\gamma \in \Gamma$.

(i) Suppose $\psi^{GS}[\gamma]$ is not individually rational for some $\gamma \in \Gamma$, then there exists $i \in I$ with $\psi^{GS}[\gamma](i) \neq \varnothing$ such that $C_i(\psi^{GS}[\gamma](i)) = \varnothing$. This implies that in the associated proxy admissions problem $\varnothing \; P_i \; \hat{\psi}^{GS}[f(\gamma)](i)$. Thus contradicting that $\psi^{\hat{G}S}$ is stable for every proxy admissions problem (Proposition 7).

(ii) Suppose there exist $\gamma \in \Gamma$ such that $\psi^{GS}[\gamma]$ has a blocking pair (i, s), i.e., $C_i(\psi^{GS}[\gamma](i) \cup s) = s$ and $i \pi_s i'$ for some $i' \in \psi^{GS}[\gamma](s)$, or $|\psi^{GS}[\gamma](s)| < q_s$ with $i \pi_s \varnothing$. Note that, $C_i(\psi^{GS}[\gamma](i) \cup s) = s$ implies that in the associated proxy admissions problem $s \; P_i \; \hat{\psi}^{GS}[f(\gamma)]$. Thus contradicting that $\hat{\psi}^{GS}$ is stable for every proxy admissions problem (Proposition 7).

(iii) Suppose ψ^{GS} is not incentive compatible for some $\gamma \in \Gamma$. That is, there exist $\gamma = \langle I, S, q, C, \pi \rangle$ and $\hat{\gamma} = \langle I, S, q, (\hat{C}_i, C_{-i}), \pi \rangle$ such that $\psi^{GS}[\gamma](i) \neq \psi^{GS}[\hat{\gamma}](i)$ and $C_i(\psi^{GS}[\hat{\gamma}](i) \cup \psi^{GS}[\gamma](i)) = \psi^{GS}[\hat{\gamma}](i)$. By construction, in the proxy admissions problem we have that $\hat{\psi}^{GS}[f(\hat{\gamma})](i) \ P_i \ \hat{\psi}^{GS}[f(\gamma)](i)$, thus contradicting that $\hat{\psi}^{GS}$ is incentive compatible (Proposition 8).

The "only if" part.

Part 1. Weak Acyclicity is necessary.

Suppose that for some C_i we have a cycle for $t \geq 3$ and distinct acceptable alternatives $s^1, s^2, \ldots, s^t$ in S such that $C_i(\{s^1, s^2\}) = \{s^2\}, \ldots C_i(\{s^{t-1}, s^t\}) = \{s^t\}$, and $C_i(\{s^t, s^1\}) = \{s^1\}$.

Consider an admissions problem $\tilde{\gamma} = \langle I, S, q, C, \tilde{\pi} \rangle$ such that $i \ \tilde{\pi}_s \ i'$ for all $i' \in I \setminus \{i\}$ and $s \in \{s^1, s^2, \ldots, s^t\}$ and $\varnothing \ \tilde{\pi}_s \ i$ for all $s \in S \setminus \{s^1, s^2, \ldots, s^t\}$. Moreover, $\tilde{\pi}$ is such that $|\{i' \in I \setminus \{i\} : i' \ \tilde{\pi}_s \ \varnothing\}| < q_s$ for all $s \in \{s^1, s^2, \ldots, s^t\}$ — which there is at least one such priority order as $q_s \geq 1$.

If $\psi[\gamma](i) = \{s\}$ for $s \in S \setminus \{s^1, s^2, \ldots, s^t\}$ we have a violation of individual rationality for the appropriate institution. Suppose then that the outcome of a mechanism is $\psi[\gamma](i) = \{s\}$ for some $s \in \{s^1, s^2, \ldots, s^t\}$ or $\psi[\gamma](i) = \varnothing$. Consider the following problem $\hat{\gamma} = \langle I, S, q, (\hat{C}_i, C_{-i}), \tilde{\pi} \rangle$ with $\hat{C}_i$ as follows:

(i) $\hat{C}_i(\{s\}) = \{s\}$ for some $s \in \{s^1, s^2, \ldots, s^t\} \setminus \{\psi[\gamma](i)\}$ such that $C_i(\psi[\gamma](i) \cup \{s\}) = \{s\}$, and

(ii) $\hat{C}_i(\{s'\}) = \varnothing$ for all $s' \in \{s^1, s^2, \ldots, s^t\} \setminus \{s\}$.

By individual rationality and weak non-wastefulness, for any mechanism ψ we have $\psi[\hat{\gamma}](i) = \{s\}$. With $\psi[\gamma](i) \neq \psi[\hat{\gamma}](i)$ and $C_i(\psi[\hat{\gamma}](i) \cup \psi[\gamma](i)) = \psi[\hat{\gamma}](i)$, ψ therefore violates incentive compatibility.

Part 2. Acceptable-consistency is necessary.

Suppose that for some C_i we have $C_i(\{s^1\}) = \{s^1\}$ and $C_i(\{s^2\}) = \varnothing$ but $C_i(\{s^1, s^2\}) = \{s^2\}$.

Consider an admissions problem $\tilde{\gamma} = \langle I, S, q, C, \tilde{\pi}\rangle$ such that $i \; \tilde{\pi}_s \; i'$ for all $i' \in I \setminus \{i\}$ and $s \in \{s^1, s^2\}$ and $\varnothing \; \tilde{\pi}_s \; i$ for all $s \in S \setminus \{s^1, s^2\}$. Moreover, $\tilde{\pi}$ is such that $|\{i' \in I \setminus \{i\} : i' \; \tilde{\pi}_s \; \varnothing\}| < q_s$ for all $s \in \{s^1, s^2\}$.

If $\psi[\gamma](i) = \{s\}$ for $s \in S \setminus \{s^1, s^2\}$ we have a violation of individual rationality for the appropriate institution. By weak non-wastefulness we have that for any mechanism $\psi[\gamma](i) \neq \varnothing$ and by individual rationality we have $\psi[\gamma](i) \neq s^2$. Therefore we have $\psi[\gamma](i) = \{s^1\}$.

Consider the following problem $\hat{\gamma} = \langle I, S, q, (\hat{C}_i, C_{-i}), \tilde{\pi}\rangle$ with $\hat{C}_i$ as follows:

(i) $\hat{C}_i(\{s^2\}) = \{s^2\}$, and

(ii) $\hat{C}_i(\{s\}) = \varnothing$ for all $s \in S \setminus \{s^2\}$

By individual rationality and weak non-wastefulness, for any mechanism ψ we have $\psi[\hat{\gamma}](i) = \{s^2\}$. With $\psi[\gamma](i) \neq \psi[\hat{\gamma}](i)$ and $C_i(\psi[\hat{\gamma}](i) \cup \psi[\gamma](i)) = \psi[\hat{\gamma}](i)$, ψ therefore violates incentive compatibility. ∎

Proposition 5

Lemma 4. *Consider a path independent choice function C_i, then choice from a set is pairwise preferred to alternatives in the set, that is,*

$$C_i(S) = \{s\} \implies C_i(\{s, s'\}) = \{s\} \quad \textit{for all} \quad s' \in S \setminus \{s\}.$$

Proof. Suppose $C_i(S) = \{s\}$ and $C_i(\{s, s'\}) \neq \{s\}$ for some $s' \in S \setminus \{s\}$. Then by path independence we have that, $C_i(S) = C_i\big(C_i(\{s, s'\}) \cup (S \setminus \{s, s'\})\big) \neq \{s\}$ since $C_i\{s, s'\} \neq \{s\}$, a contradiction.

■

We now prove Proposition 5.

Proof. **The "if" part:** If choice functions are path independent, the simultaneous cut-offs mechanism ψ^{sim} yields a stable matching for every admissions problem.

We start with individual rationality. By definition, a institution s never announces cut-off c_s lower than m_s the lowest merit scores at institution s. Hence, for all $\gamma \in \Gamma$, $s \in S$, we have that $i \in \psi^{sim}[\gamma](s)$ implies $i \; \pi_s \; \varnothing$. Similarly for individuals, due to step 0 where individuals apply to only acceptable institutions, individual i is assigned only acceptable institution s. Hence, for all for all $\gamma \in \Gamma$, $i \in I$, we have that $\psi^{sim}[\gamma](i) \neq \varnothing$ implies $C_i(\{\psi^{sim}[\gamma](i)\}) = \{\psi^{sim}[\gamma](i)\}$.

Moving on to the blocking pairs. First, an individual i can never block with a institution $s \in S$ that never proposed to her during the simultaneous cut-offs algorithm, as in that case either the institution has filled its capacity with individuals that have a higher merit score than i and/or individual i is unacceptable. Second, due to path independence, individual i will never block an acceptable institution with an unacceptable one. Combining both observations, it suffices to consider the sequence of sets of (acceptable) institutions proposing to individual i during the simultaneous cut-offs algorithm. For some admission problem $\gamma \in \Gamma$ and individual i let $(S^1, \ldots, S^K)[\gamma](i)$ denote a sequence of institutions proposing to i during the sequential cut-offs mechanism. That is, S^1 proposes first and so on and so forth until S^K, which proposes last.

If the sequence is empty, then there is no institution willing to block with i. Otherwise by Lemma 4, individual i holds a C-maximal institution at every step of the simultaneous cut-offs algorithm. Therefore, the outcome $\psi^{sim}[\gamma](i) \in (S^1 \cup \cdots \cup S^K)$ is such that $C_i(\{\psi^{sim}[\gamma](i), s'\}) = \{\psi^{sim}[\gamma](i)\}$ for all $s' \in (S^1 \cup \cdots \cup S^K) \setminus \{\psi^{sim}[\gamma](i)\}$.

The "only if" part: The simultaneous cut-offs mechanism ψ^{sim} is stable for every admissions problem only if the choice functions are path independent. The contrapositive is, if some choice functions are not path independent then the simultaneous cut-offs mechanism ψ^{sim} is not stable, that is, there exists at least one admission problem $\gamma \in \Gamma$ for which the outcome $\psi^{sim}[\gamma] \in \mathcal{M}$ is not stable.

Consider a violation of path independence $C_i(S' \cup S'') \neq C_i(C_i(S') \cup S'')$ for individual i. Let $C_i(S' \cup S'') \equiv s^1 \in S \cup \{\varnothing\}$ and $C_i(C_i(S') \cup S'') \equiv s^2 \in S \cup \{\varnothing\}$, with $s^1 \neq s^2$. Moreover, consider the following admission problems:

(i) admissions problem $\tilde{\gamma}^1 \in \Gamma$ such that $i \ \tilde{\pi}^1{}_s \ i'$ for all $i' \in I \setminus \{i\}$ and $s \in S' \cup S''$ and $\varnothing \ \tilde{\pi}^1{}_s \ i$ for all $s \in S \setminus (S' \cup S'')$;

(ii) admissions problem $\tilde{\gamma}^2 \in \Gamma$ such that $i \ \tilde{\pi}^2{}_s \ i'$ for all $i' \in I \setminus \{i\}$ and $s \in C_i(S') \cup S''$ and $\varnothing \ \tilde{\pi}^2{}_s \ i$ for all $s \in S \setminus (C_i(S') \cup S'')$; and

(iii) admissions problem $\tilde{\gamma}^3 \in \Gamma$ such that $i \ \tilde{\pi}^3{}_s \ i'$ for all $i' \in I \setminus \{i\}$ and $s \in S'$ and $\varnothing \ \tilde{\pi}^3{}_s \ i$ for all $s \in S \setminus S'$.

Case 1. Suppose $C_i(\{s^1, s^2\}) = \{s^2\}$.

We have $\psi^{sim}[\tilde{\gamma}^1](i) = \{s^1\}$ as all institutions in $S' \cup S''$ propose to i in the first round and no other institution proposes. Moreover, as $s^2 \in S' \cup S''$ individual i has the highest priority at s^2. Slightly abusing notation, note that there are three possibilities:

(a) $s^1, s^2 \in S$, or

(b) $s^1 = \varnothing$ and $s^2 \in S$, or

(c) $s^2 = \varnothing$ and $s^1 \in S$.

Under case (a) and (b) we have a blocking pair (i, s^2), while under (c) we have a violation of individual rationality as s^1 is unacceptable.

This distinction holds for the remaining cases — instead, we will simply write that some (i, s) constitutes a blocking pair.

Case 2. Suppose $C_i(\{s^1, s^2\}) = \{s^1\}$ and $s^1 \in C_i(S') \cup S''$.

We have $\psi^{sim}[\tilde{\gamma}^2](i) = \{s^2\}$ as all institutions in $C_i(S') \cup S''$ propose to i in the first round and no other institution proposes. Moreover, as $s^1 \in C_i(S') \cup S''$ — and therefore i has the highest priority at s^1 — we have a blocking pair (i, s^1).

Case 3. Suppose $C_i(\{s^1, s^2\}) = \{s^1\}$ and $s^1 \in S' \setminus C_i(S')$ and $s^2 = C_i(S')$.

We have $\psi^{sim}[\tilde{\gamma}^3](i) = \{s^2\}$ as all institutions in S' propose to i in the first round and no other institution proposes. Moreover, as $s^1 \in S'$ - we have a blocking pair (i, s^1).

Case 4. Suppose $C_i(\{s^1, s^2\}) = \{s^1\}$ and $s^1 \in S'$ and $s^2 \in S''$.

Let $C_i(S') \equiv s^3$.

Case 4.1. Suppose $C_i(\{s^1, s^3\}) = \{s^1\}$.

We have $\psi^{sim}[\tilde{\gamma}^3](i) = \{s^3\}$ as all institutions in S' propose to i in the first round and no other institution proposes. Moreover, as $s^1 \in S'$ we have a blocking pair (i, s^1).

Case 4.2. Suppose $C_i(\{s^1, s^3\}) = \{s^3\}$.

Recall that $\psi^{sim}[\tilde{\gamma}^1](i) = \{s^1\}$ as all institutions in $S' \cup S''$ propose to i in the first round and no other institution proposes. Similarly, as $s^3 \in S' \cup S''$ we have a blocking pair (i, s^3). ■

Proposition 6

Lemma 5. *Consider an acyclic and acceptable-consistent choice function C_i and a finite sequence of acceptable alternatives $s^1, \ldots s^K$ that propose to individual i (one at a time) under the sequential cut-offs algorithm. That is, s^1 proposes first and so on and so forth until s^K, which proposes last. If the individual's final choice is s^*, then $C_i(\{s^k, s^*\}) \neq \{s^k\}$ for any $s^k \in \{s^1, \ldots s^K\} \setminus \{s^*\}$.*

Proof. Let $s^{(j)}$ denote the tentative assignment of individual i when it receives proposal from institution s^j. Therefore, when institution s^j proposes individual i's choice menu is $\{s^j, s^{(j)}\}$.

Let individual's final choice be s^*, that is, $C_i(\{s^K, s^{(K)}\}) = s^*$. Since $s^1, \dots s^K$ are acceptable alternatives, by acceptable-consistency $s^* \neq \varnothing$. Suppose there exists $s^k \in \{s^1, \dots s^K\} \setminus \{s^*\}$ such that $C_i(\{s^k, s^*\}) = \{s^k\}$. There are two possibilities to consider.

(i) Suppose $C_i(\{s^k, s^{(k)}\}) = s^k$. Since s^k is not the final choice there must exist $s^{k'} \in \{s^{k+1}, \dots s^K\}$ such that $C_i(\{s^k, s^{k'}\}) = \{s^{k'}\}$. If $s^{k'} = s^*$ we have a contradiction. Otherwise, if $s^{k'}$ is not the final choice there must exists $s^{k''} \in \{s^{k'+1}, \dots s^K\}$ such that $C_i(\{s^{k'}, s^{k''}\}) = \{s^{k''}\}$. By acyclicity, $C_i(\{s^k, s^{k'}\}) = \{s^{k'}\}$ and $C_i(\{s^{k'}, s^{k''}\}) = \{s^{k''}\}$ imply that $C_i(\{s^k, s^{k''}\}) = \{s^{k''}\}$. If $s^{k''} = s^*$ we have a contradiction. Otherwise, if $s^{k''}$ is not the final choice we can finitely repeat the same steps until we arrive at the final choice, which due to acyclicity will lead to a contradiction.

(ii) Suppose $C_i(\{s^k, s^{(k)}\}) \neq s^k$. If $s^{(k)} = s^*$ we have a contradiction. Otherwise, if $s^{(k)}$ is not the final choice there must exist $s^{k'} \in \{s^{k+1}, \dots s^K\}$ such that $C_i(\{s^{(k)}, s^{k'}\}) = \{s^{k'}\}$. If $s^{k'} = s^*$, then by acyclicity $C_i(\{s^k, s^*\}) = \{s^k\}$ and $C_i(\{s^{(k)}, s^{k'}\}) = \{s^{k'}\}$ imply that $C_i(\{s^k, s^{(k)}\}) = s^k$, therefore we have a contradiction. Otherwise, if $s^{k'}$ is not the final choice there must exists $s^{k''} \in \{s^{k'+1}, \dots s^K\}$ such that $C_i(\{s^{k'}, s^{k''}\}) = \{s^{k''}\}$. By acyclicity, $C_i(\{s^{(k)}, s^{k'}\}) = \{s^{k'}\}$ and $C_i(\{s^{k'}, s^{k''}\}) = \{s^{k''}\}$ imply that $C_i(\{s^{(k)}, s^{k''}\}) = \{s^{k''}\}$. If $s^{k''} = s^*$, then by acyclicity $C_i(\{s^k, s^*\}) = \{s^k\}$ and $C_i(\{s^{(k)}, s^{k''}\}) = \{s^{k''}\}$ imply that $C_i(\{s^k, s^{(k)}\}) = s^k$, therefore we have a contradiction. Otherwise, if $s^{k''}$ is not the final choice we can finitely repeat the same steps until we arrive at the final choice, which due to acyclicity will lead to a contradiction.

■

We now prove Proposition 6.

Proof. **The "if" part:** If choice functions are weakly acylic and acceptable-consistent, the sequential cut-offs mechanism ψ^{seq} yields a stable matching for every admissions problem.

We start with individual rationality. By definition, a institution s never announces cut-off c_s lower than m_s the lowest merit scores at institution s. Hence, for all $\gamma \in \Gamma$, $s \in S$, we have that $i \in \psi^{seq}[\gamma](s)$ implies $i \ \pi_s \ \varnothing$. Similarly for individuals, due to step 0 where individuals apply to only acceptable institutions, individual i is assigned only acceptable institution s. Hence, for all for all $\gamma \in \Gamma$, $i \in I$, we have that $\psi^{seq}[\gamma](i) \neq \varnothing$ implies $C_i(\{\psi^{seq}[\gamma](i)\}) = \{\psi^{seq}[\gamma](i)\}$.

Moving on to the blocking pairs. First, an individual i can never block with a institution $s \in S$ that never proposed to her during the sequential cut-offs algorithm, as in that case either the institution has filled its capacity with individuals that have a higher merit score than i and/or individual i is unacceptable. Second, due to acceptable-consistency, individual i will never block an acceptable institution with an unacceptable one. Combining both observations, it suffices to consider the sequence of (acceptable) institutions proposing to individual i during the sequential cut-offs algorithm. For some admission problem $\gamma \in \Gamma$ and individual i let $(s^1, \ldots, s^K)[\gamma](i)$ denote a sequence of institutions proposing to i during the sequential cut-offs mechanism. That is, s^1 proposes first and so on and so forth until s^K, which proposes last.

If the sequence $K = 2$, then there is no institution willing to block with i. Otherwise by Lemma 5, individual i holds a C-maximal institution at every step of the sequential cut-offs algorithm. Therefore, the outcome $\psi^{seq}[\gamma](i) \in \{s^1, \ldots, s^K\}$ is such that $C_i(\{\psi^{seq}[\gamma](i), s'\}) = \{\psi^{seq}[\gamma](i)\}$ for all $s' \in \{s^1, \ldots, s^K\} \setminus \{\psi^{seq}[\gamma](i)\}$.

The "only if" part: The sequential cut-offs mechanism ψ^{seq} is stable for every admissions problem only if the choice functions are weakly acylic and acceptable-consistent. The contrapositive is, if some choice functions are not weakly acylic and acceptable-consistent then the sequential cut-offs mechanism ψ^{seq} is not stable, that is, there exists at least one admission problem $\gamma \in \Gamma$ for which the outcome $\psi^{seq}[\gamma] \in \mathcal{M}$ is not stable.

Part 1. Acylicity is necessary.

Case 1: Suppose that for some C_i we have a cycle for $t \geq 3$ and distinct alternatives $s^1, s^2, \ldots, s^t$ in S such that $C_i(\{s^1, s^2\}) = \{s^2\}, \ldots C_i(\{s^{t-1}, s^t\}) = \{s^t\},$ and $C_i(\{s^t, s^1\}) = \{s^1\}$.

Consider an admission problem $\tilde{\gamma} \in \Gamma$ such that $i \ \tilde{\pi}_s \ i'$ for all $i' \in I \setminus \{i\}$ and $s \in \{s^1, s^2, \ldots, s^t\}$ and $\varnothing \ \tilde{\pi}_s \ i$ for all $s \in S \setminus \{s^1, s^2, \ldots, s^t\}$.

Given the definition of the sequential cut-offs mechanism, we have that $\psi^{seq}[\tilde{\gamma}](i) \in \{s^1, s^2, \ldots, s^t\}$, regardless of outcome we have a blocking pair proving the claim. If $\psi^{seq}[\tilde{\gamma}](i) = \varnothing$ the same holds, as all institutions in $\{s^1, s^2, \ldots, s^t\}$ are acceptable, leading to a blocking pair.

Case 2: Suppose that for some C_i we have a cycle for $t \geq 3$ and distinct alternatives $s^1, s^2, \ldots, s^t$ in S such that $C_i(\{s^1, s^2\}) = \{s^2\}, \ldots C_i(\{s^{t-1}, s^t\}) = \{s^t\},$ and $C_i(\{s^t, s^1\}) = \varnothing$.

Consider an admission problem $\tilde{\gamma} \in \Gamma$ such that $i \ \tilde{\pi}_s \ i'$ for all $i' \in I \setminus \{i\}$ and $s \in \{s^1, s^2, \ldots, s^t\}$ and $\varnothing \ \tilde{\pi}_s \ i$ for all $s \in S \setminus \{s^1, s^2, \ldots, s^t\}$.

In the sequential cut-offs mechanism, $s^1, s^2, \ldots, s^t$ will propose to i and no other institution will propose to i. Suppose s^1 proposes before s^t and s^t proposes before s^2 and s^2 proposes before s^3 and so on until s^{t-1}. We get $\psi^{seq}[\tilde{\gamma}](i) = \{s^{t-1}\}$. But in this case there is a blocking pair as $C_i(\{s^{t-1}, s^t\}) = \{s^t\}$.

Part 2. Acceptable-consistency is necessary.

Suppose that for some C_i we have $C_i(\{s\}) = \{s\}$ and $C_i(\{s'\}) = \varnothing$ but $C_i(\{s, s'\}) = \{s'\}$.

Consider an admission problem $\tilde{\gamma} \in \Gamma$ such that $i \ \tilde{\pi}_s \ i'$ for all $i' \in I \setminus \{i\}$ and $s \in \{s^1, s^2\}$ and $\varnothing \ \tilde{\pi}_s \ i$ for all $s \in S \setminus \{s^1, s^2\}$.

In the sequential cut-offs mechanism, both s^1 and s^2 will propose to i and no other institution will propose to i. If s^1 proposes before s^2 we get $\psi^{seq}[\tilde{\gamma}](i) = \{s^2\}$ which violates individual rationality as $C_i(\{s^2\}) = \varnothing$. If s^2 proposes before s^1 we get $\psi^{seq}[\tilde{\gamma}](i) = \{s^1\}$. In this case there is a blocking pair as $C_i(\{s^1, s^2\}) = \{s^2\}$ and i has the highest priority at

s^2.

■

Proposition 9

Recall that, from an institutional viewpoint there are no complementarities between individuals, so the priority order π_s and capacity q_s of an institution s translate into a (partial order) preference over sets of individuals in a straightforward way. Formally, let $\succsim_s$ denote the preferences of institution s over 2^I, and $\succ_s$ denote strict preferences derived from it. We assume that college preferences are responsive. Formally, $\succsim_s$ is **responsive** (Roth (1985)) if,

(i) for any $I' \subset I$ with $|I'| < q_s$ and any $i \in I \setminus I'$,

$$(I' \cup \{i\}) \succ_s I' \iff \{i\}\ \pi_s\ \varnothing,$$

(ii) for any $I' \subset I$ with $|I'| < q_s$ and any $i, i' \in I \setminus I'$,

$$(I' \cup \{i\}) \succ_s (I' \cup \{i'\}) \iff \{i\}\ \pi_s\ \{i'\}.$$

We say matching μ is **blocked by a coalition** T of individuals and institutions, if there exists another matching ν and coalition T, such that for all $i \in T$ and $s \in T$,

(i) $\nu(i) \in T$,

(ii) $C_i(\{\nu(i), \mu(i)\}) = \nu(i)$ for all $i \in T$,

(iii) $\nu(s) \succ_s \mu(s)$ for all $s \in T$, and

(iv) if $j \in \nu(s)$, then $j \in T \cup \mu(s)$.

The first condition states that every individual in T who is matched by ν is matched to some institution in T. The second condition states that every individual in T chooses its assignment under ν over her assignment under μ. The third condition states that every institution in T strictly prefers its set of individuals under ν to that under μ. The last condition states that any new individual matched to an institution in the coalition must be a member of T.

A **group stable** matching is one that is not blocked by any coalition. Pairwise stability is equivalent to group stability in the standard setup of many-to-one matching markets (see Roth and Sotomayor (1990)). The following result shows that the same result holds in our setup.

Proposition 9. *A matching is group stable if and only if it is stable.*

Proof. Suppose μ is not stable due to an unacceptable individual (institution) assigned to an institution (individual), or a blocking pair. Then it is not group stable because it is blocked by the coalition consisting of the individual (institution), or the blocking pair respectively.

In the other direction, if μ is blocked by coalition T and matching ν. Then there must be an individual $i \in \nu(s) \setminus \mu(s)$, and a $j \in \mu(s) \setminus \nu(s)$ such that $i \succ_s j$ for some $s \in T$. If not, then $j \succ_s i$ for $j \in \mu(s) \setminus \nu(s)$ and $i \in \nu(s) \setminus \mu(s)$, which implies that $\mu(s) \succsim_s \nu(s)$ since preferences are responsive. So $i \in T$, and $C_i(\{s, \mu(i)\}) = s$, which shows that (i, s) is a blocking pair. ■

Chapter 2

Affirmative Action in Two Dimensions: A Multi-Period Apportionment Problem

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2.1 Introduction

In many countries affirmative action policies take the form of reserved seats or positions, for which only eligible candidates compete. For instance, in India beneficiary groups are entitled to their proportion of reserved seats in government jobs and publicly funded institutions. However, because of the indivisible nature of positions, the policy prescribed percentage of seats can almost never be met in practice. The fractional seats that arise in literal calculation nearly always need to be adjusted in some manner to yield whole numbers. The question then becomes: what are the ideal whole number counterparts of an affirmative action policy prescribed fractional seats?

The problem gets more complex when positions are heterogeneous. For instance, faculty positions (say assistant professors) in a university are listed under various departments.[1] Each faculty position, therefore, simultaneously represents two units, a department and the

[1]Bureaucrats of a country are posted in different states.

university. If both the department and the university adhere to the affirmative action policy, both must reserve the prescribed percentage of seats. In this paper we ask, how many seats *should* the departments and the university reserve in such cases? We term this novel problem as the problem of *reservations in two dimensions*.

An ideal solution to the problem of reservations in two dimensions should ensure that in each period as well as over time the seat allocations stay "close" to the prescribed fractional seats both (i) at the department level, and (ii) at the university level. However, delivering this is not easy. In fact each of the two solutions seen in practice in India fails to do so. Both existing solutions use a tool called *roster* that lays down the number of positions to be reserved for every number of total positions.[2] The debate in India revolves around whether the individual departments should follow the roster, or whether the university as a whole should follow it. If the departments follow the roster the solution fails to deliver the benefit of reservations at the university level. Whereas if the university as a whole follows the roster, the reserved positions could get allocated to merely a few departments in the university.

The problem with existing solutions is that they do not account for the interdependence of the departments and the university in calculating reserved seats. The reason is that each solution either operates at the department level or the university level, but not at both simultaneously. This is the main source of various shortcomings that we will document in Section 2.4. Not surprisingly, both these solutions are met with several petitions and protests leading to subsequent and frequent changes in the law. The noteworthy debates about these solutions (from Indian courts to public protests) that inspired us to write this paper are summarized in Section 2.2. We wish to design a tool, similar to a roster, that satisfactorily deals with the problem of reservations in two dimensions.

Reservations in two dimensions give rise to matrix problems, with input data in the form of a fair share table X. Its entries x_{ij} signify the fraction of seats beneficiary j is entitled in

[2] See Figure 2.17 and Figure 2.18 for the sequence in which the beneficiary groups take turns in claiming a position in India.

department i as per the affirmative action policy. The rows represent the first subdivision of the university into departments. The columns accommodate the several beneficiaries, and therefore present a second subdivision. The university is assumed to be broken down either way, providing department sizes as row sums, and overall (university level) beneficiary claims as column sums. The task is to find a two-way apportionment, with seat allocations (whole numbers, not fractions!) $\bar{x}_{ij}$ summing row-wise to the pre-specified row sums, while remaining "as near as may be" to the fractional seats x_{ij}.

The fair share table would be the ideal seat allocations if only the seats were divisible. Therefore, it is natural to consider integral seat allocations with entries that are rounded to an adjacent integer of entries of the fair share table as an ideal solution. That is, ideal seat allocations $\bar{x}_{ij}$ would consist of entries x_{ij} of the fair share table rounded up or down to the nearest integer. In fact, this is one of the most appealing and natural apportionment ideas, known as *staying within the quota* (see Balinski and Young (2010) for a fascinating discussion). The problem of reservations in two dimensions can therefore be viewed as a rounding problem of translating a matrix of fair shares to a matrix of seat allocations obtained by rounding the fair shares up or down.

Such matrix problems are not unique to the implementation of affirmative action policies. Biproportional apportionment methods introduced by Balinski and Demange (1989a,b) deal with such problems while translating electoral votes into parliamentary seats. Controlled rounding procedures introduced by Cox and Ernst (1982) also deal with such matrix problems in maintaining census data anonymity. What is unique about the problem we analyze in the affirmative action context is their multi-period aspect. For example, a department with only one new faculty position each year cannot reserve the position for the same beneficiary group each year. In such cases, to ensure that each beneficiary group gets its prescribed percentage of positions over a period of time, the beneficiary groups must take turns in claiming positions. Matrix problems with such multi-period considerations are unique to reservations in two dimensions.

The aim of this paper is twofold. The first objective is to present a comprehensive evaluation of existing solutions in light of staying within the quota property and the multi-period considerations. We do so theoretically in Section 2.4 and empirically in Section 2.6. The second objective is to check whether a solution exists to the problem of reservations in two dimensions that stays "close" to the prescribed fractional seats both (i) at the department level, and (ii) at the university level. The answer is affirmative.

Our first results that deal with the problem of reservations in two dimensions without the multi-period considerations are straightforward. The rounding problem has an elegant solution, called *controlled rounding*, that stays within quota and is simple enough to be implemented by hand. The technique was introduced by Cox (1987) to make slight perturbations in two-dimensional census data to ensure confidentiality of aggregate statistics while maintaining a good approximation of the original data. Adaptation of Cox's controlled rounding technique to our problem is summarized in Section 2.8. In addition to providing a solution that stays within quota, Cox's controlled rounding procedure provides an *unbiased* lottery solution, that is, entries of the fair share matrix are rounded up or down so that ex-ante positive and negative biases balance to yield zero bias.

The main theoretical contributions of our article address the multi-period problem of reservations in two dimensions, and are presented in Section 2.5.2. We show that there does not exist a solution for the problem of reservations in two dimensions that stays within quota at both the university and the department level simultaneously (Proposition 2). We give an even stronger result: There does not exist a solution for which the reservation table deviates from the fair share table bounded by a finite number (Proposition 3). These results justify the struggle in figuring out a solution in real-life practice as discussed in Section 2.2. Since the two constraints that staying within quota property imposes cannot be satisfied simultaneously, we ask: can these constraints be satisfied *approximately*? By approximately we mean, the probability of violating that constraint is exponentially decreasing with the size of the constraint. The answer is affirmative.

The main results of the article, stated in Theorem 1 and Theorem 2, show that there exists an unbiased solution that stays within quota at the department level and approximately stays within quota at the university level. The proof of Theorem 1 involves constructing a lottery solution that stays within quota at the department level and is unbiased at both the department and the university levels. An overview of the proof is presented in Section 2.5.2. The key technique is to design a procedure that takes the fractions of reservations and generates a roster that lays down the number of positions to be reserved for every number of total positions. For a roster, the staying within quota constraint regulates the cumulative number of positions for each category. Since there could be many rosters that would stay within quota, the procedure generates a *random roster* by assigning each solution roster a probability. Our solution to the problem of reservation in two dimensions assigns a roster to each department adhering to the probabilities dictated by the procedure.

The procedure of constructing a random roster is built around a network flow algorithm that takes a flow network as input and randomly constructs another flow network with fewer fractional flows as its output. By iterative application of this algorithm, a flow network with integral flows is generated. The random flow network has the following two properties: the expected value of each flow after the next iteration is the same as its current value and each constraint (imposed by the stay within quota property) remains satisfied. Since each flow network with integral flows can be mapped to a roster, this procedure generates a random roster. We next show, in Theorem 2, that the approximation errors are small. We do so by applying the multiplicative form of Chernoff concentration bounds to our solution in order to prove that, in addition to staying within quota at the department level, the solution approximately stays within quota at the university level. Moreover, we show that our bounds on the approximation errors are tight.

Lastly, in Section 2.6, we present an empirical case study of a two dimensional reservations problem from India using recruitment advertisement data. The objective is twofold. The first objective is to document the shortcomings of existing procedures empirically. In particular, we highlight the severity of the problem by documenting the instances and magnitude of violations. Our second objective is to quantify the performance of our proposed solutions. We do so by running simulations on the recruitment data, thus creating reservation tables per the procedures advocated in this article, and comparing the outcomes with existing (advertised) solutions.

Contributions with respect to the Related Literature

With a recent surge of research interest in implementation of affirmative action schemes, unnoticed issues in implementation of nation-wide affirmative action policies are coming to light. A considerable number of recent papers have documented such shortcomings and have also proposed practical alternatives to better implement such policies (Abdulkadiroğlu and Sönmez (2003b), Kojima (2012), Hafalir et al. (2013), Ehlers et al. (2014), Echenique and Yenmez (2015), Dur et al. (2018b), Dur et al. (2019), Sönmez and Yenmez (2019, 2021) and many others). Ours is another paper in this class. While the focus of the contemporary market design literature has been the design and analysis of assignment mechanisms given reserved seats and quotas, our paper looks at another side of affirmative action schemes: how many seats to reserve?

Distributing indivisible objects among a group of claimants in proportion to their claims, known as the *apportionment problem*, is the center point of the seminal work of Young (1995) and Balinski and Young (2010). The two-dimensional version, the *biproportional apportionment problem*, gives rise to similar matrix problems as ours, but has been investigated in a different context (Gassner (1988), Balinski and Demange (1989a), Balinski and Demange (1989b), Maier et al. (2010), Lari et al. (2014)).[3] In their context of translating electoral votes into parliamentary seats, the foremost criteria for desirability of a solution is "proportionality". However, in our context (of affirmative action) searching for the "closest" solution is better suited. More importantly, the multi-period constraints that the prob-

[3] See Pukelsheim (2017) for detailed results and insights on biproportional apportionment problems.

lem reservation in two dimensions introduces have not been featured in the literature on biproportional apportionment problem.

Lastly, our paper is related to the literature on rounding techniques. The *controlled rounding* procedure introduced in Cox (1987) suffices to solve the problem of reservations in two dimensions for a special case of the model (see Proposition 1). For the general case, our rounding approach is similar to the ones developed in the literature on *approximation algorithms* from computer science (Ageev and Sviridenko (2004), Gandhi et al. (2006) and others). These techniques are not new to market designers. The literature on implementation of random and therefore fractional assignments solves such problems in the presence of a very rich "bihierarchical" structure on the set of constraints (Budish et al. (2013), Pycia and Ünver (2015), and Akbarpour and Nikzad (2020)). In particular, Budish et al. (2013) and Akbarpour and Nikzad (2020) build implementation methods for random allocation mechanisms based on techniques from deterministic and randomized rounding developed in Edmonds (2003) and Gandhi et al. (2006). Our constraints, in addition to following a "bihierarchical" structure, also extend in the time dimension in order to accommodate the multi-period considerations. It is this multi-period aspect of our problem that renders existing solutions inadequate. A rounding procedure for a multi-period model with a "bihierarchical" constraint structure (upper and lower quotas at the department level and approximate constraints at the university level) is a theoretical contribution of our paper (Theorem 1 and Theorem 2).

2.2 Motivating Debate from India

The 1950 Constitution of India provides a clear basis for positive discrimination in favor of disadvantaged groups, in the form of *reservation policies*. India's reservation policies mandate exclusive access to a fixed percentage of government jobs and seats in publicly funded institutions to the members of Scheduled Castes (SC, 15%), Scheduled Tribes (ST, 7.5%), Other Backward Classes (OBC, 27%) and Economically Weaker Sections (EWS, 10%). For the sake of transparency, the number of reserved seats for each category are explicitly and publicly advertised in advance of any admissions or recruitment cycle.

The procedures used to calculate the number of reserved seats in various settings are also explicit and public. But they have nowhere been more contentious than in the case of universities. Unlike other government jobs, for the same faculty position in a university (say assistant professor), the eligibility and selection criteria changes with the department. Thus the faculty positions in different departments are not interchangeable across a university. Each faculty position, therefore, simultaneously represents two units, a department and the university, where each unit is subject to the reservation policy. It is this feature of faculty positions that led to complications which made all three arms of the Indian government – the executive, the judiciary and the legislative – intervene.

The Executive. In August 2006, the University Grants Commission (UGC) issued *Guidelines for Strict Implementation of Reservation Policy of the Government in Universities* to all government educational institutions in India.[4,5] Through this document the UGC prohibited the practice of treating *department as the unit* for application of the reservation scheme, that is, for calculating the proportion of seats to be reserved (see clause 6(c) in the guidelines). Instead, UGC mandated *university as the unit* for the purpose of reservation. That is, the positions in a university shall be clubbed together across departments as three separate categories: professors, associate professors (or readers), and assistant professors (or lecturers), for the application of the rule of reservation (see clause 8(a)(v) in the guidelines). However, UGC's order was challenged in the court.

[4]UGC is a statutory autonomous organization responsible for implementation of policy of the Central Government in the matter of admissions as well as recruitment to the teaching and non-teaching posts in central universities, state universities and institutions which are deemed to be universities.

[5]Document last accessed on 12 June 2021 at https://www.ugc.ac.in/pdfnews/7633178_English.pdf

The Judiciary. In April 2017, the Allahabad High Court allowed a petition demanding reservations in faculty posiitons treating department as the unit, and quashed clauses 6(c) and 8(a)(v) of the UGC Guidelines of 2006.[6] The court argued that treating the university as the unit "would be not only impracticable, unworkable but also unfair and unreasonable" for the following two reasons stated in the judgment:

> Merely because Assistant Professor, Reader, Associate Professor, and Professor of each subject or the department are placed on the same pay-scale, but their services are neither transferable nor they are in competition with each other. It is for this reason also that clubbing of the posts for the same level treating the University as a `Unit' would be completely unworkable and impractical. It would be violative of Article 14 and 16 of the Constitution.
>
> If the University is taken as a `Unit' for every level of teaching and applying the roster, it could result in some departments/subjects having all reserved candidates and some having only unreserved candidates. Such proposition again would be discriminatory and unreasonable. This, again, would be violative of Article 14 and 16 of the Constitution.

Following the court order, universities advertised vacancies with a sharp fall in the number of reserved positions. This is apparent in the case of Banaras Hindu University, presented in Table 2.1, where the number of unreserved seats increased from 1188 under government's quashed solution to 1562 under court's proposed solution.[7] The reason was that many departments had a small number of faculty positions (fewer than six). Given that each department followed the same fixed sequence in which categories take turns in claiming a position, the court's solution led to a small number of positions for the reserved categories at the university level.[8] This sparked a series of teachers' unions led protests across India.

[6] Judgement last accessed on 12 June 2021 at https://indiankanoon.org/doc/177500970/

[7] Last accessed on 12 June 2021 at https://indianexpress.com/article/explained/hrd-ministry-ordinance-teacher-quota-university-prakash-javadekar-5616157/

[8] See Figure 2.17 and Figure 2.18 for the sequence in which the beneficiary groups take turns in claiming a position in India.

Table 2.1: NUMBER OF RESERVED POSITIONS IN BANARAS HINDU UNIVERSITY

	University as a Unit (Government's Solution)					Department as a Unit (Court's Solution)				
Position	General	SC	ST	OBC	Total	General	SC	ST	OBC	Total
Professor	197	38	18	0	253	250	3	0	0	253
Associate Professor	410	79	39	0	528	500	25	3	0	528
Assistant Professor	581	172	86	310	1149	812	91	26	220	1149
Total	1188	289	143	310	1930	1562	119	29	220	1930

Notes: Data shared in government's Special Leave Petition filed in the Supreme Court of India.

The Legislative. The protests compelled the government to file a petition in the Supreme Court against the Allahabad High Court verdict. "How can the post of professor of Anatomy be compared with the professor of Geography? Are you clubbing oranges with apples?" questioned the Supreme Court rejecting the appeal and terming the Allahabad high court judgment as "logical".[9] Facing a huge aggrieved vote bank, three days prior to announcement of Lok Sabha election, in March 2019, the government promulgated an ordinance that considered the university as the unit. This ordinance is now an Act of Parliament, and therefore the law in India.[10]

Today, university is the unit for application of the reservation scheme. The court's objection that "it could result in some departments/subjects having all reserved candidates and some having only unreserved candidates" inspired us to write this paper.

2.3 Model and the Primitives

We provide a model in this section to formulate the problem of reservation in two dimensions. Since our primary application is the reservation of teaching positions in Indian universities, the terminology used is appropriate for that application.

[9]Last accessed on 12 June 2021 at https://main.sci.gov.in/supremecourt/2019/5495/5495_2019_Order_27-Feb-2019.pdf

[10]Last accessed on 12 June 2021 at http://egazette.nic.in/WriteReadData/2019/206575.pdf

2.3.1 Model

A **problem of reservation in two dimensions** in period $t \in \{1, 2, \dots, T\}$ is a quadruple $\Lambda^t = (\mathcal{D}, \mathcal{C}, \boldsymbol{\alpha}, (\mathbf{q}^s)_{s=1}^t)$. $\mathcal{D}$ and $\mathcal{C}$ are finite sets of **departments** and **categories** where $m := |\mathcal{D}| \geq 2$ and $n := |\mathcal{C}| \geq 2$. The **reservation scheme** is defined by a vector of fractions $\boldsymbol{\alpha} = [\alpha_j]_{j\in\mathcal{C}}$. For each category $j \in \mathcal{C}$, $\alpha_j \in (0, 1)$ fraction of vacancies are to be reserved so that $\sum_{j\in\mathcal{C}} \alpha_j = 1$. $\mathbf{q}^s = [q_i^s]_{i\in\mathcal{D}}$ represents the **vector of vacancies** associated with the departments in period $s \in \{1, 2, \dots, t\}$. Let $Q_i^t := \sum_{s\leq t} q_i^s$ denote **period-t cumulative sum of vacancies in department i.**

A **period-t fair share table** for problem Λ^t is a two-way table

$$X^t = \begin{array}{c|c} (x_{ij}^t)_{m\times n} & (x_{i,n+1}^t)_{m\times 1} \\ \hline (x_{m+1,j}^t)_{1\times n} & (x_{m+1,n+1}^t)_{1\times 1} \end{array}$$

with rows indexed by $i \in \mathcal{D} \cup \{m + 1\}$ and columns by $j \in \mathcal{C} \cup \{n + 1\}$, such that internal entries $x_{ij}^t = \alpha_j Q_i^t$ for all $i \in \mathcal{D}$ and $j \in \mathcal{C}$, row total entries $x_{i,n+1}^t = Q_i^t$ for all $i \in \mathcal{D}$, column total entries $x_{m+1,j}^t = \alpha_j \sum_{i\in\mathcal{D}} Q_i^t$ for all $j \in \mathcal{C}$, and grand total entry $x_{m+1,n+1}^t = \sum_{i\in\mathcal{D}} Q_i^t$. Fair shares specify the fraction of seats a category is entitled to receive as per the reservation scheme until period t. The internal entry x_{ij}^t represents the **period-t fair share for category j in department i.** The **period-t fair share for a category c_j in the university** is denoted by column total entry $x_{m+1,j}^t$. The grand total entry $x_{m+1,n+1}^t$ represents the cumulative sum of vacancies at the university.

For instance, consider a problem $\Lambda^2 = (\{d_1,\ d_2\}, \{c_1,\ c_2\}, \boldsymbol{\alpha} = [0.1, 0.9], (\mathbf{q}^1, \mathbf{q}^2) = ([9, 8], [17, 7]))$. Figure 2.1 illustrates its period-1 and period-2 fair share tables. There are two departments $\mathcal{D} = \{d_1,\ d_2\}$, corresponding to rows in the tables, and two categories $\mathcal{C} = \{c_1,\ c_2\}$, corresponding to columns. The reservation scheme reserves 10% positions in the university for members of category c_1. In period-1, department d_1 has 9 and department d_2 has 8 positions, represented by the column 3 of X^1. In period-2, department d_1 has 17 and department d_2 has 7 positions. Therefore, period-2 cumulative sums of vacancies in departments d_1 and d_2 are 26 and 15, represented by the column 3 of X^2. The first column of table X^1 (X^2) represents the period-1 (period-2) fair shares associated with the category c_1 and the second column represents the period-1 (period-2) fair shares associated with category c_2. The first row of X^1 (X^2) represents the period-1 (period-2) fair shares associated with the department d_1 and the second row represents the period-1 (period-2) fair shares associated with department d_2.

Figure 2.1: FAIR SHARE TABLES

$$X^1 = \begin{array}{cc|c} 0.9 & 8.1 & 9 \\ 0.8 & 7.2 & 8 \\ \hline 1.7 & 15.3 & 17 \end{array}$$

(a) PERIOD-1 FAIR SHARE TABLE

$$X^2 = \begin{array}{cc|c} 2.6 & 23.4 & 26 \\ 1.5 & 13.5 & 15 \\ \hline 4.1 & 36.9 & 41 \end{array}$$

(b) PERIOD-2 FAIR SHARE TABLE

A two-way table is **additive** if entries add along the rows and columns to all corresponding totals. A **period-t reservation table** for the problem Λ^t is a $(m+1) \times (n+1)$ non-negative integer two-way table $\bar{X}^t = (\bar{x}^t_{ij})$, with rows indexed by $i \in \mathcal{D} \cup \{m+1\}$ and columns by $j \in \mathcal{C} \cup \{n+1\}$, such that $\bar{X}^t$ is additive and $\bar{x}^t_{i,n+1} = x^t_{i,n+1}$ for all $i \in \mathcal{D}$. The internal entry $\bar{x}^t_{ij}$ represents the **period-t reservation for category j in department** i. The **period-t reservation for a category j in the university** is denoted by column total entry $\bar{x}^t_{m+1,j}$. We denote by $\bar{\mathcal{X}}$ the set of reservation tables.

A **period-t sequence of fair share tables** for the problem Λ^t is a sequence of two-way tables $Y^t = (X^1, \ldots, X^t)$, where table X^s is the period-s fair share table for all $s \in \{1, 2, \ldots, t\}$. We denote by $\mathcal{Y}^t$ the set of all period-t sequences of fair share tables. Given a sequence of tables Y^t, if $Y^t = (Y^{t-1}, X^t)$, then we say that Y^t **follows** Y^{t-1}.

2.3.2 Deterministic Solutions and Properties

A **deterministic solution** $R : \cup_{s=1}^T \mathcal{Y}^s \to \bar{\mathcal{X}}$ maps each sequence of fair share tables to a reservation table such that, for any $Y^t \in \cup_{s=1}^T \mathcal{Y}^s$,

1. $R(Y^t)$ is a period-t reservation table, and

2. $R(Y^t) \geq R(Y^{t-1})$ for all Y^t that follow Y^{t-1}.[11]

Part 2 of definition incorporates the idea that reservations are irreversible. We denote by $\mathcal{R}^T$ the set of deterministic solutions for reservation problems of length T.

For instance, Figure 2.2 illustrates two possible deterministic solutions for the problem depicted in Figure 2.1.

Figure 2.2: TWO DETERMINISTIC SOLUTIONS

$$X^1 = \begin{array}{cc|c} 0.9 & 8.1 & 9 \\ 0.8 & 7.2 & 8 \\ \hline 1.7 & 15.3 & 17 \end{array}$$

(a) PERIOD-1 FAIR SHARE TABLE

$$X^2 = \begin{array}{cc|c} 2.6 & 23.4 & 26 \\ 1.5 & 13.5 & 15 \\ \hline 4.1 & 36.9 & 41 \end{array}$$

(b) PERIOD-2 FAIR SHARE TABLE

$$R_1(Y^1) = \begin{array}{cc|c} 1 & 8 & 9 \\ 1 & 7 & 8 \\ \hline 2 & 15 & 17 \end{array}$$

(c) PERIOD-1 RESERVATION TABLE

$$R_1(Y^2) = \begin{array}{cc|c} 3 & 23 & 26 \\ 1 & 14 & 15 \\ \hline 4 & 37 & 41 \end{array}$$

(d) PERIOD-2 RESERVATION TABLE

$$R_2(Y^1) = \begin{array}{cc|c} 0 & 9 & 9 \\ 0 & 8 & 8 \\ \hline 0 & 17 & 17 \end{array}$$

(e) PERIOD-1 RESERVATION TABLE

$$R_2(Y^2) = \begin{array}{cc|c} 3 & 23 & 26 \\ 1 & 14 & 15 \\ \hline 4 & 37 & 41 \end{array}$$

(f) PERIOD-2 RESERVATION TABLE

We denote by $R(y^t)$ and x^t the internal and totals entries of $R(Y^t)$ and X^t, respectively. The ideal solution would be the fair share table if we were allowed to reserve fractional seats. Therefore, it is natural to consider integral seat allocations with entries rounded to an adjacent integer of the fair share table entries as an ideal solution. We next formulate this idea.

A deterministic solution R **stays within quota** if, for any Y^t,

1. R **stays within department quota**: each internal entry $R(y^t) = \lceil x^t \rceil$ or $\lfloor x^t \rfloor$, and

2. R **stays within university quota**: each total entry $R(y^t) = \lceil x^t \rceil$ or $\lfloor x^t \rfloor$.

[11] The relation "is greater than or equal to", denoted "$\geq$", compares tables entry-wise; that is, $X \geq X'$ if, for all $(1 \leq i \leq m+1,\ 1 \leq j \leq n+1)$, $x_{ij} \geq x'_{ij}$.

Our property formulates the idea that a deterministic solution should not deviate from its cumulative fair share by more than one seat. In this way, everyone gets either the ceiling of its cumulative fair share or the floor of its cumulative fair share.[12] There are two dimension of staying within quota: (1) each internal entry $R(y^t_{ij})$ $(1 \leq i \leq m,\ 1 \leq j \leq n)$ is either x^t_{ij} rounded up or rounded down and (2) each total entry $R(y^t_{m+1,j})$ $(1 \leq j \leq n)$ is either $x^t_{m+1,j}$ round up or rounded down. If a solution satisfies the former one for any problem, we say that it stays within department quota. If a solution satisfies the later one for any problem, we say that it stays within university quota. For instance, in Figure 2.2, the solution R_1 stays within both department and university quota; however, the solution R_2 stays within department quota only.

2.3.3 Lottery Solutions and Properties

Randomization is the most natural and common mechanism to use in resource allocation problems when in doubt which of two or more agents should get an indivisible object. We next introduce a function to adapt this idea.

A **lottery solution** is a probability distribution ϕ over the set of deterministic solutions, where $\phi(R)$ denotes the probability of solution R. We denote by φ^T the set of lottery solutions for reservation problems of length T.

For any sequence of fair share tables Y^t, a lottery solution ϕ induces a **period-t expected reservation table** $E_\phi(Y^t) := \sum_R \phi(R)R(Y^t)$. The internal entry (i, j) in this table represents the expected fraction of seats that category j receives at department i under ϕ. The column total entry $(m + 1, j)$ represents the expected fraction of seats that category j receives in the university under ϕ.

Our next two properties make sure that in expectation a lottery solution always achieves the fair shares as well as in implementation it picks a reservation table that is as close as to

[12] For any $x \in \mathbb{R}$, $\lfloor x \rfloor$ and $\lceil x \rceil$ are the largest integer no larger than x, i.e., floor of x, and the smallest integer no smaller than x, i.e., ceiling of x, respectively.

fair shares for each departments in every period.

Definition 1. A lottery solution ϕ is unbiased if, for any $Y^t \in \cup_{s=1}^{T} \mathcal{Y}^s$,

$$E_\phi(Y^t) = X^t.$$

This property formulates the idea that a lottery solution should implement the fair share tables in an expected sense; that is, for any Y^t, $\sum_R \phi(R)R(Y^t) = X^t$. An unbiased lottery solution promotes ex-ante "fairness". Such solutions, on the other hand, may result in an "unfair" outcome ex-post, in which one category receives all seats, while others receive none. In other words, the ex-post outcome can differ greatly from the fair share tables. To avoid this, we next extend the staying within quota property to lottery solutions.

Definition 2. A lottery solution ϕ **stays within quota** if, for any R such that $\phi(R) > 0$,

1. R stays within department quota, and

2. R stays within university quota.

We study lottery solutions ϕ that only pick deterministic solutions that stay within quota. There are two dimension of staying within quota. We say that a lottery solution stays within department quota if it only gives positive probabilities to deterministic solutions that stays within department quota. We say that a lottery solution stays within university quota if it only gives positive probabilities to deterministic solutions that stays within university quota.

2.4 Solutions from India and their shortcomings

There are two solutions seen in practice in India, the Government's solution and the Court's solution. Both solutions use a tool called roster to determine the number of positions to be reserved. Formally, a **roster** $\sigma : \{1, 2, \dots\} \to \mathcal{C}$ is an ordered list over the set of categories $\mathcal{C}$. A roster assigns each position a category so that for any number of total positions, the number of positions to be reserved are clearly laid out. Since only a few seats might arise every period, the objective of maintaining a roster is to ensure that, over a period of time, each category gets its affirmative action policy prescribed percentage of seats.

Maintaining rosters is central to implementation of reservations in India.[13] It makes uniform and transparent implementation of the reservation policy across various government departments possible. However, maintaining rosters for educational institutions raises additional complications. Does each department in a university maintain its own roster? Or does the university as a whole maintain a roster? These questions gave rise to two solutions in India.

Before illustrating the solutions, we first introduce an example that makes the solutions easier to comprehend. The example will also be sufficient to demonstrate the various shortcomings of the two solutions.[14]

Example 1. Consider a problem $\Lambda^3 = (\{d_1,\ d_2,\ d_3,\ d_4\}, \{c_1,\ c_2\}, \boldsymbol{\alpha} = [1/3, 2/3], (\mathbf{q}^1, \mathbf{q}^2, \mathbf{q}^3) = ([2, 1, 2, 1], [2, 1, 2, 1], [2, 1, 2, 1]))$. Figure 2.3 illustrates its period-1, period-2, and period-3 fair share tables. The reservation scheme reserves 1/3 of the positions in the university for members of category c_1. Each period, department d_1, d_2, d_3, and d_4 have 2, 1, 2, and 1 positions, respectively. Therefore, period-2 cumulative sums of vacancies in departments are 4, 2, 4, and 2, respectively. And, period-3 cumulative sums of vacancies in departments are 6, 3, 6, and 3, respectively. The roster is

$$\sigma(k) = \begin{cases} c_1, & \text{if } k \text{ is a multiple of } 3 \\ c_2, & \text{otherwise} \end{cases}$$

We will see that the choice of the roster in Example 1 is not the source of the shortcomings of the Government's and Court's solutions. The source of problem is that they do not

[13] See Figure 2.17 and Figure 2.18 for the rosters prescribed by Government of India.

[14] An example with two categories and two department is also sufficient to demonstrate the shortcomings. Example 1 is constructed so that it not only illustrates the shortcomings of the both solutions, but it also demonstrates the differences between the Court's and the Government's solutions.

Figure 2.3: FAIR SHARE TABLES

(a) PERIOD-1 FAIR SHARE TABLE

$$X^1 = \begin{array}{cc|c} 2/3 & 4/3 & 2 \\ 1/3 & 2/3 & 1 \\ 2/3 & 4/3 & 2 \\ 1/3 & 2/3 & 1 \\ \hline 2 & 4 & 6 \end{array}$$

(b) PERIOD-2 FAIR SHARE TABLE

$$X^2 = \begin{array}{cc|c} 4/3 & 8/3 & 4 \\ 2/3 & 4/3 & 2 \\ 4/3 & 8/3 & 4 \\ 2/3 & 4/3 & 2 \\ \hline 4 & 8 & 12 \end{array}$$

(c) PERIOD-3 FAIR SHARE TABLE

$$X^3 = \begin{array}{cc|c} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \\ \hline 6 & 12 & 18 \end{array}$$

account for interdependence of the departments and the university in calculating reserved seats.

2.4.1 Government's Solution and its Shortcomings

The Government's solution treats the *university as the unit*. That is, positions across all departments are pooled together and the roster is maintained at the university level.

For the problem in Example 1, in period-1, department d_1 has two positions: The number of positions reserved for department d_1 is determined by the 1st and 2nd positions in the roster (i.e., $\sigma(1) = c_2,\ \sigma(2) = c_2$). Department d_2 has one position: The number of positions reserved for department d_2 is determined by the 3th position in the roster (i.e., $\sigma(3) = c_1$).[15] Department d_3 has two positions: The number of positions reserved for department d_3 is determined by the 4th and 5th positions in the roster (i.e., $\sigma(4) = c_2,\ \sigma(5) = c_2$). Department d_4 has one position: The number of positions reserved for department d_4 is determined by the 6th position in the roster (i.e., $\sigma(6) = c_1$). The period-1 reservation table is illustrated by $R_G(Y^1)$ in Figure 2.4.

In period-2, department d_1 has two positions: The number of positions reserved for department d_1 is determined by the 7th and 8th positions in the roster (i.e., $\sigma(7) = c_2,\ \sigma(8) = c_2$). Department d_2 has one position: The number of positions reserved for department d_2 is determined by the 9th positions in the roster (i.e., $\sigma(9) = c_1$). Department d_3 has two positions: The number of positions reserved for department d_3 is determined by the 10th

[15] When pooling positions across departments, a fixed order over departments is required to apply to the roster. In India, the alphabetic order over departments is used.

and 11th positions in the roster (i.e., $\sigma(10) = c_2,\ \sigma(11) = c_2$). Department d_4 has one position: The number of positions reserved for department d_4 is determined by the 12th position in the roster (i.e., $\sigma(12) = c_1$). The period-2 reservation table is illustrated by $R_G(Y^2)$ in Figure 2.4. We apply this solution for the next period. The period-3 reservation table is illustrated by $R_G(Y^3)$ in Figure 2.4.

Figure 2.4: COURT'S AND GOVERNMENT'S SOLUTION

$$X^1 = \begin{array}{cc|c} 2/3 & 4/3 & 2 \\ 1/3 & 2/3 & 1 \\ 2/3 & 4/3 & 2 \\ 1/3 & 2/3 & 1 \\ \hline 2 & 4 & 6 \end{array}$$

(a) PERIOD-1 FAIR SHARE TABLE

$$X^2 = \begin{array}{cc|c} 4/3 & 8/3 & 4 \\ 2/3 & 4/3 & 2 \\ 4/3 & 8/3 & 4 \\ 2/3 & 4/3 & 2 \\ \hline 4 & 8 & 12 \end{array}$$

(b) PERIOD-2 FAIR SHARE TABLE

$$X^3 = \begin{array}{cc|c} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \\ \hline 6 & 12 & 18 \end{array}$$

(c) PERIOD-3 FAIR SHARE TABLE

$$R_G(Y^1) = \begin{array}{cc|c} 0 & 2 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 0 & 1 \\ \hline 2 & 4 & 6 \end{array}$$

(d) PERIOD-1 RESERVATION TABLE

$$R_G(Y^2) = \begin{array}{cc|c} 0 & 4 & 4 \\ 2 & 0 & 2 \\ 0 & 4 & 4 \\ 2 & 0 & 2 \\ \hline 4 & 8 & 12 \end{array}$$

(e) PERIOD-2 RESERVATION TABLE

$$R_G(Y^3) = \begin{array}{cc|c} 0 & 6 & 6 \\ 3 & 0 & 3 \\ 0 & 6 & 6 \\ 3 & 0 & 3 \\ \hline 6 & 12 & 18 \end{array}$$

(f) PERIOD-3 RESERVATION TABLE

$$R_C(Y^1) = \begin{array}{cc|c} 0 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \\ \hline 0 & 6 & 6 \end{array}$$

(g) PERIOD-1 RESERVATION TABLE

$$R_C(Y^2) = \begin{array}{cc|c} 1 & 3 & 4 \\ 0 & 2 & 2 \\ 1 & 3 & 4 \\ 0 & 2 & 2 \\ \hline 2 & 10 & 12 \end{array}$$

(h) PERIOD-2 RESERVATION TABLE

$$R_C(Y^3) = \begin{array}{cc|c} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \\ \hline 6 & 12 & 18 \end{array}$$

(i) PERIOD-3 RESERVATION TABLE

Period-3 reservation for category c_1 in department d_1 and department d_3 is 0, however, the fair share is 2 positions. Moreover, period-3 reservation for category c_1 in department d_2 and department d_4 is 3, however, the fair share is 1 position. Therefore, the Government's solution R_G does not stay within department quota. Moreover, in Example 1, if the departments had the same number of positions for the next periods, department d_1 and department d_3 would not reserve any seats for category c_1, and department d_2 and department d_4 would not reserve any seats for category c_2.

Two shortcomings of the Government's solution R_G are revealed by Example 1:

1. The Government's solution R_G does not stay within quota.
2. The Government's solution R_G allows for large deviations in seat allocations from fair shares at the department level.

Essentially, Example 1 shows that treating university as the unit can lead to outcomes that fail to follow the reservation policy at the department level.[16]

2.4.2 Court's Solution and its Shortcomings

The Court's solution treats *department as the unit*. That is, positions are not pooled across departments. Instead, each department independently maintains a roster.

For the problem in Example 1, in period-1, department d_1 has two positions: The number of positions reserved for department d_1 is determined by the 1st and 2nd positions in its roster (i.e., $\sigma(1) = c_2,\ \sigma(2) = c_2$). Department d_2 has one position: The number of positions reserved for department d_2 is determined by the 1st position in its roster (i.e., $\sigma(1) = c_2$). Department d_3 has two positions: The number of positions reserved for department d_3 is determined by the 1st and 2nd positions in its roster (i.e., $\sigma(1) = c_2,\ \sigma(2) = c_2$). Department d_4 has one position: The number of positions reserved for department d_4 is determined by the 1st position in its roster (i.e., $\sigma(1) = c_2$). The period-1 reservation table is illustrated by $R_C(Y^1)$ in Figure 2.4.

In period-2, department d_1 has two positions: The number of positions reserved for department d_1 is determined by the 3th and 4th positions in its roster (i.e., $\sigma(3) = c_1,\ \sigma(4) = c_2$). Department d_2 has one position: The number of positions reserved for department d_2 is determined by the 2nd positions in its roster (i.e., $\sigma(2) = c_1$). Department d_3 has two positions: The number of positions reserved for department d_3 is determined by the 3th and 4th positions in its roster (i.e., $\sigma(3) = c_1,\ \sigma(4) = c_2$). Department d_4 has one position: The number of positions reserved for department d_4 is determined by the 2nd position in its

[16] In fact, in Proposition 3, we show that for any solution that stays within university quota, the deviations in seat allocations from fair shares at the department level can not be limited by a fixed number.

roster (i.e., $\sigma(2) = c_1$). The period-2 reservation table is illustrated by $R_C(Y^2)$ in Figure 2.4. We apply this solution for the next period. The period-3 reservation table is illustrated by $R_C(Y^3)$ in Figure 2.4.

Period-1 reservation for category c_1 in the university is 0, however, the fair share is 2 positions. Moreover, period-2 reservation for category c_1 in the university is 2, however, the fair share is 4 positions. Therefore, the Court's solution R_C does not stay within university quota. Moreover, in Example 1, if there were 4 more departments d_5, d_6, d_7, and d_8, with the same number of positions as department d_1, d_2, d_3, and d_4, respectively, period-1 reservation for category c_1 in the university would still be 0. And, period-2 reservation for category c_1 in the university would still be 4 while the fair share was 8 positions.

Two shortcomings of the Court's solution R_C are revealed by Example 1:

1. The Court's solution R_C does not stay within quota.
2. The Court's solution R_C allows for large deviations in seat allocations from fair shares at the university level.

Essentially, Example 1 shows that treating department as the unit can lead to outcomes that fail to follow the reservation policy at the university level.

2.5 Designing Reserves in Two-Dimensions: Results

2.5.1 Single Period Results

One way to approach the problem of reservation in two dimensions is to ignore the time dimension, that is, the problem can be treated as an independent problem in each period.[17] In that case, a lottery solution that is unbiased and stays within quota always exists.

Proposition 1. There exists a lottery solution $\phi \in \varphi^1$ that is unbiased and stays within quota.

[17] This is analogous to biproportional apportionment problems. In some proportional electoral systems with more than one constituency the number of seats must be allocated to parties within territorial constituencies, as well as, the number of seats that each party has to receive at a national level.

The proof, presented in Section 2.8, uses an adaptation of the Cox (1987) controlled rounding procedure to construct a unbiased lottery solution that stays within quota. By Proposition 1, any period-1 fair share table is implemented by a lottery solution that only gives positive probability to period-1 reservation tables that do not deviate from fair shares by more than one seat. The following corollary directly follows Proposition 1.

Corollary 1. There exists a deterministic solution $R \in \mathcal{R}^1$ that stays within quota.

Corollary 1 implies that for any problem of length $T = 1$, there always exists a reservation table that stays within quota. That is, there is a satisfactory solution to the problem of reservation in two dimensions if in each period the problem is treated independently.

2.5.2 Multi Period Results

Treating each period's problem independently can lead to adverse outcomes over time. In particular, since integer seat allocations differ from the fair share tables in every period, accumulation of these differences can result in large deviation from fair shares over time. We next show this issue in an example.

Example 2. Consider a problem depicted in the following fair share table, with two departments d_1, d_2 having 2 and 7 positions, respectively, and two categories c_1, c_2, and the reservation scheme vector $\boldsymbol{\alpha} = [0.1, 0.9]$.

The following deterministic solution stays within quota, but it does not give any positions to category c_1.

$$X = \begin{array}{cc|c} 0.2 & 1.8 & 2 \\ 0.7 & 6.3 & 7 \\ \hline 0.9 & 8.1 & 9 \end{array}$$

(a) FAIR SHARE TABLE

$$R(X) = \begin{array}{cc|c} 0 & 2 & 2 \\ 0 & 7 & 7 \\ \hline 0 & 9 & 9 \end{array}$$

(b) RESERVATION TABLE

Example 2 suggests that not reserving any seats is a solution that stays within quota. For instance, a university can repeatedly apply this solution to each period's problem and does not reserve a single seat.

In general case, a lottery solution ϕ that treats each period's problem independently rounds up or rounds down each fair share with some probabilities. Therefore, in the range of the lottery solution ϕ, there exists an outcome that rounds down a particular entry in every period. That is, the lottery solution ϕ can result in seat allocations with sizeable deviations from fair shares.

We next examine how our single period results extend to the multi-period problem. We first show that for every problem, a deterministic solution that stays within quota does not always exists.

Proposition 2. There does not exist a deterministic solution $R \in \mathcal{R}^T$ that stays within quota for $T > 1$.

Proposition 2 implies that for every problem of length $T > 1$, unlike single period, solutions deviate from fair shares by more than one seat. It also implies that it is impossible to both stay within university quota and stay within department quota. We next generalize staying within quota property to allow for some differences in fair shares and seat allocations.

A **bias of a deterministic solution** R at Y^t is a two-way table *bias*$(R(Y^t))$, with each entry *bias*$(R(y^t)) := R(y^t) - x^t$. The bias of a solution is the difference between the solution and the fair share table. With this definition, a solution stays within quota if, for any Y^t, each entry $|\text{bias}(R(y^t))| < 1$, that is, for any problem, the bias of the solution is always less than 1 in absolute value. Our next property allows a solution to deviate from fair shares up to a constant number.

Definition 3. A deterministic solution $R \in \mathcal{R}^T$ has a **finite bias** if there exists a constant $b > 0$ such that, for any $Y^t \in \cup_{s=1}^T \mathcal{Y}^s$,

$$|\text{bias}(R(y^t))| < b.$$

One might be tempted to think that there would be solutions that allow for larger deviations in seat allocations from fair shares at the department level but stays within university quota. We show that such solutions do not exist.

Proposition 3. There does not exist a deterministic solution $R \in \mathcal{R}^T$ that has a finite bias and stays within university quota for $T > 1$.

The proof is in Section 2.8. Proposition 2 is a corollary of Proposition 3. By Proposition 2 we learn that any procedure that stays within university quota cannot stay within department quota. By Proposition 3 we learn that any procedure that stays within university quota can lead to departments to grow in size over time without reserving a single seat.

Proposition 2 and Proposition 3 have a stronger implication: there is no deterministic solution to the problem of reservation in two dimensions that stays within quota. This negative result provides yet another reason to use lottery solutions to address the problem of reservation in two dimensions.

We next present the main existence result: the set of lottery solutions that are unbiased and stay within department quota is non-empty.

Theorem 1. There exists a lottery solution ϕ that is unbiased and stays within department quota.

A formal proof of Theorem 1 is presented in Section 2.8. The proof utilizes a network flow to construct a lottery over rosters. Each department is then assigned a roster drawn independently from the constructed lottery. This two-step procedure induces a lottery solution, denoted ϕ^* and defined formally in Section 2.8. The lottery solution is shown to be unbiased and stays within department quota, that is, each category gets (i) ex-ante its fair share, and (ii) ex-post its fair share either rounded up or down in every department.[18]

Theorem 1 implies that there is a lottery solution that ensures that each department sticks to the reservation scheme while the university, as a whole, respects the fair shares in

[18]One can show that the set of lottery solutions that are unbiased and stay within university quota is also non-empty. However, staying within the department quota property better suits our applications because one goal of affirmative policies is to increase diversity in all sub-units (departments and university as a whole), and the smallest sub-units in our setup are departments.

an expected sense. By Proposition 3, however, we know that such solutions can result in biases greater than one at the university level. To show that our lottery solution limits the probability of these occurrences, we modify the staying within university quota property.

We denote the outcome of a lottery solution ϕ at a sequence of fair share tables Y^t by the random variable Z^t and its entries by z^t_{ij}. The deviation of the outcome of lottery solution ϕ for a category $j \in \mathcal{C}$ in the university is $z^t_{m+1,j} - x^t_{m+1,j}$. This random variable measures the deviation of the seat allocation at the university level from its fair share.

Definition 4. A lottery solution ϕ **approximately stays within university quota** if, for any Y^t, for any category $j \in \mathcal{C}$ and for any $b > 0$, we have

$$Pr(z^t_{m+1,j} - x^t_{m+1,j} \geq b) \leq e^{-\frac{b^2}{3x^t_{m+1,j}}},$$

$$Pr(z^t_{m+1,j} - x^t_{m+1,j} \leq -b) \leq e^{-\frac{b^2}{2x^t_{m+1,j}}}.$$

We establish probabilistic concentration bounds on the deviations for our lottery solution ϕ^* and show that ϕ^* approximately stays within university quota.

Theorem 2. The lottery solution ϕ^* is unbiased, stays within department quota, and approximately stays within university quota.

Theorem 2 follows from a Chernoff-type concentration bound. We establish the probability bounds in a fashion similar to Gandhi et al. (2006). By this property, the probability of deviating from university quota by a value greater than b decays exponentially with b^2. Therefore, there is a procedure that ensures that each department obeys the reservation scheme, while the university as a whole approximately follows the reservation scheme.

We next show that the bounds in Definition 4 are tight (up to a multiplicative constant in the exponent) and thus rules out any improvement of the deviation of the seat allocation at the university level from its fair share.

Proposition 4. Consider a lottery solution that is unbiased, stays within department quota and limits the probability of deviation of the seat allocation at university level in following way: for any Y^t, for any category $j \in \mathcal{C}$ and for any $b > 0$, the lottery satisfies

$$Pr(z^t_{m+1,j} - x^t_{m+1,j} \geq b) \leq f(x^t_{m+1,j}, b) \ ,$$

$$Pr(z^t_{m+1,j} - x^t_{m+1,j} \leq -b) \leq f(x^t_{m+1,j}, b) \ .$$

Then, there exists a constant $k > 0$ such that for any $b > 0$,

$$\lim_{x^t_{m+1,j} \to \infty} \frac{e^{-\frac{b^2}{x^t_{m+1,j}} k}}{f(x^t_{m+1,j}, b)} = 0 \ .$$

Proposition 4 shows that there exists a constant $k > 0$ such that any lottery that is unbiased and stays within department quota can approximately stays within university quota (in the sense of Definition 4) with a probabilistic guarantee no better than $e^{-\frac{b^2}{x^t_{m+1,j}} k}$. A proof of Proposition 4 is presented in Section 2.8.

2.6 Empirical Study of Reservation in Two Dimensions

Here we present a comprehensive evaluation of recruitment advertisements to highlight the severity of shortcomings in the existing solutions and to reflect the benefits of adopting our proposed solutions. Specifically, we evaluate the general quality of the advertised two-way apportionments with respect to the instances and magnitude of quota violations, and present the advantage our proposed solution exhibits.

Our data comprises 60 advertisements released in the following five recruitment settings where two-dimensional reservation problems are seen in practice.

1. Assistant Professors of University of Delhi
2. Officers of Indian Administrative Services
3. Officers of Indian Forest Services

Table 2.2: OVERVIEW OF RECRUITMENT ADVERTISEMENTS

Institution	Ads	Departments Avg.	Departments Min-Max	Dept. Vacancies Avg.	Dept. Vacancies Min-Max	Total Vacancies Avg.	Total Vacancies Min-Max
University of Delhi	23	19.7	8-50	4.2	1-10	94.8	21-405
Indian Administrative Services	15	24.7	24-26	9.47	5-15	148.9	87-180
Indian Forest Services	7	25.1	24-26	3.6	3-4	95.4	78-110
Indian Police Services	8	25.3	24-26	12.8	10-16	150.1	148-153
Reserve Bank of India	7	17	17-17	23.7	13-30	648.1	500-1000

4. Officers of Indian Police Services

5. Assistants of Reserve Bank of India

In the preceding sections we presented and analyzed the problem in the context of a university. Therefore, we will continue to use the same terminology for all advertisements. The term *departments* refers to departments in a university for the assistant professors advertisements. However, for other advertisements the departments correspond to the states (in India) where an officer or an assistant shall be recruited. Similarly, the term *university* corresponds to the country (India) in the latter advertisements.

An overview of the recruitment advertisement data is presented in Table 2.2. The advertisements provide a variety of two-dimensional reservation problems with the number of departments varying from 8 to 50; the number of vacancies in a department varying from 1 to 30; and the number of vacancies in the university varying from 21 to 1000. The advantage of using data from different institutions is that the variety of procedures used at these institutions help highlight the robustness of shortcomings we discussed in Section 2.4.

2.6.1 Single Period Analysis

First consider the problem of reservations as a single period problem. Thus in this subsection each advertisement is treated as an independent single period two-dimensional

Table 2.3: SINGLE PERIOD QUOTA VIOLATIONS – STATISTICS

	Instances of Violations				Magnitude of Bias	
	Avg.	Min-Max	Total	Percentage	Avg.	Min-Max
University of Delhi						
Department Quota	6.8	0-24	156	8%	1.3	1-4
University Quota	2.6	1-5	60	59.4%	2.7	1-13
Indian Administrative Services						
Department Quota	28.9	2-48	434	29.2%	1.8	1-6.5
University Quota	1.9	0-4	28	46.7%	3.9	1-6.9
Indian Forest Services						
Department Quota	17.6	8-24	123	17.5%	1.5	1-2.9
University Quota	0.7	0-2	5	17.9%	1.4	1.3-1.5
Indian Police Services						
Department Quota	32.8	27-38	262	32.5%	1.8	1-5.1
University Quota	2.9	1-4	23	71.9%	2.3	1.2-5.3
Reserve Bank of India						
Department Quota	40.1	34-49	281	59%	4.1	1-35.8
University Quota	3.7	3-4	26	92.9%	20.8	2.5-60.6

reservations problem. In line with our theoretical analysis, we use the department and university quota violations in judging the quality of solutions advertised.

Table 2.3 shows that the instances of both the department quota and the university quota violations are pervasive in the advertised solutions of all the institutions. The percentage of instances of violations, obtained by dividing the number of violations that occurred by the maximum number of violations possible, is an informative summary measure. Based on this measure, the probability that a typical category would witness a department quota violation in a typical department ranges from 0.08 in University of Delhi to 0.59 in Reserve Bank of India. The probability that a typical category would witness a university quota violation ranges from 0.18 in India Forest Services to 0.93 in Reserve Bank of India.

In order to provide a complete picture of the severity of shortcomings, we present the magnitude of bias (in cases of quota violation) in Table 2.3. The magnitude of bias is the absolute value of bias as defined in Section 2.5.2. At the department level, this measure shows that, in case of quota violation, the average deviation from fair shares for a typical category ranges from 1.3 in University of Delhi to 4.1 in Reserve Bank of India. At the university level, this measure shows that, in case of quota violation, the average deviation from fair shares for a typical category ranges from 1.4 in Indian Forest Services to 20.8 in Reserve Bank of India.

As a single period problem, the two-dimensional reservations problem has been shown to admit an elegant solution called *controlled rounding* that stays within quota (see section Section 2.5.1). If each reservation problem were to be treated independently, adopting controlled rounding procedure for making reservation tables would lead no quota violations. Therefore making it possible to achieve simultaneously the prescribed percentage of reservations at both the department and the university level in single period problems (as shown in Proposition 1).

2.6.2 Multi Period Analysis

In Section 2.2, with emphasis on maintaining rosters, the intent of India's policymakers is clear. In the face of the indivisibility of seats, their policies aim to achieve the prescribed percentage of reservations not in a single period but over time. Therefore, analysis of the recruitment data is incomplete without checking whether the quota and biases cancel out and consequently disappear over time. For this purpose we need to look at sequences of consecutive advertisements that share the same set of departments and the same reservation policy. There are seven such sequences in our data.

Results from the last period of these seven sequences of consecutive advertisements in Table 2.4 show that the single period violations are not cancelling over time, rather they are adding up. Both the instances of violations and the magnitude of bias are now higher than the numbers reported in Table 2.3 for single period problems. The probability that a typical category would witness a department quota violation in a typical department ranges

Table 2.4: MULTI PERIOD QUOTA VIOLATIONS – STATISTICS

	Instances of Violations		Magnitude of Bias	
	Total	Percentage	Avg.	Min-Max
Indian Administrative Services: 2005 to 2013				
Department Quota	76	79.2%	3.5	1-14.1
University Quota	4	100%	16.6	5.2-28
Indian Administrative Services: 2014 to 2018				
Department Quota	79	75.9%	4.1	1-18.1
University Quota	2	50%	4	3.5-4.4
Indian Forest Services: 2011 to 2013				
Department Quota	35	36.5%	1.8	1-3.1
University Quota	2	50%	1.5	1.2-1.8
Indian Forest Services: 2015 to 2018				
Department Quota	54	51.9%	2.8	1.1-6.9
University Quota	3	75%	3.1	2-4.9
Indian Police Services: 2010 to 2011				
Department Quota	45	46.9%	2.2	1-4.2
University Quota	4	100%	2.6	1.7-3.5
Indian Police Services: 2014 to 2018				
Department Quota	73	70.2%	3.4	1.1-12.6
University Quota	3	75%	5	2-7.7
Reserve Bank of India: 2012 to 2017				
Department Quota	60	88.2%	11.7	1-35.5
University Quota	4	100%	83.2	10-166.4

from 0.36 in Indian Forest Services to 0.88 in Reserve Bank of India. The probability that a typical category would witness a university quota violation ranges from 0.50 in Indian Forest Services to 1 in Reserve Bank of India. At the department level, in case of quota violation, the average deviation from fair shares for a typical category ranges from 1.8 in Indian Forest Services to 11.7 in Reserve Bank of India. At the university level, in case of quota violation, the average deviation from fair shares for a typical category ranges from 1.5 in Indian Forest Services to 83.2 in Reserve Bank of India.

The findings suggest that the problem worsens with time in that there are more instances of violations and larger deviations from policy prescribed percentage of reservations. This is not surprising given the negative results presented in Proposition 2 and Proposition 3. However, the scope of improvement is clear. Theorem 1 and Theorem 2 show that there exists an unbiased solution that stays within quota at the department level and approximately stays within quota at the university level. A comparison of this proposed solution with the existing solution is the point of our next simulation exercise. For this exercise we will consider the longest sequence of consecutive advertisements in our data: the advertisement of Indian Administrative Services from 2005 to 2013.

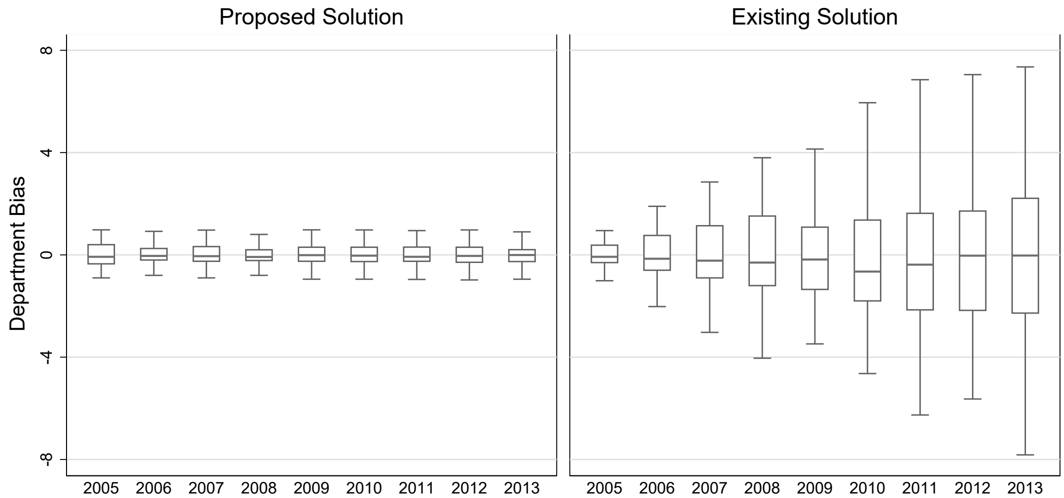
The objective of the simulation exercise is to compare the evolution of bias over time under the existing solution with the solution proposed in this paper. For this purpose, we simulate a set of 50 advertisements adhering to the proposed solution and plot the bias at each time period in Figure 2.6. The top-left panel shows that, for the proposed solution's advertisements, the department bias stays well within the $[-1, 1]$ interval, that is, there are no quota violations at the department level. In contrast, under the existing (advertised) solution presented in the top-right panel, the bias accumulates over time at the department level. The bottom-left panel shows that though the university violations occur under the proposed solution, the bias does not add up over time. The significance is apparent when one compares it to the evolution of bias under the existing solution presented in the bottom-right panel.

2.7 Conclusion

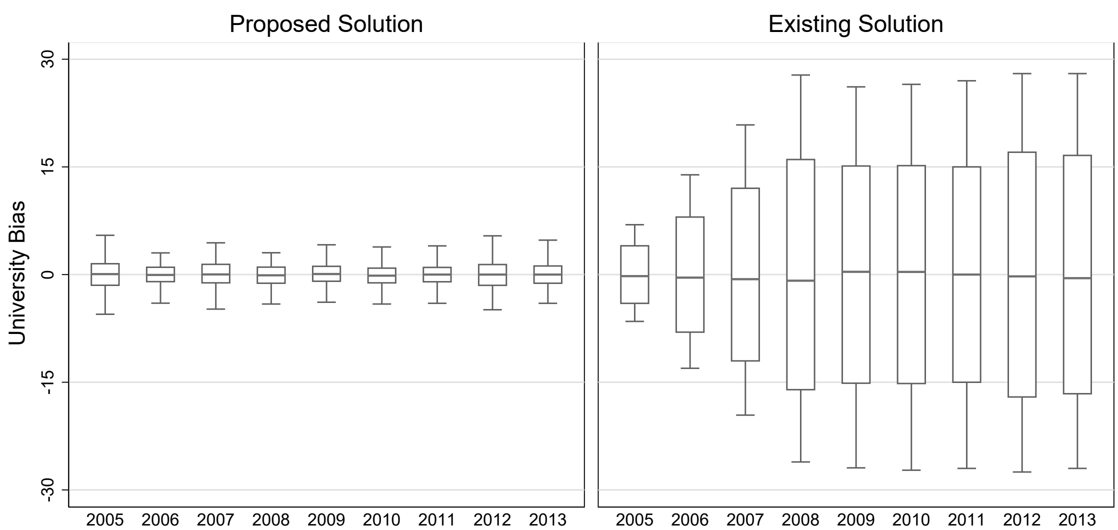
This paper has offered an analysis of two-dimensional reservation problems using the theory of apportionment and rounding problems. We have theoretically and empirically documented the shortcomings of existing solutions and proposed a solution with demonstrable advantages over the existing solutions. From a broader perspective, even though our search for quality solutions is limited to the *staying within quota* property, the analysis here can be viewed as illustrative of substantial scope for improvement in existing procedures

Figure 2.6: BIASES OF PROPOSED AND EXISTING SOLUTIONS

(a) DEPARTMENT BIAS OVER TIME



(b) UNIVERSITY BIAS OVER TIME



Note: Box plots show medians, quartiles, and adjacent values of bias distributions over time.

for two-dimensional reservation problems.

Our approach is obviously limited and the problem is open to several alternative approaches that deserve extra work. A particular one that deserves mention is the error minimization approach that has yielded a class of methods to solve biproportional apportionment problems (Ricca et al. (2012) and Serafini and Simeone (2012)). These methods take a fractional matrix as the target (fair share table in our case) and solve a constrained optimization problem where the objective corresponds to a measure of the error between the solution and the target matrix. Such an approach may pave the way to a richer study of defining and finding appealing solutions to two-dimensional reservation problems.

Our problem also suggests possible extensions in the theory of apportionment. We believe that the multi-period considerations introduced in this paper could be worth exploring in the classic biproportional apportionment problem context of translating electoral votes into parliamentary seats.

2.8 Mathematical Appendix

Proof of Theorem 1

The proof is constructive and has two parts. We first define the Roster-Finding Algorithm, which takes a reservation scheme vector as inputs and generates a random roster as an output, that is, a lottery over rosters.We then assign the random roster to each department independently. The random roster is constructed such that if every department follows it, the induced solution stays within the department quota. We denote this solution as our lottery solution ϕ^*. We, lastly, show that the lottery solution ϕ^* is unbiased.

Proof of Theorem 1. Let $\mathcal{C}$ be the set of categories and $\boldsymbol{\alpha} = [\alpha_j]_{j\in\mathcal{C}}$ be the reservation scheme. Let P represent the given reservation scheme as a $k \times n$ two-way table, where the rows denote the index of the seats and the columns denote the categories. The internal entry p_{ij} equals to α_j for every (i, j). Let assume that for each column, entries sum up to an integer (if there is a common multiplier for fractions in the reservation scheme vector, then such k exists).[19] The output of the algorithm will be an integral table that define how a department

[19]The generalization to non-integer sums is made by constructing an extended table P' in a way that is equivalent to P except the last row. The last row of P' is generated by taking 1- fractional part of the column totals (similar to how the extended table is created in the algorithm given for proof of Proposition 1).

reserves its positions over time, i.e., a roster. We next construct a set of constraints that bounds the elements of the table P.

For each constraints K, let $\underline{p_K}$ and $\bar{p_K}$ be the floor and ceiling of the constraint. That is, $\underline{p_K} = \lfloor \sum_{(i,j) \in K} p_{ij} \rfloor$ and $\bar{p_K} = \lceil \sum_{(i,j) \in K} p_{ij} \rceil$. We will consider tables P' that satisfying, for each K,

$$\underline{p_K} \leq \sum_{(i,j) \in K} p'_{ij} \leq \bar{p_K}.$$

We have three types of constraints. Internal constraints make sure that each internal entry can be either 1 or 0. Row sums are required to be one since every position is assigned to exactly one category. Column constraints make sure that difference between cumulative some of positions given to a category and cumulative fair shares is less than one.

Let $\mathcal{K}_I$ be the internal constraints, i.e., $0 \leq p'_{ij} \leq 1$ for every (i, j). Let $k_{ij} := \{(i, j)\}$ denote such constraint. Let $\mathcal{K}_R$ be the set of row constraints, i.e., $\sum_{j \in \mathcal{C}} p'_{ij} = 1$ for every i. Let $R_i := \{(i, j) | j \in \mathcal{C}\}$ denote such constraint. Let $\mathcal{K}_C$ be the set of column constraints, i.e., $\lfloor \sum_{i \leq l} p_{ij} \rfloor \leq \sum_{i \leq l} p'_{ij} \leq \lceil \sum_{i \leq l} p_{ij} \rceil$ for every $2 \leq l \leq n$ and $j \in \mathcal{C}$. Let $C_{lj} := \{(i, j) | i \leq l\}$ denote such constraint.

We next create a flow network. The set of vertices consists of the source, the sink, vertices for each $k \in \mathcal{K}_I$, each $R \in \mathcal{K}_R$, and for each $C \in \mathcal{K}_C$. The following rule governs the placement of directed edges:

1. A directed edge from source C_{nj} for every $j \in \mathcal{C}$.

2. A directed edge from C_{lj} to k_{lj} and C_{l-1j} for every $l \geq 3$ and $j \in \mathcal{C}$.

3. A directed edge from C_{2j} to k_{2j} and k_{1j} for every $j \in \mathcal{C}$.

4. A directed edge from k_{ij} to R_i for every (i, j).

5. A directed edge from R_i to sink for every i.

Note that the constraint structure for $\mathcal{K}_C \cup \mathcal{K}_I$ and $\mathcal{K}_R \cup \mathcal{K}_I$ are hierarchical. A set of constraints $\mathcal{K}$ is **hierarchical** if, for every pair of constraints K' and K'', we have that $K' \subset K''$ or $K'' \subset K'$ or $K' \cap K'' = \varnothing$.

We next associate flow with each edge. Notice that there is only one incoming edge for each vertex $K \in \mathcal{K}_C \cup \mathcal{K}_I$. And, there is only one outgoing edge for each vertex $K \in \mathcal{K}_R \cup \mathcal{K}_I$. Observe that it is because of the hierarchical sets of constraints. Therefore, it is sufficient to associate incoming flows for each vertex $K \in \mathcal{K}_C \cup \mathcal{K}_I$ and outgoing flows for each vertex $K \in \mathcal{K}_R \cup \mathcal{K}_I$. For each vertex $K \in \mathcal{K}_C \cup \mathcal{K}_I$, the incoming flow is equal to $\sum_{(i,j)\in K} p_{ij}$. For each vertex $K \in \mathcal{K}_R \cup \mathcal{K}_I$, the outgoing flow is equal to $\sum_{(i,j)\in K} p_{ij}$. Furthermore, the flow association ensures that the amount of incoming flow is equal to the amount of outgoing flow for each vertex.

Notice that we map table P with the constraint structures to a flow network. In addition, the mapping is injective. As long as the constraints are still satisfied after the transformation, every transformation in the flow network can be mapped back to table P.

Definition 5. We call the pair of tables (P^1, P^2) a **decomposition** of table P, if

1. there exists $\beta \in (0, 1)$ such that $P = \beta P^1 + (1 - \beta)P^2$,
2. for each constraint K, $\underline{p_K} \leq \sum_{(i,j)\in K} p^l_{ij} \leq \bar{p_K}$ for $l = 1, 2$, and
3. table P^1 and P^2 have more number of integral entries than table P.

The following constructive algorithm has two parts. We first find a cycle of fractional edges in the network flow. We then alter the flow of edges in two different ways until one edge becomes integral. It will provide us a decomposition of table P.

Roster-Finding Algorithm

Repeat the following as long as the flow network contains a fractional edge:

Step 1: Choose any edge that has fractional flow. Since the total inflow equals to total outflow for each vertex, there will an adjacent edge that has fractional flow. Continue to add new edges with fractional flows until a cycle is formed.

Step 2: Modify the flows in the cycle in two ways to create P^1 and P^2:

1. First way: the flow of each forward edge is increased and the flow of each backward edge is decreased at the same rate until at least one flow reaches an integer value. Record the amount of adjustment as d_-. Map back the resulting flow network to a two way table. Denote the table as P^1.
2. Second way: the flow of each forward edge is decreased and the flow of each backward edge is increased at the same rate until at least one flow reaches an integer value. Record the amount of adjustment as d_+. Map back the resulting flow network to a two way table. Denote the table as P^2.
3. Set $\beta = \frac{d_-}{d_- + d_+}$.
4. The pair of tables (P^1, P^2) is a decomposition of table P, where $P = \beta P^1 + (1-\beta)P^2$.

The algorithm creates a lottery over integral two-way tables that share the same constraint structure as table P.[20] Assume that $\bar{P}$ is an integral table constructed by the algorithm, and its compound probability is γ. We construct a roster by each of these integral tables as follows. For each internal entry of table $\bar{P}$, if $\bar{p}_{ij} = 1$ then assign $\sigma(i) = c_j$. We next assign probability γ to roster σ. Thus, we obtain a random roster.

Notice that the expected number seats for each category j in the first q seats equals to $q\alpha_j$ for $q = 1, 2, \ldots$. We next create the induced lottery solution ϕ^* for the problem of reservation in two dimensions as follows. We assign the random roster to each department. Each department then reserves positions according to the roster realized from the lottery. For example, if roster σ is realized for department i then, the number of positions reserved in department i in period-1 is determined by $\sigma(1), \ldots, \sigma(q_i^1)$. The number of positions reserved in department i in period 2 is determined by $\sigma(q_i^1 + 1), \ldots, \sigma(q_i^1 + q_i^2)$.

We next show that the lottery solution ϕ^* is unbiased. Given the lottery solution ϕ^* and a a sequence of fair share tables $Y^t = (X^1, \ldots, X^t)$, we denote the outcome of the lottery solution by the random variable Z^t. We know that the expected number of positions

[20] Moreover, the expected table equals to table P.

reserved to category j in department i until period-t is $E(z_{ij}^t) = \sum_{s \leq t} q_i^s \alpha_j$. Moreover, the internal entry x_{ij}^t of fair share table X^t also equals to $\sum_{s \leq t} q_i^s \alpha_j$. Thus, the lottery solution ϕ^* is unbiased.

This proves the theorem. ■

An example for Theorem 1

To make the Roster-Finding Algorithm easier to understand and show the whole procedure that constructs the lottery solution ϕ^*, we show an example.

Consider a university where there are two categories $\mathcal{C} = \{c_1, c_2\}$ and the reservation scheme is $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) = (1/3, 2/3)$. Suppose we wish to implement the reservation scheme in a problem of reservation in two dimensions. We represent the given reservation scheme as a two-way table P, where the rows denote the index of the positions and the columns denote the categories. Each internal entry $p_{ij} = \alpha_j$. The output of the algorithm will be an integral table that define how a department reserves its positions over time, i.e., a roster.

There are three positions for easy illustration.[21] However, this method works for more general (total $3k$ positions, where $k = 1, 2, \dots$) cases. The example table P is

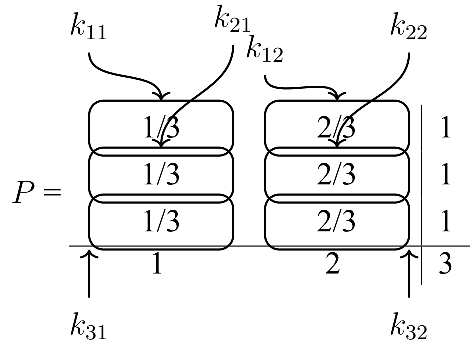
$$P = \begin{array}{cc|c} 1/3 & 2/3 & 1 \\ 1/3 & 2/3 & 1 \\ 1/3 & 2/3 & 1 \\ \hline 1 & 2 & 3 \end{array}$$

Figure 2.7 illustrates the constraint structure. Column constraints are $C_{31} = \{k_{11}, k_{21}, k_{31}\}$, $C_{21} = \{k_{11}, k_{21}\}$, $C_{32} = \{k_{12}, k_{22}, k_{32}\}$, and $C_{22} = \{k_{12}, k_{22}\}$, and row constraints are $R_1 = \{k_{11}, k_{12}\}$, $R_2 = \{k_{21}, k_{22}\}$, and $R_3 = \{k_{31}, k_{32}\}$.

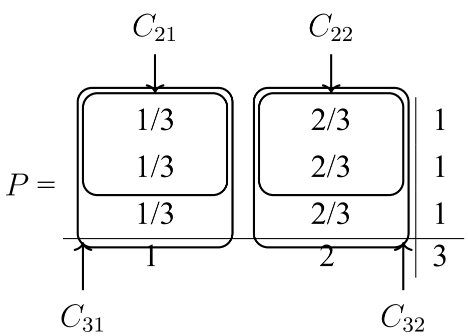
The two-way table P with the constraints is then represented as a network flow. Starting from the source, the flows first pass through the sets in column constraints, which are

[21] Using the common multiple of the fractions in the reservation scheme, three in our case, also helps to understand.

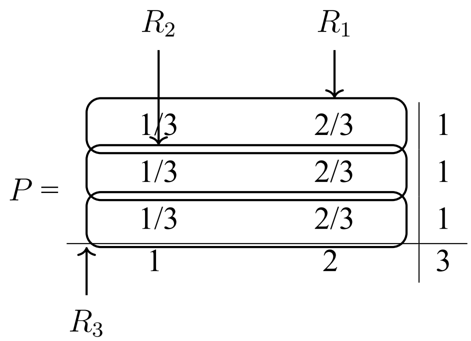
Figure 2.7: CONSTRAINT STRUCTURE OF THE EXAMPLE P



(a) INTERNAL CONSTRAINTS

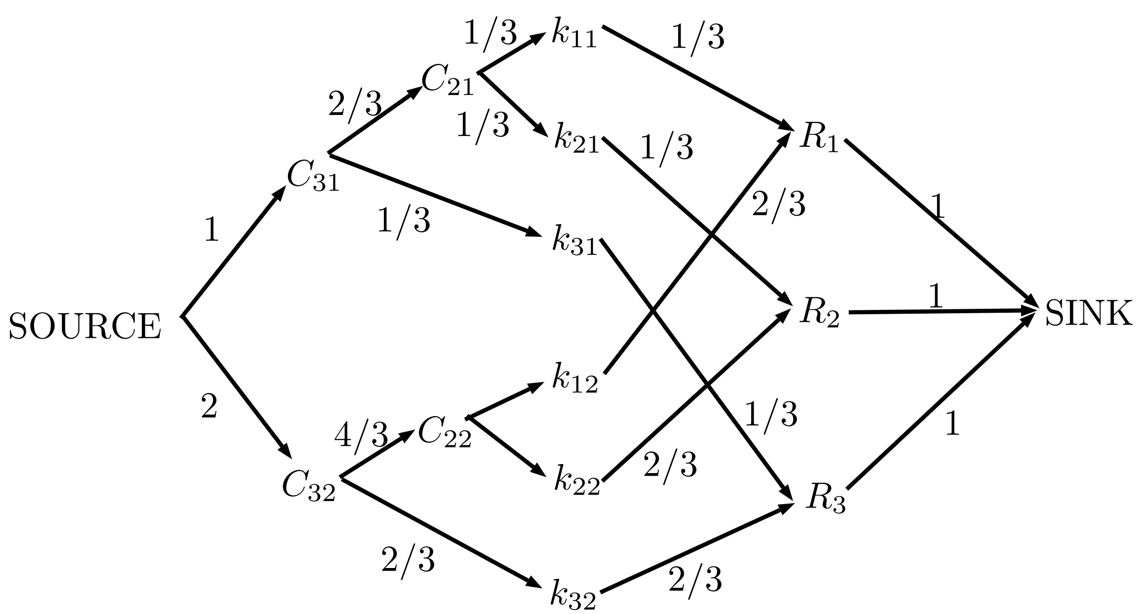


(b) COLUMN CONSTRAINTS



(c) ROW CONSTRAINTS

Figure 2.8: FLOW NETWORK REPRESENTATION OF THE EXAMPLE P



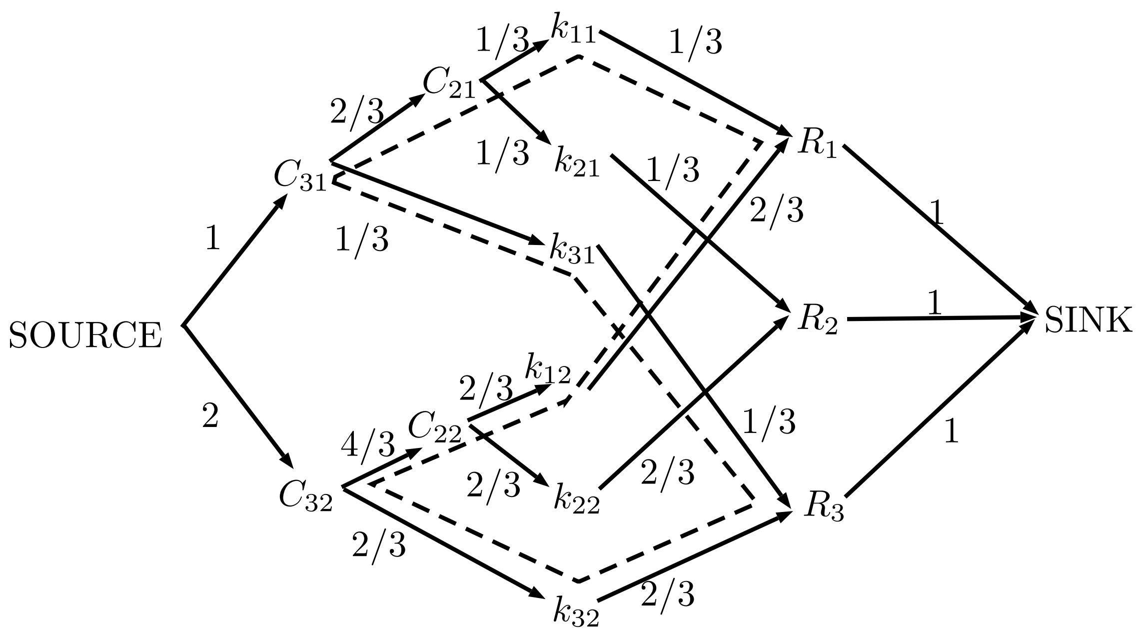
arranged in descending order of set-inclusion. That is, for example, $C_{31} \supset C_{21} \supset k_{11}$. This explains the flow network on the left side of Figure 2.8, where the numbers on the edges represent the flows. The flows then proceed along the directed edges that represent the set-inclusion tree, eventually reaching the singleton sets. That is, for example, $k_{11} \subset R_1$. This explains the flow network on the right side of Figure 2.8.

In the flow network, note that the flow associated with each edge reflects the totals of elements in the corresponding set. And, the flow arriving at each vertex equals the flow leaving that vertex. Now we are ready to present the algorithm. The algorithm will conserve these two properties while constructively find new flow network with fewer fractional elements.

We first identify a cycle of edges with fractional flows. Choosing any fractional edge, say (C_{31}, k_{31}), we find another fractional edge that is neighbor to k_{31}. If a vertex has a fractional edge then it has to have another fractional edge: since total inflow equals to outflow for every vertices(except source and sink), we would have a contradiction. We continue to add new fractional edges until we form a cycle. In our example, the cycle of fractional edges is $C_{31} \rightarrow^{1/3} k_{31} \rightarrow^{1/3} R_3 \leftarrow^{2/3} k_{32} \leftarrow^{2/3} C_{32} \rightarrow^{4/3} C_{22} \ldots \leftarrow^{2/3} C_{31}$. We illustrates this cycle in Figure 2.9 with dashed lines.

Next, we alter the cycle's edge flows. We first increase the flow of each forward edge while decreasing the flow of each backward edge at the same time until at least one flow reaches an integer value. A table P_1 is created as a result of the resulting network flow. In the example, flows along all forward edges increase from 2/3 to 1, 1/3 to 2/3, and 4/3 to 5/3, while flows along all backward edges decrease from 1/3 to 0 and 2/3 to 1/3. The adjustment is $d_+ = 1/3$. Next, the flows of the edges in the cycle are readjusted in the opposite direction, increasing those with backward edges and lowering those with forward edges in an analogous way, resulting in a new table P_2. In the example, flows along all forward edges decrease from 2/3 to 1/3, 1/3 to 0, and 4/3 to 1, while flows along all backward edges increase from 1/3 to 2/3 and 2/3 to 1. The adjustment is $d_- = 1/3$.

Figure 2.9: AN EXAMPLE OF CYCLE WITH FRACTIONAL EDGES



Now, we can decompose P into these two tables, i.e., $P = \frac{d_-}{d_- + d_+} P_1 + \frac{d_+}{d_- + d_+} P_2 = \frac{1}{2} P_1 + \frac{1}{2} P_2$. The algorithm picks P_1 with probability 0.5 and P_2 with probability 0.5. We reiterate the decomposition process until no fractions left.

At each iteration, at least one fraction in P is converted to an integer, while all current integers remain constant. Each fraction must appear in at least one iteration. As a result, the process must converge to an integer table in less iterations than the initial number of fractions in table P.

Since only the fractions along one cycle in the flow network are modified in each iteration, the expected change at this iteration for entries not on this cycle is 0, i.e., the expected change in corresponding entries in P is 0. For those fractional edges that are modified, the probabilities are picked so that the expected adjustment in each iteration is 0.

Fractional edges that are adjusted multiple times will have a variety of intermediate adjustment probabilities, but because our procedure keeps the expected change at 0 in each iteration, the compound probabilities will also keep the expected change at 0.

Proof of Theorem 2

Here, we prove that lottery solution ϕ^* in Theorem 1 approximately stays university quota. In words, the lottery solution ϕ^* is designed in such a way such that it hardly ever round up (or round down) most of the entries in each column of X^t. We show the approximately staying university quota property by proving two lemmas. We first show that entries of each column of Z^t are "independent". We next prove the approximately staying university quota by applying Chernoff concentration bounds.

Lemma 1. For any subset of $S \subset \{1, 2, \dots, m\}$ and any $j \in \{1, 2, \dots, n\}$, we have

$$\text{Pr}\Big[\bigwedge_{i\in S} z_{ij}^t = \lceil x_{ij}^t \rceil\Big] = \prod_{i\in S} \text{Pr}\Big[z_{ij}^t = \lceil x_{ij}^t \rceil\Big] ,$$

$$\text{Pr}\Big[\bigwedge_{i\in S} z_{ij}^t = \lfloor x_{ij}^t \rfloor\Big] = \prod_{i\in S} \text{Pr}\Big[z_{ij}^t = \lfloor x_{ij}^t \rfloor\Big] .$$

Proof. Notice that the random roster is assigned to each department independently. Consequently, for any pair (i, i'), random variables z_{ij}^t and $z_{i'j}^t$ become independent, which proves the lemma. ■

Lemma 2. For any subset of $S \subset \{1, 2, \dots, m\}$ and any $j \in \{1, 2, \dots, n\}$ with $\sum_{i\in S} x_{ij}^t = \mu$, and for any $\epsilon > 0$, we have

$$\text{Pr}\Big[\sum_{i\in S} z_{ij}^t - \mu > \epsilon\mu\Big] \leq e^{-\mu\frac{\epsilon^2}{3}} ,$$

$$\text{Pr}\Big[\sum_{i\in S} z_{ij}^t - \mu < -\epsilon\mu\Big] \leq e^{-\mu\frac{\epsilon^2}{2}} .$$

Proof. We begin by recalling a result of Chernoff et al. (1952), which demonstrates that the independence property has the following large deviations result. Chernoff bounds are well-known concentration inequalities that limit the deviation of a weighted sum of Bernoulli random variables from their mean. We now use the multiplicative form of Chernoff concentration bound.

Theorem 3. Chernoff bound: Let $A_1, A_2, \ldots, A_m$ be m independent random variables taking values in $\{0, 1\}$. Let $\mu = \sum_{i=1}^{m} E[A_i]$. Then, for any $\epsilon \geq 0$,

$$\Pr\Big[\sum_{i=1}^{m} A_i \geq (1+\epsilon)\mu\Big] \leq e^{-\mu\frac{\epsilon^2}{3}} \ ,$$

$$\Pr\Big[\sum_{i=1}^{m} A_i \leq (1-\epsilon)\mu\Big] \leq e^{-\mu\frac{\epsilon^2}{2}} \ .$$

The random variable z_{ij}^t can take two values, either $\lceil x_{ij}^t \rceil$ or $\lfloor x_{ij}^t \rfloor$. If we subtract the fix number $\lfloor x_{ij}^t \rfloor$ from z_{ij}^t, then we obtain a Bernoulli distribution. Lemma 1 says that the set of random variables in each column of Z^t are independent, which means Chernoff concentration bounds hold for each column of Z^t. ■

Proof of Theorem 2. We can now prove Theorem 2. In Lemma 2, if we choose $S = \{1, \ldots, m\}$, then $\sum_{i=1}^{m} x_{ij}^t = x_{m+1,j}^t$. This fact along with Lemma 2 yields our result for Theorem 2. ■

Proof of Proposition 4

Proof. For any $x_{m+1,j}^t := \mu > 0$ and any constant $b := \epsilon\mu$, we construct a problem instance. For the rest of the proof we fix category j, μ, and ϵ. This instance contains n departments, m categories. The vacancies are as follows: $q_i^s = 0$ vacancies for all $s < t$ and $q_i^t = 1$ for any $i \in \mathcal{D}$. Choose a constant $\underline{\epsilon}, \bar{\epsilon} \in (0, 1)$ such that $\epsilon \in (\underline{\epsilon}, \bar{\epsilon})$. Choose $\alpha \in (0, 1/(1+\bar{\epsilon}))$ such that μ/α is an integer. Let $m = \mu/\alpha$. For category j, α fraction of vacancies are to be reserved. Note that, by definition, $x_{ij}^t = \alpha$ for all $i \in \mathcal{D}$.

Consider a lottery solution that is unbiased and stays within department quota. Let z_{ij}^t denote the the outcome of such lottery for category j in department i. Note that, by definition of such lottery, $Pr(z_{ij}^t = 1) = \alpha$ and $Pr(z_{ij}^t = 0) = 1 - \alpha$ must fold for all

$i \in \mathcal{D}$. And, by definition, the random variable $z^t_{m+1,j} = \sum_{i=1}^{m} z^t_{i,j}$ is a sum of independent Bernoulli trials. Hence, $z^t_{m+1,j}$ has a binomial distribution. That is,

$$Pr(z^t_{m+1,j} = c) = \binom{m}{c} \alpha^c (1-\alpha)^{m-c}$$

.

Let $B_\alpha(m, \lambda)$ be the (upper) tail of the binomial distribution from λm to m. That is,

$$B_\alpha(m, \lambda) = \sum_{c=\lambda m}^{m} \binom{m}{c} \alpha^c (1-\alpha)^{m-c}$$

where λm is an integer and $\alpha < \lambda < 1$. When $\lambda = (1+\epsilon)\alpha$, by definition, the probability of $z^t_{m+1,j}$ is at least $b + x^t_{m+1,j} = (1+\epsilon)\mu$ is

$$B := Pr(z^t_{m+1,j} \geq (1+\epsilon)\mu) = B_\alpha(m, (1+\epsilon)\alpha).$$

The goal is to show that B is at least $e^{-\mu\epsilon^2 l}$, where $l > 0$ is a constant independent of μ and ϵ. This would imply that $f(\mu, \epsilon\mu) \geq e^{-\mu\epsilon^2 l}$. Hence, setting k to be any constant larger than l would prove the proposition.

To show lower bounds on the tail distribution, we use the following lemma.

Lemma 3. Ahle (2017). When $\lambda \geq 0.5$,

$$B_\alpha(m, \lambda) \geq \frac{1}{\sqrt{2m}} e^{-mH(\lambda;\alpha)}$$

where $H(\lambda; \alpha) = \lambda \log \frac{\lambda}{\alpha} + (1-\lambda) \log \frac{1-\lambda}{1-\alpha}$.

Applying this lemma for $m = \mu/\alpha$ and $\lambda = (1+\epsilon)\alpha$ implies:

$$\begin{aligned}
B &\geq \frac{1}{\sqrt{2\mu/\alpha}} e^{-\frac{\mu}{\alpha}H((1+\epsilon)\alpha;\alpha)} \\
&= \frac{1}{\sqrt{2\mu/\alpha}} e^{-\frac{\mu}{\alpha}[(1+\epsilon)\alpha\log(1+\epsilon)+(1-(1+\epsilon)\alpha)\log\frac{1-(1+\epsilon)\alpha}{1-\alpha}]} \\
&= \frac{1}{\sqrt{2\mu/\alpha}} e^{-\mu[(1+\epsilon)\log(1+\epsilon)+\frac{1-(1+\epsilon)\alpha}{\alpha}\log(1-\alpha\epsilon/(1-\alpha))]} \\
&\geq \frac{1}{\sqrt{2\mu/\alpha}} e^{-\mu[(1+\epsilon)\epsilon+\frac{1-(1+\epsilon)\alpha}{1-\alpha}\epsilon)]} \qquad (2.1) \\
&= \frac{1}{\sqrt{2\mu/\alpha}} e^{-\mu\epsilon^2(1+\frac{1}{\epsilon}+\frac{1-(1+\epsilon)\alpha}{(1-\alpha)\epsilon})} \qquad (2.2)
\end{aligned}$$

where (2.1) holds since $\log(1+\epsilon) < \epsilon$ and $\log(1-\alpha\epsilon/(1-\alpha)) < -\frac{\alpha\epsilon}{1-\alpha}$ for all $\epsilon \in (0,1)$.

The proof is complete when we observe that the right-hand side of (2.2) is larger than $e^{-\mu\epsilon^2 l}$ for any $l \geq 1 + 2/\underline{\epsilon}$ and sufficiently large μ.[22]

■

Proof of Proposition 1

In this section, we present the complete proof of Proposition 1. The proof is an adaptation of the procedure of Cox (1987).[23]

Proof. We present a constructive proof of Proposition 1 using following algorithm. The rounding algorithm takes a fair share table as input and generates a (random) reservation table as output. To make the algorithm easier to understand, after each step we demonstrate the algorithm on an example depicted in Figure 2.10.

Rounding Algorithm

Step 1: Given a fair share table X, we construct an extended table V by adding

[22] The proof is symmetric for the lower tail since $Pr(z^t_{m+1,j} \leq (1-\epsilon)\mu) = B_{1-\alpha}(m,(1-\epsilon)\alpha)$

[23] An alternative proof utilizes network flow approach, very similar to the one in proof Theorem 1. However, for its simplicity and ease of use by hand, we show a modified version of the procedure of Cox (1987).

an extra row to table X. The last row of V is generated by taking 1 - fraction part of the column totals of table X.

In our example, shown in Figure 2.10, table V is equivalent to table X except the last row. Adding this extra row makes the column totals integers.

Figure 2.10: STEP 1 OF PROCEDURE

$X =$

0.5	0.5	1	2
0.25	0.25	0.5	1
0.75	0.75	1.5	3
1.5	1.5	3	6

(a) FAIR SHARE TABLE

$V =$

0.5	0.5	1	2
0.25	0.25	0.5	1
0.75	0.75	1.5	3
0.5	0.5	0	1
2	2	3	7

(b) EXTENDED TABLE

We focus on the internal entries of table V. The procedure involves iterative adjustment of the fractions in table V until all fractions have been replaced by integers.

Step 2: If table V contains no fractions, then skip to Step 8.

Step 3: Choose any fraction v_{ij} in table V. At (i, j) begin an alternating row-column (or column-row) path of fractions. A cycle will be formed (all edges fractions).

Figure 2.11: STEP 3 OF PROCEDURE

$V=$

0.5	0.5	1	2
0.25	0.25	0.5	1
0.75	0.75	1.5	3
0.5	0.5	0	1
2	2	3	7

In our example, shown in Figure 2.11, the cycle of fractions is $(i_1, j_1) \rightarrow (i_1, j_2) \rightarrow (i_2, j_2) \rightarrow (i_2, j_3) \rightarrow (i_3, j_3) \rightarrow (i_3, j_2) \rightarrow (i_4, j_2) \rightarrow (i_4, j_1) \rightarrow (i_1, j_1)$.

Step 4: Modify the cycle. First, raise the odd edges and reduce the even edges at the same rate until at least one edge reaches an integer value. The resulting table then gives rise to a table V_1.

In our example, the odd edges rise by 0.5 and even edges reduce by 0.5 ($d_+ = 0.5$). The resulting table V_1 is shown in Figure 2.12.

Step 5: Next, readjust the edges in the cycle in the reverse direction, raising even edges and reducing odd edges in an analogous manner, which gives rises to another table V_2.

In our example, the even edges rise by 0.25 and odd edges reduce by 0.25 ($d_- = 0.25$). The resulting table V_2 is shown in Figure 2.12.

Step 6: Select either V_1 or V_2 with probabilities $p_1 = \frac{d_-}{d_- + d_+}$ and $\frac{d_+}{d_- + d_+}$, respectively.

Figure 2.12: STEP 6 OF PROCEDURE

$$V_1 = \begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0.25 & 0.75 & 0 & 1 \\ 0.75 & 0.25 & 2 & 3 \\ 0 & 1 & 0 & 1 \\ \hline 2 & 2 & 3 & 7 \end{array} \qquad V_2 = \begin{array}{ccc|c} 0.25 & 0.75 & 1 & 2 \\ 0.25 & 0 & 0.75 & 1 \\ 0.75 & 1 & 1.25 & 3 \\ 0.75 & 0.25 & 0 & 1 \\ \hline 2 & 2 & 3 & 7 \end{array}$$

In our example, table V is decomposed into table V_1 and table V_2 where $V = \frac{1}{3}V_1 + \frac{2}{3}V_2$. There are few fraction elements in both tables.

Step 7: Reiterate Step 6 until no fractional elements left.

Step 8: Delete the last row of the table and report it as the outcome of the algorithm.

The algorithm must end in finite steps (at most the number of fractions in share table V) and, at the end we must have an integer table.

Lemma 4. The outcome of the Rounding Algorithm stays within quota.

Proof. In Step 4 and 5, after each adjustment the row and column sums remains the same. Moreover, after adjustments every element v_{ij} in table V always remains less than or equal to $\lceil v_{ij} \rceil$ and greater than or equal to $\lfloor v_{ij} \rfloor$. Therefore, the outcome of the algorithm will stay within quota. ■

Lemma 5. The Rounding Algorithm satisfies the following property: For any iteration and for any entry of the table,

$$E(v_{ij}|V) = v_{ij}$$

Proof. Note that in Step 4, v_{ij} raises by d_+ and in Step 5, it reduces by d_-. In Step 6, the probabilities of raising and decreasing are assigned as $\frac{d_-}{d_-+d_+}$ and $\frac{d_+}{d_-+d_+}$. Therefore, the expected adjustment will be $d_+\frac{d_-}{d_-+d_+} + d_-\frac{d_+}{d_-+d_+} = 0$. ■

In words, Lemma 5 proves that entries of the fair share table X are rounded up or down so that ex-ante positive and negative biases balance to yield zero bias.

Lemma 4 and Lemma 5 prove Proposition 1. ■

Proof of Proposition 2 and Proposition 3

Since Proposition 2 is a special case of Proposition 3, we prove the latter. We prove the proposition by contradiction.

Proof. Suppose a deterministic solution R stays within university quota. We show an example of a problem of reservation in two dimensions that the solution R can not have a finite bias. That is, for any constant $b > 0$, there exist a Y^t and an internal entry y^t such that $|\text{bias}(R(y^t))| > b$.

Example 3. Consider a problem with three departments $d_1, d_2,$ and d_3, two categories c_1, c_2, the reservation scheme vector $\boldsymbol{\alpha} = [0.5, 0.5]$. The departments $d_1, d_2,$ and d_3 have $\mathbf{q}^1 = [0, 0, 1]$ positions in period-1 and $\mathbf{q}^2 = [1, 0, 0]$ positions in period-2.

Notice that staying within university quota is equivalent to reserving exactly k positions for c_1 and c_2 in every 2k cumulative sum of vacancies in the university, where $k = 1, 2, 3, \ldots$. In period-1, department d_3 can reserve positions to either categories. Without loss of generality, we assume that it reserves 1 position for c_1. In period-2, since there are 2 cumulative sum of vacancies in the university, there should be exactly 1 position reserved for c_1. Department d_1 should reserve 1 position for category c_2. The period-1 and period-2 reservation tables are shown in Figure 2.13.

Figure 2.13: PERIOD-1 AND PERIOD-2 RESERVATION TABLES

$$X^1 = \begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.5 & 0.5 & 1 \\ \hline 0.5 & 0.5 & 1 \end{array}$$

(a) PERIOD-1 FAIR SHARE TABLE

$$X^2 = \begin{array}{cc|c} 0.5 & 0.5 & 1 \\ 0 & 0 & 0 \\ 0.5 & 0.5 & 1 \\ \hline 1 & 1 & 2 \end{array}$$

(b) PERIOD-2 FAIR SHARE TABLE

$$R(Y^1) = \begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \\ \hline 1 & 0 & 1 \end{array}$$

(c) PERIOD-1 RESERVATION TABLE

$$R(Y^2) = \begin{array}{cc|c} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \\ \hline 1 & 1 & 2 \end{array}$$

(d) PERIOD-2 RESERVATION TABLE

If departments have $\mathbf{q}^3 = [0, 0, 1]$ positions in period-3, department d_3 can reserve its position to either categories. These two cases are show in Figure 2.15.

Figure 2.14: TWO CASES FOR PERIOD-3 RESERVATION TABLES

$$X^3 = \begin{array}{cc|c} 0.5 & 0.5 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 2 \\ \hline 1.5 & 1.5 & 3 \end{array}$$

(a) PERIOD-3 FAIR SHARE TABLE

$$R_1(Y^3) = \begin{array}{cc|c} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 2 & 0 & 2 \\ \hline 2 & 1 & 3 \end{array}$$

(b) PERIOD-3 RESERVATION TABLE

$$R_2(Y^3) = \begin{array}{cc|c} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 2 \\ \hline 1 & 2 & 3 \end{array}$$

(c) PERIOD-3 RESERVATION TABLE

Case 1: We assume that the solution is $R = R_1$. If the departments have $\mathbf{q}^4 = [1, 0, 0]$ positions in period-4, department d_1 should reserve 1 position for category c_2. Otherwise, the solution R would violate staying within university quota property. Period-4 fair share table and the period-4 reservation table are illustrated by X_1^4 and $R_1(X_1^4)$ in Figure 2.15.

Case 2: We assume that the solution is $R = R_2$. If the departments have $\mathbf{q}^4 = [0, 1, 0]$ positions in period-4, department d_2 should reserve 1 position for category c_1. Otherwise, the solution R would violate staying within university quota property. Period-4 fair share table and the period-4 reservation table are illustrated by X_2^4 and $R_2(X_2^4)$ in Figure 2.15.

If departments have $\mathbf{q}^5 = [0, 0, 1]$ positions in period-3, department d_3 can reserve its position to either categories. These two cases are show in Figure 2.16.

Figure 2.15: TWO CASES FOR PERIOD-4 RESERVATION TABLES

$$X_1^4 = \begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 1 & 1 & 2 \\ \hline 2 & 2 & 4 \end{array}$$

(a) CASE 1: PERIOD-4 FAIR SHARE TABLE

$$R_1(X_1^4) = \begin{array}{cc|c} 0 & 2 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 2 \\ \hline 2 & 2 & 4 \end{array}$$

(b) CASE 1: PERIOD-4 RESERVATION TABLE

$$X_2^4 = \begin{array}{cc|c} 0.5 & 0.5 & 1 \\ 0.5 & 0.5 & 1 \\ 1 & 1 & 2 \\ \hline 2 & 2 & 4 \end{array}$$

(c) CASE 2: PERIOD-5 FAIR SHARE TABLE

$$R_2(X_2^4) = \begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \\ \hline 2 & 2 & 4 \end{array}$$

(d) CASE 2: PERIOD-4 RESERVATION TABLE

Figure 2.16: TWO CASES FOR PERIOD-5 RESERVATION TABLES

$$X_1^5 = \begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 1.5 & 1.5 & 3 \\ \hline 2.5 & 2.5 & 5 \end{array}$$

(a) PERIOD-5 FAIR SHARE TABLE

$$R_{1.1}(Y_1^5) = \begin{array}{cc|c} 0 & 2 & 2 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \\ \hline 3 & 2 & 5 \end{array}$$

(b) PERIOD-5 RESERVATION TABLE

$$R_{1.2}(Y_1^5) = \begin{array}{cc|c} 0 & 2 & 2 \\ 0 & 0 & 0 \\ 2 & 1 & 3 \\ \hline 2 & 3 & 5 \end{array}$$

(c) PERIOD-5 RESERVATION TABLE

$$X_2^5 = \begin{array}{cc|c} 0.5 & 0.5 & 1 \\ 0.5 & 0.5 & 1 \\ 1.5 & 1.5 & 3 \\ \hline 2.5 & 2.5 & 5 \end{array}$$

(d) PERIOD-5 FAIR SHARE TABLE

$$R_{2.1}(Y_2^5) = \begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \\ \hline 3 & 2 & 5 \end{array}$$

(e) PERIOD-5 RESERVATION TABLE

$$R_{2.2}(Y_2^5) = \begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \\ \hline 2 & 3 & 5 \end{array}$$

(f) PERIOD-5 RESERVATION TABLE

In Example 3 for each case, period-5 reservation for category c_1 in department d_1 is 0 and period-5 reservation for category c_2 in department d_2 is 0. We can extend these example for more periods analogously. The idea is following. In each period, the university has only one position. Department d_3 has always one position in odd periods and in the following period either department d_1 or department d_2 has one position according to these following cases.

- Case I: If department d_3 reserves 1 position to category c_1, department d_1 has one position in the next period.

- Case II: If department d_3 reserves 1 position to category c_2, department d_2 has one position in the next period.

In case I, department d_1 should reserve 1 position for category c_2, otherwise, solution would violate staying university quota property. In case II, department d_2 should reserve 1 position for category c_1, otherwise, solution would violate staying university quota property. Example 3 shows that if a solution stays within university quota, departments can grow in size without giving a seat to one category, i.e., the solution violates finite bias.[24]

This proves the proposition. ■

[24] An example for any number of categories and departments can be constructed in a similar way. Example 3 is constructed so that it not only illustrates the failure, but it also demonstrates any solution can fail to have finite bias in all categories.

2.9 Tables and Figures

Figure 2.17: 200-POINT ROSTER PRESCRIBED BY THE GOVERNMENT OF INDIA

FOR DIRECT RECRUITMENT

Model Roster of Reservation with reference to posts for Direct recruitment on All India Basis by Open Competition

Sl. No. of Post	Share of Entitlement				Category for which the posts should be earmarked
	SC @15%	ST @7.5%	OBC @27%	EWS @10%	
1	0.15	0.08	0.27	0.10	UR
2	0.30	0.15	0.54	0.20	UR
3	0.45	0.23	0.81	0.30	UR
4	0.60	0.30	1.08	0.40	OBC-1
5	0.75	0.38	1.35	0.50	UR
6	0.90	0.45	1.62	0.60	UR
7	1.05	0.53	1.89	0.70	SC-1
8	1.20	0.60	2.16	0.80	OBC-2
9	1.35	0.68	2.43	0.90	UR
10	1.50	0.75	2.70	1.00	**EWS-1**
11	1.65	0.83	2.97	1.10	UR
12	1.80	0.90	3.24	1.20	OBC-3
13	1.95	0.98	3.51	1.30	UR
14	2.10	1.05	3.78	1.40	ST-1
15	2.25	1.13	4.05	1.50	SC-2
16	2.40	1.20	4.32	1.60	OBC-4
17	2.55	1.28	4.59	1.70	UR
18	2.70	1.35	4.86	1.80	UR
19	2.85	1.43	5.13	1.90	OBC-5
20	3.00	1.50	5.40	2.00	SC-3
21	3.15	1.58	5.67	2.10	**EWS-2**
22	3.30	1.65	5.94	2.20	UR
23	3.45	1.73	6.21	2.30	OBC-6
24	3.60	1.80	6.48	2.40	UR
25	3.75	1.88	6.75	2.50	UR
26	3.90	1.95	7.02	2.60	OBC-7
27	4.05	2.03	7.29	2.70	SC-4
28	4.20	2.10	7.56	2.80	ST-2
29	4.35	2.18	7.83	2.90	UR
30	4.50	2.25	8.10	3.00	OBC-8
31	4.65	2.33	8.37	3.10	**EWS-3**

Source: https://dopt.gov.in/sites/default/files/ewsf28fT.PDF

Figure 2.18: 13-POINT ROSTER PRESCRIBED BY THE GOVERNMENT OF INDIA

FOR DIRECT RECRUITMENT

Roster for Direct Recruitment otherwise than through Open Competition for cadre strength upto 13 posts

Cadre Strength	Initial Recruit-ment	Replacement No.												
		1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th	11th	12th	13th
1	UR	UR	UR	OBC	UR	UR	SC	OBC	UR	**EWS**	UR	OBC	SC	ST
2	UR	UR	OBC	UR	UR	SC	OBC	UR	**EWS**	UR	OBC	SC	ST	
3	UR	OBC	UR	UR	SC	OBC	UR	**EWS**	UR	OBC	SC	ST		
4	OBC	UR	UR	SC	OBC	UR	**EWS**	UR	OBC	SC	ST			
5	UR	UR	SC	OBC	UR	**EWS**	UR	OBC	SC	ST				
6	UR	SC	OBC	UR	**EWS**	UR	OBC	SC	ST					
7	SC	OBC	UR	**EWS**	UR	OBC	SC	ST						
8	OBC	UR	**EWS**	UR	OBC	SC	ST							
9	UR	**EWS**	UR	OBC	SC	ST								
10	**EWS**	UR	OBC	SC	ST									
11	UR	OBC	SC	ST										
12	OBC	SC	ST											
13	SC	ST												

Source: https://dopt.gov.in/sites/default/files/ewsf28fT.PDF

Chapter 3

Ordinally Efficient Testing Policies to Identify COVID-19 Infection Rate

3.1 Introduction

How many air passengers that arrived in the United States yesterday were infected with COVID-19? A credible answer to this simple question is not available. The reason is there is no country-wide testing mandate for entry into the United States. Airport authorities face the same logistical and financial issues that countries, counties and campuses have been facing since the beginning of the pandemic. Testing all individuals is just not feasible.

A common policy response to the lack of testing infrastructure has been to test only those persons that are suspected of having COVID-19.[1] However, given the possibility of asymptotic infections and gaps in contact tracing systems, it is impossible to arrive at a reliable estimate of the number of infected persons by only testing suspected persons. As such, most expert commentators from around the world have advised against following such selective testing policies.[2]

[1] These includes persons that either display COVID-19 symptoms, or have come into contact with a COVID-19 infected person. See https://ourworldindata.org/grapher/covid-19-testing-policy for an overview of testing policies around the world.

[2] See Padula (2020) and the following opinion articles for the arguments:

A straightforward alternative to estimate how many infected people there are is *random testing*. Yet most countries have not adopted it. This is because in the face of testing constraints, random testing would not test many suspected persons who (ideally) should be tested, while testing many others that are not suspected of carrying the virus. Since random testing does not prioritize testing of suspected persons it might seem inefficient to policymakers.

In this note we propose a middle course to generate reliable estimates of the number of persons infected while prioritizing testing of suspected persons. By formalizing testing as a resource allocation problem and using tools from microeconomic theory, we analyze the design of various testing policies. We provide a novel comparison of various testing policies in terms of ordinal efficiency. Our efficiency comparisons, combined with the objective to generate credible estimates of the numbers of persons infected, enables a better informed choice of testing policies.

We focus on three testing policies: *selective testing*, *random testing* and *mixed testing*. Under selective testing, the highest possible number of suspected persons are tested. Under random testing, both suspected and non-suspected persons have the same chances of getting tested. Mixed testing lies somewhere between selective and random testing.

To facilitate efficiency comparisons across the three policies, we describe a model in which the policymaker is endowed with a preference of testing suspected persons, and not testing the non-suspected ones. Such preferences are in line with the current practice of testing only suspected persons in many countries. We show that these preferences induce a partial ordering over testing policies, namely the (first order) stochastic dominance relation. The Pareto (partial) ordering induced by the stochastic dominance relations enables efficiency comparisons of the three testing policies (in Proposition 5 and 6). We show that, in terms of efficiency, selective testing fares better than both mixed and random testing. Mixed testing fares better than random testing.

https://www.nytimes.com/2020/04/24/opinion/coronavirus-testing.html;
https://www.wsj.com/articles/its-dangerous-to-test-only-the-sick-11584288494?mod.

The choice between the three policies becomes apparent on analyzing their econometric properties. Though selective testing is the best in terms of efficiency, there are instances where reliable estimates of the numbers of persons infected cannot be generated under selective testing. In those instances, mixed testing is the next best alternative (in terms of efficiency). We provide a convenient estimator to conclude our analysis, noting that the three testing policies cannot be compared in terms of the precision of estimates generated under them.

Our work is related to that of many notable economists. Like Manski and Molinari (2021), we are interested in credible estimates of COVID-19 infection rate. However, instead of making clever use of the available data, we focus on designing testing policies that would automatically facilitate such estimation. Like Pathak et al. (2021), Kasy and Teytelboym (2020) and Ely et al. (2021), we are also interested in allocation of testing kits. However, the objective is different. Pathak et al. (2021) provides a framework for pandemic rationing of scarce medical resources, whereas Kasy and Teytelboym (2020) and Ely et al. (2021) formalize the trade-offs involved in testing to derive optimal testing policies. We, instead, focus on generating reliable estimates of infection rates while prioritizing testing of suspected persons.

In the following section we formalize the COVID-19 testing problem and define the various testing policies. In Section 3.3 and 3.4, we outline their efficiency and econometric properties. Section 3.5 is a conclusion. Proofs are relegated to appendix 3.6.

3.2 Testing Policies

In this section, we define a testing problem, a testing policy and the various testing policies that we will analyze further in the following sections.

3.2.1 The Problem

Definition 6. A **COVID-19 testing problem** is a triple $\Gamma = (N, s, q)$ where N represents a finite set of n persons, s is the number of suspected persons, and q is the testing capacity.

We are interested in the problem where testing *all* persons is not feasible, that is, $n > q$, and not *all* persons are suspected , that is, $n > s$.

Let $T \coloneqq \{t_1, t_0\}$ be the set of treatments associated with a problem Γ, where subjects receiving t_1 get tested while those receiving t_0 do not get tested. Let q_t denote the capacity of treatment t, i.e, the maximum number of subjects that can receive treatment t. By definition of the problem $q_{t_1} = q$. To minimize the number of subjects that do not get tested we let $q_{t_0} = n - q$.

3.2.2 Testing Policies

Definition 7. A **testing policy** for problem Γ is a matrix (p_{it}), with $p_{it} \geq 0$ for all i and t, $\sum_{t \in T} p_{it} = 1$ for all i, and $\sum_{i \in N} p_{it} \leq q_t$ for all t; p_{it} specifies the probability that person i is assigned treatment t.

We need additional notation to define the testing policies. Let s_i be a binary variable that identifies each subject $i \in N$ as either suspected ($s_i = 1$) or non-suspected ($s_i = 0$).

The current objective of testing policies, in many countries, is to test the maximum number of suspected persons possible. This policy, that we call selective testing, can be defined as follows.

Definition 8. **Selective testing** is a testing policy that assigns suspected persons the maximum possible probability of testing. Probability of testing under such policy can be written as $p^S_{it_1} = s_i * \min\{{}^{q}/_{s}, 1\} + (1 - s_i) * \max\{0, \frac{q-s}{n-s}\}$ for all $i \in N$.[3]

Selective testing resembles current practice in that it allocates the maximum possible probability of testing to suspected persons. However, unlike the current practice, selective

[3] To understand why, see table 3.1 in appendix 3.7 that expands upon this formula.

testing allocates positive probability of testing to unsuspected persons whenever possible.

As is the case in many countries, if the number of suspected persons exceeds the testing capacity (that is, $s > q$), it is impossible to generate reliable estimates of COVID-19 infection rate under selective testing.[4] The first alternative that comes to mind in such case is random testing.

Definition 9. **Random testing** is a testing policy that assigns each person an equal probability of testing, that is, $p^R_{it_1} = {}^q/_n$ for all $i \in N$.

Random testing allows the experimenter to estimate the infection rate, however, it leaves a significant proportion of suspected persons, that are believed to be infected, untested. Hence we propose another alternative that lies in-between selective and random testing.

For $\epsilon \in (0, \min_t p^R_{it})$, define treatment assignment probabilities $p^M_{it}(\epsilon)$ as

$$p^M_{it}(\epsilon) \equiv \delta p^R_{it} + (1-\delta)p^S_{it},$$

where $\delta \equiv \inf\{\delta' \in [0,1) \mid \delta' p^R_{it_1} + (1-\delta')p^S_{it_1} \in [\epsilon, 1] \text{ for all } i\}$.

Definition 10. **Mixed testing** is a testing policy that assigns person i probability $p^M_{it_1}(\epsilon)$ of testing, where $\epsilon \in (0, \min_t p^R_{it})$.

Comparing the testing probabilities of the three policies discussed above is straightforward. For suspected persons, $p^R_{it_1} < p^M_{it_1}(\epsilon) \leq p^S_{it_1}$, whereas, for non-suspected persons, $p^R_{it_1} > p^M_{it_1}(\epsilon) \geq p^S_{it_1}$. Hence mixed testing is a middle ground between selective and random testing. In the following two sections we compare these three policies in terms of their efficiency and econometric properties.

[4]This is the case in countries facing severe shortage of testing kits and other testing related infrastructure. For instance, India's current policy is to test only suspected persons and even then it exhausts its daily testing capacity. This implies that India must be receiving more suspected cases than their capacity.

3.3 Ordinal Efficiency

In this section we extend a well-known efficiency concept to our setting. This concept will allow for comparison between the three testing policies that are under scrutiny.

We assume that the policymaker prefers testing suspected subjects and not testing the non-suspected ones.[5] That is, for each subject i the experimenter is endowed with a strict preference $\succ_i$ over T such that

$$\succ_i = \begin{cases} t_1 \succ_i t_0 & \text{if } s_i = 1 \\ t_0 \succ_i t_1 & \text{if } s_i = 0 \end{cases}.$$

Such a preference follows naturally from the premise that the testing capacity is limited, and that the suspected subjects are likely to be infected, while non-suspected subjects are not. For random assignment mechanisms based solely on ordinal preferences, *ordinal efficiency* is an appropriate notion of efficiency.[6] We now extend the concept of ordinal efficiency to our setting.

Preference order $\succ_i$ on T induces a partial ordering of the set of testing policies that is known in the economics literature as the stochastic dominance relation. A testing policy (p_{it}) *stochastically dominates* another testing policy (p'_{it}) if

$$\sum_{t \succ_i t'} p_{it} \geq \sum_{t \succ_i t'} p'_{it}, \quad \text{for all } i \in N,\ t' \in T,$$

with strict inequality for some i, t. For $\lambda \in [0, 0.5]$, a testing policy is λ-*efficient* if it is not stochastically dominated by any other testing policy with $p_{it} \in [\lambda, 1]$.

[5] With such preferences, the treatment assignment probabilities under selective testing can be viewed as the outcome of the random priority mechanism (Abdulkadiroğlu and Sönmez (1998)), or the probabilistic serial mechanism (Bogomolnaia and Moulin (2001)), adapted to our setting. Since the number of treatments is two, both the mechanisms yield the same assignment probabilities. Our proposal shares some of its spirit with Narita (2021)'s experiment-as-market idea.

[6] Since the number of treatments is two, ordinal efficiency coincides with other two well-known notions of efficiency – *ex-ante efficiency* and *ex-post efficiency*. See Lemma 2 in Bogomolnaia and Moulin (2001).

Before efficiency comparisons, first consider the efficiency properties of the policies:

Proposition 5. For any problem Γ, the following are true:

(a) Selective testing (p_{it}^{S}) is 0-efficient.

(b) Mixed testing $(p_{it}^{M}(\epsilon))$ is ϵ-efficient, where $\epsilon \in (0, \min_t p_{it}^{R})$.

(c) Random testing (p_{it}^{R}) is $\bar{\epsilon}$-efficient, where $\bar{\epsilon} = \min_t p_{it}^{R}$.

We now define another property to compare the three testing policies. A testing policy (p_{it}) is *more efficient than* (p'_{it}) if (p_{it}) stochastically dominates (p'_{it}) for some problem Γ, and (p_{it}) is not stochastically dominated by (p'_{it}) for any problem Γ.

The three testing policies can be compared as follows:

Proposition 6. (a) Selective testing is more efficient than mixed and random testing.

(b) Mixed testing is more efficient than random testing.

3.4 Econometric Properties

Knowledge of the quality of infection rate estimates is important to guide the choice of testing policies. The purpose of this section is to detail conditions under which reliable estimates of the infection rate can be generated under various testing policies.

For each person i, for $i \in N$, let w_i indicate whether the person was tested, with $w_i = 1$ if person i is tested and $w_i = 0$ otherwise. Let y_i denote whether person i is infected, with $y_i = 1$ if person i is infected and $y_i = 0$ otherwise.

We are interested in estimating infection rate, that is, the population average of variable y, $\beta = E(y)$, given a random sample of size n of the triple $(w_i, s_i, w_i y_i)$. The sample is such that w_i and s_i are observed for all persons in the sample, but y_i is only observed if $w_i = 1$, that is, if person i is tested.

Notice that each of the three testing policies (selective, mixed and random) is stratified on observable variable s. Therefore, each policy's allocation of testing is independent of potential outcomes conditional on whether the person is suspected or not, which is observable to the experimenter:[7]

$$w \perp\!\!\!\perp y \,|\, s.$$

Conditional independence combined with the additional condition that the testing probability is positive, that is, for $s \in \{0, 1\}$,

$$p(s) = E[w|s] = Pr(w = 1|s) > 0,$$

allows the experimenter to generate reliable estimates of the infection rate.[8] Though random testing policy always satisfies this additional condition, in terms of efficiency, random testing is the least efficient of the three policies (see Proposition 6). The most efficient policy, selective testing, satisfies this additional condition only when $s < q < n$. For $q \leq s < n$, there is no better alternative than mixed testing which always satisfies this additional condition.

To conclude this subsection we provide a consistent and statistically efficient estimator of infection rate. Let n_{ws} denote the number of observations with $w_i = w$ and $s_i = s$, for $w, s \in \{0, 1\}$. In this setting the nonparametric estimate of testing probability is simply the proportion of persons for a given value of s. For $s_i = s$ the proportion of persons tested is $n_{1s}/(n_{0s} + n_{1s})$. Thus the estimated testing probability is

$$\hat{p}(s) = \begin{cases} n_{11}/(n_{01} + n_{11}) & \text{if } s = 1, \\ n_{10}/(n_{00} + n_{10}) & \text{if } s = 0. \end{cases}$$

Under conditional independence and the condition that $p(s) > 0$ for $s \in \{0, 1\}$, among

[7] That is, conditional on observables, the y_i are Missing At Random (MAR; Rubin (1976)).
[8] See section 4 of Hirano et al. (2000).

many other estimators, a consistent and statistically efficient estimator of β (discussed in Horvitz and Thompson (1952) and Hirano et al. (2003)) is

$$\hat{\beta} = \frac{1}{n}\sum_{i=1}^{n}\frac{w_i y_i}{\hat{p}(s_i)}.$$

The normalized variance of this estimator is

$$V(\hat{\beta}) = E[V(y|s)/p(s)] + V(E[y|s]).$$

Notice the testing probability term in the denominator. As noted immediately before section 3.3, among selective, random and mixed testing, the testing probability for suspected persons is the lowest under random testing, and the highest under selective testing. For non-suspected persons, the opposite is true. Therefore, the three testing policies cannot be compared in terms of the precision of estimates generated under them.

However, if only suspected persons are likely to be infected, mixed testing leads to more precise estimates than random testing. To see why, notice that since y is binary, $V(y|s) = Pr(y = 1|s)Pr(y = 0|s)$ and $E(y|s) = Pr(y = 1|s)$. If only suspected persons are likely to be infected, that is, $Pr(y = 1|s = 0)$ is close to zero and $Pr(y = 1|s = 1)$ is not, then $V(y|s = 0) < V(y|s = 1)$. Thus, allocating more tests to suspected persons ($s = 1$) lowers the variance of the estimator.[9]

3.5 Recommended Testing Policies

Every COVID-19 testing center (in cities, at airports, on campuses) faces a different COVID-19 testing problem $\Gamma = (N, s, q)$, every day. N represents the center's target population, s is the number of suspected persons expected to arrive at the center,[10] and q is the center's testing capacity for the day.

[9] Thanks to an anonymous referee at *Management Science* for pointing this out.

[10] For instance, the number of suspected persons on the previous day is a good estimate of s.

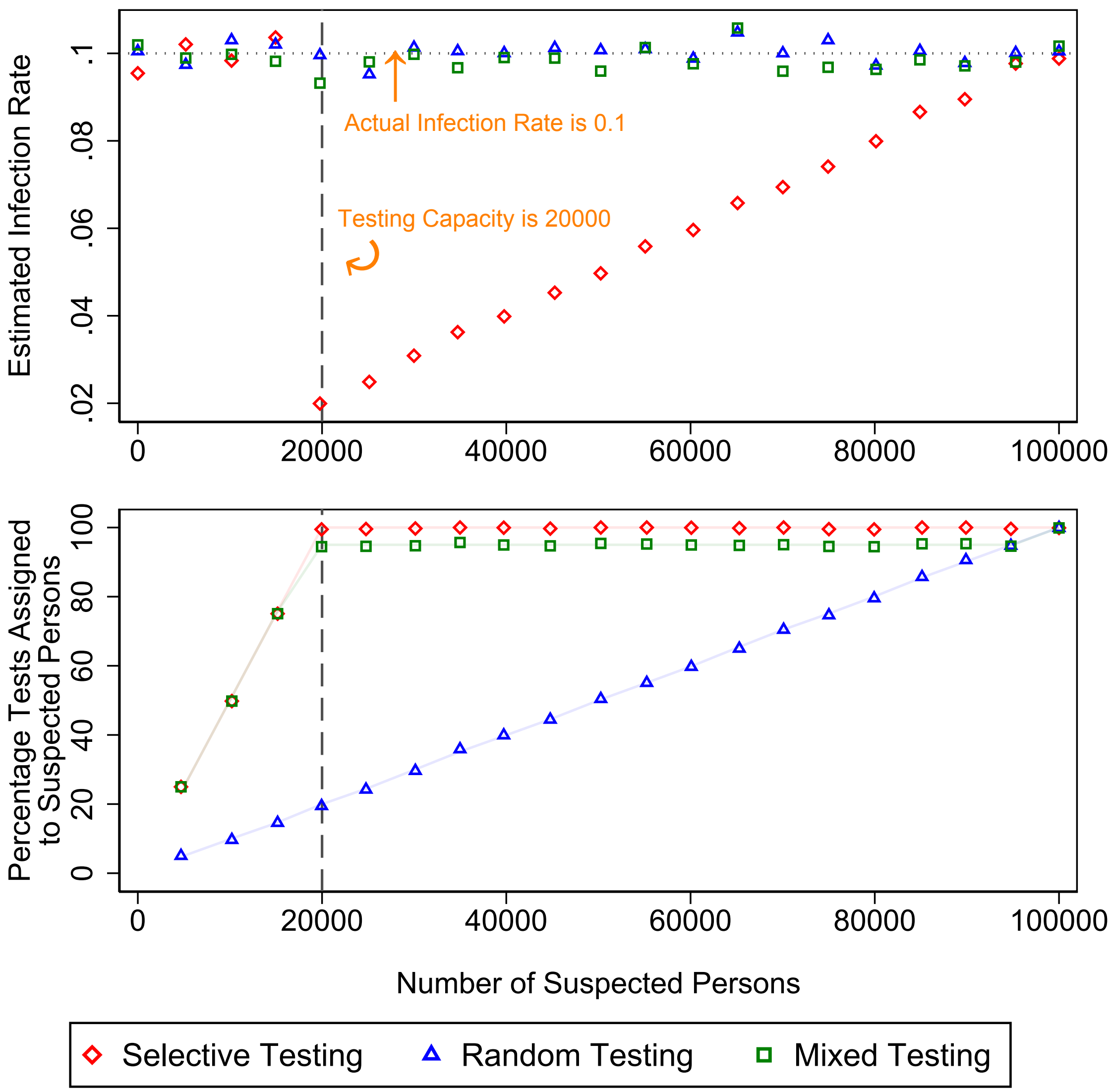
An Illustration. Consider a hypothetical testing center with a target population of size $n = 100000$ and testing capacity $q = 20000$. For the sake of illustration, assume that (1) the true infection rate in the center's target population is 0.1, that is, 10% of population is coronavirus infected; (2) both suspected and non-suspected persons are equally likely to be infected; and (3) the test is completely accurate. Under this hypothetical setup, we compare the three testing policies of random testing, selective testing, and mixed testing. In particular, we consider how estimates of infection rate and the share of tests that are assigned to suspected persons evolve with the number of suspected persons in the population under each testing policy (see figure 3.1).[11]

In the upper panel of figure 3.1, we plot the estimated infection rates. Notice the failure of selective testing in identifying the infection rate. For sufficiently high numbers of suspected persons ($q \leq s$), the estimated infection rate under selective testing is off by as much as 80% that is, the estimate suggests that only 2% of the population is infected while the actual infection rate is 10%. As discussed in section 3.4, for such testing problems, mixed testing is better suited as it allows us to generate reliable estimates of infection rate. Indeed the estimates of infection rate generated under mixed testing (plotted for $\epsilon = 0.01$) are almost identical to those generated under random testing, both being satisfactorily close to the actual infection rate of 0.1.

In the lower panel of figure 3.1, we plot the percentage of total tests that are assigned to suspected persons. Under selective testing, the highest possible number of suspected persons are tested. In contrast, under random testing, that assigns an equal chance of being tested to both suspected and non-suspected persons, and therefore does not prioritize testing of suspected persons, a much lower percentage of total tests are assigned to suspected persons. Mixed testing (plotted for $\epsilon = 0.01$) on the other hand, prioritizes testing of suspected persons, and therefore appears fairly close to selective testing in terms of percentage of tests assigned to suspected persons.

[11] STATA do file for this simulation is available on request.

Figure 3.1: AN ILLUSTRATION



Recommendation. Although random testing generates reliable estimates of infection rates, it is shown to be the least efficient (in Proposition 6). The most efficient policy, selective testing, is recommended for centers with reasonably high testing capacities, that is centers with $s < q < n$. For centers facing severe shortage of testing infrastructure, that is centers with $q \leq s < n$, we recommend mixed testing.[12] For centers with $q \leq s < n$, mixed testing is that most efficient policy that generates reliable estimates of infection rates while prioritizing testing of suspected persons.

3.6 Mathematical Appendix

Proposition 5: Supporting Lemmas and Proof

Lemma 6. For any testing policy (p_{it}), $\sum_{i \in N} p_{it} = q_t$ for all $t \in T$.

Proof. Proof of Lemma 6 Since $\sum_{i \in N} \sum_{t \in T} p_{it} = n$, $\sum_{t \in T} q_t = n$, and $\sum_{i \in N} p_{it} \leq q_t$ for all $t \in T$, it follows that $\sum_{i \in N} p_{it} = q_t$ for all $t \in T$. ■

Lemma 7. For problem Γ and $\lambda \in [0, 0.5]$, testing policy (p_{it}) with, either:

(i) $p_{it_0} = \lambda$ for all $i \in N$ with $s_i = 1$; or,

(ii) $p_{it_1} = \lambda$ for all $i \in N$ with $s_i = 0$,

is λ-efficient.

Proof. Proof of Lemma 7 Suppose that $p_{it_0} = \lambda$ for all $i \in N$ with $s_i = 1$, and (p_{it}) is not λ-efficient. Then there must exist a testing policy (p'_{it}) with $p'_{it} \in [\lambda, 1]$ such that:

(i) $p'_{it_1} \geq p_{it_1}$ for all $i \in N$ with $s_i = 1$; and,

(ii) $p'_{it_0} \geq p_{it_0}$ for all $i \in N$ with $s_i = 0$,

[12] However, in some countries stochastic testing is not an acceptable alternative. In such places, with both mixed and random testing infeasible, selective testing is the only practical testing policy.

with strict inequality for some $i \in N$. Notice that the strict inequality cannot hold for any $i \in N$ with $s_i = 1$, because then $1 - p'_{it_1} = p'_{it_0} < p_{it_0} = \lambda$, which is a contradiction. Suppose that the strict inequality holds for some $j \in N$ with $s_j = 0$, that is, $p'_{jt_0} > p_{jt_0}$. Then $1 - p'_{jt_0} = p'_{jt_1} < p_{jt_1}$. Combined with Lemma 6, this implies that there must exist some $k \in N$ with $p'_{kt_1} > p_{kt_1}$. But, this cannot happen because:

(i) if $s_k = 1$, $p'_{kt_1} > p_{kt_1}$ implies $1 - p'_{it_1} = p'_{it_0} < p_{it_0} = \lambda$, which is a contradiction; and,

(ii) if $s_k = 0$, $p'_{kt_1} > p_{kt_1}$ implies $1 - p'_{it_1} = p'_{it_0} < p_{it_0}$, which is again a contradiction.

By the same kind of reasoning, a testing policy with $p_{it_1} = \lambda$ for all $i \in N$ with $s_i = 0$, is λ-efficient. ∎

Lemma 8. A λ-efficient testing policy is also λ'-efficient for $\lambda' \in [\lambda, 0.5]$.

Proof. Proof of Lemma 8 By definition. ∎

Lemma 9. Under mixed testing, either:

(i) $(p^M_{it}(\epsilon)) = (p^S_{it})$; or,

(ii) $p^M_{it_1} = \epsilon$ for all $i \in N$ with $s_i = 0$.

Proof. Proof of Lemma 9 For $\epsilon \leq \min_i p^S_{it_1}$, δ (in the definition of $p^M_{it}(\epsilon)$) would be equal to 0, and therefore $(p^M_{it}(\epsilon)) = (p^S_{it})$. Whereas, for $\epsilon > \min_i p^S_{it_1}$, since $p^S_{it_1}$ is lower for i with $s_i = 0$ (than for i with $s_i = 1$), definition of $p^M_{it}(\epsilon)$ implies that $p^M_{it_1} = \epsilon$ for all $i \in N$ with $s_i = 0$. ∎

Proof. Proof of Proposition 5 Under selective testing, for testing problems with $q \leq s$, we have $p^S_{it_1} = 0$ for all $i \in N$ with $s_i = 0$. Whereas, for testing problems with $q > s$, we have $p^S_{it_0} = 0$ for all $i \in N$ with $s_i = 1$. Therefore, part (a) follows from Lemma 7.

Under mixed testing, due to Lemma 9, there are two cases to consider: (1) $p^M_{it_1} = \epsilon$ for all $i \in N$ with $s_i = 0$; and, (2) $(p^M_{it}(\epsilon)) = (p^S_{it})$. In case (1), $(p^M_{it}(\epsilon))$ is ϵ-efficient by Lemma 7. In case (2), $(p^M_{it}(\epsilon))$ is 0-efficient, and therefore ϵ-efficient by Lemma 8.

Under random testing, either $p^R_{it_1} = \min_t p^R_{it}$, or, $p^R_{it_0} = \min_t p^R_{it}$. Therefore, part (c) follows from Lemma 7. ■

Proposition 6: Supporting Lemmas and Proof

Lemma 10. For any problem Γ, the following are true:

(a) Selective testing (p^S_{it}) stochastically dominates random testing (p^R_{it}).

(b) Selective testing (p^S_{it}) stochastically dominates mixed testing $(p^M_{it}(\epsilon))$ for $\epsilon > \min_i p^S_{it_1}$.

(c) Mixed testing $(p^M_{it}(\epsilon))$ stochastically dominates random testing (p^R_{it}).

Proof. Proof of Lemma 10 Since:

(i) $p^S_{it_1} = \min\{\frac{q}{s}, 1\} > \frac{q}{n} = p^R_{it_1}$ for all $i \in N$ with $s_i = 1$; and,

(ii) $p^S_{it_0} = \min\{\frac{n-q}{n-s}, 1\} > \frac{n-q}{n} = p^R_{it_0}$ for all $i \in N$ with $s_i = 0$,

the result in part (a) follows.

Recall that, for $\epsilon \in (0, \min_t p^R_{it})$,

$$p^M_{it}(\epsilon) \equiv \delta p^R_{it} + (1-\delta) p^S_{it},$$

where $\delta \equiv \inf\{\delta' \in [0,1) \mid \delta' p^R_{it_1} + (1-\delta') p^S_{it_1} \in [\epsilon, 1] \text{ for all } i\}$. For $\epsilon > \min_i p^S_{it_1}$ it must be the true that $\delta > 0$. Combined with the two inequalities established in proof of part (a) this implies that:

(i) $p^S_{it_1} > p^M_{it_1}(\epsilon)$ for all $i \in N$ with $s_i = 1$; and,

(ii) $p^S_{it_0} > p^M_{it_0}(\epsilon)$ for all $i \in N$ with $s_i = 0$.

Therefore, the result in part (b) follows.

The two inequalities established in proof of part (a), and that, $\delta \in [0,1)$, imply that:

(i) $p^M_{it_1}(\epsilon) > p^R_{it_1}$ for all $i \in N$ with $s_i = 1$; and,

(ii) $p^M_{it_0}(\epsilon) > p^R_{it_0}$ for all $i \in N$ with $s_i = 0$.

Therefore, part (c) follows. ■

Proof. Proof of Proposition 6 Since selective testing is 0-efficient, it is not stochastically dominated by any other testing policy (by Lemma 8). However, selective testing stochastically dominates mixed testing when $\epsilon > \min_i p^S_{it_1}$, and random testing in all testing problems (by Lemma 10). Therefore, selective testing is more efficient than mixed and random testing.

Mixed testing is ϵ-efficient, where $\epsilon \in (0, \min_t p^R_{it})$. It follows from Lemma 8 that mixed testing is not dominated by random testing. However, mixed testing stochastically dominates random testing in all testing problems (by Lemma 10). Therefore, mixed testing is more efficient than random testing. ■

3.7 Tables and Figures

Table 3.1: PROBABILITIES UNDER SELECTIVE AND RANDOM TESTING

	$q \leq s$		$q > s$	
	p^S_{it}	p^R_{it}	p^S_{it}	p^R_{it}
Probability of Testing				
$s_i = 1$	$\frac{q}{s}$	$\frac{q}{n}$	1	$\frac{q}{n}$
$s_i = 0$	0	$\frac{q}{n}$	$\frac{q-s}{n-s}$	$\frac{q}{n}$
Probability of Not Testing				
$s_i = 1$	$\frac{s-q}{s}$	$\frac{n-q}{n}$	0	$\frac{n-q}{n}$
$s_i = 0$	1	$\frac{n-q}{n}$	$\frac{n-q}{n-s}$	$\frac{n-q}{n}$

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