Connectivity of the space of pointed hyperbolic surfaces

Sangsan Warakkagun

A dissertation

submitted to the Faculty of

the department of mathematics

in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

Boston College
Morrisey College of Arts and Sciences
Graduate School

July 2021

Connectivity of the space of pointed hyperbolic surfaces

Sangsan Warakkagun

Advisor: Ian Biringer, Ph.D.

Abstract

We consider the space $\mathcal{H}^2_\bullet$ of all complete hyperbolic surfaces without boundary with a basepoint equipped with the pointed Gromov-Hausdorff topology. Continuous paths within $\mathcal{H}^2_\bullet$ arising from certain deformations on a hyperbolic surface and concrete geometric constructions are studied. These include changing some Fenchel-Nielsen parameters of a subsurface, pinching a simple closed geodesic to a cusp, and inserting an infinite strip along a proper bi-infinite geodesic. We then use these paths to show that $\mathcal{H}^2_\bullet$ is path-connected and that it is locally weakly connected at points whose underlying surfaces are either the hyperbolic plane or hyperbolic surfaces of the first kind.

Contents

Acknowledgments

Six years after I started graduate school at Boston College, I successfully defended this dissertation in July 2021. Throughout this yearslong endeavor, many have given me help. First, I want to thank my thesis advisor, Ian Biringer, who introduces me to the mathematics in this work, which I find absolutely fascinating. I am grateful for his guidance and patience. His valuable comments on earlier drafts of this thesis have made it much more accurate and readable. Thanks to Martin Bridgeman and Rob Meyerhoff, who were on my thesis committee and provided useful feedback on the work. Thanks also to my friend and colleague, Tommaso Cremaschi, for many helpful conversations and motivational pushes.

My thanks extend to the BC math department. The kindness of the professors and the graduate students is unmatched and has made it a good place to be.

In the past six years, my life has also progressed—sometimes in conjunction with math and sometimes outside of math. I thank my friends, many of whom were fellow Ph.D. students. They are (in alphabetical order) Ryan Contreras, Tom Cremaschi, Ross Goluboff, Earth Jaikla, Risa Kuroda, Can Naowarojna, Fa Ngamnithiporn, Nat Pombubpa, Matt Sarmiento, and Champ Somboonpanyakul. Thank you all for having given me nourishment—intellectual, social, and oftentimes nutritional!

My thanks go to Matt Lord for his constant thoughtfulness and unwavering support. He has kept my morale high since I have known him.

My parents have always cheered me on, even when they do not understand much about what I write here. Their sacrifice has made it possible for me to do this work. The dissertation is dedicated to them.

Chapter 1

Introduction

For a fixed $g \geq 2$, let $\mathcal{M}_g$ be the set of closed hyperbolic surfaces of genus g. Early motivating questions in surface geometry, like Riemann's moduli problem, concerned its structure.[1] Much is now known about this so-called *moduli space* $\mathcal{M}_g$ and its universal cover $\mathcal{T}_g$, called the *Teichmüller space*. Most notably, in 1960, Bers [10] gave an analytic parametrization for $\mathcal{T}_g$, showing that it is an open bounded domain in $\mathbb{C}^{3g-3}$. As for $\mathcal{M}_g$, it is also non-compact with one end [21]. A more comprehensive list of topological properties of $\mathcal{M}_g$ and $\mathcal{T}_g$ can be found in [21, Part 2].

There are many possible 'generalizations' of moduli and Teichmüller spaces. Using a decomposition of a surface by a maximal collection of disjoint simple closed curves [4], we can endow any non-compact surface with a complete hyperbolic metric [7]. Thus, one generalization is to fix a hyperbolic surface of infinite type and study its various Teichmüller spaces, as in [2, 7]. In this thesis, however, we will be interested in larger generality. We ask:

[1]Riemann's moduli problem asks for an analytic parametrization of $\mathcal{M}_g$, viewed as the set of biholomorphism classes of compact Riemann surfaces of fixed genus g. We have a dictionary between Riemann surfaces and hyperbolic surfaces since the conformal class of metrics defined by the complex structure of a Riemann surface contains a unique complete hyperbolic metric so long as $g \geq 2$ by the uniformization theorem. See [1] for historical development of moduli and Teichmüller spaces and [19] for a quick introduction of $\mathcal{M}_g$.

what is the structure of the space of all hyperbolic surfaces of all possible diffeomorphism types?

A connection between the geometry of a compact surface and that of a non-compact one already becomes apparent when we compactify $\mathcal{M}_g$ with noded surfaces, in which some simple closed curves on a hyperbolic surface are pinched into punctures. Indeed, as a simple closed geodesic shrinks, the collar lemma [24] asserts that it has an increasingly longer annular neighborhood, and a cusp is introduced in the 'limit' as the length goes to zero. To capture such a transition, we need a suitable framework that allows for comparisons between surfaces of different diffeomorphism types.

Gromov introduces a metrizable topology on the space of pointed proper metric spaces, with respect to which two such spaces are close if large closed balls around their respective basepoints are almost isometric [23, 25]. In the setting of Riemannian manifolds, it is the language used in Cheeger-Gromov compactness theorem and Perelman's proof of the Geometrization conjecture, see [34]. Following Canary-Epstein-Green [16], who expand the idea of Thurston [37], we specialize this so-called pointed Gromov-Hausdorff topology (see Definition 2.2.1) to the space of (isometry classes of) pointed hyperbolic surfaces:

$$\mathcal{H}^2_\bullet = \{(X, p) : X \text{ hyperbolic surface, } p \in X\}/\text{basepoint-preserving isometry.}$$

All underlying surfaces in $\mathcal{H}^2_\bullet$ are assumed throughout to be connected, oriented, and metrically complete without boundary.

Gromov's turns out to be a natural and useful notion for hyperbolic geometry, too. This $\mathcal{H}^2_\bullet$ and its analogues in higher dimensions are key ingredients used to determine the volume spectrum of hyperbolic manifolds. They also have close connections to the Chabauty topology on the space of closed subgroups of the isometry group of the hyperbolic n-spaces, as we will explain more in details later in Section 1.2.

1.1 Summary of results

In this thesis, I study both global and local path-connectivity of $\mathcal{H}^2_\bullet$. The main results, Theorems 4.1.2 and 4.2.2, are numbered as they appear in subsequent chapters.

Theorem 4.1.2. *The space $\mathcal{H}^2_\bullet$ is path-connected.*

It is a natural next step to ask whether this property holds locally:

Question 1. *Is $\mathcal{H}^2_\bullet$ is locally path-connected? Equivalently, is $\mathcal{H}^2_\bullet$ is weakly locally path-connected at every point?*

The second main theorem will give a partial answer to this question. Finding a complete answer will perhaps constitute a future project. Recall that a topological space $\mathcal{X}$ is *weakly locally path-connected* at a point $x \in \mathcal{X}$ provided that if $V \subset \mathcal{X}$ is an open set containing x there exists an open neighborhood $U \subset V$ of x such that any two points in U are on the same path component of V. Following common (albeit nondescriptive) terminology found in literature, we say that a complete hyperbolic surface without boundary is *of the first kind* if its convex core is the entire surface. Otherwise, it is *of the second kind.*

Theorem 4.2.2. *The space $\mathcal{H}^2_\bullet$ is weakly locally path-connected at*

- $(\mathbb{H}^2, z_0)$ *for any choice of* $z_0 \in \mathbb{H}^2$
- (X, p) *where* X *is of the first kind.*

Our proofs of the two main theorems make use of continuous paths arising from certain modifications of surfaces. To first understand the geometric structure on a hyperbolic surface, we break down its convex core into topologically simpler pieces, appealing to the general and geometric version of a pants decomposition, as formulated in [7], which we cite below. By a *geodesic pair of pants*, we refer to a complete hyperbolic surface with geodesic boundary diffeomorphic to a sphere with a combination of three disks or points removed.

Theorem 2.1.2. *([7]) For any complete hyperbolic surface X with geodesic boundary that is not a sphere with three punctures or an infinite cylinder, there exists a collection $\mathcal{P}$ of mutually disjoint, pairwise non-homotopic simple closed geodesics in the convex core $\mathsf{CC}(X)$ such that each component of $\mathsf{CC}(X) - \mathcal{P}$ is an open geodesic pair of pants (possibly with cusps). Moreover, the closure of each component of $X - \mathsf{CC}(X)$ is either a half-infinite cylinder bounded by a simple closed geodesic or a half-plane bounded by an infinite simple geodesic.*

As a result, the geometry of X is completely determined by its convex core. In particular, the structure of the individual geodesic pairs of pants and how they are glued together specify the isometry class of a surface. In Sections 3.2 and 3.3, we show that, with respect to a fixed collection of pants curves and transverse seams, modifying the associated Fenchel-Nielsen length and twist parameters yields a continuous path in $\mathcal{H}^2_\bullet$, all while keeping track of the basepoint. We owe much to a family of 'canonical' markings on a finite-type surface as defined by Buser in Chapter 6 of [14].

One other class of continuous paths comes from growing an infinite strip along a geodesic. Let $(X,p) \in \mathcal{H}^2_\bullet$ and a proper infinite geodesic α in X with $a \in \alpha$ be given. For any $s > 0$, an *s-strip* is a region in $\mathbb{H}^2$ bounded by two hyperparallel geodesics whose common perpendicular τ_s has length s. We construct a new hyperbolic surface X_s by first cutting X along α and gluing a s-strip along the boundary components of $X - \alpha$ so that a is identified with the endpoints of τ_s and the pasting scheme respects the unit speed parametrization of α where $\alpha(0) = a$. We show that this construction is continuous with respect to the topology of $\mathcal{H}^2_\bullet$.

Proposition 3.4.3. *Given (X, p), α, and a as above, the strip insertion map*

$$\begin{aligned} \mathsf{Strip} : \mathbb{R}_+ &\to \mathcal{H}^2_\bullet \\ s &\mapsto (X_s, p) \end{aligned}$$

is continuous, where $p \in X_s$ is chosen to be the image of $p \in X$ under the natural isometric embedding $X - \alpha \to X_s$.

Growing a strip is a crucial construction in the proofs of our main theorems, as we apply it to create a path that increases the injectivity radius at the basepoint (in Theorem 4.1.2) as well as to 'blow up' cusps (as in lemmas 3.4.4 and 3.4.5).

1.2 Related results

1.2.1 Chabauty topology

Let G be a locally compact metrizable group. The *Chabauty topology* on the space of closed subgroups of G, denoted by $\mathrm{Sub}(G)$, has the following convergence criteria: a sequence of subgroups $\{H_n\}$ converges to $H \leq G$ if and only if

(a) if $h_n \in H_n$ is a sequence such that $h_n \to h$ in G, then $h \in H$; and

(b) for each $h \in H$, there exists a sequence $h_n \in H_n$ such that $h_n \to h$ in G.

Closely related to $\mathcal{H}^2_\bullet$ is the Chabauty topology on the subspace of discrete torsion-free subgroups of $\mathrm{PSL}_2(\mathbb{R})$. We fix once and for all a basepoint $z_0 \in \mathbb{H}^2$ and an orthonormal basis $\mathbf{v}_0$ of the tangent space $T_{z_0}(\mathbb{H}^2)$. One can provide additional local data to each $(X, p) \in \mathcal{H}^2_\bullet$ by selecting an orthonormal basis $\mathbf{v}$ for the tangent space $T_p(X)$. But to the now *framed hyperbolic surface* $(X, p, \mathbf{v})$, one can associate a unique discrete torsion-free subgroup Γ of $\mathrm{PSL}_2(\mathbb{R})$ such that $\mathbb{H}^2/\Gamma \cong X$, $\pi(z_0) = x$, and $d\pi(\mathbf{v}_0) = \mathbf{v}$, where $\pi : \mathbb{H}^2 \to \mathbb{H}^2/\Gamma$ is the natural projection. It is well-known that this correspondence is a homeomorphism between the space of framed hyperbolic surfaces, endowed with the pointed Gromov-Hausdorff topology (with extra requirements for tangent vectors, see Definition 2.2.5), and the Chabauty space of discrete torsion-free subgroups of $\mathrm{PSL}_2(\mathbb{R})$. The proof of this fact can be found in [16].

In [5], Baik and Clavier explicitly parametrize and classify the Chabauty limits of one-generator subgroups of $\mathrm{PSL}_2(\mathbb{R})$, showing that its Chabauty closure is simply connected. For lack of a global coordinate system, it is not possible to do so for the entire $\mathrm{Sub}(\mathrm{PSL}_2(\mathbb{R}))$, at least from the point of view of subgroups. With the dictionary between discrete-torsion free subgroups and framed hyperbolic surfaces, we may interpret the main theorems in this work as connectivity results for (a quotient of) a subset of $\mathrm{Sub}(\mathrm{PSL}_2(\mathbb{R}))$.

While the Chabauty space of closed subgroups of a Lie group is an interesting mathematical object in itself, only few other examples of Chabauty spaces have been completely described including when $G = \mathbb{R}^n$ [32, 27] and when G is the Heisenberg group [11]. The reader may consult an unpublished note by de la Harpe [18], which summarizes these results as well as provides a list of references.

1.2.2 In higher dimensions

It is natural and possible to extend the framework of $\mathcal{H}^2_\bullet$ to the space of pointed n-dimensional hyperbolic manifolds $\mathcal{H}^n_\bullet$ for a higher n. Let us first contrast the topology of $\mathcal{H}^n_\bullet$ with our results. Unlike in dimension 2, when $n \geq 3$, Mostow's rigidity theorem states that the isometry class of a finite-volume hyperbolic n-manifolds is determined by its fundamental group. This implies that the space $\mathcal{H}^n_\bullet$ is no longer connected, as $\{(X, p) : p \in X\}$ forms a component if X is compact.

It is in dimension 3 where the pointed Gromov-Hausdorff convergence (also called the *pointed geometric convergence* in this setting) has yielded its richest applications.

It appears in Minsky's and Brock-Canary-Minsky's program to build models for ends of hyperbolic 3-manifolds, settling Thurston's Ending Lamination Conjecture [29, 12]. By works of Thurston and Jørgensen, using continuity of the volume function on the subspace $\mathcal{F}^3_\bullet \subset \mathcal{H}^3_\bullet$ of pointed finite-volume hyperbolic 3-manifolds, it is shown that the volume spectrum $\mathrm{vol}(\mathcal{F}^3_\bullet) \subset \mathbb{R}$ is closed with a positive minimum and that non-trivial limits arise only from hyperbolic Dehn surgery constructions. The interested reader may refer to Chapter E of [9] for a detailed exposition.

1.3 Organization

The subsequent chapters of this dissertation are organized as follows. Chapter 2 covers basic backgrounds on surface topology and hyperbolic geometry. We also formulate a precise definition of the pointed Gromov-Hausdorff topology on $\mathcal{H}^2_\bullet$. In Chapter 3, we construct continuous paths in $\mathcal{H}^2_\bullet$ arising from certain deformations of surfaces. Finally, we give proofs of our main results concerning global and local path-connectivity about $\mathcal{H}^2_\bullet$ in Chapter 4.

Chapter 2

Preliminaries

In this chapter, we lay out relevant backgrounds as well as recall well-known facts about hyperbolic surfaces and the pointed Gromov-Hausdorff topology.

2.1 Surface topology and geometry

A topological surface, possibly with boundary, is a two-dimensional manifold with coordinate charts mapping into $\mathbb{R} \times \mathbb{R}_{\geq 0}$. A surface is of *finite type* if its fundamental group is finitely generated. Otherwise, it is of *infinite type*, in which case the fundamental group is necessarily free with countably many generators. The homeomorphism classes of finite-type surfaces are determined by their orientation, their genera, as well as the number of punctures and boundary components. For a complete classification of surfaces, including those of infinite type, we refer the reader to [33]. In this work, all surfaces are assumed to be connected and oriented.

In what follows, we discuss a decomposition of a surface into simpler pieces in both topological and geometric settings.

2.1.1 Topological pants decomposition

A *pair of pants* is a surface homeomorphic to a sphere with three disjoint open disks or points removed so that its boundary is a union of up to three circles. Here, we make a distinction between boundary components that are circles and punctures. Pairs of pants prove to be fundamental building blocks of surfaces, as gluing them together along their boundary components results in a surface of higher complexity. As a reverse of this combinatorial construction, it is well-known that a maximal collection of pairwise disjoint, essential simple closed curves in a closed surface (except for the sphere, and the torus) cuts the surface into a union of pairs of pants. This is indeed true in more generality, and we record the following result, due to Álvarez and Rodríguez, for reference.

Theorem 2.1.1 ([4]). *Let S be a surface whose fundamental group is nonabelian and whose boundary is a (possibly empty) disjoint union of simple closed curves. Then S has a locally finite collection $\mathcal{P} = \{c_i\}_{i \in I}$ of pairwise disjoint, homotopically distinct and nontrivial, simple closed curves such that the closure of each component of $S - \bigcup c_i$ is either a pair of pants or a cylinder.*

Here, by a *cylinder*, we mean a subsurface homeomorphic to $S^1 \times [0, 1)$. A collection $\mathcal{P}$ as in Theorem 2.1.1 is called a *topological pants decomposition*. We note that there can be at most countably many curves in $\mathcal{P}$ since the surface is second-countable.

2.1.2 Hyperbolic surfaces

To introduce a suitable metric on a surface included in Theorem 2.1.1, we will use hyperbolic geometry, which we now give a brief review as necessary. The reader is advised to consult [14] and [9] for a detailed exposition of hyperbolic surfaces and hyperbolic geometry in general, respectively.

For $z \in \mathbb{C}$, denote its real part by $\operatorname{Re}(z)$, its imaginary part by $\operatorname{Im}(z)$, and its modulus by $|z|$. We consider two models of the 2-dimensional hyperbolic space:

1. the upper-half plane model $\mathbb{H}^2 = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$, equipped with the metric $ds_{\mathbb{H}} = \frac{|dz|}{\operatorname{Im}(z)}$, and
2. the Poincaré disk model $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with the metric $ds_{\mathbb{D}} = \frac{2\,|dz|}{1-|z|^2}$.

The two models are indeed isometric via the map $\mathbb{H}^2 \to \mathbb{D}$ given by $z \mapsto \frac{z-i}{z+i}$. By the uniformization theorem, $\mathbb{H}^2$ (or $\mathbb{D}$) is the unique simply connected Riemann surface with constant curvature -1, up to isometry. We use "the hyperbolic plane" as a blanket term to refer to either model, though we will mainly use $\mathbb{H}^2$ for computations and $\mathbb{D}$ for illustrations. The ideal boundary of the hyperbolic plane is identified as $\partial_\infty \mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$ for $\mathbb{H}^2$ and as $\partial_\infty \mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$ for $\mathbb{D}$.

In $\mathbb{H}^2$, the geodesics are vertical straight lines and semicircles with centers on $\mathbb{R}$, while the geodesics in $\mathbb{D}$ are diameters and circular arcs perpendicular to $\partial\mathbb{D}$. It follows that the hyperbolic plane is uniquely geodesic—that is, any two points can be joined via a unique geodesic path. The orientation-preserving isometries of $\mathbb{H}^2$ are identified as the elements of $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm I\}$ by fractional linear transformations, and they take geodesics to geodesics.

By a *hyperbolic metric*, we mean a Riemannian metric of constant negative curvature -1. A surface equipped with a hyperbolic metric is called a *hyperbolic surface*, and it is *complete* if it is complete as a metric space. We say that a topological surface is *hyperbolizable* if it can be equipped with a complete hyperbolic metric, possibly with geodesic boundary. It is well-known that any topological surface S of finite type with $\chi(S) < 0$ is hyperbolizable. That any topological non-compact surface is hyperbolizable follows from Theorem 2.1.1 above and [7, Theorem 1].

A complete hyperbolic surface X can be obtained as the quotient of a convex subset $C(X) \subset \mathbb{H}^2$ by the action of the orientation-preserving isometries in some discrete (*Fuchsian*) torsion-free subgroup Γ of $\mathrm{PSL}(2,\mathbb{R})$, where $\Gamma \cong \pi_1(X)$. This gives rise to a Γ-equivariant universal covering map $\pi : C(X) \to X$, and X is locally modeled by $\mathbb{H}^2$. Since Γ acts by isometry, geodesics in X are precisely the images under π of geodesics in $C(X)$.

The action of Γ extends to $\partial_\infty C(X)$ and defines the limit set $\Lambda(\Gamma) \subset \partial_\infty C(X)$. Let $\mathsf{CH}(\Lambda(\Gamma))$ be the convex hull of $\Lambda(\Gamma)$ in $\mathbb{H}^2$. The *convex core* of X, denoted by $\mathsf{CC}(X)$, is the quotient of $\mathsf{CH}(\Lambda(\Gamma))$ by Γ. It is the smallest closed convex geodesic subsurface of X that is homotopy equivalent to X. By [7, Proposition 3.1], the boundary of the closure of $\mathsf{CC}(X)$ is a countable union of disjoint simple closed geodesics and simple infinite geodesics.

There is also another important notion of completeness. A hyperbolic surface is said to be *geodesically complete* if all geodesic rays, starting at any point and going off in any direction, extend indefinitely far. For hyperbolic surfaces, the property is equivalent to the induced metric being complete and the surface having no boundary. These are the hyperbolic surfaces that we will study in this thesis.

Theorem 3.4 of [7] shows that any (not necessarily complete) hyperbolic surface has a canonical geodesically complete extension obtained by attaching appropriate hyperbolic cylinders and half-planes to the simple closed geodesics and infinite geodesic components of $\partial\mathsf{CC}(X)$, respectively. The latter case can occur only as limits of simple closed geodesics. (Here, we say that a sequence of geodesics $\{\beta_n\}_{n=1}^{\infty}$ in X *converges* to β if their lifts in $\mathbb{H}^2$ converge with respect to the Hausdorff distance, see the definition in section 2.2.)

2.1.3 Geodesic pants decomposition

Before we discuss the hyperbolic geometry analogue of Theorem 2.1.1, we need to define three types of geometric building blocks for a hyperbolic surface:

(1) A *(generalized) geodesic pair of pants* is a hyperbolic surface with boundary diffeomorphic to a sphere with three disks or points removed such that each of its boundary components is a simple closed geodesic. The hyperbolic structure of a pair of pants is completely determined by the lengths of its boundary geodesics (called *cuffs*). A boundary curve of zero length is understood to be a puncture, and each of which has a corresponding cusp neighborhood. A *cusp* is a non-compact cylinder with one boundary component, and it is isometric to

$$\{z \in \mathbb{H}^2 : 0 \leq \mathrm{Re}(z) \leq 1, \mathrm{Im}(z) > h\} \big/ \langle z \mapsto z + 1 \rangle$$

for some $h > 0$. In this case, the boundary is a *horocycle* with length $1/h$.

(2) A *funnel* is a half-infinite hyperbolic cylinder with one simple closed geodesic boundary component. It is isometric to

$$\{z \in \mathbb{H}^2 : \mathrm{Re}(z) \geq 0\} \big/ \langle z \mapsto e^l z \rangle$$

for some $l > 0$. In this case, the boundary has length l.

(3) A *half-plane* is a region of $\mathbb{H}^2$ bounded by a simple infinite geodesic; it is isometric to

$$\{z \in \mathbb{H}^2 : \mathrm{Re}(z) \geq 0\}.$$

We are now ready to state the geometric counter part of the topological pants decomposition in Theorem 2.1.1, originally due to Álvarez and Rodríguez [4]. Following Basmajian and Šăríc's formulation in [7], we have:

Theorem 2.1.2 ([4, 7]). *Let X be a complete hyperbolic surface whose boundary is a (possibly empty) union of disjoint simple closed geodesics. Assume that X is not a sphere with three punctures or an infinite cylinder. Let $\{\alpha_j\}_{j \in J}$ be the set of boundary components of $\mathsf{CC}(X)$ which are open infinite geodesics (if there are any).*

Then, $\mathsf{CC}(X) - \bigcup \alpha_j$ *admits a topological pants decomposition* $\mathcal{P} = \{c_i\}_{i \in I}$ *and any such collection* $\mathcal{P}$ *satisfies the following properties:*

(1) every curve $c_i \in \mathcal{P}$ *is freely homotopic in* X *to a unique simple closed geodesic* γ_i *so that each component of* $\mathsf{CC}(X) - \bigcup \gamma_i$ *is a generalized geodesic pair of pants, and*

(2) the components of $X - \mathsf{CC}(X)$ *(if there are any) are funnels and half-planes.*

A collection of simple closed geodesics $\{\gamma_i\}$ as in Theorem 2.1.2 is called a *geodesic pants decomposition* of the surface X. Sometimes, we simply call it a pants decomposition, as the context should make it clear. With Theorem 2.1.2 in mind, we introduce the following terminology, which will be used later.

Definition 2.1.3. A geodesically complete hyperbolic surface $X = \mathbb{H}^2/\Gamma$ is said to be *of the first kind* if $X = \mathsf{CC}(X)$ (or equivalently $\Lambda(\Gamma) = \partial_\infty \mathbb{H}^2$). It is *of the second kind* otherwise.

On one hand, Theorem 2.1.2 implies that a complete hyperbolic surface of the first kind contains no funnels or half-planes, and it is obtained by gluing together generalized geodesic pair of pants along their boundary components in some way. This is Theorem 4.5 of [3], where the authors use the term *Nielsen-convex* to mean "complete with no funnels and half-planes". On the other hand, simply gluing together geodesic pairs of pants does not always yield a complete hyperbolic surface, see Example 2.1.6 below.

2.1.4 Fenchel-Nielsen coordinates

A hyperbolizable surface can admit many different hyperbolic metrics. To understand the underlying hyperbolic structure of a given surface, we describe the geometry of the pairs of pants comprising a fixed pants decomposition, along with their gluing patterns, in terms of the lengths and the so-called twists of the pants curves. These are the Fenchel-Nielsen coordinates, which we will now briefly recall.

Start with a hyperbolizable topological surface S that is not a pair of pants with three cusps. A *hyperbolic structure* on S is a diffeomorphism $f : S \to X$, where X is a complete hyperbolic surface with geodesic boundary (if it is non-empty). The pair (X, f), or simply X if the marking is unimportant for the context, is called a *marked hyperbolic surface* with the *marking* f.

By Theorem 2.1.1, there is a collection of simple closed curves, which we call *pants curves*, $\mathcal{P} = \{c_i\}_{i \in I}$ in S such that the closure of each component of $S - \bigcup c_i$ is a pair of pants or a cylinder. (The index set I is finite if S is of finite type and is countably infinite otherwise.) We assume that $\mathcal{P}$ includes all the boundary curves. We also fix a set of disjoint simple closed curves $\{b_j\}_{j \in J}$ called *seams* so that the intersection of any pair of pants in the decomposition determined by $\mathcal{P}$ with $\cup b_j$ is a union of disjoint arcs connecting each pair of its boundary components.

Let (X, f) be a marked hyperbolic surface. The X-*length parameter* of a curve $c \in \mathcal{P}$, denoted by $\ell_X(c)$, is the length of the unique geodesic representative of the homotopy class of $f(c)$ in X. Furthermore, if $c \in \mathcal{P}$ does not bound a cylinder in S, or if c is not homotopic to a component of ∂S, then we also associate to c the X-*twist parameter*, denoted by $\theta_c(X)$, which is defined as follows. First, after isotoping, we may assume that $f(c)$ is a geodesic. Orient it so that it is a common boundary component of two (not necessarily distinct) geodesic pairs of pants, say, Y_L to its left and Y_R to its right. For $i \in \{L, R\}$, choose a seam $f(b_i)$ that intersects $f(c)$ in Y_i. The geodesic arc $f(b_i) \cap Y_i$ joins two boundary components of Y_i: $f(c)$ and, say, γ_i. We let δ_i be the common perpendicular geodesic between these two components. If N_i and M_i are regular metric neighborhoods of $f(c)$ and γ_i in Y_i, respectively, we can isotope $f(b_i)$ to agree with δ_i outside of $N_i \cup M_i$, leaving the endpoints fixed. Set t_i

to be the signed displacement of the endpoints $f(b_i) \cap \partial N_i$. Finally, we define

$$\theta_X(c) = 2\pi \frac{t_L - t_R}{\ell_X(c)}.$$

This is well-defined regardless of the choice of seams we chose in Y_i, see [21, section 10.6.1]. We remark that the twist parameter is omitted for a closed geodesic bounding a funnel because a funnel is rotationally symmetric.

Definition 2.1.4. Fix a pair of pants decomposition $\mathcal{P} = \{c_i\}_{i\in I}$ and seams on a hyperbolizable surface S. The *Fenchel-Nielsen coordinates* (with respect to $\mathcal{P}$) of a complete hyperbolic surface X diffeomorphic to S is the collection

$$FN(X) = ((\ell_X(c_i), \theta_X(c_i)))_{i\in I}$$

where $\ell_X(c_i) \in (0,\infty)$, $\theta_X(c_i) \in (-\infty,\infty)$, and $\theta_X(c_i)$ is omitted if c_i is homotopic to ∂S or if c_i bounds a cylinder in S.

Fenchel-Nielsen coordinates enable us to understand the hyperbolic structure of a given surface, since they specify the geometry of each of the building blocks as well as how they are glued together.

Definition 2.1.5. Let S is a hyperbolizable surface of finite-type possibly with boundary. The *Teichmüller space* of S is the set of complete hyperbolic structures on S considered up to homotopy (or *marking equivalence*):

$$\mathcal{T}(S) = \{(X,f) : f \text{ a hyperbolic structure on } S\}/\sim$$

where $(X_1,f_1) \sim (X_2,f_2)$ if there is an isometry $I : X_1 \to X_2$ such that $f_2 \circ f_1^{-1}$ are homotopic to I.

If S is of finite-type, it is well-known that Fenchel-Nielsen coordinates parametrize and topologize $\mathcal{T}(S)$. That is,

$$FN : \mathcal{T}(S) \to (\mathbb{R}_+ \times \mathbb{R})^{\mathcal{P}}$$

$$(X, f) \to FN(X)$$

is a homeomorphism, with the convention that boundary curves are not assigned a twist parameter, see [21, Chapter 10]. We contrast this fact with the infinite-type case in which some Fenchel-Nielsen parameters may correspond to an incomplete surface, as the example below shows.

Example 2.1.6. Consider a hyperbolic surface X obtained by gluing geodesic pairs of pants $\{Y_n\}_{n=1}^{\infty}$ along a pair of boundaries of common length in a sequence, where unglued components are punctures. This is an example of a *tight flute surface*; see Figure 2.1 below. Let γ_n be the geodesic boundary component of Y_n incident to Y_{n+1}. The collection $\mathcal{P} = \{\gamma_n\}_{n=1}^{\infty}$ gives a pants decomposition of X. Let θ_n be the twist parameter of γ_n and let $d_n = d_X(\gamma_n, \gamma_{n+1})$.

Basmajian shows in [6, Theorem 4] that X is complete if $\sum d_n$ diverges (regardless of the twists). On the other hand, if both $\sum d_n$ and $\sum |\theta_n|$ converge, then γ_n limits to an open geodesic, and the geodesic completion of X contains a half-plane.

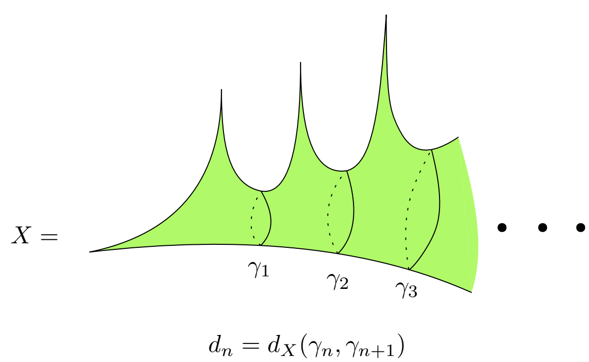

FIGURE 2.1: X is a tight flute surface with no twists in the gluing.

Detailed studies of various types of the Teichmüller space for a surface of infinite type include [3], [2], and [7], for example.

2.2 Spaces of hyperbolic surfaces

The main goal of this section is to define various spaces of hyperbolic surfaces and introduce appropriate topologies on them. Intuitively, a surface can be approximated by a sequence of larger and larger compact subsurfaces. We would also like to consider two hyperbolic surfaces near in these spaces if they appear "similar" on large neighborhoods of *their respective basepoints*.

2.2.1 The pointed Gromov-Hausdorff topology

We begin with a general theory on the space of metric spaces first developed by Edwards in [20] and later widely popularized by Gromov in [22]. First, we recall that the *Hausdorff distance* between two subsets A and B of a metric space X is

$$d_H(A, B) = \inf\{\epsilon > 0 : A \subset N_\epsilon(B) \text{ and } B \subset N_\epsilon(A)\}, \tag{2.1}$$

where $N_\epsilon(A) = \{x : d(x, A) < \epsilon\}$ denotes the ϵ-neighborhood of A. The Hausdorff distance measures "metric similarity" between two subsets, and actually defines a metric on the space of closed subsets of X.

Next, instead of working with subsets of one fixed metric space, we would like to compare two different metric spaces in a meaningful way that generalizes the Hausdorff distance. One method is to construct another metric space in which both spaces can be isometrically embedded, and then calculate the Hausdorff distance in the new space. This is a cumbersome and somewhat unnatural task, as it involves passing through a new metric. We will take an equivalent approach that directly relates the two metrics by finding points in one metric space that "correspond metrically" to points in the other and then measuring a distortion of distances. A precise formulation is Definition 2.2.1 below.

Given a metric space X and $p \in X$, a *pointed metric space* is the pair (X, p).

Definition 2.2.1. Two pointed metric spaces (X, x_0) and (Y, y_0) are said to be (ϵ, R)-*related* if there are compact subsets $X_1 \subset X$ and $Y_1 \subset Y$, where $B_X(x_0, R) \subset X_1$ and $B_Y(y_0, R) \subset Y_1$, together with a relation $\mathfrak{R} \subset X_1 \times Y_1$ satisfying the following conditions

1) $(x_0, y_0) \in \mathfrak{R}$;
2) for every $x \in X_1$, there is a point $y \in Y_1$ such that $(x, y) \in \mathfrak{R}$;
3) for every $y \in Y_1$, there is a point $x \in X_1$ such that $(x, y) \in \mathfrak{R}$;
4) the *distortion* of $\mathfrak{R}$

$$\operatorname{dis} \mathfrak{R} := \sup \{|d_Y(y_1, y_2) - d_X(x_1, x_2)| : (x_1, y_1), (x_2, y_2) \in \mathfrak{R}\}$$

is less than ϵ.

Such a relation $\mathfrak{R}$ is called an (ϵ, R)-*relation.* We also write $x\mathfrak{y}$ for $(x, y) \in \mathfrak{R}$.

We will use (ϵ, R)-relations to define a topology on the space of pointed proper metric spaces. A metric space is *proper* if all closed balls of finite radius are compact.

Definition 2.2.2. Let $\mathcal{M}_\bullet$ be the set of all (isometry classes of) pointed proper metric spaces. The *pointed Gromov-Hausdorff topology* on $\mathcal{M}_\bullet$ is generated by neighborhoods of the form

$$\mathcal{N}(X, p, \epsilon, R) = \{(Y, q) \in \mathcal{M}_\bullet : (X, p) \text{ and } (Y, q) \text{ are } (\epsilon, R)\text{-related}\}$$

for every $(X, p) \in \mathcal{M}_\bullet$, $\epsilon > 0$, and $R > 0$.

The space $\mathcal{M}_\bullet$ is a complete metric space, see [25] in which an explicit metric is given and the details for many more results about $\mathcal{M}_\bullet$ are worked out. For a discussion of the space of metric spaces, we recommend [13, Chapters 7 and 8].

2.2.2 (ϵ, R)-relations

While the definitions below generalize verbatim to higher dimensions, we shall restrict our attention to spaces of hyperbolic surfaces. The interested reader may refer to [16, Chapter I.3] for the definitions of spaces of n-dimensional hyperbolic manifolds and for proofs of any results recorded here.

In this section, all hyperbolic surfaces are oriented, connected, complete, and have empty boundary, unless otherwise specified.

We begin by attaching additional information to each surface. Let X an oriented hyperbolic surface. If $p \in X$ is a basepoint and $\mathbf{e}$ is a positively oriented orthonormal basis of the tangent space $T_p(X)$, then the pair (X, p) is called a *pointed hyperbolic surface*, and the triple $(X, p, \mathbf{e})$ is called a *framed hyperbolic surface*. We call the choice $\mathbf{e}$ a *baseframe* for (X, p).

Definition 2.2.3. The *space of (isometry classes of) pointed hyperbolic surfaces* is

$$\mathcal{H}^2_\bullet = \{(X, p) : (X, p) \text{ is a pointed hyperbolic surface}\} / \sim_{\mathrm{p}},$$

where $(X, p) \sim_{\mathrm{p}} (Y, q)$ if there is an isometry $f : X \to Y$ such that $f(p) = q$, endowed with the pointed Gromov-Hausdorff topology.

Since $\mathcal{H}^2_\bullet$ is a subspace of $\mathcal{M}_\bullet$, the space of pointed proper metric spaces, it is Hausdorff. We can then consider limits of sequences of pointed surfaces. In fact, it will often be more convenient to understand the topology of $\mathcal{H}^2_\bullet$ in terms of Gromov-Hausdorff convergence, which formalizes the notion of surface approximation by large compact subsurfaces that we mentioned in the introduction. The following proposition is immediate from the definition. By abuse of language, we will not distinguish a pointed surface from its pointed-isometry class.

Proposition 2.2.4. *A sequence of pointed hyperbolic surfaces* $\{(X_n, p_n)\}_{n=1}^{\infty}$ *converges to* $(X, p) \in \mathcal{H}^2_\bullet$ *if there exist sequences* $\epsilon_n \to 0$ *and* $R_n \to \infty$ *such that* (X_n, p_n) *and* (X, p) *are* (ϵ_n, R_n)*-related.*

Similarly, we let $\mathcal{H}^2_{\mathrm{f}}$ be the set of framed hyperbolic surfaces up to baseframe-preserving isometry:

$$\mathcal{H}^2_{\mathrm{f}} = \{(X, p, \mathbf{v}) : (X, p, \mathbf{v}) \text{ is a framed hyperbolic surface}\}/\sim_{\mathrm{f}},$$

where $(X, p, \mathbf{v}) \sim_{\mathrm{f}} (Y, q, \mathbf{w})$ if there is an isometry $f : X \to Y$ such that $f(p) = q$ and $df(\mathbf{v}) = \mathbf{w}$. Here, df denotes the differential of f. Note that an isometry between Riemannian manifolds is guaranteed to be smooth by a theorem of Myers-Steenrod, see [31, Chapter 5, Theorem 18].

To topologize $\mathcal{H}^2_{\mathrm{f}}$, we introduce a framed version of (ϵ, R)-relations that also controls the behavior of baseframes.

Definition 2.2.5. For $\epsilon > 0$ and $R > 0$, two framed surfaces $(X, p, \mathbf{e})$ and $(X', p', \mathbf{e}')$ in $\mathcal{H}^2_{\mathrm{f}}$ are *framed* (ϵ, R)*-related* if there is an (ϵ, R)-relation $\mathfrak{R}$ between (X, p) and (X', p') which satisfies the additional requirement:

5) if $\mathbf{v} = \sum r_i e_i$ and $\mathbf{v}' = \sum r_i e_i'$ are tangent vectors at p and at p' written as a combinations of vectors in $\mathbf{e}$ and $\mathbf{e}'$, respectively, and $\|\mathbf{v}\| < R$, then $(\exp \mathbf{v})\mathfrak{R}(\exp \mathbf{v}')$.

Definition 2.2.6. The *pointed Gromov-Hausdorff topology* on $\mathcal{H}^2_{\mathrm{f}}$ is generated by neighborhoods of the form

$$\mathfrak{N}(X, p, \mathbf{v}, \epsilon, R) = \left\{ (Y, q, \mathbf{w}) \in \mathcal{H}^2_{\mathrm{f}} : \begin{array}{c} (X, p, \mathbf{v}) \text{ and } (Y, q, \mathbf{w}) \\ \text{are framed } (\epsilon, R)\text{-related} \end{array} \right\} \tag{2.2}$$

where we vary $\epsilon, R > 0$ and $(X, p, \mathbf{v}) \in \mathcal{H}^2_{\mathrm{f}}$.

Similarly, $\mathcal{H}^2_{\mathrm{f}}$ is Hausdorff. Moreover, the natural projection $\mathcal{H}^2_{\mathrm{f}} \to \mathcal{H}^2_{\bullet}$ obtained from forgetting the baseframe is open and continuous, and is thus a quotient map.

2.2.3 Quasi-isometries

Taking advantage of the fact that hyperbolic surfaces have more structure than general metric spaces, we can require more regularity when defining the topologies of $\mathcal{H}^2_{\bullet}$ and $\mathcal{H}^2_{\mathrm{f}}$. What we present below, while seemingly much stronger, will turn out to be equivalent to the (ϵ, R)-relation definitions above in our settings. We follow [16] in conventions.

Definition 2.2.7. Let $(X, p, \mathbf{v})$ and $(Y, q, \mathbf{w})$ be framed hyperbolic surfaces. For any $K > 0$, a *framed* (K, r)*-quasi-isometry* (or a *framed* (K, r)*-approximate isometry*) between $(X, p, \mathbf{v})$ and $(Y, q, \mathbf{w})$ is a diffeomorphism between framed subsurfaces $(X_1, p, \mathbf{v}) \subset (X, p, \mathbf{v})$ and $(Y_1, q, \mathbf{w}) \subset (Y, q, \mathbf{w})$

$$f : (X_1, p) \to (Y_1, q)$$

such that

(1) $B_X(p, r) \subset (X_1, p)$ and $B_Y(q, r) \subset (Y_1, q)$;

(2) $f(p) = q$ and $df(\mathbf{v}) = \mathbf{w}$;

(3) f distorts the distance by less than a factor of K, i.e.,

$$\frac{1}{K} d(x_1, x_2) \leq d(f(x_1), f(x_2)) \leq K d(x_1, x_2) \quad \text{for all } x_1, x_2 \in X_1.$$

Analogously, we can define a (K, r)*-quasi-isometry* between two pointed hyperbolic surfaces in $\mathcal{H}^2_{\bullet}$, and a K*-quasi-isometry* between two hyperbolic surfaces. The last case coincides with the notion of a K-bilipschitz map.

As in Definition 2.2.1, we use quasi-isometries to topologize $\mathcal{H}^2_\bullet$ and $\mathcal{H}^2_{\mathrm{f}}$ via the open sets of the form

$$\mathfrak{N}_{q.i.}(X,p,\mathbf{v},K,r) = \left\{ (Y,q,\mathbf{w}) \in \mathcal{H}^2_{\mathrm{f}} : \begin{array}{l} \text{there is a framed } (K,r)\text{-quasi-isometry} \\ \text{betweeen } (X,p,\mathbf{v}) \text{ and } (Y,q,\mathbf{w}) \end{array} \right\} \tag{2.3}$$

as we vary $K > 1$ and $r > 0$ and $(X,p,\mathbf{v}) \in \mathcal{H}^2_{\mathrm{f}}$. (Baseframes are simply ignored for $\mathcal{H}^2_\bullet$.) We record the following fact for reference. It is Corollary I.3.2.11 in [16].

Proposition 2.2.8. *The pointed Gromov-Hausdorff topology on $\mathcal{H}^2_f$ defined using (ϵ, r)-relations is equivalent to the topology on $\mathcal{H}^2_f$ induced by framed (K,r)-approximate isometries. The same holds for the pointed version in $\mathcal{H}^2_\bullet$.*

The equivalence of the pointed (or framed) Gromov-Hausdorff topology defined by the rather 'weak' (ϵ, R)-relation to a strong version given by approximate isometries affords us flexibility in choosing any notion of convergence 'in between'.

Chapter 3

Tools: Continuous Paths

Unless specified otherwise, a hyperbolic surface is always assumed to be connected, oriented, and complete without boundary—that is, it is the underlying surface of an element in $\mathcal{H}^2_\bullet$. By abuse of language, we will not distinguish a pointed surface from its pointed isometry class.

To examine global and local path-connectivity of $\mathcal{H}^2_\bullet$, we construct continuous paths from the following deformations of a pointed hyperbolic surface:

1) moving a basepoint on a fixed hyperbolic surface;
2) changing the Fenchel-Nielsen length and twist parameters of a subsurface;
3) pinching a simple closed geodesic to a cusp; and
4) inserting a strip along an infinite geodesic.

We remark that while 1) and 2) preserve the topology of a base surface, 3) and 4) deal with deformations that alter the topology. In this chapter, we give precise definitions and prove continuity of the above procedures.

3.1 Moving a basepoint

The first lemma allows us to move a basepoint continuously on a fixed surface in $\mathcal{H}^2_\bullet$.

Lemma 3.1.1. *For any hyperbolic surface X, the basepoint map $X \to \mathcal{H}^2_\bullet$ defined by $p \mapsto (X, p)$ is continuous.*

Proof. Let $\{p_n\}_{n=1}^{\infty}$ be a sequence of points in X converging to $p \in X$. For each $n \in \mathbb{N}$, define a bijection $f_n : X \to X$ by swapping the points p and p_n and fixing all the other points. Then, the relation $\mathfrak{R}_n = \{(x, f_n(x)) : x \in X\}$ is a (ϵ_n, ∞)-relation between (X, p_n) and (X, p), where $\epsilon_n = d(x, x_n)$. Then, $\epsilon_n \to 0$ as $n \to \infty$. This proves that $(X, p_n) \to (X, p)$. ■

3.2 Modifying lengths and twists

In this section, we claim that we can create a continuous deformation of a pointed hyperbolic surface by changing a finite number of length and twist parameters in the Fenchel-Nielsen coordinates and moving the basepoint accordingly. This will be stated more precisely as Proposition 3.2.6.

Below, S denotes an orientable hyperbolizable topological surface. Fix a pants decomposition $\mathcal{P}$ of S as in Theorem 2.1.1, and let $\mathcal{P}_b \subset \mathcal{P}$ be the subset of boundary curves and curves bounding cylinders. Also fix a collection $\mathcal{S}$ of *seams* with respect to $\mathcal{P}$ as defined in Section 2.1.4.

3.2.1 Good markings

To adjust Fenchel-Nielsen parameters consistently with the pointed Gromov-Hausdorff topology, we need some 'canonical' identifications of hyperbolic structures. We will use Buser's explicit markings of hyperbolic surfaces. What we now outline follows his treatment of the Teichmüller space in Chapter 6 of [14]. The interested reader can find more details there.

Any triple $(\ell_1, \ell_2, \ell_3) \in (\mathbb{R}_{\geq 0})^3$ specifies up to isometry a unique geodesic pair of pants Y with cuffs γ_1, γ_2, and γ_3 such that $\ell_Y(\gamma_i) = \ell_i$. The length of zero is understood to be that of a puncture. The common perpendiculars between pairs of boundary components decompose Y into two isometric hyperbolic polygons (or two right-angled hexagons in a case with no cusps). We say that Y is in *standard form* if each γ_i is parametrized on $\mathbb{S}^1 = \mathbb{R}/\langle t \mapsto t+1\rangle$ with constant speed so that it starts and ends at a point where a common perpendicular meets γ_i (which we call a *standard parametrization* of γ_i). By a *model pair of pants*, we mean a generalized geodesic pair of pants, possibly with cusps, given in standard form such that all cuffs have length 1.

Assume for now that S contains all of its boundary components that are circles. First, we consider a special hyperbolic surface J obtained by replacing the pairs of pants in $S \setminus \mathcal{P}$ with model pairs of pants and gluing them back with no twists so that the seam arcs match up. We call J the *model surface* for S, and we will consider $\mathcal{P}$ to be a geodesic pants decomposition of J. For our purpose, we will use J as a base for a marking of any marked hyperbolic surface in $\mathcal{T}(S)$.

Let $\mathcal{R}^{\mathcal{P}} = (\mathbb{R}_+ \times \mathbb{R})^{\mathcal{P}-\mathcal{P}_b} \times \mathbb{R}_+^{\mathcal{P}_b}$. For $\mathbf{a} = (\ell_{\mathbf{a}}(\gamma), \theta_{\mathbf{a}}(\gamma))_{\gamma \in \mathcal{P}-\mathcal{P}_b}, (\ell_{\mathbf{a}}(\beta))_{\beta \in \mathcal{P}_b} \in \mathcal{R}^{\mathcal{P}}$, we construct a hyperbolic surface $X_{\mathbf{a}}$ as follows. For any pants component Y of $J - \mathcal{P}$ with cuffs $\gamma_1, \gamma_2, \gamma_3$, let $Y^{\mathbf{a}}$ be a geodesic pair of pants with cuffs $\gamma_1^{\mathbf{a}}, \gamma_2^{\mathbf{a}}, \gamma_3^{\mathbf{a}}$ in standard form in which $\ell(\gamma_i^{\mathbf{a}}) = \ell_{\mathbf{a}}(\gamma_i)$. Let $X_{\mathbf{a}}$ be the quotient

$$X_{\mathbf{a}} = \left(\bigcup Y^{\mathbf{a}}\right) / \text{pasting scheme},$$

where the pasting scheme is such that if two pants $Y_1^{\mathbf{a}}$ and $Y_2^{\mathbf{a}}$ are glued along simple closed geodesics $\gamma_1 \subset \partial Y_1^{\mathbf{a}}$ and $\gamma_2 \subset \partial Y_2^{\mathbf{a}}$ of the same length so that both project to

γ in $X_{\mathbf{a}}$, then the identification is

$$\gamma_1(t) = \gamma_2(\theta_{\mathbf{a}}(\gamma) - t) =: \gamma(t) \quad \text{for all } t \in \mathbb{S}^1. \tag{3.1}$$

In his model of the Teichmüller space, Buser describes how to construct a marking $\varphi_{\mathbf{a}} : J \to X_{\mathbf{a}}$ for each $\mathbf{a} \in \mathcal{R}^{\mathcal{P}}$ as the composition of two maps: the first stretches each model pair of pants in J, and the second twists a collar neighborhood of each pants curve to attain the desired values for the twist parameters.[1] Furthermore, if γ is as in (3.1), then $\varphi_{\mathbf{a}}$ reparametrizes γ as $\gamma_{\mathbf{a}}$ with

$$xs\varphi_{\mathbf{a}} \circ \gamma(t) = \gamma_{\mathbf{a}}(t + \frac{1}{2}\theta_{\mathbf{a}}(\gamma)) \quad \text{for all } t \in \mathbb{S}^1. \tag{3.2}$$

In particular, the construction of $\varphi_{\mathbf{a}}$ also ensures that the standard parametrization of the boundary geodesics of J are preserved. The details of these constructions can be found in Section 6.2 in [14].

Definition 3.2.1. Let J be a model surface for S, and let X be a hyperbolic surface such that $\mathsf{CC}(X)$ is diffeomorphic to J. Then, a diffeomorphism $f : J \to \mathsf{CC}(X)$ is called a *good marking* if there exist a marking $\varphi_{\mathbf{a}} : J \to X_{\mathbf{a}}$ as defined by Buser for some $\mathbf{a} \in \mathcal{R}^{\mathcal{P}}$ and an isometry $I : \mathsf{CC}(X) \to X_{\mathbf{a}}$ such that $I \circ f$ and $\varphi_{\mathbf{a}}$ are isotopic.

We record the main consequences of Buser's markings, applied to our setting.

Proposition 3.2.2 (Theorem 6.2.7 in [14])**.** *Fix a model hyperbolic surface J with finite area. Let (X, g) be a marked hyperbolic surface with $g : J \to X$ as a marking with its Fenchel-Nielsen parameters $FN(X) = \mathbf{a} \in \mathcal{R}^{\mathcal{P}}$. Then, (X, g) and $(X_{\mathbf{a}}, \varphi_{\mathbf{a}})$ are marking equivalent. In other words, there is a choice of good marking for every marked surface.*

[1] Although Buser only defines these markings for compact hyperbolic surfaces, the construction readily generalizes to our current setting.

Theorem 6.4.2 in [14] asserts that, for a closed surface S, the Fenchel-Nielsen coordinates of $\mathcal{T}(S)$ form a real-analytic global atlas compatible with the topology defined by the *quasi-isometry distance* between two marked surfaces (assumed with good markings)

$$d_{\text{q.i.}}((X_1, f_1), (X_2, f_2)) = \inf \log K \tag{3.3}$$

where the infimum is taken over all $K > 1$ such that there is a K-quasi-isometry $X_1 \to X_2$ in the isotopy class of $f_2 \circ f_1^{-1}$.

In fact, this compatibility holds so long as S is of finite type—regardless of whether S has cusps or includes boundary circles, see Proposition 3.2.4 below—using the existence of a quasi-isometry between two hyperbolic funnels, which we now show. This is Exercise 4.6.15 in [35].

Lemma 3.2.3. *For each $\ell > 0$, let $F_\ell = \{z \in \mathbb{H}^2 : \operatorname{Re}(z) \geq 0\} / \langle z \mapsto e^\ell z \rangle$ be a hyperbolic funnel with a simple closed geodesic boundary of length ℓ. If $0 < \ell_1 < \ell_2$, then there exists a K-quasi-isometry $f_{\ell_1,\ell_2} : F_{\ell_1} \to F_{\ell_2}$ where $K = \ell_2/\ell_1$.*

Proof. For $i = 1, 2$, let γ_i be the boundary curve of F_{ℓ_i} and $\ell_i = \ell(\gamma_i)$. We will work with Fermi coordinates with respect to the γ_i. Fixing a point $x_i \in \gamma_i$, we parametrize γ_i by unit speed so that $\gamma_i(0) = x_i$. Then, the Fermi coordinate $(t, \rho) \in [0, \ell_i) \times [0, \infty)$ on F_i represents the unique point $p \in F_i$ whose perpendicular from p to γ_i has length ρ and meets γ_i at $\gamma_i(t)$.

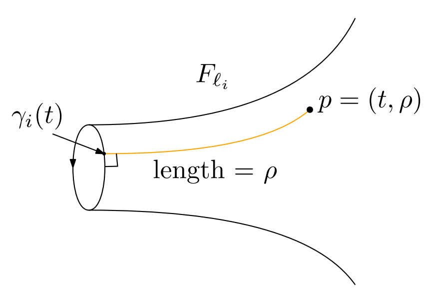

FIGURE 3.1: A point on a funnel F_{ℓ_i} shown with a Fermi coordinate.

Consider the map $f_{\ell_1,\ell_2} : F_{\ell_1} \to F_{\ell_2}$ given by $f(t, \rho) = (Kt, \rho)$. That is, f_{ℓ_1,ℓ_2} sends the γ_1-equidistant curve of height ρ to the γ_2-equidistant curve of height ρ. We claim that this f_{ℓ_1,ℓ_2} is a K-quasi-isometry. The calculations are straightforward, but we include them here for completeness.

The hyperbolic metric with respect to the Fermi coordinate (t, ρ) can be expressed as $ds^2 = \cosh^2 \rho \, dt^2 + d\rho^2$. Let $\gamma : [0, 1] \to F_{\ell_1}$ be a differentiable path in F_{ℓ_1} given by $\gamma(u) = (t(u), \rho(u))$. Then,

$$\frac{d}{du} f_{\ell_1,\ell_2}(\gamma(u)) = \frac{d}{du}(Kt(u), \rho(u)) = (Kt'(u), \rho'(u)).$$

Thus,

$$\begin{aligned}
\ell_{F_2}(f_{\ell_1,\ell_2}(\gamma)) &= \int_0^1 \sqrt{\cosh^2 \rho(u)\, (Kt'(u))^2 + \rho'(u)^2} du \\
&\leq K \int_0^1 \sqrt{\cosh^2 \rho(u)\, t'(u)^2 + \rho'(u)^2} du \\
&= K \cdot \ell_{F_{\ell_1}}(\gamma).
\end{aligned}$$

It is clear that f_{ℓ_1,ℓ_2} is distance non-decreasing. So, the map f_{ℓ_1,ℓ_2} is indeed a K-quasi-isometry between F_{ℓ_1} and F_{ℓ_2} ■

Let J be a model surface for a finite-type surface S with geodesic boundary. By a *model funnel*, we refer to the hyperbolic funnel F_1 with its boundary geodesic parametrized with constant speed on $\mathbb{S}^1$. If we glue model funnels to the boundary components of J so that the boundary parametrizations line up, we can also consider the Teichmüller space of a geodesically complete hyperbolic surface. We extend good markings in an obvious way by pasting together $\varphi_{\mathbf{a}}$ on the convex core with appropriate $f_{1,\ell}$ (as defined in lemma 3.2.3 above) along the boundary.

The proposition below is a generalized, paraphrased version of Theorem 6.4.2 in [14]. We modify Buser's proof slightly.

Proposition 3.2.4. *If S is of finite type that is not a sphere with three points removed, then the Fenchel-Nielsen coordinate map*

$$FN : \mathcal{T}(S) \to \mathcal{R}^{\mathcal{P}}$$
$$(X, f) \mapsto FN(\mathsf{CC}(X))$$

is a diffeomorphism, where $\mathsf{CC}(X)$ is the convex core of X. Here, the Teichmüller space is considered equipped with the quasi-isometry distance.

Proof. We view $S = J$ as a model surface. That is, S is equipped with a hyperbolic structure in which all pairs of pants and funnels in the decomposition determined by $\mathcal{P}$ are in model form and each gluing has no twist. Suppose that the funnel components of $S - \mathcal{P}$ are bounded by $c_1, \ldots, c_k \in \mathcal{P}_b$. Any marked hyperbolic structure (X, f) is realized by gluing hyperbolic funnels along the geodesic boundary components of its convex core. The isometry classes of these funnels are uniquely determined by $\ell_X(c_i)$. Since funnels are rotationally symmetric, the geometry of X is completely specified by $\mathsf{CC}(X)$. Thus, the natural inclusion $\mathcal{T}(\mathsf{CC}(S)) \to \mathcal{T}(S)$ is in fact a bijection, and we may regard $\mathcal{T}(S)$ as the set

$$\mathcal{T}(S) = \{(X_{\mathbf{a}}, \varphi_{\mathbf{a}}) : \mathbf{a} \in \mathcal{R}^{\mathcal{P}}\},$$

where φ_a is a good marking.

We will now prove that both FN and FN^{-1} are continuous. Assume $\mathbf{a_n} \to \mathbf{a}$ in $\mathcal{R}^{\mathcal{P}}$. First, notice that, since all funnels are quasi-isometric and there are only finitely many funnels in S, the notion of the q.i.-distance (3.3) still holds in $\mathcal{T}(S)$. Indeed, for $(X_{\mathbf{a}_n}, \varphi_{\mathbf{a}_n})$ and $(X_{\mathbf{a}}, \varphi_{\mathbf{a}})$, the map $\varphi_{\mathbf{a}_n}|_{\mathsf{CC}(S)} \circ \varphi_{\mathbf{a}}|^{-1}_{\mathsf{CC}(S)}$ between the convex cores is a K_n-quasi-isometry by construction, where $K_n \to 1$. Together with lemma 3.2.3, the quasi-isometry constant of $\varphi_{\mathbf{a}_n} \circ \varphi_{\mathbf{a}}^{-1}$ is

$$\max\{K_n, \frac{\ell_{\mathbf{a}_n}(\gamma)}{\ell_{\mathbf{a}}(\gamma)}, \frac{\ell_{\mathbf{a}}(\gamma)}{\ell_{\mathbf{a}_n}(\gamma)} : \gamma \in \mathcal{P}_b\} < \infty$$

which goes to 1 as $n \to \infty$. This shows that $d_{\text{q.i.}}((X_{\mathbf{a}_n}, \varphi_{\mathbf{a}_n}), (X_{\mathbf{a}}, \varphi_{\mathbf{a}})) \to 0$.

Now, suppose that $d_{\text{q.i.}}(X_n, X) \to 0$. We may assume that all of X_n and X come equipped with good markings φ_n and φ, respectively. Let γ be an essential simple closed geodesic in S. If $f : X_n \to X$ is a K-quasi-isometry in the homotopy class of $\varphi \circ \varphi_n^{-1}$, then the simple closed curve $f \circ \varphi_n(\gamma)$ is in the same homotopy class as the simple closed geodesic $\varphi(\gamma)$. This implies that

$$\frac{1}{K}\ell_X(\varphi(\gamma)) \leq \frac{1}{K}\ell_X(f \circ \varphi_n(\gamma)) \leq \ell_{X_n}(\varphi_n(\gamma)),$$

so $\ell_X(\varphi(\gamma)) \leq K\ell(\varphi_n(\gamma))$. Similarly, $\ell_{X_n}(\varphi_n(\gamma)) \leq K\ell_X(\varphi(\gamma))$. Thus, setting

$$d_n = d_{\text{q.i.}}(X_n, X),$$

we have

$$\exp(-d_n) \leq \frac{\ell_{X_n}(\varphi_n(\gamma))}{\ell_X(\varphi(\gamma))} \leq \exp(d_n).$$

In particular, $\ell_{X_n}(\varphi_n(\gamma)) \to \ell_X(\varphi(\gamma))$. Moreover, all essential simple closed geodesics in S, X, and X_n are contained in their convex cores. By equipping $\mathcal{T}(\mathsf{CC}(S))$ with the length spectrum metric (see Proposition 3.3 in [28]), it follows that $FN(\mathsf{CC}(X_n))$ converges to $FN(\mathsf{CC}(X))$. This concludes the proof. ■

3.2.2 Adjusting parameters

In this subsection, a topological hyperbolizable S need not be of finite type.

Definition 3.2.5. We call $R \subset S$ a $\mathcal{P}$-*subsurface* if R is a connected subsurface and all boundary components of R (if they exist in S) belong to $\mathcal{P}$. Similarly, if X is a hyperbolic surface diffeomorphic to S with a marking $f : S \to X$, we say $Z \subset X$ is a $\mathcal{P}$-*(geodesic) subsurface* if it is connected and all boundary components are simple closed geodesics belonging to the homotopy classes of $f(\mathcal{P})$. The context should clearly indicate whether it is topological or geometric.

Notation. *With the same setup as in Definition 3.2.5 above, we write* $FN|_R(X)$ *for the restriction of the Fenchel-Nielsen parameters of* X *to those belonging to the geodesics in the homotopy classes of* $f(\mathcal{P} \cap R)$*. Set*

$$\mathcal{FN}(R) := \{FN|_R(X) : (X, f) \text{ is a marked hyperbolic surface diffeomorphic to } S\}.$$

We are ready to state the main proposition of this section: varying finitely many Fenchel-Nielsen parameters is continuous in $\mathcal{H}^2_\bullet$. Let $R \subset S$ be a finite-type $\mathcal{P}$-subsurface. Fix $\mathbf{a}_0 \in \mathcal{FN}(R)$. Suppose that (X_0, f_0) is a marked hyperbolic surface diffeomorphic to S with Fenchel-Nielsen coordinates $FN(X_0)$ where $FN|_R(X_0) = \mathbf{a}_0$.

Proposition 3.2.6. *Fix a basepoint* $p_0 \in X_0$*. If* $t \mapsto \mathbf{a}_t$ *is a continuous path in* $\mathcal{FN}(R)$ *for* $t \in [0, 1]$*, then there is a continuous path* $t \mapsto (X_t, p_t)$ *in* $\mathcal{H}^2_\bullet$*, where* $p_t \in X_t$ *and* $\psi_t : S \to X_t$ *is a marking for which* $FN|_R(X_t) = \mathbf{a}_t$ *at all time* $0 \leq t \leq 1$*.*

Proof. First, enlarge R to the finite-type $\mathcal{P}$-subsurface $R' \supset R$ so that no boundary component of R is still a boundary component of R'. Viewing R' as a model hyperbolic surface, let $b_1, \ldots, b_n$ be the boundary geodesics of R'. We also record the twist parameter $\theta_{X_0}(b_j)$ for all $j = 1, \ldots, n$.

Set $\mathbf{a}'_0 = FN|_{R'}(X_0)$. For $t \in [0, 1]$, create a continuous path $t \mapsto \mathbf{a}'_t$ in $\mathcal{FN}(R')$, where the coordinates that do not come from R are all held constant, and those from R take corresponding values from $\mathbf{a}_t$.

For all $t \geq 0$, consider $(Z_t, \varphi_t) \in \mathcal{T}(R')$ such that $\varphi_t : R' \to Z_t$ is a good marking and $FN(Z_t) = \mathbf{a}'_t$. Temporarily parametrize each boundary component of $X_0 - f_0(R')$

by constant speed on $\mathbb{S}^1$. Since $\ell(\varphi_t(b_j)) = \ell(f_0(b_j)$ for all $j = 1, \dots, n$, we can set

$$X_t = Z_t \cup_\partial (X_0 - f_0(R'))/\text{pasting scheme}$$

where the pasting scheme follows (3.1):

$$\varphi_t \circ b_j(s) = f_0 \circ b_j(\theta_{X_0}(b_j) - s) =: \beta_j(s) \quad \text{for all } s \in \mathbb{S}^1$$

and β_j is the projection of $\varphi_t(b_j)$ in X_t. We now introduce a family of markings $\psi_t : S \to X_t$ by setting ψ_t to be a good marking that extends φ_t on a regular neighborhood of R' and smoothing it to f_0 on the complement. With respect to ψ_t, $FN(X_t) = \mathbf{a}_t$ for all t. We remark that the original marked surface (X_0, f_0) and the new (X_0, ψ_0) are marking equivalent.

Since we only alter the Fenchel-Nielsen coordinates corresponding to $R \subset R'$, we may assume that $\psi_t|_{X_t - Z_t} \circ \psi_s^{-1}|_{X_s - Z_s}$ is an isometry for all $s, t \in [0, 1]$. As in the proof of Proposition 3.2.6, our choice of good markings shows that $\psi_t \circ \psi_s^{-1}$ is a $K = K(s,t)$-quasi-isometry $X_s \to X_t$ where $K \to 1$ as $|s - t| \to 0$. Thus, the family X_t is a continuous path in the quasi-isometry distance. Finally, choose the basepoint $p_t = (\psi_t \circ \psi_0^{-1})(p_0) \in X_t$. It follows that the (X_t, p_t) vary continuously in $\mathcal{H}^2_\bullet$. ■

Repeatedly applying proposition 3.2.6 to modify finite-type subsurfaces, we can construct a continuous path from a given pointed hyperbolic surface to any other with the same diffeomorphism type, where the basepoints are suitably chosen. However, changing some collection of infinitely many Fenchel-Nielsen parameters at once may end up producing a surface with an incomplete hyperbolic metric as in Example 2.1.6.

3.3 Pinching a simple closed curve to a cusp

We now record a continuous deformation in which a simple closed curve is shrunk to have length 0, so its neighborhood becomes a cusp. We note that this is not a subcase of the previous section, as we did not allow a geodesic pants curve to degenerate into a puncture. First, we need a generalized (and pointed) version of lemma 3.2.6 in [14].

Lemma 3.3.1. *Let $\{Y_n\}_{n=1}^{\infty}$ be a sequence of generalized pairs of pants with boundary geodesics (or punctures) γ_1^n, γ_2^n, and γ_3^n. Let Y be another generalized pair of pants with boundary geodesics (or punctures) γ_1, γ_2, and γ_3. Suppose that $\ell(\gamma_k^n) \to \ell(\gamma_k)$ as $n \to \infty$ for every $k = 1, 2, 3$.*

Then, for any sequence of points $p_n \in Y_n$ whose injectivity radii are uniformly bounded away from zero, there is a point $p \in Y$ such that, by possibly passing to a subsequence, $(Y_n, p_n) \to (Y, p)$ in the pointed Gromov-Hausdorff topology.

Proof. If all geodesic boundaries of Y have positive lengths, the result follows immediately from lemma 3.2.6 in [14]. In this case, Buser's stretch homeomorphisms $\sigma_n : Y \to Y_n$ are desired quasi-isometries (illustrated below in Figure 3.2) and we take $p \in Y$ to be an accumulation point of $\{\sigma_n^{-1}(p_n)\}$.

It remains to prove the statement when at least one boundary component of Y is a puncture. We show this when Y has three punctures—i.e. $\ell(\gamma_k) = 0$ for all $k = 1, 2, 3$. We shall assume that $\ell(\gamma_k^n) > 0$ for a sufficiently large n. The proof can be easily modified for the other cases.

Let α_1, α_2, and α_3 be the disjoint infinite geodesics joining the punctures of Y. Let $\Delta \subset Y$ be the closure of a component of $Y \setminus \{\alpha_1, \alpha_2, \alpha_3\}$. Then, Δ is an ideal triangle. In Y_n, let β_k^n be the common perpendicular of γ_k^n and γ_{k+1}^n, where the subscripts are read modulo 3. Let $H_n \subset Y_n$ be the closure of a component of $Y_n \setminus \{\beta_1^n, \beta_2^n, \beta_3^n\}$ containing p_n. So H_n is a right-angled geodesic hexagon, with alternating sides of

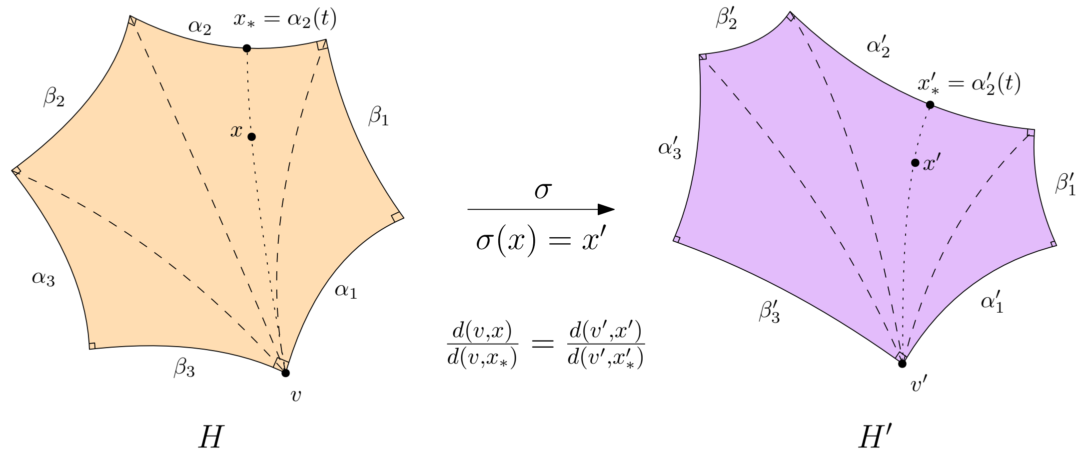

FIGURE 3.2: A stretch homeomorphism between two right-angled geodesic hexagons H and H' (with distinguished vertices v and v', respectively), which is affine on each boundary geodesic segment. A concatenation of two identical σ defines a stretch map between two geodesic pairs of pants. See Definition 3.2.4 in [14].

lengths $\ell(\gamma_k^n)/2$.

Fix Δ as an ideal triangle in $\mathbb{D}$. Consider H_n as a subset of $\mathbb{D}$ as follows. Extend each β_k^n to an infinite geodesic $\tilde{\beta}_k^n$. Together, $\tilde{\beta}_1^n, \tilde{\beta}_2^n$, and $\tilde{\beta}_3^n$ bound a convex region F_n containing H_n. After rotating F_n and possibly relabeling the sides, we may assume that $\Delta \subset F_n$ and that α_k and α_{k+1} intersect γ_k. Choose p to be an accumulation point of $\{p_n\} \subset \mathbb{D}$, possibly after passing to a subsequence. Since $\ell(\gamma_k^n) \to 0$, it must be that $p \in Y$.

Parametrize the geodesic segments β_k^n and $\alpha_k^n = \alpha_k \cap H_n$ with unit speed so that both start on the side γ_k^n and end on the side γ_{k+1}^n, where $k \in \{1,2,3\}$ and the subscript is read cyclically. See Figure 3.3 below.

Let $\epsilon_n := \max\{\ell(\gamma_k^n)/2 : k = 1,2,3\}$. Then, the endpoints of β_k^n and α_k^n, being on γ_k^n, are at a distance less than ϵ_n apart. It is a hyperbolic geometry fact that the function $(s,t) \mapsto d(\beta_k(s), \alpha_k^n(t))$ is strictly convex (see Proposition II.2.2 in [16]). This shows that $\beta_k^n \subset N_{\epsilon_n}(\alpha_k^n)$ for all $k = 1,2,3$, and hence H_n is in the ϵ_n-neighborhood of $\Delta \cap H_n$.

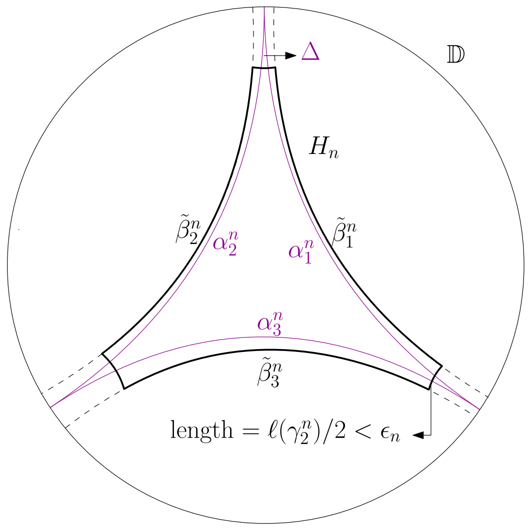

FIGURE 3.3: A right-angled geodesic hexagon H_n (bold) with its extension F_n (dashed) and an ideal triangle Δ (in magenta).

Let $r_n : H_n \to \Delta$ be the nearest point retraction. Since Δ is convex, the map r_n is well-defined [16, Lemma I.2.3.1]. The retraction map r_n gives a (ϵ_n, R_n)-relation between (H_n, p_n) and (Δ, p), where $R_n = \min\{d(p_n, \gamma_k^n) : k = 1, 2, 3\}$. We then use it to define a (ϵ_n, R_n)-relation between (Y_n, p_n) and (Y, p), individually between the right-angle hexagon components of $Y_n \backslash \{\beta_1^n, \beta_2^n, \beta_3^n\}$ and the ideal triangle components of $Y \setminus \{\alpha_1, \alpha_2, \alpha_3\}$.

As $n \to \infty$, we have that $\epsilon_n \to 0$ and also $R_n \to \infty$, since $\ell(\gamma_k^n) \to 0$ while the injectivity radii of p_n are uniformly bounded away from zero. Certainly, these relations give the desired convergence $(Y_n, p_n) \to (Y, p)$. ■

Using the same argument as above, we obtain a similar result for funnels, whose proof is straightforward and we now omit. Let F_0 be the infinite cylinder which is the quotient of $\mathbb{H}^2$ by a parabolic isometry. (Since all parabolic isometries are conjugate, we may take $F_0 = \mathbb{H}^2/\langle z \mapsto z+1\rangle$ for concreteness.) For $\ell > 0$, F_ℓ is the funnel whose boundary geodesic has length ℓ.

Corollary 3.3.2. *Let $\{(F_{\ell_n}, p_n)\}_{n=1}^{\infty}$ be a sequence of funnels with a basepoint with $\ell_n \to 0$. Suppose that the injectivity radii of p_n in F_{ℓ_n} are uniformly bounded both below and above. Then, $(F_{\ell_n}, p_n) \to (F_0, p)$ in the pointed Gromov-Hausdorff topology, for some suitably chosen $p \in F_0$.*

Remark 1. Let us contrast Corollary 3.3.2 with the fact that there is no global quasi-isometry between F_ℓ and F_0 for any $\ell > 0$, for they are not diffeomorphic. So, $d_{\text{q.i.}}(F_0, F_\ell) = \infty$. Unlike a quasi-isometry which captures the distortion of the corresponding metrics globally, the pointed Gromov-Hausdorff topology only detects such change on large compact subsets.

We are now positioned to state that shrinking a collection of simple closed geodesics to punctures is continuous in the pointed Gromov-Hausdorff topology of $\mathcal{H}^2_\bullet$.

For a hyperbolic surface X, if $H_1, H_2, \ldots$ are the half-plane components of $X - \mathsf{CC}(X)$, we let $\mathsf{CC}_F(X) = X - \bigcup H_n$.

Proposition 3.3.3. *Let S be a model hyperbolic surface that admits a geodesic pants decomposition $\mathcal{P}$, possibly with model funnels. Fix a subset $\mathcal{Q} \subset \mathcal{P}$. If (X, p) is a pointed hyperbolic surface with a good marking $\varphi : S \to \mathsf{CC}_F(X)$, then there is a finite-time continuous path from (X, p) to a pointed hyperbolic surface $(X_\mathcal{Q}, q)$, where $X_\mathcal{Q}$ is obtained by pinching all boundary geodesics of $X - \varphi(\mathcal{Q})$ to punctures and $q \in X_\mathcal{Q}$.*

Proof. Denote by $\bar{S}$ the component of $S - \mathcal{Q}$ that contains $\varphi^{-1}(p)$. For time $t \in [0, 1)$, construct a surface X_t by modifying S as in the previous section (and completing the metric by adding half-planes if necessary) so that $\ell_{X_t}(\gamma) = (1-t)\ell_X(\gamma)$ in $FN(X_t)$ for all boundary curves γ of $\bar{S}$, while all other Fenchel-Nielsen parameters are kept constant. We obtain a family of good markings $\varphi_t : S \to \mathsf{CC}_F(X_t)$ accordingly. Choose a basepoint $p_t = \varphi_t \circ \varphi_0^{-1}(p)$ in X_t.

Consider a surface $X_{\mathcal{Q}}$ obtained as follows. Let S' be a new model surface obtained by changing the boundary curves of $\bar{S}$ into punctures. We keep the same pants decomposition $\mathcal{P}$ so that the $\mathcal{P}$-pants in S with boundary in $\partial\bar{S}$ are replaced with pants with cusps in $\bar{S}$. By pasting together the relations as in lemma 3.3.1 (or corollary 3.3.2), we conclude that $(\varphi_t(\bar{S}), p_t) \to (X_{\mathcal{Q}}, p')$ for some choice of $p' \in S_{\mathcal{Q}}$ in the pointed Gromov-Hausdorff topology. Finally, by the collar lemma, the radius R_t for which $B_{X_t}(p_t, R_t) \subset \varphi_t(\bar{S})$ approaches infinity as $t \to 1$, we deduce that $(X_t, p_t) \to (X_{\mathcal{Q}}, p')$ in $\mathcal{H}^2_\bullet$. We can apply lemma 3.1.1 to move from $(X_{\mathcal{Q}}, p')$ to the desired $(X_{\mathcal{Q}}, q)$. ■

Remark 2. With Proposition 3.3.3, we have that shrinking a simple closed curve introduces a new cusp to a base surface in the topology of $\mathcal{H}^2_\bullet$. By allowing the length of these degenerate curves as zero, we can now strengthen Proposition 3.2.6 by allowing length parameters to take values from $\mathbb{R}_{\geq 0}$ rather than just $\mathbb{R}_+$.

3.4 Inserting an infinite strip

Our last class of continuous paths in $\mathcal{H}^2_\bullet$ comes from strip deformations. This is a geometric construction in which an infinite strip is inserted along a properly embedded infinite geodesic in a hyperbolic surface. Thurston first observed in [36] that removing an infinite strip from the geodesic completion of a finite-type bordered hyperbolic surface shortens all closed geodesics in the convex core. Papadopoulos and Théret [30] later give a proof and show that the lengths decrease by at least a positive constant depending on the width of the strip. In [17], Danciger-Guéritaud-Kassel characterize all proper deformations of a convex co-compact hyperbolic surface that lengthens all simple closed geodesics as arising in some unique way from so-called infinitesimal strip deformations along a collection of finitely many geodesic arcs.

The main tools in this section are two specific strip deformations of a cusped pair of pants that turn the cusps into funnels in a way that is continuous in $\mathcal{H}^2_\bullet$. We begin by formulating a strip deformation in a concrete way and show that it is compatible with the pointed Gromov-Hausdorff topology.

Definition 3.4.1. For any $s > 0$, an *(infinite) s-strip* is the region in $\mathbb{H}^2$ bounded by two hyperparallel infinite geodesics, whose unique common perpendicular τ between them has length s. The geodesic arc τ is called the *waist* of A_s. The *core* of the s strip is the infinite perpendicular bisector of τ. The *center* of the s-strip is the point of intersection between τ and the core.

Each s-strip A_s comes foliated with equidistant arcs of the waist τ as leaves. These arcs are $\{y \in A_s : d(\tau, y) = h\}$ indexed by $h > 0$. This foliation induces an isometry between the boundary geodesics of A_s by identifying the two endpoints of each leaf. This is referred to as the *canonical isometry* between the two geodesics.

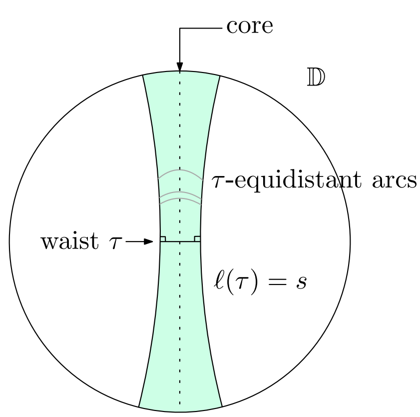

FIGURE 3.4: An infinite s-strip (shaded) with its waist τ, its core (dotted). Some τ-equidistant arcs are also shown.

Definition 3.4.2 (Inserting a strip to a surface)**.** Given a complete hyperbolic surface X with a properly embedded simple bi-infinite geodesic α and $a \in \alpha$, we can cut open X along α and then glue an s-strip along the boundary of the closure of $X \setminus \alpha$ via the isometry which respects the parametrization with unit speed of α in such a way that both endpoints of τ are identified with a. The resulting surface depends on the choices of X, α, a, and s, and we denote it by $\mathsf{Strip}(X, \alpha, a, s)$.

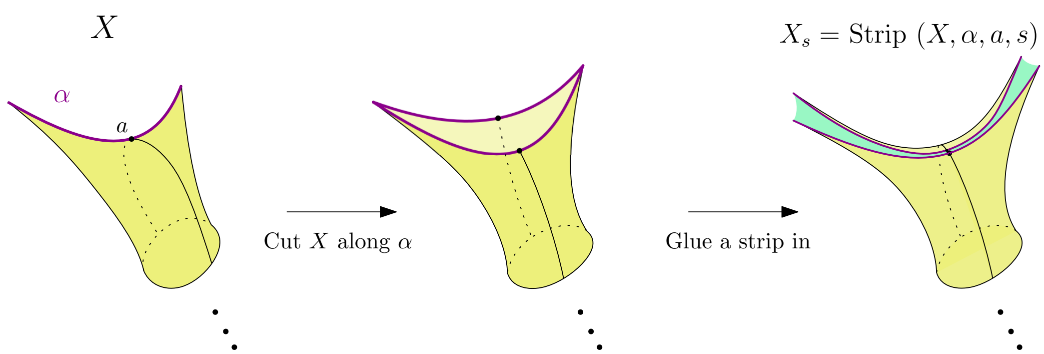

FIGURE 3.5: Inserting a strip (green) along α (purple) in X.

By construction, the resulting surface $\mathsf{Strip}(X, \alpha, a, s)$ has a complete hyperbolic structure. We will now see that inserting a strip and varying its width are continuous deformations of a pointed surface in $\mathcal{H}^2_\bullet$.

Let X, α, and a be given as in Definition 3.4.2. Choose a basepoint $p \in X - \alpha$. Let $X_s = \mathsf{Strip}(X, \alpha, a, s)$. For any $s \geq 0$, denote the inserted s-strip in X_s by A_s and its waist by τ_s. Orient and parametrize α by unit speed so that $\alpha(0) = a$. This also determines the orientation of the left and the right boundary components of A_s, which we call $\alpha_{s,L}$ and $\alpha_{s,R}$, respectively. There is an isometric embedding $X_s - A_s \hookrightarrow X_t - A_t \subset X_t$ that extends the canonical isometry between $\alpha_{s,L}$ and $\alpha_{t,L}$ and also between $\alpha_{s,R}$ and $\alpha_{t,R}$, which we refer to as the *canonical isometric embedding.*

Theorem 3.4.3. *The strip insertion map*

$$\begin{aligned} \mathsf{Strip} : \mathbb{R}_+ &\to \mathcal{H}^2_\bullet \\ s &\mapsto (X_s, p) \end{aligned}$$

is continuous, where $p \in X_s$ is the image of p under the canonical isometric embedding $X - \alpha \to X_s$.

Proof. First, if $s > t > 0$, we define a map $f_{s,t} : (X_s, p) \to (X_t, p)$ by mapping $X_s - A_s$ to $X_t - A_t$ via the canonical isometric embedding and proportionally shrinking A_s to A_t along the foliated leaves (we can work with Fermi coordinates to give an explicit formula). The same calculations as in lemma 3.2.3 can be used to show that $f_{s,t}$ is a (s/t)-quasi-isometry. This proves that Strip is continuous away from 0.

Now, define $g_s : X \to X_s$ by sending $X - \alpha$ to $X_s - A_s$ and α to $\alpha_{s,L}$ via the canonical isometric embedding. Fixing $R > 0$, we claim that g_s distorts the distance in $B_X(p, R)$ by no more than some $M_R(s) > 0$ and that $M_R(s) \to 0$ as $s \to 0$. It is clear that g_s does not change the distance between two points of X if they lie on the same side of $X - \alpha$. So, we consider two points $x, y \in B_X(p, R)$ such that the shortest geodesic segment joining them $[x, y]$ intersects α, say at $\alpha(t)$. Without loss of generality, assume that x is on the left of α. Refer to Figure 3.6 below. For each $h \in \mathbb{R}$, let $\eta_{s,h}$ denote the τ-equidistant arc in A_s that joins $\alpha_{s,L}(h)$ and $\alpha_{s,R}(h)$. Observe that there is a maximal value $|T| > 0$ such that $\alpha(T) \cap \overline{B_X(p, R)} \neq \emptyset$, and set $M_R(s)$ to be the length of $\eta_{s,T}$.

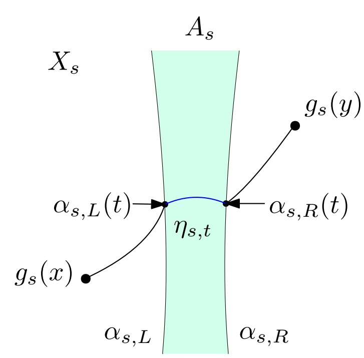

FIGURE 3.6: The images under x and y under g_s in X_s.

Concatenating together $[g_s(x), \alpha_L(t)] \cup \eta_{s,t} \cup [\alpha_R(t), g_s(y)]$, we have that

$$\begin{aligned} d_{X_s}(g_s(x), g_s(y)) - d_X(x,y) &\le d_{X_s}(g_s(x), \alpha_L(t)) + \ell(\eta_{s,t}) \\ &\quad + d_{X_s}(\alpha_R(t), g_s(y)) - d_X(x,y) \\ &= \ell(\eta_{s,t}) \le M_R(s). \end{aligned}$$

Moreover, Lemma 2.1 in [30] shows that g_s is distance non-decreasing. Together, we have

$$0 \le d_{X_s}(g_s(x), g_s(y)) - d_X(x,y) \le M_R(s)$$

for $x, y \in B_X(p, R)$. Clearly, $M_R(s) \to 0$ as $s \to 0$, so we have our claim.

To prove continuity of Strip at 0, we note that for any $\epsilon, R > 0$, we can choose the width $s > 0$ sufficiently small so that $M_R(s) < \epsilon$. This gives an (ϵ, R)-relation between (X, p) and (X_s, p_s), and we are done. ■

Remark 3. If $\{\alpha_1, \ldots, \alpha_n\}$ is a finite collection of disjoint proper bi-infinite geodesics in a hyperbolic surface X with $a_i \in \alpha_i$, then we can simultaneously insert strips of width t_i along α_i at a_i, which we denote the resulting hyperbolic surface by

$$\mathsf{Strip}(X, (\alpha_1, \cdots, \alpha_n), (a_1, \ldots, a_n), (t_1, \ldots, t_n)).$$

The same proof applies to show that growing a finite number of strips simultaneously is a continuous deformation in $\mathcal{H}^2_\bullet$; that is,

$$\begin{aligned} &\mathsf{Strip} : (\mathbb{R}_+)^n \to \mathcal{H}^2_\bullet \\ &\quad (t_1, \ldots, t_n) \mapsto (\mathsf{Strip}(X, (\alpha_1, \cdots, \alpha_n), (a_1, \ldots, a_n), (t_1, \ldots, t_n)), p) \end{aligned}$$

where p is chosen to be the image of the basepoint $p \in X$ under the canonical isometric embedding of $X - \bigcup \alpha_i$ into the new surface.

By a *hyperbolic cylinder* of length h bounded by a simple closed geodesic γ, we mean a cylinder isometric to a one-sided metric neighborhood of γ of width h in the funnel $\mathbb{H}^2/\langle z \mapsto e^{\ell(\gamma)}z\rangle$. We will analyze cylinders in the following specific constructions involving strip insertion in a single pair of pants with cusps.

Scenario 1: Both ends of the infinite geodesic exit a single cusp.

Let Y^1 be a pair of pants with two closed geodesic boundary components γ_1 and γ_2 and one cusp. Denote by σ the common perpendicular of γ_1 and γ_2. In this case, we take α to be the unique simple bi-infinite geodesic that exits the cusp in both ends and intersects σ once, say at $a \in \alpha$. For each $s > 0$, let $Y^1_s := \mathsf{Strip}(Y^1, \alpha, a, s)$. Geometrically, the cusp of Y^1 is replaced with a funnel in Y^1_s, see Figure 3.7 below. We consider what happens to Y^1_s as we increase the width s.

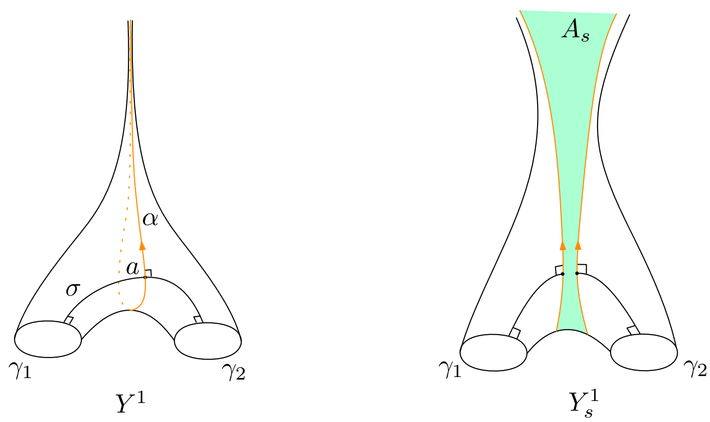

FIGURE 3.7: A pair of pants with one cusp Y^1 (left) and the surface Y^1_s (right) with the inserted strip A_s shaded.

Lemma 3.4.4. *The lengths of embedded hyperbolic cylinders bounded by γ_1 and γ_2 in Y^1_s both approach infinity as $s \to \infty$.*

Proof. Let C_s be the core of A_s. Each component of $Y^1_s - C_s$ is topologically a cylinder and contains a hyperbolic cylinder with boundary γ_i of length at least $d_{Y^1}(a, \gamma_i) + s/2$. The result immediately follows. ■

Scenario 2: The two ends of the infinite geodesic exit different cusps.

Consider a pair of pants Y^2 with two cusps and one geodesic boundary component γ. Take α to be the simple bi-infinite geodesic in Y^2 joining the two cusps as in Figure 3.8 below. Let σ be the shortest geodesic with endpoints in γ that intersects α exactly once, say at a. For each $s > 0$, let $Y_s^2 = \mathsf{Strip}(Y^2, \alpha, a, s)$. Then, the two cusps in Y^2 is replaced by two funnels in Y_s^2. As an analog of lemma 3.4.4, we analyze Y_s^2 as s increases to infinity.

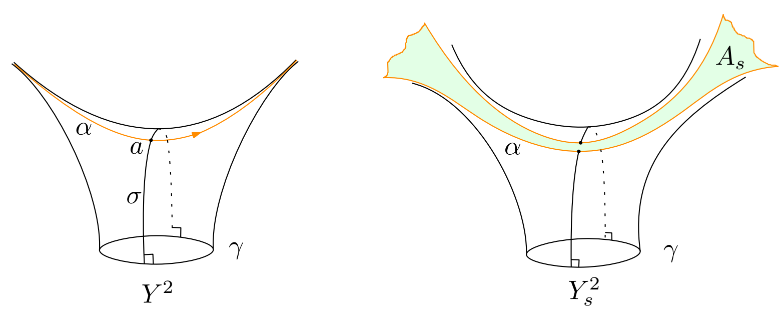

FIGURE 3.8: A pair of pants with two cusp Y^2 (left) and the surface Y_s^2 (right) with the inserted strip A_s shaded.

Lemma 3.4.5. *The length of the embedded hyperbolic cylinder bounded by γ in Y_s^2 approaches infinity as $s \to \infty$.*

Proof. This is also immediate: if C_s is the core of A_s, then $Y_s^2 - C_s$ is topologically a cylinder, and it contains a hyperbolic cylinder bounded by γ of length at least $d_{Y^2}(a, \gamma) + s/2$. ■

We will use lemmas 3.4.4 and 3.4.5 to modify parts of a surface which are pairs of pants into funnels, while keeping the remaining components unchanged. This surface modification will be continuous with respect to the Gromov-Hausdorff topology.

Proposition 3.4.6 (Capping with funnels)**.** *Let X_0 be a complete hyperbolic surface without boundary, which is not diffeomorphic to a sphere with three points removed* *or a cylinder. Fix a geodesic pants decomposition $\mathcal{P}$. Suppose that $Z_0 \subset X_0$ is a $\mathcal{P}$-subsurface such that no three components of ∂Z_0 bound a common pair of pants in $X_0 - Z_0$. Let $\widehat{Z}$ be the geodesic completion of Z_0 obtained by gluing funnels to each boundary component of Z_0. Fix a basepoint $x_0 \in \mathrm{int}(\mathsf{CC}(Z_0))$ of X_0. By viewing $Z_0 \subset \widehat{Z}$, we take $x_0 \in Z_0$ as the basepoint of $\widehat{Z}$.*

Then, there is a continuous path $t \mapsto (X_t, x_t)$ in $\mathcal{H}^2_\bullet$ (for $t \geq 0$) from (X_0, x_0) limiting to $(\widehat{Z}, x_0)$ such that there is a pointed subsurface $(Z_t, x_t) \subset (X_t, x_t)$ isometric to (Z_0, x_0) for all $t \geq 0$.

Proof. We describe how to construct X_t and the desired properties will be evident. First, let $\mathcal{Q} \subset \mathcal{P}$ be the pants geodesics not intersecting Z_0. By Proposition 3.3.3, these simple closed geodesics in $\mathcal{Q}$ can be shrunk simultaneously to punctures in finite time. We may then assume that each component C of $X_0 - Z_0$ is either a funnel or a pair of pants necessarily with one or two cusps.

For each $t \geq 0$, we define the components C_t as follows. If C is a funnel, set $C_t = C$. If C is a pair of pants, let $C_t = \mathsf{Strip}(C, \alpha, a, t)$, where α and a are as chosen as in the assumption of lemma 3.4.4 or lemma 3.4.5, depending on the number of cusps in C. Let X_t be the surface obtained by gluing the C_t to Z_0, keeping the same gluing identifications and twists. Clearly, there is an isometric copy of Z_0 in X_t; call it Z_t. We may now choose the basepoint $z_t \in Z_t$ accordingly. Each boundary geodesic of $X_t - Z_t$ bounds a hyperbolic cylinder in the corresponding component C_t. By lemma 3.4.4 and lemma 3.4.5, the lengths of these cylinders approach infinity as $t \to \infty$. This proves that $\lim_{t\to\infty}(X_t, x_t) = (\widehat{Z}, x_0)$ ■

Chapter 4

Results on Global and Local Path Connectivity of $\mathcal{H}^2_\bullet$

Our main results concerning path connectivity of $\mathbb{H}^2$, both global and local, are discussed in this chapter. Again, unless specified otherwise, a hyperbolic surface is assumed to be connected, oriented, and complete without boundary.

4.1 Path-connectivity

In this section, we deduce that $\mathcal{H}^2_\bullet$ is path-connected using continuous paths laid out in Chapter 3.

Recall that the injectivity radius of a point x in a hyperbolic surface X, denoted $\operatorname{injrad}_X(x)$, is the radius of the largest isometrically embedded hyperbolic disk centered at x. We first characterize sequences of pointed surfaces converging to $\mathbb{H}^2$ (with a basepoint).

Lemma 4.1.1. *If $\{(X_n, p_n)\}_{n=1}^{\infty}$ is a sequence of pointed hyperbolic surfaces in $\mathcal{H}^2_\bullet$ such that $\operatorname{injrad}_{X_n}(p_n) \to \infty$ as $n \to \infty$, then the limit of the sequence exists and it is $(\mathbb{H}^2, z_0)$ for any (equivalent) choice of basepoint $z_0 \in \mathbb{H}^2$.*

Proof. For each $n \in \mathbb{N}$, let $R_n = \mathrm{injrad}_{X_n}(p_n)$. Then, any choice of isometry between $B_{X_n}(p_n, R_n)$ and $B_{\mathbb{H}^2}(z_0, \mathbb{H}^2)$ gives a $(0, R_n)$-relation. Since $R_n \to \infty$ by assumption, the result follows. ■

Notation. *For two hyperbolic surfaces X and Y, we write $X \sim Y$ if there exist $p \in X$ and $q \in Y$ such that (X, p) and (Y, q) belong to the same path component of $\mathcal{H}^2_\bullet$. Clearly, $\sim$ is reflexive and symmetric.*

Fact. *$\sim$ is also transitive.*

Proof. Suppose $X \sim Y$ and $Y \sim Z$. That is, (X, x) and (Y, y) are in some path component C_1 of $\mathcal{H}^2_\bullet$ and (Y, y') and (Z, z) in some path component C_2 for some choices of $x \in X$, $y, y' \in Y$, and $z \in Z$. By lemma 3.1.1, a path joining y to y' in Y gives a path between (Y, y) and (Y, y') in $\mathcal{H}^2_\bullet$, so $C_1 = C_2$ and $X \sim Z$. ■

Theorem 4.1.2. *The space $\mathcal{H}^2_\bullet$ is path-connected.*

Proof. Our strategy is to show that $X \sim \mathbb{H}^2$ for any hyperbolic surface X. We are done if X is simply connected, in which case X is isometric to $\mathbb{H}^2$. If X is topologically a cylinder, then X is isometric to a quotient of $\mathbb{H}^2$ by some cyclic subgroup generated by either a single parabolic isometry or a single hyperbolic isometry in $\mathrm{PSL}_2(\mathbb{R})$. In either case, we can move the basepoint along a path which increases the injectivity radius without bound. Using lemma 4.1.1, we see that $X \sim \mathbb{H}^2$.

Now, we can suppose that X admits a (possibly empty) geodesic pants decomposition $\mathcal{P}$ as in Theorem 2.1.2 such that there is a component Y of $X - \mathcal{P}$ whose closure is a generalized geodesic pair of pants. Let Y_0 be the geodesic pair of pants with three cusps. Without changing the path component of (X, p) in $\mathcal{H}^2_\bullet$, we may move the basepoint (lemma 3.1.1) and assume that $p \in Y \subset X$. Applying lemma 3.3.3 to shrink all boundary components of Y to punctures, we obtain an path from (X, p) to (Y_0, y) for some choice of $y_0 \in Y_0$. Thus, $X \sim Y_0$.

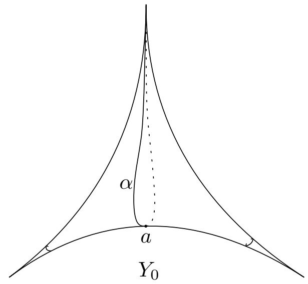

FIGURE 4.1: The pair of pants Y_0 with three cusps and our choice of a bi-infinite geodesic.

It remains to show that $Y_0 \sim \mathbb{H}^2$. To do this, we choose a simple bi-infinite α geodesic whose ends exit a single cusp of Y_0 (see Figure 4.1) and a point $a \in \alpha$. We also fix a basepoint $y_0 \in Y_0$ away from α. For each $t > 0$, consider the surface $Y_t = \mathsf{Strip}(Y, \alpha, a, t)$ with the basepoint y_t chosen to be the image of y_0 under the canonical isometric embedding $Y_0 \hookrightarrow Y_t$. By Proposition 3.4.2, $t \to (Y_t, y_t)$ is a continuous path from $t = 0$ to $t = 1$, which implies that $Y_0 \sim Y_1$. Now, since Y_1 is a hyperbolic surface that contains a funnel, there is a ray $\gamma : R_{\geq 0} \to Y_1$ starting y_1 that goes off the funnel such that the $\text{injrad}_{Y_1}(\gamma(t))$ eventually increases without bound. Thus, $(Y_1, \gamma(t)) \to (\mathbb{H}^2, z_0)$ by lemma 4.1.1, and so $Y_0 \sim Y_1 \sim \mathbb{H}^2$. This finishes the proof. ∎

4.2 Local path-connectivity

We first recall the notion of a weakly locally path-connected space from point-set topology, see [38]. Its relevance to our purpose will be apparent in the fact that follows. We thank Nir Lazarovich and Arielle Leitner for helpful conversations and the idea of this approach.

Definition 4.2.1. A topological space $\mathcal{X}$ is *weakly locally path-connected* at $x \in \mathcal{X}$ if for every open set V containing x there exists an open neighborhood $U \subset V$ of x such that any two points in U are on some path component of V.

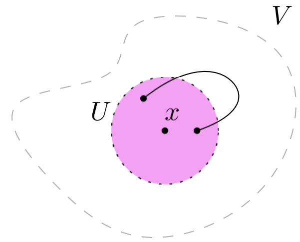

FIGURE 4.2: Two points in the smaller open set U are in a path-component (in this case, a path) of the larger open set U.

Fact. *If X is weakly locally path-connected at every point $x \in X$, then X is locally path-connected.*

Proof. Let $x \in X$. For any open set V containing x, consider a path-component C_V of V which contains x. Then, there is an open neighborhood $U_x \subset V$ such that any two points in U_x lie in some path-component of V. In particular, $U_x \subset C_V$. Thus, C_V is open and we are done. ∎

Towards the goal proving local path-connectivity of $\mathcal{H}^2_\bullet$, we will work with this point of view. The remainder of this chapter is devoted to the proof of the second main theorem below:

Theorem 4.2.2. *The space $\mathcal{H}^2_\bullet$ is weakly locally path-connected at the following points:*

1) $(\mathbb{H}^2, z_0)$ *for any choice of* z_0

2) (X, p) *where X is of the first kind (in fact, it is locally connected here).*

Each case will be examined separately in subsequent subsections.

4.2.1 Weakly local path connectivity at $(\mathbb{H}^2, z_0)$

Fix once and for all a basepoint $z_0 \in \mathbb{H}^2$. Any choice of z_0 is equivalent as $\mathrm{Isom}(\mathbb{H}^2)$ acts transitively on $\mathbb{H}^2$.

Definition 4.2.3. For each $r > 0$, we let $\mathcal{U}(r) = \{(X, x) \in \mathcal{H}^2_\bullet : \mathrm{injrad}_X(x) > r\}$.

Fact. *The collection* $\{\mathcal{U}(r) : r > 0\}$ *is a basis of neighborhoods of* $(\mathbb{H}^2, z_0)$ *in* $\mathcal{H}^2_\bullet$.

Proof. The fact that each $\mathcal{U}(r)$ is open follows from continuity of the injectivity radius function on $\mathcal{H}^2_\bullet$ (lemma I.3.2.6 in [16]). It is clear that any open set containing $(\mathbb{H}^2, z_0)$ defined using quasi-isometries (see Definition 2.2.7) contains $\mathcal{U}(r)$ for some $r > 0$. ■

The proof of weakly local connectivity below will use these basis sets $\mathcal{U}(r)$. It will also rely on two key facts, which we state as lemma 4.2.4 and proposition 4.2.5 below.

By an infinite polygon, we refer to a convex region in $\mathbb{H}^2$ bounded by a finite or countable family of pairwise disjoint infinite geodesics.

Lemma 4.2.4. *Let X be a non-compact complete hyperbolic surface that is not topologically a disk or a cylinder. Then, there exists a collection $\mathcal{A}$ of mutually disjoint, pairwise non-homotopic, simple,proper infinite geodesics such that each component of $X - \mathcal{A}$ is an open infinite polygon.*

Proof. Let S be a topological surface homeomorphic to the convex core $\mathsf{CC}(X)$, say with a marking $f : S \to \mathsf{CC}(X)$. By an algorithm of Bavard and Walker (lemma 2.3.2 in [8]), we can produce a collection of mutually disjoint essential proper arcs $\mathcal{W}$ in S that cuts S into two simply connected pieces. Consider the collection $f(\mathcal{W})$ in X and extend each arc with an endpoint on $\partial\mathsf{CC}(X)$ into the adjacent funnels or half-planes in X. By the same argument as in lemma A.0.1 of the same paper, each proper arc (after extension) in $f(\mathcal{W})$ can be isotoped to a proper infinite geodesic (this representative is not unique unless the arc joins two cusps). Set $\mathcal{A}$ to be the set of geodesics representatives (one for each) of the arcs in $f(\mathcal{W})$. The result follows since $X - \mathcal{A}$ is a union of two infinite polygons. ■

Proposition 4.2.5. *Let X be a complete hyperbolic surface with no cusps and $x \in X$. For each $r > 0$, there exist $R' > R > r$ such that if* $\operatorname{injrad}_X(x) > R'$*, then there exists a separating simple closed geodesic α such that $d(y, \alpha) > r$ for some $y \in B(x, R)$ and* $\operatorname{injrad}_X(y) > r$ *with $B_X(y, r) \subset B_X(x, R)$. For practical purposes, we may choose $R = 4(r + 5)$ and $R' = 9R$.*

Proof. We postpone a proof, which is somewhat computationally involved, until after the next proposition for readability. ∎

The main result of this section is:

Proposition 4.2.6. *$\mathcal{H}^2_\bullet$ is weakly locally connected at $(\mathbb{H}^2, z_0)$.*

Proof. Our strategy will be to prove that for any $r > 0$, there exists $R' > r$ (whose value is determined in Proposition 4.2.5) such that any pointed surface (X, p) in $\mathcal{U}(R') \subset \mathcal{U}(r)$ lies in the same path component as $(\mathbb{H}^2, z_0)$ in the larger open set $\mathcal{U}(r)$. In fact, we will produce a path from (X, p) to (or limiting to) $(\mathbb{H}^2, z_0)$ that is contained entirely within $\mathcal{U}(r)$.

Given $r > 0$, assume that we have already chosen such an $R' > 0$. Let $(X, p) \in \mathcal{U}(R')$. If X is simply connected, we are done since $X \cong \mathbb{H}^2$. If X is a quotient $\mathbb{H}^2/\langle g \rangle$, where g is either a parabolic or a hyperbolic isometry, then we can simply move the basepoint along a path in X that strictly increases the injectivity radius without bound, and lemma 4.1.1 implies that such a path limits to $(\mathbb{H}^2, z_0)$.

Now, let us suppose that X is non-compact and is none of the surfaces already considered. We can work with the assumption that $\operatorname{injrad}_X(p) > r$ here. Choose a finite or countable collection $\mathcal{A} = \{\alpha_n\}_{n \in \mathbb{N}}$ of proper infinite geodesics as in lemma 4.2.4. By moving the basepoint slightly, we can assume that p is not in $\bigcup \alpha_n$. For each $n \in \mathbb{N}$, let a_n be the point on α_n closest to p. Define a path $\Psi : \mathbb{R}_{\geq 0} \to \mathcal{H}^2_\bullet$ piecewise as follows. From time $t = 0$ to $t = 1$, let $X_t = \mathsf{Strip}(X, \alpha_1, a_1, t)$ be obtained by adding the infinite strip of width t along α_1, see Definition 3.4.2 for how to perform this construction. Once X_t has already been defined for $t \in [0, n]$, we set

$$X_t = \mathsf{Strip}(X, (\alpha_1, \ldots, \alpha_n, \alpha_{n+1}), (a_1, \ldots, a_n, a_{n+1}), (t, t-1, \ldots, t-n))$$

from time $t = n$ to $t = n + 1$. In other words, we increase the width of each strip linearly, but we only start inserting a strip along α_n at time $t = n-1$. At all time t, we pick the basepoint $p_t \in X_t$ to be the image under the canonical isometric embedding $X - \bigcup \alpha_n \hookrightarrow X_t$, and let $\Psi(t) = (X_t, p_t)$. Since we insert only finitely many strips at any given time, the path Ψ is continuous by the remark following proposition 3.4.3.

Claim. *For any $M > 0$, there exists $T > 0$ such that* $\operatorname{injrad}_{X_t}(p_t) > M$ *for all $t > T$. That is,* $\operatorname{injrad}_{X_t}(p_t) \to \infty$ *as $t \to \infty$.*

Proof of Claim. Let $M > 0$. Since the action of $\pi_1(X)$ on $\mathbb{H}^2$ is properly discontinuous, there are only finitely many simple geodesic loops in X based at p with length at most $2M$, say $b_1, \ldots, b_K$. Each of these loops must cross some geodesics in $\mathcal{A}$, for each component of $X - \mathcal{A}$ is simply connected. So, there is $N \in \mathbb{N}$ such that

$$b_j \cap (\alpha_1 \cup \ldots \cup \alpha_N) \neq \emptyset$$

for all $j = 1, \ldots, K$. Set $T = N + 2M$. If $t > T$, consider a shortest geodesic loop c_t in X_t based at p_t so that $\ell_{X_t}(c_t) = 2 \operatorname{injrad}_{X_t}(p_t)$. There are two possibilities:

Case 1. if c_t crosses any of the strips added along $\alpha_1, \ldots, \alpha_N$ in X_t, then $\ell_{X_t}(c_t) > 2M$ since all these strips are wider than $2M$ in X_t. Notice that if c_t intersects the inserted strip along α_i, then it must cross both boundary components of the strip, since there would be a geodesic bigon otherwise, which is impossible.

Case 2. c_t avoids all the strips inserted along $\alpha_1, \ldots, \alpha_N$. Consider the strip collapsing map $f : X_t \to X$ defined by mapping the complement of the added strips isometrically onto $X - \mathcal{A}$, and projecting a point on each strip onto its core along equidistant arcs to the waist. The image $f(c_t)$ is a piecewise geodesic loop in X based at $f(p_t) = p$, which does not intersect any geodesics in $\{\alpha_1, \ldots, \alpha_N\}$. Let c be the geodesic representative of $f(c_t)$ relative to p. Then, $c \cap (\alpha_1 \cup \ldots \cup \alpha_N) = \emptyset$ and, in particular, $c \neq b_j$ for any $j = 1, \ldots, K$. Thus,

$$\ell_{X_t}(c_t) \geq \ell_X(f(c_t)) \geq \ell_X(c) > 2M,$$

where the first inequality holds since f is length non-increasing (Proposition 2.2 in [30]).

In any possibility, it follows that $\text{injrad}_{X_t}(p_t) = \frac{1}{2}\ell_{X_t}(c_t) > M$ for all $t > T$ and we have established the claim. ■

Applying lemma 4.1.1, $(X_t, p_t) \to (\mathbb{H}^2, z_0)$ as $t \to \infty$. Furthermore, Proposition 2.2 in [30] implies that inserting strips does not decrease the injectivity radius at any given point, so $\text{injrad}_{X_t}(p_t) \geq \text{injrad}_X(p) > r$. Hence, Ψ is indeed a path in $\mathcal{U}(r)$ joining (X, p) and $(\mathbb{H}^2, z_0)$ when X is non-compact.

It then remains to consider when X is compact with $\text{injrad}_X(p) > R'$. By Proposition 4.2.5, there exists a point $y \in B_{R'}(X)$ with $\text{injrad}_X(q) > r$ as well as a separating simple closed geodesic α with $d(q, \alpha) > r$. Since $B_X(q, r) \subset B_X(p, R')$, there is a continuous path within $B_X(p, R')$ from p to q passing through points whose injectivity radii never go below r. By lemma 3.1.1, we can move from (X, p) to (X, q) while still remaining in $\mathcal{U}(r)$. Choose a simple closed geodesic γ in the component of $X - \alpha$ that does not contain y. By lemma 3.3.3, we have a path from (X, q) to a cusped surface in which γ is shrunk to have length 0. Finally, $B_X(q, r)$ is contained entirely in a different component in $X - \alpha$ than the shrunken curve γ, so the basepoint always has injectivity radius at least r. This shows that there is a path from (X, p) to a a cusped surface within $\mathcal{U}(r)$. We now return to the non-compact case and proceed as above. This finishes our proof. ∎

4.2.1.1 Towards the Proof of Proposition 4.2.5

We are now left to show Proposition 4.2.5. First, we need a lemma about the width of a collar neighborhood of a shortest simple closed geodesic on a surface.

Lemma 4.2.7. *Let X be a complete hyperbolic surface with no cusp. If γ is a shortest simple closed geodesic in X, then $N_{\ell(\gamma)/4}(\gamma) = \{x \in X : d(x, \gamma) < \ell(\gamma)/4\}$ is an embedded collar neighborhood of γ in X.*

Proof. Let m be the maximal width for which $N_m(\gamma)$ is an embedded cylinder in X. Then, $\overline{N_m(\gamma)}$ intersects itself, say at a point p. There are two geodesic arcs η_1, η_2 emanating from p and ending on γ which meet γ perpendicularly on different sides. Let η be the geodesic arc homotopic to the concatenation $\eta_1 \cup \eta_2^{-1}$, relative the endpoints. Then, η separates γ into two subarcs. Let γ' be the shorter arc of the two. Consider the closed curve α obtained by concatenating η and γ'. Then, α is simple and essential (for otherwise it would bound a geodesic bigon, which is impossible). Moreover, since X does not have any cusp, α does not bound a punctured disk. Let α' be the geodesic representative in the homotopy class of α. It follows that

$$\ell(\alpha') \leq \ell(\eta) + \ell(\gamma') \leq 2m + \ell(\gamma)/2.$$

But since γ is a shortest simple closed geodesic in X, $\ell(\alpha') \geq \ell(\gamma)$, and so it must be the case that $m \geq \ell(\gamma)/4$. ∎

Remark 4. We can prove a similar statement when a surface X with no cusps has no shortest simple closed geodesic, but has a non-zero lower bound on the lengths.

Define

$$\ell_0 := \inf\{\ell(\gamma) : \gamma \text{ is a simple closed geodesic in } X\}.$$

If $\ell_0 > 0$, then the same argument in the proof of lemma 4.2.7 above can be used to show that a simple closed geodesic of length $\ell_0 + \epsilon$ admits a collar neighborhood of width $\frac{1}{4}(\ell_0 - \epsilon)$ for a sufficiently small $\epsilon > 0$.

We turn our attention now to some computations involving hyperbolic trigonometry. These will be used to show that the geodesic representative of a path, which is a certain concatenation of geodesic arcs, stays within a bounded distance from the path. To simplify notations in what follows, for any geodesic arc β, we write β instead of $\ell(\beta)$ as the argument for any hyperbolic trigonometric function, e.g. $\sinh \beta := \sinh(\ell(\beta))$.

Lemma 4.2.8. *Consider a geodesic quadrilateral with two right angles with the sides labeled counterclockwise as α, c, β, and a such that α is adjacent to both right angles. See Figure 4.3 below. If $\ell(\alpha) \geq 1$, then there exists a constant $C_1 \approx 3.729$ such that $\alpha \subset N_{C_1}(\beta)$.*

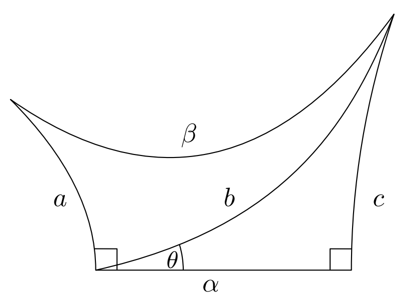

FIGURE 4.3

Proof. Let b be a geodesic segment joining the vertex where a meets α and the vertex where β meets c, and let θ be the angle between α and b. It is well-known that geodesic triangles in the hyperbolic plane are δ-thin for $\delta = \log(1 + \sqrt{2}) \approx 0.881$. So, we conclude that $\alpha \subset N_\delta(b \cup c)$. Since $\measuredangle(\alpha, c) = \pi/2$, any point $x \in \alpha \cap N_\delta(c)$ is within distance δ of the complement $(\alpha - \alpha \cap (N_\delta(c))) \subset N_\delta(b)$. Thus, $\alpha \subset N_{2\delta}(b)$.

We again apply the δ-thin condition to obtain that $b \subset N_\delta(a \cup \beta)$. By a hyperbolic identity in a geodesic right triangle (Formula 2.2.6 (vi) in the Formula Glossary of [14]), we have

$$\cos\theta = \tanh\alpha \coth b > \tanh 1,$$

since $\coth b > 1$ and $\tanh\alpha \geq \tanh 1$. Thus, $\theta < \theta_0 := \cos^{-1}(\tanh 1)$. We use this fact to estimate the length of $b \cap N_\delta(a)$. Let d be a geodesic segment of length δ between a and b and orthogonal to a. Consider the geodesic right triangle bounded by d with its opposite angle $\angle(a, b)$, see Figure 4.4.

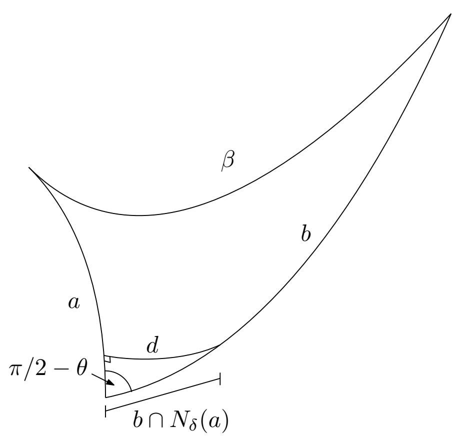

FIGURE 4.4: Since the angle $\measuredangle(a, b) \approx \pi/2$, the length of $b \cap N_\delta(a)$ is bounded. By δ-thinness, it must be that most of b is covered by $N_\delta(\beta)$.

Using Formula 2.2.2 (iii) (*ibid.*), we obtain

$$\begin{aligned}\sinh \ell(b \cap N_\delta(a) &= \sinh d \csc(\angle(a, b)) \\ &\leq \sinh d \csc(\pi/2 - \theta_0).\end{aligned}$$

Setting $\delta' = \operatorname{arcsinh}(\sinh d \csc(\pi/2 - \theta_0)) \approx 1.086$, we conclude that any point $x \in b \cap N_\delta(a)$ is at most δ' away from the complement $(b - b \cap N_\delta(a)) \subset N_\delta(\beta)$. Thus,

$b \subset N_{\delta+\delta'}(\beta)$. Together with the fact that $\alpha \subset N_{2\delta}(b)$ from the first paragraph, it follows that $\alpha \subset N_{3\delta+\delta'}(b)$. ■

Lemma 4.2.9. *For any geodesic right-angled hyperbolic pentagon with the sides labeled counterclockwise as α, a, b, c, d, if $\ell(\alpha) \geq 1$, then there exists a constant $K \approx 4.61$ such that $\alpha \subset N_K(b \cup c)$.*

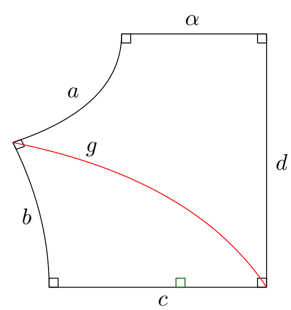

FIGURE 4.5

Proof. Let g be the geodesic segment joining the vertex where a meets b and the vertex where c meets d. Then, since the geodesic triangle with sides b, c, and g are δ-thin (for $\delta = \log(1+\sqrt{2})$), $g \subset N_\delta(b \cup c)$. Combining with lemma 4.2.8, we have that $\alpha \subset N_{C_1+\delta}(b \cup c)$. ■

Lemma 4.2.10. *Let Q be a hyperbolic one-holed torus with geodesic boundary α. Let b be a simple closed geodesic in Q and c be the common perpendicular of the boundary components of the closure of $Q \setminus b$ which are not α.*

If $\ell(\alpha) \geq 4$, then $\alpha \subset N_K(b \cup c)$, where $K \approx 4.61$ is from Lemma 4.2.9 above.

Proof. Cutting Q along b, c, and the two common perpendiculars of α and b results in two isometric geodesic right-angled hexagons. Divide each hexagon into two isometric right-angled pentagons by the common perpendicular of α and c, as in Figure 4.6 below.

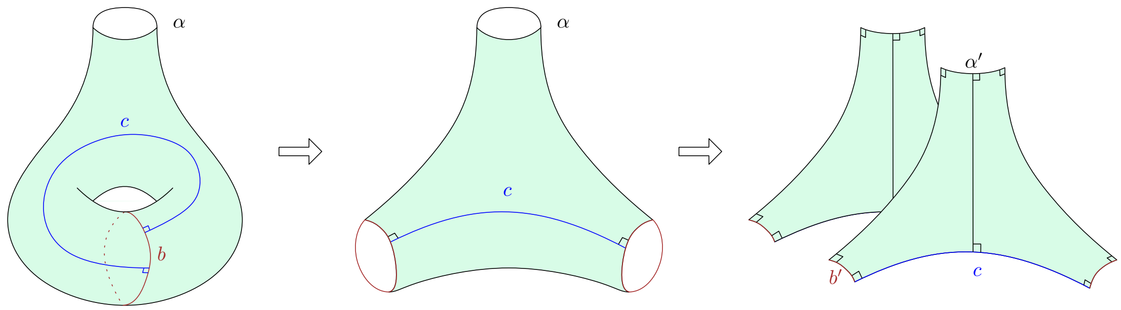

FIGURE 4.6: Decomposing a one-holed torus into four right-angled geodesic pentagons

Here, α', b', and c' denote the sides of each pentagon that are geodesic subarcs of α, b, and c, respectively. By assumption, $\ell(\alpha') \geq 1$, so lemma 4.2.9 implies that $\alpha' \subset N_K(b' \cup c')$ in each pentagon. Putting together the four pentagons that make up Q, we conclude that $\alpha \subset N_K(b \cup c)$. ■

We are now ready to prove Proposition 4.2.5. Since we only deal with a single surface X, we drop the subscripts and write $\ell(c) = \ell_X(c)$ for any geodesic arc c and $B(x, R) = B_X(x, R)$.

Proof of Proposition 4.2.5 . Before we prove the proposition, we record two easy facts which we use repeatedly throughout the proof.

Fact 1. For any isometrically embedded disk $B(x, R)$, if η is a geodesic segment joining two points on $\partial B(x, R)$, then there exists $y \in B(x, R)$ such that $B(y, R/2) \subset B(x, R)$ and $\eta \cap B(y, R/2) = \emptyset$. This is because one component of $B(x, R) - \eta$ must contain an open half disk.

Fact 2. Any simple closed geodesic crossing $B(x, R)$, where $\mathrm{injrad}(x) > R' > R$, must contain at least two subarcs connecting $\partial B(x, R)$ and $\partial B(x, R')$, which implies that its length must be more than $2(R' - R)$.

Set $R = 4(r + K)$ and $R' = 9R$ where $K \approx 4.61$ is the constant from lemma 4.2.9.

We will show that these values of R and R' work, though they are not claimed to be optimal. Our proof can be broken down into three main claims as follows.

Claim 1. There exists a simple closed geodesic b such that there is $z \in B(x, R)$ satisfying $B(z, R/2) \subset B(x, R)$ and $d(z, b) > R/2$.

Proof of Claim 1. Using Fact 2, we may assume that

$$\ell_0 := \inf\{\ell(\gamma) : \gamma \text{ is a simple closed geodesic in } X\} \geq 2(R' - R) = 16R$$

since otherwise we are done, as there is a simple closed geodesic disjoint from $B(x, R)$. Consider the following cases.

Case 1: ℓ_0 is realized by a simple closed geodesic b. If b is disjoint from $B(x, R)$, we are done. If not, b has an embedded collar neighborhood of width $4R$ by lemma 4.2.7. In particular, b only crosses $\cap B(x, R)$ in one component. By Fact 1, we can find $z \in B(x, R)$ with $B(z, R/2) \subset B(x, R)$ and $b \cap B(z, R/2) = \emptyset$.

Case 2: ℓ_0 is not achieved. Let $\epsilon < \max\{1, \ell_0 - 16R\}$ be such that there is a simple closed geodesic b in X with length $\ell_0 + \epsilon$. Again, we are done if $b \cap B(x, R) = \emptyset$. So, suppose that b goes through $B(x, R)$. By the remark following lemma 4.2.7, b has a collar of width $\frac{1}{4}(\ell_0 - \epsilon) \geq 4R - 1/4 > 3R$. This implies that $b \cap B(x, R)$ has exactly one component. We can now similarly apply Fact 1 to find a point z in the claim. ∎

If such a closed geodesic b in Claim 1 is separating, the lemma is proved. We now assume that b is non-separating. Cut X along b to create a surface Y with two geodesic boundary components. We identify the interior of Y with $X - b$. Let c be the shortest simple geodesic arc joining the two boundary components of Y. Note that c is necessarily the unique common perpendicular between the boundary components.

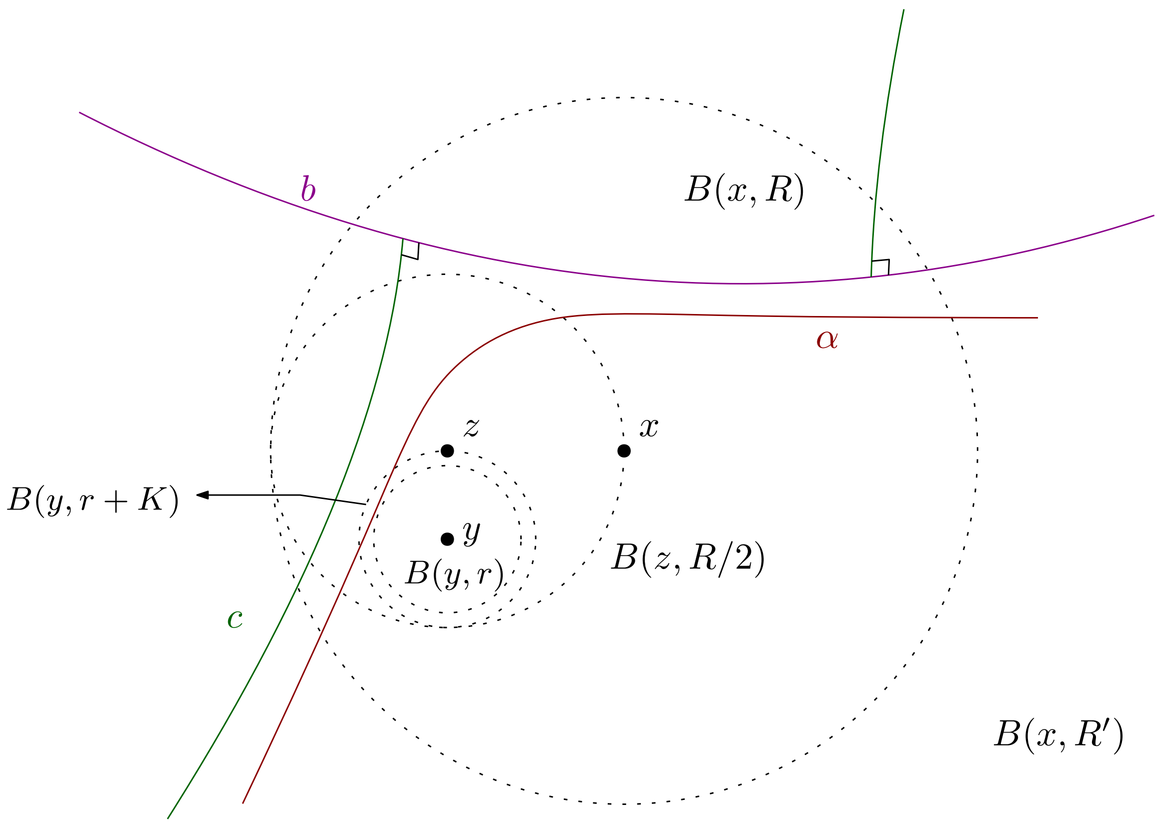

FIGURE 4.7: A schematic for the proof of proposition 4.2.5. The goal is to find a simple separating closed geodesic in a closed surface far away from the basepoint. After possibly moving the basepoint to z, we locate a simple closed geodesic b at least $R/2$ away from z. If b is not separating, we again possibly move the basepoint to y and find a geodesic arc c joining the two boundary components of $X - b$. The geodesic representative of the band sum of the arc c with b stays far from y.

Claim 2. $c \cap B(z, R/2)$ has at most one component.

Proof of Claim 2. Suppose not. Parametrize c by unit speed. Then, there are $0 < t_1 < t_2 < t_3 < t_4 < \ell(c)$ such that the geodesic subarcs $c|_{(t_1,t_2)}$ and $c|_{(t_3,t_4)}$ are disjoint components of $c \cap B(z, R/2)$ and that the geodesic arc η joining $c(t_1)$ and $c(t_4)$ is disjoint from c, except at the endpoints.

In this case, $c|_{[t_1,t_4]}$ must contain a subarc joining $\partial B(x,R)$ and $\partial B(x,R')$. Thus, $\ell(c|_{[t_1,t_4]}) > R' - R = 8R > R > \ell(\eta)$. Consider the piecewise geodesic path c' in

Y obtained by concatenating $c|_{[0,t_1]} \cdot \eta \cdot c|_{[t_4,\ell(c)]}$. Then, c' is a simple and essential arc joining the two boundary components of Y. The geodesic representative γ in the homotopy class of c' has length

$$\ell(\gamma) \leq \ell(c') = \ell(c|_{[0,t_1]}) + \ell(\eta) + (\ell(c|_{[t_4,\ell_0]}) < \ell(c),$$

contradicting that c is the shortest such arc. ■

Applying Fact 1 to the result of Claim 2, we have that there exists $y \in B(z, R/2)$ such that $d(y,c) > R/4 = r + K$ and $B(y, r+K) \subset B(z, R/2)$. In particular, the latter shows that $d(y,b) > r + K$. Consider the piecewise geodesic loop a obtained by concatenating together $c \cdot b \cdot c^{-1} \cdot b^{-1}$. Let α be the unique simple closed geodesic homotopic to a in X. Then, α is a simple separating closed geodesic bounding a subsurface $Q \subset X$ diffeomorphic to a one-holed torus.

Claim 3. $d(y,\alpha) > r$.

Proof of Claim 3. We only need to consider when $\alpha \cap B(y,r) \neq \emptyset$. So, by Fact 2, $\ell(\alpha) \geq \ell_0 \geq 16R > 4$. By lemma 4.2.10, $\alpha \subset N_K(b \cup c)$. Since $d(y, b \cup c) > r + K$, it follows that $d(y,\alpha) > r$. ■

In conclusion, we have shown that there is a point $y \in B(z, R/2) \subset B(x,R)$ with $d(y,\alpha) > r$ where α is a separating simple closed geodesic in X. This finishes the proof of Proposition 4.2.5. ■

4.2.2 Local connectivity at (X,p) when X is of the first kind

We now turn our attention to points in $\mathcal{H}^2_\bullet$, whose underlying hyperbolic surfaces are of the first kind. That is, we will consider $(X,p) \in \mathcal{H}^2_\bullet$ when $\mathsf{CC}(X) = X$.

4.2.2.1 Open Sets

In this subsection, $S = S_{g,c,p}$ is a hyperbolizable surface of finite type of genus g, with c punctures and n boundary components (which are all included in S). Fixing a pants decomposition $\mathcal{P}$ and a collection of seams, we will actually view S as a model surface with geodesic boundary. By virtue of Proposition 3.2.2, we may assume that any marked hyperbolic surface X in $\mathcal{T}(S)$ comes equipped with a good marking $\varphi : S \to X$ and that the boundary components of X are given standard parametrization, see Section 3.2.1 for details.

We rely on the material in [15] in setting up the definitions below. For each puncture ξ of X, we fix a cusp neighborhood $\mathcal{C}_\xi$ that is isometric to

$$\{z \in \mathbb{H}^2 : \mathrm{Im}(z) > 1/2\}/\langle z \mapsto z + 1\rangle.$$

If C_ξ is a subset of a $\mathcal{P}$-pair of pants $Y \subset X$, let β be a half-infinite geodesic seam that emanates from a boundary component of Y perpendicularly and exits C_ξ. Then, $\mathcal{C}_\xi$ is foliated by horocycles h_ξ^ϵ of length $0 < \epsilon < 2$, which we parametrize by constant speed on $\mathbb{S}^1$ with positive orientation so that it begins and ends on the seam β.

For each $\epsilon > 0$, define the *ϵ-restriction* of X, denoted by X^ϵ, to be the subsurface of X with the part of C_ξ beyond h_ξ^ϵ is deleted for every puncture (if any) ξ of X. The components of ∂X^ϵ inherit the parametrization from X, which we call the *standard parametrization.*

If X is marked surfaces in $\mathcal{T}(S)$ and Z is a hyperbolic surface with geodesic boundary, then a diffeomorphism between (possibly restricted) surfaces $f : X^\epsilon \to Z^\delta$ is said to be *boundary coherent* if it preserves the standard parametrization of the boundary components. That is, if f sends a boundary curve $\gamma \subset \partial X^\epsilon$ to $\gamma' \subset \partial Z^\delta$, then $f(\gamma)(t) = \gamma'(t)$ in standard parametrization.

Definition 4.2.11. Let X be a marked hyperbolic surface in $\mathcal{T}(S)$ with a good

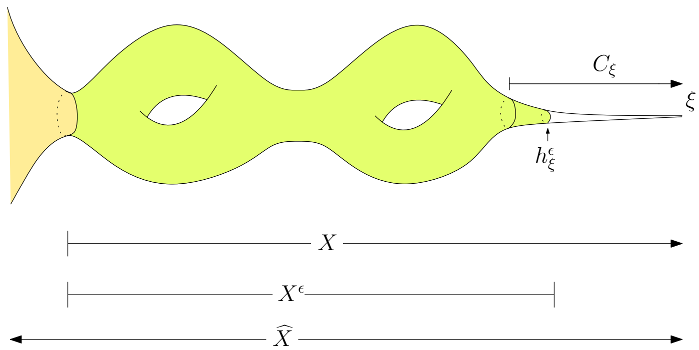

FIGURE 4.8: A finite-type surface with boundary X shown with its ϵ-restriction X^ϵ and its Nielsen extension $\widehat{X}$.

marking φ and $p \in X$. For $\epsilon > 0$ and $K > 1$, let $\mathcal{N}(X, p, \epsilon, K)$ be the set of all $(Z, q) \in \mathcal{H}^2_\bullet$ such that there is a subsurface with geodesic boundary $X_Z \subset Z$, where $q \in X_Z$, together with a smooth embedding $f : (X^\epsilon, p) \to (X_Z, q)$ satisfying

1) $f(X^\epsilon) = X_Z^{\epsilon'}$ for some $\epsilon' > 0$;
2) $f : X^\epsilon \to X_Z^{\epsilon'}$ is a boundary coherent K'-quasi-isometry for some $1 < K' < K$;
3) no three simple closed geodesics in ∂X_Z simultaneously bound a common geodesic pair of pants in $Z - X_Z$.

For brevity, we will refer to the pair $(X_Z; f)$ (or just X_Z), which satisfies 1) and 2), as an X*-like subsurface* in Z, assuming that ϵ and K have been fixed. In the notion of the quasi-isometry distance as in (3.3), condition 2) implies that $d_{\text{q.i.}}(X^\epsilon, X_{Z'}^{\epsilon'}) < \log K$.

Lemma 4.2.12. *The set $\mathcal{N} = \mathcal{N}(X, p, \epsilon, K)$ as defined above is open in $\mathcal{H}^2_\bullet$.*

Proof. To show that $\mathcal{N}$ is open, we argue that requirements 1) and 2) in Definition 2.3 are open conditions and that the negation of 3) is a closed property in $\mathcal{H}^2_\bullet$.

Take $(Z, q) \in \mathcal{N}$. Let $(X_Z; f)$ be an X-like subsurface in Z. Since $f(X^\epsilon) = X_Z^{\epsilon'}$ is compact, by virtue of the quasi-isometry definition of $\mathcal{H}^2_\bullet$, any pointed surface (Z', q') in a small enough $\mathfrak{N}_q = \mathfrak{N}_q(Z, q, \delta, R)$ as in (2.3)—that is, with a sufficiently small δ and a large enough R—contains a geodesic subsurface $X_{Z'} \subset Z'$ so that $d_{\text{q.i.}}(X_Z^{\epsilon'}, X_{Z'}^{\epsilon''}) < \delta$ for some $\epsilon', \epsilon'' > 0$.

We then regard $X_{Z'}$ as a point in $\mathcal{T}(S')$ for an appropriate model surface $S' = S_{g',c',n'}$ such that $g = g'$ and $c + n = c' + n'$. Notice that the model surface S' is obtained from S only by changing some punctures of to boundary components (or vice versa), so the curves in the pants decomposition $\mathcal{P}$ of S together with its punctures can be modified into a decomposition $\mathcal{P}'$ of S' in the obvious way. We can then construct a diffeomorphism $f : X^\epsilon \to X_{Z'}^{\epsilon''}$ by pasting together a combination of Buser's stretch and twist maps that take the (possibly restricted) $\mathcal{P}$-pants in X^ϵ to the (possibly restricted) $\mathcal{P}'$-pants in $X_{Z'}^{\epsilon''}$. By design, f is boundary coherent. (One can find the explicit definitions of these maps in [14, Chapters 3 and 6] and [15].) By fixing a small enough δ in our choice of $\mathfrak{N}_q$ at the beginning, we can guarantee that $X_{Z'}$ is X^ϵ-like. Hence, both 1) and 2) hold on an open subset $\mathfrak{N}_q \subset \mathcal{N}$.

We claim now that the negation $\neg 3)$ is a closed condition. Let $(Z_k, q_k) \to (\bar{Z}, \bar{q})$ be a convergent sequence in $\mathcal{N}$. From the preceding paragraph, we will suppose that $(X_k; f_k)$ are X-like subsurfaces in Z_k, but that 3) fails for all k.

Since there are only finitely many combinations of $c+n$ punctures and boundary components, after passing to a subsequence, we may choose a suitable model surface $S' = S'_{g,c',n'}$ with $c' + n' = c + n$ such that the X_k are marked surfaces in $\mathcal{T}(S')$ with $\varphi_k : S' \to X_k$ as good markings. After restricting to yet another subsequence, we may label by c_1, c_2, and c_3 three simple closed geodesics in $\partial S'$, whose corresponding curves $\varphi_k(c_1)$, $\varphi_k(c_2)$, and $\varphi_k(c_3)$ bound a common geodesic pair of pants $Y_k \subset Z_k - X_k$. As the sequence (X_k, φ_k) converges in the q.i.-distance, the lengths $\ell_{Z_k}(c_j)$ for $j = 1, 2, 3$ are universally bounded. Thus, the diameters of $Y_k \subset Z_k$ are also universally bounded, which implies that there is $R > 0$ such that $(X'_k)^\epsilon = f_k(X^\epsilon) \cup Y_k \subset B_{Z_k}(q_k, R)$ for all k. So, there exist a geodesic subsurface X containing $\bar{q}$ and some $\bar{\epsilon} > 0$ for which

$d_{\text{q.i.}}((X_k')^{\epsilon}, \bar{X}^{\bar{\epsilon}}) \to 0$. In particular, $(X_k')^{\epsilon}$ and $\bar{X}^{\bar{\epsilon}}$ are diffeomorphic. Therefore, for any embedding $f : (X^{\epsilon}, p) \to (\bar{X}, \bar{q})$ satisfying 1) and 2), we have that a component of $\bar{X} - f(X^{\epsilon})$ must be a geodesic pair of pants. This concludes the proof. ∎

4.2.2.2 Path-Connectivity

We let $S = S_{g,n}$ be a model finite-type hyperbolic surface of genus g with n boundary components with a geodesic decomposition $\mathcal{P}$, including the boundary geodesics, and a collection of seams $\mathcal{S}$. Let $\mathcal{P}_b \subset \mathcal{P}$ be the subset of boundary curves. Set

$$\mathcal{FN}(S) = (\mathbb{R}_+ \times \mathbb{R})^{\mathcal{P}-\mathcal{P}_b} \times \mathbb{R}_{\geq 0}^{\mathcal{P}_b}$$

to be the set of possible Fenchel-Nielsen coordinates of a hyperbolic surface based on S, whose boundary curves (those in $\mathcal{P}_b$) are allowed to have length 0 (as punctures). Such a surface has c' cusps and n' boundary components such that $n = c' + n'$.

Recall that we can extend a finite-type hyperbolic surface X with geodesic boundary uniquely to a geodesic completion by attaching a funnel of appropriate length to each boundary component of ∂X. The resulting surface is called the *Nielsen extension* of X, denoted by $\widehat{X}$. There is an obvious isometric embedding $X \to \widehat{X}$, and our convention is to continue to refer to $p \in X$ as a basepoint for $\widehat{X}$ so that $(\widehat{X}, p) \in \mathcal{H}^2_{\bullet}$.

Proposition 4.2.13. *Each open set $\mathcal{N} = \mathcal{N}(X, p, \epsilon, K)$ defined as in Definition 4.2.11 is path-connected for any K sufficiently close to* 1.

Proof. To show that $\mathcal{N}$ is path-connected, we begin with any $(Z, q) \in \mathcal{N}$ and show that it belongs to the same path component in $\mathcal{N}$ as $(\widehat{X}, p)$, where $\widehat{X}$ is the Nielsen extension of X, for clearly $(\widehat{X}, p) \in \mathcal{N}$. The length and twist parameters of X are recorded as $\mathbf{b}(\text{ase}) = FN(X) \in \mathcal{FN}(S)$. Let $(X_Z; f)$ be an X-like subsurface in Z so that no three components of ∂X_Z it bound a common geodesic pair of pants in $Z - X_z$. Since there are pointed surfaces in $\mathcal{H}^2_{\bullet}$ arbitrarily close to (Z, q) whose

X-like subsurfaces have no cusp and for which 3) holds, we may assume that X_Z has no cusp by Proposition 3.3.3. Note that this is equivalent to moving backwards in time before these curves degenerate. Regarding X_Z as a marked surface of S, we let $\mathbf{a} = FN(X_Z) \in \mathcal{FN}(S)$ be its Fenchel-Nielsen parameters.

A path in $\mathcal{N}$ from (Z, q) limiting to $(\widehat{X}, p)$ can be constructed in steps as follows.

Step 1. *Make X_Z isometric to X.* By Proposition 3.2.4, the q.i. distance on $\mathcal{T}(S)$ is compatible with the topology of $\mathcal{FN}(S)$. Consider the open subset

$$U_K = \left\{ \mathbf{v} \in \mathcal{FN}(S) : \begin{array}{l} \text{if } X' \in \mathcal{T}(S) \text{ with } FN(X') = \mathbf{v}, \\ \text{then } d_{\text{q.i.}}(X^\epsilon, X') < \log K \end{array} \right\}$$

for some $K > 0$. By decreasing K (hence shrinking U_K) as necessary, we may assume that U is connected. By assumption, $\mathbf{a}, \mathbf{b} \in U$. Let $t \mapsto \mathbf{a}_t$, for $t \in [0, 1]$, be a continuous path from $\mathbf{a}$ to $\mathbf{b}$ in U. By Proposition 3.2.6, there is a continuous path in $\mathcal{H}^2_\bullet$ from (Z_t, q_t), whose X-like subsurface $X_t \subset Z_t$ has $FN(X_{Z_t}) = \mathbf{a}_t$. Our choices of U and $\mathbf{a}_t$ also ensure that all (Z_t, q_t) stay in $\mathcal{N}$. After moving the basepoint slightly, we can now work with (Z_1, q_1), where the X-like subsurface (X_{Z_1}, q_1) and (X, p) are isometric as pointed surfaces.

Step 2. *Pinch pants curves outside of X_{Z_1} to cusps.* Let $\mathcal{Q}$ be the geodesic pants decomposition of the interior of $Z_1 - X_{Z_1}$. Using Proposition 3.3.3, we create a continuous path from (Z_1, q_1) to $(Z_{\mathcal{Q}}, q_1)$, which is obtained by shrinking all simple closed geodesics in $\mathcal{Q}$ to cusps, while keeping the X-like subsurface X_{Z_1} unchanged and viewing q_1 still as the basepoint. Thus, this continuous path lies entirely in $\mathcal{N}$. We let $X_{\mathcal{Q}} \subset Z_{\mathcal{Q}}$ be the X-like subsurface in this construction.

Step 3. *Cap off each boundary of $X_\mathcal{Q} \subset Z_\mathcal{Q}$ with a funnel to get $\widehat{X}$.* Note first that X and $X_\mathcal{Q}$ are isometric, and no component of $Z_\mathcal{Q} = X_\mathcal{Q}$ is a geodesic pair of pants whose all three boundary components come from $\partial X_\mathcal{Q}$. We are now in the position to apply Proposition 3.4.6. This creates a path from $(Z_\mathcal{Q}, q_1)$ to $(\widehat{X}_\mathcal{Q}, q_1)$ along which the X-like subsurfaces remain unchanged. Finally, $(\widehat{X}_\mathcal{Q}, q_1)$ is isometric as a pointed surface to $(\widehat{X}, p)$

Concatenating these subpaths, we create a continuous path from (Z, q) to $(\widehat{X}, p)$ that lies entirely in $\mathcal{N}$. This concludes the proof. ∎

We illustrate the proof of Proposition 4.2.13 in Figure 4.9 below.

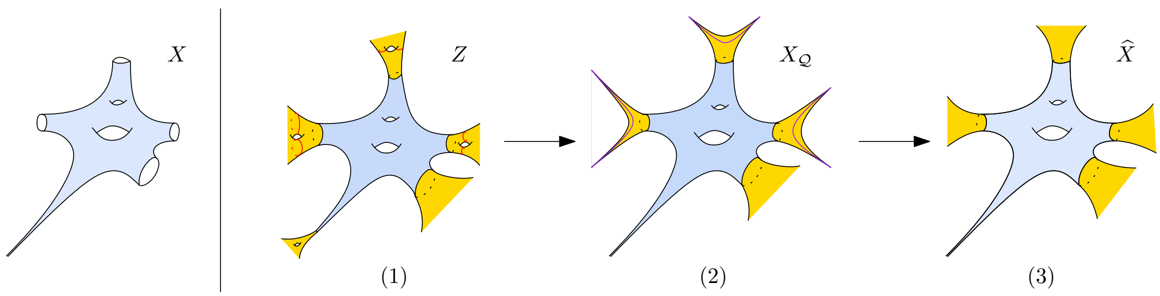

Figure 4.9: (1) We begin with a surface Z with an X-like subsurface X_Z. (2) After making X_Z isometric to X, we pinch the pants curves outside (in red) to obtain $X_\mathcal{Q}$ by Proposition 3.3.3. (3) Via inserting and widening strips along the infinite geodesics joining the new cusps of $X_\mathcal{Q}$ (in purple), this path limits to $\widehat{X}$ by Proposition 3.4.6. We keep track of all X-like subsurfaces in blue.

Proposition 4.2.14. *If X is a hyperbolic surface of the first kind, then $\mathcal{H}^2_\bullet$ is locally path-connected at (X, p) for any choice of $p \in X$.*

Proof. Since X is of the first kind, X admits a geodesic pants decomposition with no funnel or half-plane components. Let $X_1 \subset X_2 \subset \cdots$ be an exhaustion of X by finite-type surfaces with geodesic boundary such that $p \in X_1$. Consider the bases $\{\mathcal{N}(X_n, p, \epsilon, K)\}$as we range $\epsilon > 0$, $K > 1$, and $n \in \mathbb{N}$. Each of these sets is path-connected by Proposition 4.2.13, and we are done. ∎

Bibliography

[1] Norbert A'Campo, Lizhen Ji, and Athanase Papadopoulos. On the early history of moduli and Teichm{ü}ller spaces. *arXiv e-prints*, page arXiv:1602.07208, February 2016.

[2] Daniele Alessandrini, Lixin Liu, Athanase Papadopoulos, and Weixu Su. On various Teichmüller spaces of a surface of infinite topological type. *Proc. Amer. Math. Soc.*, 140(2):561–574, 2012.

[3] Daniele Alessandrini, Lixin Liu, Athanase Papadopoulos, Weixu Su, and Zongliang Sun. On Fenchel-Nielsen coordinates on Teichmüller spaces of surfaces of infinite type. *Ann. Acad. Sci. Fenn. Math.*, 36(2):621–659, 2011.

[4] Venancio Álvarez and José M. Rodríguez. Structure theorems for Riemann and topological surfaces. *J. London Math. Soc. (2)*, 69(1):153–168, 2004.

[5] Hyungryul Baik and Lucien Clavier. The space of geometric limits of one-generator closed subgroups of $\mathrm{PSL}_2(\mathbb{R})$. *Algebr. Geom. Topol.*, 13(1):549–576, 2013.

[6] Ara Basmajian. Hyperbolic structures for surfaces of infinite type. *Trans. Amer. Math. Soc.*, 336(1):421–444, 1993.

[7] Ara Basmajian and Dragomir Šarić. Geodesically complete hyperbolic structures. *Math. Proc. Cambridge Philos. Soc.*, 166(2):219–242, 2019.

[8] Juliette Bavard and Alden Walker. Two simultaneous actions of big mapping class groups. *arXiv e-prints*, page arXiv:1806.10272, Jun 2018.

[9] Riccardo Benedetti and Carlo Petronio. *Lectures on Hyperbolic Geometry*. Springer-Verlag, Berlin, 1992.

[10] Lipman Bers. Spaces of Riemann surfaces as bounded domains. *Bulletin of the American Mathematical Society*, 66(2):98 – 103, 1960.

[11] Martin R. Bridson, Pierre de la Harpe, and Victor Kleptsyn. The Chabauty space of closed subgroups of the three-dimensional Heisenberg group. *Pacific J. Math.*, 240(1):1–48, 2009.

[12] Jeffrey F. Brock, Richard D. Canary, and Yair N. Minsky. The classification of Kleinian surface groups, II: The ending lamination conjecture. *Ann. of Math. (2)*, 176(1):1–149, 2012.

[13] Dmitri Burago, Yuri Burago, and Sergei Ivanov. *A Course in Metric Geometry*. American Mathematical Society, Providence, RI, 2001.

[14] Peter Buser. *Geometry and spectra of compact Riemann surfaces*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2010. Reprint of the 1992 edition.

[15] Peter Buser, Eran Makover, Björn Mützel, and Robert Silhol. Quasiconformal embeddings of y-pieces. *Computational Methods and Function Theory*, 14(2):431–452, Oct 2014.

[16] R. D. Canary, D. B. A. Epstein, and P. L. Green. Notes on notes of Thurston [mr0903850]. In *Fundamentals of hyperbolic geometry: selected expositions*, volume 328 of *London Math. Soc. Lecture Note Ser.*, pages 1–115. Cambridge Univ. Press, Cambridge, 2006. With a new foreword by Canary.

[17] Jeffrey Danciger, François Guéritaud, and Fanny Kassel. Margulis spacetimes via the arc complex. *Invent. Math.*, 204(1):133–193, 2016.

[18] Pierre de la Harpe. Spaces of closed subgroups of locally compact groups. https://arxiv.org/abs/0807.2030, 2008. Online; accessed 16 July 2021.

[19] Norman Do. Moduli spaces of hyperbolic surfaces and their weil-petersson volumes, 2011.

[20] David A. Edwards. The structure of superspace. In *Studies in topology (Proc. Conf., Univ. North Carolina, Charlotte, N. C., 1974; dedicated to Math. Sect. Polish Acad. Sci.)*, pages 121–133, 1975.

[21] Benson Farb and Dan Margalit. *A primer on mapping class groups*, volume 49 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2012.

[22] Mikhael Gromov. Groups of polynomial growth and expanding maps. *Inst. Hautes Études Sci. Publ. Math.*, (53):53–73, 1981.

[23] Misha Gromov. *Metric structures for Riemannian and non-Riemannian spaces*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, english edition, 2007. Based on the 1981 French original, With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates.

[24] Noemi Halpern. A proof of the collar lemma. *Bull. London Math. Soc.*, 13(2):141–144, 1981.

[25] David A. Herron. Gromov–hausdorff distance for pointed metric spaces. *The Journal of Analysis*, 24(1):1–38, Jun 2016.

[26] Lizhen Ji. The story of Riemann's moduli space. *ICCM Not.*, 3(2):46–73, 2015.

[27] Benoît Kloeckner. The space of closed subgroups of $\mathbb{R}^n$ is stratified and simply connected. *J. Topol.*, 2(3):570–588, 2009.

[28] Lixin Liu, Athanase Papadopoulos, Weixu Su, and Guillaume Théret. On length spectrum metrics and weak metrics on Teichmüller spaces of surfaces with boundary. *Ann. Acad. Sci. Fenn. Math.*, 35(1):255–274, 2010.

[29] Yair Minsky. The classification of Kleinian surface groups. I. Models and bounds. *Ann. of Math. (2)*, 171(1):1–107, 2010.

[30] Athanase Papadopoulos and Guillaume Théret. Shortening all the simple closed geodesics on surfaces with boundary. *Proc. Amer. Math. Soc.*, 138(5):1775–1784, 2010.

[31] Peter Petersen. *Riemannian geometry*, volume 171 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2006.

[32] Ibrahim Pourezza and John Hubbard. The space of closed subgroups of $\mathbf{R}^2$. *Topology*, 18(2):143–146, 1979.

[33] Ian Richards. On the classification of noncompact surfaces. *Trans. Amer. Math. Soc.*, 106:259–269, 1963.

[34] Christina Sormani. How Riemannian manifolds converge. In *Metric and differential geometry*, volume 297 of *Progr. Math.*, pages 91–117. Birkhäuser/Springer, Basel, 2012.

[35] William P. Thurston. *Three-Dimensional Geometry and Topology, Volume 1: Volume 1*. Princeton University Press, 1997.

[36] William P. Thurston. Minimal stretch maps between hyperbolic surfaces. *arXiv Mathematics e-prints*, page math/9801039, Jan 1998.

[37] William P. Thurston. The geometry and topology of three-manifolds, Mar 2002.

[38] Stephen Willard. *General topology*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1970.