Unconventional Superconductivity Mediated by the Higgs Amplitude Mode in Itinerant Ferromagnets

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Roy Thomas Forestano

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Dr. Kevin S. Bedell, Advisor
Dr. Michael Graf, Physics Department Chairperson
Dr. Robert Meyerhoff, Mathematics Department Chairperson

Abstract

Over 20 years ago, Blagoev et. al. predicted an s-wave pairing instability in a ferromagnetic Fermi liquid (FFL) as a consequence of spin fluctuations [5]. Shortly after, it was discovered that, when magnetic interactions in the ferromagnetic superconductor UGe2 dominate, quasiparticles with parallel spin form pairs in odd-parity orbitals; i.e., a form of spin-triplet p-wave superconductivity emerges, in contrast to Blagoev et. al.'s prediction [6]. In this work, we return to this issue by introducing the effects of a gapped amplitude (or "Higgs") mode on the vertex corrections and subsequent form of Cooper pairing. As the Higgs mode only propagates in the presence of a finite spin current, such an amplitude mode results in strong momentum-dependence in the many-body vertex. This results in the emergence of an unconventional form of superconductivity mediated by unconventional low-energy modes in a weak itinerant ferromagnet.

To my mom,
for endless love, being the
most resilient and inspiring
person I know, and believing
in my potential

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Chapter 1

Overview

Superconductivity is thought to arise from an attractive interaction that overwhelms the Coulomb repulsion between pairs of electrons. In the standard model due to to Bardeen, Cooper and Schrieffer (BCS), an attraction between electrons is caused by the indirect effects of the crystalline lattice of ions as electrons move across it, where an electron moving through the lattice creates a positive ripple, thereby, attracting another electron to it. This BCS picture of cooper pairing is now believed to account well for traditional superconductors, but there is a growing number of metallic compounds, including the high-T_C superconductors, in which superconductivity appears anomalous and where the precise mechanism of electron pairing remains controversial. Therefore, it is reasonable to look at other states, namely magnetic states such as paramagnetism where spins are randomly aligned, ferromagnetism where spins align, and anti-ferromagnetism where spins are oppositely aligned, which may contribute to electron-electron pairing. An attraction between Fermion quasiparticles near the Fermi surface, arises, at first sight, from the effects of electrons moving over the lattice, but unlike the bare Coulomb repulsion, which is independent of the electron spin, a part of the interaction between quasiparticles can depend on the relative orientation of the spins, and thus, on the magnetic moments of the carriers. In the simplest case of nearly ferromagnetic metals, pairs of quasiparticles with parallel spins attract while pairs with antiparallel spins tend to repel, showing ferromagnetism may contribute to electron-electron pairing.

This search for ferromagnetic superconductors dates back to the 60's when superconducting materials with magnetic impurities were studied by Abrikosov and Gorkov, which inspired Larkin and Ovchinnikov to study a model of effective field theory of superconducting fermions coupled to magnetic impurities where they described the phase diagram of such a system [1]. While at first, phonon modes may seem detrimental to superconductivity, Thomsen, Cardona, Friedl, et al. reported quantitative evidence from Raman scattering for the applicability of stroing-coupling

theory for phonon self-energies to high T_C superconductors. They found optical phonon frequencies in the superconducting state of $YBa_2Cu_3O_{\gamma-\delta}$ which aligned with the phonon self-energy shift obtained from string coupling theory. They used this to determine the interaction's effect on T_C and found that some form of coupling of the lattice to the electronic system exists in the superconducting state of high-T_C superconductors [2]. Later it was found that a collective acoustic phonon, or Nambu-Goldstone mode, and optical phonon, or Higgs mode, acted similar to phonons in a lattice [3]. As s-wave pairing in the precessional Goldstone mode case must overcome the large Stoner gap which suppresses superconductivity, or decreases the magnetization amplitude, the prescence of the Higgs in a ferromagnetic state was found to induce magnetization amplitude fluctuations, which made the Higgs mode seem more likely to have a stronger effect on the Cooper instability in ferromagnetic superconductors.

In 1999, an Itinerant ferromagnet, Rutehnium 1212, undergoing a High temp superconducting transition was studied which prompted question of a possible many-body itinerant fermionic system supporting both types of broken symmetry, i.e. a system supporting both ferromagnetism and superconductivity [4]. In this work, the self consistent equations for the superconducting gap and the magnetization, which was tied to the doped layers of this copper oxide compound, were solved simultaneously for the first time. Their calculations were tied to the doped layers of this copper oxide compound where the magnetism becomes itinerant, and they concluded that the induced superconductivity arose from spin-singlet d-wave pairing in the nearly antiferromagnetic CuO2 layers. In 1998, Blagoev and Bedell solved for the zero-temperature vertex corrections for both the longitudinal and Goldstone phonon propagators [5]. They found that very similar logarithmic behavior would emerge from including an additional term with the second momentum p_σ term in the vertex. This implied that the self-energy was weakly momentum dependent close to the phase transition and allowed for the realization of a local ferromagnetic Fermi-liquid theory could be used to describe weak ferromagnetic metals where the magnetization is sufficiently small. This confirmed a s-wave pairing instability in this phase.

Shortly after this proposed theory, an experimental approach was taken by Saxena et al. on the border of itinerant-electron ferromagnetism in UGe_2, suggesting the real possibility for the existence of a pure ferromagnetic superconductor[6]. They proposed three conditions that must be satisfied for specimens to exhibit this phenomenon, of which the most important being that the material should be close to the border of ferromagnetism, i.e. either in a strongly paramagnetic or a weakly ferromagnetic state at low temperature. Now, whereas in Bedell's 1999 paper, we saw the induced superconductivity arose from spin-singlet d-wave pairing

in the nearly antiferromagnetic copper oxide layers, and in his 1998 paper, we saw spin-singlet s-wave pairing, Saxena found that when magnetic interactions in this compound dominate over other types of quasiparticle interactions, parallel-spin quasiparticles tend to form pairs that must necessarily be in odd-parity orbitals. In other words, a form of spin-triplet, or p-wave, magnetically mediated, superconductivity emerged. A year later, Wang, Mao and Bedell, studied weak and nearly ferromagnetic metals exhibiting p-wave superconductivity mediated by spin fluctuations [7]. They determined T_C for the $l = 1$ p-wave case from the gap function, and in doing so, found a suppressed, nonzero, T_C value near the quantum critical point further supporting that pairing due to ferromagnetic spin fluctuations may be spin triplet in nature.

From these proposals, it is reasonable to assume that a ferromagnetic superconductor will exhibit some form of unconventional pairing, namely spin-triplet pairing or p-wave superconductivity, induced by an overlooked amplitude mode within the ferromagnetic Fermi liquid. In this work, we introduce the effects of the gapped amplitude, or Higgs, mode on the zero-temperature vertex corrections, which is the first time this calculation has ever been done.[1] It was found that the Higgs mode, which requires a finite valued spin-current to propagate, introduces strong momentum dependence in the many-body vertex. In turn, this increased momentum dependence allows for an unconventional form of superconductivity mediated by unconventional low-energy amplitude modes to arise in a weak itinerant ferromagnet.

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Chapter 2

Second Quantization

We can begin with an approach toward understanding the BCS theory of superconductivity through the methods of second quantization.[1] This method allows us to determine the appropriate creation and destruction operators for the BCS Hamiltonian and perform a Bogoliubov transformation of this Hamiltonian to determine T_C from the gap function that arises.[2]

2.1 Issues with First Quantization

We begin with the case of many quantum particles, where if we know the Hilbert space H_1 of a single particle along with a basis $|\alpha\rangle$, we know that for $\mathcal{N}$ particles we have a Hilbert space of

$$H_{\mathcal{N}} = \bigotimes_{i=1}^{\mathcal{N}} H_i \tag{2.1}$$

where a complete basis becomes

$$|\alpha, \beta \ldots, \omega) = |\alpha\rangle \otimes |\beta\rangle \cdots |\gamma\rangle \tag{2.2}$$

Following this, we must solve a Schrödinger equation that admits a wavefunction depending on $\mathcal{N}$ variables for each of the $\mathcal{N}$ particles. However, for many interacting fermions or bosons, this procedure is inadequate due to the indiscernibility of the particles. Wavefunctions of the form (2) are no longer directly applicable to the system. The functions can become complicated even without interactions in the system as they must be properly (anti-)symmetrized and normalized. For instance, with two particles we have

$$|\alpha, \beta\rangle = \frac{1}{\sqrt{2}} \left[|\alpha, \beta) \pm |\beta, \alpha) \right] \tag{2.3}$$

[1] For a review of quantum statistics, see Appendix *A*[8-9].

[2] These notes on second quantization have been adapted from Giamarchi [10].

where + denotes bosons and − fermions. The $|\alpha, \beta\rangle$ denotes properly symmertized and normlaized kets for indistinguishable particles. For *ordered* kets by

$$|\alpha, \beta) = |\alpha) \otimes |\beta) \tag{2.4}$$

where the first particle is in state α and the second in β. Certainly, as the number of particles increases the need to use a symmetrized wavefunction becoems extremely heavy as the terms grow as $\mathcal{N}!$. A general wavefunction can be expressed as

$$\psi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \ldots, \mathbf{r}_\mathcal{N}) = C \sum_P (\pm 1)^{s(P)} \psi_1(\mathbf{r}_{P(1)}) \psi_2(\mathbf{r}_{P(2)}) \cdots \psi_\mathcal{N}(\mathbf{r}_{P(\mathcal{N})}) \tag{2.5}$$

where P represent the permutations of the numbers from 1 to $\mathcal{N}$, $s(P)$ is the signature of the permutation P, and + is for bosons and − is for fermions. For a fermionic system, we can write this wavefunction as the Slater determinant

$$\psi(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_\mathcal{N}) = \begin{vmatrix} \psi_1(\mathbf{r}_1) & \cdots & \psi_1(\mathbf{r}_\mathcal{N}) \\ \vdots & & \vdots \\ \psi_\mathcal{N}(\mathbf{r}_1) & \cdots & \psi_\mathcal{N}(\mathbf{r}_\mathcal{N}) \end{vmatrix} \tag{2.6}$$

A second issue arises from the way operators are represented in the standard expression of quantum mechanics. Based on the fact that operators depend on sums of terms, one would have to change the entire calculation simply based on the number of particles, which is not ideal. Therefore, the standard quantum mechanical representation, or *first quantization*, of systems with many indistinguishable particles is inadequate here. We need a system that permits these two things:

(a) Antisymmetrization is easily accomplished without having to deal with $\mathcal{N}!$ terms.

(b) The description of the system is independent of the number or particles in the system.

Note that the second method will allow us the take the thermodynamic limit, where $\mathcal{N} \to \infty$ and generalize to more situations where the number of particles can change. These requirements can be accomplished through the *second quantization* method provided in the following.

2.2 Fock Space

When particles are indistinguishable from one another, all we need to know is how many particles are in a given quantum state. We can assume we have

a basis $|\alpha\rangle$ of states for a single particle. This basis is generally infinite, but states can be quantized if we place the system in a box resulting in a finite number of states $|\alpha_1\rangle, |\alpha_2\rangle, \ldots, |\alpha_\Omega\rangle$. We can see that the length of this basis is independent of the number of particles. For Bosons, one can have a basis of length two with 1000 bosons present due to the fact that they can be in the same quantum state. However, for fermions, the total number of particles is always smaller than the possible states due to the Pauli principle. Once we know the number of particles in a given state $|\alpha_i\rangle$, we can fully describe the system and generate its wavefunction, thus, fully characterizing the wave function by the set $n_1, n_2, \ldots, n_\Omega$, where the total number of particles becomes $\mathcal{N} = n_1 + n_2 + \cdots + n_\Omega$. We can define a space where an arbitrary number of particles can exist. The Fock space can be defined as the direct sum of all Hilbert spaces with $0, 1, 2, etc.$ particles, where if we let $H_\mathcal{N}$ be the Hilbert space with $\mathcal{N}$ particles, then

$$\mathfrak{F} = \bigoplus_{j=0}^{+\infty} H_j \tag{2.7}$$

We can define the state $|n_1, n_2, \ldots, n_\Omega\rangle$. It is clear that two states with a differeing number of particles $\mathcal{N}$ belong to different Hilbert spaces, and are orthogonal in the Fock space. Using the wave function, for systems with the same number of particles, one can see if the states form an orthogonal and normalized basis

$$\langle n_1, n_2, \ldots, n_\Omega | n'_1, n'_2, \ldots, n'_\Omega \rangle = \delta_{n_1,n'_1}\delta_{n_2,n'_2}\cdots\delta_{n_\Omega,n'_\Omega} \tag{2.8}$$

The basis can thus be used to determine evert operator and matrix element in the Fock space. The basis chosen is convenient because it uses the minimal amount of information to describe the system as the number of n_i needed does not grow with the number of particles in the system. [3]

2.3 Bogoliubov Transformation

In principle, we know how to solve Hamiltonians of the form

$$H = \sum_{\alpha,\beta} A_{\alpha,\beta} c^\dagger_\alpha c_\beta \tag{2.9}$$

Here we have a quadratic, but not diagonal, Hamiltonian. We can bring the matrix $A_{\alpha,\beta}$ into a diagonal form by making the appropriate linear combination of the operators c_α. The operators d_α, which are linear combinations

[3] For a discussion on creation and destruction operators, one-body operators, and two-body operators see Appendices $B.1, B.2, B.3$ respectively.

of the c_α, bring the matrix under diagonal form which allows us to get all the eigenvectors of the problem. In general, the matrix $A_{\alpha,\beta}$ would be of size $\Omega \times \Omega$ and the diagnonalization would be very difficult to carry out. There are, however, simple cases where the physics of the problem aids in in this process. We can illustrate this with the tight binding Hamiltonian. We will also be able to write this in second quantized form. A complete basis is provided by the states on each site $|i\rangle$ and we can thus define the creation and destruction operators associated with it. In other words, $c_i^\dagger$ is the operator creating a particle on site i. The second quantization form reads

$$H = \sum_{i,j} (i|H^{(1)}|j) c_i^\dagger c_j \tag{2.10}$$

where $H^{(1)}$ is the Hamiltonian. We thus immediately obtain

$$H = E_0 \sum_i c_i^\dagger c_i - t \sum_{\langle t,j \rangle} c_i^\dagger c_j \tag{2.11}$$

The second term describes a process where a particle on site i is to reappear on the site j and vice versa. This is quadratic but not diagonal. What this actually is, is a tridiagonal matrix. To diagonalize this matrix, we can use the physical input in that since the Hamiltonian is invariant by translation, momentum must be a good quantum number. Thus, we want to deal with creation and destruction operators that are linear combinations of the c_i that correspond to a Fourier transform. Here, operators will correspond to the state $|k\rangle$. But let us simply treat this as a linear combination of operators. We can define

$$d_k^\dagger = \frac{1}{\sqrt{N_s}} \sum_{j=0}^{N_s-1} e^{ikr_j} c_j^\dagger \tag{2.12}$$

where we have used a different name d to emphasize that these are new operators, which obey the required Fermionic operator properties. There are exactly N_s different operators (the size of the Hilbert space cannot change) and k is confined inside the first Brillouin zone $k \in [-\pi/a, \pi/a]$ as discussed for the first quantization solution. Thus, $|\emptyset_d\rangle = |\emptyset_c\rangle$. The transformation is easily inverted

$$c_j^\dagger = \frac{1}{\sqrt{N_s}} \sum_k e^{-ikr_j} d_k^\dagger \tag{2.13}$$

and thus replacing the c_j in (78) and after some algebra, one obtains

$$H = E_0 \sum_k d_k^\dagger d_k - \sum_k 2t \cos(ka) d_k^\dagger d_k \tag{2.14}$$

Since the Hamiltonian is now diagonal we can use the operators d_k to express simply the ground state and various averages. On the physical level we have used the fact that since momentum is conserved one can diagonalize simultaneously the momentum operators and the Hamiltonian. Thus, the Hamiltonian is block diagonal in the basis of eigenvectors of the momentum operator. Since this basis is of size N_s (N_s different k values in the first Brillouin zone) we are left for each value of k with a 1×1 matrix to diagonalize. [4]

The Bogoliubov transformation is a very useful tool to solve quadratic but non diagonal Hamiltonians in second quantization. One of its remarkable uses is that the one is not limited to Hamiltonians of the form $c^\dagger c$, but can treat forms that contain $c^\dagger c^\dagger$ terms as well. For example, the Hamiltonian

$$H_{BCS} = \sum_k \xi(k)(c^\dagger_{k\uparrow} c_{k\uparrow} + c^\dagger_{k\downarrow} c_{k\downarrow}) + \delta \sum k(c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} + c_{-k\downarrow} c_{k\uparrow}) \tag{2.15}$$

is the so called Bardeen Cooper Schrieffer Hamiltonian which is the basis for the theory of superconductivity. The first term is simply the kinetic energy, the second term represents the creation and destruction of pairs of electrons. Note that this Hamiltonian does not conserve the number of particles and has thus no simple expression in first quantization. However, since it is quadratic one can treat this Hamiltonian by the Bogoliubov transformation. In fact, using the canonical transformation

$$\begin{aligned} c^\dagger_{k\uparrow} &= \alpha^\dagger_k \\ c^\dagger_{-k\downarrow} &= \beta_k \end{aligned} \tag{2.16}$$

Note that this transformation mixes creation and destruction operators. This one great advantage of the second quantization to allow easily for such operations. In the transformation it is also important to remember that

$$|\emptyset_{\alpha,\beta}\rangle = \Pi_k c^\dagger_{k\downarrow} |\emptyset_{c\uparrow,c\downarrow}\rangle \tag{2.17}$$

In terms of the operators α and β the BCS Hamiltonian becomes

$$H_{BCS} = \sum_k \xi(k)(\alpha^\dagger_k \alpha_k - \beta^\dagger_k \beta_k) + \delta \sum_k (\alpha^\dagger_k \beta_k + \beta^\dagger_k \alpha_k) + \sum_k \xi(k) \tag{2.18}$$

since for most dispersions $\xi(k) = \xi(-k)$. This is exactly the Hamiltonian we already examined up to a simple constant, and therefore, it can be solved by exactly the same transformation.

[4] For a more complicated problem using the Bogoliubov Transformation to transform the tight binding Hamiltonian, see Appendix $B.5$.

2.4 Bardeen-Cooper-Schrieffer (BCS) Theory

Here, we will utilize the methods of second quantization to find T_C from the gap function, which contains components of s and p wave pairing, defined in the BCS Hamiltonian.These notes on BCS Theory have been adapted from Henley [11] and Bukov [12].

2.4.1 Derivation of T_C

The Fermi-Hubbard Hamiltonian, which describes fermions of spin $\sigma =\uparrow,\downarrow$ on a lattice, is given by

$$H = -t_f \sum_{\langle i,j\rangle,\sigma} \left(c^\dagger_{i\sigma} c_{j\sigma} + h.c. \right) - \mu \sum_i m_i + U_{ff} \sum_i m_{i\uparrow} m_{i\downarrow} \tag{2.19}$$

Here, we assume that each fermion can hop to the nearest-neighbor lattice sites, which gives them an energy of t_f. If they are on the same site, then we must provide an energy U_{ff}, which is included in the second or interaction term on the right hand side. Also, the fermions obey the commutation relation $\{c^\dagger_{i\sigma}, c_{i\sigma}\} = \delta_{\sigma,\sigma'}\delta_{ij}$. The number operator is given by $m_i = m_{i\uparrow} + m_{i\downarrow}$, where $m_{i\sigma} = c^\dagger_{i\sigma} c_{i\sigma}$. For half-filling, we have $\langle m_i \rangle = 1$, or $\mu = E_F = 0$, thus simplifying the above to

$$H = -t_f \sum_{\langle i,j\rangle,\sigma} \left(c^\dagger_{i\sigma} c_{j\sigma} + h.c. \right) + U_{ff} \sum_i m_{i\uparrow} m_{i\downarrow} \tag{2.20}$$

From this, the Hamiltonian can now be written as

$$H_{BCS} = \sum_{k\sigma} \epsilon_k c^\dagger_{k\sigma} c_{k\sigma} + \frac{U_{ff}}{N_s} \sum_{k,k',q} c^\dagger_{k+q\uparrow} c^\dagger_{-k\downarrow} c_{-k'\downarrow} c_{k'+q\uparrow} \tag{2.21}$$

where ϵ_k is the individual energy of the k-th state. As this appears unsolvable because it is not quadratic, let us cast the interaction part of the BCS Hamiltonian in a quadratic term with the fermionic operators c and $c^\dagger$. To do this, we utilize a mean-field (MF) description to decouple the quartic term which allows us to define the gap function Δ_q as[5]

$$\Delta_q = \frac{U_{ff}}{N_s} \sum_{k\in BZ} \langle c^\dagger_{k+q\uparrow} c^\dagger_{-k\downarrow} \rangle \tag{2.22}$$

This acts to break a Cooper pair with energy U_{ff} only to create a new, subsequent Cooper pair. In other words, it defines the superfluid order parameter while representing half the energy needed in order to break a

[5]The derivation of T_C from the gap has been largely assisted by Bukov [12].

pair of Fermions. In general, the gap is a function of the quasi-momentum q, and its different components correspond to s, p, and d wave pairing if we expand to into the basis with respect to the point group of the underlying lattice. Therefore, s wave pairing is defined by Δ_0 in the Fourier expansion. Let us limit our investigation to s wave pairing. To find the mean-filed decoupling of the interaction portion of the Hamiltonian, we utilize the mean-field approximation for an operator $\hat{A}\hat{B} \approx \langle\hat{A}\rangle\hat{B} + \hat{A}\langle\hat{B}\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle$. This assumes that $\langle\delta\hat{A}\delta\hat{B}\rangle$ is to be ignored, where $\delta\hat{A} = \hat{A} - \langle\hat{A}\rangle$ and $\delta\hat{B} = \hat{B} - \langle\hat{B}\rangle$. Here, we make the Bogoliubov replacement for the four fermion operator, which is given as

$$c_4^\dagger c_3^\dagger c_2 c_1 \approx \langle c_4^\dagger c_3^\dagger\rangle c_2 c_1 + c_4^\dagger c_3^\dagger\langle c_2 c_1\rangle - \langle c_4^\dagger c_3^\dagger\rangle\langle c_2 c_1\rangle \tag{2.23}$$

Thus, the mean field decoupling is given by the form

$$\frac{U_{ff}}{N_s}\sum_{k,k',q} c_{k+q\uparrow}^\dagger c_{-k\downarrow}^\dagger c_{-k'\downarrow}c_{k'+q\uparrow} \approx \Delta_0\sum_k c_{-k\downarrow}c_{k+q\uparrow} + \Delta_0^*\sum_k c_{k+q\uparrow}^\dagger c_{-k\downarrow}^\dagger - \frac{N_s}{U_{ff}}|\Delta_0|^2 \tag{2.24}$$

Since $|\Delta_0|$ only enters the equations for determining the phase boundary, we can take Δ_0 as a real number and write our BCS Hamiltonian as,

$$H \approx -\frac{N_0}{U_{ff}}|\Delta_0|^2 + \sum_{k\in BZ}\epsilon_k\sum_{k\in BZ}\begin{pmatrix}c_{k\uparrow}^\dagger & c_{-k\downarrow}\end{pmatrix}\begin{pmatrix}\epsilon_k & \Delta_0\\ \Delta_0 & -\epsilon_k\end{pmatrix}\begin{pmatrix}c_{k\uparrow}\\ c_{-k\downarrow}\end{pmatrix} \tag{2.25}$$

Due to the tight-binding dispersion, we have $\sum_{k\in BZ}\epsilon_k = 0$, and thus, we introduce the Bogoliubov transformation

$$M = \begin{pmatrix}\sqrt{\frac{1}{2}\left(1 + \frac{\epsilon_k}{\sqrt{\epsilon_k^2+|\Delta_0|^2}}\right)} & -\sqrt{\frac{1}{2}\left(1 - \frac{\epsilon_k}{\sqrt{\epsilon_k^2+|\Delta_0|^2}}\right)}\\ \sqrt{\frac{1}{2}\left(1 - \frac{\epsilon_k}{\sqrt{\epsilon_k^2+|\Delta_0|^2}}\right)} & \sqrt{\frac{1}{2}\left(1 + \frac{\epsilon_k}{\sqrt{\epsilon_k^2+|\Delta_0|^2}}\right)}\end{pmatrix} \tag{2.26}$$

with $M_{11} = u, M_{21} = v$,and define operators $\vec{a}_k = M\vec{c}_k$ and $\vec{c}_k = (c_{k\uparrow}c_{-k\downarrow}^\dagger)^t$ to obtain [6]

$$H \approx E_{gs} + \Sigma_{k\in BZ,\sigma}E_k a_{k\sigma}^\dagger a_{k\sigma} \tag{2.27}$$

Here, E_{gs} is the ground state energy, and $E_k = \sqrt{\epsilon_k^2 + |\Delta_0|^2}$. Now that this is derived, we procede to the gap function. We know that we have

$$\Delta_q = \frac{U_{ff}}{N_s}\sum_{k\in BZ}\langle c_{k+q\uparrow}^\dagger c_{-k\downarrow}^\dagger\rangle \implies \Delta_0 = \frac{U_{ff}}{N_s}\sum_{k\in BZ}\langle\vec{c}_k^\dagger\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\vec{c}_k\rangle \tag{2.28}$$

[6]For the following calculations, see Appendix B.6.

Implementing the Bogoliubov transformation, we obtain

$$\Delta_0 = \frac{U_{ff}}{N_s} \sum_{k \in BZ} \left(uv \langle a^\dagger_{k\uparrow} a_{k\uparrow} \rangle + u^2 \langle a^\dagger_{k\uparrow} a^\dagger_{k\downarrow} \rangle - v^2 \langle a_{-k\downarrow} a^\dagger_{-k\downarrow} \rangle - uv \langle a_{-k\downarrow} a^\dagger_{-k\downarrow} \rangle \right) \tag{2.29}$$

The averages above were calculated in the notes by Giamarchi, except there, we used γ instead of a, and thus we obtain

$$\Delta_0 = -\frac{U_{ff}}{N_s} \sum_{k \in BZ} (uv(1 - 2f_F(E_k))) \tag{2.30}$$

where

$$uv = \frac{1}{2} \frac{|\Delta_0|}{\sqrt{\epsilon_k^2 + |\Delta_0|^2}} \tag{2.31}$$

Therefore, the finite-temperature gap equation is given as

$$\Delta_0 = -\frac{U_{ff}}{2N_s} \sum_{k \in BZ} \frac{\Delta_0}{\sqrt{\epsilon_k^2 + |\Delta_0|^2}} (1 - 2f_F(E_k)) \tag{2.32}$$

If we consider the zero-temperature limit, then $f_F(E_k) \to 0$, and we obtain the zero-temperature gap equation

$$\Delta_0 = -\frac{U_{ff}}{2N_s} \sum_{k \in BZ} \frac{\Delta_0}{\sqrt{\epsilon_k^2 + |\Delta_0|^2}} \tag{2.33}$$

This is a nonlinear integral equation for a whole unknown function Δ_0, which in general could only be solved by numeric integration. To calculate the value of T_c, the new gap equation looks like this function with an extra factor of $\tanh(E_{\mathbf{k}'}/2T)$ on the right side as $\tanh(\epsilon\beta) = 1 - 2F(\epsilon)$. If we adopt Cooper's toy potential, we obtain

$$1 = \frac{1}{2N} V_0 \sum_{\mathbf{k}} \frac{1}{E_{\mathbf{k}}} \tanh\left(\frac{E_{\mathbf{k}}}{2T}\right) \tag{2.34}$$

where the potential U_{ff} V_0. Raising T, the tanh factor suppresses the logarithmic divergence, which was handled by Δ in $E_{\mathbf{k}} \equiv \sqrt{\epsilon_{\mathbf{k}}^2 + |\Delta|^2}$ in the denominator. Therefore, the value of Δ will decrease. By definition, T_c is the temperature where $\Delta = 0$ and thus $E_{\mathbf{k}} = |\epsilon_{\mathbf{k}}|$. At the critical $\beta_c \equiv 1/T_c$, we get

$$1 = \frac{1}{2N} V_0 \sum_{\mathbf{k}} \frac{1}{|\epsilon_{\mathbf{k}}|} \tanh\left(\frac{1}{2}\beta_c |\epsilon_{\mathbf{k}}|\right) = \frac{1}{2} \mathcal{N}(0) V_0 \int_{\hbar\omega_c}^{+\hbar\omega_c} d\epsilon \frac{1}{|\epsilon|} \tanh \frac{1}{2}\beta_c |\epsilon| \tag{2.35}$$

using the same approximations as in Sec. 7.3 D. We can estimate this integral by the "poor man's approximation" as in Sec. 7.3 D. The upper cutoff is $\hbar\omega_c$ while the lower cutoff is roughly $\epsilon_{\mathbf{k}} \approx T_c$, since that is where the tanh fucntion in (121) crosses over from unity to a linear behavior that cancels the $1/|\epsilon|$ divergence. Thus, T_c is playing the same role in (121) that Δ played in the gap equation in Sec. 7.3 D; we have $1 \approx \mathcal{N}(0)V_0[\ln(\hbar\omega_c/T_c)$ + const.] and $T_c \sim \Delta(0)$. In fact, you can solve (2.23) exactly,

$$T_c = 1.14\hbar\omega_c e^{1/\mathcal{N}(0)V_0} \tag{2.36}$$

which is written in terms of the density of states $N(0)$ times the potential V_0. This is the BCS result for the superconducting transition temperature T_C.

Chapter 3

Fermi Liquid Theory

A Fermi Liquid is a quantum many-body state that involves a group of interacting fermions, whether interactions are small or large. It generally is a low temperature state of a metal characterized by well defined Landau quasiparticles, or low lying excitations of an interacting electron system with the same spin, charge, and k vectors as the electrons. The quasiparticles can have as many degrees of freedom as electrons and be labeled by k. As they still obey the exclusion principle, a quasiparticle is fermionic.

A collective excitation is what becomes bosonic. A collective excitation is a quantized mode in a many body system which occurs because of oscillatory (cooperative) motion of the whole system as a result of interactions between particles. These obey Bose-Einstein statistics and examples are plasmons and phonons in solids. A cooper pair is a loosely bound pair of fermions with opposite or aligned spins and moving with the same speed in opposite directions. They are believed to be responsible for superconductivity. As the pair is composed of total spin 0 or 1, these are bosonic.

A Fermi Liquid exhibits two types of symmetry, namely gauge $U(1)$ symmetry and spatial translational symmetry. Gauge symmetry is geometrically the rotational symmetry of a circle, or a set of 1×1 matrices, i.e. a symmetry group from electromagnetic interactions where fermions act individually. Furthermore, a Fermi liquid is highly entangled. This can be quantified and the entanglement entropy for free fermions scales as

$$S_L \sim L^{d-1} \ln L \tag{3.1}$$

with the Fermi momentum k_F allowing the units to match. Using $L = 2\pi k_F$, we have

$$S_L \sim (2\pi k_F)^{d-1} \ln 2\pi k_F \tag{3.2}$$

which is isotropic, or exhibiting the same properties in all directions.

Really, metals themselves are not Fermi Liquids, however, the sea of electrons within them is, which is also dependent on the lattice of the metal

in some cases. It is really the interactions between these electrons that form a Fermi Liquid. However, Fermi liquids only describe a metal in its normal state. Therefore, superconductors and orthogonal metals, where charge carriers are orthogonal to electrons, do not exhibit the necessary properties to be Fermi Liquids.

3.1 Quantum Fermi Liquid

Once the temperature of a liquid is low enough, the de Broglie wavelength corresponding to the thermal motion of the atoms becomes comparable with the distances between the atoms, and thus, the macroscopic properties of it are determined by quantum phenomena.[1] We will need to understand a macroscopic body's energy level spectrum in order to find relevant thermodynamic quantities. When calculating the partition function at sufficiently low temperatures, we will only consider energy levels that are weakly excited within the liquid, i.e. those lying close to the ground state. We can then regard these weakly excited states of a macroscopic body as a collection of separate elementary excitations acting as quasiparticles with energies ϵ and momenta $\mathbf{p}$. Here, we will take $\epsilon(p)$ as the dispersion relation for the elementary excitations.

Noting that energy spectrums within quantum liquids can vary, consider a liquid with a Fermi spectrum. The structure of this energy spectrum will take a similar form to that of a Fermi gas. However, in a liquid, there are no quantum states for individual particles. Thus, for a Fermi liquid, we can consider a system of fermions where we turn on interactions slowly enough to build a one-to-one correspondence between the free Fermi gas and the interacting Fermi liquid. Here, the elementary excitations, or quasiparticles, relate the the gas particles where the number of quasiparticles, which obey Fermi statistics, is equivalent to the number of atoms. This spectrum is specific to the nature of interactions between atoms in this system and cannot be applied in general to all liquids. Each of the quasiparticles has momentum p. Take $n(p)$ as the momentum distribution function of the quasiparticles, which is normalized by

$$\int n d\tau = \frac{N}{V} \tag{3.3}$$

where $d\tau = \frac{d^3p}{(2\pi\hbar)^3}$. Classifying the type of spectrum involves supposing that the energy E of the liquid is uniquely determined, where the ground state corresponds to the distribution in which all states are occupied within

[1] The notes in this section have been adapted from E.M. Lifshitz and L.P. Pitaevskii [13].

the Fermi sphere, whose radius p_F is related to the density of the liquid by

$$\frac{N}{V} = 2 \cdot \frac{4\pi p_F^3}{3(2\pi\hbar)^3} = \frac{p_F^3}{2\pi^2\hbar^3} \tag{3.4}$$

which is the same as seen for an ideal gas. As the total energy E of the liquid is a functional of the distribution function, which does not reduce to the same form as for an ideal gas in $\int n\epsilon d\tau$, the total energy E is not the sum of energies ϵ of the quasiparticles. In order to write an expression for the energy of the quasipartciles allowing for their interaction, we can consider the change in E due to an infinitesimal change in the distribution, i.e the free energy functional,

$$\frac{\delta E}{V} = \int \epsilon(p)\delta n d\tau \tag{3.5}$$

where ϵ is the functional derivative of the energy E with respect to the distribution function. This expression accounts for the change in energy of the system when a single quasiparticle with momentum p is added, and is a functional of the distribution function as ϵ depends on the distribution of particles in the liquid.

Thus far, we have ignored the spin of the quasi-particles, and since spin is quantum-mechanical, the distribution function must be a regarded as a statistical matrix or an operator $\widehat{n}(p)$ with respect to the spin. The energy ϵ of an elementary excitation is a function of both the momentum and an operator with respect to the spin variables expressed in terms of the quasi-particle spin operator $\widehat{s}$. Stating that a quasi-particle has spin expresses the possibility that all energy levels could become doubly degenerate, as in the case of a homogeneous isotropic liquid where the quasi-particle energy becomes independent of the spin operator. This allows for the spin of the quasi-particles to be labelled as $\frac{1}{2}$, regardless of the spin of the particles within the liquid, as any spin s other than a $\frac{1}{2}$ would simply create branches of ϵ with each branch corresponding to quasiparticles with spin $\frac{1}{2}$. Now, the distribution operator $\widehat{n}(p)$, which arises from the existence of spin, can be written as a Hermitian statistical matrix $n_{\alpha\beta}(p)$, where $\alpha, \beta = \pm\frac{1}{2}$ are spin matrix indices. Here, the diagonal matrix elements determine the numbers of quasi-partciles in each spin state, and the normalization condition for the distribution is written as

$$\begin{aligned} \mathrm{tr} \int \widehat{n} d\tau \equiv \int n_{\alpha\alpha} d\tau = \frac{N}{V} \\ d\tau = \frac{d^3p}{(2\pi\hbar)^3} \end{aligned} \tag{3.6}$$

where tr denotes the trace of the matrix with respect to the spin indices.

The quasi-partcile energy ϵ is then also an operator expressed as

$$\frac{\delta E}{V} = \mathrm{tr} \int \epsilon \delta n d\tau \equiv \int \epsilon_{\alpha\beta} \delta n_{\beta\alpha} d\tau \tag{3.7}$$

and as there is no spin dependence of the distribution function and the energy, $n_{\alpha\beta}, \epsilon_{\alpha\beta}$ reduce to unit matrices $n_{\alpha\beta} = n\delta_{\alpha\beta}, \epsilon_{\alpha\beta} = \epsilon\delta_{\alpha\beta}$. Then, taking the trace in the previous expressions multiplies each by a factor of 2, i.e.

$$2\int n d\tau = \frac{N}{V} \quad \text{and} \quad \frac{\delta E}{V} = 2\int \epsilon \delta n d\tau \tag{3.8}$$

In statistical equilibrium, the quasi-particle distribution is a normal Fermi distribution with the energy as defined by equation (3.7). Thus, as the energy levels of the Fermi gas and liquid are distinguished in the same way, the entropy S of the liquid can be found from a like combinatorial expression in

$$\frac{S}{V} = -\mathrm{tr} \int [n \log n + (1-n)\log(1-n)] d\tau \tag{3.9}$$

with total particle number conservation $\frac{\delta N}{V} = \mathrm{tr} \int \delta n d\tau = 0$ and total energy conservation $\frac{\delta E}{V} = \mathrm{tr} \int \epsilon \delta n d\tau = 0$,

$$\implies n = \frac{1}{e^{\frac{\epsilon-\mu}{T}} + 1} \tag{3.10}$$

we obtain the desired distribution of the liquid where μ is the chemical potential. When the energy becomes independent of the spin, we have

$$n = \frac{1}{e^{(\epsilon-\mu)T} + 1} \tag{3.11}$$

where at $T = 0$ the chemical potential is equal to the Fermi energy, or the limiting energy on the surface of the Fermi sphere, i.e. $\mu \,|_{T=0} = \epsilon_F \equiv \epsilon(p_F)$.

For the assumption that a definite momentum can be assigned to each quasi-particle to be valid, we require that the uncertainty in the momentum due to the finite mean free path of the quasi-particle must be small in comparison with both the momentum itself and the width Δp of the transitional zone of the distribution. This condition takes the form

$$\theta(p) = \begin{cases} 1 & p < p_F \\ 0 & p > p_F \end{cases} \tag{3.12}$$

where it is satisfied if the distribution $n(p)$ differs from the above expression only in a small region near the surface of the Fermi sphere. This is

due to the fact that, by the exclusion principle, only quasi-particles in the transitional zone of the distribution can take part in mutual scattering, and as a result, they must enter free states in that zone. Therefore, as the probability of collision is proportional to $(\Delta p)^2$, the uncertainty in the energy, and thus, the momentum are both proportional to this quantity. When Δp is small, this allows for the condition to be satisfied, where the uncertainty in the momentum will be small in comparison with p_F and Δp. This method described can be adequately applied to to excited states of the liquid which are classified by a distribution differieng from a step-function in just a narrow region near the Fermi surface. Since thermodynamic equilibrium distributions require sufficiently low temperatures and the width in this distribution has order T, the condition for the theory to be valid is

$$\frac{\hbar}{\tau} << T \tag{3.13}$$

where the quantum uncertainty in the energy of a quasi-particle is of order $\frac{\hbar}{\tau}$, where τ is defined as the mean free time of the quasi-particle. We can see that the time τ is inversely proportional to the squared width of this zone in τT^{-2} where the condition on $\frac{\hbar}{\tau}$ is certainly satisfied as $T \to 0$, and as we are considering weak interactions, $T << |\epsilon_F|$. For distributions close to $T = 0$, i.e. near step-functions, we can make a first approximation wjere we can replace the functional ϵ by its value for $n(p) = \theta(p)$. Then ϵ becomes a definite function of the magnitude of the momentum, and $n = \frac{1}{e^{\frac{\epsilon-\mu}{T}})+1}$ becomes the normal Fermi distribution.

As the function $\epsilon(p)$ has explicit physical significance near the Fermi sphere's surface, we can expand it in powers of $p - p_F$, i.e. $\epsilon - \epsilon_F \approx v_F(p - p - F)$ where $v_F = \left[\frac{\partial \epsilon}{\partial p}\right]_{p=p_F}$ is the velocity of the quasiparticles on the Fermi surface. In an ideal Fermi gas, where quasi-particles are the same as the actual particles, $\epsilon = \frac{p^2}{2m} \implies v_F = \frac{p_F}{v_F}$. Thus, for a Fermi liquid we can define

$$m^* = \frac{p_F}{v_F} \tag{3.14}$$

called the effective mass of the quasi-particle. We can now write as the condition for the theory to be applicable $T << v_F p_F$, where only quasi-particles with momenta $p, |p - p_F| << p_F$ hold value. This effective mass allows us to write the entropy S and specific heat C of the liquid at low temperatures as

$$S = C = V\gamma T \tag{3.15}$$

where $\gamma = \frac{m^* p_F}{3\hbar^3} = (\frac{1}{3}\pi)^{\frac{2}{3}}(\frac{m^*}{\hbar^2})(\frac{N}{V})^{\frac{1}{3}}$, which is the same expression as for an ideal gas with the insertion of the effective mass m^* for the particle

mass m. This can also be seen from equation (3.9), which is the same in the gas and liquid cases, and as we are only considering momenta near p_F to be important, the quasi-particle distribution of the liquid and particle distribution of the gas can be expressed as in equation (3.11).

Now, in order to determine the Landau parameters in Fermi liquid theory, we must use a perturbative approach to write the difference in the Free energy as

$$\delta F = \frac{1}{V}\sum(\epsilon_{p\sigma} - \mu)\delta n_{p\sigma} + \frac{1}{2V}\sum_{p\sigma} f_{\substack{pp' \\ \sigma\sigma'}} \delta n_{p\sigma}\delta n_{p'\sigma} \tag{3.16}$$

where δn describes density fluctuations about equilibrium and the landau paramter f contains all interactions,

$$f_{\substack{pp' \\ \sigma\sigma'}} = \frac{\delta^2 E[n_{p\sigma}]}{\delta n_{p\sigma}\delta n_{p'\sigma'}} \tag{3.17}$$

In the following section we will show that there is a lower bound to the Landau parameter, which can be shown by performing a transformation of Free energy. From this theory, we also have the linearized Landau Kinetic equation,

$$\frac{\partial \delta n_p}{\partial t} + v_p \cdot \frac{\partial}{\partial r}\left(\delta n_p - \frac{\partial n_p^0}{\partial \epsilon}\delta\epsilon_p\right) = I[n_{p'}] \tag{3.18}$$

where v_p and the partial derivative of the density fluctuations with respect to ϵ are equilibrium functions, and I is the collision integral in term of the density fluctuations. This function describes the dynamics of quasiparticle disturbances within our system, and therefore, the collective modes of the Fermi liquid can be derived from this equation.

An important theorem that arises from Fermi Liquid Theory includes Luttinger's theorem, which states that the volume enclosed in the Fermi surface is independent of the interaction strength so long as the density is held fixed. Fermi liquids are not the only things to obey Luttinger's theorem. Things like dilute $2D$ materials such as low-disordered silicon metal-oxide semiconductor field-effect transistors, where there appears to be a strongly-correlated metallic ground state with a lack of a Landau Fermi liquid. Strange metals obey this and pseudogaps violate this theorem. This theorem is profound as it can be utilized in theoretical models of correlated electrons, such as the high-temperature superconductors, and in photoemission, where a metal's Fermi surface can be observed directly.

Another reminder is that a superconductor is not a Fermi liquid because in the normal BCS state there is an attractive interaction between fermions to create bosonic cooper pairs. The Fermi liquid description fails here because the quantum numbers of the noninteracting Fermi gas are no longer good, i.e. particle number conservation is violated, and thus, superconductors are not Fermi Liquids.

3.2 The Pomeranchuk Instability Condition

We can now look at a transformation of the free energy to derive the Pomeranchuck instability condition.. This instability condition imposes a lower bound on the Landau-Fermi Liquid parameter F_0^s by imposing that the free energy of the Fermi liquid cannot become negative. To derive the Pomeranchuk instability condition, we can begin with the difference in the distribution of fermions,

$$\delta n_{p\sigma} = n_{p\sigma} - n^o_{p\sigma} = \Theta(p_F(\theta,\phi) - p) - \Theta(p^o_F - p) \tag{3.19}$$

where $n_{p\sigma}$ is the distribution function, p is the momenta, $p_F = k_F\hbar$ is the Fermi momentum, and Θ is the Heaviside step function. We can then input this difference into the expression for the free energy

$$\sum_{p\sigma}(\epsilon_p - \mu)\delta n_{p\sigma} = V_F \sum_{p\sigma}(p - p_F)\delta n_{p\sigma} \tag{3.20}$$

and expand via spherical harmonics to obtain

$$\begin{aligned} 1 + F_0 - \left(\frac{2}{3} + F_0\right)\left(\frac{1}{p_F\sqrt{\pi}}\right)\mu_{00} = 0 \\ \implies 1 + F_0 \geq \frac{\mu_{00}}{p_F\sqrt{\pi}}\left(\frac{2}{3} + F_0\right) \end{aligned} \tag{3.21}$$

where we require the final expression to be ≥ 0 for this condition to remain unbroken.[2] In general, when this condition is violated, the Fermi liquid description is no longer valid. Here, if $\mu_{00} \leq \frac{1+F_0}{\frac{2}{3}+F_0} p_F\sqrt{\pi}$, when μ_{00} is less than this, a Fermi liquid approach is no longer valid. In fact, when this condition is broken, it allows for ferromagnetic order to arise within the system.

3.3 The Ferromagnetic Fermi Liquid (FFL)

We can now extend the Theory of the Fermi Liquid to the Ferromagnetic Fermi Liquid. In the Ferromagnetic Fermi liquid, spin $SU(2)$ symmetry is conserved. This is geometrically the rotational symmetry of a sphere, or the set of 2×2 matrices with unit determinant. This can be thought of as analogous to the weak nuclear attraction between a pair of fermions and set of bosons. Here, we are considering an itinerant Ferromagnetic Fermi Liquid where spins can move, and thus, magnetization fluctuations in this system are permitted.

[2]For the full calculation, see Appendix $C.1$.

3.3.1 FFL Self Energy Expansion and Linearized Spin-Magnetic Equation

In order to aid our extension of Fermi Liquid Theory to Ferromagnetic Fermi Liquid Theory, we can look at the work of Abrikosov, Gorkov, and Dzyaloshinski. We can write the free energy functional in terms of the magnetization, which is our order parameter in the FFL,

$$\Delta F = \frac{1+F_0^a}{2N(0)} m_0^2 + g\frac{1}{N(0)^3} m_0^4 + \cdots \tag{3.22}$$

where m_0 is the equilibrium magnetization, $N(0)$ the density of states, g is some positive constant, and $F_l^{a,s} = fN(0)$ are the dimensionless anti-symmetric Landau parameters. In this Landau-Ginzburg free energy, when $1+F_0^a$ is negative, we break the Pomeranchuck instability condition allowing ferromagnetic order to arise and the possibility for spontaneous symmetry breaking to occur. In order to study the collective modes in the FFL, we use the Linearized Landau Kinetic Equation in the spin channel, which describes the free oscillation of the momentum dependent magnetization, or δm_p which describes magnetization fluctuations about equilibrium,

$$\begin{aligned} &\frac{\partial \delta m_p(r,t)}{\partial t} + v_p \cdot \nabla \left(\delta m_p(r,t) - \frac{\partial n_p^0}{\partial \epsilon_p^0} \delta h_p(r,t) \right) \\ &= -2\left(m_p^0(r,t) \times \delta h_p(r,t) + \delta m_p(r,t) \times h_p^0(r,t) \right) + I[m_p] \end{aligned} \tag{3.23}$$

where

$$\begin{aligned} h_p^0 &= -B + 2\sum_{p'} f_{pp'}^a m_p' \\ \delta h_p &= -\delta B + 2\sum_{p'} f_{pp'}^a \delta m_p' \end{aligned}$$

where m_p is the spin density distribution function, v_p is some velocity, h_p is the effective equilibrium field, and δh_p describes the fluctuations of the effective equilibrium field. If one were to compare this to the Fermi Liquid case, we see an additional term on the right side along with the collision integral which accounts for the interaction between the magnetization and the effective equilibrium field, i.e. the coupling between the electrons and the magnetic and electric fields [14]. Here, a small magnetic field delta B is set transverse to the equilibrium magnetization, and one can find the dispersions by taking the limit of free oscillations of the kinetic equation by setting $B = 0$ and $\delta B = 0$, and analyzing the low temperature limit for which the collision integral $I[m_p]$ can be ignored. Through this, one can then study the collective modes of the FFL by studying the magnetization amplitude fluctuations.

3.3.2 Collective Modes Derived from the Kinetic Equation

The emergence of low-energy excitations in systems with spontaneously broken symmetry plays an important role in our fundamental understanding of these systems. The two types of fundamental excitations, or particles, that are present in the field theory description of these spontaneously broken symmetries include the Nambu-Goldstone bosons, also known as magnons or phase modes, and massive Higgs bosons, or amplitude modes.

$$\begin{aligned} \text{Nambu-Goldstone} \quad & \phi(x) = \rho_0 e^{i\theta(x)} \text{ with } \rho_0 \text{ constant} \\ \text{Higgs} \quad & \phi(x) = \rho(x) e^{i\theta_0} \text{ with } \theta_0 \text{ constant} \end{aligned}$$

In the simplest case, we can let the order parameter that describes a state with a spontaneously broken symmetry be $\phi(x) = \rho_0 e^{i\theta(x)}$, or an amplitude, ρ, times a phase factor, $e^{i\theta}$. This results in a term quadratic in the amplitude ρ, but not in the phase, which leads to the two fundamental excitations: the massless Nambu Goldstone boson, or gapless collective mode, which corresponds to fluctuations of the phase, $\theta(x)$, with fixed amplitude, ρ_0, and the massive Higgs boson, or a gapped collective mode, which corresponds to fluctuations of the amplitude, $\rho(x)$, with the phase, θ_0, held constant. Here, when I say gapped mode, I am referring to a mode with a gap in the excitation spectrum. A precessing or phase mode is the analog of the acoustic phonon, or Goldstone mode, in our system, and the Higgs in high energy physics is the analog of the optical phonon, or Higgs amplitude mode.

In the context of high energy physics, three of the four fundamental forces, the strong, weak, and electromagnetic forces, are the result of gauge symmetries. When one of these gauge symmetries spontaneously breaks, one obtains a massive Higgs mode in the system. This is in stark contrast to the breaking of a physical global symmetry, where spontaneous symmetry breaking results in a massless Nambu-Goldstone mode.

One can understand the fundamentals of the Higgs mechanism by considering the case of a complex ϕ^4 theory:

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi^*) - m^2 \phi^* \phi - \lambda(\phi^* \phi)^2 \tag{3.24}$$

We ignore the cubic term from symmetry arguments, where we only consider small deviations from the equilibrium in order to truncate at quartic order [15]. Here, m is some "mass-like" parameter and λ is the self-interaction which is quartic in the field ϕ. Note that the system has a minimum for $\phi^* = \phi = 0$ when $m^2 > 0$. However, if $m^2 < 0$, then the system has a local maximum at $\phi = 0$ and local minima given by $|\phi|^2 = -m^2/2\lambda$. This is visualized below, with the blue curve corresponding to the Ginzburg-Landau like potential $V(\phi) = m^2 \phi * \phi + \lambda(\phi^* \phi)^2$ when $m^2 > 0$ and the orange curve corresponds to $V(\phi)$ when $m^2 < 0$.

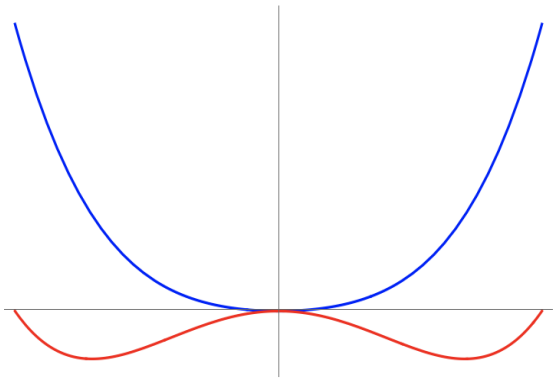

Figure 3.1: Ginzburg-Landau potential where the cases of $m^2 > 0$ (blue) and $m^2 < 0$ (red) are plotted. Here, we are interested in the $m^2 < 0$ case as, in analogy with our system, the Pomeranchuck instability is broken in this regime, which allows for ferromagnetic order to arise.

With the potential now defined, let us consider some small amplitude mode. In other words, letting $\phi = \phi_1 + i\phi_2$, we now let $\phi(x) = (\rho'(x) + a)e^{i\theta(x)}$. This ultimately results in a term quadratic in the amplitude ϕ', but no quadratic term in the phase $\theta(x)$. Thus, the spontaneous breaking of a continuous global symmetry results in the appearance of a gapless collective mode; i.e., the Goldstone boson.

In the presence of a gauge symmetry, we demand invariance of the Lagrangian given above under the transformation $\phi \to e^{i\Lambda(x)}\phi$. The resulting gauge-invariant Lagrangian is then given by

$$\mathcal{L} = (\partial_\mu + ieA_\mu)\phi(\partial^\mu - ieA^\mu)\phi^* - m^2\phi^*\phi - \lambda(\phi^*\phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad (3.25)$$

where $\partial_\mu + ieA_\mu$ defines the covariant derivative and $F_{\mu\nu}$ is the electromagnetic field tensor. Under an amplitude fluctuation of the order parameter ϕ, the system contains two massive fields. In this sense, we say that the spontaneous breaking of a local gauge symmetry results in the photon mode becoming massive[1][16]. Therefore, we can say that the Goldstone mode is consumed by the gauge bosons to give the photon mass.

The spontaneous breaking of gauge symmetry has implications in the context of high-energy physics, where the spontaneous breaking of a $U(1) \times SU(2)$ symmetry results in electroweak symmetry breaking. The breaking of such a symmetry ensures that electrons, muons, and τ-neutrinos (along with the gauge bosons) remain massive while the photon remains massless. Additionally, the discovery of the Higgs boson at the Large Hadron Collider resulted the wide-spread popularization of the Higgs boson.

In condensed matter and many-body physics, the Higgs mechanism has similarly gained prominence. The Landau-Ginzburg free energy describing

Bardeen-Cooper-Schrieffer (BCS) theory follows the same basic form of a ϕ^4 theory. In the presence of a gauge field, a Higgs mechanism emerges, which relates to the London equations for superconductors and the Meissner effect. In particular, Bedell and Blagoev found a new massive mode in the ferromagnetic metal by constructing a spin-hydrodynamic analog of the quasi-classical Landau-Silin kinetic equation [16]. More recently, Zhang, Farinas, and Bedell identified this mode as the Higgs mode [15]. However,

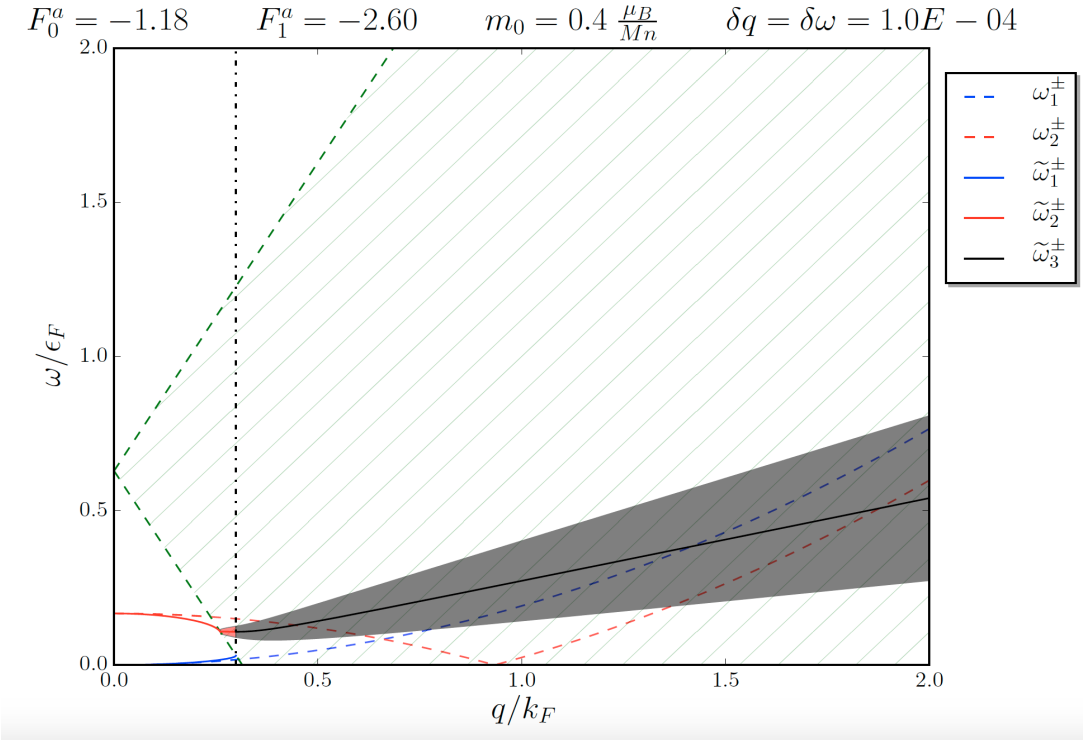

Figure 3.2: Dispersion relations of the Goldstone and Higgs modes in the FFL for specific parameters are given in the title. The blue and red dotted lines represent Yi Zhang's analytical solution assuming $s = \omega/qv_F << 1$. The solid lines represent J.T. Heath's analytical predictions. The blue lines represent the Goldstone, or gapless mode, and the red lines represent the Higgs, or gapped mode. The shaded region is the imaginary region [17]. Since one mode lies in the particle hole continuum but two lie outside of this region, or as two modes outside transition to one inside, this implies that leaving or entering the particle hole continuum breaks some symmetry meaning that the Higgs mode could mediate some pairing effect on our system.

there is still confusion as to what a "Higgs mode" means in many-body physics. The Higgs mode is often called an amplitude mode even in the ab-

sence of a gauge field, which leads to such an ambiguity. Unlike the localized ferromagnet, the itinerant ferromagnet, as described by FFL, supports a "true" Higgs mechanism. In this work, we show that the Higgs mechanism in the ferromagnetic Fermi liquid may result in a highly unconventional form of superconductivity.

From the kinetic equation (3.23), the dispersions of the gapless Goldstone and gapped Higgs mode are given by the following [15], respectively,

$$\omega_1^{\pm}(p) = \frac{c_s^2}{\omega^{\pm}} p^2$$
$$\omega_2^{\pm}(p) = \omega^{\pm} - \frac{c_s^2}{\omega^{\pm}} p^2$$

where

$$\omega^{\pm} = \pm 2m_0 |F_0^a - F_1^a/3|$$
$$c_s^2 = |1 + F_0^a|(1 + F_1^a/3)v_F^2/3$$

As the Higgs mode arises due to the breaking of $SU(2)$, or spin symmetry, i.e. the relative spin phases remain constant but spin amplitude fluctuations are permitted, this causes it to exhibit an initial mass like effect as seen in Figure (3.3.2). This is possible when considering an itinerant ferromagnetic fermi liquid where spins can move. In contrast to the Stoner gap $\omega^{\pm}(F_0^a)$ that must be overcome in the Goldstone case, the gap $\omega^{\pm}(F_0^a, F_1^a)$ quantifies the mass of the Higgs mode in the itinerant ferromagnet. This mode sits close to the Stoner gap, is propagating at small momentum p, and becomes Landau damped at larger momentum p.

The introduction of the higher order Fermi liquid parameter F_1^a, here, is responsible for the propagation of the Higgs mode. This parameter couples the momentum of the quasi-particle to its spin and is responsible for pushing the mode out of the particle hole continuum. The Higgs only propagates in the presence of a finite-valued F_1^a, i.e. with a finite spin current, otherwise, there is no propagation of the Higgs mode as it becomes stuck in the particle-hole continuum. This is equivalent to stating that the mode lies in the spin orbit magnetic state in the F_1^a channel. By reconstructing the Fermi liquid theory in a gauge-invariant form, the massive propagating mode becomes the result of spontaneous breaking of a spin gauge symmetry in the FFL. This is apparent in figure (3.3.2), where numerical calculations show the Goldstone and Higgs mode merging in the particle-hole continuum, appearing to mirror the mass generation of $W^{\pm}$ and Z bosons by breaking an $SU(2) \times U(1)$ symmetry. This, in turn, could mediate some pairing effect on our system in analogy with High energy physics.

Chapter 4

FFL Modes' Effects on the Cooper Instability

4.1 The Green's Function

We can now proceed by taking a Green's function approach to better understand the many-body interaction between particles in our system. This approach should allow us to determine what effects the FFL phonon modes have on the Cooper Instability.[1]

4.1.1 The Fermionic Green's Function

The one particle Green's function in the free field Heisenberg operator representation can be written as[2]

$$G_{\alpha\beta} = -i\langle T(\tilde{\psi}_\alpha(x)\tilde{\psi}_\beta(x'))\rangle \tag{4.1}$$

where

$$\tilde{\psi}(\vec{r},t) = \frac{1}{\sqrt{V}}\sum_p a_p e^{i[\vec{p}\vec{r}-\epsilon_0(\vec{p})t]} \tag{4.2}$$

Thus, we can rewrite in term of the fermionic operators $c, c^\dagger$

$$G = -i\langle T(c(x,t)c^\dagger(x',t'))\rangle \tag{4.3}$$

We can then take the forward and backward in time components, to obtain

$$\begin{aligned} G &= -i\langle c(x,t)c^\dagger(x',t')\Theta(t-t') - c^\dagger(x,t)c(x',t')\Theta(t'-t)\rangle \\ &= \int c_\sigma(x,t)c^\dagger_\sigma(x,t)c_{k\sigma}(t)c^\dagger_{k\sigma}(t)e^{-ikr}dr \end{aligned} \tag{4.4}$$

[1] For all substantial calculations in this section and a more thorough analysis on the results presented here, see Appendix D.

[2] The understanding behind Green's functions was developed from [18],[19],[20].

We can now perform a Fourier transform to momentum space while setting $t' = 0$. We also desire a form $c_k(t) = f(t)c_k$. Now, recall $H = \sum_{k\sigma} \epsilon_k c^\dagger_{k\sigma} c_{k\sigma}$ and the fermionic anti-commutation relation $\{c, c^\dagger\} = cc^\dagger + c^\dagger c = \delta_{ij}$ where for one state we have $1 - c^\dagger c = cc^\dagger$. Using this, we require $c(t) = ce^{\frac{-i\epsilon}{\hbar}}$. One can then remove the time components by performing a temporal Fourier transform with an additional $e^{-\delta|t|}$ term added to describe behavior at infinite (∞) time,

$$G_0(k,\omega) = \int e^{i\omega t - \delta t} G_\sigma(k,t)dt = \frac{1 - n_{k\sigma}}{\omega - \frac{\epsilon_k}{\hbar} + i\delta} + \frac{n_{k\sigma}}{\omega - \frac{\epsilon_k}{\hbar} - i\delta} \tag{4.5}$$

The first term becomes zero for $\epsilon_k < \mu = \epsilon_F$ and nonzero for $\epsilon_k > \mu = \epsilon_F$. This term describes the propagation of the electron (positive time representation). The second term becomes zero for $\epsilon_k > \mu = \epsilon_F$ and nonzero for $\epsilon_k < \mu = \epsilon_F$. This term describes the propagation of the hole (negative time representation). Now, choose $\delta_k = sgn(\epsilon_k - \epsilon_F)\delta)$ and take $\hbar = 1$, measure energy relative to Fermi level μ.

$$G_0(k,\omega) = \frac{1}{\omega - \xi_k + i\delta_k} \tag{4.6}$$

where $\xi_k = \epsilon_k - \mu$.

4.1.2 The Bosonic Green's Function

We can begin with the same formalism as we used in the derivation for the one particle fermionic Green's function by writing the phonon Green's function as

$$D = -i\langle T(\tilde{\psi}(x,t)\psi(x',t')\rangle \tag{4.7}$$

where we define

$$\psi(r,t) = \frac{i}{\sqrt{V}} \sum_k \sqrt{\frac{\omega_0(k)}{2}} \left\{ b_k e^{i[k\cdot r - \omega_0(k)t]} - b_k^\dagger e^{-i[k\cdot r - \omega_0(k)t]} \right\} \tag{4.8}$$

as we need more than the displacement operator ϵ if we eventually want to include the interaction between phonons and electrons in a metal. Here, we take $|k| < k_0$, and as there are no phonons in the ground state, this implies $x', t' \to 0$. In this step, we have also expanded the normal position $\mathbf{q}$ in plane waves to obtain $\mathbf{q_k}$ and translated these into new operators $b_k = q_k\sqrt{2\rho\omega_0(k)}$ which obey the usual bosonic commutation relations. Substituting this expression $(D.12)$ into expression $(D.11)$ to obtain

$$D^{(0)}(r,t) = -i\langle T(\tilde{\psi}(x,t)\psi(x',t')\rangle = \frac{i}{V}\sum_k \frac{\omega_0(k)}{2} \begin{cases} e^{i[k\cdot r - \omega_0(k)t]} & t > 0 \\ e^{-i[k\cdot r - \omega_0(k)t]} & t < 0 \end{cases} \tag{4.9}$$

as because we are in the ground state $b_k^\dagger b_k = n = 0$ and $b_k b_k^\dagger = 1 - n = 1$. Further, we have that $b_{k'} b_k, b_{k'}^\dagger b_k, b_k b_{k'}^\dagger, b_{k'}^\dagger b_k, b_{k'} b_k^\dagger \to 0$. The remaining terms $b_{k'}^\dagger b_k^\dagger$ also go to 0. Taking the limit as $k \to \infty$ and performing a temporal transform where the sum becomes an integral we obtain

$$D^{(0)}(k,\omega) = \int e^{-\delta t} D^{(0)}(r,t)dt = \frac{\omega_0^2(\vec{k})}{\omega^2 - \omega_0^2(\vec{k}) + i\delta} \tag{4.10}$$

4.2 Goldstone Mode's Effect on s-Wave Pairing

We can now take a Feynman diagrammatic approach to determine what the FFL mode's affects on particle particle pairing are.[3] The vertex describes the many body interaction of the itinerant ferromagnetic fermi liquid, which is connected to the scattering amplitude. In particular, it describes the coupling between a photon and an electron beyond the leading order of perturbation theory.

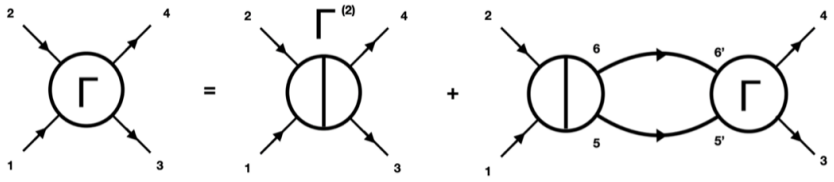

Figure 4.1: Vertex function for conventional BCS theory of superconductivity. This diagram describes the vertex integral we will need to solve in order to determine what the affect on particle particle pairing is. The first term, $\Gamma^{(2)}$, is the two-particle irreducible vertex, and both terms describe the interaction between particles within our system.

4.2.1 Zero temperature Vertex Function

It is important to note that here, p is the quasiparticle momentum, p_0 is the bare particle momentum, and p_F is the Fermi momentum. The integral

[3]This derivation was largely aided by Professor Bedell's *Notes on Superfluid FLuctuations*[21].

equation is

$$\Gamma(1,2;3,4) = \Gamma^{(2)}(1,2;3,4) \tag{4.11}$$
$$+\frac{1}{2}\sum_{5,5';6,6'} \Gamma^{(2)}(1,2;5,6)G(5,5')G(6,6')\Gamma(5',6',3,4)$$

where $\Gamma^{(2)}$ is the two-particle irreducible vertex which describes the non-interacting contribution and Γ describes the interacting vertex. Here the labels $1,2$ etc. are defined by $1 = (p_1, \epsilon_1, \sigma)$, etc. Now, let us introduce new variables $p, p+k$ with $p = (p, \epsilon)$ and $k = (k, \Gamma)$. In this section, p is the quasiparticle momentum, p_0 is the bare particle momentum, and p_F is the Fermi momentum. We also have that $G(5,5') = G(5)\delta_{5,5'} = G(5)\delta_{p_5,p_5'}\delta_{\sigma_5,\sigma_5'}$ for paramagnetic systems. We now use the labels $1,2$, etc. for spin labels. For convenience under the integrals we have made a change of variables, $p" \to p" - \frac{k}{2}$. We will show that the integral equation $D.16$ decomposes into two separate integral equations; one for spin singlet Γ_s and one for triplet spins Γ_t. Now, making use of these transformations the integral equation becomes

$$\Gamma_{1,2;3,4} = \Gamma^{(2)}_{1,2;3,4} + \frac{1}{2}\sum_{p"} G(p")G(k-p")\sum_{5,6}\Gamma^{(2)}_{1,2;5,6}\Gamma_{5,6;3,4} \tag{4.12}$$

For paramagnetic systems both Γ and $\Gamma^{(2)}$ have the same spin structure, thus, $\Gamma_{1,2;3,4} = \Gamma^s\delta_{13}\delta_{24} + \Gamma^a\sigma_{24}\sigma_{13}$ and $\Gamma^{(2)}_{1,2;3,4} = \Gamma^{(2)s}\delta_{13}\delta_{24} + \Gamma^{(2)a}\sigma_{24}\sigma_{13}$ where $\Gamma_e = -\Gamma_e$ is standard notation. The triplet amplitude Γ_t is defined by

$$\Gamma_t = \Gamma_{\uparrow\uparrow;\uparrow\uparrow} = \Gamma_{\uparrow\uparrow} = \Gamma^s + \Gamma^a \tag{4.13}$$

The singlet amplitude Γ_s is

$$\Gamma_s = \Gamma^s - 3\Gamma^a = 2\Gamma_{\uparrow\downarrow} - \Gamma_{\uparrow\uparrow} = \Gamma_d + \Gamma_e \tag{4.14}$$

Identical expressions can be written for $\Gamma^{(2)}$. We can then proceed by considering the the equation for $\Gamma_{\uparrow\uparrow}$ and sum on 5 and 6. Once the integral equation for $\Gamma_{\uparrow\downarrow}$ is constructed and using the definition $\Gamma_s = 2\Gamma_{\uparrow\downarrow} - \Gamma_{\uparrow\uparrow}$ we have that,

$$\Gamma_s = \Gamma^{(2)}_s + \frac{1}{2}\sum_{p"}\Gamma^{(2)}_s G(p")G(k-p")\Gamma_s \tag{4.15}$$

Following this, we can study the singularities of Γ as $k \to 0$. The analysis is identical for spin singlet and triplet, and thus, we can drop the subscripts such that

$$\Gamma(p,p';k) = \Gamma^{(2)}(p,p';k) \tag{4.16}$$
$$+\frac{1}{2}\sum_{p"}\Gamma^{(2)}(p,p" - \frac{k}{2};k)G(p")G(k-p")\Gamma(p" - \frac{k}{2};k)$$

Before we analyze the singularities of this function and compute the integral equations for the vertex, we expand $\Gamma^{(2)}(p,p')$ and $\Gamma(p,p';k)$ in partial waves. For small k we have that $p\cdot p" = p\cdot p" - \frac{p\cdot k}{2}$ and $\frac{p\cdot k}{2} = 0$ for $|p_n| = k_F$. Thus, in the limit $k\to 0$ we have,

$$\sum_{l,l'}\int \frac{d\Omega"}{4\pi}\Gamma^{(2)}_l\Gamma_{l'}(\lambda)P_l(\hat{p},\hat{p}")P_l(\hat{p}"\cdot\hat{p}') = \sum_l \Gamma^{(2)}_l\Gamma_l\frac{P_l(\hat{p}\cdot\hat{p}')}{2l+1} \tag{4.17}$$

Note here that

$$P_l(\hat{p}\cdot\hat{p}") = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_l^{*m}(\theta,\phi)Y_l^m(\theta",\phi") \tag{4.18}$$

where $Y_l^m(\theta,\phi) = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}P_l^m(\cos\theta)e^{im\phi}$ and $\int d\Omega Y_l^m(\theta,\phi)Y_{l'}^{m'}(\theta,\phi) = \delta_{l,l'}\delta_{m,m'}$. This equation decouples into partial waves,

$$\sum_{l'}\int\frac{d\Omega"}{4\pi}P_l(\hat{p}\cdot\hat{p}")P_l(\hat{p}"\cdot\hat{p}') = \frac{1}{2l+1}P_l(\hat{p}\cdot\hat{p}') \tag{4.19}$$

Note that we carried this step out first as it is more convenient. We can now look at the second term of our vertex. Now that we have established the proper form of our function, let us go back to focusing on the singularities of the vertex function. In equation (4.16), it is important to note again that we are working in the limit of small k, and that,

(a) $\Gamma^{(2)}$ is slowly varying function of k on the scale set by k_F. Therefore, we can set $k=0$ in $\Gamma^{(2)}$.

(b) The most important contribution comes out of $G(p")G(k-p")$ as $k\to 0$.

(c) The most significant of this being the vicinity of $|p"|\approx k_F$ and $\epsilon"\approx 0$. In this region, $G(p")G(k-p")$ can be written.

Continuing, we can see that

$$G(p")G(k-p") \tag{4.20}$$
$$\to \phi(p")\left\{\frac{Z_{p"}}{\epsilon" - \xi_{p"} + i\delta sgn(p"-k_F)}\frac{Z_{p"}}{\lambda-\epsilon"-\xi_{p"}+v_{p"}\cdot k + i\delta sgn(|p"-k|-k_F)}\right\}$$

where the quasiparticle residue, or weight, $Z_{p"}$ has been introduced into each Green's function and $\xi_{p"} = v_p(p"-k_F)$ for $p"\approx k_F$. Now, fully expanding the summation in $(D.25)$ and using the above, we can write the integral equations for the vertex as

$$\Gamma(p,p';k) = \Gamma^{(2)}(p,p') + \frac{i}{2}\int\frac{d^4p"}{(2\pi)^4}\Gamma^{(2)}(p,p")\times \tag{4.21}$$
$$\left\{\phi(p") + \frac{Z_{p"}}{\lambda-\epsilon"-\xi_{p"-k}+i\delta sgn(|p"-k|-k_F)}\frac{Z_{p"}}{\epsilon"-\xi_{p"}+i\delta sgn(p"-k_F)}\right\}\Gamma(p",p';k)$$

with $\xi_{p-k} = \epsilon_{p-k} - \epsilon_F \approx -v_p \cdot k$. At small k, $v_p = \frac{\partial \epsilon_p}{\partial p}$. We can now linearize the expression where for $p \approx k_F$, $v_p = \frac{k_F}{m^*}$ and $\xi_p = v_F(p - p_F)$. The largest contribution to the integral will come from the neighborhood of $\epsilon" = 0$ and $\xi_p" = 0$. If we assume $\Gamma + \Gamma^{(2)}$ vary slowly in this region we can perform the frequency and momentum integrals where we know that $p"$ remains finite. Moreover, on the Fermi surface $p" \approx k_F$ with $\epsilon" = 0$ the excitations become pure quasiparticle excitations, which implies that $\phi(p" = k_F, \epsilon" = 0) = 0$. The integral is also concentrated around k_F. We can proceed to pull $Z_{p"} = Z$ out and write the integral as

$$- iZ^2 k_F^2 \int \frac{d\Omega"}{(2\pi)^2} \int_0^\infty \frac{dp"}{2\pi} \int_{-\infty}^\infty \frac{d\epsilon"}{2\pi} \times \tag{4.22}$$
$$\frac{1}{\epsilon" - \xi_{p"} + i\delta sgn(p" - k_F)} \frac{-1}{[\lambda - \epsilon" + \xi_{k-p"} - i\delta sgn(|k - p"| - k_F)]}$$

Only those terms will survive in which, (a) $p" \geq k_F$, $|p" - k| \geq k_F$, which will correspond to the $(1 - n_{p"})(1 - n_{k-p"})$ terms, and (b) $p" \leq k_F$, $|p" - k| \leq k_F$, which will correspond to the $n_{p"} n_{k-p"}$ terms. For (a) we have a closed contour in the lower half plane and for (b) in the upper half plane. Therefore, the above expression reduces to the following:

$$- iZ^2 k_F^2 \int \frac{d\Omega"}{(2\pi)^2} \int_0^\infty \frac{dp"}{2\pi} \int_{-\infty}^\infty \frac{d\epsilon"}{2\pi} \times \tag{4.23}$$
$$\left(\frac{n_{p"} n_{k-p"}}{\epsilon" - \xi_{p"} - i\delta} \frac{1}{[\lambda - \epsilon" + \xi_{k-p"} + i\delta]} + \frac{(1 - n_{p"})(1 - n_{k-p"})}{\epsilon" - \xi_{p"} + i\delta} \frac{1}{[\lambda - \epsilon" + \xi_{k-p"} - i\delta]} \right)$$

We will now proceed by first calculating the ϵ integral, then the k integral, and then the Ω integral over the function. First, let us simplify the ϵ integral. Using the Sokhotski-Plemelj formula, $\lim_{\delta \to 0} \frac{1}{x - x_0 \pm i\delta} = P\frac{1}{x - x_0} \mp i\pi\delta(x - x_0)$, the above ϵ integral reduces the integrand to

$$i \left(\frac{(1 - n_{p"})(1 - n_{k-p"})}{\xi_{p"} + \xi_{k-p"} - \lambda - i\delta} - \frac{n_{p"} n_{k-p"}}{\xi_{p"} + \xi_{k-p"} - \lambda + i\delta} \right) \tag{4.24}$$

in the limits $\lim_{\delta sgn(p" - k_F) \to 0}, \lim_{\delta sgn(|p" - k| - k_F) \to 0}$ [22][23]. We just solved a closed contour in the first integral in the upper half plane and a closed contour in the second integral in the lower half plane. Plugging equation $(D.44)$ into equation $(D.45)$, we obtain the simplified integral term of

$$Z^2 k_F^2 \int \frac{d\Omega"}{(2\pi)^2} \int_0^\infty \frac{dp"}{2\pi} \left(\frac{(1 - n_{p"})(1 - n_{k-p"})}{\xi_{p"} + \xi_{k-p"} - \lambda - i\delta} - \frac{n_{p"} n_{k-p"}}{\xi_{p"} + \xi_{k-p"} - \lambda + i\delta} \right) \tag{4.25}$$

We will now compute the momentum integral, followed by the x component of the Ω integral, and lastly the ϕ component of the Ω integral. In order to compute the momentum integrals we must consider the phase space

restrictions imposed by the Fermi factors: For $n_{p"}n_{p"-k}$ we have $p" \leq k_F$ and $|\vec{p}" - \vec{k}| \leq k_F$ or $p"^2 - 2kxp" + k^2 \leq k_F^2$. Now, if we integrate freely over x we must restrict $p"$, $p"^2 - 2kxp" - (k_F^2 - k^2) \leq 0 \implies p_+" \approx k_F + kx$ to order $\frac{k^2}{k_F^2}$ where we can see $p_+"^2 - 2kxp_+" - (k_F^2 - k^2) = 0$, and therefore, $p" \leq p_+"$. We can use the equality $\xi_{p"} + \xi_{k-p"} = 2\xi_{p"} - v_{p"} \cdot k = 2v_F(p" - k_F) - v_F kx"$ and employ a change of variables by setting $\xi = \xi_{p"} = v_F(p" - k_F)$ and $d\xi = v_F dp"$ so that $\xi_{p"} + \xi_{k-p"} = 2\xi_{p"} - v_F kx"$. We also intend to reduce the first integral by introducing a cutoff, U, in the ξ integrand where $k_B T_c << U << \epsilon_F = k_B T_F$. Therefore, using these substitutions and adjusting the limits of each integral, we can write the integral over x and p as

$$\int_{-1}^{1} dx" \int_{0}^{\infty} dp" \left(\frac{(1 - n_{p"})(1 - n_{k-p"})}{\xi_{p"} + \xi_{k-p"} - \lambda - i\delta} - \frac{n_{p"} n_{k-p"}}{\xi_{p"} + \xi_{k-p"} - \lambda + i\delta} \right)$$

$$= \int_{-1}^{1} \frac{dx"}{2} \left\{ \ln \left| \frac{2U - \lambda - v_F kx"}{v_F kx" - \lambda} \right| + \ln \left| \frac{2U + \lambda - v_F kx"}{v_F kx" + \lambda} \right| + i\pi \right\} \quad (4.26)$$

which admits

$$\Gamma(p, p'; k) = \Gamma^{(2)}(p, p') + \frac{i}{2} \left(\frac{-ik_F^2}{2\pi^2} \right) \left(\frac{Z^2}{2v_F} \right) \int \frac{d\Omega"}{4\pi} \times \quad (4.27)$$

$$\Gamma^{(2)}(p, p")\Gamma(p", p'; k) \underbrace{\left\{ \ln \left| \frac{2U - \lambda - v_F kx"}{v_F kx" - \lambda} \right| + \ln \left| \frac{2U + \lambda - v_F kx"}{v_F kx" + \lambda} \right| + i\pi \right\}}_{R(k,x")}$$

We can look back at our expansion of the vertex function in partial waves, equation $(D.34)$ and see that the vertex function is now complete with the addition of one more step. We must compute the integral over Ω, which decomposes into an integral over x and ϕ, and produces the result

$$\Gamma_m(k) = \Gamma_m^{(2)} + \frac{1}{2} \left(\frac{k_F^2}{2\pi^2} \right) \left(\frac{Z^2}{2v_F} \right) \Gamma_m^{(2)} \int_{-1}^{1} \frac{dx"}{2} R(k, x") \Gamma_{m'}(k) \quad (4.28)$$

Here, we set $k = 0$ in $R(k, x")$, which leaves no x dependence and leaves the vertex as

$$\Gamma_l = \frac{\Gamma_l^{(2)}}{1 + \frac{Z^2 N(0)}{4(2l+1)} \Gamma_l^{(2)} \left\{ \ln \frac{2U}{\lambda} + i\frac{\pi}{2} \right\}} \quad (4.29)$$

It is important to note that this vertex will take the same form whether you are working in the spin singlet or triplet case. We can now make some useful observations. For $\Gamma_l^{(2)} > 0$, Γ_l has no poles. For $\Gamma_l^{(2)} < 0$, Γ_l has a pole on the positive imaginary axis. Further, we can analytically continue to $\lambda \to i\lambda_c$, which admits

$$\ln \frac{2U}{i\lambda_c} = \ln \frac{-i2U}{\lambda_c} = \ln \frac{2U}{\lambda_c} - i\frac{\pi}{2} \quad (4.30)$$

Thus, we have a pole when

$$1 + \frac{Z^2 N(0)}{4(2l+1)} \Gamma_l^{(2)} \ln \frac{2U}{\lambda_c} = 0 \implies \lambda_c = 2U \exp\left\{ \frac{1}{\frac{N(0) Z^2 \Gamma_l^{(2)}}{4(2l+1)}} \right\} \tag{4.31}$$

which will only happen for the case when $\Gamma_l^{(2)}$ is attractive since $2U >> \lambda_c$.

4.2.2 Finite Temperature Vertex Function

We have just derived the zero temperature vertex function and found that is was singular for $k = 0$ and $i\lambda \to \lambda_c$. To determine the critical temperature of the vertex we must find the finite temperature vertex function $\mathcal{T}$. The spin structure is identical to the zero temperature case so we have singlet and triplet terms.

$$\begin{aligned} \mathcal{T}(p_1, p_2; p_3, p_4) &= \mathcal{T}^{(2)}(p_1, p_2; p_3, p_4) \\ &\quad - \frac{T}{2} \sum_{p_5, p_6} \mathcal{T}^{(2)}(p_1, p_2; p_5, p_6) \mathcal{G}(p_5) \mathcal{G}(p_6) \mathcal{T}(p_4, p_6; p_5, p_4) \end{aligned} \tag{4.32}$$

We can set $k_B = 1, p_i = (p_i, \omega_{n_i}), \omega_{n_i} = (2n_i + 1)\pi T$. The sum we must solve becomes

$$\sum_{p_N} = \sum_{\omega_{n_i}} \int \frac{d^3 p_i}{(2\pi)^3} \tag{4.33}$$

New variables can be defined, as we did for the zero temperature Green's function, by setting $p" \to p" - \frac{k}{2}$, the integral equation becomes

$$\begin{aligned} \mathcal{T}(p, \omega_n, p', \omega_{n'}; k, \lambda_n) &= \mathcal{T}^{(2)}(p, \omega_n, p', \omega_{n'}; k, \lambda_n) \\ &\quad - \frac{T}{2} \sum_{\omega_{n"}} \int \frac{d^3 p"}{(2\pi)^3} \mathcal{T}^{(2)}(p, \omega_n, p", \omega_{n"}; k, \lambda_n) \mathcal{G}(p", \omega_{n"}) \times \\ &\quad \mathcal{G}(k - p", \lambda_n - \omega_{n"}) \mathcal{T}(k - p", \lambda_n - \omega_{n"}, p', \omega_{n'}; k, \lambda_n) \end{aligned} \tag{4.34}$$

Since we want to study this function in the limit $\lambda \to 0$ and $k \to 0$, it is necessary to construct an analytic continuation of this function. The spectral representation for the $\mathcal{G}$'s is,

$$\mathcal{G}(p, \omega_n) = \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \frac{p(p, \epsilon)}{i\omega_n - \epsilon} \qquad \mathcal{G}_{qp}(p, \omega_n) = \frac{Z_p}{i\omega_n - \xi_p} \tag{4.35}$$

where $\xi_p = v_F(p - k_F)$, $|p| \approx k_F$, and $T << T_F$. Moreover, in the limit $k \to 0$, $|p| = |p'| = k_F$ the function $\mathcal{T}$ will depend only on k and λ_n. We can write the spectral representation as

$$\mathcal{T}(k, \lambda_n) = \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \frac{\sigma(k, \epsilon)}{i\lambda_n - \epsilon} \tag{4.36}$$

Note $\mathcal{T}$ will actually depend on the angle between p and p' but not on their magnitudes. As the interest is in the limit $T << T_F$ amd $k \to 0$, $\mathcal{T}^{(2)}$ can be replaced by $\Gamma^{(2)}$ at $k = 0$, which is the zero temperature value of $\mathcal{T}^{(2)}$. $\mathcal{T}^{(2)}$ is not anomalous as $k \to 0$ and is slowly varying as a function of T. Furthermore, most of the weight in the integrand of equation $(D.62)$ comes from $|p_i| \approx k_F$ since, in this region, the $\mathcal{G}$'s$\sim \frac{1}{i\omega_n}$, $T << T_F$. Assuming that $\mathcal{T}^{(2)}$ and $\mathcal{T}$ are slowly varying functions in this region, we can bring them out of the frequency sum. Once expanding the product of $\mathcal{G}'s$ in the sum, the finite temperature vertex reduces to

$$\mathcal{T} = \Gamma^{(2)} - \frac{1}{2}\int \frac{d\Omega"}{4\pi}\frac{Z^2 k_F^2}{2\pi^2 v_F}\Gamma^{(2)}\mathcal{T}\int_0^U d\xi \frac{\tanh\frac{\beta\xi}{2}}{2\xi - i\lambda_n} \tag{4.37}$$

where we restrict ourselves to the region near k_F so that $d^3p" = d\Omega" k_F^2 dp" = d\Omega" k_F \frac{d\xi_{p"}}{v_F}$. Using $\mathcal{T} \to \Gamma^{(+)}$, or the retarded function defined in the upper half plane, where $i\lambda_n \to \lambda + i\delta$, we can write

$$\Gamma_l^{(+)}(\lambda, T) = \frac{\Gamma_l^{(2)}}{1 + \frac{Z^2 N(0)\Gamma_l^{(2)}}{4(2l+1)}\int_0^U d\xi \frac{\tanh\frac{\beta\xi}{2}}{2\xi - (\lambda + i\delta)}} \tag{4.38}$$

which is found from the same decomposition as in the $T = 0$ case. For $\lambda \to 0$ we can determine the temperature T_c at which $\Gamma_l^{(+)}$ is singular. First, we must reduce the ξ integral in order to analyze the poles of this function. Employ a change of variables where $x = \frac{\beta\xi}{2} \implies \xi = \frac{2x}{\beta}$ and $dx = \frac{\beta}{2}d\xi \implies \frac{2}{\beta}dx = d\xi$. We also define $x_0 = \frac{\beta U}{2}$, such that

$$\int_0^U d\xi \frac{\tanh\frac{\beta\xi}{2}}{2\xi} = \ln\frac{2\gamma\beta U}{\pi} \tag{4.39}$$

The final form of the vertex for finite temperature becomes

$$\Gamma_l^{(+)}(\lambda, T) = \frac{\Gamma_l^{(2)}}{1 + \frac{Z^2 N(0)\Gamma_l^{(2)}}{4(2l+1)}\ln\frac{2\gamma U}{T\pi}} \tag{4.40}$$

where we get a pole as $T \to T_c$ for $\Gamma_l^{(2)} < 0$ and $U >> T_c$, i.e.

$$1 + \frac{Z^2 N(0)\Gamma_l^{(2)}}{4(2l+1)}\ln\frac{2\gamma U}{T\pi} = 0 \implies T_c^l = 1.13 U e^{\frac{1}{g_l}} \tag{4.41}$$

where $g_l = \frac{Z^2 N(0)\Gamma_l^{(2)}}{4(2l+1)}$ with $N(0) = \frac{k_F^2}{\pi^2 v_F} = \frac{k_F m^*}{\pi^2}$. As in the zero-temperature case, it is important to note that the finite temperature vertex will take the same form whether you are working in the spin singlet or triplet case. For Helium-3,

$$T_c^l = 1.13\alpha T_F e^{\frac{1}{g_l}} \tag{4.42}$$

as T_F is our scale of energies in ^3He, therefore, we set $U = \alpha T_F$. This cutoff is more accurately determined from the cutoff frequency of the particular phonon mode of interest.

$$\begin{aligned} \textbf{Normal Phonon} &\implies U \sim \omega_D, \textbf{Debye frequency} \omega_D \\ \textbf{Goldstone Phonon} &\implies U \sim \omega^+(F_0^a), \textbf{Stoner gap} \\ \textbf{Higgs Phonon} &\implies U \sim \omega^+(F_0^a, F_1^a), \textbf{Higgs gap} \end{aligned}$$

The cutoff U for a normal phonon mode goes like the debye frequency, the Goldstone mode goes like the Stoner gap, which is not a lot of energy to excite, and the Higgs mode goes like the Higgs gap, which is a larger gap to overcome, as illustrated above. This shows that the Higgs mechanism may lead to stronger coupling of electrons, and thus, a higher T_C value.

4.2.3 Corrections to the Description of s-Wave Superconductivity Induced by Weak Ferromagnetism

We will now solve for the finite temperature vertex corrections to the three-point vertex in the weak ferromagnetic metal. These can be written as

$$\begin{aligned} \Gamma^{(1)}_{\uparrow\uparrow}(k, k+q) &= \Gamma^{(1)}_{\uparrow\uparrow l} + \Gamma^{(1)}_{\uparrow\uparrow G} \\ \Gamma^{(1)}_{\uparrow\downarrow}(k, k+q) &= \Gamma^{(1)}_{\uparrow\downarrow l} + \Gamma^{(1)}_{\uparrow\downarrow G} \end{aligned} \tag{4.43}$$

In the above expression, the terms can be defined as

$$\begin{aligned} \Gamma^{(1)}_{\uparrow\uparrow l} &= i g_0^2 \int dp G_\uparrow(p) D_l(p-k) G_\uparrow(k+q) \\ \Gamma^{(1)}_{\uparrow\uparrow} &= i g_0^2 \int dp G_\uparrow(p) D_G(p-q) G_\downarrow(k+q) \\ \Gamma^{(1)}_{\uparrow\downarrow l} &= i g_0^2 \int dp G_\downarrow(p) D_l(p-q) G_\uparrow(k+q) \\ \Gamma^{(1)}_{\uparrow\downarrow G} &= i g_0^2 \int dp G_\downarrow(p) D_G(p-q) G_\downarrow(k+q) \end{aligned} \tag{4.44}$$

We can generalize these down to two expression

$$\begin{aligned} \Gamma^{(1)}_{\sigma\sigma' l} &= i g_0^2 \int dp G_\sigma(p) D_l(p-k) G_{\sigma'}(p+q) \\ \Gamma^{(1)}_{\sigma\sigma' G} &= i g_0^2 \int dp G_\sigma(p) D_G(p-k) G_{\sigma'}(p+q) \end{aligned} \tag{4.45}$$

We also know that

$$\begin{aligned} G_\sigma(k,\omega) &= \frac{Z}{\omega - v_F(|k|-k_\sigma) + i\delta_\sigma(k)} \\ D_G(p,\omega) &= -\frac{\Delta N(0) v_F}{2} \frac{\Omega_0(p)}{(\omega+i\delta)^2 - \Omega_0^2(p)} \\ D_l(p,\omega) &= -\frac{N(0)p_F^2}{2} \frac{1}{\xi^{-2} + |p|^2 - \frac{i\pi p_F^2 \omega}{2v_F|p|}} \end{aligned} \tag{4.46}$$

where k is the three dimensional momentum of the particle, k_σ is the Fermi momentum of the spin-σ electrons, $N(0)$ is the average density of states over the Fermi Surfaces, p is the quasiparticle momentum, v_F is the Fermi velocity, and $\delta_\sigma = \delta \times \text{ sign}(|k|-k_\sigma)$, with δ an infinitesimal real number. The exchange splitting Δ takes the form $\Delta = k_\uparrow - k_\downarrow$ and $\Omega_0(p) = D|p|^2$ where $D = \frac{v_F \Delta}{k_F^2}$ is the spin stiffness. The electron Green's function G describes a system of quasiparticles with spontaneous magnetization given by Dzyaloshinskii's theorem,

$$m_0 = \frac{1}{12\pi^2}(k_\uparrow^3 - p_\downarrow^3) = \frac{n_\uparrow - n_\downarrow}{2} \tag{4.47}$$

where m_0 is the uniform, static magnetization, and n_σ is the density of spin-σ particles [5]. When calculating the vertex corrections, we first set the frequency ω to zero and then proceed to take the limit for the momentum. The phonon Green's function $D_{G/l}$ is the spin wave analog of the phonon propagator. The phonon dispersion Ω_0 represents the phonon mode's dispersion of interest. Note that in the itinerant FFL case, the Fermi momentum, k_σ of the electrons located inside the electron Green's function G, depends on the spin, σ, of the electrons in contrast to traditional BCS theory.

As we are working in the broken symmetry phase, we must make a distinction between corrections involving particles on one of two Fermi surfaces and corrections involved with particles on different Fermi surfaces. In the former and latter case we have the limits, $\Gamma_{\sigma\sigma}(|k| \to k_\sigma, |k| \to k_\sigma)$ and $\Gamma_{\sigma\sigma'}(|k| \to k_\sigma, |k| \to k_{\sigma'} + \Delta)$, respectively. For these cases, we use the spectral representation for the propagators $D_{l/G}$,

$$D_{l/G}(p,\omega) = \frac{2}{\pi}\int_0^\infty dz \frac{z \text{Im} D_{l/G}(p,z)}{z^2 - \omega^2 - i\delta} \tag{4.48}$$

Writing $G_\sigma(k+p, \epsilon+\omega)$, we can expand this as

$$\begin{aligned} G_\sigma(k+p,\epsilon+\omega) = &\frac{Z f_{k'}}{\epsilon + \omega - v_F(|k+p|-k_\sigma) + i\delta_\sigma(k+p)} \\ &+ \frac{Z(1-f_{k'})}{\epsilon + \omega - v_F(|k+p|-k_\sigma) - i\delta_\sigma(k+p)} \end{aligned} \tag{4.49}$$

where $f_{k'} = \theta(|k+p|-k_F)$, which allows us to expand $G_\sigma(k+p, \epsilon+\omega)G_{\sigma'}(k+p+q, \epsilon+\omega)$ and set $q = 0$. In these expressions, we use a Sokhotski-Plemelj expansion where all $\frac{1}{\epsilon+\omega}\delta \to 0$ as these are far from the Fermi surface and we set $\xi_{k_\sigma} = v_F(|k+p|-k_\sigma)$. Spins were also insignificant here, which is why we can combine terms. Therefore, the two integrals we must solve for the vertex corrections at zero temperature are

$$\Gamma^{(1)}_{\sigma\sigma' l/G} = \frac{i2Zg_0^2}{v_F}\int\frac{d\Omega}{(2\pi)^2}\int_0^\infty\frac{dp}{2\pi}\int_{-\infty}^\infty\frac{d\epsilon}{2\pi}\int_0^\infty\frac{dz}{\pi}\times$$
$$\frac{z\mathrm{Im}D_{l/G}(p,z)}{z^2-\omega^2-i\delta}[G_\sigma(k+p,\epsilon+\omega)G_{\sigma'}(k+p,\epsilon+\omega)]$$

$$= \frac{i2Z^2g_0^2}{v_F}\int\frac{d\Omega}{(2\pi)^2}\int_0^\infty\frac{dp}{2\pi}\int_{-\infty}^\infty\frac{d\epsilon}{2\pi}\int_0^\infty\frac{dz}{\pi}\times$$
$$\mathrm{Im}D_{l/G}(p,z)\left(\frac{1}{z-\omega+i\delta}-\frac{1}{z+\omega-i\delta}\right)\times \tag{4.50}$$
$$\left[\frac{f_{k'}}{[\epsilon-\xi_{k_\sigma}+\omega+i\delta_\sigma(k+p)]^2}+\frac{(1-f_{k'})}{[\epsilon-\xi_{k_\sigma}+\omega-i\delta_\sigma(k+p)]^2}+\pi^2\delta(\epsilon-\xi_{k_\sigma}+\omega)\right]$$

As before, we will first solve the ϵ integral. Recalling the Sokhotski-Plemelj formula, $\lim_{\delta\to0}\frac{1}{x-x_0\pm i\delta} = P\frac{1}{x-x_0}\mp i\pi\delta(x-x_0)$. We can then continue to set $\epsilon \to 0$, consider z small, and take the integral over $d\omega$. It is important to note here that the principal value terms become negligible in the bosonic expansion because we are considering z small. Therefore, we can set $\omega = 0$, $|k_\sigma| = k_F$, which allows the integral to become

$$\Gamma^{(1)}_{\sigma\sigma' G} = -\frac{2Z^2g_0^2}{v_F}\int\frac{d\Omega}{(2\pi)^2}\int_0^\infty\frac{dp}{2\pi}\int_0^\infty\frac{dz}{2\pi}\mathrm{Im}D_{l/G}(p,z)\times$$
$$\left[\frac{f_{k'}}{(\xi_{k_F}-z)^2}+\frac{(1-f_{k'})}{(\xi_{k_F}+z)^2}+\pi^2\left(\delta(z-\epsilon_{k_F})-\delta(z+\epsilon_{k_F})\right)\right] \tag{4.51}$$

and the imaginary bosonic term for the Goldstone mode [24] becomes

$$\mathrm{Im}\left[\frac{\Omega_0}{(\omega+i\delta)^2-\Omega_0^2}\right] = \delta(|\omega|-\Omega_0) \tag{4.52}$$

Plugging this expression back into the above integral, setting $\Omega_0 = Dp^2+g$, and taking $\phi \to 0$ we come to the expression

$$\Gamma^{(1)}_{\sigma\sigma' G} = \frac{Z^2g_0^2\Delta N(0)}{2}\int_{-1}^1\frac{dx}{(2\pi)^2}\int_0^\infty\frac{dp}{2\pi}\times \tag{4.53}$$
$$\left[\frac{f_{k'}}{(\xi_{k_F}-(Dp^2+g))^2}+\frac{(1-f_{k'})}{(\xi_{k_F}+(Dp^2+g))^2}+\pi^2\left[\delta(Dp^2+g-\epsilon_{k_F})-\delta(Dp^2+g+\epsilon_{k_F})\right]\right]$$

I will now proceed to compute the momentum and space integrals over the respective heaviside step functions while setting $R = \frac{Z^2 g_0^2 \Delta N(0)}{(2\pi)^3}$. recall that $\xi_{k_F} = v_F(|k+p|-k_F)$. As before we will consider $p \approx k_F + px$. Setting the gapped term to $g = 0$ and reimplementing the cutoff p_C where $k_B T_c << p_c << \epsilon_F = k_B T_F$ the vertex [25] becomes

$$\Gamma^{(1)}_{\sigma\sigma' G} = R \int_{-1}^{1} dx \int_{0}^{\infty} dp \left[\frac{f_{k'}}{(\xi_{k_F} - Dp^2)^2} + \frac{(1 - f_{k'})}{(\xi_{k_F} + Dp^2)^2} \right] \tag{4.54}$$

and reduces to

$$\Gamma^{(1)}_{\sigma\sigma';G} = \frac{g_0^2 N^2(0) Z^2}{4} \log \left[1 + \frac{2Dmp_C}{k_F} \right] \tag{4.55}$$

with $m \to 0$, $p_C << k_F$, and $D = \frac{v_F \Delta}{k_F^2}$. This is the final form for the Goldstone mode's effect on the zero-temperature vertex, where significant momentum dependence lies in the log. Very similar logarithmic behavior would emerge from including an additional term with the a second p_sigma term. This implies that the self-energy is weakly momentum dependent close to the phase transition and therefore a local ferromagnetic Fermi-liquid theory can be used to describe weak ferromagnetic metals where the magnetization is sufficiently small. This confirms an s-wave pairing instability induced by the Goldstone mode within the FFL.

4.3 Higgs Mode's Effect on p-Wave Pairing

A similar procedure as in the Goldstone case can be carried out while including the Higgs modes's dispersion in the zero-temperature vertex function. We will begin from equation (4.53), where, rather than set $g = 0$, we can set $g = \omega^{\pm}$ as in the Higgs dispersion.

4.3.1 Zero-Temperature Vertex Calculation

I will now proceed by computing the imaginary bosonic term for the Gapped, or Higgs, mode. Setting g as the gap, we have

$$\mathrm{Im} \left[\frac{(\Omega_0 + g)}{(\omega + i\delta)^2 - (\Omega_0 + g)^2} \right] = \delta(|\omega| - (\Omega_0 + g)) \tag{4.56}$$

In the expansion of equation (4.53), an additional gapped term does that change the integral over x. However, we will see an additonal term arise from the momentum dependence. Therefore, the Higgs result takes the

form,

$$\Gamma^{(1)}_{\sigma\sigma' H} = \frac{g_0^3 m}{(2\pi)^2 k_F}\int_0^{p_C} pdp\times \left[\frac{1}{v_F p + Dp^2 + g - \frac{p^2}{2m}} + \frac{1}{v_F p + Dp^2 + g + \frac{p^2}{2m}}\right] \tag{4.57}$$

For the Higgs case, it is appropriate to assume a small mass limit, however, we cannot take $m \to 0$ due to the fact that the propagation of the Higgs in the spin-channel arises from spin amplitude fluctuations that induce an initial mass-like effect. Carrying out the indefinite integration over p, we obtain

$$\begin{aligned}\Gamma^{(1)}_{\sigma\sigma' H} = &\frac{g_0^3 m^2}{(2\pi)^2 k_F}\left[\log\left[\frac{2gm}{k_F} + 2k_F - p + 2Dmp\right] + \log[\frac{2gm}{p} + 2k_F - p + 2Dmp]\right]|_0^{p_c} \\ &+ \frac{g_0^3 m^2}{(2\pi)^2 k_F}\left[\log\left[\frac{2gm}{k_F} - 2k_F - p - 2Dmp\right] + \log[\frac{2gm}{p} + 2k_F + p + 2Dmp]\right]|_0^{p_c}\end{aligned} \tag{4.58}$$

using the small mass limit for $\frac{2gm}{k_F p}$ as the gap $g \to 0$ when the momentum $p \to 0$. Where the expression for the Higgs effect on the many body vertex becomes

$$\Gamma^{(1)}_{\sigma\sigma';H} = \frac{g_0^2 N^2(0) Z^2}{8}\times \log\left[1 + \frac{8Dk_F^2 m p_C^3 + 12D(\omega^{\pm})p_C^2 k_F m^2 + (\omega^{\pm})m p_C^3}{2k_F^3 p_C^2 + 2k_F(\omega^{\pm})^2 m^2 + 4(\omega^{\pm})k_F^2 m p_C}\right] \tag{4.59}$$

with $p_C << k_F$, $g = \omega^{\pm}$, and $D = \frac{v_F \Delta}{k_F^2}$. The above is the final form of the Higgs effect on the zero temperature vertex, where I have added the additional term from the gapped mode's dispersion. This is the first time that this calculation has ever been done.

In the Higgs case, it is evident that the momentum dependence is much greater than in the case of the Goldstone mode, and the presence of a finite gap also increases this effect. The Higgs only exists in the case where $l = 0$ and when there is momentum dependence which leads to higher order Fermi liquid parameters. Furthermore, as F_1^a must must be finite valued while going beyond the Pomeranchuck instability constraint allowing ferromagnetic order to arise and the possibiity for spontaneous symmetry breaking to occur, the induced pairing effect cannot be s-wave in nature. In the traditional Stoner model to describe ferromagnetic metals, the self-energy is local for the Goldstone mode which leads to logarithmic dependence of the quasiparticle residue on the magnetizations which implies a s-wave pairing instability arises. However, in the Higgs case, the self-energy is

not local, as the Higgs propagates in the spin channel, which will likely lead to triplet pairing. This effect will produce the realization of some unconventional form of superconductivity induced by the Higgs mode, and therefore, the propgation of the Higgs will affect the Cooper instability in the itinerant FFL. A calculation of the four point vertex, as shown in Fig-

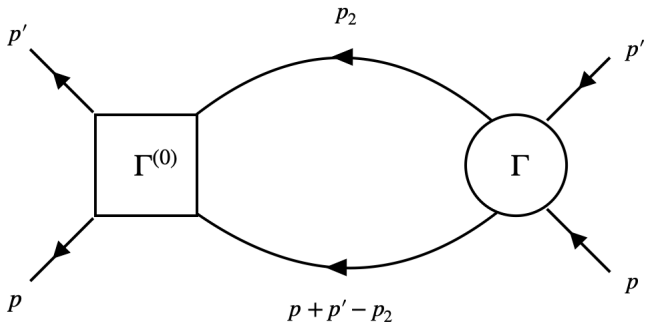

Figure 4.2: This is the four point vertex, which includes the effects of both the singlet and triplet propagating terms.

ure 4.2 will determine the total pairing interaction by both modes. This vertex includes both the singlet and triplet terms, and thus, this should be able to show which mode, the Goldstone or Higgs, has a stronger effect on particle-particle pairing in the FFL. Hopefully, this will lead to further understanding on the proposed coexistence of superconductivity and ferromagnetism, as well as, the collective mode effects on the Cooper instability in these systems.

Appendix A

Quantum Statistics

This review of quantum statistics has been adapted from and elaborated on from Richard Fitzpatrick's *Thermodynamics and Statistical Physics* [6] and L.D. Landau's and L.M. Lifshitz *Statistical Physics: 3rd Edition Part 1* [7].

A.1 Fermi-Dirac Statistics

In Fermi-Dirac statistics, particles are considered to be indistinguishable and obey the principle that no more than one particle can occupy a given quantum state. Now, consider a gas made up of two identical particles, denoted by A, where each particle can be in one of three possible quantum states, x=1,2,3. Using Fermi-Dirac statistics, we can write the possible states of the entire gas as Using the expression

1	2	3
A	A	...
A	...	A
...	A	A

Table A.1: Fermi-Dirac statistics applied to a system of two particles with three possible states.

$$X_i = \frac{\text{probability of particles being in the same state}}{\text{probability of particles being in different staes}} \tag{A.1}$$

we know that for Fermi-Dirac statistics, $X_{FD} = 0$. For Maxwell-Boltzmann statistics $X_{MB} = \frac{1}{2}$ and for Bose-Einstein statistics $X_{BE} = 1$, which tells us that in Fermi-Dirac statistics, there is a lesser relative tendency for particles to cluster in the same state than in classical statistics.

A.1.1 Fermi-Dirac Distribution Function

The average number of particles in quantum state x can be written

$$\overline{n_x} = \frac{\sum_{n_x} n_x e^{-\beta n_x \epsilon_x} \sum^{(x)}_{n_1,n_2,\ldots} e^{-\beta(n_1\epsilon_1+n_2\epsilon_2+\cdots)}}{\sum_{n_x} e^{-\beta n_x \epsilon_x} \sum^{(x)}_{n_1,n_2,\cdots} e^{-\beta(n_1\epsilon_1+n_2\epsilon_2+\cdots)}} \tag{A.2}$$

where n_i are the number of particles in state i with energy ϵ_i and $\beta = \frac{1}{kT}$. The first sums involve all possible values of the number of particles occupying the quantum state x, whereas the latter sums omit the particular state x as indicated by the superscript (x). These sums range over all values of the numbers $n_1, n_2, \cdots$ such that $n_i = 0$ and 1 for each i with the constraint that

$$\sum_i n_i = N \tag{A.3}$$

Using Fermi-Dirac statistics, weintroduce the partition function for N particles distributed over all quantum states except state x as

$$Z_x(N) = \sum^{(x)}_{n_1,n_2,\cdots} e^{-\beta(n_1\epsilon_1+n_2\epsilon_2+\cdots))} \tag{A.4}$$

Due to the fact that state x is either occupied or unoccupied, we can now consider the summation over $n_x = 0$ and 1, where equation $A.2$ becomes

$$\overline{n_x} = \frac{\sum_{n_x} n_x e^{-\beta n_x \epsilon_x} \sum^{(x)}_{n_1,n_2,\ldots} e^{-\beta(n_1\epsilon_1+n_2\epsilon_2+\cdots)}}{\sum_{n_x} e^{-\beta n_x \epsilon_x} \sum^{(x)}_{n_1,n_2,\cdots} e^{-\beta(n_1\epsilon_1+n_2\epsilon_2+\cdots)}}$$

$$= \frac{(0 + e^{-\beta\epsilon_x}) \sum^{(x)}_{n_1,n_2,\ldots} e^{-\beta(n_1\epsilon_1+n_2\epsilon_2+\cdots)}}{(0 + e^{-\beta\epsilon_x}) \sum^{(x)}_{n_1,n_2,\cdots} e^{-\beta(n_1\epsilon_1+n_2\epsilon_2+\cdots)}}$$

$$= \frac{0 + e^{-\beta\epsilon_x} Z_x(N-1)}{0 + e^{-\beta\epsilon_x} Z_x(N-1)}$$

$$= \frac{1}{\frac{Z_X(N)}{Z_x(N-1)} e^{\beta\epsilon_x} + 1} \tag{A.5}$$

In order to find the form of the Fermi-Dirac distribution, we must relate $Z_x(N-1)$ to $Z_x(N)$. If we assume $\Delta N \ll N$, then $\ln Z_x(N - \Delta N)$ can be Taylor expanded as

$$\ln Z_x(N - \Delta N) \simeq \ln Z_x(N) - \frac{\partial \ln Z_x}{\partial N} \Delta N = \ln Z_x(N) - \alpha_x \Delta N \tag{A.6}$$

where

$$\alpha_x = \frac{\partial \ln Z_x}{\partial N} \tag{A.7}$$

Taylor expanding the slowly-varying function $\ln Z_x(N)$, rather than the rapidly-varying function $Z_x(N)$, is important as the latter Taylor series has a radius of convergence that is too small to be of use. Rearranging equation $A.6$ gives

$$Z_x(N - \Delta N) = Z_x(N)e^{-\alpha_x \Delta N} \tag{A.8}$$

As $Z_x(N)$ is a sum over many different quantum states, the logarithm of this function will not be sensitive to the state x that is excluded. Thus, we can approximate α_x as being independent of x for all x and write $\alpha_x \simeq \alpha$. It follows that the derivative in equation $A.7$ can be approximated as the derivative of the entire partition function Z, where the N particles are distributed over all quantum states. In other words,

$$\alpha \simeq \frac{\partial \ln Z}{\partial N} \tag{A.9}$$

Setting $\Delta N = 1$ in $A.8$, equation $A.5$ reduces to the Fermi-Dirac distribution in

$$\overline{n_x} = \frac{1}{e^{\alpha+\beta\epsilon_x} + 1} \tag{A.10}$$

where the parameter α is determined by the constraint that $\sum_i \overline{n_i} = N$, i.e.

$$N = \frac{1}{e^{\alpha+\beta\epsilon_i} + 1} \tag{A.11}$$

Taking the limit as $\epsilon_x \to \infty \implies \overline{n_x} \to 0$. Taking the limit as $\epsilon_x \to 0 \implies \overline{n_x} \to 1$. Therefore, the average number of particles in quantum state x is limited by the constraint

$$0 \leq \overline{n_x} \leq 1 \tag{A.12}$$

as expected by the Pauli exclusion principle.

A.1.2 Density of States

Consider a gas of electrons confined to a $3D$ space with sides of length L_x, L_y, L_z. Set $V = L_x L_y L_z$. We can write the wavevector of each electron as, e.g. for the x direction,

$$k_x = \frac{2\pi}{\lambda_x} = \frac{2\pi}{L_x} \tag{A.13}$$

The energy of each electron can be written as

$$\epsilon(k) = \frac{\hbar^2 k^2}{2m} \tag{A.14}$$

which gives the wavevector in terms of ϵ as

$$k(\epsilon) = \left(\frac{2m}{\hbar^2}\right)^{\frac{1}{2}} \epsilon^{\frac{1}{2}} \tag{A.15}$$

Here, there are $\frac{V}{2\pi^3}$ allowable transitional states per unit volume of $\mathbf{k}$-space. At $T = 0$, the number of electrons in this space can be described as

$$\begin{aligned}
N &= \frac{\text{volume of filled } k\text{-states}}{\text{volume per } k\text{-state}} \\
&= 2 \times \frac{\frac{4}{3}\pi r^3}{k_x k_y k_z} \\
&\quad \text{the 2 takes into account the two} \\
&\quad \text{possibilities for the electron spin} \\
&= \frac{\frac{8}{3}\pi k_F^3}{\frac{2\pi}{L_x}\frac{2\pi}{L_y}\frac{2\pi}{L_z}} \\
&\quad \text{where we take the Fermi wavevector } k_F \\
&\quad \text{as the radius of the gas of electrons at } T = 0 \\
&= \frac{8\pi k_F^3}{\frac{3(2\pi)^3}{V}} \\
&= \frac{V k_F^3}{3\pi^2} \\
&= \frac{V}{3\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \epsilon^{\frac{3}{2}}
\end{aligned} \tag{A.16}$$

since the total number of occupied states, i.e. the total number of quantum states inside the Fermi sphere, will be equivalent to the total number of gas partciles. The density of states can be written as the derivative of the particle number N with respect to the energy ϵ as

$$\rho(\epsilon) = \frac{dN}{d\epsilon} = \frac{d}{d\epsilon}\left[\frac{V}{3\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \epsilon^{\frac{3}{2}}\right] = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \epsilon^{\frac{1}{2}} \tag{A.17}$$

A.1.3 Conduction Electrons in Metal

In a metal, conduction electrons are non-localized meaning not tied to any particular atom, and in conventional metals, each of these atoms contributes a certain fixed number of electrons. We could treat these electrons as an ideal gas by neglecting the mutual interaction of the conduction electrons due to the fact that this interaction is largely shielded by the stationary ions. However, classical statistics normally cannot be used to analyze conduction electrons as the concentration of electrons in a metal is much greater than the concentration of particles in a gas. Electrons are subject to Fermi-Dirac statistics and can be referred to as fermions. According to the Fermi-Dirac distribution, the mean number of particles occupying state x with energy ϵ_x is given by

$$\overline{n_x} = \frac{1}{e^{\alpha+\beta\epsilon_x}+1} = \frac{1}{e^{\beta(\epsilon_x-\mu)}+1} \tag{A.18}$$

where $\mu(T) = -\frac{\alpha}{\beta}$ is the Fermi energy, also known as the chemical potential, of the system and is determined by the condition on the total number of particles N by

$$N = \sum_i \overline{n_i} = \sum_i \frac{1}{e^{\beta(\epsilon_i-\mu)}+1} \tag{A.19}$$

contained in a volume V. Let us now explore the behavior of the Fermi function

$$F(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)}+1} \tag{A.20}$$

The energy ϵ is measured from $\epsilon = 0$, or its lowest possible value. Thus, if the Fermi energy, μ, is $\beta\mu \ll -1$, then $\beta(\epsilon-\mu) \gg 1$ and F reduces to the Maxwell-Boltzmann distribution. In the case of the electrons, the opposite limit is more interesting, where $\beta\mu \gg 1$. Here, if $0 < \epsilon \ll \mu$ then $\beta(\epsilon-\mu \ll -1$, admitting $F(\epsilon) = 1$. We can also take the limit where $\epsilon \gg \mu$, then $\beta(\epsilon-\mu) \gg 1$, implying that $F(\epsilon) = e^{-\beta(\epsilon-\mu)}$ decreases exponentially with increasing energy ϵ, just as a classical distribution would. The transition region when F moves from a value near unity to a value of nearly zero occurs at a value of $F = \frac{1}{2}$, i.e. when $\epsilon = \mu$ and corresponds to an energy interval of order kT. The Fermi Function's behavior is described in Figure A.1.3.

Clearly, it can be seen that in the limit as $T \to 0$, the transition region width, kT, becomes extremely small. For this case, $F = 1$ when $\epsilon < \mu$, and $F = 0$ for $\epsilon > \mu$. In other words, at $T = 0$,

$$F(\epsilon) = \begin{cases} 1 & \epsilon < \epsilon_F \\ 0 & \epsilon > \epsilon_F \end{cases} \tag{A.21}$$

This makes sense because when $T = 0$, the conduction electrons are at their lowest energy level, or their ground-state. By the Pauli exclusion principle, this lowest energy formation is obtained by placing electrons into the lowest available unoccupied states until all electrons have been placed. As all lower energy states are filled, the final electron placed in has a significant energy, $\epsilon = \mu$, which represents that a Fermi-Dirac gas always possesses a large mean energy, even at absolute zero.

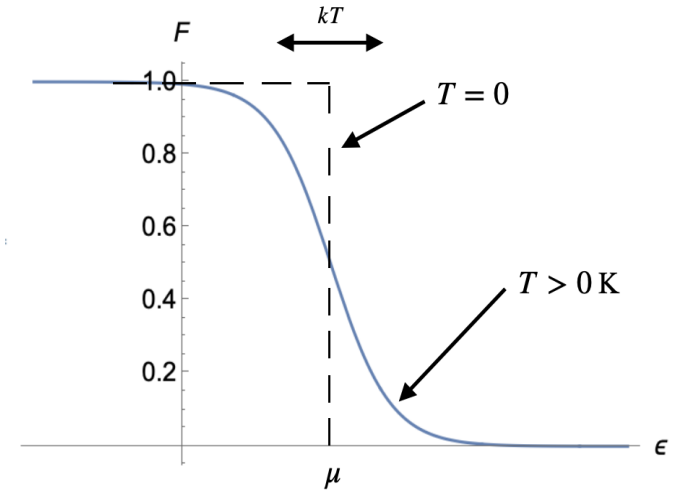

Figure A.1: Graph of the Fermi Function's behavior. The solid blue line represents the gas of electrons at a general temperature $T > 0$. The dotted line represents the gas at $T = 0$, where the Fermi function takes the form of equation A.21. Notice that $F = \frac{1}{2}$ at $\epsilon = \mu$, $F = \frac{3}{4}$ at $\epsilon = \mu - (\ln 3)kT$, and $F = \frac{1}{4}$ at $\epsilon = \mu + (\ln 3)kT$.

Now, we can calculate the Fermi energy, $\mu = \mu_F = \epsilon_F$ of a Fermi-Dirac gas at $T = 0$. The energy of each electron can be written as

$$\epsilon = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m} \tag{A.22}$$

where $\mathbf{p} = \hbar\mathbf{k}$, $\mathbf{k}$ is the de Broglie wavevector, and m is the mass of an electron. All quantum states with energy less than the Fermi energy, ϵ_F, are filled at $T = 0$. From above, since the energy is a function of the wavevector k of the electron, it is clear to see that the Fermi energy will be a function of the Fermi wavevector, k_F, or the Fermi momentum, $p_F = \hbar k_F$, such that

$$\epsilon_F = \frac{p_F^2}{2m} = \frac{\hbar^2 k_F^2}{2m} \tag{A.23}$$

In particular, we can say that all quantum states with $k < k_F$ are filled, and those with $k > k_F$ are empty at $T = 0$. From equation $A.16$, we have

$$\begin{aligned} N &= g\frac{\frac{4}{3}\pi k_F^3}{\frac{(2\pi)^3}{V}} \\ \implies N &= \frac{Vk_F^3}{3\pi^2} \\ &\text{where } g = 2s+1 = 2 \text{ for spin } s = \frac{1}{2} \\ \implies k_F &= \left(3\pi^2\frac{N}{V}\right)^{\frac{1}{3}} \\ \implies \lambda_F &\equiv \frac{2\pi}{k_F} = 2\pi\left(\frac{V}{3\pi^2 N}\right)^{\frac{1}{3}} \end{aligned} \tag{A.24}$$

showing that at the Fermi energy, the de Broglie wavelength, λ_F, is of order the mean separation between particles in $\left(\frac{V}{N}\right)^{\frac{1}{3}}$, i.e. quantum states with $\lambda > \lambda_F$ are occupied, and those with $\lambda < \lambda_F$ are empty at $T = 0$. Therefore, from equations $A.23$ and $A.24$, the Fermi energy at $T = 0$ becomes

$$\epsilon_F = \frac{\hbar^2}{2m}\left(3\pi^2\frac{N}{V}\right)^{\frac{2}{3}} \tag{A.25}$$

To put this in perspective, it can be shown that $\epsilon_F \gg kT$ for typical metals at room temperature.

A.1.4 Particle Number and Average Energy

The number of quantized states within a region of length $d\mathbf{k}$ in k-space can be written as

$$N = \int \overline{n}(\mathbf{k})\rho(\mathbf{k})d\mathbf{k} \tag{A.26}$$

where $\overline{n}$ is the distribution function and ρ is the density of states. Thus, the total number of particles is simply

$$N = \int_0^\infty \overline{n}(k)\rho(k)dk \tag{A.27}$$

where, for Fermi-Dirac statistics, the distribution $\overline{n}$ is $\overline{n} = F$, the Fermi-function, and $\rho = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}}\epsilon^{\frac{1}{2}}$. Therefore,

$$\begin{aligned} N &= \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}}\int_0^\infty \frac{\epsilon^{\frac{1}{2}}}{e^{\frac{\epsilon-\mu}{k_BT}}+1}d\epsilon \\ &= \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}}\int_0^\infty \frac{\epsilon^{\frac{1}{2}}}{e^{\frac{\epsilon}{k_BT}}e^{\frac{-\mu}{k_BT}}+1}d\epsilon \\ &= \frac{V}{2\pi^2}\left(\frac{2mk_BT}{\hbar^2}\right)^{\frac{3}{2}}\int_0^\infty \frac{x^{\frac{1}{2}}}{ce^x+1}dx \\ &\text{where } c = e^{\frac{-\mu}{k_BT}} \text{ and } x = \frac{\epsilon}{k_BT} \end{aligned}$$

$$\begin{aligned} &= -\frac{1}{2}\pi^{\frac{1}{2}}\frac{V}{2\pi^2}\left(\frac{2mk_BT}{\hbar^2}\right)^{\frac{3}{2}} Li_{\frac{3}{2}}(-e^{\frac{\mu}{k_BT}}) \\ &= -\frac{V}{4}\left(\frac{2mk_BT}{\hbar^2\pi}\right)^{\frac{3}{2}} Li_{\frac{3}{2}}(-e^{\frac{\mu}{k_BT}}) \end{aligned} \tag{A.28}$$

where $Li_{\frac{3}{2}}(-e^{\frac{\mu}{k_BT}})$ is the polylogarithmic function defined by

$$Li_{\frac{3}{2}}(-e^{\frac{\mu}{k_BT}}) = \sum_{k=1}^{\infty}\frac{(-e^{\frac{\mu}{k_BT}})^k}{k^{\frac{3}{2}}} = \sum_{k=1}^{\infty}\frac{(-1)^k e^{\frac{k\mu}{k_BT}}}{k^{\frac{3}{2}}} \tag{A.29}$$

To first order, we have

$$N = \frac{V}{4}\left(\frac{2mk_BT}{\hbar^2\pi}\right)^{\frac{3}{2}} e^{\frac{\mu}{k_BT}} \tag{A.30}$$

which is the total particle number in a gas of electrons under Fermi-Dirac statistics.The average energy calculation is similar to the particle number

calculation. The average energy of the gas of electrons is

$$\begin{aligned}
\overline{E} &= \int \epsilon \overline{n}(\epsilon)\rho(\epsilon)d\epsilon \\
&= \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \int_0^\infty \frac{\epsilon^{\frac{3}{2}}}{e^{\frac{\epsilon-\mu}{k_BT}}+1} d\epsilon \\
&= \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \int_0^\infty \frac{\epsilon^{\frac{3}{2}}}{e^{\frac{\epsilon}{k_BT}}e^{\frac{-\mu}{k_BT}}+1} d\epsilon \\
&= \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} (k_BT)^{\frac{5}{2}} \int_0^\infty \frac{x^{\frac{3}{2}}}{ce^x+1} dx \\
&\quad \text{where } c = e^{\frac{-\mu}{k_BT}} \text{ and } x = \frac{\epsilon}{k_BT}
\end{aligned}$$

$$\begin{aligned}
&= -\frac{3}{4}\pi^{\frac{1}{2}}(k_BT)^{\frac{5}{2}} \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} Li_{\frac{5}{2}}(-e^{\frac{\mu}{k_BT}}) \\
&= -(k_BT)^{\frac{5}{2}} \frac{3V}{8}\left(\frac{2m}{\hbar^2\pi}\right)^{\frac{3}{2}} Li_{\frac{5}{2}}(-e^{\frac{\mu}{k_BT}})
\end{aligned} \tag{A.31}$$

where $Li_{\frac{5}{2}}(-e^{\frac{\mu}{k_BT}})$ is the polylogarithmic function defined by

$$Li_{\frac{5}{2}}(-e^{\frac{\mu}{k_BT}}) = \sum_{k=1}^{\infty} \frac{(-e^{\frac{\mu}{k_BT}})^k}{k^{\frac{5}{2}}} = \sum_{k=1}^{\infty} \frac{(-1)^k e^{\frac{k\mu}{k_BT}}}{k^{\frac{5}{2}}} \tag{A.32}$$

To first order, we have

$$\overline{E} = (k_BT)^{\frac{5}{2}} \frac{3V}{8}\left(\frac{2m}{\hbar^2\pi}\right)^{\frac{3}{2}} e^{\frac{\mu}{k_BT}} \tag{A.33}$$

which is the average energy of a gas of electrons under Fermi-Dirac statistics. The potential Ω is the same as the expression for the total energy of the gas with an additional factor of $-\frac{2}{3}$, i.e. $\Omega = -\frac{2}{3}E$. One can find the equation of state from $\Omega = -PV \implies PV = \frac{2}{3}E$.

A.1.5 Electron Gas at Zero-Temperature

We will now continue to discuss the electron gas, and its significance at absolute zero, which can be classified as a completely degenerate Fermi gas. Electrons at this temperature will distribute themselves throughout the quantum states so as to obtain the lowest total energy of the gas. Thus, depending on the number of electrons, electrons will first fill states with

energies at zero, continue to fill the next lowest energy states, and so on, until the number of electrons runs out.

With $g = 2$ spin-degeneracy of the levels, as before, we will now derive the particle number and energy in momentum space. In a volume V, the number of quantum states of an electron with magnitude of momentum between $p, p + dp$ becomes

$$2 \cdot \frac{4\pi p^2 V dp}{(2\pi\hbar)^3} = V\frac{p^2 dp}{\pi^2\hbar^3} \tag{A.34}$$

In the case of $T = 0$, electron states occupied have a limit, i.e. a momentum limit ,$p = p_F$, denoted as the radius of the Fermi sphere in momentum space. The total number of electrons then becomes

$$\begin{aligned} N &= \frac{V}{\pi^2\hbar^3}\int_0^{p_F} p^2 dp = \frac{Vp_F^3}{3\pi^2\hbar^3} \\ \implies p_F &= \hbar(3\pi^2)^{\frac{1}{3}}\left(\frac{N}{V}\right)^{\frac{1}{3}} \\ \implies \epsilon_F &= \frac{p_F^2}{2m} = (3\pi^2)^{\frac{2}{3}}\left(\frac{\hbar^2}{2m}\right)\left(\frac{N}{V}\right)^{\frac{2}{3}} \end{aligned} \tag{A.35}$$

where p_F and ϵ_F are the Fermi momentum and energy, respectively. A diagram of the behavior of the Fermi distribution was shown in Figure A.1.3. Recall that in the limit as $T \to 0$, the Fermi distribution $\overline{n_p}$ becomes a step function, with a value of unity for $\epsilon < \mu$ and zero for $\epsilon > \mu$ where $\mu = \epsilon_F$.

We can find the total energy simply by multiplying the number of states from equation A.34 by $\frac{p^2}{2m}$ and integrating over momentum space,

$$E = \frac{V}{2m\pi^2\hbar^3}\int_0^{p_F} p^4 dp = \frac{Vp_F^5}{10m\pi^2\hbar^3}$$

$$\implies E = \frac{3(3\pi^2)^{\frac{2}{3}}}{10}\frac{\hbar^2}{m}\left(\frac{N}{V}\right)^{\frac{2}{3}} N$$

inserting the value of p_F

$$\implies P = \frac{(3\pi^2)^{\frac{2}{3}}}{5}\frac{\hbar^2}{m}\left(\frac{N}{V}\right)^{\frac{5}{3}} \tag{A.36}$$

using the equation of state $P = \frac{2}{3}\frac{E}{V}$

we find the pressure of a Fermi gas at absolute zero is proportional to the its density to the power of $\frac{5}{3}$. For these equations to be valid representations

of strongly degenerate electron gases, it is reasonable to assume that the temperature T should be small in comparison to the limiting energy ϵ_F, i.e. $T << \left(\frac{\hbar^2}{2m}\right)\left(\frac{N}{V}\right)^{\frac{2}{3}} \cong \epsilon_F$. The temperature defined by $T_F \cong \epsilon_F$ is called the degeneracy temperature.

A property of the degenerate electron gas is that it approaches the ideal gas state as its density increases. This can be easily understood from the condition that in an ideal gas, the mean kinetic energy of electrons must be large compared to the Coulomb interaction between the electrons and nuclei, $\cong \frac{Ze^2}{a}$, where Ze is the nuclear charge and $a \sim \left(\frac{ZV}{N}\right)^{\frac{1}{3}}$ is the mean distance between the electrons and the nuclei. As the mean kinetic energy can be approximated as having the same order of magnitude as the Fermi energy, ϵ_F, and since $\epsilon_F \cong \left(\frac{\hbar^2}{2m}\right)\left(\frac{N}{V}\right)^{\frac{2}{3}}$, then

$$\frac{Ze^2}{a} << \epsilon_F \implies \left(\frac{2me^2}{\hbar^2}\right)^3 Z^2 << \frac{N}{V} \tag{A.37}$$

showing this condition is certainly met as the density of the gas, $\frac{N}{V}$ increases.

A.2 Bose-Einstein Statistics

In Bose-Einstein statistics, particles are considered to be indistinguishable with no limitations on the number of particles within a state. Now, consider a gas made up of two identical particles, denoted by A, where each particle can be in one of three possible quantum states, x=1,2,3, as before. Using Bose-Einstein statistics, we can write the possible states of the entire gas as Using the expression X_i defined in the discussion on Fermi-Dirac statistics,

1	2	3
AA	...	...
...	AA	...
...	...	AA
A	A	...
A	...	A
...	A	A

Table A.2: Bose-Einstein statistics applied to a system of two particles with three possible states.

we have $X_{BE} = 1$ as opposed to $X_{FD} = 0$ and $X_{MB} = \frac{1}{2}$ for Fermi-Dirac and Maxwell-Boltzmann statistics, respectively. This tells us that in Bose-Einstein statistics, there is a greater relative tendency for particles to cluster in the same state than in classical statistics.

A.2.1 Bose-Einstein Distribution Function

The particles in this system are assumed to be massive, therefore, we take the total number of particles, N, to be a fixed number. The average number of particles in quantum state s can be written

$$\overline{n_x} = \frac{\sum_{n_x} n_x e^{-\beta n_x \epsilon_x} \sum_{n_1,n_2,\ldots}^{(x)} e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + \cdots)}}{\sum_{n_x} e^{-\beta n_x \epsilon_x} \sum_{n_1,n_2,\cdots}^{(x)} e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + \cdots)}} \tag{A.38}$$

where n_i are the number of particles in state i with energy ϵ_i and $\beta = \frac{1}{kT}$, as seen before with Fermi-Dirac Statistics. For massive bosons, the particle numbers $n_1, n_2, \ldots$ in each state assume values of $n_i = 0, 1, 2, \ldots$ for each i where $\sum n_i = N$. Taking the sum over n_x, the above expression becomes

$$\overline{n_x} = \frac{0 + e^{-\beta\epsilon_x} Z_x(N-1) + 2e^{-2\beta\epsilon_x} Z_x(N-2) + \cdots}{Z_x(N) + e^{-\beta\epsilon_x} Z_x(N-1) + e^{-2\beta\epsilon_x} Z_x(N-2) + \cdots} \tag{A.39}$$

where $Z_x(N)$ is the partition function for N particles distributed over all states except state x in accordance with Bose-Einstein statistics. Utilizing equation $A.8$ and taking $\alpha_x \simeq \alpha$, we can write

$$\begin{aligned}
\overline{n_x} &= \frac{Z_x(N)[0 + e^{-(\alpha+\beta\epsilon_x)} + 2e^{-2(\alpha+\beta\epsilon_x)} + \cdots]}{Z_x(N)[e^{\alpha} + e^{-(\alpha+\beta\epsilon_x)} + e^{-2(\alpha+\beta\epsilon_x} + \cdots]} \\
&= \frac{\sum_x n_x e^{-n_x(\alpha+\beta\epsilon_x)}}{\sum_x e^{-n_x(\alpha+\beta\epsilon_x)}} \\
&= -\frac{1}{\beta}\frac{\partial}{\partial\epsilon_x}\left(\ln \sum_{n_x} e^{-(\alpha+\beta n_x\epsilon_x)}\right) \\
&= -\frac{1}{\beta}\frac{\partial}{\partial\epsilon_x} \ln\left(\frac{1}{1-e^{-(\alpha+\beta\epsilon_x)}}\right)
\end{aligned}$$

$$\text{as} \sum_{n_x=0}^{\infty} e^{-n_x(\alpha+\beta\epsilon_x)} = 1 + e^{-(\alpha+\beta\epsilon_x)} + e^{-2(\alpha+\beta\epsilon_x)} + \cdots = \frac{1}{1-e^{-(\alpha+\beta\epsilon_x)}}$$

$$\begin{aligned}
&= \frac{1}{\beta}\frac{\partial}{\partial\epsilon_x} \ln\left(1 - e^{-(\alpha+\beta\epsilon_x)}\right) \\
&= \frac{e^{-(\alpha+\beta\epsilon_x)}}{1-e^{-(\alpha+\beta\epsilon_x)}} \\
&= \frac{1}{e^{\alpha+\beta\epsilon_x} - 1}
\end{aligned} \tag{A.40}$$

which is known as the Bose-Einstein distribution. Here, it is important to note that $\overline{n_x}$ can be large. The parameter α is determined by the expression

for the total number of particles in

$$N = \sum_i \frac{1}{e^{\alpha+\beta\epsilon_x} - 1} \tag{A.41}$$

As for the case of electrons in Fermi-Dirac statistics, we will take $\alpha = -\beta\mu$ where μ the chemical potential for the bosonic system.

A.2.2 Density of States

For bosons we have $\epsilon > \mu$, where μ can be found from the distribution when the particle number N is known. Given $\epsilon(k) = \hbar v k$ for a d dimensional system of bosons, we must change the integral $gV \int \frac{d^d k}{(2\pi)^d}$ to an integral over ϵ. The coupling constant g with the volume V will be denoted as $\Omega_d = gV_d$. Therefore, the density of states for bosons in d dimensions per unit volume of k space is

$$\begin{aligned} \rho(\epsilon) &= \int \frac{d^d k}{(2\pi)^d} \delta(\epsilon - \epsilon(k)) \\ &= \int \frac{d^d k}{(2\pi)^d} \delta(\epsilon - \hbar v k) \\ &= \frac{\Omega_d}{(2\pi)^d} \frac{\epsilon^{d-1}}{(\hbar v)^d} \end{aligned} \tag{A.42}$$

The density of states in three dimensions for a bosons obeying the dispersion $\epsilon(k) = \hbar v k$ becomes

$$\rho(\epsilon) = \frac{\Omega_3}{(2\pi)^3} \frac{\epsilon^2}{(\hbar v)^3} \tag{A.43}$$

where $\Omega_3 = gV_3$ where V_3 is the volume is three dimensional space.

A.2.3 T_c for Bose-Einstein Condensation

The number of quantized states within a region of length $d\mathbf{k}$ in k-space can be written as

$$N = \int \overline{n}(\mathbf{k}) \rho(\mathbf{k}) d\mathbf{k} \tag{A.44}$$

where $\overline{n}$ is the distribution function and ρ is the density of states. Thus, the total number of particles is simply

$$N = \int_0^\infty \overline{n}(k) \rho(k) dk \tag{A.45}$$

where, for Bose-Einstein statistics, the distribution $\overline{n}$ is the Bose-Einstein distribution, and $\rho = \frac{\Omega_3}{(2\pi)^3}\frac{\epsilon^2}{(\hbar v)^3}$. Therefore,

$$\begin{aligned} N &= \frac{\Omega_3}{(2\pi)^3(\hbar v)^3}\int_0^\infty \frac{\epsilon^2}{e^{\frac{\epsilon-\mu}{k_BT}}-1}d\epsilon \\ &= \frac{1}{2\pi^2(\hbar v)^3}\int_0^\infty \frac{\epsilon^2}{e^{\frac{\epsilon}{k_BT}}-1}d\epsilon \\ &\qquad \text{where we set } \Omega_3 \approx 4\pi \text{ and } \mu = 0 \\ &= \frac{\zeta(3)}{\pi^2}\left(\frac{k_BT_c}{\hbar v}\right)^3 \end{aligned} \tag{A.46}$$

where the zeta function $\zeta(3) = 1 + 2^{-3} + +3^{-3}4^{-3}$. The Bose-Einstein condensation critical temperature in three dimensions for a system of non-interacting bosons becomes

$$k_BT_c = \left(\frac{N\pi^2}{\zeta(3)}\right)^{\frac{1}{3}}\hbar v \tag{A.47}$$

A.2.4 The Bose Gas at Zero Temperature

A Bose gas obeys different properties than that of a Fermi gas at low temperatures. One example of this can be seen from the lowest energy state of the Bose gas, which is occupied at $T = 0$, where $E = 0$, whereas this energy is non-zero for the Fermi gas. If we maintain constant density $\frac{N}{V}$ while lowering the temperature of the gas, the chemical potential μ will increase (become less negative) and reach zero at a temperature determined by

$$\frac{N}{V} = \frac{gT^{\frac{1}{2}}m^{\frac{3}{2}}}{2^{\frac{1}{2}}\pi^2\hbar^3}\int_0^\infty \frac{\epsilon^{\frac{1}{2}}dx}{e^{\frac{\epsilon}{T}}-1} \tag{A.48}$$

$$\implies \frac{N}{V} = \frac{g(mT)^{\frac{3}{2}}}{2^{\frac{1}{2}}\pi^2\hbar^3}\int_0^\infty \frac{x^{\frac{1}{2}}d\epsilon}{e^x-1} \tag{A.49}$$

where $x = \frac{\epsilon}{T}$ and the integral can simply be expressed as a Riemann zeta function, $\zeta(\frac{3}{2}) = \sum_{n=1}^{\infty}\frac{1}{n^x} = 2.612$ times a Gamma function $\Gamma(\frac{3}{2}) = (\frac{3}{2}-1)! = 0.886$. Rearranging we find

$$T_0 = \frac{3.31\hbar^2}{mg^{\frac{2}{3}}}\left(\frac{N}{V}\right)^{\frac{2}{3}} \tag{A.50}$$

Here, we must consider the case where $T < T_0$, but not quite $T = T_0$ in order to maintain finite terms in the sum and obtain negative solutions for the chemical potential μ, which is required at all temperatures in Bose

statistics. For particles with energy $\epsilon > 0$, we can write the distribution with $\mu = 0$ as,

$$dN_\epsilon = \frac{gm^{\frac{3}{2}}V}{2^{\frac{1}{2}}\pi^2\hbar^3}\frac{\epsilon^{\frac{1}{2}}d\epsilon}{e^{\frac{\epsilon}{T}}-1} \tag{A.51}$$

$$\implies N_{\epsilon>0} = \int dN_\epsilon = \frac{gV(mT)^{\frac{3}{2}}}{2^{\frac{1}{2}}\pi^2\hbar^3}\int_0^\infty \frac{x^{\frac{1}{2}}dx}{e^x-1} = N\left(\frac{T}{T_0}\right)^{\frac{3}{2}} \tag{A.52}$$

total number of particles
with energies $\epsilon > 0$

$$\implies N_{\epsilon=0} = N\left[1-\left(\frac{T}{T_0}\right)^{\frac{3}{2}}\right] \tag{A.53}$$

with the remaining particles at $\epsilon = 0$ being in the lowest energy state. The energy of the gas is only determined by the particles with $\epsilon > 0$, and hence, setting $\mu = 0$, we obtain

$$E = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}}\int_0^\infty \frac{\epsilon^{\frac{3}{2}}}{e^{\frac{\epsilon}{T}}+1}d\epsilon \tag{A.54}$$

$$= \frac{gV(mT)^{\frac{3}{2}}T}{2^{\frac{1}{2}}\pi^2\hbar^3}\int_0^\infty \frac{x^{\frac{3}{2}}}{e^x+1}dx \tag{A.55}$$

$$= \frac{gV(mT)^{\frac{3}{2}}T}{2^{\frac{1}{2}}\pi^2\hbar^3}\Gamma(\frac{5}{2})\zeta(\frac{5}{2}) \tag{A.56}$$

$$= 0.770NT\left(\frac{T}{T_0}\right)^{\frac{3}{2}} \tag{A.57}$$

$$= 0.128\frac{gVm^{\frac{3}{2}}T^{\frac{5}{2}}}{\hbar^3} \tag{A.58}$$

with $\zeta(\frac{5}{2}) = 1.341$ and $\Gamma(\frac{5}{2}) = 1.329$. The pressure becomes $P = -\frac{\partial F}{\partial V}_T = 0.0851\frac{gm^{\frac{3}{2}}T^{\frac{5}{2}}}{\hbar^3}$, and thus, for $T < T_0$, the pressure is proportional to $T^{\frac{5}{2}}$ and exhibits no volume dependence. This is valid due to the fact that particles in a state with $\epsilon = 0$ have no momentum, thereby, contributing nothing to the pressure.

A.3 Non-equilibria Fermi and Bose Gases

The aim is to find the entropy for Fermi and Bose gases which are not in equilibria, and then, rederive the Fermi and Bose distribution functions with the condition of maximum entropy. We will consider N_j identical particles and G_j states. As discussed previously, states in a Fermi gas can

only be occupied by at most one particle, and in fact, the numbers N_j are of the same order of magnitude as the G_j.

The number of ways to fill G_j states with no more than one of each N_j identical particles is simply the number of possible ways to select N_j of the G_j states,

$$\Delta\Gamma_j = \frac{G_j!}{N_j!\,(G_j - N_j)!}$$

$$\implies S = \log(\Delta\Gamma_j) = \log(G_j!\,) - \log(N_j!\,) - \log[(G_j - N_j)!\,] \tag{A.59}$$

Since the N_j are large, we can utilize the approximation $\log N! \cong N\log(\frac{N}{e})$.

$$S = \sum_j \{G_j \log G_j - N_j \log N_j - (G_j - N_j)\log(G_j - N_j)\} \tag{A.60}$$

Using the mean occupation numbers of the quantum states, $\overline{n_j} = \frac{N_j}{G_j}$, we can write the entropy of a Fermi gas in a non-equilibrium state as

$$S = -\sum_j G_j[\overline{n_j}\log\overline{n_j} + (1 - \overline{n_j})\log(1 - \overline{n_j})] \tag{A.61}$$

Using the method of Lagrange's undetermined multipliers, we can find the criteria for this function to be a maximum by setting the derivative to zero

$$\frac{\partial(S + \alpha N + \beta E)}{\partial\overline{n_j}} = 0 \tag{A.62}$$

where α, β are constants. This admits

$$\begin{aligned} G_j(log\frac{\overline{n_j}}{1 - \overline{n_j}} + \alpha + \beta\epsilon_j) &= 0 \\ \implies \frac{1 - \overline{n_j}}{\overline{n_j}} &= e^{\alpha + \beta\epsilon_j} \\ \implies \frac{1}{\overline{n_j}} &= e^{\alpha + \beta\epsilon_j} + 1 \\ \implies \overline{n_j} &= \frac{1}{e^{\alpha + \beta\epsilon_j} + 1} \end{aligned} \tag{A.63}$$

which is the expected Fermi distribution with $\alpha = \frac{\mu}{T}, \beta = -\frac{1}{T}$.

For Bose statistics, the analysis is similar. Each quantum state can contain any number of particles, allowimng the statistical weight $\Delta\Gamma_j$ to simply be the total number of ways to distribute N_j particles throughout G_j states, or

$$\Delta\Gamma_j = \frac{(G_j + N_j - 1)!}{(G_j - 1)!\,N_j!} \tag{A.64}$$

We can then take the logarithm of this expression and note that $G_j + N_j, G_j >> 1$ to write,

$$\begin{aligned} S &= \sum_j \{(G_j + N_j)\log(G_j + N_j) - N_j \log N_j - G_j \log G_j\} \\ &= \sum_j G_j[(1+\overline{n_j})\log(1+\overline{n_j}) - \overline{n_j}\log\overline{n_j}] \end{aligned} \tag{A.65}$$

Carrying out the same procedure as above for the maximum, we find that

$$\begin{aligned} \frac{\partial(S+\alpha N+\beta E)}{\partial\overline{n_j}} = 0 & \\ \implies & G_j(log\frac{\overline{n_j}}{1+\overline{n_j}} + \alpha + \beta\epsilon_j) = 0 \\ \implies & \frac{1+\overline{n_j}}{\overline{n_j}} = e^{\alpha+\beta\epsilon_j} \\ \implies & \frac{1}{\overline{n_j}} = e^{\alpha+\beta\epsilon_j} - 1 \\ \implies & \overline{n_j} = \frac{1}{e^{\alpha+\beta\epsilon_j}-1} \end{aligned} \tag{A.66}$$

which is the expected Bose distribution with $\alpha = \frac{\mu}{T}, \beta = -\frac{1}{T}$. One can show that in the limiting case $N_j << G_j$, the Fermi and Bose formulas for entropy tend to the Boltzmann formula for an ideal gas not in equilibrium, $S = \sum_j N_j \log\left(\frac{eG_j}{N_j}\right)$, and each of the statistical weights in this limit tend to the expression $\Delta\Gamma_j = \frac{G_j^{N_j}}{N_j!}$. In the case where, for a Bose gas, each quantum state is occupied by a large number of particles, i.e. $N_j >> G_j \implies \overline{n_j} >> 1$, this corresponds to the classical wave description of the field. Here, the statistical weight becomes $\Delta\Gamma_j = \frac{N_j^{G_j-1}}{(G_j-1)!}$, and the entropy can be written as $S = \sum_j G_j \log\left(\frac{eN_j}{G_j}\right)$.

Appendix B

Second Quantization

The notes on second quantization have been adapted from Giamarchi [8]. The notes on BCS theory have been adapted from Henley [9] and Bukov [10].

B.1 Creation and destruction operators

Here, we will discuss operators that will allow us to form all the elements of the basis mentioned above. A creation and destruction operator can be defined that will increase or decrease, respectively, the number of particles in a state α_i. These operators can be used to alter n_i, and will therefore span the entire Fock space. The definitions of the operators is modified based on the statistics of the particles.

B.1.1 Bosons

We can define the creation $a_i^\dagger$ and destruction a_i operators by what they do to the states in a complete basis in the Fock space

$$\begin{aligned} a_i^\dagger|n_1,\ldots,n_i,\ldots,n_\Omega\rangle &= \sqrt{n_i+1}|n_1,\ldots,n_i+1,\ldots,n_\Omega\rangle \\ a_i|n_1,\ldots,n_i,\ldots,n_\Omega\rangle &= \sqrt{n_i}|n_1,\ldots,n_i-1,\ldots,n_\Omega\rangle \end{aligned} \tag{B.1}$$

We must now verify that the operators $a_i^\dagger$ and a_i are indeed hermitian conjugate. The only nonzero matrix element for $a_i^\dagger$ is $\langle n_1,\ldots,n_i+1,\ldots,n_\Omega|a_i^\dagger$ $|n_1,\ldots,n_i,\ldots,n_\Omega\rangle = \sqrt{n_i+1}$. Taking the complex conjugate of the above admits $\langle n_1,\ldots,n_i,\ldots,n_\Omega|a_i^\dagger|n_1,\ldots,n_i+1,\ldots,n_\Omega\rangle = \sqrt{n_i+1}$, which is the same as the definition of the operator a_i in (9). These operators only span the Fock space. Although it appears a_i can operate on a state that has $n_i = 0$ particles in the state α_i, the prefactor ensures the matrix element will be zero $a_i|n_1,\ldots,n_i=0,\ldots,n_\Omega\rangle = 0$, and therefore, applying the

destruction operator on a state with no particle in the quantum state cannot admit unphysical states. We can define the vacuum as the state that contains no particles in any of the quantum states, or

$$|\emptyset\rangle = |n_1 = 0, n_2 = 0, \ldots, n_\Omega = 0\rangle \tag{B.2}$$

From this expression and the operators $a_i^\dagger$ we can construct all the vectors of the complete basis of the Fock space, sinc3

$$|n_1, \ldots, n_i, \ldots . n_\Omega\rangle = \frac{(a_1^\dagger)^{n_1} \cdots (a_\Omega^\dagger)^{n_\Omega}}{\sqrt{n_1!} \cdots \sqrt{n_\Omega!}} |\emptyset\rangle \tag{B.3}$$

From this, it is clear that one can completely describe the Fock space from the single state $|\emptyset\rangle$ and the creation and destruction operators. The definitions of $a_i^\dagger$ and a_i imply they have commutation relations, and conversely, if these commutation relations are obeyed, then the corresponding operators and vacuum will define a Fock space from (11) in which they will contain the matrix elements (9). We can see from $(i \neq j)$

$$\begin{aligned} a_i^\dagger a_j^\dagger |n_1, \ldots, n_i, \ldots, n_\Omega\rangle &= a_i^\dagger \sqrt{n_j + 1} |n_1, \ldots, n_i, \ldots, n_j + 1, \ldots, n_\Omega\rangle \\ &= \sqrt{n_i + 1}\sqrt{n_j + 1} |n_1, \ldots, n_i + 1, \ldots, n_j + 1, \ldots, n_\Omega\rangle \\ &= a_j^\dagger a_i^\dagger |n_1, \ldots, n_i, \ldots, n_\Omega\rangle \end{aligned} \tag{B.4}$$

Thus,

$$[a_i^\dagger, a_j^\dagger] |n_1, \ldots, n_i, \ldots, n_j, \ldots, n_\Omega\rangle = 0 \tag{B.5}$$

which admits

$$[a_i^\dagger, a_j^\dagger] = 0 \tag{B.6}$$

Because of commutation, this holds for $i = j$. The Hermitian conjugate of (14) implies

$$[a_i, a_j] = 0 \tag{B.7}$$

For the case where we use $a_i^\dagger a_j$ as the operator, our result is the same so that $[a_i^\dagger, a_j] = 0$ when $i \neq j$. For this operator, the $i = j$ case is special where

$$\begin{aligned} a_i^\dagger a_i |n_1, \ldots, n_i, \ldots, n_\Omega\rangle &= a_i^\dagger \sqrt{n_i} |n_1, \ldots, n_i, \ldots, n_j - 1, \ldots, n_\Omega\rangle \\ &= \sqrt{(n_i - 1) + 1}\sqrt{n_i} |n_1, \ldots, n_i, \ldots, n_\Omega\rangle \\ &= n_i |n_1, \ldots, n_i, \ldots, n_\Omega\rangle \end{aligned} \tag{B.8}$$

and using the operator $a_i a_i^\dagger$ we obtain $a_i a_i^\dagger |n_1, \ldots, n_i, \ldots, n_\Omega\rangle = (n_1 + 1)|n_1, \ldots, n_i, \ldots, n_\Omega\rangle$ which implies that

$$[a_i, a_i^\dagger] |n_1, \ldots, n_i, \ldots, n_\Omega\rangle = |n_1, \ldots, n_i, \ldots, n_{|Omega}\rangle \tag{B.9}$$

Therefore,

$$[a_i, a_j^\dagger] = \delta_{ij} \tag{B.10}$$

and the properties pf the creation and destruction operators can be explained by

$$\begin{aligned} [a_i, a_j^\dagger] &= \delta_{ij} \\ [a_i^\dagger, a_j^\dagger] &= 0 \\ [a_i, a_j] &= 0 \end{aligned} \tag{B.11}$$

Thus, once we obtain a complete basis α_i of the single particle states along with the wave functions $|\alpha_i\rangle$, creation and destruction operators a_i for each state that obeys the canonical commutation relations, and a vacuum $|\emptyset\rangle$ destroyed by the destruction operators $a_i|\emptyset\rangle = 0$, we will be able to Fully construct a Fock space for bosons. Using the above properties and canonical commutation relations between the bosons to compute physical properties rather than using the wavefunctions is the process of second quantization.

B.1.2 Fermions

We can similarly construct the operators for fermions, as we did for bosons, in

$$\begin{aligned} c_i^\dagger|n_1, \ldots, n_i, \ldots, n_\Omega\rangle &= (1 - n_i)(-1)^{\epsilon_i}|n_1, \ldots, n_i + 1, \ldots, n_\Omega \\ c_i|n_1, \ldots, n_i, \ldots, n_\Omega\rangle &= n_i(-1)^{\epsilon_i}|n_1, \ldots, n_i - 1, \ldots, n_\Omega\rangle \end{aligned} \tag{B.12}$$

where $\epsilon_i = \sum_{j=1}^{i-1} n_j$ and $\epsilon_1 = 0$. because the Pauli exclusion principle applies to fermions, the n_i are restricted top be either 0 or 1. From this, we know the creation operator should not be able to generate two particles in the same state, which is why the factor of $1 - n_i$ is used. Also, the n_i factor ensures that the destruction operator cannot destroy a particle in the $n_i = 0$ state. The procedure here is the same of that with the bosons. We first check that the operators $c_i^\dagger$ and c_i are hermitian conjugate. The only nonzero matrix element for the operator $c_i^\dagger$ is

$$\langle n_1, \ldots, n_i = 1, \ldots, n_\Omega|c_i^\dagger|n_1, \ldots, n_i = 0, \ldots, n_\Omega\rangle = (-1)^{\epsilon_i} \tag{B.13}$$

and for c_i we have

$$\langle n_1, \ldots, n_i = 0, \ldots, n_\Omega|c_i^\dagger|n_1, \ldots, n_i = 1, \ldots, n_\Omega\rangle = (-1)^{\epsilon_i} \tag{B.14}$$

We can now examine the action of $c_i c_i^\dagger$, and since this only affects the state α_i, we only need to consider the action on the two states $n_i = 0$ and $n_i = 1$

$$\begin{aligned} c_i c_i^\dagger|n_1, \ldots, n_i = 0, \ldots, n_\Omega\rangle &\\ &= (-1)^{\epsilon_i} c_i|n_1, \ldots, n_i = 1, \ldots, n_\Omega\rangle \\ &= (-1)^{2\epsilon_i}|n_1, \ldots, n_i = 0, \ldots, n_\Omega \\ &= |n_1, \ldots, n_i = 0, \ldots, n_\Omega\rangle \end{aligned} \tag{B.15}$$

where, on the other hand, $c_i^\dagger c_i|n_1,\ldots,n_i=0,\ldots,n_\Omega\rangle = 0$. Here, $(-1)^{\epsilon_i}$ terms do not play any role. We also have

$$\begin{aligned} c_i c_i^\dagger|n_1,\ldots,n_i=1,\ldots,n_\Omega\rangle &= 0 \\ c_i^\dagger c_i|n_1,\ldots,n_i=1,\ldots,n_\Omega\rangle &= |n_1,\ldots,n_i=1,\ldots.n_\Omega\rangle \end{aligned} \tag{B.16}$$

Clearly, $[c_i, c_i^\dagger]$ has no simple expression. However, for the anticommutator,

$$[c_i, c_i^\dagger]_+ = c_i c_i^\dagger + c_i^\dagger c_i \tag{B.17}$$

admits

$$[c_i, c_i^\dagger]_+|n_1,\ldots,n_i,\ldots,n_\Omega\rangle = |n_1,\ldots,n_i,\ldots,n_\Omega\rangle \tag{B.18}$$

and therefore,

$$[c_i, c_i^\dagger]_+ = 1 \tag{B.19}$$

which shows that the anticommutator will play an important role. The factor $(-1)^{\epsilon_i}$ ensures that the other combinations give simple results for the anticommutator. This leads to the further result that

$$[c_i, c_j^\dagger] = 0 \tag{B.20}$$

where $i \neq j$ and $i < j$ which is due to the phase factors ϵ_j admitting a minus sign between terms. This will allow us to define c_i operators only in terms of the their anticommutators. Thus, we can write

$$\begin{aligned} [c_i, c_j^\dagger]_+ &= \delta_{i,j} \\ [c_i^\dagger, c_j^\dagger]_+ &= 0 \\ [c_i, c_j]_+ &= 0 \end{aligned} \tag{B.21}$$

where c_i and $c_i^\dagger$ can be used to construct all the states of the Fock space from a vacuum $|\emptyset\rangle$ which gets destroyed by all the destroyers c_i using the same relation (11) as before. The wavefunctions and averages can also be computed using the same techniques as with the bosons. The antisymmetric form of the wavefunction for Fermions can be seen directly through the operators without even viewing the wavefunction. Using $[c_1, c_2]_+ = 0$, admits $c_1^\dagger c_2^\dagger|\emptyset\rangle = -c_2^\dagger c_1^\dagger|\emptyset\rangle$, showing the wavefucntion $|\psi\rangle$ is antisymmetric by permutation of the particles.

B.2 One body operators

Now that we can construct a Fock space using operators, we want to use the same operators to construct the physical observables. Due to the fact that

the physical observables have to act on indistinguishable particles, there are constraints as to what they can be. For an expression of the observables in second quantization, we must characterize observables in terms of how many particles are involved. One body operators are those physical observables that measure only the quantum numbers of one particle at a time, and two body operators are those observables that measure the quantum numbers of two particles at a time to determine the matrix elements.

B.2.1 Definition

We can begin by letting O be an operator which involves only the measurement of one particle. Let $O^{(1)}$ be the operator acting in the Hilbert space of a single particle, then O must be

$$O = O_1^{(1)} \otimes 1_2 \otimes \cdots \otimes 1_{\mathcal{N}} + 1_1 \otimes O_2^{(1)} \otimes \cdots \otimes 1_{\mathcal{N}} + \cdots + 1_1 \otimes \cdots \otimes O_{\mathcal{N}}^{(1)} \tag{B.22}$$

where $O_i^{(1)}$ is the operator acting on the i-th particle. This admits the most general form of a one body operator for indistinguishable particles. To express (30) in second quantization, we can look at a system with one particle. In this case, $O = O^{(1)}$ and using the complete basis α admits

$$O = \sum_{\alpha,\beta} |\alpha\rangle\langle\alpha|O^{(1)}|\beta\rangle\langle\beta| \tag{B.23}$$

and with $|\alpha\rangle = c_\alpha^\dagger|\emptyset\rangle$, we have

$$O = \sum_{\alpha,\beta} \langle\alpha|O^{(1)}|\beta\rangle c_\alpha^\dagger|\emptyset\rangle\langle\emptyset|c_\beta \tag{B.24}$$

From this, we know that the destroyer operator c_β destroys a particle in a state β. Going through the vacuum, the particle will change its quantum state from the state β to the state α. With an arbitrary number of particles, we must do the same thing for each particle which requires an operator

$$O = \sum_{\alpha,\beta} \langle\alpha|O^{(1)}|\beta\rangle c_\alpha^\dagger c_\beta \tag{B.25}$$

which is identical to (32) except we do not have to go through the vacuum after destruction. With the operator $c_\alpha^\dagger c_\beta$, c_β will act on all particles. This expression (33) allows us to represent any single-body operator in second quantization. When the matrix elements $\langle\alpha|O^{(1)}|\beta\rangle$ are computed, the entire operator reduces to a linear combination of creation and destruction operators. Therefore, all physical averages can be computed by the techniques described previously.

B.2.2 Examples

We can begin by writing the operator measuring the density of particles at a point $\mathbf{r}_0$ for one particle

$$\rho^{(1)}(\mathbf{r}_0) = |\mathbf{r}_0\rangle\langle\mathbf{r}_0| \tag{B.26}$$

For first quantization, we have $\langle\psi|\rho^{(1)}(\mathbf{r}_0)|\psi\rangle = |\psi(\mathbf{r}_0)|^2$. In second quantization, the form of the operator will depend on the choice of basis α. We can start by taking the position basis $|\mathbf{r}\rangle$, where $c^\dagger_{\mathbf{r}}$ is the operator creating a particle at point $\mathbf{r}$. With the basis and (33), we have

$$\begin{aligned}\rho(\mathbf{r}_0) &= \sum_{\mathbf{r}\mathbf{r}'}\langle\mathbf{r}|\mathbf{r}_0\rangle\langle\mathbf{r}_0|\mathbf{r}'\rangle c^\dagger_{\mathbf{r}}c_{\mathbf{r}'} \\ &= \sum_{\mathbf{r}\mathbf{r}'}\delta(\mathbf{r}-\mathbf{r}_0)\delta(\mathbf{r}_0-\mathbf{r}')c^\dagger_{\mathbf{r}}c_{\mathbf{r}'} \\ &= c^\dagger_{\mathbf{r}_0}c_{\mathbf{r}_0}\end{aligned} \tag{B.27}$$

where the operator $c^\dagger_{\mathbf{r}_0}c_{\mathbf{r}_0}$ destroys and recreates a particle in the same quantum state. This operator will give zero if there is no particle in the corresponding quantum state and one with one particle. This gives a number of particles at the point $\mathbf{r}_0$. In general, $c^\dagger_\alpha c_\alpha$ counts the number of particles in the state α. We can get the total number of particles by integrating over position

$$N = \int d\mathbf{r} c^\dagger_{\mathbf{r}}c_{\mathbf{r}} \tag{B.28}$$

We can generalize this further to the case of particles with spins, where the complete basis becomes $\alpha = (\mathbf{r}, \text{œ})$. Here, the density operator only acts on the spatial part which admits

$$\rho^{(1)}(\mathbf{r}_0) = |\mathbf{r}_0\rangle\langle\mathbf{r}_0|\otimes 1_{spin} \tag{B.29}$$

giving

$$\begin{aligned}\rho(\mathbf{r}_0) &= \sum_{\mathbf{r}\sigma,\mathbf{r}'\sigma'}\langle\mathbf{r}\sigma|\mathbf{r}_0\rangle\langle\mathbf{r}_0|\mathbf{r}'\sigma'\rangle c^\dagger_{\mathbf{r}\sigma}c_{\mathbf{r}'\sigma'} \\ &= \sum_{\mathbf{r}\sigma,\mathbf{r}'\sigma'}\delta(\mathbf{r}-\mathbf{r}_0)\delta(\mathbf{r}_0-\mathbf{r}')\delta_{\sigma\sigma'}c^\dagger_{\mathbf{r}}c_{\mathbf{r}',\sigma'} \\ &= c_{\mathbf{r}_0\uparrow}c_{\mathbf{r}_0\uparrow} + c^\uparrow_{\mathbf{r}_0\downarrow}c_{\mathbf{r}_0\downarrow}\end{aligned} \tag{B.30}$$

For particles with spin, we can compute the spin density along z at the point $\mathbf{r}_0$. The operator in this case becomes

$$\sigma^{(1)}_z(\mathbf{r}_0) = |\mathbf{r}_0\rangle\langle\mathbf{r}_0|\otimes S_z \tag{B.31}$$

Thus, using (33) we come to

$$\begin{aligned}\sigma_z(\mathbf{r}_0) &= \sum_{\mathbf{r}\sigma,\mathbf{r}'\mathbf{o}\mathbf{e}'} \langle \mathbf{r}\sigma|\mathbf{r}_0\rangle\langle\mathbf{r}_0|\otimes S_z|\mathbf{r}'\sigma'\rangle c^\dagger_{\mathbf{r}\sigma}c_{\mathbf{r}'\sigma'} \\ &= \sum_{\sigma\sigma'}\langle\sigma|S_z|\sigma'\rangle c^\dagger_{\mathbf{r}_0\sigma}c_{\mathbf{r}_0\sigma'} \\ &= \frac{1}{2}(c^\dagger_{\mathbf{r}_0\uparrow}c_{\mathbf{r}_0\uparrow} - c^\dagger_{\mathbf{r}_0\downarrow}c_{\mathbf{r}_0\downarrow}) \end{aligned} \tag{B.32}$$

in a similar way for the x-direction we have

$$\begin{aligned}\sigma_x(\mathbf{r}_0) &= \sum_{\mathbf{r}\sigma,\mathbf{r}'\sigma'} \langle \mathbf{r}\sigma|\mathbf{r}_0\rangle\langle\mathbf{r}_0|\otimes S_x|\mathbf{r}'\sigma'\rangle c^\dagger_{\mathbf{r}\sigma}c_{\mathbf{r}'\sigma'} \\ &= \sum_{\sigma\sigma'}\langle\sigma|S_x|\sigma'\rangle c^\dagger_{\mathbf{r}_0\sigma}c_{\mathbf{r}_0\sigma'} \\ &= \frac{1}{2}(c^\dagger_{\mathbf{r}_0\uparrow}c_{\mathbf{r}_0\downarrow} - c^\dagger_{\mathbf{r}_0\downarrow}c_{\mathbf{r}_0\uparrow}) \end{aligned} \tag{B.33}$$

We could have used the basis of the eigenstates of the momentum $|\mathbf{k}\rangle$ where

$$\langle\mathbf{r}|\mathbf{k}\rangle = \frac{1}{\sqrt{\Omega}}e^{i\mathbf{k}\mathbf{r}} \tag{B.34}$$

Due to the independence between the spin and orbital part, we will give expressions for the spinless case. The operator $c_\mathbf{k}$ now destroys a particle with momentum $\mathbf{k}$. Using (33) one obtains

$$\begin{aligned}\rho(\mathbf{r}_0) &= \sum_{\mathbf{k}_1\mathbf{k}_2}\langle\mathbf{k}_1|\mathbf{r}_0\rangle\langle\mathbf{r}_0|\mathbf{k}_2\rangle c^\dagger_{\mathbf{k}_1}c_{\mathbf{k}_2} \\ &= \frac{1}{\Omega}\sum_{\mathbf{k}_1\mathbf{k}_2} e^{-i\mathbf{k}_1\mathbf{r}_0}e^{i\mathbf{k}_2\mathbf{r}_0}c^\dagger_{\mathbf{k}_1}c_{\mathbf{k}_2} \end{aligned} \tag{B.35}$$

Here, as in contrast to (35), the density operator is not diagonal in the momentum basis. However, both (35) and (43) represent the same operator. By comparing these expressions, we can obtain a connection between the operators creating a particle at point $\mathbf{r}$ and one creating a particle with momentum $\mathbf{k}$,

$$c_\mathbf{r} = \frac{1}{\sqrt{\Omega}}\sum e^{i\mathbf{k}\mathbf{r}}c_\mathbf{k} \tag{B.36}$$

Using (43) we find the total number of particles in the system,

$$\begin{aligned}N &= \int \mathbf{r}\frac{1}{\Omega}\sum_{\mathbf{k}_1\mathbf{k}_2} e^{-i\mathbf{k}_1\mathbf{r}}e^{i\mathbf{k}_2\mathbf{r}}c^\dagger_{\mathbf{k}_1}c_{\mathbf{k}_2} \\ &= \sum_{\mathbf{k}_1\mathbf{k}_2}\delta_{\mathbf{k}_1\mathbf{k}_2}c^\dagger_{\mathbf{k}_1}c_{\mathbf{k}_2} \\ &= \sum_\mathbf{k} c^\dagger_\mathbf{k}c_\mathbf{k} \end{aligned} \tag{B.37}$$

As $c^\dagger_{\mathbf{k}} c_{\mathbf{k}}$ counts the number of particles in the quantum state $\mathbf{k}$, the total number of particles is the sum of all numbers of particles in all possible quantum states. We can attain the Fourier transform of the density using (43) giving

$$\begin{aligned}\rho(\mathbf{q}) &= \int \underset{\cdot}{\mathbf{r}} e^{-i\mathbf{q}\mathbf{r}} \rho(\mathbf{r}) \\ &= \int \underset{\cdot}{\mathbf{r}} e^{-i\mathbf{q}\mathbf{r}} \frac{1}{\Omega} \sum_{\mathbf{k}_1\mathbf{k}_2} e^{-i\mathbf{k}_1\mathbf{r}} e^{i\mathbf{k}_2\mathbf{r}} c^\dagger_{\mathbf{k}_1} c_{\mathbf{k}_2} \\ &= \sum_{\mathbf{k}_1\mathbf{k}_2} \delta_{\mathbf{k}_2,\mathbf{k}_1+\mathbf{q}} c^\dagger_{\mathbf{k}_1} c_{\mathbf{k}_2} \\ &= \sum_{\mathbf{k}} c^\dagger_{\mathbf{k}-\mathbf{q}} c_{\mathbf{k}} \end{aligned} \tag{B.38}$$

Another operator we must consider id the kinetic energy of the particles. For one particle, we have $H^{(1)} = \frac{p^2}{2m}$. In general, we can have any function of the momentum $H^{(1)} = E(P)$, thus, it is best to use the momentum basis. The kinetic energy can be written as

$$\begin{aligned} H &= \sum_{\mathbf{k}_1\mathbf{k}_2} \langle \mathbf{k}_1 | \epsilon(P) | \mathbf{k}_2 \rangle c^\dagger_{\mathbf{k}_1} c_{\mathbf{k}_2} \\ &= \sum_{\mathbf{k}_1\mathbf{k}_2} \delta_{\mathbf{k}_1\mathbf{k}_2} \epsilon(\mathbf{k}_2) c^\dagger_{\mathbf{k}_1} c_{\mathbf{k}_2} \\ &= \sum_{\mathbf{k}} \epsilon(\mathbf{k}) c^\dagger_{\mathbf{k}} c_{\mathbf{k}} \end{aligned} \tag{B.39}$$

To generalize in the case of a system with spin, we have

$$H = \sum_{\mathbf{k}\sigma} \epsilon(\mathbf{k}) c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} \tag{B.40}$$

where we assumed the kinetic energy does not depend on spin. Since the total number of particles $N = \sum_{\mathbf{k}} c^\dagger_{\mathbf{k}} c_{\mathbf{k}}$ adding a chemical potential $-\mu N$ does not change the form of the Hamiltonian

$$H = \sum_{\mathbf{k}\sigma} \zeta(\mathbf{k}) c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} \tag{B.41}$$

where $\epsilon(\mathbf{k})$ is replaced by $\zeta(\mathbf{k}) = \epsilon(\mathbf{k}) - \mu$. At zero temperature the energy $\zeta(\mathbf{k})$ is zero at the Fermi level, negative below, and positive above.

B.3 Two body operators

We can now look at operators that involve two particles in order to define their matrix elements. This is the case of the interaction between two particles.

B.3.1 Definition

We can denote $O^{(2)}$ as the corresponding operator that operates in the space of only two particles. The two body operator that will operate with $\mathcal{N}$ particles must have the form

$$\begin{aligned} O &= \sum_{i\leq j} O^{(2)}_{i,j} \bigotimes_{k\neq i,j} 1_k \\ &= \frac{1}{2}\sum_{i\neq j} O^{(2)}_{i,j} \bigotimes_{k\neq i,j} 1_k \end{aligned} \tag{B.42}$$

where the $O^{(2)}$ operator can operate on each pairs of particles in the system. For second quantization, we must look at the case with two particles. The operator O must be defined by its matrix elements in the physical space of the (anti)symmetrized functions $|\alpha,\beta\rangle$. This corresponds to knowing all the elements of

$$\langle\alpha,\beta|O^{(2)}|\gamma,\delta\rangle \tag{B.43}$$

We can express $|\alpha,\beta\rangle$ in terms of the ordered kets and use the fact that we are dealing with indistinguishable particles to obtain

$$(\alpha,\beta|O^{(2)}|\gamma,\delta) = (\beta,\alpha|O^{(2)}|\delta,\gamma) \tag{B.44}$$

which is just the exchange of two particles, thus

$$\langle\alpha,\beta|O^{(2)}|\gamma,\delta\rangle = (\alpha,\beta|O^{(2)}|\gamma,\delta) \pm (\alpha,\beta|O^{(2)}|\delta,\gamma) \tag{B.45}$$

In second quantization, there should be an operator that reproduces these elements and works for any number $\mathcal{N}$ of particles. We can verify that

$$O = \frac{1}{2}\sum_{\alpha,\beta\gamma,\delta} (\alpha,\beta|O^{(2)}|\gamma,\delta) c^\dagger_\alpha c^\dagger_\beta c_\delta c_\gamma \tag{B.46}$$

work for fermions and bosons. Here, we will check that it works for 2 particles. Using (54) the matrix elements become

$$\langle\alpha_0,\beta_0|O|\gamma_0,\delta_0\rangle = \frac{1}{2}\sum_{\alpha,\beta,\gamma,\delta} (\alpha,\beta|O^{(2)}|\gamma,\delta)\langle\alpha_0,\beta_0|c^\dagger_\alpha c^\dagger_\beta c_\delta c_\gamma|\gamma_0,\delta_0\rangle \tag{B.47}$$

and since $|\alpha_0,\beta_0\rangle = c^\dagger_{\alpha_0} c^\dagger_{\beta_0}$, we must compute averages of the form

$$\langle\emptyset|c_{\beta_0} c_{\alpha_0} c^\dagger_\alpha c^\dagger_\beta c_\delta c_\gamma c^\dagger_{\gamma_0} c^\dagger_{\delta_0}|\emptyset\rangle \tag{B.48}$$

which can be computed using the technique where the destruction operators act on the vacuum. This admits

$$\langle\emptyset|c_{\beta_0} c_{\alpha_0} c^\dagger_\alpha c^\dagger_\beta c_\delta c_\gamma c^\dagger_{\gamma_0} c^\dagger_{\delta_0}|\emptyset\rangle = [\delta_{\alpha_0,\alpha}\delta_{\beta_0,\beta} \pm \delta_{\alpha_0,\beta}\delta_{\beta_0,\alpha}][\delta_{\gamma_0,\gamma}\delta_{\delta_0,\delta} \pm \delta_{\gamma_0,\delta}\delta_{\delta_0,\gamma}] \tag{B.49}$$

where the + is for bosons and − for fermions. The physical meaning of this is that when the destruction operators act

$$c_\delta c_\gamma |\gamma_0, \delta_0\rangle \tag{B.50}$$

they must destroy the two particles in the two possible quantum states, δ and γ, with the correctly assigned sign depending on the (anti)symmetry of the wavefunction.

B.4 Solving with second quantization

We will now examine how to solve a problem in second quantization when we know the Hamiltonian. We will be solving for the eigenvalues and eigenstates.

Let us begin by writing a general quadratic Hamiltonian

$$H = \sum_\alpha A_\alpha c_\alpha^\dagger c_\alpha \tag{B.51}$$

where α is some complete basi, and all A_α are arbitrary numbers. For quadratic and diagonal Hamiltonians of this form, the problem is solved. Each vector of the form

$$c_{\alpha_1}^\dagger c_{\alpha_2}^\dagger c_{\alpha_3}^\dagger \cdots c_{\alpha_p}^\dagger |\emptyset\rangle \tag{B.52}$$

is an eigen vector of H with an eigenvalue

$$E = \sum_{i=1}^{p} A_i \tag{B.53}$$

We will illustrate this with two terms $|\psi\rangle = c_{\alpha_1}^\dagger c_{\alpha_2}^\dagger |\emptyset\rangle$ for fermions, where a similar calculation can be done for bosons.

$$\begin{aligned}
H c_{\alpha_1}^\dagger c_{\alpha_2}^\dagger |\emptyset\rangle &= \left(\sum_\alpha A_\alpha c_\alpha^\dagger c_\alpha \right) c_{\alpha_1}^\dagger c_{\alpha_2}^\dagger |\emptyset\rangle \\
&= \sum_\alpha A_\alpha c_\alpha^\dagger (\delta_{\alpha,\alpha_1} - c_{\alpha_1}^\dagger c_\alpha) c_{\alpha_2}^\dagger |\emptyset\rangle \\
&= A_{\alpha_1} |\psi\rangle - \sum_\alpha A_\alpha c_\alpha^\dagger c_{\alpha_1}^\dagger c_\alpha c_{\alpha_2}^\dagger |\emptyset\rangle \\
&= A_{\alpha_1} |\psi\rangle - \sum_\alpha A_\alpha c_\alpha^\dagger c_{\alpha_1}^\dagger (\delta_{\alpha,\alpha_2} - c_{\alpha_2}^\dagger c_\alpha) |\emptyset\rangle \\
&= A_{\alpha_1} |\psi\rangle - A_{\alpha_2} c_{\alpha_2}^\dagger c_{\alpha_1}^\dagger |\emptyset\rangle \\
&= A_{\alpha_1} |\psi\rangle + A_{\alpha_2} |\psi\rangle
\end{aligned} \tag{B.54}$$

The operator $c^\dagger_\alpha c_\alpha$ counts the particles in the state α. If there is a particle in $|\psi\rangle$, it will return 1, where the corresponding energy will be counted in H. At zero temperature the ground state will simply consist (for fermions) in occupying all the states with the minimum energy according to the number of particles in the system.

$$|F\rangle = \Pi_{i=1}^{\mathcal{N}} c^\dagger_{\alpha_i}|\emptyset\rangle \tag{B.55}$$

if $E_1 \leq E_2 \leq \cdots \leq E_\Omega$. At finite temperature we can also compute the averages of many operators.

$$\begin{aligned}\langle c^\dagger_{\alpha_p}, c_{\alpha_p}\rangle &= \frac{Tr[e^{-\beta H} c^\dagger_{\alpha_p} c_{\alpha_p}]}{Tr[e^{-\beta H}]} \\ &= \frac{\sum_{n_1,\ldots,n_\Omega}\langle n_1,\ldots,n_\Omega|e^{-\beta H}c^\dagger_{\alpha_p}c_\alpha|n_1,\ldots,n_\Omega\rangle}{\sum_{n_1,\ldots,n_\Omega}\langle n_1,\ldots,n_\Omega|e^{\beta H}|n_1,\ldots,n_\Omega\rangle}\end{aligned} \tag{B.56}$$

Since $[c^\dagger_\alpha, c_\alpha, c_\gamma] = 0$ if $\alpha =\neq \gamma$ and a similar relation for $c^\dagger_\gamma$, we see that the term $e^{\beta H}$ factorizes

$$e^{-\beta H} = \Pi_{j=1}^{\Omega} e^{-\beta A_{\alpha_j} c^\dagger_{\alpha_j} c_{\alpha_j}} \tag{B.57}$$

Since in the trace each term n_i is independent, the average factorizes. The numerator becomes

$$\left(\sum_{n_{\alpha_p}}\langle n_{\alpha_p}|e^{-\beta A_{\alpha_p} c^\dagger_{\alpha_p} c_{\alpha_p}} c^\dagger_{\alpha_p} c_{\alpha_p}|n_{\alpha_p}\rangle\right)\Pi_{j\neq p}\left(\sum_{n_{\alpha_j}}\langle n_{\alpha_j}|e^{-\beta_{\alpha_j} c^\dagger_{\alpha_j} c_{\alpha_j}}|n_{\alpha_j}\rangle\right) \tag{B.58}$$

All the terms for $j \neq p$ are identical in the numerator and denominator and cancel each other. The trace thus reduces to

$$\langle c^\dagger_{\alpha_p} c_{\alpha_p}\rangle = \frac{\sum_{n_{\alpha_p}}\langle n_{\alpha_p}|e^{-\beta A_{\alpha_p} c^\dagger_{\alpha_p} c_{\alpha_p}} c^\dagger_{\alpha_p} c_{\alpha_p}|n_{\alpha_p}\rangle}{\sum_{n_{\alpha_p}}\langle n_{\alpha_p}|e^{-\beta A_{\alpha_p} c^\dagger_{\alpha_p} c_{\alpha_p}}|n_{\alpha_p}} \tag{B.59}$$

Since the Hamiltonian being diagonal in α only the state α_p can contribute to the average of an operator only involving the state α_p. Since $c^\dagger_{\alpha_p} c_{\alpha_p}|n_p\rangle = n_p|n_p\rangle$ one obtains

$$\langle c^\dagger_{\alpha_p}, c_{\alpha_p}\rangle = \frac{\sum_{n_{\alpha_p}} e^{-\beta A_{\alpha_p} n_p} n_p}{\sum_{n_{\alpha_p}} e^{-\beta A_{\alpha_p} n_p}} \tag{B.60}$$

Now, for Fermions, only $n_p = 0$ and $n_p = 1$ are in the sum. Therefore,

$$\langle c^\dagger_{\alpha_p}, c_{\alpha_p}\rangle = \frac{e^{-\beta A_{\alpha_p}}}{1 + e^{-\beta A_{\alpha_p}}} = \frac{1}{1 + e^{\beta A_{\alpha_p}}} \tag{B.61}$$

which gives the Fermi factor. This is a totally general result (not limited to eigenstates of momentum) as soon as one has a quadratic Hamiltonian and one is in thermal equilibrium. For bosons, $n_p = 0, \ldots, +\infty$, and thus the sum becomes

$$\begin{aligned}\langle c^\dagger_{\alpha_p}, c_{\alpha_p}\rangle &= -\frac{\partial}{\partial\beta} log\left[\sum_{n_p=0}^{\infty} e^{-\beta n_p A_{\alpha_p}}\right] \\ &= -\frac{\partial}{\partial\beta} log\left[\frac{1}{1-e^{-\beta A_{\alpha_p}}}\right] \\ &= \frac{e^{\beta A_{\alpha_p}}}{1-e^{-\beta A_{\alpha_p}}} \\ &= \frac{1}{e^{\beta A_{\alpha_p}}-1}\end{aligned} \tag{B.62}$$

which is precisely the Bose factor. However, although a quadratic diagonal Hamiltonian gives all the physical quantities we need, this is usually not the case. Thus, solving in second quantization means that we have to find a transformation of the operators c and $c^\dagger$ that put the Hamiltonian in a quadratic diagonal form. We want the new operators d and $d^\dagger$ that are the results of the transformation to still spawn the Fock space. It means that We must use a transformation that preserves the canonical commutation relations. Even without solving the Hamiltonian one can exploit the freedom of choosing different creation and destruction operators to use a more convenient representation. We can begin with a simple example in the particle-hole transformation

$$\begin{aligned}c^\dagger_\alpha &= d_\alpha \\ c_\alpha &= d^\dagger_\alpha\end{aligned} \tag{B.63}$$

For fermions it is easy to check, by substitution of the d operators that they verify the canonical anticommutation relations. For example,

$$[d_\alpha, d^\dagger_\beta]_+ = [c^\dagger_\alpha, c_\beta]_+ = \delta\alpha, \beta \tag{B.64}$$

Here, d_α destroys a hole with state α (which is identical to creating an electron) and the operator $d^\dagger_\alpha$ creates a hole (which is the same thing as destroying an electron). Remember also that when making the transformation one must modify the vacuum as well. Indeed the vacuum for the particles d is not the same as for the vacuum for particles c. We define $|\emptyset_c\rangle$ and $|\emptyset_d\rangle$. The vacuum of d particles is defined as usual by

$$d_\alpha|\emptyset_d\rangle = 0 \tag{B.65}$$

for all states α. It is easy to check using the relation that

$$|\emptyset_d\rangle = \Pi_\alpha c^\dagger_\alpha|\emptyset_c\rangle \tag{B.66}$$

B.5 Bogoliubov Transformation

This is a problem of the tight binding Hamiltonian to which we have added a periodic potential. The potential is in first qunatization

$$H_{pot}^{(1)} = V \sum_i (-1)^i |i\rangle\langle i| \tag{B.67}$$

which in second quantization gives

$$H_{pot} = V \sum_i (-1)^i c_i^\dagger c_i \tag{B.68}$$

Since we know how to diagonalize (78) using the operators d_k, it is natural to express H_{pot} in this basis. One has, using that $(-1)^j = e^{i\frac{\pi}{a}r_j}$

$$H_{pot} = V \sum_j e^{i\frac{\pi}{a}r_j} \frac{1}{N_s} \sum_{k_1,k_2} e^{-ik_1 r_j} e^{-ik_2 r_j} d_{k_1}^\dagger d_{k_2} \tag{B.69}$$

where the two vectors k_1 and k_2 belong to the first Brillouin zone $k \in [-\pi/a, \pi/a]$. The sum over j can be easily performed and gives the constraint that $k_1 = k_2 + \pi/a$ modulo $2\pi/a$. the potential becomes

$$H_{pt} = V \sum_k c_{k+\pi(\mod 2\pi)/a}^\dagger c_k \tag{B.70}$$

the momentum $k+\pi/a$ has to be brought back in the first Brillouin zone by the proper translation in $\pm 2\pi/a$. the potential thus couples two different k values. The tight binding Hamiltonian is diagonal in neither basis. In the real space basis the tight binding Hamiltonian is non diagonal while the potential is, while in the k basis the tight binding Hamiltonian is diagonal but the potential is not. To diagonalize the total Hamiltonian we can notice that although a state k is coupled to a state $k + \pi/a$ the coupling stops there since the state $k + \pi/a$ is coupled to $k + 2\pi/a \equiv k$ and thus linear combinations of d_k and $d_{k+\pi}$ should be able to diagonalize the full Hamiltonian. We can also use the more physical approach in using the symmetries of the problem. Even in the presence of the potential, the system is still invariant by translation provided that one considers a unit cell with two atoms per unit cell. For such a lattice system where the lattice spacing is $2a$ the Brillouin zone $Z'_B = [-\pi/(2a), \pi/(2a)]$. One thus has two equivalent ways of viewing this problem

(a) The original lattice spacing.

(b) The reduced zone scheme.

Let us continue with the reduced zone scheme. We need to express the operators d_k with $k \in Z_B$ in terms of new operators expressing the degrees of freedom in the reduced zone. We can consider the mapping

$$\begin{aligned} d_k &= \alpha_k \quad k \in [-\pi/(2a), \pi/(2a)] \\ d_k &= \beta_{k-\pi/a} \quad k \in [\pi/(2a), \pi/a] \\ d_k &= \beta_{k+\pi/a} \quad k \in [-\pi/a, -\pi/(2a)] \end{aligned} \tag{B.71}$$

It is easy to check that the operators α_k and β_k are defined for $k \in Z'_B$. they also obey the canonicaql commutation rules, and are thus faithful representations of fermions. In terms of this operator one can rewrite the Hamiltonian. Let us begin with the tight binding part where $\xi(k) = -2t\cos(ka)$

$$\begin{aligned} \sum_{k\in Z_B} \xi(k) x_k^\dagger c_k &= \sum_{k\in Z'_B} \xi(k) c_k^\dagger c_k + \sum_{k\in[\pi/(2a),\pi/a]} \xi(k) c_k^\dagger c_k + \sum_{k\in[-\pi/a,-\pi/(2a)]} \xi(k) c_k^\dagger c_k \\ &= \sum_{k\in Z'_B} \xi(k) \alpha_k^\dagger \alpha_k + \sum_{k\in[-\pi/(2a),0]} \xi(k+\pi/a) \beta_k^\dagger \beta_k + \sum_{k\in[0,\pi/(2a)]} \xi(k-\pi/a) \beta_k^\dagger \beta_k \\ &= \sum_{k\in Z'_B} \xi(k) (\alpha_k^\dagger \alpha_k - \beta_k^\dagger \beta_k) \end{aligned} \tag{B.72}$$

having used $\xi(k \pm \pi/a) = -\xi(k)$. In the above formula, the two bands are clearly visible. Since the tight binding Hamiltonian is also invariant by a one unit cell translation the energy dispersion has no gap as the zone boundary, since the folding zone is merely an arbitrary choice here.

$$\sum_{k\in Z_B} c^\dagger_{k+\pi/a(\mod 2\pi/a)} c_k = \sum_{k\in Z'_B} (\alpha_k^\dagger \beta_k + \beta_k^\dagger \alpha_k) \tag{B.73}$$

The final result becomes

$$H = \sum_{k\in Z'_B} [A(k)(\beta_k^\dagger \beta_k - \alpha_k^\dagger \alpha_k) + V(\alpha_k^\dagger \beta_k + \beta_k^\dagger \alpha_k)] \tag{B.74}$$

where we have introduced $A(k) = -\xi(k)$ which is a positive quantity for $k \in Z'_B$. We see that the Hamiltonian is diagonal in k in the reduced zone, as it should since the system is invariant by translation. However, it is not fully diagonal since the lower band α is coupled to the upper band β by the potential V. We can rewrite the Hamiltonian in a matrix form

$$H = \sum_{k\in Z'_B} \begin{pmatrix} \alpha_k^\dagger & \beta_k^\dagger \end{pmatrix} \begin{pmatrix} -A(k) & V \\ V & A(k) \end{pmatrix} \begin{pmatrix} \alpha_k \\ \beta_k \end{pmatrix} \tag{B.75}$$

This structure makes it clear that the Hamiltonian is diagonal in k, but that there are two residual degrees of freedom that a coupled by the potential

making the matrix of the quadratic form non-diagonal. It also makes it obvious that all we have to do is to diagonalize the matrix by the appropriate linear combination of the operators α and *beta*. This type of Hamiltonian where 2 degrees of freedom are coupled is specially important and arises in several occasions, for example as a result of a mean field approximation of the real hamiltonian as we will see. It is thus important to see the general procedure to solve it. Is known as the Bogoliubov transformation and is nothing but the above mentioned diagonalization of the matrix. On the form of the above expression, we can directly read the eigenvalues. They are given by the equation

$$(A(k) - E)(-A(k) - E) - V^2 = 0 \tag{B.76}$$

and thus

$$E(k) = \pm\sqrt{A(k)^2 + V^2} \tag{B.77}$$

There is now a gap $2V$ that opens at the reduced zone boundary. To determine the eigenvectors we must make combinations of the operators α and β.

$$\begin{pmatrix} \gamma_{k-} \\ \gamma_{k+} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_k \\ \beta_k \end{pmatrix} \tag{B.78}$$

which defines new operators γ_{k+} and γ_{k-}. These operators result from the combination of the top and bottom band or in the original language of states with momentum k and momentum $k+\pi/a$ and describe the interference effect between these two states due to the scattering potential. Since the new operators γ must verify the canonical commutation relations, the coefficients a, b, c, d cannot be arbitrary. In order to ensure $[\gamma_{k+}, \gamma_{k-}^\dagger]_+ = 0$ and the other commutators, one can check that it is necessary to take the coefficients as

$$\begin{pmatrix} \gamma_{k-} \\ \gamma_{k+} \end{pmatrix} = \begin{pmatrix} u_k & -v_k \\ v_k & u_k \end{pmatrix} \begin{pmatrix} \alpha_k \\ \beta_k \end{pmatrix} \tag{B.79}$$

moreover one must have $u_k^2 + v_k^2 = 1$. Note that here we have chosen real coefficients but one can generalize the transformation to complex ones. The last condition comes for example from

$$\begin{aligned} [\gamma_{k-}, \gamma_{k-}^\dagger]_+ &= [u_k]alpha_k - v_k\beta_k, u_k\alpha_k^\dagger - v_k\beta_k^\dagger]_+ \\ &= u_k^2[\alpha_k, \alpha_k^\dagger]_+ + v_k^2[\beta_k, \beta_k^\dagger]_+ + u_k v_k([\beta_k, \alpha_k^\dagger]_+ + [\alpha_k, \beta_k^\dagger]_+) \\ &= u_k^2 + v_k^2 \end{aligned} \tag{B.80}$$

This is quite natural since the transformation (the Bogoliubov transformation) is an orthogonal transformation with two vectors that are orthogonal

and normed. It is thus a "rotation" in the space spawned by the vectors α_k and β_k. Note that in general the coefficients of the transformation depend on k. Given the constraint one often uses the parametrization

$$u_k = \cos\theta_k \quad , \quad v_k = \sin\theta_k \tag{B.81}$$

The transformation is easily inverted

$$\begin{pmatrix} \alpha_k \\ \beta_k \end{pmatrix} = \begin{pmatrix} u_k & v_k \\ -v_k & u_k \end{pmatrix} \begin{pmatrix} \gamma_{k-} \\ \gamma_{k+} \end{pmatrix} \tag{B.82}$$

Inserting (98) into (91) one obtains

$$H = \sum_{k \in Z'_B} \begin{pmatrix} \gamma^\dagger_{k-} & \gamma^\dagger_{k+} \end{pmatrix} \begin{pmatrix} -[A(k)(u_k^2 - v_k^2) + 2V u_k v_k] & V(u_k^2 - v_k^2) - 2A(k) u_k v_k \\ V(u_k^2 - v_k^2) - 2A(k) u_k v_k & +[A(k)(u_k^2 - v_k^2) + 2V u_k v_k] \end{pmatrix} \begin{pmatrix} \gamma_{k-} \\ \gamma_{k+} \end{pmatrix} \tag{B.83}$$

which leads to

$$V(u_k^2 - v_k^2) - 2A(k) u_k v_k = 0 \tag{B.84}$$

to diagonalize the matrix, while

$$E_\pm(k) = \pm[A(k)(u_k^2 - v_k^2) + 2V u_k v_k] \tag{B.85}$$

are the two eigenvalues. To solve (100), one can use the parametrization (97), which admits

$$\begin{aligned} u_k &= \left[\frac{1}{2} \left(1 + \frac{A(k)}{\sqrt{A(k)^2 + V^2)}} \right) \right]^{1/2} \\ v_k &= \left[\frac{1}{2} \left(1 - \frac{A(k)}{\sqrt{A(k)^2 + V^2)}} \right) \right]^{1/2} \end{aligned} \tag{B.86}$$

which also gives

$$E_\pm(k) = \pm[A(k)(u_k^2 - v_k^2) + 2V u_k v_k] = \pm[A(k)^2 + V^2]^{1/2} \tag{B.87}$$

which are of course the same values than the one we obtained by computing directly eigenvalues. The expressions (102) and the energies (103) are shown in Fig. 3.7. One sees that far from the zone boundary where $A(k) = 0$ one has $u_k \sim 1$ and $v_k \sim 0$. The new operators are basically the old ones $\gamma_{k-} \simeq \alpha_k$ and $\gamma_{k+} \simeq \beta_k$. For these wavevectors the potential is not able to act efficiently and produce interferences. The wavefunctions of the eigenstates are essentially the old plane waves. On the other hand, close to the zone boundary the interferences produced by the potential will

be maximal. One can look at the wavefunctions corresponding to the new operators

$$\begin{aligned}\langle r_i|\gamma_{k-}^\dagger|\emptyset\rangle &= \langle r_i|(u_k\alpha_k^\dagger - v_k\beta_k^\dagger)|\emptyset\rangle \\ &= \langle r_i|(u_k c_k^\dagger - v_k c_{k+\pi/a}^\dagger)|\emptyset\rangle \\ &= u_k\frac{1}{\sqrt{\Omega}}e^{ikr_i} - v_k\frac{1}{\sqrt{\Omega}}e^{i(k+\pi/a)r_i} \\ &= \frac{1}{\Omega}e^{ikr_i}(u_k - (-1)^i v_k) \end{aligned} \tag{B.88}$$

and

$$\begin{aligned}\langle r_i|\gamma_{k+}^\dagger|\emptyset\rangle &= \langle r_i|(v_k\alpha_k^\dagger + u_k\beta_k^\dagger)|\emptyset\rangle \\ &= \frac{1}{\sqrt{\Omega}}e^{ikr_i}(v_k + (-1)^i u_k) \\ &= \frac{1}{\sqrt{\Omega}}e^{i(k+\pi/a)r_i}(u_k + (-1)^i v_k) \end{aligned} \tag{B.89}$$

we therefore see that when $u_k \sim 1$ and $v_k \sim 0$ the two operators γ_{k-} and γ_{k+} are essentially creating a plane wave with momentum k and $k + \pi/a$, and the potential has little action. However, when the wavevector of these plane waves approaches $\pm\pi/(2a)$, i.e. the zone boundary in the reduced scheme zone for which $A(k) = 0$, the potential that has a wavevector $Q = \pi/a$ can induce strong interferences between the two states $\pm\pi/(2a)$. The eigenstates are thus transformed because of these interferences into stationary waves. The amplitude of the wave caused by γ_{k-} is essentially zero on the even sites and maximal on the odd sites. Since the potential is $V(r_i) = V(-1)^i$ we see that such a wave gains an energy of order $-V$, which is indeed what its eigenvalue is. On the contrary, the stationary wave created by γ_{k+} is maximal on the even sites and thus loses the energy $+V$. The Bogoliubov transformation allows us to go beyond this cartoon and gives the full eigenstates for all values of k. To complete this, let us illustrate how we can use the transformation to compute any observable. The observables are easily expressed in terms of the original operators c, while we know the ground state or other thermodynamic averages in terms of the operators γ. The strategy can work two ways: either we express the ground state back in terms of the operators c, and then compute the observables by the usual way, or we do the opposite an reexpress the observables in terms of the operators γ. The second method is usually simpler, but both give of course equivalent results. Let us illustrate it with the calculation of the average, at finite temperature, of the density of particles at the point $r_i = 0$. The corresponding operator is

$$\rho(r = 0) = \frac{1}{\Omega}\sum_{k_1,k_2} c_{k_1}^\dagger c_{k_2} \tag{B.90}$$

the sum over momentum runs in Z_B. We now restrict the momenta to Z'_B and express the operators c in terms of the operators α and β. One has

$$\begin{aligned}\rho(r=0) &= \frac{1}{\Omega}\sum_{(k_1,k_2)\in Z'_B}(\alpha^\dagger_{k_1}\alpha_{k_2}+\beta^\dagger_{k_1}\beta_{k_2}+\alpha^\dagger_{k_1}\beta_{k_2}+\beta^\dagger_{k_1}\alpha_{k_2})\\ &= \frac{1}{\Omega}\sum_{(k_1,k_2)\in Z'_B}\begin{pmatrix}\alpha^\dagger_{k_1} & \beta^\dagger_{k_1}\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}\alpha_{k_2}\\ \beta_{k_2}\end{pmatrix}\\ &= \frac{1}{\Omega}\sum_{(k_1,k_2)\in Z'_B}\begin{pmatrix}\gamma^\dagger_{k_1-} & \gamma^\dagger_{k_1+}\end{pmatrix}\begin{pmatrix}(u_{k_1}-v_{k_1})(u_{k_2}-v_{k_2}) & (u_{k_1}-v_{k_1})(u_{k_2}+v_{k_2})\\ (u_{k_1}+v_{k_1})(u_{k_2}-v_{k_2}) & +(u_{k_1}+v_{k_1})(u_{k_2}+v_{k_2})\end{pmatrix}\begin{pmatrix}\gamma_{k_2-}\\ \gamma_{k_2+}\end{pmatrix}\end{aligned} \tag{B.91}$$

Computing the average of the operator is now reduced to computing averages of the form $\gamma^\dagger_{k_1\pm}\gamma_{k_2\pm}$ which we know well since the Hamiltonian is diagonal in terms of the operators γ. Let us begin with the case of $T=0$. Given the eigenvalues (103) the ground state of the system consists in filling first the states γ_{k-} and then the states γ_{k+}. The filling depends on the total number of particles. For example, for a case when one has one particle every two sites (i.e. and half filled band since we did not put spin for the particles) all states γ_{k-} will be filled and all states γ_{k+} will be empty. The ground state becomes

$$|G\rangle = \Pi_{k\in Z'_B}\gamma^\dagger_{k-}|\emptyset\rangle \tag{B.92}$$

In that case the only nonzero average for the operators $\gamma^\dagger_{k_1\pm}\gamma_{k_2\pm}$ is

$$\langle G|\gamma^\dagger_{k_1\pm}\gamma_{k_2\pm}|G\rangle = \delta_{k_1,k_2} \tag{B.93}$$

and the average of the operator

$$\begin{aligned}\langle G|\rho(r=0)|G\rangle &= \frac{1}{\Omega}\sum_{k\in Z'_B}(u_k-v_k)^2\\ &= \frac{1}{\Omega}\sum_{k\in Z'_B}\left[1-\frac{V}{\sqrt{A(k)^2+V^2}}\right]\end{aligned} \tag{B.94}$$

which we can compute explicitly. For $V=0$, we get back that $\rho(r=0)=1/2$ as it should since the number of k values in Z'_B is half of the ones in the total zone. We also see that when $V\to\infty$, $\rho(r=0)\to 0$ since the particles tend to avoid the site where there is the gigantic potential $+V$. Since γ are eigenstates of the Hamiltonian, we can even compute easily the averages at finite temperature. In that case, we see that the only non zero averages are

$$\begin{aligned}\langle\gamma^\dagger_{k_1-}\gamma_{k_2-}\rangle &= \delta_{k_1,k_2}f_F(-E(k))\\ \langle\gamma^\dagger_{k_1+}\gamma_{k_2+}\rangle &= \delta_{k_1,k_2}f_F(+E(k))\end{aligned} \tag{B.95}$$

The average of the density at $r = 0$ becomes

$$\begin{aligned}\langle\rho(r=0)\rangle &= \frac{1}{\Omega}\sum_{k]inZ'_B}(u_k - v_k)^2 f_F(-E(k)) + (u_k + v_k)^2 f_F(+E(k)) \\ &= \frac{1}{\Omega}\sum_{k]inZ'_B}\left(1 - \frac{V}{\sqrt{A(k)^2 + V^2}}\tanh\left(\frac{\beta E(k)}{2}\right)\right) \end{aligned} \tag{B.96}$$

When $\beta \to \infty$ we obviously recover the previous result, while when $\beta \to 0$ we see that $\langle\rho(r=0)\rangle = 1/2$ since the potential or kinetic energy do not matter any more when compared to the thermal energies.

B.6 Derivation of T_C

This is a more descriptive analysis of the derivation carried out when solving for T_C from the gap function. Due to the tight-binding dispersion, we have $\sum_{k\in BZ}\epsilon_k = 0$, and thus, we introduce the Bogoliubov transformation

$$\begin{aligned} M &= \begin{pmatrix} u & -v \\ v & u \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{\frac{1}{2}\left(1 + \frac{\epsilon_k}{\sqrt{\epsilon_k^2 + |\Delta_0|^2}}\right)} & -\sqrt{\frac{1}{2}\left(1 - \frac{\epsilon_k}{\sqrt{\epsilon_k^2 + |\Delta_0|^2}}\right)} \\ \sqrt{\frac{1}{2}\left(1 - \frac{\epsilon_k}{\sqrt{\epsilon_k^2 + |\Delta_0|^2}}\right)} & \sqrt{\frac{1}{2}\left(1 + \frac{\epsilon_k}{\sqrt{\epsilon_k^2 + |\Delta_0|^2}}\right)} \end{pmatrix} \end{aligned} \tag{B.97}$$

and define operators $\vec{a}_k = M\vec{c}_k$ and $\vec{c}_k = (c_{k\uparrow}c^\dagger_{-k\downarrow})^t$ to obtain

$$\begin{aligned} H &\approx -\frac{N_0}{U_{ff}}|\Delta_0|^2 + \sum_{k\in BZ}\epsilon_k \sum_{k\in BZ}\begin{pmatrix} c^\dagger_{k\uparrow} & c_{-k\downarrow} \end{pmatrix}\begin{pmatrix} \epsilon_k & \Delta_0 \\ \Delta_0 & -\epsilon_k \end{pmatrix}\begin{pmatrix} c_{k\uparrow} \\ c_{-k\downarrow} \end{pmatrix} \\ &= -\frac{N_0}{U_{ff}}|\Delta_0|^2 - \sum_{k\in BZ} E_k + \sum_{k\in BZ,\sigma} E_k a^\dagger_{k\sigma}a_{k\sigma} \\ &= E_{gs} + \Sigma_{k\in BZ,\sigma} E_k a^\dagger_{k\sigma}a_{k\sigma} \end{aligned} \tag{B.98}$$

Here, E_{gs} is the ground state energy, and $E_k = \sqrt{\epsilon_k^2 + |\Delta_0|^2}$. Now that this is derived, we procede to the gap function. This is a very important equation, so let's start from the top. We know that we have

$$\Delta_q = \frac{U_{ff}}{N_s}\sum_{k\in BZ}\langle c^\dagger_{k+q\uparrow}c^\dagger_{-k\downarrow}\rangle \tag{B.99}$$

Therefore,

$$\Delta_0 = \frac{U_{ff}}{N_s} \sum_{k \in BZ} \langle c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} \rangle \tag{B.100}$$

We may rewrite this as

$$\begin{aligned}
\Delta_0 &= \frac{U_{ff}}{N_s} \sum_{k \in BZ} \langle \begin{pmatrix} c_{k\uparrow} & c^\dagger_{-k\downarrow} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c^\dagger_{-k\downarrow} \end{pmatrix} \rangle \\
&= \frac{U_{ff}}{N_s} \sum_{k \in BZ} \langle \vec{c}^\dagger_k \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \vec{c}_k \rangle
\end{aligned} \tag{B.101}$$

Implementing the Bogoliubov transformation, we obtain

$$\begin{aligned}
\Delta_0 &= \frac{U_{ff}}{N_s} \sum_{k \in BZ} \langle \vec{a}^\dagger_k M^\dagger \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} M \vec{a}_k \rangle \\
&= \frac{U_{ff}}{N_s} \sum_{k \in BZ} \langle \begin{pmatrix} a^\dagger_{k\uparrow} & a_{-k\downarrow} \end{pmatrix} \begin{pmatrix} u & v \\ -v & u \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u & -v \\ v & u \end{pmatrix} \begin{pmatrix} a_{k\uparrow} \\ a^\dagger_{-k\downarrow} \end{pmatrix} \rangle \\
&= \frac{U_{ff}}{N_s} \sum_{k \in BZ} \langle \begin{pmatrix} a^\dagger_{k\uparrow} & a_{-k\downarrow} \end{pmatrix} \begin{pmatrix} u & v \\ -v & u \end{pmatrix} \begin{pmatrix} v & u \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_{k\uparrow} \\ a^\dagger_{-k\downarrow} \end{pmatrix} \rangle \\
&= \frac{U_{ff}}{N_s} \sum_{k \in BZ} \langle \begin{pmatrix} a^\dagger_{k\uparrow} & a_{-k\downarrow} \end{pmatrix} \begin{pmatrix} uv & u^2 \\ -v^2 & -uv \end{pmatrix} \begin{pmatrix} a_{k\uparrow} \\ a^\dagger_{-k\downarrow} \end{pmatrix} \rangle \\
&= \frac{U_{ff}}{N_s} \sum_{k \in BZ} \langle \begin{pmatrix} a^\dagger_{k\uparrow} & a_{-k\downarrow} \end{pmatrix} \begin{pmatrix} a_{k\uparrow} uv + a^\dagger_{k\downarrow} u^2 \\ -a^\dagger_{-k\downarrow} v^2 - a^\dagger_{-k\downarrow} uv \end{pmatrix} \rangle \\
&= \frac{U_{ff}}{N_s} \sum_{k \in BZ} \langle a^\dagger_{k\uparrow} (a_{k\uparrow} uv + a^\dagger_{k\downarrow} u^2) + a_{-k\downarrow} (-a^\dagger_{-k\downarrow} v^2 - a_{-k\downarrow}) \rangle \\
&= \frac{U_{ff}}{N_s} \sum_{k \in BZ} \left(uv \langle a^\dagger_{k\uparrow} a_{k\uparrow} \rangle + u^2 \langle a^\dagger_{k\uparrow} a^\dagger_{k\downarrow} \rangle - v^2 \langle a_{-k\downarrow} a^\dagger_{-k\downarrow} \rangle - uv \langle a_{-k\downarrow} a^\dagger_{-k\downarrow} \rangle \right)
\end{aligned} \tag{B.102}$$

The averages above were calculated in the notes by Giamarchi (except there, we used γ instead of a), and thus we obtain

$$\begin{aligned}
\Delta_0 &= \frac{U_{ff}}{N_s} \sum_{k \in BZ} \left(uv(f_F(E_k)) + u^2(0) - v^2(0) - uv(1 - f_F(E_k)) \right) \\
&= \frac{U_{ff}}{N_s} \sum_{k \in BZ} \left(uv(f_F(E_k)) - uv(1 - f_F(E_k)) \right) \\
&= \frac{U_{ff}}{N_s} \sum_{k \in BZ} \left(uv(2 f_F(E_k) - 1) \right) \\
&= -\frac{U_{ff}}{N_s} \sum_{k \in BZ} \left(uv(1 - 2 f_F(E_k)) \right)
\end{aligned} \tag{B.103}$$

Calculating uv, we obtain

$$\begin{aligned}
uv &= \sqrt{\frac{1}{2}\left(1+\frac{\epsilon_k}{\sqrt{\epsilon_k^2+|\Delta_0|^2}}\right)}\sqrt{\frac{1}{2}\left(1-\frac{\epsilon_k}{\sqrt{\epsilon_k^2+|\Delta_0|^2}}\right)} \\
&= \sqrt{\frac{1}{2}\left(1+\frac{\epsilon_k}{\sqrt{\epsilon_k^2+|\Delta_0|^2}}\right)\left(1-\frac{\epsilon_k}{\sqrt{\epsilon_k^2+|\Delta_0|^2}}\right)} \\
&= \frac{1}{2}\sqrt{1-\frac{\epsilon_k}{\sqrt{\epsilon_k^2+|\Delta_0|^2}}+\frac{\epsilon_k}{\sqrt{\epsilon_k^2+|\Delta_0|^2}}-\left(\frac{\epsilon_k}{\sqrt{\epsilon_k^2+|\Delta_0|^2}}\right)^2} \\
&= \frac{1}{2}\sqrt{1-\frac{\epsilon_k^2}{\epsilon_k^2+|\Delta_0|^2}} \\
&= \frac{1}{2}\sqrt{\frac{\epsilon_k^2+|\Delta_0|^2-\epsilon_k^2}{\epsilon_k^2+|\Delta_0|^2}} \\
&= \frac{1}{2}\sqrt{\frac{|\Delta_0|^2}{\epsilon_k^2+|\Delta_0|^2}} \\
&= \frac{1}{2}\frac{|\Delta_0|}{\sqrt{\epsilon_k^2+|\Delta_0|^2}}
\end{aligned} \tag{B.104}$$

Appendix C

Fermi Liquid Theory

C.1 Pomeranchuck Instability Condition

Here is the in-depth derivation the Pomeranchuk instability condition to third order.

$$\begin{aligned}
\delta n_{p\sigma} &= n_{p\sigma} - n^o_{p\sigma} = \Theta(p_F(\theta,\phi) - p) - \Theta(p^o_F - p) \\
&= \Theta(p_F(\theta,\phi) - p)|_{p=p^o_F} + \frac{\partial}{\partial p}\Theta(p_F(\theta,\phi) - p)|_{p=p^o_F}(p - p_F) \\
&+ \frac{1}{2}\frac{\partial^2}{\partial p^2}\Theta(p_F(\theta,\phi) - p)|_{p=p^o_F}(p - p^o_F)^2 \\
&+ \frac{1}{6}\frac{\partial^3}{\partial p^3}\Theta(p_F(\theta,\phi) - p)|_{p=p^o_F}(p - p^o_F)^3 - \frac{1}{2} \\
&= -\delta(p^o_F - p)\delta p_F - \frac{1}{2}\frac{\partial}{\partial p}\delta(p^o_F - p)\delta p^2_F - \frac{1}{6}\frac{\partial^2}{\partial p^2}\delta(p^o_F - p)\delta p^3_F \quad \text{(C.1)}
\end{aligned}$$

where $n_{p\sigma}$ is the distribution function, p is the momenta, $p_F = k_F\hbar$ is the Fermi momentum, Fstands for Fermi, and Θ is the Heaviside step function. $\delta p_F = p_F(\theta,\phi) - p^o_F$. Then,

$$\begin{aligned}
\sum_{p\sigma}(\epsilon_p - \mu)\delta n_{p\sigma} &= V_F \sum_{p\sigma}(p - p_F)\delta n_{p\sigma} \\
&= \frac{2V_F}{(2\pi\hbar)^3}\int d\Omega \int p^2(p - p_F)dp \\
&\left(-\delta(p^o_F - p)\delta p_F - \frac{1}{2}\frac{\partial}{\partial p}\delta(p^o_F - p)\delta p^2_F - \frac{1}{6}\frac{\partial^2}{\partial p^2}\delta(p^o_F - p)\delta p^3_F\right) \\
&= \frac{V_F p^2_F}{(2\pi\hbar)^3}\sum_{lm}|\mu_{lm}|^2 - \frac{8V_F p_F}{6(2\pi\hbar)^3}\sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}} \\
&\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}\mu_{l_1 m_2}\mu_{l_2,m_2}\mu_{l_3 m_3} \quad \text{(C.2)}
\end{aligned}$$

where here, δp_F is expanded via spherical harmonics, and we have done integration by parts for the quadratic and cubic terms by $-\int p^2(p-p_f)\frac{\partial}{\partial p}\delta(p-p_F) = \int(3p^2-2pp_F)\delta(p_F-p)dp = p_F^2$ and $-\int p^2(p-p_f)\frac{\partial^2}{\partial p^2}\delta(p_F^o-p)dp = \int\delta(p-p_F)(6p-2p_F)dp = 4p_F$. We also have the relations $\delta p_F = \sum Y_{lm}(\theta,\phi)\mu_{lm}$, $\delta p_F^2 = \sum_{lm} Y_{l_1m_1}(\theta,\phi)Y_{l_2,m_2}(\theta,\phi)\mu_{l_1m_1}\mu_{l_2,m_2}$ and applying $\int d\Omega$ we obtain $\delta_{l_1,l_2}\delta_{m_1,m_2}(-1)^{m_2}\sum_{lm}\mu_{l_1m_1}\mu_{l_2,m_2} = \sum_{lm}|\mu_{lm}|^2$. For the cubic term, it is the same with the addition of l_3, m_3 terms and the result is shown above. Then, we can write

$$
\begin{aligned}
\sum\sum_{lm} f_l P_l(\cos\theta)\delta n_p\delta n_{p'} &= \sum\sum_{lm} f_l\frac{4\pi}{2l+1}Y_{lm}^*(\theta,\phi)Y_{lm}(\theta',\phi')\delta n_p\delta n_{p'} \\
&= \frac{p_F^2}{(2\pi\hbar)^3}\sum_l\frac{F_l}{2l+1}|\mu_{lm}|^2 \\
&+ \frac{1}{2(2\pi\hbar)^6}\delta p_F\int p^2\frac{\partial}{\partial p}\delta(p_F^o-p)dp\int p'^2\delta(p_F^o-p')dp'\int d\Omega\int d\Omega' \\
&\sum\mu_{l_1m_1}\mu_{l_2m_2}Y_{l_1m_1}(\theta',\phi')Y_{l_2m_2}(\theta',\phi') \\
&\sum_{lm} f_l\left(\frac{4\pi}{2l+1}\right)Y_{lm}^*(\theta,\phi)Y_{lm}(\theta',\phi') \\
&+ \frac{1}{2(2\pi\hbar)^6}\delta p_F'\int p'^2\frac{\partial}{\partial p'}\delta(p_F^o-p')dp'\int p^2\delta(p_F^o-p)dp \\
&\sum\mu_{l_1m_1}\mu_{l_2m_2}Y_{l_1m_1}(\theta,\phi)Y_{l_2m_2}(\theta,\phi) \\
&\sum_{lm} f_l\left(\frac{4\pi}{2l+1}\right)Y_{lm}^*(\theta,\phi)Y_{lm}(\theta',\phi') \\
&= \frac{p_F^2}{(2\pi\hbar)^3}\sum_l\frac{F_l}{2l+1}|\mu_{lm}|^2 \\
&+ \left[\sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}}\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3\end{pmatrix}\begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0\end{pmatrix}\right] \\
&\frac{2p_F^2}{(2\pi\hbar)^3}\sum_l\frac{F_l}{2l+1}\delta_{l_1,l_2}\delta_{m_1,m_2}\mu_{l_1m_2}\mu_{l_2,m_2}\mu_{l_3m_3}
\end{aligned}
\tag{C.3}
$$

where $F_l = f_l N(0)$ and $N(0) = \frac{m^* p_F}{\pi^2 \hbar^3}$. Now, assuming $l_1 = l_2 = l_3 = 0$, we have

$$\begin{aligned}
\delta F &= \sum(\epsilon - \mu)\delta n_p + \sum f_{pp'}\delta n_p \delta n_{p'} = \frac{V_F p_F^2}{(2\pi\hbar)^3}\sum_{lm}|\mu_{lm}|^2 \\
&- \frac{8V_F p_F}{6(2\pi\hbar)^3}\frac{1}{\sqrt{4\pi}}\mu_{l_1m_1}\mu_{l_2m_2}\mu_{l_3m_3} + \frac{p_F^4}{(2\pi\hbar)^6}\sum_l f_l 4\pi|\mu_{lm}|^2 \\
&- \frac{2p_F^2}{(2\pi\hbar)^6}\frac{1}{\sqrt{4\pi}}\sum_l f_l 4\pi\delta_{l_1l_2}\delta_{m_1m_2}\mu_{l_1m_1}\mu_{l_2m_2}\mu_{l_3m_3} \\
&= \frac{V_F p_F^2}{(2\pi\hbar)^3}\mu_{00}^2 - \frac{4V_F p_F}{3(2\pi\hbar)^3}\frac{1}{\sqrt{4\pi}}\mu_{00}^3 + \frac{4\pi p_F^4}{(2\pi\hbar)^6}\mu_{00}^2 f_0 - \frac{8\pi p_F^3}{(2\pi\hbar)^6}\frac{1}{\sqrt{4\pi}}\mu_{00}^3 f_0 \\
&= \frac{V_F p_F^2}{(2\pi\hbar)^3}\left[\mu_{00}^2 - \frac{2}{3p_F\sqrt{\pi}}\mu_{00}^3 + F_0\mu_{00}^2 - F_0\frac{1}{p_F\sqrt{\pi}}\mu_{00}^3\right] \qquad \text{(C.4)}
\end{aligned}$$

We want this expression to be greater than or equal to 0. Therefore, we can solve for μ_{00},

$$\begin{aligned}
&1 + F_0 - \left(\frac{2}{3} + F_0\right)\left(\frac{1}{p_F\sqrt{\pi}}\right)\mu_{00} = 0 \\
&\implies \mu_{00} = \frac{1+F_0}{\frac{2}{3}+F_0}p_F\sqrt{\pi} \qquad\qquad \implies 1 + F_0 \geq \frac{\mu_{00}}{p_F\sqrt{\pi}}\left(\frac{2}{3} + F_0\right)
\end{aligned} \qquad \text{(C.5)}$$

In these expressions, F_0 is the Landau parameter, ϵ is the energy, μ_{00} is the distortion of the Fermi surface, n is the distribution function, $d\Omega$ is the solid angle, and l and m are quantum numbers.

Appendix D

FFL Modes' Effects on the Cooper Instability

D.1 The Green's Function

The methods developed here have been guided by [16-21].

D.1.1 The Fermionic Green's Function

The one particle Green's function in the free field Heisenberg operator representation can be written as

$$G_{\alpha\beta} = -i\langle T(\tilde{\psi}_\alpha(x)\tilde{\psi}_\beta(x'))\rangle \tag{D.1}$$

where

$$\tilde{\psi}(\vec{r},t) = \frac{1}{\sqrt{V}}\sum_p a_p e^{i[\vec{p}\vec{r}-\epsilon_0(\vec{p})t]} \tag{D.2}$$

Thus, we can rewrite 138 in term of the fermionic operators $c, c^\dagger$

$$G = -i\langle T(c(x,t)c^\dagger(x',t'))\rangle \tag{D.3}$$

Taking the forward and backward in time components

$$\begin{aligned} G &= -i\langle c(x,t)c^\dagger(x',t')\Theta(t-t') - c^\dagger(x,t)c(x',t')\Theta(t'-t)\rangle \\ &= \int c_\sigma(x,t)c^\dagger_\sigma(x,t)c_{k\sigma}(t)c^\dagger_{k\sigma}(t)e^{-ikr}dr \end{aligned} \tag{D.4}$$

$$\tag{D.5}$$

Taking the Fourier transform to momentum space and setting $t' = 0$, we obtain

$$\begin{aligned} G(\vec{k}) &= \int G(\vec{r})e^{i(\vec{k}\cdot\vec{r})}d\vec{r} \\ &= -i\langle c_{k\sigma}(t)c^\dagger_{k\sigma}(t)\Theta(t) - c^\dagger_{k\sigma}(t)c_{k\sigma}(t)\Theta(-t)\rangle e^{\frac{-i\epsilon_k t}{\hbar}} \end{aligned} \tag{D.6}$$

Now, before we proceed we want a form $c_k(t) = f(t)c_k$. Recall $H = \sum_{k\sigma} \epsilon_k c^\dagger_{k\sigma} c_{k\sigma}$ and the fermionic anticommutation relation $\{c, c^\dagger\} = cc^\dagger + c^\dagger c = \delta_{ij}$ where for 1 state we have $1 - c^\dagger c = cc^\dagger$. We want

$$\begin{aligned}
\frac{d}{dt}c &= \frac{i}{\hbar}[H, c] = \frac{i}{\hbar}\epsilon[c^\dagger c, c] \\
&= \frac{i}{\hbar}\epsilon(c^\dagger \underbrace{\{c, c\}}_{0} - \underbrace{\{c^\dagger, c\}}_{1} c) \text{ where we have used } [AB, C] = A\{B, C\} - \{A, C\}B \\
&= \frac{i}{\hbar}\epsilon(-c) \\
\implies c(t) &= ce^{\frac{-i\epsilon}{\hbar}} \qquad \text{(D.7)}
\end{aligned}$$

Plug this expression back into 143 to obtain

$$\begin{aligned}
G(\vec{k}) &= -i\langle ce^{\frac{-i\epsilon}{\hbar}} c^\dagger e^{\frac{i\epsilon}{\hbar}} \Theta(t) - c^\dagger e^{\frac{i\epsilon}{\hbar}} ce^{\frac{-i\epsilon}{\hbar}} \Theta(-t)\rangle e^{\frac{-i\epsilon_k t}{\hbar}} \\
&= -i\langle (1 - n_{k\sigma})\Theta(t) - n_{k\sigma}\Theta(-t)\rangle e^{\frac{-i\epsilon_k t}{\hbar}} \qquad \text{(D.8)}
\end{aligned}$$

To remove the explicit time components, we will perform a temporal Fourier transform with an additional $e^{-\delta|t|}$ term added to describe behavior at infinite (∞) time.

$$\begin{aligned}
G_0(k, \omega) &= \int e^{i\omega t - \delta t} G_\sigma(k, t) dt \\
&= \int e^{-i\omega t - \delta t} \left\{ -i\{\Theta(t)(1 - n_{k\sigma}) - \Theta(-t)n_{k\sigma}\} e^{\frac{-i\epsilon_k t}{\hbar}} \right\} dt \\
&= \int e^{it(\omega - \frac{\epsilon_k}{\hbar} + i\delta)} \left\{ -i\{\Theta(t)(1 - n_{k\sigma}) - \Theta(-t)n_{k\sigma}\} \right\} dt \\
&= -i \int e^{it(\omega - \frac{\epsilon_k}{\hbar} + i\delta)} \Theta(t)(1 - n_{k\sigma}) dt - i \int e^{it(\omega - \frac{\epsilon_k}{\hbar} + i\delta)} \Theta(-t) n_{k\sigma} dt \\
&= -i \int_0^\infty e^{it(\omega - \frac{\epsilon_k}{\hbar} + i\delta)} (1 - n_{k\sigma}) dt - i \int_{-\infty}^0 e^{it(\omega - \frac{\epsilon_k}{\hbar} + i\delta)} n_{k\sigma} dt \\
&= -i(1 - n_{k\sigma}) \int_0^\infty e^{it(\omega - \frac{\epsilon_k}{\hbar} + i\delta)} dt + i n_{k\sigma} \int_0^{-\infty} e^{it(\omega - \frac{\epsilon_k}{\hbar} + i\delta)} dt \\
&= i(1 - n_{k\sigma}) \frac{1}{i} \left(\frac{e^{it(\omega - \frac{\epsilon_k}{\hbar} + i\delta)}}{\omega - \frac{\epsilon_k}{\hbar} + i\delta)} \right) \Big|_\infty^0 + i n_{k\sigma} \frac{1}{i} \left(\frac{e^{it(\omega - \frac{\epsilon_k}{\hbar} + i\delta}}{\omega - \frac{\epsilon_k}{\hbar} + i\delta} \right) \Big|_0^{-\infty} \\
&= (1 - n_{k\sigma}) \left(\frac{e^{it(\omega - \frac{\epsilon_k}{\hbar} + i\delta)}}{\omega - \frac{\epsilon_k}{\hbar} + i\delta)} \right) \Big|_\infty^0 + n_{k\sigma} \left(\frac{e^{it(\omega - \frac{\epsilon_k}{\hbar} + i\delta}}{\omega - \frac{\epsilon_k}{\hbar} - i\delta} \right) \Big|_0^{-\infty} \\
&= \frac{1 - n_{k\sigma}}{\omega - \frac{\epsilon_k}{\hbar} + i\delta} + \frac{n_{k\sigma}}{\omega - \frac{\epsilon_k}{\hbar} - i\delta} \qquad \text{(D.9)}
\end{aligned}$$

The first term becomes zero for $\epsilon_k < \mu = \epsilon_F$ and nonzero for $\epsilon_k > \mu = \epsilon_F$. This term describes the propagation of the electron (positive time

representation). The second term becomes zero for $\epsilon_k > \mu = \epsilon_F$ and nonzero for $\epsilon_k < \mu = \epsilon_F$. This term describes the propagation of the hole (negative time representation). Now, choose $\delta_k = sgn(\epsilon_k - \epsilon_F)\delta)$ and take $\hbar = 1$, measure energy relative to Fermi level μ.

$$G_0(k,\omega) = \frac{1}{\omega - \xi_k + i\delta_k} \tag{D.10}$$

where $\xi_k = \epsilon_k - \mu$.

D.1.2 The Bosonic Green's Function

We can begin with the same formalism as we used in the derivation for the one particle fermionic Green's function by writing the phonon Green's function as

$$D = -i\langle T(\tilde{\psi}(x,t)\psi(x',t')\rangle \tag{D.11}$$

where we define

$$\psi(r,t) = \frac{i}{\sqrt{V}}\sum_k \sqrt{\frac{\omega_0(k)}{2}}\left\{b_k e^{i[k\cdot r - \omega_0(k)t]} - b_k^\dagger e^{-i[k\cdot r - \omega_0(k)t]}\right\} \tag{D.12}$$

as we need more than the displacement operator ϵ if we eventually want to include the interaction between phonons and electrons in a metal. Here, we take $|k| < k_0$, and as there are no phonons in the ground state, this implies $x', t' \to 0$. In this step, we have also expanded the normal position $\mathbf{q}$ in plane waves to obtain $\mathbf{q_k}$ and translated these into new operators $b_k = q_k\sqrt{2\rho\omega_0(k)}$ which obey the usual bosonic commutation relations. Substituting this expression $(D.12)$ into expression $(D.11)$ to obtain

$$D^{(0)}(r,t) = -i\langle T(\tilde{\psi}(x,t)\psi(x',t')\rangle = \frac{i}{V}\sum_k \frac{\omega_0(k)}{2}\begin{cases} e^{i[k\cdot r - \omega_0(k)t]} & t > 0 \\ e^{-i[k\cdot r - \omega_0(k)t]} & t < 0 \end{cases} \tag{D.13}$$

as because we are in the ground state $b_k^\dagger b_k = n = 0$ and $b_k b_k^\dagger = 1 - n = 1$. Further, we have that $b_{k'}b_k, b_{k'}^\dagger b_k, b_k b_{k'}^\dagger, b_{k'}^\dagger b_k, b_{k'}b_k^\dagger \to 0$. The remaining terms $b_{k'}^\dagger b_k^\dagger$ also go to 0. Taking the limit as $k \to \infty$ and performing a

temporal transform where the sum becomes an integral we obtain

$$\begin{aligned}
D^{(0)}(k,\omega) &= \int e^{-\delta t} D^{(0)}(r,t)dt \\
&= \int i\frac{\omega_0(k)}{2V}\left\{\Theta(t)e^{it[\omega-\omega_0(k)+i\delta]} - \Theta(-t)e^{-it[\omega+\omega_0(k)-i\delta]}\right\}dt \\
&= i\frac{\omega_0(k)}{2V}\left\{\int_0^\infty e^{it[\omega-\omega_0(k)+i\delta]}dt - \int_{-\infty}^0 e^{-it[\omega+\omega_0(k)-i\delta]}dt\right\} \\
&= i\frac{\omega_0(k)}{2V}\left\{\frac{1}{i}\left(\frac{e^{it[\omega-\omega_0(k)+i\delta]}}{\omega-\omega_0(k)+i\delta}\right)\Big|_0^\infty - \frac{1}{i}\left(\frac{e^{it[\omega-\omega_0(k)-i\delta]}}{\omega-\omega_0(k)-i\delta}\right)\Big|_{-\infty}^0\right\} \\
&= \frac{\omega_0(k)}{2V}\left\{\left(\frac{e^{it[\omega-\omega_0(k)+i\delta]}}{\omega-\omega_0(k)+i\delta}\right)\Big|_0^\infty - \left(\frac{e^{it[\omega-\omega_0(k)-i\delta]}}{\omega-\omega_0(k)-i\delta}\right)\Big|_{-\infty}^0\right\} \\
&= \frac{\omega_0(\vec{k})}{2}\left[\frac{1}{\omega-\omega_0(\vec{k})+i\delta} - \frac{1}{\omega+\omega_0(\vec{k})-i\delta}\right] \text{ set } V=1 \\
&= \frac{\omega_0(\vec{k})}{2}\left[\frac{\omega+\omega_0(\vec{k})-i\delta-(\omega-\omega_0(\vec{k})+i\delta)}{\omega^2-\omega_0^2(\vec{k})+i\delta}\right] \\
&= \frac{\omega_0(\vec{k})}{2}\left[\frac{2\omega_0(\vec{k})}{\omega^2-\omega_0^2(\vec{k})+i\delta}\right] \\
&= \frac{\omega_0^2(\vec{k})}{\omega^2-\omega_0^2(\vec{k})+i\delta}
\end{aligned} \tag{D.14}$$

D.2 The Goldstone Mode's Effect on an s-Wave Pairing Instability

D.2.1 Singularities of the Vertex Part When the Total Momentum of the Colliding Particles is Small; The Zero temperature Vertex Function

It is important to note that here, p is the quasiparticle momentum, p_0 is the bare particle momentum, and p_F is the Fermi momentum. The integral equation is

$$\Gamma(1,2;3,4) = \Gamma^{(2)}(1,2;3,4) + \frac{1}{2}\sum_{5,5';6,6'}\Gamma^{(2)}(1,2;5,6)G(5,5')G(6,6')\Gamma(5',6',3,4) \tag{D.15}$$

where $\Gamma^{(2)}$ is the two-particle irreducible vertex which describes the non-interacting contribution and Γ describes the interacting vertex. Here the

labels $1, 2$ etc. are defined by $1 = (p_1, \epsilon_1, \sigma)$, etc. Now, let us introduce new variables $p, p + k$ where,

$$\begin{aligned} p_1 + p_2 = p_3 + p_4 &= k \\ p_1 &= \frac{1}{2}k + p \\ p_2 &= \frac{1}{2}k - p \\ p_3 &= \frac{1}{2}k + p' \\ p_4 &= \frac{1}{2}k - p' \\ p_5 &= \frac{1}{2}k + p" \\ p_6 &= \frac{1}{2}k - p" \end{aligned} \tag{D.16}$$

with $p = (p, \epsilon)$ and $k = (k, \Gamma)$. Here, p is the quasiparticle momentum, p_0 is the bare particle momentum, and p_F is the Fermi momentum. We also have that

$$G(5, 5') = G(5)\delta_{5,5'} = G(5)\delta_{p_5,p_5'}\delta_{\sigma_5,\sigma_5'} \tag{D.17}$$

for paramagnetic systems. Now, making use of the transformations the integral equation becomes

$$\begin{aligned} \Gamma_{1,2;3,4}(p, p'; k) = \Gamma^{(2)}_{1,2;3,4}(p.p'; k) \\ + \frac{1}{2}\sum_{p"}\sum_{5,6,5',6'} \Gamma^{(2)}_{1,2;5,6}(p, p" - \frac{k}{2}; k)G(k - p")\delta_{5,5'}G(p")\delta_{6,6'}\Gamma_{5',6';3,4}(p" - \frac{k}{2}, p'; k) \end{aligned} \tag{D.18}$$

We now use the labels $1, 2$, etc. for spin labels. For convenience under the integrals we have made a change of variables, $p" \to p" - \frac{k}{2}$. We will show that the integral equation $D.16$ decomposes into two separate integral equations; one for spin singlet and one for triplet spins. This admits

$$\Gamma_{1,2;3,4} = \Gamma^{(2)}_{1,2;3,4} + \frac{1}{2}\sum_{p"} G(p")G(k - p")\sum_{5,6}\Gamma^{(2)}_{1,2;5,6}\Gamma_{5,6;3,4} \tag{D.19}$$

For paramagnetic systems both Γ and $\Gamma^{(2)}$ have the same spin structure, thus,

$$\begin{aligned}
\Gamma_{1,2;3,4} &= \Gamma^s\delta_{13}\delta_{24} + \Gamma^a\sigma_{24}\sigma_{13} \\
&= (\Gamma^s - \Gamma^a)\delta_{13}\delta_{24} + 2\Gamma^a\delta_{14}\delta_{23} \quad \text{using the identity } \frac{1}{2}\sigma_{24}\sigma_{13} = \delta_{41}\delta_{23} - \frac{1}{2}\delta_{24}\delta_{13} \\
&= \Gamma_d\delta_{13}\delta_{24} - \Gamma_e\delta_{14}\delta_{23} \\
\Gamma^{(2)}_{1,2;3,4} &= \Gamma^{(2)s}\delta_{13}\delta_{24} + \Gamma^{(2)a}\sigma_{24}\sigma_{13} \\
&= (\Gamma^{(2)s} - \Gamma^{(2)a})\delta_{13}\delta_{24} + 2\Gamma^{(2)a}\delta_{14}\delta_{23} \\
&= \Gamma^{(2)}_d\delta_{13}\delta_{24} - \Gamma^{(2)}_e\delta_{14}\delta_{23}
\end{aligned} \tag{D.20}$$

where $\Gamma_e = -\Gamma_e$ is standard notation. The triplet amplitude Γ_t is defined by

$$\Gamma_t = \Gamma_{\uparrow\uparrow;\uparrow\uparrow} = \Gamma_{\uparrow\uparrow} = \Gamma^s + \Gamma^a \tag{D.21}$$

For the singlet amplitude Γ_s, we need $\Gamma_{\uparrow,\downarrow}$ where

$$\Gamma_{\uparrow\downarrow} = \Gamma_{\uparrow\downarrow;\uparrow\downarrow} = \Gamma^s - \Gamma^a \tag{D.22}$$

We also have that $\Gamma_{\uparrow\uparrow} = \Gamma_d - \Gamma_e = \Gamma_t$ and $\Gamma_{\uparrow\downarrow} = \Gamma_d$. The singlet amplitude Γ_s is

$$\Gamma_s = \Gamma^s - 3\Gamma^a = 2\Gamma_{\uparrow\downarrow} - \Gamma_{\uparrow\uparrow} = \Gamma_d + \Gamma_e \tag{D.23}$$

Identical expressions can be written for $\Gamma^{(2)}$. Consider first the equation for $\Gamma_{\uparrow\uparrow}$,

$$\begin{aligned}
\Gamma_{\uparrow\uparrow} &= \Gamma^{(2)}_{\uparrow\uparrow} + \frac{1}{2}\sum_{p"} G(p")G(k-p")\sum_{5,6}\Gamma^{(2)}_{\uparrow\uparrow;56}\Gamma_{5,6;\uparrow\uparrow} \\
&= \Gamma^{(2)}_{\uparrow\uparrow} + \frac{1}{2}\sum_{p"} G(p")G(k-p")\left\{\sum_{5,6}\left(\Gamma^{(2)}_d\delta_{\uparrow5}\delta_{\uparrow6} - \Gamma^{(2)}_e\delta_{\uparrow6}\delta_{\uparrow5}\right)\left(\Gamma_d\delta_{5\uparrow}\delta_{6\uparrow} - \Gamma_e\delta_{5\uparrow}\delta_{6\uparrow}\right)\right\}
\end{aligned} \tag{D.24}$$

Summing on 5 and 6 we have, $\Gamma_{\uparrow\uparrow} = \Gamma^{(2)}_{\uparrow\uparrow} + \frac{1}{2}\sum_{p"} G(p")G(k-p")(\Gamma^{(2)}_d - \Gamma^{(2)}_e)(\Gamma_d - \Gamma_e)$,

$$\Gamma_t = \Gamma^{(2)}_t + \frac{1}{2}\sum_{p"}\Gamma^{(2)}_t G(p")G(k-p")\Gamma_t \tag{D.25}$$

$$\tag{D.26}$$

For Γ_s we need first construct the integral equation for $\Gamma_{\uparrow\downarrow}$,

$$\Gamma_{\uparrow\downarrow} = \Gamma_{\uparrow\downarrow;\uparrow\downarrow} = \Gamma^{(2)}_{\uparrow\downarrow;\uparrow\downarrow} + \frac{1}{2}\sum_{p"} G(p")G(k-p") \sum_{5,6} \Gamma^{(2)}_{\uparrow\downarrow;5,6}\Gamma^{(2)}_{5,6;\uparrow\downarrow}\Gamma_{\uparrow\downarrow;\uparrow\downarrow} + \Gamma_{\uparrow\downarrow;\uparrow\downarrow} + \Gamma^{(2)}_{\uparrow\downarrow;\downarrow\uparrow} + \Gamma_{\downarrow\uparrow;\uparrow\downarrow} \tag{D.27}$$

Then,

$$\Gamma_{\uparrow\downarrow} = \Gamma^{(2)}_{\uparrow\downarrow} + \frac{1}{2}\sum_{p"} G(p")G(k-p") \left[\Gamma^{(2)}_d\Gamma_d + \Gamma^{(2)}_e\Gamma_e\right] \tag{D.28}$$

Using the definition of $\Gamma_s = 2\Gamma_{\uparrow\downarrow} - \Gamma_{\uparrow\uparrow}$ we have that,

$$\begin{aligned}
\Gamma_s &= 2\Gamma^{(2)}_{\uparrow\downarrow} - \Gamma^{(2)}_{\uparrow\uparrow} + 2(\frac{1}{2})\sum_{p"} G(p")G(k-p")(\Gamma^{(2)}_d\Gamma_d + \Gamma^{(2)}_e\Gamma_e - \frac{1}{2}\sum_{p"} G(p")G(k-p")\Gamma^{(2)}_t\Gamma_t \\
&= \Gamma^{(2)}_s + \frac{1}{2}\sum_{p"} G(p")G(k-p")2(\Gamma^{(2)}_d\Gamma_d + \Gamma^{(2)}_e\Gamma_e) - (\Gamma^{(2)}_d - \Gamma^{(2)}_e)(\Gamma_d - \Gamma_e) \\
&= \Gamma^{(2)}_s + \frac{1}{2}\sum_{p"} \Gamma^{(2)}_s G(p")G(k-p")\Gamma_s
\end{aligned} \tag{D.29}$$

Following this, we can study the singularities of Γ as $k \to 0$. The analysis is identical for spin singlet and triplet, thus, we can drop the subscripts s, t:

$$\Gamma(p,p';k) = \Gamma^{(2)}(p,p';k) + \frac{1}{2}\sum_{p"} \Gamma^{(2)}(p,p"-\frac{k}{2};k)G(p")G(k-p")\Gamma(p"-\frac{k}{2};k) \tag{D.30}$$

Before we analyze the singularities of this function and compute the integral equations for the vertex, let us expand $\Gamma^{(2)}(p,p')$ and $\Gamma(p,p';k)$ in partial waves,

$$\begin{aligned}
\Gamma^{(2)}(p,p') &= \sum_l \Gamma^{(2)}_l P_l(\hat{p}\cdot\hat{p}') \\
\Gamma(p,p';k) &= \sum_l \Gamma_l(k) P_l(\hat{p},\hat{p}')
\end{aligned} \tag{D.31}$$

It is important to remember that $p" \to p" - \frac{k}{2}$ so that when expanding under the integral we expand in the angles between $p + p" - \frac{k}{2}$ and $p' + p" - \frac{k}{2}$. For small k we have that $p \cdot p" = p \cdot p" - \frac{p\cdot k}{2}$ and $\frac{p\cdot k}{2} = 0$ for $|p_n| = k_F$. Finally in the limit $k \to 0$ we have,

$$\sum_{l,l'} \int \frac{d\Omega"}{4\pi} \Gamma^{(2)}_l \Gamma_{l'}(\lambda) P_l(\hat{p},\hat{p}") P_l(\hat{p}"\cdot\hat{p}') = \sum_l \Gamma^{(2)}_l \Gamma_l \frac{P_l(\hat{p}\cdot\hat{p}')}{2l+1} \tag{D.32}$$

Note here that

$$P_l(\hat{p}\cdot\hat{p}") = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_l^{*m}(\theta,\phi)Y_l^m(\theta",\phi") \tag{D.33}$$

where $Y_l^m(\theta,\phi) = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}P_l^m(\cos\theta)e^{im\phi}$ and $\int d\Omega Y_l^m(\theta,\phi)Y_{l'}^{m'}(\theta,\phi) = \delta_{l,l'}\delta_{m,m'}$. This equation decouples into partial waves,

$$\begin{aligned}
\sum_{l'}\int\frac{d\Omega"}{4\pi}P_l(\hat{p}\cdot\hat{p}")P_l(\hat{p}"\cdot\hat{p}') &= \sum_{l'}\frac{\delta_{ll'}}{2l'+1}P_{l'}(\hat{p}\cdot\hat{p}') \\
&= \left(\frac{4\pi}{2l+1}\right)\left(\frac{4\pi}{2l'+1}\right)\sum_{m=-l}^{l}\sum_{m'=-l'}^{l'} \\
&\left\{\int\frac{d\Omega"}{4\pi}Y_l^{*m}(\theta,\phi)Y_l^{*m}(\theta",\phi")Y_{l'}^{*m'}(\theta",\phi")Y_{l'}^{m'}(\theta',\phi')\right\} \\
&= \left(\frac{4\pi}{2l+1}\right)\left(\frac{4\pi}{2l'+1}\right)\times \\
&\sum_{m=-l}^{l}\sum_{m'=-l'}^{l'}\left(\frac{1}{4\pi}\right)Y_l^{*m}(\theta,\phi)Y_{l'}^{m'}(\theta',\phi')\delta_{l,l'}\delta_{m,m'} \\
&= \left(\frac{1}{2l+1}\right)\left(\frac{4\pi}{2l'+1}\right)\delta_{l,l'}\sum_{m=-l}^{l}Y_l^{*m}(\theta,\phi)Y_{l'}^{m}(\theta',\phi') \\
&= \sum_{l'}\frac{\delta_{l,l'}}{2l+1}\frac{4\pi}{2l'+1}\sum_{m=-l}^{l}Y_l^{*m}(\theta,\phi)Y_{l'}^{m}(\theta',\phi') \\
&= \frac{1}{2l+1}\left\{\frac{4\pi}{2l+1}\sum_{m=-l}^{l}Y_l^{*m}(\theta,\phi)Y_l^{m}(\theta',\phi')\right\} \\
&= \frac{1}{2l+1}P_l(\hat{p}\cdot\hat{p}')
\end{aligned} \tag{D.34}$$

Note that we carried this step out first as it is more convenient. We can now look at the second term of our vertex. Now that we have established the proper form of our function, let us go back to focusing on the singularities of the vertex function. Let me bring you back to the form of the vertex function where we have dropped the singlet and triplet subscripts,

$$\Gamma(p,p';k) = \Gamma^{(2)}(p,p';k) + \frac{1}{2}\sum_{p"}\Gamma^{(2)}(p,p"-\frac{k}{2};k)G(p")G(k-p")\Gamma(p"-\frac{k}{2};k) \tag{D.35}$$

Noting again that we are working in the limit of small k, note that,

(a) $\Gamma^{(2)}$ is slowly varying function of k on the scale set by k_F. Therefore, we can set $k=0$ in $\Gamma^{(2)}$.

(b) The most important contribution comes out of $G(p")G(k-p")$ as $k \to 0$.

(c) The most significant of this being the vicinity of $|p"| \approx k_F$ and $\epsilon" \approx 0$. In this region, $G(p")G(k-p")$ can be written.

Continuing, we can see that

$$G(p")G(k-p") \tag{D.36}$$
$$\to \phi(p") \left\{ \frac{Z_{p"}}{\epsilon" - \xi_{p"} + i\delta sgn(p" - k_F)} \frac{Z_{p"}}{\lambda - \epsilon" - \xi_{p"} + v_{p"} \cdot k + i\delta sgn(|p" - k| - k_F)} \right\}$$

where $\xi_{p"} = v_p(p" - k_F)$ for $p" \approx k_F$. Now, fully expanding the summation in $D.25$ and using the above, we can write the integral equations for the vertex as

$$\Gamma(p, p'; k) = \Gamma^{(2)}(p, p') + \frac{i}{2} \int \frac{d^4p"}{(2\pi)^4} \Gamma^{(2)}(p, p") \times \tag{D.37}$$
$$\left\{ \phi(p") + \frac{Z_{p"}}{\lambda - \epsilon" - \xi_{p"-k} + i\delta sgn(|p" - k| - k_F)} \frac{Z_{p"}}{\epsilon" - \xi_{p"} + i\delta sgn(p" - k_F)} \right\} \Gamma(p", p'; k)$$

with $\xi_{p-k} = \epsilon_{p-k} - \epsilon_F \approx -v_p \cdot k$. At small k, $v_p = \frac{\partial \epsilon_p}{\partial p}$. We can now linearize the expression where for $p \approx k_F$, $v_p = \frac{k_F}{m^*}$ and $\xi_p = v_F(p - p_F)$. The largest contribution to the integral will come from the neighborhood of $\epsilon" = 0$ and $\xi_p" = 0$. If we assume $\Gamma + \Gamma^{(2)}$ vary slowly in this region we can perform the frequency and momentum integrals where we know that $p"$ remains finite.

Moreover, on the Fermi surface $p" \approx k_F$ with $\epsilon" = 0$ the excitations become pure quasiparticle excitations, which implies that $\phi(p" = k_F, \epsilon" = 0) = 0$. Thus, the integral equation reduces to

$$\Gamma(p, p'; k) = \Gamma^{(2)}(p, p') + \frac{i}{2} \int \frac{d^4p"}{(2\pi)^4} \Gamma^{(2)}(p, p") \times \tag{D.38}$$
$$\left\{ \frac{Z_{p"}}{\lambda - \epsilon" - \xi_{p"-k} + i\delta sgn(|p" - k| - k_F)} \frac{Z_{p"}}{\epsilon" - \xi_{p"} + i\delta sgn(p" - k_F)} \right\} \Gamma(p", p'; k)$$

Let us now study the integral. Here, we can expand the integral into the following integral terms

$$\int \frac{d^4p"}{(2\pi)^4} = \int \frac{d^3p}{(2\pi)^3} \int \frac{d\epsilon"}{2\pi} = k_F^2 \int dp" \int \frac{d\Omega"}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\epsilon"}{2\pi} \tag{D.39}$$

which is valid as the integral is concentrated around k_F. Thus, we can pull $Z_{p"} = Z$ out,

$$Z^2 k_F^2 \int dp" \int \frac{d\Omega"}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\epsilon"}{2\pi} \times \tag{D.40}$$
$$\left(\frac{1}{\epsilon" - \xi_{p"} + i\delta sgn(p" - k_F)} \right) \left(\frac{1}{\lambda - \epsilon" - \xi_{p"-k} + i\delta sgn(|p" - k| - k_F)} \right)$$

We can rewrite this as

$$-iZ^2k_F^2\int\frac{d\Omega"}{(2\pi)^2}\int_0^\infty\frac{dp"}{2\pi}\int_{-\infty}^\infty\frac{d\epsilon"}{2\pi}\times \tag{D.41}$$
$$\frac{1}{\epsilon" - \xi_{p"} + i\delta sgn(p" - k_F)}\frac{-1}{[\lambda - \epsilon" + \xi_{k-p"} - i\delta sgn(|k-p"|-k_F)]}$$

Only those terms will survive in which,

(a) $p" \geq k_F$ and $|p" - k| \geq k_F$, $(1 - n_{p"})(1 - n_{k-p"})$

(b) $p" \leq k_F$ and $|p" - k| \leq k_F$, $n_{p"}n_{k-p"}$

or

$$n_{p"} = \begin{cases} 1 & p" \leq k_F \\ 0 & p" > k_F \end{cases} \tag{D.42}$$

$$n_{k-p"} = \begin{cases} 1 & |p" - k| \leq k_F \\ 0 & |p" - k| > k_F \end{cases} \tag{D.43}$$

where for (a) we have a closed contour in the lower half plane and for (b) in the upper half plane. Therefore, the above expression reduces to the following:

$$= -iZ^2k_F^2\int\frac{d\Omega"}{(2\pi)^2}\int_0^\infty\frac{dp"}{2\pi}\int_{-\infty}^\infty\frac{d\epsilon"}{2\pi}\times \tag{D.44}$$
$$\left(\frac{n_{p"}n_{k-p"}}{\epsilon" - \xi_{p"} - i\delta}\frac{1}{[\lambda - \epsilon" + \xi_{k-p"} + i\delta]} + \frac{(1-n_{p"})(1-n_{k-p"})}{\epsilon" - \xi_{p"} + i\delta}\frac{1}{[\lambda - \epsilon" + \xi_{k-p"} - i\delta]}\right)$$

We will now proceed by first calculating the ϵ integral, then the k integral, and then the Ω integral over the function. First, let us simplify the ϵ integral. Using the Sokhotski-Plemelj formula, $\lim_{\delta\to 0}\frac{1}{x-x_0\pm i\delta} = P\frac{1}{x-x_0} \mp$

$i\pi\delta(x-x_0)$, the above ϵ integral reduces to

$$\lim_{\delta sgn(p"-k_F)\to 0}\lim_{\delta sgn(|p"-k|-k_F)\to 0} n_{p"}n_{k-p"}$$
$$\int \frac{d\epsilon"}{2\pi}\left(\frac{1}{\epsilon" - \xi_{p"} - i\delta sgn(p" - k_F)}\right)\left(\frac{1}{\lambda - \epsilon" - \xi_{p"-k} + i\delta sgn(|p" - k|-k_F)}\right)$$
$$+ \lim_{\delta sgn(p"-k_F)\to 0}\lim_{\delta sgn(|p"-k|-k_F)\to 0}(1-n_{p"})(1-n_{k-p"})$$
$$\int \frac{d\epsilon"}{2\pi}\left(\frac{1}{\epsilon" - \xi_{p"} + i\delta sgn(p" - k_F)}\right)\left(\frac{1}{\lambda - \epsilon" - \xi_{p"-k} - i\delta sgn(|p" - k|-k_F)}\right)$$

$$= \int \frac{d\epsilon"}{2\pi} n_{p"}n_{k-p"}\left[\left(P\frac{1}{\epsilon" - \xi_{p"}} + i\pi\delta(\epsilon" - \xi_{p"})\right)\left(P\frac{1}{\lambda - \epsilon" - \xi_{p"-k}} - i\pi\delta(-\epsilon" + \lambda - \xi_{p"-k})\right)\right]$$
$$+ \int \frac{d\epsilon"}{2\pi}(1-n_{p"})(1-n_{k-p"})\left[\left(P\frac{1}{\epsilon" - \xi_{p"}} - i\pi\delta(\epsilon" - \xi_{p"})\right)\left(P\frac{1}{\lambda - \epsilon" - \xi_{p"-k}} + i\pi\delta(-\epsilon" + \lambda - \xi_{p"-k})\right)\right]$$

$$= i\pi \int \frac{d\epsilon"}{2\pi} n_{p"}n_{k-p"}\left[\left(P\frac{1}{\lambda - \epsilon" - \xi_{p"-k}}\delta(\epsilon" - \xi_{p"})\right) - \left(P\frac{1}{\epsilon" - \xi_{p"}}\delta(-\epsilon" + \lambda - \xi_{p"-k})\right)\right]$$
$$+ i\pi \int \frac{d\epsilon"}{2\pi}(1-n_{p"})(1-n_{k-p"})\left[\left(P\frac{1}{\epsilon" - \xi_{p"}}\delta(-\epsilon" + \lambda - \xi_{p"-k})\right) - \left(P\frac{1}{\lambda - \epsilon" - \xi_{p"-k}}\delta(\epsilon" - \xi_{p"})\right)\right]$$

$$= \frac{i}{2}\left[\frac{n_{p"}n_{k-p"}}{\lambda - \xi_{p"} - \xi_{k-p"}} + \frac{n_{p"}n_{k-p"}}{\lambda - \xi_{p"} - \xi_{k-p"}}\right] + \frac{i}{2}\left[\frac{-(1-n_{p"})(1-n_{k-p"})}{\lambda - \xi_{p"} - \xi_{k-p"}} - \frac{(1-n_{p"})(1-n_{k-p"})}{\lambda - \xi_{p"} - \xi_{k-p"}}\right]$$

$$= i\frac{n_{p"}n_{k-p"}}{\lambda - \xi_{p"} - \xi_{k-p"} - i\delta} - i\frac{(1-n_{p"})(1-n_{k-p"})}{\lambda - \xi_{p"} - \xi_{k-p"} + i\delta} \text{ where we tack on } i\delta \text{ for analyticity}$$

$$= i\left(\frac{n_{p"}n_{k-p"}}{\lambda + \xi_{p"} - \xi_{k-p"} - i\delta} + \frac{(1-n_{p"})(1-n_{k-p"})}{\lambda - \xi_{p"} - \xi_{k-p"} + i\delta}\right)$$

$$= i\left(\frac{(1-n_{p"})(1-n_{k-p"})}{\xi_{p"} + \xi_{k-p"} - \lambda - i\delta} - \frac{n_{p"}n_{k-p"}}{\xi_{p"} + \xi_{k-p"} - \lambda + i\delta}\right) \tag{D.45}$$

We just solved a closed contour in the first integral in the upper half plane and a closed contour in the second integral in the lower half plane. Plugging equation $(D.44)$ above back into equation $(D.45)$, we obtain the simplified integral term of

$$Z^2k_F^2\int \frac{d\Omega"}{(2\pi)^2}\int_0^\infty \frac{dp"}{2\pi}\left(\frac{(1-n_{p"})(1-n_{k-p"})}{\xi_{p"} + \xi_{k-p"} - \lambda - i\delta} - \frac{n_{p"}n_{k-p"}}{\xi_{p"} + \xi_{k-p"} - \lambda + i\delta}\right) \tag{D.46}$$

We will now compute the momentum integral, followed by the x component of the Ω integral, and lastly the ϕ component of the Ω integral. In order to compute the momentum integrals we must consider the phase space restrictions imposed by the Fermi factors: For $n_{p"}n_{p"-k}$ we have $p" \leq k_F$ and $|\vec{p}" - \vec{k}| \leq k_F$ or $p"^2 - 2kxp" + k^2 \leq k_F^2$. Now, if we integrate freely over x we must restrict $p"$,

$$\begin{aligned}
&p"^2 - 2kxp" - (k_F^2 - k^2) \leq 0 \\
&\implies p" = \frac{2kx \pm \sqrt{4k^2x^2 + 4(k_F^2 - k^2)}}{2} = kx \pm \sqrt{k_F^2 - k^2(1-x^2)} \\
&\implies p" \approx kx + k_F \left(1 - \underbrace{\frac{1}{2}\frac{k^2}{k_F^2}(1-x^2)}_{k<<k_F} \right) \\
&\implies p_+" \approx k_F + kx
\end{aligned} \tag{D.47}$$

to order $\frac{k^2}{k_F^2}$ where we can see $p_+"^2 - 2kxp_+" - (k_F^2 - k^2) = 0$, and therefore, $p" \leq p_+"$. We can use the equality $\xi_{p"} + \xi_{k-p"} = 2\xi_{p"} - v_{p"} \cdot k = 2v_F(p" - k_F) - v_F kx"$ and employ a change of variables by setting $\xi = \xi_{p"} = v_F(p" - k_F)$ and $d\xi = v_F dp"$ so that $\xi_{p"} + \xi_{k-p"} = 2\xi_{p"} - v_F kx"$. We also intend to reduce the first integral by introducing a cutoff, U, in the ξ integrand where $k_B T_c << U << \epsilon_F = k_B T_F$. Therefore, using these substitutions and adjusting the limits of each integral, we can write the integral over x and p as

$$\int_{-1}^{1} dx" \int_{0}^{\infty} dp" \left(\frac{(1-n_{p"})(1-n_{k-p"})}{\xi_{p"} + \xi_{k-p"} - \lambda - i\delta} - \frac{n_{p"}n_{k-p"}}{\xi_{p"} + \xi_{k-p"} - \lambda + i\delta} \right)$$

$$= \int_{-1}^{1} dx" \int_{0}^{\infty} dp" \left(\frac{(1-n_{p"})(1-n_{k-p"})}{2\xi_{p"} - v_F kx" - \lambda - i\delta} - \frac{n_{p"}n_{k-p"}}{2\xi_{p"} - v_F kx" - \lambda + i\delta} \right)$$

$$= \int_{-1}^{1} dx" \int_{0}^{\infty} dp" \left(\frac{(1-n_{p"})(1-n_{k-p"})}{2\xi - v_F kx" - \lambda - i\delta} - \frac{n_{p"}n_{k-p"}}{2\xi - v_F kx" - \lambda + i\delta} \right)$$

$$
\begin{aligned}
&= \left[\int_0^1 dx" \int_{k_F+kx"}^{\infty} dp" + \int_{-1}^0 dx" \int_{k_F}^{\infty} dp"\right] \frac{1}{2\xi - v_F kx" - \lambda - i\delta} \\
&- \left[\int_{-1}^0 dx" \int_0^{k_F+kx"} dp" + \int_0^1 dx" \int_0^{k_F} dp"\right] \frac{1}{2\xi - v_F kx" - \lambda + i\delta}
\end{aligned}
$$

$$
\begin{aligned}
&= \left[\int_0^1 dx" \int_{x_F kx"}^{U} \frac{d\xi}{v_F} + \int_{-1}^0 dx" \int_0^U \frac{d\xi}{v_F}\right] \left[\frac{1}{2\xi - \lambda - v_F kx" - i\delta}\right] \\
&- \left[\int_{-1}^0 dx" \int_{-U}^{v_F kx"} \frac{d\xi}{v_F} + \int_0^1 dx" \int_{-U}^0 \frac{d\xi}{v_F}\right] \left[\frac{1}{2\xi - \lambda - v_F kx" + i\delta}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \underbrace{\int_0^1 dx" \int_{v_F kx"}^{U} d\xi \frac{1}{2\xi - \lambda - v_F kx" - i\delta}}_{(1)} + \underbrace{\int_{-1}^0 dx" \int_0^U d\xi \frac{1}{2\xi - \lambda - v_F kx" - i\delta}}_{(2)} \\
&- \underbrace{\int_{-1}^0 dx" \int_{-U}^{v_F kx"} d\xi \frac{1}{2\xi - \lambda - v_F kx" + i\delta}}_{(3)} - \underbrace{\int_0^1 dx" \int_{-U}^0 d\xi \frac{1}{2\xi - \lambda - v_F kx" + i\delta}}_{(4)}
\end{aligned}
$$

$$
\begin{aligned}
&= \underbrace{\int_0^1 dx" \int_{v_F kx"}^{U} d\xi \left\{ \underbrace{\frac{1}{2\xi - \lambda - v_F kx" - i\delta}}_{\text{sign change at } \bar{\xi} = \frac{1}{2}(\lambda + v_F kx")} + \underbrace{\frac{1}{2\xi + \lambda - v_F kx" - i\delta}}_{\text{No sign change, remains positive}} \right\}}_{(1)+(3)} \\
&+ \underbrace{\int_{-1}^0 dx" \int_0^U d\xi \left\{ \underbrace{\frac{1}{2\xi - \lambda - v_F kx" - i\delta}}_{\text{sign change at } \bar{\xi} = \frac{1}{2}(\lambda + v_F kx")} + \underbrace{\frac{1}{2\xi + \lambda - v_F kx" - i\delta}}_{\text{No sign change, remains positive}} \right\}}_{(2)+(4)}
\end{aligned}
\tag{D.48}
$$

where in integral (3) and (4) we let $x" \to -x"$ and $\xi \to -\xi$ so term (1) and (3) have the same integral, as do (2) and (4). Note that we have also pulled out a v_F^{-1} which we will reintroduce later. Since there is a sign change at $\bar{\xi} = \frac{1}{2}(\lambda + v_F kx")$, we can introduce a small parameter ϵ and form $\xi_\pm = \frac{1}{2}(\lambda + v_F kx" \pm \epsilon)$, which allows us to expand the integral into a

distinct form with no sign changes within any integral,

$$= \int_0^1 dx" \left(\int_{v_F k x"}^{\xi_-} d\xi + \int_{\xi_+}^{U} d\xi \right) \left\{ \frac{1}{2\xi - \lambda - v_F k x" - i\delta} + \frac{1}{2\xi + \lambda - v_F k x" - i\delta} \right\}$$
$$+ \int_{-1}^0 dx" \left(\int_0^{\xi_-} d\xi + \int_{\xi_+}^{U} d\xi \right) \left\{ \frac{1}{2\xi - \lambda - v_F k x" - i\delta} + \frac{1}{2\xi + \lambda - v_F k x" - i\delta} \right\}$$

$$= \int_0^1 dx" \left(\int_{v_F k x"}^{\xi_-} d\xi + \int_{\xi_+}^{U} d\xi \right) \left\{ \frac{1}{2\xi - \lambda - v_F k x"} + \frac{1}{2\xi + \lambda - v_F k x"} \right\}$$
$$+ \int_{-1}^0 dx" \left(\int_0^{\xi_-} d\xi + \int_{\xi_+}^{U} d\xi \right) \left\{ \frac{1}{2\xi - \lambda - v_F k x"} + \frac{1}{2\xi + \lambda - v_F k x"} \right\}$$

$$= \int_0^1 dx" \frac{1}{2} \left[\ln(2\xi - 2\bar{\xi}) \Big|_{v_F k x"}^{\xi_-} + \ln(2\xi - 2\bar{\xi}) \Big|_{\xi_+}^{U} \right]$$
$$+ \int_0^1 dx" \frac{1}{2} \left[\ln(2\xi + 2\bar{\xi} - 2v_F k x") \Big|_{v_F k x"}^{\xi_-} + \ln(2\xi + 2\bar{\xi} - 2v_F k x") \Big|_{\xi_+}^{U} \right]$$
$$+ \int_{-1}^0 dx" \frac{1}{2} \left[\ln(2\xi - 2\bar{\xi}) \Big|_{0}^{\xi_-} + \ln(2\xi - 2\bar{\xi}) \Big|_{\xi_+}^{U} \right]$$
$$+ \int_{-1}^0 dx" \frac{1}{2} \left[\ln(2\xi + 2\bar{\xi} - 2v_F k x") \Big|_{0}^{\xi_-} + \ln(2\xi + 2\bar{\xi} - 2v_F k x") \Big|_{\xi_+}^{U} \right]$$

$$= \int_0^1 dx" \frac{1}{2} \left[\ln(2\xi_- - 2\bar{\xi}) - \ln(2v_F k x" - 2\bar{\xi}) + \ln(2U - 2\bar{\xi}) - \ln(2\xi_+ - 2\bar{\xi}) \right]$$
$$+ \int_0^1 dx" \frac{1}{2} \left[\ln(2\xi_- + 2\bar{\xi} - 2v_F k x") - \ln(2\bar{\xi}) + \ln(2U + 2\bar{\xi} - 2v_F k x") - \ln(2\xi_+ + 2\bar{\xi} - 2v_F k x") \right]$$
$$+ \int_{-1}^0 dx" \frac{1}{2} \left[\ln(2\xi_- - 2\bar{\xi}) - \ln(-2\bar{\xi}) + \ln(2U - 2\bar{\xi}) - \ln(2\xi_+ - 2\bar{\xi}) \right]$$
$$+ \int_{-1}^0 dx" \frac{1}{2} \left[\ln(2\xi_- + 2\bar{\xi} - 2v_F k x") - \ln(2\bar{\xi} - 2v_F k x") + \ln(2U + 2\bar{\xi} - 2v_F k x") - \ln(2\xi_+ + 2\bar{\xi} - 2v_F k x") \right]$$

as $\ln(2\xi_- - 2\bar{\xi}) - \ln(2\xi_+ - 2\bar{\xi}) = i\pi$ because
$2\xi_\pm - 2\bar{\xi} = \pm\epsilon$ and $\ln(-\epsilon) = \ln(\epsilon) + i\pi$

$$= \int_0^1 dx" \left\{ \frac{1}{2} \ln \left| \frac{2U - \lambda - v_F k x"}{v_F k x" - \lambda} \right| + i\frac{\pi}{2} + \frac{1}{2} \ln \left| \frac{2U + \lambda - v_F k x"}{v_F k x" + \lambda} \right| \right\}$$
$$+ \int_{-1}^0 dx" \left\{ \frac{1}{2} \ln \left| \frac{2U - \lambda - v_F k x"}{v_F k x" + \lambda} \right| + i\frac{\pi}{2} + \frac{1}{2} \ln \left| \frac{2U + \lambda - v_F k x"}{v_F k x" - \lambda} \right| \right\} \tag{D.49}$$

where we have sent ϵ to 0 in the terms $\ln(2\xi_- + 2\bar{\xi})$ and $-\ln(2\xi_+ + 2\bar{\xi})$ so that when combined using logarithmic identities, they term simply goes to zero which is valid since they are additive, thus positive, within the log. Using logarithmic identities, we can rearrange the denominators $v_F k x" + \lambda$ and $v_F k x" - \lambda$ so that the integrand of the integral from 0 to 1 equals that of the integral from -1 to 0. We can now combine the integrals as they have the same integrands

$$\int_{-1}^1 \frac{dx"}{2} \left\{ \ln \left| \frac{2U - \lambda - v_F k x"}{v_F k x" - \lambda} \right| + \ln \left| \frac{2U + \lambda - v_F k x"}{v_F k x" + \lambda} \right| + i\pi \right\} \tag{D.50}$$

If we put this back together in Equation (166), we obtain

$$\Gamma(p, p'; k) = \Gamma^{(2)}(p, p') + \frac{i}{2} \left(\frac{-ik_F^2}{2\pi^2} \right) \left(\frac{Z^2}{2v_F} \right)$$
$$\int \frac{d\Omega"}{4\pi} \Gamma^{(2)}(p, p") \Gamma(p", p'; k) \underbrace{\left\{ \ln \left| \frac{2U - \lambda - v_F k x"}{v_F k x" - \lambda} \right| + \ln \left| \frac{2U + \lambda - v_F k x"}{v_F k x" + \lambda} \right| + i\pi \right\}}_{R(k, x")} \tag{D.51}$$

We can look back at our expansion of the vertex function in partial waves, equation $(D.34)$ and see that the vertex function is now complete with the addition of one more step. We must compute the integral over Ω, which decomposes into an integral over x and ϕ. With $\hat{p} \cdot \hat{p}' = \cos\phi$,

$$\Gamma^{(2)}(p, p') = \sum_m \Gamma_m^{(2)} e^{im\phi}$$
$$\Gamma_m(p, p'; k) = \sum_m \Gamma_m(k) e^{im\phi} \tag{D.52}$$

where now we can compute the ϕ integral.

$$\begin{aligned}
\sum_m \Gamma_m(k)e^{im\phi} &= \sum_m \Gamma_m^{(2)}e^{im\phi} + \frac{1}{2}\left(\frac{k_F^2}{2\pi^2}\right)\left(\frac{Z^2}{2v_F}\right)\int_{-1}^{1}\frac{dx"}{2}\int_0^{2\pi}\frac{d\phi"}{2\pi} \\
&\quad \sum_{m'}\sum_{m"} e^{im'\phi_{pp"}}e^{im"\phi_{p"p'}}\Gamma_m^{(2)}R(k,x")\Gamma_{m"}(k) \\
&= \sum_m \Gamma_m^{(2)}e^{im\phi} + \frac{1}{2}\left(\frac{k_F^2}{2\pi^2}\right)\left(\frac{Z^2}{2v_F}\right)\int_{-1}^{1}\frac{dx"}{2}\int_0^{2\pi}\frac{d\phi"}{2\pi} \\
&\quad \sum_{m'}\sum_{m"} e^{i(m'\phi_{pp"}+m"\phi_{p"p'})}\Gamma_m^{(2)}R(k,x")\Gamma_{m"}(k) \\
&= \sum_m \Gamma_m^{(2)}e^{im\phi} + \frac{1}{2}\left(\frac{k_F^2}{2\pi^2}\right)\left(\frac{Z^2}{2v_F}\right)\int_{-1}^{1}\frac{dx"}{2}\int_0^{2\pi}\frac{d\phi"}{2\pi} \\
&\quad \sum_{m'}\sum_{m"} e^{i(m'\phi_p - m"\phi_{p'})}e^{-i\phi_{p"}(m'-m")}\Gamma_m^{(2)}R(k,x")\Gamma_{m"}(k) \\
&\qquad \text{as } \phi_{pp"} = \phi_p - \phi_{p"} \text{ and } \phi_{p"p'} = \phi_{p"} - \phi_{p'} \\
&= \sum_m \Gamma_m^{(2)}e^{im\phi} + \frac{1}{2}\left(\frac{k_F^2}{2\pi^2}\right)\left(\frac{Z^2}{2v_F}\right)\int_{-1}^{1}\frac{dx"}{2}\int_0^{2\pi}\frac{d\phi"}{2\pi} \\
&\quad \sum_{m'}\sum_{m"} e^{i(m'\phi_p - m"\phi_{p'})}\delta_{m',m"}\Gamma_m^{(2)}R(k,x")\Gamma_{m"}(k) \\
&\qquad \text{as } \int_0^{2\pi}\frac{d\phi"}{2\pi}e^{-i\phi_{p"}(m'-m")} = \delta_{m',m"} \\
&= \sum_m \Gamma_m^{(2)}e^{im\phi} - \frac{1}{2}\left(\frac{k_F^2}{2\pi^2}\right)\left(\frac{Z^2}{2v_F}\right)\sum_{m'} e^{im'\phi}\int_{-1}^{1}\frac{dx"}{2}\Gamma_{m'}^{(2)}R(k,x")\Gamma_{m'}(k)
\end{aligned}$$

$$\implies \Gamma_m(k) = \Gamma_m^{(2)} + \frac{1}{2}\left(\frac{k_F^2}{2\pi^2}\right)\left(\frac{Z^2}{2v_F}\right)\Gamma_m^{(2)}\int_{-1}^{1}\frac{dx"}{2}R(k,x")\Gamma_{m'}(k) \tag{D.53}$$

Here, we set $k = 0$ in $R(k, x")$, which leaves no x dependence. Therefore, the integral over $x"$ equals 2 and the $\frac{1}{2}$ embedded in the integral dissolves.

This leaves the vertex as

$$\Gamma_l = \Gamma_l^{(2)} + \frac{1}{4}\frac{k_F m^*}{\pi^2}\frac{Z^2}{2l+1}\Gamma_l^{(2)}\frac{1}{2}\underbrace{\left\{\ln\left|\frac{2U-\lambda}{\lambda}\right| + \ln\left|\frac{2U+\lambda}{\lambda}\right| + i\pi\right\}}_{\approx \ln\frac{(2U)^2-\lambda^2}{\lambda^2}+i\pi \approx 2\ln\frac{2U}{\lambda}+i\pi}$$

$$\implies \Gamma_l = \frac{\Gamma_l^{(2)}}{1+\frac{Z^2N(0)}{4(2l+1)}\Gamma_l^{(2)}\left\{\ln\frac{2U}{\lambda}+i\frac{\pi}{2}\right\}} \tag{D.54}$$

We can now make some useful observations. For $\Gamma_l^{(2)} > 0$, Γ_l has no poles. For $\Gamma_l^{(2)} < 0$, Γ_l has a pole on the positive imaginary axis. Further, we can analytically continue to $\lambda \to i\lambda_c$, which admits

$$\ln\frac{2U}{i\lambda_c} = \ln\frac{-i2U}{\lambda_c} = \ln\frac{2U}{\lambda_c} - i\frac{\pi}{2} \tag{D.55}$$

Thus, we have a pole when

$$1 + \frac{Z^2N(0)}{4(2l+1)}\Gamma_l^{(2)}\ln\frac{2U}{\lambda_c} = 0 \tag{D.56}$$

which will only happen for the case when $\Gamma_l^{(2)}$ is attractive since $2U >> \lambda_c$,

$$\ln\frac{2U}{\lambda_c} = \frac{-1}{\frac{Z^2N(0)}{4(2l+1)}\Gamma_l^{(2)}} \tag{D.57}$$

Therefore, we obtain a pole when

$$\lambda_c = 2U\exp\left\{\frac{1}{\frac{N(0)Z^2\Gamma_l^{(2)}}{4(2l+1)}}\right\} \tag{D.58}$$

D.2.2 The Finite Temperature Vertex Function and Calculation of T_c

We have just derived the zero temperature vertex function and found that is was singular for $k = 0$ and $i\lambda \to \lambda_c$. To determine the critical temperature of the vertex we must find the finite temperature vertex function $\mathcal{T}$. The spin structure is identical to the zero temperature case so we have singlet and triplet terms.

$$\mathcal{T}(p_1,p_2;p_3,p_4) = \mathcal{T}^{(2)}(p_1,p_2;p_3,p_4) - \frac{T}{2}\sum_{p_5,p_6}\mathcal{T}^{(2)}(p_1,p_2;p_5,p_6)\mathcal{G}(p_5)\mathcal{G}(p_6)\mathcal{T}(p_4,p_6;p_5,p_4) \tag{D.59}$$

We can set $k_B = 1, p_i = (p_i, \omega_{n_i}), \omega_{n_i} = (2n_i + 1)\pi T$. The sum we must solve becomes

$$\sum_{p_N} = \sum_{\omega_{n_i}} \int \frac{d^3 p_i}{(2\pi)^3} \tag{D.60}$$

We will now introduce new variables as we did for the zero temperature Green's function

$$\begin{aligned} p &= \frac{p_1 - p_2}{2} \\ p' &= \frac{p_3 - p_4}{2} \\ p" &= \frac{p_5 - p_6}{2} \\ p_5 &= p" + \frac{1}{2}k \\ p_6 &= \frac{1}{2}k - p" \\ k = p_1 + p_2 &= p_3 + p_4 = p_5 + p_6 \text{ momentum conservation} \\ k &= (k, \lambda_n) \end{aligned} \tag{D.61}$$

Setting $p" \to p" - \frac{k}{2}$ and taking into account the terms defined above, the integral equation becomes

$$\begin{aligned} \mathcal{T}(p, \omega_n, p', \omega_{n'}; k, \lambda_n) = \mathcal{T}^{(2)}&(p, \omega_n, p', \omega_{n'}; k, \lambda_n) \\ &- \frac{T}{2} \sum_{\omega_{n"}} \int \frac{d^3 p"}{(2\pi)^3} \mathcal{T}^{(2)}(p, \omega_n, p", \omega_{n"}; k, \lambda_n) \mathcal{G}(p", \omega_{n"}) \\ &\mathcal{G}(k - p", \lambda_n - \omega_{n"}) \mathcal{T}(k - p", \lambda_n - \omega_{n"}, p', \omega_{n'}; k, \lambda_n) \end{aligned} \tag{D.62}$$

Since we want to study this function in the limit $\lambda \to 0$ and $k \to 0$, it is necessary to construct an analytic continuation of this function. The spectral representation for the $\mathcal{G}$'s is,

$$\begin{aligned} \mathcal{G}(p, \omega_n) &= \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \frac{p(p, \epsilon)}{i\omega_n - \epsilon} \\ \mathcal{G}_{qp}(p, \omega_n) = \frac{Z_p}{i\omega_n - \xi_p} &\qquad \xi_p = v_F(p - k_F) \\ \text{where } |p| &\approx k_F \text{ and } T << T_F \end{aligned} \tag{D.63}$$

Moreover, in the limit $k \to 0$, $|p| = |p'| = k_F$ the function $\mathcal{T}$ will depend only on k and λ_n. We can write the spectral representation as

$$\mathcal{T}(k, \lambda_n) = \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \frac{\sigma(k, \epsilon)}{i\lambda_n - \epsilon} \tag{D.64}$$

Note $\mathcal{T}$ will actually depend on the angle between p and p' but not on their magnitudes. As the interest is in the limit $T << T_F$ amd $k \to 0$, $\mathcal{T}^{(2)}$ can be replaced by $\Gamma^{(2)}$ at $k = 0$, which is the zero temperature value of $\mathcal{T}^{(2)}$. $\mathcal{T}^{(2)}$ is not anomalous as $k \to 0$ and is slowly varying as a function of T. Furthermore, most of the weight in the integrand of equation $(D.62)$ comes from $|p_i| \approx k_F$ since, in this region, the $\mathcal{G}$'s$\sim \frac{1}{i\omega_n}$, $T << T_F$. Assuming that $\mathcal{T}^{(2)}$ and $\mathcal{T}$ are slowly varying functions in this region, we can bring them out of the frequency sum. Now, this means we need to solve

$$\sum_{\omega_{n"}} \mathcal{G}_{qp}(k - p", \lambda_n - \omega_{n"}) \mathcal{G}_{qp}(p", \omega_{n"})$$

$$= Z^2 \sum_{\omega_{n"}} \frac{1}{i(\lambda_n - \omega_{n"}) - \xi_{k-p"}} \frac{1}{i\omega_{n"} - \xi_{p"}}$$

$$= Z^2 \sum_{\omega_{n"}} \left\{ \left[\frac{1}{i\omega_{n"} + (\xi_{p"} - v_{p"} \cdot k - i\lambda_n)} \right] \left[\frac{1}{i\omega_{n"} - \xi_{p"}} \right] \right\}$$

where for small k, $\xi_{k-p"} = \xi_{p"} - v_{p"} \cdot k, \xi_{p"} = v_F(p" - k_F)$

$$= Z^2 \sum_{\omega_{n"}} \left[\frac{1}{2\xi_{p"} - v_{p"} \cdot k - i\lambda_n} \right] \left[\frac{1}{i\omega_{n"} + (\xi_{p"} - v_{p"} \cdot k - i\lambda_n)} - \frac{1}{i\omega_{n"} - \xi_{p"}} \right]$$

$$= \frac{Z^2 \beta}{2\xi_{p"} - v_{p"} \cdot k - i\lambda_n} \left[\frac{1}{e^{-\beta(\xi_{p"} - v_{p"} \cdot k - i\lambda_n)} + 1} - \frac{1}{e^{\beta \xi_{p"}} + 1} \right]$$

as $\sum_{\omega_{n"}} \frac{1}{i\omega_{n"} \pm x} = \frac{\beta}{e^{\mp \beta x} + 1}$ with $\beta = \frac{1}{T}, k_B = 0$

$$= \frac{Z^2 \beta}{2\xi_{p"} - v_{p"} \cdot k - i\lambda_n} \left[\frac{1}{e^{-\beta(\xi_{p"} - v_{p"} \cdot k)} + 1} - \frac{1}{e^{\beta \xi_{p"}} + 1} \right]$$

where $\lambda_n = 2\pi n T \implies e^{i\beta\lambda_n} = 1$

$$= \frac{Z^2\beta}{2\xi_{p"} - v_{p"}\cdot k - i\lambda_n}\left[\frac{e^{\beta\xi_{p"}}}{e^{\beta(v_{p"}\cdot k)} + e^{\beta\xi_{p"}}} - \frac{e^{-\beta(\xi_{p"} - v_{p"}\cdot k)}}{e^{\beta(v_{p"}\cdot k)} + e^{-\beta(\xi_{p"} - v_{p"}\cdot k)}}\right]$$

$$= \frac{Z^2\beta}{2\xi_{p"} - i\lambda_n}\left[\frac{e^{-\beta\xi_{p"}/2}}{e^{-\beta\xi_{p"}/2}}\frac{e^{\beta\xi_{p"}}}{1+e^{\beta\xi_{p"}}} - \frac{e^{-\beta\xi_{p"}}}{1+e^{-\beta\xi_{p"}}}\frac{e^{\beta\xi_{p"}/2}}{e^{\beta\xi_{p"}/2}}\right]$$

where $k \to 0$

$$= \frac{Z^2\beta}{2\xi_{p"} - i\lambda_n}\left[\frac{e^{\beta\xi_{p"}/2} - e^{-\beta\xi_{p"}/2}}{e^{\beta\xi_{p"}/2} + e^{-\beta\xi_{p"}/2}}\right]$$

$$= \frac{Z^2\beta}{2\xi_{p"} - i\lambda_n}\left[\tanh\frac{\beta\xi_{p"}}{2}\right] \tag{D.65}$$

Plugging the above expression back into equation $(D.62)$, we find that the equation reduces to

$$\mathcal{T} = \Gamma^{(2)} - \frac{T}{2}\int\frac{d^3p"}{(2\pi)^3}\Gamma^{(2)}\mathcal{T}\frac{Z^2\beta}{2\xi_{p"} - i\lambda_n}\tanh\frac{\beta\xi_{p"}}{2}$$

$$= \Gamma^{(2)} - \frac{1}{2}\int\frac{d\Omega"}{4\pi}\frac{Z^2k_F^2}{2\pi^2 v_F}\Gamma^{(2)}\mathcal{T}\int_0^U d\xi\frac{\tanh\frac{\beta\xi}{2}}{2\xi - i\lambda_n} \tag{D.66}$$

where we restrict ourselves to the region near k_F so that $d^3p" = d\Omega" k_F^2 dp" = d\Omega" k_F\frac{d\xi_{p"}}{v_F}$. Using $\mathcal{T} \to \Gamma^{(+)}$, or the retarded function defined in the upper half plane, where $i\lambda_n \to \lambda + i\delta$, we can write

$$\Gamma^{(+)} = \Gamma^{(2)} - \frac{1}{2}\int\frac{d\Omega"}{4\pi}\frac{Z^2k_F^2}{2\pi^2 v_F}\Gamma^{(2)}\Gamma^{(+)}\int_0^U d\xi\frac{\tanh\frac{\beta\xi}{2}}{2\xi - (\lambda + i\delta)}$$

$$\implies \Gamma_l^{(+)}(\lambda, T) = \frac{\Gamma_l^{(2)}}{1 + \frac{Z^2N(0)\Gamma_l^{(2)}}{4(2l+1)}\int_0^U d\xi\frac{\tanh\frac{\beta\xi}{2}}{2\xi - (\lambda+i\delta)}} \tag{D.67}$$

which is found from the same decomposition as in the $T = 0$ case. For $\lambda \to 0$ we can determine the temperature T_c at which $\Gamma_l^{(+)}$ is singular. First, we must reduce the ξ integral in order to analyze the poles of this function. Employ a change of variables where $x = \frac{\beta\xi}{2} \implies \xi = \frac{2x}{\beta}$ and

$dx = \frac{\beta}{2} d\xi \implies \frac{2}{\beta} dx = d\xi$. We also define $x_0 = \frac{\beta U}{2}$.

$$\begin{aligned}
\int_0^U d\xi \frac{\tanh \frac{\beta\xi}{2}}{2\xi} &= \int_0^{x_0} \frac{2dx}{\beta} \frac{\tanh x}{2(\frac{2x}{\beta})} \\
&= \int_0^U dx \frac{\tanh x}{2x} \\
&= \ln x \tanh x \Big|_0^{x_0} - \int_0^{x_0} dx \frac{\ln x}{\cosh^2 x} \\
&= \ln x_0 - \int_0^{x_0} dx \frac{\ln x}{\cosh^2 x} \\
&= \ln x_0 - \ln \frac{\pi}{4\gamma} \\
&= \ln \frac{2\gamma\beta U}{\pi} \qquad \text{(D.68)}
\end{aligned}$$

The final form of the vertex for finite temperature becomes

$$\Gamma_l^{(+)}(\lambda, T) = \frac{\Gamma_l^{(2)}}{1 + \frac{Z^2 N(0)\Gamma_l^{(2)}}{4(2l+1)} \ln \frac{2\gamma U}{T\pi}} \qquad \text{(D.69)}$$

where we get a pole as $T \to T_c$ for $\Gamma_l^{(2)} < 0$ and $U >> T_c$, i.e.

$$\begin{aligned}
&1 + \frac{Z^2 N(0)\Gamma_l^{(2)}}{4(2l+1)} \ln \frac{2\gamma U}{T\pi} = 0 \\
&\implies \frac{2\gamma}{\pi} \frac{U}{T_c} = e^{\left[\frac{-1}{\frac{Z^2 N(0)\Gamma_l^{(2)}}{4(2l+1)}} \right]} \\
&\implies T_c^l = \frac{2U\gamma}{\pi} e^{\left[\frac{1}{\frac{Z^2 N(0)\Gamma_l^{(2)}}{4(2l+1)}} \right]} \\
&\implies T_c^l = 1.13 U e^{\frac{1}{g_l}} \text{ where } r = e^c \text{ and } c = 0.577215 \text{ (Euler's constant)} \\
&\implies g_l = \frac{Z^2 N(0)\Gamma_l^{(2)}}{4(2l+1)} \text{ with } N(0) = \frac{k_F^2}{\pi^2 v_F} = \frac{k_F m^*}{\pi^2} \qquad \text{(D.70)}
\end{aligned}$$

We only have T_F as our scale of energies in ^3He, therefore, we set $U = \alpha T_F$ to obtain

$$T_c^l = 1.13\alpha T_F e^{\frac{1}{g_l}} \tag{D.71}$$

D.2.3 Corrections to the Description of s-Wave Superconductivity Induced by Weak Ferromagnetism

We will now solve for the finite temperature vertex corrections to the three-point vertex in the weak ferromagnetic metal. These can be written as

$$\begin{aligned}\Gamma^{(1)}_{\uparrow\uparrow}(k,k+q) &= \Gamma^{(1)}_{\uparrow\uparrow l} + \Gamma^{(1)}_{\uparrow\uparrow G}\\ \Gamma^{(1)}_{\uparrow\downarrow}(k,k+q) &= \Gamma^{(1)}_{\uparrow\downarrow l} + \Gamma^{(1)}_{\uparrow\downarrow G}\end{aligned} \tag{D.72}$$

In the above expression, the terms can be defined as

$$\begin{aligned}\Gamma^{(1)}_{\uparrow\uparrow l} &= ig_0^2 \int dp G_\uparrow(p) D_l(p-k) G_\uparrow(k+q)\\ \Gamma^{(1)}_{\uparrow\uparrow} &= ig_0^2 \int dp G_\uparrow(p) D_G(p-q) G_\downarrow(k+q)\\ \Gamma^{(1)}_{\uparrow\downarrow l} &= ig_0^2 \int dp G_\downarrow(p) D_l(p-q) G_\uparrow(k+q)\\ \Gamma^{(1)}_{\uparrow\downarrow G} &= ig_0^2 \int dp G_\downarrow(p) D_G(p-q) G_\downarrow(k+q)\\ dp = \frac{d^4p}{(2\pi)^4} &= \int \frac{d^3p}{(2\pi)^3}\int\frac{d\epsilon}{2\pi} = k_F^2\int dp\int\frac{d\Omega}{(2\pi)^3}\int\frac{d\epsilon}{2\pi}\end{aligned} \tag{D.73}$$

We can generalize these down to two expression

$$\begin{aligned}\Gamma^{(1)}_{\sigma\sigma' l} &= ig_0^2 \int dp G_\sigma(p) D_l(p-k) G_{\sigma'}(p+q)\\ \Gamma^{(1)}_{\sigma\sigma' G} &= ig_0^2 \int dp G_\sigma(p) D_G(p-k) G_{\sigma'}(p+q)\end{aligned} \tag{D.74}$$

We also know that

$$\begin{aligned}G_\sigma(k,\omega) &= \frac{Z}{\omega - v_F(|k|-k_\sigma) + i\delta_\sigma(k)}\\ D_G(p,\omega) &= -\frac{\Delta N(0) v_F}{2}\frac{\Omega_0(p)}{(\omega+i\delta)^2 - \Omega_0^2(p)}\\ D_l(p,\omega) &= -\frac{N(0)p_F^2}{2}\frac{1}{\xi^{-2} + |p|^2 - \frac{i\pi p_F^2\omega}{2v_F|p|}}\end{aligned} \tag{D.75}$$

where k is the three dimensional momentum of the particle, k_σ is the Fermi momentum of the spin-σ electrons, v_F is the Fermi velocity, and $\delta_\sigma =$

$\delta\times$ sign$(|k|-k_\sigma)$, with δ an infinitesimal real number. The term $\Delta = k_\uparrow - k_\downarrow$ and $\Omega_0(p) = D|p|^2$ where $D = \frac{v_F\Delta}{k_F^2}$ is the spin stiffness. When calculating the vertex corrections, we first set the frequency ω to zero and then proceed to take the limit for the momentum. As we are working in the broken symmetry phase, we must make a distinction between corrections involving particles on one of two Fermi surfaces and corrections involved with particles on different Fermi surfaces. In the former and latter case we have the limits,

$$\begin{aligned}\Gamma_{\sigma\sigma}(|k|\to k_\sigma, |k|\to k_\sigma)\\ \Gamma_{\sigma\sigma'}(|k|\to k_\sigma, |k|\to k_{\sigma'}+\Delta)\end{aligned} \tag{D.76}$$

respectively. For these cases, we use the spectral representation for the propagators $D_{l/G}$,

$$D_{l/G}(p,\omega) = \frac{2}{\pi}\int_0^\infty dz \frac{z\mathrm{Im}D_{l/G}(p,z)}{z^2-\omega^2-i\delta} \tag{D.77}$$

Writing $G_\sigma(k+p,\epsilon+\omega)$ as,

$$\begin{aligned}G_\sigma(k+p,\epsilon+\omega) = &\frac{Zf_{k'}}{\epsilon+\omega-v_F(|k+p|-k_\sigma)+i\delta_\sigma(k+p)}\\ &+\frac{Z(1-f_{k'})}{\epsilon+\omega-v_F(|k+p|-k_\sigma)-i\delta_\sigma(k+p)}\end{aligned} \tag{D.78}$$

where $f_{k'} = \theta(|k+p|-k_F)$, which allows us to expand $G_\sigma(k+p,\epsilon+\omega)G_{\sigma'}(k+p+q,\epsilon+\omega)$ and set $q=0$ as,

$$\begin{aligned}&G_\sigma(k+p,\epsilon+\omega)G_{\sigma'}(k+p,\epsilon+\omega) =\\ &= \left[\frac{Zf_{k'}}{\epsilon+\omega-v_F(|k+p|-k_\sigma)+i\delta_\sigma(k+p)}+\frac{Z(1-f_{k'})}{\epsilon+\omega-v_F(|k+p|-k_\sigma)-i\delta_\sigma(k+p)}\right]\\ &\times\left[\frac{Zf_{k'}}{\epsilon+\omega-v_F(|k+p|-k_\sigma)+i\delta_\sigma(k+p)}+\frac{Z(1-f_{k'})}{\epsilon+\omega-v_F(|k+p|-k_\sigma)-i\delta_\sigma(k+p)}\right]\end{aligned}$$

$$
\begin{aligned}
&= Z[\frac{f_{k'}}{\epsilon+\omega-v_F(|k+p|-k_\sigma)+i\delta_\sigma(k+p)} \\
&+ \left(\frac{(1-f_{k'})}{\epsilon+\omega-v_F(|k+p|-k_\sigma)-i\delta_\sigma(k+p)} + i\pi\delta(\epsilon+\omega-v_F(|k+p|-k_\sigma))\right)] \\
&\times Z[\left(\frac{f_{k'}}{\epsilon+\omega-v_F(|k+p|-k_\sigma)+i\delta_\sigma(k+p)} - i\pi\delta(\epsilon+\omega-v_F(|k+p|-k_\sigma))\right) \\
&+ \frac{(1-f_{k'})}{\epsilon+\omega-v_F(|k+p|-k_\sigma)-i\delta_\sigma(k+p)}] \\
&= Z^2[\frac{f_{k'}}{[\epsilon+\omega-v_F(|k+p|-k_\sigma)+i\delta_\sigma(k+p)]^2} \\
&+ \frac{(1-f_{k'})}{[\epsilon+\omega-v_F(|k+p|-k_\sigma)-i\delta_\sigma(k+p)]^2} + \pi^2\delta(\epsilon+\omega-v_F(|k+p|-k_\sigma)] \\
&= Z^2\left[\frac{f_{k'}}{[\epsilon-\xi_{k_\sigma}+\omega+i\delta_\sigma(k+p)]^2} + \frac{(1-f_{k'})}{[\epsilon-\xi_{k_\sigma}+\omega-i\delta_\sigma(k+p)]^2} + \pi^2\delta(\epsilon-\xi_{k_\sigma}+\omega)\right]
\end{aligned}
\tag{D.79}
$$

where all $\frac{1}{\epsilon+\omega}\delta \to 0$ as these are far from the Fermi surface and we set $\xi_{k_\sigma} = v_F(|k+p|-k_\sigma)$. We have also expanded the second and third terms using the Sokhotski-Plemelj formula while keeping the $i\delta$ for now as this will be a key term when multiplying this term by the expanded imaginary bosonic propagator. Spins were also insignificant here, which is why we can combine terms. Therefore, the two integrals we must solve for the vertex corrections at zero temperature are

$$
\begin{aligned}
\Gamma^{(1)}_{\sigma\sigma' l/G} &= \frac{i2Zg_0^2}{v_F}\int\frac{d\Omega}{(2\pi)^2}\int_0^\infty\frac{dp}{2\pi}\int_{-\infty}^\infty\frac{d\epsilon}{2\pi}\times \\
&\int_0^\infty\frac{dz}{\pi}\frac{z\mathrm{Im}D_{l/G}(p,z)}{z^2-\omega^2-i\delta}[G_\sigma(k+p,\epsilon+\omega)G_{\sigma'}(k+p,\epsilon+\omega)]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{i2Z^2g_0^2}{v_F}\int\frac{d\Omega}{(2\pi)^2}\int_0^\infty\frac{dq}{2\pi}\int_{-\infty}^\infty\frac{d\epsilon}{2\pi}\int_0^\infty\frac{dz}{\pi}z\mathrm{Im}D_{l/G}(p,z)\frac{1}{z^2-\omega^2-i\delta} \\
&\times\left[\frac{f_{k'}}{[\epsilon-\xi_{k_\sigma}+\omega+i\delta_\sigma(k+p)]^2} + \frac{(1-f_{k'})}{[\epsilon-\epsilon_{k_\sigma}+\omega-i\delta_\sigma(k+p)]^2} + \pi^2\delta(\epsilon-\xi_{k_\sigma}+\omega)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{i2Z^2g_0^2}{v_F}\int\frac{d\Omega}{(2\pi)^2}\int_0^\infty\frac{dp}{2\pi}\int_{-\infty}^\infty\frac{d\epsilon}{2\pi}\int_0^\infty\frac{dz}{\pi}\mathrm{Im}D_{l/G}(p,z)\left(\frac{1}{z-\omega+i\delta}-\frac{1}{z+\omega-i\delta}\right) \\
&\times\left[\frac{f_{k'}}{[\epsilon-\xi_{k_\sigma}+\omega+i\delta_\sigma(k+p)]^2} + \frac{(1-f_{k'})}{[\epsilon-\xi_{k_\sigma}+\omega-i\delta_\sigma(k+p)]^2} + \pi^2\delta(\epsilon-\xi_{k_\sigma}+\omega)\right]
\end{aligned}
\tag{D.80}
$$

As before, we will first solve the ϵ integral. Recalling the Sokhotski-Plemelj formula, $\lim_{\delta\to0}\frac{1}{x-x_0\pm i\delta} = P\frac{1}{x-x_0}\mp i\pi\delta(x-x_0)$. We will set $\epsilon\to0$, consider

z small, and take the integral over $d\omega$. This integral then reduces to

$$\lim_{\delta\to 0}\int \frac{d\omega}{2\pi}[\frac{f_{k'}}{[\xi_{k_\sigma}+\omega+i\delta_\sigma(k+p)]^2}\left(\frac{1}{z-\omega+i\delta}\right)-\frac{(1-f_{k'})}{[\xi_{k_\sigma}+\omega-i\delta_\sigma(k+p)]^2}\left(\frac{1}{z+\omega-i\delta}\right)$$
$$+\pi^2\delta(\xi_{k_\sigma}+\omega)\left(\frac{1}{z-\omega+i\delta}-\frac{1}{z+\omega-i\delta}\right)]$$

$$=i\int\frac{d\omega}{2}\left[\frac{f_{k'}}{[\xi_{k_\sigma}+\omega]^2}(-\delta(z-\omega))+\frac{(1-f_{k'})}{[\xi_{k_\sigma}+\omega]^2}(-\delta(z+\omega))+\pi^2\delta(\xi_{k_\sigma}+\omega)\left(\delta(z+\omega)-\delta(z-\omega)\right)\right]$$

$$=\frac{i}{2}\left[\frac{f_{k'}}{[\xi_{k_\sigma}-z]^2}+\frac{(1-f_{k'})}{[\xi_{k_\sigma}+z]^2}\right]+\frac{i\pi^2}{2}\left(\delta(z-\epsilon_{k_\sigma})-\delta(z+\epsilon_{k_\sigma})\right) \tag{D.81}$$

It is important to note here that the principal value terms become negligible in the bosonic expansion because we are considering z small. Therefore, we can set $\omega=0$, $|k_\sigma|=k_F$, which allows the integral to become

$$\Gamma^{(1)}_{\sigma\sigma' G}=-\frac{2Z^2g_0^2}{v_F}\int\frac{d\Omega}{(2\pi)^2}\int_0^\infty\frac{dp}{2\pi}\int_0^\infty\frac{dz}{2\pi}\mathrm{Im}D_{l/G}(p,z)\times$$
$$\left[\frac{f_{k'}}{(\xi_{k_F}-z)^2}+\frac{(1-f_{k'})}{(\xi_{k_F}+z)^2}+\pi^2\left(\delta(z-\epsilon_{k_F})-\delta(z+\epsilon_{k_F})\right)\right] \tag{D.82}$$

I will now proceed by first computing the imaginary bosonic term for the Goldstone mode. Setting $C=\frac{\Delta N(0)v_F}{2}$, we have

$$\begin{aligned}
\mathrm{Im}\left[\frac{C\Omega_0}{(\omega+i\delta)^2-\Omega_0^2}\right]&=\mathrm{Im}\left[\frac{C\Omega_0}{\omega^2+2i\delta\omega-\delta^2-\Omega_0^2}\right]\\
&=\lim_{\delta\to0}\mathrm{Im}\left[\frac{C\Omega_0(\omega^2-\Omega_0^2-\delta^2-2i\delta\omega)}{(\omega^2-\Omega_0^2-\delta^2)^2+(2\delta\omega)^2}\right]\\
&=\lim_{\delta\to0}\frac{-2C\Omega_0\delta\omega}{(\omega^2-\Omega_0^2-\delta^2)^2+(2\delta\omega)^2}\\
&=C\begin{cases}0 & \omega^2\neq\Omega_0^2\\ \infty & \omega^2=\Omega_0^2\end{cases}\\
&=C\delta(\omega^2-\Omega_0^2)\\
&=C\delta(|\omega|-\Omega_0)
\end{aligned} \tag{D.83}$$

Plugging this expression back into the above integral, setting $\Omega_0 = Dp^2+g$, and taking $\phi \to 0$ we come to the expression

$$\Gamma^{(1)}_{\sigma\sigma' G} = Z^2 g_0^2 \Delta N(0) \int \frac{d\Omega}{(2\pi)^2} \int_0^\infty \frac{dp}{2\pi} \int_0^\infty \frac{dz}{2} \delta(|z|-\Omega_0) \times \left[\frac{f_{k'}}{(\xi_{k_F}-z)^2} + \frac{(1-f_{k'})}{(\xi_{k_F}+z)^2} + \pi^2 \left(\delta(z-\epsilon_{k_F}) - \delta(z+\epsilon_{k_F})\right) \right]$$

$$\Gamma^{(1)}_{\sigma\sigma' G} = \frac{Z^2 g_0^2 \Delta N(0)}{2} \int_{-1}^{1} \frac{dx}{(2\pi)^2} \int_0^\infty \frac{dp}{2\pi} \times \left[\frac{f_{k'}}{(\xi_{k_F}-(Dp^2+g))^2} + \frac{(1-f_{k'})}{(\xi_{k_F}+(Dp^2+g))^2} + \pi^2 \left[\delta(Dp^2+g-\epsilon_{k_F}) - \delta(Dp^2+g+\epsilon_{k_F})\right] \right] \tag{D.84}$$

I will now proceed to compute the momentum and space integrals over the respective heaviside step functions while setting $R = \frac{Z^2 g_0^2 \Delta N(0)}{(2\pi)^3}$. recall that $\xi_{k_F} = v_F(|k+p|-k_F)$. As before we will consider $p \approx k_F + px$. Setting the gapped term to $g = 0$ and reimplementing the cutoff p_C where $k_B T_c << p_c << \epsilon_F = k_B T_F$ the vertex becomes

$$\Gamma^{(1)}_{\sigma\sigma' G} = R \int_{-1}^{1} dx \int_0^\infty dp \left[\frac{f_{k'}}{(\xi_{k_F}-Dp^2)^2} + \frac{(1-f_{k'})}{(\xi_{k_F}+Dp^2)^2} \right] \tag{D.85}$$

We will reintroduce the constant R later, but for now, setting $\Omega_0 = Dp^2$, we have,

$$\Gamma = -g^3 \int \frac{d^2p}{(2\pi)^2} [\frac{f_{k'} + (\xi_{k_F} - k_0 - \Omega_0)\delta(\xi_{k_F})}{(\xi_{k_F} - k_0 - \Omega_0)(\xi_{k_F} - k_0 - \Omega_0 + q\cdot v_k + q \cdot v_p)} + \frac{(1-f_{k'}) - (\xi_{k_F} - k_0 - \Omega_0)\delta(\xi_{k_F})}{(\xi_{k_F} - k_0 + \Omega_0)(\xi_{k_F} - k_0 + \Omega_0 + q\cdot v_k + q \cdot v_p)}] \tag{D.86}$$

$$\begin{aligned} &= -g^3 \int \frac{d^2p}{(2\pi)^2} \frac{f_{k'}}{(\xi_{k_F} - k_0 - \Omega_0)(\xi_{k_F} - k_0 - \Omega_0 + q\cdot v_k + q \cdot v_p)} \\ &- g^3 \int \frac{d^2p}{(2\pi)^2} \frac{(\xi_{k_F} - k_0 - \Omega_0)\delta(\xi_{k_F})}{(\xi_{k_F} - k_0 - \Omega_0)(\xi_{k_F} - k_0 - \Omega_0 + q\cdot v_k + q \cdot v_p)} \\ &- g^3 \int \frac{d^2p}{(2\pi)^2} \frac{(1-f_{k'})}{(\xi_{k_F} - k_0 + \Omega_0)(\xi_{k_F} - k_0 + \Omega_0 + q\cdot v_k + q \cdot v_p)} \\ &+ g^3 \int \frac{d^2p}{(2\pi)^2} \frac{(\xi_{k_F} - k_0 - \Omega_0)\delta(\xi_{k_F})}{v_F(\xi_{k_F} - k_0 + \Omega_0)(\xi_{k_F} - k_0 + \Omega_0 + q\cdot v_k + q \cdot v_p)} \\ &\quad \text{take } q \to 0,\ k_0 \to 0 \\ &= \Gamma^{(1)} + \Gamma^{(2)} \end{aligned} \tag{D.87}$$

where in those limits the δ terms become 0. We will now only consider the first term of this sequence.

$$\begin{aligned}
\Gamma^{(1)} &= -g^3 \int \frac{d^2p}{(2\pi)^2} \frac{f_{k'}}{(\xi_{k_F} - \Omega_0)^2} \\
&= -g^3 \int \frac{d^2p}{(2\pi)^2} \frac{f_{k'}}{(\epsilon_{k_F} - \mu - \Omega_0)^2} \\
&= -g^3 \int \frac{d^2p}{(2\pi)^2} \frac{f_{k'}}{(\frac{|k+p|^2}{2m} - \frac{k_F^2}{2m} - \Omega_0)^2} \\
&= -g^3 \int \frac{d^2p}{(2\pi)^2} \frac{f_{k'}}{(\frac{k^2}{2m} + \frac{p^2}{2m} + \frac{kpx}{m} - \frac{k_F^2}{2m} - \Omega_0)^2} \\
&= -g^3 \int_0^{p_C} \frac{p^2 dp}{2\pi} \int_{-1}^{1} \frac{dx}{2\pi} \frac{f_{k'}}{(\frac{k^2}{2m} + \frac{p^2}{2m} + \frac{kpx}{m} - \frac{k_F^2}{2m} - \Omega_0)^2} \\
&= -\frac{g^3}{(2\pi)^2} \int_0^{p_C} p^2 dp \int_{-1}^{1} dx \frac{f_{k'}}{(-\frac{k^2}{2m} - \frac{p^2}{2m} - \frac{kpx}{m} + \frac{k_F^2}{2m} + \Omega_0)^2} \\
&= \frac{g^3 m}{(2\pi)^2 k_F} \int_0^{p_C} pdp \left[\frac{f_{k'}}{\frac{k_F p}{m} - \frac{p^2}{2m} + \Omega_0} - \frac{f_{k'}}{-\frac{k^2}{2m} - \frac{p^2}{2m} - \frac{kp}{m} + \frac{k_F^2}{2m} + \Omega_0} \right]
\end{aligned} \tag{D.88}$$

where we take $k \approx k_F$ in the first term and pull a -1 out

$$= \frac{g^3 m}{(2\pi)^2 k_F} \int_0^{p_C} pdp \left[\frac{f_{k'}}{\frac{k_F p}{m} - \frac{p^2}{2m} + \Omega_0} - \frac{f_{k'}}{\frac{1}{2m}\left(k_F - (k+p)\right)^2 + \Omega_0} \right]$$

$$= \frac{g^3 m}{(2\pi)^2 k_F} \int_0^{p_C} pdp \frac{1}{\frac{k_F p}{m} - \frac{p^2}{2m} + \Omega_0} - \frac{g^3 m}{(2\pi)^2 k_F} \int_0^{p_C} pdp \frac{1}{\Omega_0}$$

where the $(k_F - (k+p))^2$ vanishes as we are considering the total momentum $k + p \approx k_F$ for the second term containing $f_{k'}$

$$= \frac{g^3 m}{(2\pi)^2 k_F} \int_0^{p_C} pdp \frac{1}{\frac{k_F p}{m} - \frac{p^2}{2m} + \Omega_0} - \frac{g^3 (2m) k_F}{(2\pi)^2 \Omega_0}$$

$$= -\frac{g^3 N(E_f)}{\Omega_0} + \frac{g^3 m}{(2\pi)^2 k_F} \int_0^{p_C} pdp \frac{1}{\frac{k_F p}{m} - \frac{p^2}{2m} + \Omega_0} \tag{D.89}$$

Now, we can calculate the other integral term in a similar way. The second term in the expression becomes

$$\Gamma^{(2)} = -g^3 \int \frac{d^2 p}{(2\pi)^2} \frac{(1 - f_{k'})}{(\xi_{k_F} + \Omega_0)^2}$$

$$= -g^3 \int \frac{d^2 p}{(2\pi)^2} \frac{(1 - f_{k'})}{(\epsilon_{k_F} - \mu + \Omega_0)^2}$$

$$= -g^3 \int \frac{d^2 p}{(2\pi)^2} \frac{(1 - f_{k'})}{(\frac{|k+p|^2}{2m} - \frac{k_F^2}{2m} + \Omega_0)^2}$$

$$= -g^3 \int \frac{d^2 p}{(2\pi)^2} \frac{(1 - f_{k'})}{(\frac{k^2}{2m} + \frac{p^2}{2m} + \frac{kpx}{m} - \frac{k_F^2}{2m} + \Omega_0)^2}$$

$$= -g^3 \int_0^{p_C} \frac{p^2 dp}{2\pi} \int_{-1}^{1} \frac{dx}{2\pi} \frac{(1-f_{k'})}{(\frac{k^2}{2m} + \frac{p^2}{2m} + \frac{kpx}{m} - \frac{k_F^2}{2m} + \Omega_0)^2}$$

$$= -\frac{g^3}{(2\pi)^2} \int_0^{p_C} p^2 dp \int_{-1}^{1} dx \frac{(1-f_{k'})}{(\frac{k^2}{2m} + \frac{p^2}{2m} + \frac{kpx}{m} - \frac{k_F^2}{2m} + \Omega_0)^2}$$

$$= -\frac{g^3}{(2\pi)^2} \int_0^{p_C} p^2 dp \int_{-1}^{1} dx \frac{(1-f_{k'})}{(-\frac{k^2}{2m} - \frac{p^2}{2m} - \frac{kpx}{m} + \frac{k_F^2}{2m} - \Omega_0)^2}$$

$$= -\frac{g^3 m}{(2\pi)^2 k_F} \int_0^{p_C} pdp \left[\frac{(1-f_{k'})}{-\frac{k^2}{2m} - \frac{p^2}{2m} - \frac{kp}{m} + \frac{k_F^2}{2m} - \Omega_0} - \frac{(1-f_{k'})}{-\frac{k^2}{2m} - \frac{p^2}{2m} + \frac{kp}{m} + \frac{k_F^2}{2m} - \Omega_0} \right]$$

$$= \frac{g^3 m}{(2\pi)^2 k_F} \int_0^{p_C} pdp \left[\frac{(1-f_{k'})}{-\frac{k^2}{2m} - \frac{p^2}{2m} + \frac{kp}{m} + \frac{k_F^2}{2m} - \Omega_0} - \frac{(1-f_{k'})}{-\frac{k^2}{2m} - \frac{p^2}{2m} - \frac{kp}{m} + \frac{k_F^2}{2m} - \Omega_0} \right]$$

$$= \frac{g^3 m}{(2\pi)^2 k_F} \int_0^{p_C} pdp \left[\frac{(1-f_{k'})}{-\frac{k^2}{2m} - \frac{p^2}{2m} + \frac{kp}{m} + \frac{k_F^2}{2m} - \Omega_0} + \frac{(1-f_{k'})}{\frac{k_F p}{m} + \frac{p^2}{2m} + \Omega_0} \right]$$

where we take $k \approx k_F$ in the second term.

$$= \frac{g^3 m}{(2\pi)^2 k_F} \int_0^{p_C} pdp \left[\frac{-(1-f_{k'})}{\frac{1}{2m}\left((k-p) - k_F\right)^2 + \Omega_0} + \frac{(1-f_{k'})}{\frac{k_F p}{m} + \frac{p^2}{2m} + \Omega_0} \right]$$

$$= \frac{g^3 m}{(2\pi)^2 k_F} \int_0^{p_C} pdp \frac{1}{\Omega_0} + \frac{g^3 m}{(2\pi)^2 k_F} \int_0^{p_C} pdp \frac{1}{\frac{k_F p}{m} + \frac{p^2}{2m} + \Omega_0}$$

where the $((k-p) - k_F)^2$ vanishes as we are considering the the momentum $k - p \approx k_F$ for the first term containing $1 - f_{k'}$.

$$= \frac{g^3 (2m) k_F}{(2\pi)^2 \Omega_0} + \frac{g^3 m}{(2\pi)^2 k_F} \int_0^{p_C} pdp \frac{1}{\frac{k_F p}{m} + \frac{p^2}{2m} + \Omega_0}$$

$$= \frac{g^3 N(E_f)}{\Omega_0} + \frac{g^3 m}{(2\pi)^2 k_F} \int_0^{p_C} pdp \frac{1}{\frac{k_F p}{m} + \frac{p^2}{2m} + \Omega_0} \tag{D.90}$$

Therefore, the vertex becomes

$$\begin{aligned}
\Gamma_{\sigma\sigma';G} &= \Gamma^{(1)} + \Gamma^{(2)} \\
&= \frac{g^3 m}{(2\pi)^2 k_F} \int_0^{p_C} pdp \left[\frac{1}{v_F p + \Omega_0 - \frac{p^2}{2m}} + \frac{1}{v_F p + \Omega_0 + \frac{p^2}{2m}} \right] \\
&= \frac{g^3 m}{(2\pi)^2 k_F} \int_0^{p_C} pdp \left[\frac{1}{v_F p + Dp^2 - \frac{p^2}{2m}} + \frac{1}{v_F p + Dp^2 + \frac{p^2}{2m}} \right] \\
&= \frac{g^3 m}{(2\pi)^2 k_F} \left[\frac{2m \log[2k_F - p + 2Dmp]}{-1 + 2Dm} + \frac{2m \log[2k_F + p + 2Dmp]}{1 + 2Dm} \right] |_0^{p_C} \\
&= \frac{g^3 m(2m)}{(2\pi)^2 k_F} \left[\log[2k_F - p + 2Dmp] + \log[2k_F + p + 2Dmp] \right] |_0^{p_C} \\
&\quad \text{in the small mass limit} \\
&= \frac{g^3 m(2m)}{(2\pi)^2 k_F} \log \left[\frac{(2k_F - p_C + 2Dmp)(2k_F + p_C + 2Dmp)}{4k_F^2} \right] \\
&= \frac{g^3 m(2m)}{(2\pi)^2 k_F} \log \left[1 + \frac{2Dmp_C}{k_F} - \frac{p_C^2}{4k_F^2} + \frac{D^2 m^2 p_C^2}{k_F^2} \right] \\
&= \frac{g^3 m(2m)}{(2\pi)^2 k_F} \log \left[1 + \frac{2Dmp_C}{k_F} \right] \\
&= \frac{g_0^2 N^2(0) Z^2}{4} \log \left[1 + \frac{2Dmp_C}{k_F} \right] \qquad \text{(D.91)}
\end{aligned}$$

with $m \to 0$, $p_C << k_F$, and $D = \frac{v_F \Delta}{k_F^2}$.

D.3 The Higgs Mode's Effect on a p-Wave Pairing Instability

A similar procedure as in the Goldstone case can be carried out while including the Higgs modes's dispersion in the zero-temperature vertex function.

D.3.1 Zero-Temperature Vertex Calculation for the Higgs Mode

I will now proceed by first computing the imaginary bosonic term for the Gapped, or Higgs, mode. Setting $C = \frac{\Delta N(0) v_F}{2}$ and g as the gap, we have

$$\begin{aligned}
\mathrm{Im}\left[\frac{C(\Omega_0+g)}{(\omega+i\delta)^2-(\Omega_0+g)^2}\right] &= \mathrm{Im}\left[\frac{C(\Omega_0+g)}{\omega^2+2i\delta\omega-\delta^2-(\Omega_0+g)^2}\right] \\
&= \lim_{\delta\to 0}\mathrm{Im}\left[\frac{C(\omega^2-(\Omega_0+g)^2-\delta^2-2i\delta\omega)}{(\omega^2-(\Omega_0+g)^2-\delta^2)^2+(2\delta\omega)^2}\right] \\
&= \lim_{\delta\to 0}\frac{-2C(\Omega_0+g)\delta\omega}{(\omega^2-(\Omega_0+g)^2-\delta^2)^2+(2\delta\omega)^2} \\
&= C\begin{cases} 0 & \omega^2 \neq (\Omega_0+g)^2 \\ \infty & \omega^2 = (\Omega_0^2+g)^2 \end{cases} \\
&= C\delta(\omega^2-(\Omega_0+g)^2) \\
&= C\delta(|\omega|-(\Omega_0+g)) \qquad \text{(D.92)}
\end{aligned}$$

Thus for the Higgs we obtain,

$$\Gamma^{(1)}_{\sigma\sigma' H} = \frac{g_0^3 m}{(2\pi)^2 k_F}\int_0^{p_C} pdp\left[\frac{1}{v_F p+\Omega_0-\frac{p^2}{2m}}+\frac{1}{v_F p+\Omega_0+\frac{p^2}{2m}}\right]$$

$$= \frac{g_0^3 m}{(2\pi)^2 k_F}\int_0^{p_C} pdp\left[\frac{1}{v_F p+Dp^2+g-\frac{p^2}{2m}}+\frac{1}{v_F p+Dp^2+g+\frac{p^2}{2m}}\right]$$

$$= \frac{g_0^3 m^2}{(2\pi)^2 k_F}\left[-\frac{\frac{2k_F \arctan\left[\frac{k_F+(-1+2Dm)p}{\sqrt{-k_F^2+2gm(-1+2Dm)}}\right]}{\sqrt{-k_F^2+2gm(-1+2Dm)}}+\log[2gm+p(2k_F-p+2Dmp)]}{-1+2Dm}\right]\Big|_0^{p_c} \qquad \text{(D.93)}$$

$$+\frac{g_0^3 m^2}{(2\pi)^2 k_F}\left[-\frac{\frac{2k_F \arctan\left[\frac{k_F+p+2Dmp}{\sqrt{-k_F^2+2gm(1+2Dm)}}\right]}{\sqrt{-k_F^2+2gm(1+2Dm)}}+\log[2gm+p(2k_F+p+2Dmp)]}{1+2Dm}\right]\Big|_0^{p_c} \qquad \text{(D.94)}$$

$$= \frac{g_0^3 m^2}{(2\pi)^2 k_F}\left[-\frac{2k_F \arctan\left[\frac{k_F+(-1+2Dm)p}{\sqrt{-k_F^2-2gm}}\right]}{\sqrt{-k_F^2-2gm}} + \log[2gm+p(2k_F-p+2Dmp)]\right]|_0^{p_c} \tag{D.95}$$

$$+ \frac{g_0^3 m^2}{(2\pi)^2 k_F}\left[-\frac{2k_F \arctan\left[\frac{k_F+p+2Dmp}{\sqrt{-k_F^2+2gm}}\right]}{\sqrt{-k_F^2+2gm}} + \log[2gm+p(2k_F+p+2Dmp)]\right]|_0^{p_c} \tag{D.96}$$

small mass m limit

$$= \frac{g_0^3 m^2}{(2\pi)^2 k_F}[-\frac{2ik_F\left[\log\left[1-\frac{k_F+(-1+2Dm)p}{\sqrt{k_F^2+2gm}}\right]-\log\left[1+\frac{k_F+(-1+2Dm)p}{\sqrt{k_F^2+2gm}}\right]\right]}{2i\sqrt{k_F^2+2gm}} + \log[2gm+p(2k_F-p+2Dmp)]]\,|_0^{p_c} \tag{D.97}$$

$$+ \frac{g_0^3 m^2}{(2\pi)^2 k_F}\left[-\frac{2ik_F\left[\log\left[1-\frac{k_F+p+2Dmp}{\sqrt{k_F^2-2gm}}\right]-\log\left[1+\frac{k_F+p+2Dmp}{\sqrt{k_F^2-2gm}}\right]\right]}{2i\sqrt{k_F^2-2gm}} + \log[2gm+p(2k_F+p+2Dmp)]\right]|_0^{p_c} \tag{D.98}$$

using $arctan(z) = \frac{i}{2}[\log(1-iz)-\log(1+iz)]$

$$= \frac{g_0^3 m^2}{(2\pi)^2 k_F}\left[-\frac{\log\left[1-\frac{k_F-p+2Dmp}{\sqrt{k_F^2+2gm}}\right]-\log\left[1+\frac{k_F-p+2Dmp}{\sqrt{k_F^2+2gm}}\right]}{\sqrt{1+\frac{2gm}{k_F^2}}} + \log[2gm+p(2k_F-p+2Dmp)]\right]|_0^{p_c} \tag{D.99}$$

$$+ \frac{g_0^3 m^2}{(2\pi)^2 k_F}\left[-\frac{\log\left[1-\frac{k_F+p+2Dmp}{\sqrt{k_F^2-2gm}}\right]-\log\left[1+\frac{k_F+p+2Dmp}{\sqrt{k_F^2-2gm}}\right]}{\sqrt{1-\frac{2gm}{k_F^2}}} + \log[2gm+p(2k_F+p+2Dmp)]\right]|_0^{p_c} \tag{D.100}$$

(D.101)

$$= \frac{g_0^3 m^2}{(2\pi)^2 k_F} \left[-\frac{\log\left[\frac{k_F\sqrt{1+\frac{2gm}{k_F^2}}-k_F+p-2Dmp}{k_F\sqrt{1+\frac{2gm}{k_F^2}}+k_F-p+2Dmp}\right]}{\sqrt{1+\frac{2gm}{k_F^2}}} + \log[2gm+p(2k_F-p+2Dmp)] \right] |_0^{p_c} \tag{D.102}$$

$$+ \frac{g_0^3 m^2}{(2\pi)^2 k_F} \left[-\frac{\log\left[\frac{k_F\sqrt{1-\frac{2gm}{k_F^2}}-k_F-p-2Dmp}{k_F\sqrt{1-\frac{2gm}{k_F^2}}+k_F+p+2Dmp}\right]}{\sqrt{1-\frac{2gm}{k_F^2}}} + \log[2gm+p(2k_F+p+2Dmp)] \right] |_0^{p_c} \tag{D.103}$$

$$= \frac{g_0^3 m^2}{(2\pi)^2 k_F} \left[-\log\left[\frac{k_F\sqrt{1+\frac{2gm}{k_F^2}}-k_F+p-2Dmp}{k_F\sqrt{1+\frac{2gm}{k_F^2}}+k_F-p+2Dmp}\right] + \log[2gm+p(2k_F-p+2Dmp)] \right] |_0^{p_c} \tag{D.104}$$

$$+ \frac{g_0^3 m^2}{(2\pi)^2 k_F} \left[-\log\left[\frac{k_F\sqrt{1-\frac{2gm}{k_F^2}}-k_F-p-2Dmp}{k_F\sqrt{1-\frac{2gm}{k_F^2}}+k_F+p+2Dmp}\right] + \log[2gm+p(2k_F+p+2Dmp)] \right] |_0^{p_c} \tag{D.105}$$

ignore denominator in small mass limit

$$= \frac{g_0^3 m^2}{(2\pi)^2 k_F} \left[\log\left[\frac{k_F p\sqrt{1+\frac{2gm}{k_F^2}}+k_F p-p^2+2Dmp^2}{k_F\sqrt{1+\frac{2gm}{k_F^2}}-k_F+p-2Dmp}\right] + \log[\frac{2gm}{p}+2k_F-p+2Dmp] \right] |_0^{p_c} \tag{D.106}$$

$$+ \frac{g_0^3 m^2}{(2\pi)^2 k_F} \left[\log\left[\frac{k_F p\sqrt{1-\frac{2gm}{k_F^2}}+k_F p+p^2+2Dmp^2}{k_F\sqrt{1-\frac{2gm}{k_F^2}}-k_F-p-2Dmp}\right] + \log[\frac{2gm}{p}+2k_F+p+2Dmp] \right] |_0^{p_c} \tag{D.107}$$

flip log term with the negative factor and pull out p from the second term

$$= \frac{g_0^3 m^2}{(2\pi)^2 k_F}\left[\log\left[\frac{k_F(1+\frac{2gm}{k_F^2})+k_F-p+2Dmp}{\frac{k_F}{p}(1+\frac{2gm}{k_F^2})-\frac{k_F}{p}+1-2Dm}\right]+\log[\frac{2gm}{p}+2k_F-p+2Dmp]\right]|_0^{p_c} \tag{D.108}$$

$$+\frac{g_0^3 m^2}{(2\pi)^2 k_F}\left[\log\left[\frac{k_F(1-\frac{2gm}{k_F^2})+k_F+p+2Dmp}{\frac{k_F}{p}(1-\frac{2gm}{k_F^2})-\frac{k_F}{p}-1-2Dm}\right]+\log[\frac{2gm}{p}+2k_F+p+2Dmp]\right]|_0^{p_c} \tag{D.109}$$

use small mass limit to approximate $(1\pm x)^n \to 1\pm nx \approx 1\pm x$

$$= \frac{g_0^3 m^2}{(2\pi)^2 k_F}\left[\log\left[\frac{2k_F+\frac{2gm}{k_F}-p+2Dmp}{\frac{2gm}{k_F p}+1}\right]+\log[\frac{2gm}{p}+2k_F-p+2Dmp]\right]|_0^{p_c} \tag{D.110}$$

$$+\frac{g_0^3 m^2}{(2\pi)^2 k_F}\left[\log\left[\frac{2k_F-\frac{2gm}{k_F}+p+2Dmp}{-\frac{2gm}{k_F p}-1}\right]+\log[\frac{2gm}{p}+2k_F+p+2Dmp]\right]|_0^{p_c} \tag{D.111}$$

reduce like terms

$$= \frac{g_0^3 m^2}{(2\pi)^2 k_F}\left[\log\left[\frac{2gm}{k_F}+2k_F-p+2Dmp\right]+\log[\frac{2gm}{p}+2k_F-p+2Dmp]\right]|_0^{p_c} \tag{D.112}$$

$$+\frac{g_0^3 m^2}{(2\pi)^2 k_F}\left[\log\left[\frac{2gm}{k_F}-2k_F-p-2Dmp\right]+\log[\frac{2gm}{p}+2k_F+p+2Dmp]\right]|_0^{p_c} \tag{D.113}$$

using the small mass limit for $\dfrac{2gm}{k_F p}$ as $g\to 0$ as $p\to 0$

$$= \frac{g_0^3 m^2}{(2\pi)^2 k_F}\log[\frac{(\frac{2gm}{k_F}+2k_F-p_C+2Dmp_C)(\frac{2gm}{p_C}+2k_F-p_C+2Dmp_C)}{1}\times$$
$$\frac{(\frac{2gm}{k_F}-2k_F-p_C-2Dmp_C)(\frac{2gm}{p_C}+2k_F+p_C+2Dmp_C)}{(\frac{2gm}{k_F}+2k_F)(\frac{2gm}{k_F}-2k_F)(2m+2k_F)^2}] \tag{D.114}$$

$$= \frac{g_0^3 m^2}{(2\pi)^2 k_F} \log[1 + \frac{-96Dgk^2m^2 - 8gkmp - 64Dk^3mp + 8k^2p^2 - 24Dgm^2p^2}{-16k^4 + 16g^2m^2 - \frac{16g^2k^2m^2}{p^2} - \frac{32gk^3m}{p}} + \tag{D.115}$$

$$\frac{-\frac{4g^2m^2p^2}{k^2} - 96D^2k^2m^2p^2 + \frac{4gmp^3}{k} + 16Dkmp^3 - p^4 + 8D^2m^2p^4}{-16k^4 + 16g^2m^2 - \frac{16g^2k^2m^2}{p^2} - \frac{32gk^3m}{p}}] \tag{D.116}$$

expand numerator and denominator and divide the front term to get 1, and remove higher order mass terms > 2

$$= \frac{g_0^3 m^2}{(2\pi)^2 k_F} \log[1 + \frac{-96Dgp_C^2k^4m^2 - 8gk^3mp_C^3 - 64Dk^5mp_C^3 + 8k^4p_C^4 - 24Dk^2gm^2p_C^4}{-16k^6p_C^2 + 16k^2g^2m^2p_C^2 - 16g^2k^4m^2 - 32gk^5mp_C} +$$
$$\frac{-4g^2m^2p_C^4 - 96D^2k^4m^2p_C^4 + 4kgmp_C^5 + 16Dk^3mp_C^5 - k^2p_C^6 + 8k^2D^2m^2p_C^6}{-16k^6p_C^2 + 16k^2g^2m^2p_C^2 - 16g^2k^4m^2 - 32gk^5mp_C}]$$

multiply numerator and denominator by k^2 and p^2 (D.117)

$$= \frac{g_0^3 m^2}{(2\pi)^2 k_F} \log\left[1 + \frac{-96Dgp_C^2k^4m^2 - 8gk^3mp_C^3 - 64Dk^5mp_C^3}{-16k^6p_C^2 + 16k^2g^2m^2p_C^2 - 16g^2k^4m^2 - 32gk^5mp_C}\right] \tag{D.118}$$

where we keep order p_C^0, p_C^1 for $p_C << k_F$ as in the Goldstone case

$$= \frac{g_0^3 m^2}{(2\pi)^2 k_F} \log\left[1 + \frac{12Dgp_C^2k^4m^2 + gk^3mp_C^3 + 8Dk^5mp_C^3}{2k^6p_C^2 + 2k^2g^2m^2(k^2 - p_C^2) + 4gk^5mp_C}\right] \tag{D.119}$$

$$= \frac{g_0^3 m^2}{(2\pi)^2 k_F} \log\left[1 + \frac{8Dk^5mp_C^3 + 12Dgp_C^2k^4m^2 + gk^3mp_C^3}{2k^6p_C^2 + 2k^4g^2m^2 + 4gk^5mp_C}\right] \tag{D.120}$$

$$= \frac{g_0^2 N^2(0) Z^2}{8} \log\left[1 + \frac{8Dk_F^2mp_C^3 + 12D(\omega^\pm)p_C^2k_Fm^2 + (\omega^\pm)mp_C^3}{2k_F^3p_C^2 + 2k_F(\omega^\pm)^2m^2 + 4(\omega^\pm)k_F^2mp_C}\right] \tag{D.121}$$

with $m \to 0$, $p_C << k_F$, $g = \omega^\pm$, and $D = \frac{v_F\Delta}{k_F^2}$.

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