Viscosity Bound Violation in the MTZ Black Hole

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Contents

1 Abstract

Using the AdS/CFT correspondence, it has been shown that the ratio of shear viscosity to entropy density is bounded from below in strongly coupled field theories with a gravity dual. More recently, this bound has been shown to be grossly violated in novel non-Fermi liquids and the unitary Fermi gas in the presence of superfluid fluctuations above T_c. Nevertheless, a holographic approach to such systems which break the lower bound have been strongly reliant on AdS spacetimes with massive gravitons. In this work, we propose a violation of the viscosity over entropy bound in 3+1 dimensional AdS spacetimes that support stable black hole solutions with non-zero scalar field. Such a black hole is shown to be characterized by a novel phase transition at large negative mass, where the underlying thermodynamics agrees with the Larkin-Ovchinnikov-Fulde-Ferrell (LOFF)-like phase seen in the unitary Fermi gas near Tc and the bound is similarly broken. Such a work paves the way for a holographic description of strongly-entangled quantum fluids at high Reynolds number.

2 Introduction

It has been shown that strongly interacting quantum field theories in d dimensions can be described in term of $d+1$ dimensional weakly interacting gravitational systems in anti-de Sitter space. This one-to-one correspondence is known as the anti-de Sitter space/conformal field theory (AdS/CFT) correspondence [1]. In more recent years, the AdS/CFT correspondence has been used to propose a universal lower bound (the KSS bound) on the ratio of shear viscosity to entropy density in strongly coupled field theories, given as [2, 3, 4, 5]:

$$\frac{\eta}{s} \geq \frac{\hbar}{4\pi k_B} \tag{1}$$

Often, natural units are taken such that $\hbar$ and k_B are unity, and Eq 1 takes the form $\frac{\eta}{s} \geq \frac{1}{4\pi}$ In more recent years, however, this bound has been shown to be violated in multiple systems, including holographic solids [6], and the unitary Fermi gas in the presence of superfluid fluctuations [7]. It is known from [8] that cooper pairing in a conformal field theory will be dual to a scalar field in the corresponding Einstein gravity theory. Thus, due to the breaking of the bound in [7] due to a cooper pairing instability, we expect to see a violation of the KSS bound in a black hole surrounded by a non-zero scalar hair.

Typically, in space with vanishing cosmological constant Λ, this scalar hair would be disallowed by the no-hair theorem, which states that properties of black holes are restricted to mass, angular momentum, and charge [9]. However, for $\Lambda < 0$, anti-de Sitter space, scalar hair is well documented and allowable [10].

In this paper, we consider the Martínez, Tronosco, Zanelli (MTZ) black hole, a black hole solution with a minimally coupled scalar field in $3+1$ dimensional anti-de Sitter space. We show that for positive mass and small negative mass, the formation of scalar hair is given by a continuous phase transition, and for large negative mass, the formation of hair is given by a first order, ice-like, phase transition. However, by considering particles coupled to the scalar hair, we determine that this negative mass regime is dis-allowable due to a naked singularity in the scalar hair for sufficiently large negative mass. In Section 8, we plot $\frac{\eta}{s}$ as a function of black hole mass, showing that the KSS bound can be arbitrarily broken in the small negative mass regime before becoming negative and diverging, indicating a superradiant instability of the black hole [11]. This prevents the black hole from reaching large negative masses, where further superradiant instabilities are also found.

3 Background

3.1 Einstein's Field Equation for the Schwarzschild Case

So as to establish a method for finding the exact form of a metric, we will solve for the exact form of the simplest case: the Schwarzschild metric, which describes a static spherically symmetric body in free space of mass M. Thus, we can begin with the following metric ansatz using the $(+---)$ metric convention:

$$ds^2 = \sum_{\mu\nu} g_{\mu\nu} dx^\mu dx^\nu = U(r)dt^2 - V(r)dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{2}$$

We see here that without loss of generality, we have set $g_{22} = g_{33} = 1$ due to the spherical symmetry of the system (note that we are also working with natural units where $c = 1$).

Now we must obtain Einstein's field equation for the Schwarzschild case, which can be done by minimizing the Einstein Hilbert action given as [12]:

$$S = \int \mathcal{L} dV = \frac{1}{2\kappa} \int d^4x R\sqrt{-g} \tag{3}$$

Here $\mathcal{L}$ is the Lagrange density, R is the Ricci scalar, $\kappa = \frac{8\pi G}{c^4}$ is Einstein's gravitational constant, and g is the determinant of the metric tensor. Note that g must be accompanied by a negative sign $(-)$ under the square root to ensure that $\sqrt{-g} \in \mathbb{R}$ since the determinant of the metric with metric signature $(-+++)$ will be negative. To minimize this action, we will consider the variation of S with respect to the metric that vanishes at infinity (more specifically the inverse metric, $g^{\mu\nu}$), namely δS, which by the product rule yields:

$$\delta S = \frac{1}{2\kappa} \int d^4x \ \delta(R\sqrt{-g}) \tag{4}$$

By definition of the Ricci scalar, we have $R = g^{\mu\nu} R_{\mu\nu}$ where $R_{\mu\nu}$ is the Ricci curvature tensor, defined below as the trace of the Riemann curvature tensor over the first and third indices which can be written in terms of the affine connection [12]:

$$R_{\mu\nu} \equiv R^\beta_{\ \mu\beta\nu} = \Gamma^\beta_{\nu\mu,\beta} - \Gamma^\beta_{\beta\mu,\nu} + \Gamma^\beta_{\beta\alpha}\Gamma^\alpha_{\nu\mu} - \Gamma^\beta_{\nu\alpha}\Gamma^\alpha_{\beta\mu} \tag{5}$$

The Christoffel symbols above can be written as [13]:

$$\Gamma^\mu_{\nu\sigma} = \frac{1}{2} g^{\mu\lambda} \{g_{\lambda\nu,\sigma} + g_{\lambda\sigma,\nu} - g_{\nu\sigma,\lambda}\} \tag{6}$$

Note that in the definition of the Ricci tensor above, we have used the notation $\Gamma^\beta_{\nu\sigma,\rho}$, where the comma indicates differentiation of $\Gamma^\beta_{\nu\sigma}$ with respect to the ρ-th component using the normal derivative operator ∂_ρ. Thus, $\Gamma^\beta_{\nu\sigma,\rho} = \partial_\rho \Gamma^\beta_{\nu\sigma}$. We will use these notations interchangeably depending on the context. (Also note that a semicolon, i.e. $\Gamma^\beta_{\nu\sigma;\rho}$ would indicate the covariant derivative).

Thus, we can rewrite Eq 4 using the product rule as:

$$\delta S = \frac{1}{2\kappa} \int d^4x \ (R\delta(\sqrt{-g}) + \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} + \sqrt{-g} \delta g^{\mu\nu} R_{\mu\nu}) \tag{7}$$

To simplify this, we can consider this as as three separate integrals,those being:

$$A_1 = \frac{1}{2\kappa} \int d^4x \ R\delta(\sqrt{-g}) \tag{8}$$

$$A_2 = \frac{1}{2\kappa} \int d^4x \ \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} \tag{9}$$

$$A_3 = \frac{1}{2\kappa} \int d^4x \ \sqrt{-g} \delta g^{\mu\nu} R_{\mu\nu} \tag{10}$$

For A_1, calculating this integral amounts to calculating the variation $\delta\sqrt{-g}$ which, by the chain rule, gives:

$$\delta\sqrt{-g} = -\frac{\delta g}{2\sqrt{-g}} \tag{11}$$

However, since we want all terms to be in terms of variations of the metric, this needs some further manipulation, which in turn requires the following identity for some general matrix M [12]:

$$\ln(\det(M)) = \mathrm{Tr}(\ln(M)) \tag{12}$$

Thus, for $M = g_{\mu\nu}$, we have:

$$\ln(g) = \mathrm{Tr}(\ln(g_{\mu\nu})) \tag{13}$$

Taking the variation of the LHS, (using standard logarithmic differentiation rules) we obtain:

$$\delta\ln(g) = \frac{\delta g}{g} \tag{14}$$

Similarly the RHS becomes:

$$\delta\mathrm{Tr}(\ln(g_{\mu\nu})) = g^{\mu\nu}\delta g_{\mu\nu} \tag{15}$$

Note that due to the fact that $g_{\mu\nu}$ is diagonal, by Einstein summation notation, we have that $\mathrm{Tr}((g_{\mu\nu})^{-1}\delta g_{\mu\nu}) = g^{\mu\nu}\delta g_{\mu\nu}$. In general, however, we can actually move δ inside of the trace since the trace is a sum, and δ is linear. Thus, we can now equate these to results, giving:

$$\frac{\delta g}{g} = g^{\mu\nu}\delta g_{\mu\nu} \tag{16}$$

Therefore, we have te following expansion of δg:

$$\delta g = g g^{\mu\nu}\delta g_{\mu\nu} \tag{17}$$

which we can substitute into Eq 11 above, giving:

$$\delta\sqrt{-g} = -\frac{g g^{\mu\nu}\delta g_{\mu\nu}}{2\sqrt{-g}} \tag{18}$$

However, since we want each term to be in terms of $\delta g^{\mu\nu}$ instead of $\delta g_{\mu\nu}$, we can use the property of the metric tensor that it raises and lowers indices and simplifying $\frac{g}{\sqrt{-g}} = -\sqrt{-g}$ to rewrite Eq 18 as:

$$\delta\sqrt{-g} = -\frac{\sqrt{-g} g^{\mu\nu} g_{\mu\lambda} g_{\nu\sigma}\delta g^{\lambda\sigma}}{2} \tag{19}$$

Using this same property, we can simplify again, and re-index dummy indices to give:

$$\delta\sqrt{-g} = -\frac{\sqrt{-g} g_{\mu\nu}\delta g^{\mu\nu}}{2} \tag{20}$$

Therefore, we can rewrite A_1 in the desired form:

$$A_1 = -\frac{1}{2\kappa}\int d^4x\, R\frac{\sqrt{-g} g_{\mu\nu}\delta g^{\mu\nu}}{2} \tag{21}$$

Now we will rewrite A_2 in the desired for. To do this, we must calculate $\delta R_{\mu\nu}$, which by the Eq 5 above indicates that this largely amounts to calculating $\delta R^{\rho}_{\ \mu\lambda\nu}$

For this we will rewrite the Riemann tensor using the ∂_λ notation since it will be easier to keep track of in sea of Christoffel Symbols:

$$R^{\rho}_{\ \mu\lambda\nu} = \partial_\lambda\Gamma^{\rho}_{\nu\mu} + \Gamma^{\rho}_{\lambda\sigma}\Gamma^{\sigma}_{\nu\mu} - \partial_\nu\Gamma^{\rho}_{\lambda\mu} - \Gamma^{\rho}_{\nu\sigma}\Gamma^{\sigma}_{\lambda\mu} \tag{22}$$

Thus, we can calculate $\delta R^{\rho}_{\ \mu\lambda\nu}$ using the product rule:

$$\delta R^{\rho}{}_{\mu\lambda\nu} = \partial_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) + \Gamma^{\sigma}_{\nu\mu}\delta\Gamma^{\rho}_{\lambda\sigma} + \Gamma^{\rho}_{\lambda\sigma}\delta\Gamma^{\sigma}_{\nu\mu} - \partial_\nu(\delta\Gamma^{\rho}_{\lambda\mu}) - \Gamma^{\sigma}_{\lambda\mu}\delta\Gamma^{\rho}_{\nu\sigma} - \Gamma^{\rho}_{\nu\sigma}\delta\Gamma^{\sigma}_{\lambda\mu} \tag{23}$$

Now, note that since the Christoffel symbol is symmetric about its lower indices, i.e. $\Gamma^{\rho}_{\nu\mu} = \Gamma^{\rho}_{\mu\nu}$, we can write $0 = -\Gamma^{\sigma}_{\lambda\nu}\delta\Gamma^{\rho}_{\sigma\mu} + \Gamma^{\sigma}_{\nu\lambda}\delta\Gamma^{\rho}_{\sigma\mu}$. Adding this to Eq 23, we obtain:

$$\delta R^{\rho}{}_{\mu\lambda\nu} = \partial_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) + \Gamma^{\rho}_{\lambda\sigma}\delta\Gamma^{\sigma}_{\nu\mu} + \Gamma^{\sigma}_{\nu\mu}\delta\Gamma^{\rho}_{\lambda\sigma} - \partial_\nu(\delta\Gamma^{\rho}_{\lambda\mu}) - \Gamma^{\sigma}_{\lambda\mu}\delta\Gamma^{\rho}_{\nu\sigma} - \Gamma^{\rho}_{\nu\sigma}\delta\Gamma^{\sigma}_{\lambda\mu} - \Gamma^{\sigma}_{\lambda\nu}\delta\Gamma^{\rho}_{\sigma\mu} + \Gamma^{\sigma}_{\nu\lambda}\delta\Gamma^{\rho}_{\sigma\mu} \tag{24}$$

Now, let us note the definition of covariant derivative of some general tensor $T^{\mu_1\mu_2\ldots\mu_j}{}_{\nu_1\nu_2\ldots\nu_k}$, which is given as [13]:

$$\nabla_\lambda T^{\mu_1\mu_2\ldots\mu_j}{}_{\nu_1\nu_2\ldots\nu_k} = \partial_\lambda T^{\mu_1\mu_2\ldots\mu_j}{}_{\nu_1\nu_2\ldots\nu_k} + \sum_{i=1}^{k}\Gamma^{\mu_i}_{\lambda\sigma}T^{\mu_1\ldots\sigma\ldots\mu_j}{}_{\nu_1\nu_2\ldots\nu_k} - \sum_{i=1}^{k}\Gamma^{\sigma}_{\lambda\nu_i}T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\ldots\sigma\ldots\nu_j} \tag{25}$$

where in the sums given above in Eq 25, σ replaces the ith μ (as in the first sum) or the ith ν (as in the second sum). Thus, consider the covariant derivative of the variation of $\Gamma^{\rho}_{\nu\mu}$, which itself is a tensor. With respect to the λth basis element, from the definition in Eq 25 above, we have that this becomes:

$$\nabla_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) = \partial_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) + \Gamma^{\rho}_{\lambda\sigma}\delta\Gamma^{\sigma}_{\nu\mu} - \Gamma^{\sigma}_{\lambda\nu}\delta\Gamma^{\rho}_{\sigma\mu} - \Gamma^{\sigma}_{\lambda\mu}\delta\Gamma^{\rho}_{\nu\sigma} \tag{26}$$

Similarly, with respect to the νth basis element, we have

$$\nabla_\nu(\delta\Gamma^{\rho}_{\nu\mu}) = \partial_\nu(\delta\Gamma^{\rho}_{\lambda\mu}) + \Gamma^{\rho}_{\nu\sigma}\delta\Gamma^{\sigma}_{\lambda\mu} - \Gamma^{\sigma}_{\nu\lambda}\delta\Gamma^{\rho}_{\sigma\mu} - \Gamma^{\sigma}_{\nu\mu}\delta\Gamma^{\rho}_{\lambda\sigma} \tag{27}$$

Now, rearranging Eq 24 in a convenient way, we obtain:

$$\delta R^{\rho}{}_{\mu\lambda\nu} = \partial_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) + \Gamma^{\rho}_{\lambda\sigma}\delta\Gamma^{\sigma}_{\nu\mu} - \Gamma^{\sigma}_{\lambda\nu}\delta\Gamma^{\rho}_{\sigma\mu} - \Gamma^{\sigma}_{\lambda\mu}\delta\Gamma^{\rho}_{\nu\sigma} - \partial_\nu(\delta\Gamma^{\rho}_{\lambda\mu}) - \Gamma^{\rho}_{\nu\sigma}\delta\Gamma^{\sigma}_{\lambda\mu} + \Gamma^{\sigma}_{\nu\lambda}\delta\Gamma^{\rho}_{\sigma\mu} + \Gamma^{\sigma}_{\nu\mu}\delta\Gamma^{\rho}_{\lambda\sigma} \tag{28}$$

which using Eq 26 and Eq 27, simplifies nicely to:

$$\delta R^{\rho}{}_{\mu\lambda\nu} = \nabla_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^{\rho}_{\nu\mu}) \tag{29}$$

Thus, taking the trace over the first and third indices, we obtain the variation of the Ricci tensor:

$$\delta R_{\mu\nu} = \nabla_\beta(\delta\Gamma^{\beta}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^{\beta}_{\beta\mu}) \tag{30}$$

Thus, substituting this into Eq 9, we obtain:

$$A = \frac{1}{2\kappa}\int d^4x\,\sqrt{-g}g^{\mu\nu}(\nabla_\beta(\delta\Gamma^{\beta}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^{\beta}_{\beta\mu})) \tag{31}$$

Since the covariant derivative is compatible with the metric by definition, we can rewrite this with some relabeling of dummy indices as:

$$A = \frac{1}{2\kappa}\int d^4x\,\sqrt{-g}\nabla_\lambda(g^{\mu\nu}(\delta\Gamma^{\lambda}_{\nu\mu}) - g^{\mu\lambda}(\delta\Gamma^{\beta}_{\beta\mu})) \tag{32}$$

Now, since $\delta\Gamma^{\lambda}_{\mu\nu}$ is going to be a function of $\delta g_{\mu\nu}$ which we have arbitrarily set to zero at the boundary, by Stokes' Theorem, we have that A is vanishing. (Note that in general we cannot assume that $\delta g_{\mu\nu}$ vanishes at infinity, but in our case, the Schwarzschild case, this is allowable).

Finally, since A_3 is already in the desired form, we are done. Thus, we have that the final form of the action when varied with respect to the metric reads:

$$\delta S = \frac{1}{2\kappa}\int d^4x\,\sqrt{-g}\left(R_{\mu\nu} - \frac{g_{\mu\nu}}{2}R\right)\delta g^{\mu\nu} \tag{33}$$

Minimizing, we obtain Einstein's field equations for the Schwarzschild case:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0 \tag{34}$$

Note that in general there may be a non-zero stress energy tensor, $T_{\mu\nu}$, such as seen later in this work. These terms would appear explicitly in the Lagrange density, and we would have a set of non-homogeneous set of field equations instead of a homogeneous set as in Eq 34.

3.2 Explicit Forms of Einstein's Field Equations for the Schwarzschild Case

Next, our task is to write out the explicit forms of Einstein's field equations for the Schwarzschild case, which will allow us to solve for the Schwarzschild metric. Let us begin by explicitly expanding Eq 5:

$$\begin{aligned} R_{\mu\nu} = &\Gamma^0_{\mu\nu,0} - \Gamma^0_{\mu0,\nu} + \Gamma^\alpha_{\mu\nu}\Gamma^0_{\alpha0} - \Gamma^\alpha_{\mu0}\Gamma^0_{\alpha\nu} \\ &+\Gamma^1_{\mu\nu,1} - \Gamma^1_{\mu1,\nu} + \Gamma^\alpha_{\mu\nu}\Gamma^1_{\alpha1} - \Gamma^\alpha_{\mu1}\Gamma^1_{\alpha\nu} \\ &+\Gamma^2_{\mu\nu,2} - \Gamma^2_{\mu2,\nu} + \Gamma^\alpha_{\mu\nu}\Gamma^2_{\alpha2} - \Gamma^\alpha_{\mu2}\Gamma^2_{\alpha\nu} \\ &+\Gamma^3_{\mu\nu,3} - \Gamma^3_{\mu3,\nu} + \Gamma^\alpha_{\mu\nu}\Gamma^3_{\alpha3} - \Gamma^\alpha_{\mu3}\Gamma^3_{\alpha\nu}. \end{aligned} \tag{35}$$

Now, note for the cases where $\mu \neq \nu$, $R_{\mu\nu} = 0$ (See Appendix, Section 11.2, for explicit calculation of these cases, and Section 11.1 for explicit calculation of these Christoffel Symbols for the general metric ansatz given in Eq 2).

Now we must compute the four cases of $R_{\mu\nu}$ such that $\mu = \nu$. Let us begin with R_{00} (once again, each of these are explicitly calculated in Appendix, Section 11.2):

$$\begin{aligned} R_{00} = &\Gamma^0_{00,0} - \Gamma^0_{00,0} + \Gamma^\alpha_{00}\Gamma^0_{\alpha0} - \Gamma^\alpha_{00}\Gamma^0_{\alpha0} \\ &+\Gamma^1_{00,1} - \Gamma^1_{01,0} + \Gamma^\alpha_{00}\Gamma^1_{\alpha1} - \Gamma^\alpha_{01}\Gamma^1_{\alpha0} \\ &+\Gamma^2_{00,2} - \Gamma^2_{02,0} + \Gamma^\alpha_{00}\Gamma^2_{\alpha2} - \Gamma^\alpha_{02}\Gamma^2_{\alpha0} \\ &+\Gamma^3_{00,3} - \Gamma^3_{03,0} + \Gamma^\alpha_{00}\Gamma^3_{\alpha3} - \Gamma^\alpha_{03}\Gamma^3_{\alpha0} \end{aligned} \tag{36}$$

Thus, compiling our non-vanishing terms, we obtain

$$R_{00} = \Gamma^1_{00,1} - \Gamma^1_{01}\Gamma^1_{10} + \Gamma^1_{00}\Gamma^1_{11} + \Gamma^1_{00}\Gamma^2_{12} + \Gamma^1_{00}\Gamma^3_{13}$$

which explicitly becomes:

$$R_{00} = -\frac{1}{4V^2}\frac{dV}{dr}\frac{dU}{dr} + \frac{1}{2V}\frac{d^2U}{dr^2} - \frac{1}{4UV}\left(\frac{dU}{dr}\right)^2 + \frac{1}{Vr}\frac{dU}{dr} \tag{37}$$

Next we can obtain R_{11}, given as:

$$\begin{aligned} R_{11} = &\Gamma^0_{11,0} - \Gamma^0_{10,1} + \Gamma^\alpha_{11}\Gamma^0_{\alpha0} - \Gamma^\alpha_{10}\Gamma^0_{\alpha1} \\ &+\Gamma^1_{11,1} - \Gamma^1_{11,1} + \Gamma^\alpha_{11}\Gamma^1_{\alpha1} - \Gamma^\alpha_{11}\Gamma^1_{\alpha1} \\ &+\Gamma^2_{11,2} - \Gamma^2_{12,1} + \Gamma^\alpha_{11}\Gamma^2_{\alpha2} - \Gamma^\alpha_{12}\Gamma^2_{\alpha1} \\ &+\Gamma^3_{11,3} - \Gamma^3_{13,1} + \Gamma^\alpha_{11}\Gamma^3_{\alpha3} - \Gamma^\alpha_{13}\Gamma^3_{\alpha1} \end{aligned} \tag{38}$$

Compiling our non-vanishing terms for R_{11} we obtain:

$$R_{11} = -\Gamma^0_{10,1} - \Gamma^0_{10}\Gamma^0_{01} + \Gamma^1_{11}\Gamma^0_{10} + \Gamma^1_{11}\Gamma^2_{12} + \Gamma^1_{11}\Gamma^3_{13}$$

which explicitly becomes:

$$R_{11} = -\frac{1}{2U}\frac{d^2U}{dr^2} + \frac{1}{4U^2}\left(\frac{dU}{dr}\right)^2 + \frac{1}{4UV}\frac{dU}{dr}\frac{dV}{dr} + \frac{1}{Vr}\frac{dV}{dr} \tag{39}$$

Next we will obtain R_{22} which is given as:

$$\begin{aligned} R_{22} = {} & \Gamma^0_{22,0} - \Gamma^0_{20,2} + \Gamma^\alpha_{22}\Gamma^0_{\alpha 0} - \Gamma^\alpha_{20}\Gamma^0_{\alpha 2} \\ & + \Gamma^1_{22,1} - \Gamma^1_{21,2} + \Gamma^\alpha_{22}\Gamma^1_{\alpha 1} - \Gamma^\alpha_{21}\Gamma^1_{\alpha 2} \\ & + \Gamma^2_{22,2} - \Gamma^2_{22,2} + \Gamma^\alpha_{22}\Gamma^2_{\alpha 2} - \Gamma^\alpha_{22}\Gamma^2_{\alpha 2} \\ & + \Gamma^2_{22,3} - \Gamma^3_{23,2} + \Gamma^\alpha_{22}\Gamma^3_{\alpha 3} - \Gamma^\alpha_{23}\Gamma^3_{\alpha 2} \end{aligned} \tag{40}$$

Compiling the non-vanishing terms, we obtain:

$$R_{22} = \Gamma^1_{22}\Gamma^0_{10} + \Gamma^1_{22,1} + \Gamma^1_{22}\Gamma^1_{11} - \Gamma^3_{23,2} - \Gamma^3_{23}\Gamma^3_{32} + \Gamma^1_{22}\Gamma^3_{13}$$

which explicitly becomes:

$$R_{22} = -\frac{r}{2UV}\frac{dU}{dr} + \frac{r}{2V^2}\frac{dV}{dr} + 1 - \frac{1}{V} \tag{41}$$

Now we can obtain the final term, R_{33}:

$$\begin{aligned} R_{33} = {} & \Gamma^0_{33,0} - \Gamma^0_{30,3} + \Gamma^\alpha_{33}\Gamma^0_{\alpha 0} - \Gamma^\alpha_{30}\Gamma^0_{\alpha 3} \\ & + \Gamma^1_{33,1} - \Gamma^1_{31,3} + \Gamma^\alpha_{33}\Gamma^1_{\alpha 1} - \Gamma^\alpha_{31}\Gamma^1_{\alpha 3} \\ & + \Gamma^2_{33,2} - \Gamma^2_{32,3} + \Gamma^\alpha_{33}\Gamma^2_{\alpha 2} - \Gamma^\alpha_{32}\Gamma^2_{\alpha 3} \\ & + \Gamma^3_{33,3} - \Gamma^3_{33,3} + \Gamma^\alpha_{33}\Gamma^3_{\alpha 3} - \Gamma^\alpha_{33}\Gamma^3_{\alpha 3} \end{aligned} \tag{42}$$

Compiling these terms, we obtain

$$\begin{aligned} R_{33} &= \Gamma^1_{33}\Gamma^0_{10} + \Gamma^1_{33,1} - \Gamma^3_{31}\Gamma^1_{33} + \Gamma^1_{33}\Gamma^1_{11} + \Gamma^2_{33,2} + \Gamma^3_{32}\Gamma^2_{33} + \Gamma^1_{33}\Gamma^2_{12} \\ R_{33} &= -\frac{r\sin^2\theta}{2UV} + \frac{r\sin^2\theta}{2V^2}\frac{dV}{dr} + \sin^2\theta - \frac{\sin^2\theta}{V} \\ R_{33} &= \left(-\frac{r}{2UV}\frac{dU}{dr} + \frac{r}{2V^2}\frac{dV}{dr} + 1 - \frac{1}{V}\right)\sin^2\theta \end{aligned} \tag{43}$$

We can rewrite this in terms of R_{22}

$$R_{33} = R_{22}\sin^2\theta \tag{44}$$

Important to solving Einstein's field equation is the Ricci scalar, given as:

$$R = g^{\mu\nu}R_{\mu\nu} \tag{45}$$

which using the Einstein summation convention tells us to sum over all $\mu\nu$. Plugging in the above results, we obtain:

$$R = -\frac{1}{2UV^2}\frac{dV}{dr}\frac{dU}{dr} + \frac{1}{UV}\frac{d^2U}{dr^2} - \frac{1}{2U^2V}\left(\frac{dU}{dr}\right)^2 - \frac{2}{V^2r}\frac{dV}{dr} + \frac{2}{rUV}\frac{dU}{dr} - \frac{2}{r^2} + \frac{2}{r^2V} \tag{46}$$

Steps to obtaining this are given explicitly in the Appendices, Section 11.2, below.

3.3 The Schwarzschild Metric

Now that we have the Ricci curvature tensor and the Ricci scalar, we can plug these into Eq 34, giving:

$$R_{00} - \frac{1}{2}g_{00}R = 0 \tag{47}$$

$$R_{11} - \frac{1}{2}g_{11}R = 0 \tag{48}$$

$$R_{22} - \frac{1}{2}g_{22}R = 0 \tag{49}$$

$$R_{33} - \frac{1}{2}g_{33}R = 0 \tag{50}$$

All steps in the following calculation are performed explicitly in the Appendix Section 11.3 Plugging in the results above to Eq 47 gives:

$$\frac{1}{V^2}\frac{dV}{dr} + \frac{1}{r}\left(1 - \frac{1}{V}\right) = 0 \tag{51}$$

Next, plugging in the results above to Eq 48 gives:

$$-\frac{1}{U}\frac{dU}{dr} + \frac{1g}{r}(V-1) = 0 \tag{52}$$

Next, plugging in the results above to Eq 49 gives:

$$-\frac{1}{U}\frac{dU}{dr} + \frac{1}{V}\frac{dV}{dr} + \frac{r}{2UV}\frac{dU}{dr}\frac{dV}{dr} - \frac{r}{U}\frac{d^2U}{dr^2} + \frac{r}{2U^2}\left(\frac{dU}{dr}\right)^2 = 0 \tag{53}$$

This is the same result as for $\mu\nu = 33$ and thus that equation will be omitted here.

Now that we have a system of equations, we can use these to solve for the components of the metric. Rearranging the result from Eq 51,

$$-\frac{dr}{r} = \frac{dV}{V-1} - \frac{dV}{V} \tag{54}$$

Integrating both sides of this, we get

$$\ln\frac{1}{r} + K = \ln\frac{V-1}{V} \tag{55}$$

Letting $C = e^K$, we get

$$\frac{C}{r} = \frac{V-1}{V} \tag{56}$$

Which when solved for V becomes:

$$V = \frac{1}{1-\frac{C}{r}} \tag{57}$$

We can then plug this into Eq 52, giving:

$$-\frac{1}{U}\frac{dU}{dr} + \frac{1}{r}\left(\frac{\frac{C}{r}}{1-\frac{C}{r}}\right) = 0 \tag{58}$$

Rearranging, we obtain:

$$\left(\frac{1}{r-C} - \frac{1}{r}\right)dr = \frac{dU}{U} \tag{59}$$

Thus, integrating as above, we get:

$$\ln\frac{r-C}{r} = \ln U + A \tag{60}$$

Without loss of generality, here we take $A = 0$, giving:

$$U = \frac{r-C}{r}$$

Which when put in the same form as above gives

$$U = 1 - \frac{C}{r} \tag{61}$$

Plugging these results into Eq 2, we obtain:

$$ds^2 = (1-\frac{C}{r})dt^2 - \left(\frac{1}{1-\frac{C}{r}}\right)dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2 \tag{62}$$

Connecting this with Newtonian Gravity using c as the escape velocity, we find that $C = \frac{2GM}{c^2}$, giving the final form the Schwarzschild metric:

$$ds^2 = (1-\frac{2GM}{c^2 r})dt^2 - \left(\frac{1}{1-\frac{2GM}{c^2 r}}\right)dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2 \tag{63}$$

4 The MTZ Black Hole

4.1 Einstein's Field Equation for a Black Hole with Scalar Hair

We will now consider a new metric ansatz for a black hole with scalar hair, given as [14]:

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + a^2(r)d\Omega^2 \tag{64}$$

where Ω is the solid angle. Note that since we are dealing with anti de-Sitter space, $d\Omega^2 = \frac{1}{1-k\rho^2}d\rho^2 + \rho^2 d\varphi^2$ where $\rho = sin\theta$. Additionally, for the sake of convention, we have taken the metric signature $(-++\,+)$. To obtain an exact metric, we must formulate Einstein's field equations under the new conditions, which include the scalar field. These new conditions yield a new Lagrangian, and thus a new action, given as [15]:

$$S = \int d^4x\sqrt{-g}(\frac{1}{2\kappa}R - \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)) \tag{65}$$

where $\frac{1}{2\kappa}R$ is as in the Lagrangian in the case derived above, $\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$ is the kinetic term of the Lagrangian contributed by the scalar hair and $V(\phi)$ is the potential contributed by the scalar hair. Here, we include the cosmological constant, $\Lambda = -\frac{6}{\kappa l^2}$ (l is the length of the AdS space), in the potential by $V(0) = \Lambda$. Additionally, by definition of covariant derivative, since ϕ is a scalar field, the covariant derivative is equivalent to the partial derivative and the second term of Eq 65 becomes $\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. Thus, we can proceed as in the Schwarzschild case above by minimizing the action as follows:

$$\frac{\delta S}{\delta g^{\mu\nu}} = \frac{\delta}{\delta g^{\mu\nu}}\sqrt{-g}\left[\frac{1}{2\kappa}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] = 0 \tag{66}$$

We can now expand the derivative on left hand side of the equation term by term. We can set:

$$A_1 = \frac{\delta}{\delta g^{\mu\nu}}\frac{1}{2\kappa}\sqrt{-g}R \tag{67}$$

$$A_2 = -\frac{\delta}{\delta g^{\mu\nu}}\frac{\sqrt{-g}}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \tag{68}$$

$$A_3 = -\frac{\delta}{\delta g^{\mu\nu}}\sqrt{-g}V(\phi) \tag{69}$$

So then we have that $\frac{\delta S}{\delta g^{\mu\nu}} = A_1 + A_2 + A_3$. The first term is exactly the same as in the Schwarzschild case, and thus gives the same result:

$$A_1 = \frac{\sqrt{-g}}{2\kappa}R_{\mu\nu} - \frac{\sqrt{-g}}{4\kappa}g_{\mu\nu}R \tag{70}$$

Proceeding with the second term by the product rule, we obtain:

$$\frac{\delta}{\delta g^{\mu\nu}}\left[\frac{\sqrt{-g}}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] = -\frac{\sqrt{-g}}{4}g_{\mu\nu}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{\sqrt{-g}}{2}\partial_\mu\phi\partial_\nu\phi \tag{71}$$

where we have used the variation of $\sqrt{-g}$ from the Schwarzschild case above. Now, using the fact that $g_{\mu\nu}g^{\mu\nu} = \delta^\mu_\mu = 4$ (note that this is due to the Einstein summation convention where $\delta^\mu_\mu = \sum_{i=0}^{n-1}\delta^\mu_\mu = n$, where in our case, $n = 4$), Eq 71 simplifies to:

$$\frac{\delta}{\delta g^{\mu\nu}}\left[\frac{\sqrt{-g}}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] = -\sqrt{-g}\partial_\mu\phi\partial_\nu\phi + \frac{\sqrt{-g}}{2}\partial_\mu\phi\partial_\nu\phi = -\frac{\sqrt{-g}}{2}\partial_\mu\phi\partial_\nu\phi \tag{72}$$

Thus, we have that:

$$A_2 = \frac{\sqrt{-g}}{2}\partial_\mu\phi\partial_\nu\phi \tag{73}$$

Finally, the third term, again by the product rule becomes:

$$A_3 = -\frac{d}{dg^{\mu\nu}}[\sqrt{-g}V(\phi)] = \frac{\sqrt{-g}}{2}g_{\mu\nu}V(\phi) - \sqrt{-g}\frac{dV(\phi)}{dg^{\mu\nu}} \tag{74}$$

Here we see that since the potential is dependent on ϕ only, we can say:

$$A_3 = \frac{\sqrt{-g}}{2}g_{\mu\nu}V(\phi) \tag{75}$$

Thus, combining these terms, we obtain:

$$\frac{\sqrt{-g}}{2\kappa}R_{\mu\nu} - \frac{\sqrt{-g}}{4\kappa}g_{\mu\nu}R + \frac{\sqrt{-g}}{2}\partial_\mu\phi\partial_\nu\phi + \frac{\sqrt{-g}}{2}g_{\mu\nu}V(\phi) = 0 \tag{76}$$

Canceling $\sqrt{-g}$ and rearranging, we obtain Einstein's field equation:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa\Big[-\partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}V(\phi)\Big] \tag{77}$$

Eq 77 can be simplified by multiplying on the left by $g^{\mu\nu}$ and solving for R in the following steps:

$$g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}g^{\mu\nu}g_{\mu\nu}R + \kappa g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \kappa g^{\mu\nu}g_{\mu\nu}V(\phi) = 0 \tag{78}$$

By using the value of $g^{\mu\nu}g_{\mu\nu}$ from above, and the definition of the Ricci scalar from Eq 45, this simplifies to:

$$R = \kappa g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + 4\kappa V(\phi) \tag{79}$$

Thus, we can substitute this into the above equation, giving:

$$R_{\mu\nu} - \frac{\kappa}{2}g_{\mu\nu}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 2\kappa g_{\mu\nu}V(\phi) + \kappa\partial_\mu\phi\partial_\nu\phi + \kappa g_{\mu\nu}V(\phi) = 0 \tag{80}$$

Simplifying in the same fashion as above, we have the final form of Einstein's field equation for the hairy black hole (as seen in [14]):

$$R_{\mu\nu} - \kappa(\partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}V(\phi)) = 0 \tag{81}$$

Now, as above, to solve for the exact metric of the black hole, we must solve Einstein's equation. To do this, we must first calculate the Christoffel symbols, and then the Ricci tensors (explicit calculations of the Christoffel symbols and the Ricci tensors in more detail, along with the Ricci scalar for good measure, can be found in).

4.2 Ricci Curvature Tensor Components and Explicit Einstein Field Equations for the Hairy Black Hole

As a reminder, the definition of the Ricci curvature tensor is given as:

$$R_{\mu\nu} = \Gamma^0_{\mu\nu,0} - \Gamma^0_{\mu 0,\nu} + \Gamma^\alpha_{\mu\nu}\Gamma^0_{\alpha 0} - \Gamma^\alpha_{\mu 0}\Gamma^0_{\alpha\nu} \tag{82}$$
$$+\Gamma^1_{\mu\nu,1} - \Gamma^1_{\mu 1,\nu} + \Gamma^\alpha_{\mu\nu}\Gamma^1_{\alpha 1} - \Gamma^\alpha_{\mu 1}\Gamma^1_{\alpha\nu}$$
$$+\Gamma^2_{\mu\nu,2} - \Gamma^2_{\mu 2,\nu} + \Gamma^\alpha_{\mu\nu}\Gamma^2_{\alpha 2} - \Gamma^\alpha_{\mu 2}\Gamma^2_{\alpha\nu}$$
$$+\Gamma^3_{\mu\nu,3} - \Gamma^3_{\mu 3,\nu} + \Gamma^\alpha_{\mu\nu}\Gamma^3_{\alpha 3} - \Gamma^\alpha_{\mu 3}\Gamma^3_{\alpha\nu}.$$

First, let us consider R_{0i}, which is thus given as:

$$R_{0i} = \Gamma^0_{0i,0} - \Gamma^0_{00,i} + \Gamma^\alpha_{0i}\Gamma^0_{\alpha 0} - \Gamma^\alpha_{00}\Gamma^0_{\alpha i} \tag{83}$$
$$+\Gamma^1_{0i,1} - \Gamma^1_{01,i} + \Gamma^\alpha_{0i}\Gamma^1_{\alpha 1} - \Gamma^\alpha_{01}\Gamma^1_{\alpha i}$$
$$+\Gamma^2_{0i,2} - \Gamma^2_{02,i} + \Gamma^\alpha_{0i}\Gamma^2_{\alpha 2} - \Gamma^\alpha_{02}\Gamma^2_{\alpha i}$$
$$+\Gamma^2_{0i,3} - \Gamma^3_{03,i} + \Gamma^\alpha_{0i}\Gamma^3_{\alpha 3} - \Gamma^\alpha_{03}\Gamma^3_{\alpha i}$$

Since all of the above terms are vanishing, we have that:

$$R_{0i} = 0 \tag{84}$$

as was for the Schwarzschild case.
Next, we will consider R_{ij} for $i \neq j$:

$$R_{ij} = \Gamma^0_{ij,0} - \Gamma^0_{i0,j} + \Gamma^\alpha_{ij}\Gamma^0_{\alpha 0} - \Gamma^\alpha_{i0}\Gamma^0_{\alpha j} \tag{85}$$
$$+\Gamma^1_{ij,1} - \Gamma^1_{i1,j} + \Gamma^\alpha_{ij}\Gamma^1_{\alpha 1} - \Gamma^\alpha_{i1}\Gamma^1_{\alpha j}$$
$$+\Gamma^2_{ij,2} - \Gamma^2_{i2,j} + \Gamma^\alpha_{ij}\Gamma^2_{\alpha 2} - \Gamma^\alpha_{i2}\Gamma^2_{\alpha j}$$
$$+\Gamma^3_{ij,3} - \Gamma^3_{i3,j} + \Gamma^\alpha_{ij}\Gamma^3_{\alpha 3} - \Gamma^\alpha_{i3}\Gamma^3_{\alpha j}.$$

Compiling these terms we see that this component is vanishing:

$$R_{ij} = 0 \tag{86}$$

Next we will consider R_{00}, given as:

$$R_{00} = \Gamma^0_{00,0} - \Gamma^0_{00,0} + \Gamma^\alpha_{00}\Gamma^0_{\alpha 0} - \Gamma^\alpha_{00}\Gamma^0_{\alpha 0} \tag{87}$$
$$+\Gamma^1_{00,1} - \Gamma^1_{01,0} + \Gamma^\alpha_{00}\Gamma^1_{\alpha 1} - \Gamma^\alpha_{01}\Gamma^1_{\alpha 0}$$
$$+\Gamma^2_{00,2} - \Gamma^2_{02,0} + \Gamma^\alpha_{00}\Gamma^2_{\alpha 2} - \Gamma^\alpha_{02}\Gamma^2_{\alpha 0}$$
$$+\Gamma^3_{00,3} - \Gamma^3_{03,0} + \Gamma^\alpha_{00}\Gamma^3_{\alpha 3} - \Gamma^\alpha_{03}\Gamma^3_{\alpha 0}$$

Thus, compiling these, we obtain:

$$R_{00} = -\frac{1}{2}f''(r)f(r) - \frac{f'(r)f(r)a'(r)}{a(r)} \tag{88}$$

Next, consider R_{11}, given as:

$$\begin{aligned} R_{11} = {} & \Gamma^{0}_{11,0} - \Gamma^{0}_{10,1} + \Gamma^{\alpha}_{11}\Gamma^{0}_{\alpha 0} - \Gamma^{\alpha}_{10}\Gamma^{0}_{\alpha 1} \\ & +\Gamma^{1}_{11,1} - \Gamma^{1}_{11,1} + \Gamma^{\alpha}_{11}\Gamma^{1}_{\alpha 1} - \Gamma^{\alpha}_{11}\Gamma^{1}_{\alpha 1} \\ & +\Gamma^{2}_{11,2} - \Gamma^{2}_{12,1} + \Gamma^{\alpha}_{11}\Gamma^{2}_{\alpha 2} - \Gamma^{\alpha}_{12}\Gamma^{2}_{\alpha 1} \\ & +\Gamma^{3}_{11,3} - \Gamma^{3}_{13,1} + \Gamma^{\alpha}_{11}\Gamma^{3}_{\alpha 3} - \Gamma^{\alpha}_{13}\Gamma^{3}_{\alpha 1} \end{aligned} \tag{89}$$

Thus compiling these above terms nd simplifying, we obtain:

$$R_{11} = \frac{1}{2}\frac{f''(r)}{f(r)} + 2\frac{a''(r)}{a(r)} + \frac{f'(r)a'(r)}{f(r)a(r)} \tag{90}$$

Next, consider R_{22}, which is give as

$$\begin{aligned} R_{22} = {} & \Gamma^{0}_{22,0} - \Gamma^{0}_{20,2} + \Gamma^{\alpha}_{22}\Gamma^{0}_{\alpha 0} - \Gamma^{\alpha}_{20}\Gamma^{0}_{\alpha 2} \\ & +\Gamma^{1}_{22,1} - \Gamma^{1}_{21,2} + \Gamma^{\alpha}_{22}\Gamma^{1}_{\alpha 1} - \Gamma^{\alpha}_{21}\Gamma^{1}_{\alpha 2} \\ & +\Gamma^{2}_{22,2} - \Gamma^{2}_{22,2} + \Gamma^{\alpha}_{22}\Gamma^{2}_{\alpha 2} - \Gamma^{\alpha}_{22}\Gamma^{2}_{\alpha 2} \\ & +\Gamma^{2}_{22,3} - \Gamma^{3}_{23,2} + \Gamma^{\alpha}_{22}\Gamma^{3}_{\alpha 3} - \Gamma^{\alpha}_{23}\Gamma^{3}_{\alpha 2} \end{aligned} \tag{91}$$

Compiling and simplifying, we obtain:

$$R_{22} = \frac{f'(r)a'(r)a(r) + f(r)(a'(r))^2 + f(r)a(r)a''(r) - k}{1 - k\rho^2} \tag{92}$$

Finally, consider R_{33}, which is given as:

$$\begin{aligned} R_{33} = {} & \Gamma^{0}_{33,0} - \Gamma^{0}_{30,3} + \Gamma^{\alpha}_{33}\Gamma^{0}_{\alpha 0} - \Gamma^{\alpha}_{30}\Gamma^{0}_{\alpha 3} \\ & +\Gamma^{1}_{33,1} - \Gamma^{1}_{31,3} + \Gamma^{\alpha}_{33}\Gamma^{1}_{\alpha 1} - \Gamma^{\alpha}_{31}\Gamma^{1}_{\alpha 3} \\ & +\Gamma^{2}_{33,2} - \Gamma^{2}_{32,3} + \Gamma^{\alpha}_{33}\Gamma^{2}_{\alpha 2} - \Gamma^{\alpha}_{32}\Gamma^{2}_{\alpha 3} \\ & +\Gamma^{3}_{33,3} - \Gamma^{3}_{33,3} + \Gamma^{\alpha}_{33}\Gamma^{3}_{\alpha 3} - \Gamma^{\alpha}_{33}\Gamma^{3}_{\alpha 3} \end{aligned} \tag{93}$$

Thus, compiling and simplifying, we obtain:

$$R_{33} = f(r)a(r)a''(r)\rho^2 - k\rho^2 \tag{94}$$

Thus, using these values for the Ricci tensor elements and Eq 81, we obtain 4 independent differential equations, 3 of which are necessary. They are as follows:

$$R_{00} - \kappa(\partial_0\phi\partial_0\phi + g_{00}V(\phi)) = 0 \tag{95}$$

$$R_{11} - \kappa(\partial_1\phi\partial_1\phi + g_{11}V(\phi)) = 0 \tag{96}$$

$$R_{22} - \kappa(\partial_2\phi\partial_2\phi + g_{22}V(\phi)) = 0 \tag{97}$$

$$R_{33} - \kappa(\partial_3\phi\partial_3\phi + g_{33}V(\phi)) = 0 \tag{98}$$

Thus, substituting the above results into these above equations, we respectively obtain the following differential equations, which can also be found in [14]:

$$f''(r) + 2\frac{a'(r)}{a(r)}f'(r) + 2\kappa V(\phi) = 0 \tag{99}$$

$$\frac{a'(r)}{a(r)}f'(r) + \left(\left(\frac{a'(r)}{a(r)}\right)^2 + \frac{a''(r)}{a(r)}f(r)\right) - \frac{k}{a^2(r)} + \kappa V(\phi) = 0 \tag{100}$$

$$f''(r) + 2\frac{a'(r)}{a(r)}f'(r) + \left(4\frac{a''(r)}{a(r)} + 2\kappa(\phi'(r))^2\right)f(r) + 2\kappa V(\phi) = 0 \tag{101}$$

We want to eliminate the potential term, $V(\phi)$, from the above equations to have them in terms of only elements of the metric. To do this, we can use Eq 99, which gives us the following:

$$\kappa V(\phi) = -\frac{1}{2}f''(r) - \frac{a'(r)}{a(r)}f'(r) \tag{102}$$

Thus, substituting this result into Eq 100 gives:

$$f''(r) - 2\left(\left(\frac{a'(r)}{a(r)}\right)^2 + \frac{a'(r)}{a(r)}\right)f(r) + \frac{2k}{a(r)^2} = 0$$

Substituting into Eq 101 gives:

$$a''(r) - \frac{1}{2}a(r)(\phi'(r))^2 = 0$$

Thus, combining these with the Klein-Gordon equation for our scalar field, ϕ, we have the three final differential equations [14]:

$$f''(r) - 2\left(\left(\frac{a'(r)}{a(r)}\right)^2 + \frac{a'(r)}{a(r)}\right)f(r) + \frac{2k}{a(r)^2} = 0 \tag{103}$$

$$a''(r) - \frac{1}{2}a(r)(\phi'(r))^2 = 0 \tag{104}$$

$$\Box\phi = g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = \frac{dV(\phi)}{d\phi} \tag{105}$$

4.3 The MTZ Solution

A famous solution to differential equations Eq 103-105, the MTZ (Martínez, Troncoso, Zanelli) solution [14, 15], is found under the following change of coordinates [14]: $\frac{dr'}{r'^2} = \frac{dr}{(a(r))^2}$. This solution is given as:

$$ds^2 = \frac{r(r+2G\mu)}{(r+G\mu)^2}\left[-\left(\frac{r^2}{l^2} - \left(1+\frac{G\mu}{r}\right)^2\right)dt^2 + \left(\frac{r^2}{l^2} - \left(1+\frac{G\mu}{r}\right)^2\right)^{-1}dr^2 + r^2d\Omega^2\right] \tag{106}$$

Where above, G is Newton's gravitational constant, and μ is an integration constant related to mass of the black hole:

$$M = \frac{\sigma}{4\pi}\mu \tag{107}$$

Where σ is the area of the spatial 2-section, namely Σ, a 2-dimensional manifold with constant negative curvature k [14, 15]. Since we are working in anti de Sitter space, the topology of the semi-Riemannian manifold is given as: $\mathbb{R}^2 \times \Sigma$ [15]. The potential and scalar field in Eq 105 are given in this solution as:

$$V(\phi) = -\frac{3}{4\pi G l^2}\sinh^2\left(\sqrt{\frac{4\pi G}{3}}\phi\right) \tag{108}$$

$$\phi = \sqrt{\frac{3}{4\pi G}}\arctan\left(\frac{G\mu}{r + G\mu}\right) \tag{109}$$

However, it is useful to consider the following conformal transformation [15]:

$$\hat{g}_{\mu\nu} = \left(1 - \frac{4\pi G}{3}\hat{\phi}^2\right)^{-1} g_{\mu\nu} \qquad \hat{\phi} = \sqrt{\frac{3}{4\pi G}}\tanh\left(\sqrt{\frac{4\pi G}{3}}\phi\right) \tag{110}$$

This transformation yields the following forms of the metric and scalar hair:

$$d\hat{s}^2 = -\left(\frac{r^2}{l^2} - \left(1 + \frac{G\mu}{r}\right)^2\right)dt^2 + \left(\frac{r^2}{l^2} - \left(1 + \frac{G\mu}{r}\right)^2\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{111}$$

$$\hat{\phi} = \sqrt{\frac{3}{4\pi G}}\frac{G\mu}{r + G\mu} \tag{112}$$

Going forward, we will refer to the metric above as $g_{\mu\nu}$ instead of $\hat{g}_{\mu\nu}$, and similarly the line element $d\hat{s}^2$ as ds^2, and $\hat{\phi}$ as ϕ for ease of notation. Now, by solving:

$$\frac{r^2}{l^2} - \left(1 + \frac{G\mu}{r}\right)^2 = 0 \tag{113}$$

We can obtain the event horizons of the MTZ black hole. Note that since this is a quartic polynomial, we will have four independent solutions, given below as:

$$r_1 = \frac{l}{2}\left(1 + \sqrt{1 + \frac{4G\mu}{l}}\right) \tag{114}$$

$$r_2 = \frac{l}{2}\left(1 - \sqrt{1 + \frac{4G\mu}{l}}\right) \tag{115}$$

$$r_3 = \frac{l}{2}\left(-1 + \sqrt{1 - \frac{4G\mu}{l}}\right) \tag{116}$$

$$r_4 = \frac{l}{2}\left(-1 - \sqrt{1 - \frac{4G\mu}{l}}\right) \tag{117}$$

However, note that r_4 is nonphysical, and thus we will exclude this result. Below is the plot depicting these event horizons, along with the singularity in the scalar hair, Eq 112:

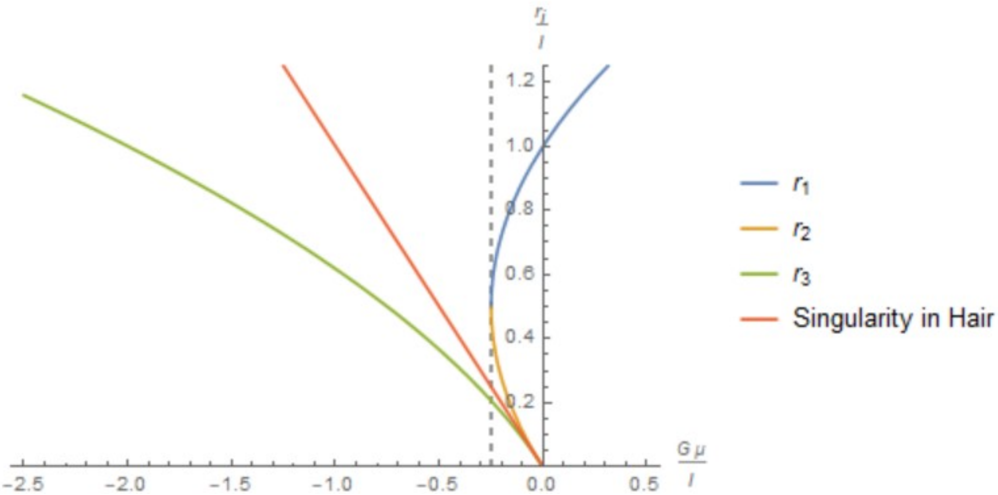

Figure 1: Event Horizons of the MTZ Black Hole as a Function of Re-normalized Black Hole Mass

Note here that these event horizons are displayed for negative mass, for reasons that will be explored in the following section. Also note that at $\frac{G\mu}{l} = -\frac{1}{4}$, the two outermost event horizons (namely those in blue and orange) become imaginary, exposing a singularity in the the scalar hair. Additionally, we can see that for positive mass, r_2 becomes negative and is nonphysical, and similarly, r_3 becomes negative and eventually acquires an imaginary component, and is similarly nonphysical. Thus, we can conclude that r_1 is the only event horizon for $\mu > 0$.

5 Thermodynamics

It is important to note that the solution to the action given in Eq 65 with vanishing scalar hair, the vacuum solution (or topological AdS black hole), is found to have the metric given as [15]:

$$ds^2 = -\left(\frac{\rho^2}{l^2} - 1 - \frac{2G\mu}{\rho}\right)dt^2 + \left(\frac{\rho^2}{l^2} - 1 - \frac{2G\mu}{\rho}\right)^{-1} d\rho^2 + \rho^2 d\Omega^2 \tag{118}$$

Note that the above metric yields only one event horizon that is real-valued for all μ. One can easily see this by solving:

$$\frac{\rho^2}{l^2} - 1 - \frac{2G\mu}{\rho} = 0 \tag{119}$$

One sees that this yields $\rho_1 \in \mathbb{R}\ \forall \mu$, and $\rho_2, \rho_3 \in \mathbb{C}$ such that $\rho_2, \rho_3 \notin \mathbb{R}\ \forall \mu \neq 0$.

It is also important to note that the scalar hair cannot be vanishing for non-zero mass, as is obvious when looking at Eq 112. Thus, for any given $\mu \neq 0$, one can determine whether the topological black hole or the hairy black hole is the stable solution which can be done through calculating the difference in free energy.

5.1 Free Energy

Recall that the Bekenstein-Hawking entropy is dependent on th outermost event horizon. Additionally, recall from Section 4.3 that for $\frac{G\mu}{l} > -s\frac{1}{4}$, the outermost event horizon is given as:

$$r_1 = \frac{l}{2}\left(1 + \sqrt{1 + \frac{4G\mu}{l}}\right) \tag{120}$$

And for $\frac{G\mu}{l} < -\frac{1}{4}$, the outermost event horizon is given as:

$$r_3 = \frac{l}{2}\left(- 1 + \sqrt{1 - \frac{4G\mu}{l}}\right) \tag{121}$$

Thus, solving for μ in each case, we obtain:

$$\mu_1 = \left(\frac{r_1^2}{Gl} - \frac{r_1}{G}\right) = \frac{r_1}{G}\left(\frac{r_1}{l} - 1\right) \tag{122}$$

$$\mu_3 = -\left(\frac{r_3^2}{Gl} + \frac{r_3}{G}\right) = -\frac{r_3}{G}\left(\frac{r_3}{l} + 1\right) \tag{123}$$

Now, recall the definition of the MTZ black hole mass:

$$M = \frac{\sigma}{4\pi}\mu \tag{124}$$

Additionally, we have the following form of the Bekenstein-Hawking entropy [15]:

$$S = \frac{\sigma}{4G}a^2(r_{1,3}) \tag{125}$$

Thus, let us first consider the region $\frac{G\mu}{l} > -\frac{1}{4}$ where r_1 is our outermost event horizon. Note that the mass in this region is given as:

$$M_1 = \frac{\sigma r_1}{4\pi G}\left(\frac{r_1}{l} - 1\right) \tag{126}$$

We can see that since $r_1 \geq 0$, $M_1 \geq 0$ for $r_1 \geq l$ and $M_1 < 0$ for $0 < r_1 < l$, which is the case for $-\frac{1}{4} < \frac{G\mu}{l} < 0$, as was desired.

Now, note that for the MTZ metric, $a^2(r_h) = \dfrac{r_h^3(r_h + 2G\mu)}{(r_h + G\mu)^2}$,(where r_h is the radius of the outermst event horizon at a given mass) thus giving that for r_1, Eq 125 becomes:

$$S_1 = \frac{\sigma}{4G}\frac{r_1^3(r_1 + 2G\mu)}{(r_1 + G\mu)^2} \tag{127}$$

Substituting Eq 122 into Eq 127:

$$\begin{aligned} S_1 &= \frac{\sigma}{4G}\frac{r_1^3(r_1 + 2G\frac{r_1}{G}(\frac{r_1}{l} - 1))}{(r_1 + G\frac{r_1}{G}(\frac{r_1}{l} - 1))^2} \\ &= \frac{\sigma}{4G}\frac{r_1^3(r_1 + \frac{2r_1^2}{l} - 2r_1)}{(r_1 - \frac{r_1^2}{l} - r_1)^2} \end{aligned}$$

Simplifying, we obtain:

$$S_1 \frac{\sigma l^2}{4G}\left(\frac{2r_1}{l} - 1\right) \tag{128}$$

Now, note that the temperature of the MTZ black hole is given as [15]:

$$T = \frac{1}{2\pi l}\left(\frac{2r_h}{l} - 1\right) \tag{129}$$

The critical temperature, T_c, is T at $r_h = l$, which evaluating T accordingly, we can see that [15]:

$$T_c = \frac{1}{2\pi l} \tag{130}$$

From these values, we can obtain the free energy of the MTZ black hole, which is given as:

$$F = M - TS \tag{131}$$

Thus, substituting the above results for the thermodynamics of the MTZ black hole into Eq 131, we obtain:

$$\begin{aligned} F_{MTZ} &= \frac{\sigma r_1}{4\pi G}\left(\frac{r_1}{l} - l\right) - \frac{\sigma l^2}{4G}\left(\frac{2r_1}{l}\right)\left[\frac{1}{2\pi l}\left(\frac{2r_1}{l} - 1\right)\right] \\ &= \frac{\sigma l}{4\pi G}\left(\frac{r_1^2}{l} - \frac{r_1}{l}\right) - \frac{\sigma l}{4\pi G}\left(\frac{2r_1^2}{l^2} - \frac{2r_1}{l} + \frac{1}{2}\right) \\ &= \frac{\sigma l}{4\pi G}\left(-\frac{r_1^2}{l^2} + \frac{2r_1}{l} - 1 - \frac{r_1}{l} + 1 - \frac{1}{2}\right) \end{aligned}$$

Note that in the final step above, we have added and subtracted a 1 to make way for further simplification in terms of T and T_c. Thus, let us calculate two values, namely $(T - T_c)$ and $(T - T_c)^2$ from the above definitions. The first is simple, and is given as:

$$T - T_c = \frac{1}{2\pi l}\left(\frac{2r_{1,3}}{l} - 1\right) - \frac{1}{2\pi l} = \frac{1}{\pi l}\left(\frac{r_{1,3}}{l} - 1\right) \tag{132}$$

Next, we can compute $(T - T_c)^2$ by squaring the above result, which becomes:

$$(T - T_c)^2 = \frac{1}{\pi^2 l^2}\left(\frac{r_{1,3}^2}{l^2} - \frac{2r_{1,3}}{l} + 1\right) \tag{133}$$

Thus, noting Eq 132 and Eq 133, we can rewrite the free energy in its final form:

$$F_{MTZ} = \frac{\sigma l}{8\pi G}(-1 - 2\pi l(T - T_c) - 2\pi^2 l^2 (T - T_c)^2) \tag{134}$$

Thus, let us now perform the same calculation for the topological black hole solution. For the sake of book-keeping, let us rename the radius of the event horizon for this black hole ρ. The temperature for the topological black hole is given as:

$$T = \frac{3}{4\pi l}\left(\frac{\rho}{l} - \frac{l}{3\rho}\right) \tag{135}$$

the critical temperature is the same as that given in Eq 130, and the entropy of the topological is given as:

$$S = \frac{\sigma \rho^2}{4G} \tag{136}$$

and the mass is given as:

$$M = \frac{\sigma\rho}{8\pi G}\left(\frac{\rho^2}{l^2} - 1\right) \tag{137}$$

Thus, given these values, we can obtain the free energy of the topological black hole:

$$\begin{aligned} F_{TBH} &= \frac{\sigma\rho}{8\pi G}\left(\frac{\rho^2}{l^2} - 1\right) - \frac{3}{4\pi l}\left(\frac{\rho}{l} - \frac{l}{3\rho}\right)\left(\frac{\sigma\rho^2}{4G}\right) \\ &= \frac{\sigma\rho^3}{8\pi G l^2} - \frac{\sigma\rho}{8\pi G} - \frac{3\sigma\rho^3}{16\pi G l^2} + \frac{\sigma\rho}{16\pi G} \\ &= -\frac{\sigma\rho}{16\pi G}\left(\frac{\rho^2}{l^2} + 1\right) \end{aligned}$$

Thus, in order to determine energetic favorability, we must also put this in terms of T and T_c. For this case, it is easiest to write ρ in terms of T and T_c and substitute this result into our above expression for F_{TBH}. Thus, consider the expression for T, in which we can substitute T_c, giving:

$$T = \frac{3lT_c}{2\rho}\left(\frac{\rho^2}{l^2} - \frac{1}{3}\right)$$

We can rearrange this as:

$$\rho^2 - \frac{2lT}{3T_c}\rho - \frac{l}{3} = 0$$

Using the quadratic formula, we can find the roots of this quadratic equation:

$$\begin{aligned} \rho &= \frac{l}{3}\frac{T}{T_c} \pm l\sqrt{\frac{T^2}{9T_c^2} + \frac{3}{9}} \\ &= \frac{l}{3}\frac{T}{T_c} \pm l\ \frac{l}{3}\sqrt{\frac{T^2}{T_c^2} + 3} \\ &= \frac{l}{3}\left(\frac{T}{T_c} \pm \sqrt{\frac{T^2}{T_c^2} + 3}\right) \end{aligned}$$

Now, note that since $\sqrt{\frac{T^2}{T_c^2} + 3} > \frac{T}{T_c}$, and since we consider only $\rho \in [0, \infty)$, we can disregard the negative sign and use only the positive sign, giving that:

$$\rho = \frac{l}{3}\left(\frac{T}{T_c} + \sqrt{\frac{T^2}{T_c^2} + 3}\right)$$

Thus, substituting this result into our current form of F_{MTZ}, we obtain:

$$F_{TBH} = -\frac{\sigma l}{16\pi G}\left(\frac{1}{3}\left[\frac{T}{T_c} + \sqrt{\frac{T^2}{T_c^2} + 3}\right]\right)\left(\frac{1}{9}\left[\frac{T}{T_c} + \sqrt{\frac{T^2}{T_c^2} + 3}\right]^2 + 1\right)$$

$$= -\frac{\sigma l}{16\pi G}\left(\frac{1}{3}\right)\left[\frac{2T^3}{9T_c^3} + \frac{2T^2}{9T_c^2}\sqrt{\frac{T^2}{T_c^2} + 3} + \frac{4T}{3T_c} + \frac{2T^2}{9T_c^2}\sqrt{\frac{T^2}{T_c^2} + 3} + \frac{2T^3}{9T_c^3} + \frac{2T}{3T_c} + \frac{4}{3}\sqrt{\frac{T^2}{T_c^2} + 3}\right]$$

Thus, combining like terms, we obtain:

$$F_{TBH} = -\frac{\sigma l}{8\pi G}\left(\frac{1}{3}\right)\left[\frac{2T^3}{9T_c^3} + \frac{T}{T_c} + \left(\frac{2T^2}{9T_c^2} + \frac{2}{3}\right)\sqrt{\frac{T^2}{T_c^2} + 3}\right] \tag{138}$$

Thus, to put this in a form that is more desirable, let us apply Taylor's theorem and expand Eq 138 about T_c to four terms, which gives:

$$F_{TBH} = -\frac{\sigma l}{8\pi G}[1 + 2\pi l(T - T_c) + 2\pi^2 l^2 (T - T_c)^2 + \pi^3 l^3 (T - T_c)^3] \tag{139}$$

Thus, with Eq 134 and Eq 138, we can calculate the difference in free energy in the region $\frac{G\mu}{l} > -\frac{1}{4}$, which is given as the difference:

$$\Delta F_1 = F_{TBH} - F_{MTZ} \tag{140}$$

Which thus gives:

$$\Delta F_1 = -\frac{\sigma\pi^3 l^4}{8\pi G}(T - T_c)^3 \tag{141}$$

Thus, by definition of continuous phase transition, we can see that the formation (or de-formation) of scalar hair around the black hole will be a continuous phase transition in this region. Additionally, we can see for $T > T_c$, $\Delta F_1 < 0$, and thus the MTZ solution in this region is unstable. Namely, the MTZ solution in this region would decay into the vacuum solution, i.e. the hair would be "absorbed" by the black hole, as described in [15]. This region, $T > T_c$ corresponds to $r_1 > l$, or $\frac{G\mu}{l} > 0$, i.e. positive black hole mass. Similarly, for $T < T_c$, $\Delta F_1 > 0$, indicating that the MTZ solution is the stable solution in this region. This region corresponds to $-\frac{1}{4} < \frac{G\mu}{l} < 0$.

Now, since r_1 disappears at $\frac{G\mu}{l} = -\frac{1}{4}$, we must repeat the above calculation using r_3 for $\frac{G\mu}{l} < -\frac{1}{4}$. Since we already have F_{TBH}, this amounts to calculating $F_M TZ$ using r_3 the outermost event horizon in this region.

Thus, we can substitute our value of μ_1 into Eq 124, which becomes:

$$M_3 = -\frac{\sigma r_3}{4\pi G}\left(\frac{r_3}{l} + 1\right) \tag{142}$$

Once again, note that for the MTZ metric, $a^2(r_3) = \dfrac{r_3^3(r_3 + 2G\mu)}{(r_3 + G\mu)^2}$,thus giving that Eq 125 becomes:

$$S = \frac{\sigma}{4G}\frac{r_3^3(r_3 + 2G\mu)}{(r_3 + G\mu)^2} \tag{143}$$

Substituting what we obtained for μ above, the entropy becomes:

$$\begin{aligned} S &= \frac{\sigma}{4G}\frac{r_3^3(r_3 - 2G\frac{r_3}{G}(\frac{r_3}{l} + 1))}{(r_3 - G\frac{r_3}{G}(\frac{r_3}{l} + 1))^2} \\ &= \frac{\sigma}{4G}\frac{r_3^3(r_3 - \frac{2r_3^2}{l} - 2r_3)}{(r_3 - \frac{r_3^2}{l} - r_3)^2} \end{aligned}$$

Which simplifies to:

$$S = -\frac{\sigma l^2}{4G}\left(\frac{2r_3}{l} + 1\right) \tag{144}$$

Again, note that the temperature of the MTZ black hole is given as:

$$T = \frac{1}{2\pi l}\left(\frac{2r_h}{l} - 1\right) \tag{145}$$

And again, the critical temperature is given as:

$$T_c = \frac{1}{2\pi l} \tag{146}$$

From these values, we can obtain the free energy of the MTZ black hole using Eq 131:

$$F_{MTZ} = -\frac{\sigma r_3}{4\pi G}\left(\frac{r_3}{l}+1\right) + \frac{\sigma l^2}{4G}\left[\frac{1}{2\pi l}\left(\frac{2r_3}{l}-l\right)\right]\left(\frac{2r_3}{l}+l\right)$$

thus, distributing we obtain:

$$\begin{aligned}F_{MTZ} &= -\frac{\sigma r_3^2}{4\pi Gl} - \frac{\sigma r_3}{4\pi G} + \frac{\sigma r_3^2}{2\pi Gl} - \frac{\sigma l}{8\pi G}\\ &= \frac{\sigma r_3^2}{4\pi Gl} - \frac{\sigma r_3}{4\pi G} - \frac{\sigma l}{8\pi G}\\ &= \frac{\sigma l}{4\pi G}\left(\frac{r_3^2}{l^2} - \frac{r_3}{l} - \frac{1}{2}\right)\end{aligned}$$

Thus, returning to the free energy calculation, for the sake of future substitution, let us rewrite $-\frac{r_3}{l}$ as $-\frac{2r_3}{l} + \frac{r_3}{l}$ and let us add and subtract 1. Thus the free energy becomes:

$$\begin{aligned}F_{MTZ} &= \frac{\sigma l}{4\pi G}\left(\frac{r_3^2}{l^2} - \frac{2r_3}{l} + \frac{r_3}{l} - \frac{1}{2} + 1 - 1\right)\\ &= \frac{\sigma l}{4\pi G}\left[\left(\frac{r_3^2}{l^2} - \frac{2r_3}{l} + 1\right) + \left(\frac{r_3}{l} - 1\right) - \frac{1}{2}\right]\end{aligned}$$

We can now substitute our above calculations:

$$F_{MTZ} = \frac{\sigma l}{4\pi G}\left(\pi^2 l^2 (T-T_c)^2 + \pi l(T-T_c) - \frac{1}{2}\right)$$

Thus, factoring out a $-\frac{1}{2}$, we obtain our final desired form of the free energy:

$$F_{MTZ} = -\frac{\sigma l}{8\pi G}(1 - 2\pi l(T-T_c) - 2\pi^2 l^2 (T-T_c)^2) \tag{147}$$

Thus, as above, let us calculate ΔF_2 using Eq 139 and Eq 147, which becomes:

$$\Delta F_2 = -\frac{\sigma l}{8\pi G}[4\pi l(T-T_c) + 4\pi^2 l^2 (T-T_c)^2 + \pi^3 l^3 (T-T_c)^3] \tag{148}$$

Thus, we can see that the formation of hair in this region of the MTZ solution is given by a first order phase transition, and is the energetically favorable solution (rather than the vacuum solution). Note that while this region was initially disregarded by Martínez, Troncoso, and Zanelli because of a naked singularity in the scalar hair, due to the fact that the formation of hair in this region is supported by a first order phase transition, this region is still of interest regarding violation of the KSS bound. The fact that the phase transition will be first order implies that the phase transition is ice-like, namely a liquid-solid phase transition. Since the KSS bound has been shown to be broken in holographic solids [6], it is worthwhile to investigate $\frac{\eta}{s}$ in this large negative mass regime.

6 Trajectories of Particles Coupled to the Scalar Field in the Large Negative Mass Regime

To determine whether or not it is valid to consider this large negative mass regime, we now consider a similar system considered by Jacob Bekenstein in [16], where considered are particles coupled to a scalar field identical to the MTZ scalar field under the conformal transformation given in Eq 110. Here, we will reproduce this calculation in an explicit fashion.

6.1 Explicit Reproduction of Bekenstein's *Black Holes with Scalar Charge* Section 3

Thus, let us consider the metric and scalar hair as provided in [16]:

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dt^2 + \left(1 - \frac{M}{r}\right)^{-2} dr^2 + r^2 d\Omega^2 \tag{149}$$

$$\psi = \frac{q}{r - M} \tag{150}$$

where M is a parameter given as:

$$M = 2q\sqrt{\frac{\pi}{3}}$$

Note that there is a singularity in Eq 150 at $r = M$; however, we claim that this singularity is not a physical. Thus, we want to show that trajectories of particles coupled to the scalar field diverge at $r = M$ at infinite proper time. For such particles, we have the action given as [16]:

$$S = -\int (m + f\phi)\left(- g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}\right) d\lambda \tag{151}$$

where m is the rest mass of the particle, f is the coupling strength of the particle to the field ϕ. Minimizing this action with respect to x^ν gives the following equation of motion [16]:

$$(m+f\phi)\frac{d^2x^\nu}{d\lambda^2} = \left[\frac{1}{2}(m+f\phi)\frac{d}{d\lambda}\ln\left(-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}\right) - f\phi_{,\alpha}\frac{dx^\alpha}{d\lambda}\right]\frac{dx^\nu}{d\lambda} - f\left(-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}\right)\phi^{,\nu} \tag{152}$$

Since Eq 151 is invariant with respect to λ, we are able to arbitrarily set [16]:

$$-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = m^{-2}(m + f\phi)^2 = \left(\frac{d\tau}{d\lambda}\right)^2 \tag{153}$$

Note that the second equality above comes from the fact that in the proper time frame, $dx_\tau = dy_\tau = dz_\tau = 0$ and thus $g_{\alpha\beta}dx^\alpha dx^\beta = ds^2 = -d\tau^2 + dx_\tau + dy_\tau + dz_\tau = -d\tau^2$

Substituting this into Eq 152, we obtain:

$$\begin{aligned}
(m + f\phi)\frac{d^2x^\nu}{d\lambda^2} &= \left[\frac{1}{2}(m + f\phi)\frac{d}{d\lambda}\ln(m^{-2}(m + f\phi)^2) - f\phi_{,\alpha}\frac{dx^\alpha}{d\lambda}\right]\frac{dx^\nu}{d\lambda} - f(m^{-2}(m + f\phi)^2)\phi^{,\nu} \\
&= \left[\frac{1}{2}(m + f\phi)(2)\frac{d}{d\lambda}(\ln(m + f\phi) - \ln(m)) - f\phi_{,\alpha}\frac{dx^\alpha}{d\lambda}\right]\frac{dx^\nu}{d\lambda} - f(m^{-2}(m + f\phi)^2)\phi^{,\nu} \\
&= \left[(m + f\phi)\frac{f\phi_{,\alpha}\frac{dx^\alpha}{d\lambda}}{m + f\phi} - f\phi_{,\alpha}\frac{dx^\alpha}{d\lambda}\right]\frac{dx^\nu}{d\lambda} - f(m^{-2}(m + f\phi)^2)\phi^{,\nu} \\
&= -fm^{-2}(m + f\phi)^2\phi^{,\nu}
\end{aligned}$$

Thus, canceling $(m + f\phi)$ from each side gives our new equation of motion:

$$\frac{d^2x^\nu}{d\lambda^2} = -fm^{-2}(m+f\phi)^2\phi^{,\nu} \tag{154}$$

Now, using the Killing equation $\xi_\alpha\phi^{,\alpha} = 0$ as given in [16], we can multiply each side of Eq 154 by ξ_ν and integrate to obtain the constant of motion for the Killing vector ξ_ν:

$$\int \xi_\nu \frac{d^2x^\nu}{d\lambda^2} d\lambda = -\int fm^{-2}(m+f\phi)^2\xi_\nu\phi^{,\nu} d\lambda$$

We see that the RHS becomes 0 due to the Killing equation given above, and thus, integrating the LHS, we obtain:

$$\xi_\nu \frac{dx^\nu}{d\lambda} + E = 0$$

We claim that since Eq 149 is time independent, we thus have time-like Killing vector given as $\xi^\alpha = \delta_0^\alpha$ where δ_0^α is the Kronecker delta. Note that $\xi_\alpha = g_{\alpha\beta}\xi^\beta$ by properties of the metric tensor.

Proof. Suppose that $\xi^\alpha = \delta_0^\alpha$ and $g_{\alpha\beta,0} = 0$. Thus, we want to show that ξ^α satisfies the Killing equation, $\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0$. First, note the Killing Condition: $\xi_{\alpha;\beta} + \xi_{\beta;\alpha} \equiv \xi_{\alpha,\beta} + \xi_{\beta,\alpha} - 2\Gamma^\rho_{\alpha\beta}\xi_\rho$. From the above identity, $\xi_\alpha = g_{\alpha\beta}\xi^\beta = g_{\alpha\beta}\delta_0^\beta = g_{\alpha 0}$, we can rewrite this as:

$$\xi_{\alpha;\beta} + \xi_{\beta;\alpha} \equiv \xi_{\alpha,\beta} + \xi_{\beta,\alpha} - 2\Gamma^\rho_{\alpha\beta}\xi_\rho = g_{\alpha 0,\beta} + g_{\beta 0,\alpha} - g^{\rho\sigma}(g_{\sigma\alpha,\beta} + g_{\sigma\beta,\alpha} - g_{\alpha\beta,\sigma})g_{\rho 0}$$

Now, note that since metric tensor components $g_{\mu\nu} \in \mathbb{R}$, and since Einstein summation notation dictates that the third term above is a summation over the components, we can rewrite this as:

$$g_{\alpha 0,\beta} + g_{\beta 0,\alpha} - g^{\rho\sigma}(g_{\sigma\alpha,\beta} + g_{\sigma\beta,\alpha} - g_{\alpha\beta,\sigma})g_{\rho 0} = g_{\alpha 0,\beta} + g_{\beta 0,\alpha} - g^{\rho\sigma}g_{\rho 0}(g_{\sigma\alpha,\beta} + g_{\sigma\beta,\alpha} - g_{\alpha\beta,\sigma})$$

$$= g_{\alpha 0,\beta} + g_{\beta 0,\alpha} - \delta_0^\sigma(g_{\sigma\alpha,\beta} + g_{\sigma\beta,\alpha} - g_{\alpha\beta,\sigma})$$

$$= g_{\alpha 0,\beta} + g_{\beta 0,\alpha} - g_{0\alpha,\beta} - g_{0\beta,\alpha} + g_{\alpha\beta,0})$$

Since the metric tensor is symmetric about the diagonal, the first four terms cancel, and we are left with

$$\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = g_{\alpha\beta,0}$$

which, by hypothesis, is equal to zero, and thus we have $\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0$, as was to be shown. □

Thus, let us rewrite our above result:

$$E = -\xi_\nu \frac{dx^\nu}{d\lambda} \tag{155}$$

which, using the result of the proof above, gives that

$$-\xi_\nu \frac{dx^\nu}{d\lambda} = -g_{\nu 0}\frac{dx^\nu}{d\lambda}$$

Since the metric tensor is represented by a diagonal matrix, we have that $\nu = 0$, and since $g_{00} = -(1 - \frac{M}{r})^2$, we can rewrite Eq 155 as:

$$E = \left(1 - \frac{M}{r}\right)^2 \frac{dt}{d\lambda} \tag{156}$$

Now, let us expand Eq 153, which is given as:

$$-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = m^{-2}(m+f\phi)^2$$

Expanding for orbits of angular momentum zero (without loss of generality)and distributing the m^{-2}, we obtain:

$$\left(1-\frac{M}{r}\right)^2\left(\frac{dt}{d\lambda}\right)^2-\left(1-\frac{M}{r}\right)^{-2}\left(\frac{dr}{d\lambda}\right)^2=\left(1+\frac{f}{m}\frac{q}{r-M}\right)^2$$

Substituting Eq 156, we obtain:

$$\left(1-\frac{M}{r}\right)^{-2}E^2-\left(1-\frac{M}{r}\right)^{-2}\left(\frac{dr}{d\lambda}\right)^2=\left(1+\frac{f}{m}\frac{q}{r-M}\right)^2$$

Multiplying by $(1-\frac{M}{r})^2$ and rearranging for $(\frac{dr}{d\lambda})^2$, we obtain:

$$\left(\frac{dr}{d\lambda}\right)^2=E^2-\left(1-\frac{M}{r}\right)^2\left(1+\frac{f}{m}\frac{q}{r-M}\right)^2$$

and thus we have:

$$\frac{dr}{d\lambda}=\pm\left(E^2-\left(1-\frac{M}{r}\right)^2\left(1+\frac{f}{m}\frac{q}{r-M}\right)^2\right)^{\frac{1}{2}}$$

To better understand this result, let us factor an r out of the denominator of the hair term and expand the parentheses:

$$\frac{dr}{d\lambda}=\pm\left[E^2-\left(1-\frac{M}{r}\right)^2\left(1+\frac{f}{mr}\frac{q}{1-M/r}\right)^2\right]^{\frac{1}{2}} \tag{157}$$

which thus becomes:

$$\frac{dr}{d\lambda}=\pm\left[E^2-\left(1-\frac{M}{r}+\frac{fq}{mr}\right)^2\right]^{\frac{1}{2}} \tag{158}$$

Now, note for $r\to M$, Eq 158 becomes:

$$\frac{dr}{d\lambda}=\pm\left[E^2-\left(\frac{fq}{mM}\right)^2\right]^{\frac{1}{2}}$$

We can thus see for $\dfrac{fq}{mM}$ sufficiently small, $\frac{dr}{d\lambda}\approx\pm E$, which indicates that $\frac{dr}{d\lambda}$ never changes sign for $r\geq M$ and thus does not change direction, and thus there certainly is no infinite potential barrier, as there would be if the singularity were physical. Additionally, we see that for $|\dfrac{fq}{mM}|\geq E$, $\frac{dr}{d\lambda}$ does change sign for $r\geq M$ and thus the particle does change direction as a result of a finite potential barrier [16].

To make more sense of this result, it would be more useful to consider $\frac{dr}{d\tau}$, where τ is proper time. To do this, we will use the chain rule:

$$\frac{dr}{d\lambda}=\frac{dr}{d\tau}\frac{d\tau}{d\lambda}$$

Rearranging, we obtain:

$$\frac{dr}{d\tau}=\frac{dr}{d\lambda}\left(\frac{d\tau}{d\lambda}\right)^{-1}$$

We know $\frac{dr}{d\lambda}$ from Eq 157, and we know $\frac{d\tau}{d\lambda}$ from Eq 153, which is stated again explicitly below:

$$\frac{d\tau}{d\lambda}=\left[\left(1+\frac{f}{m}\frac{q}{r-M}\right)^2\right]^{\frac{1}{2}}$$

Thus, the above expression becomes:

$$\frac{dr}{d\tau}=\pm\left[E^2-\left(1-\frac{M}{r}\right)^2\left(1+\frac{f}{m}\frac{q}{r-M}\right)^2\right]^{\frac{1}{2}}\left[\left(1+\frac{f}{m}\frac{q}{r-M}\right)^{-2}\right]^{\frac{1}{2}}$$

$$= \pm\left[E^2\left(1+\frac{f}{m}\frac{q}{r-M}\right)^{-2}-\left(1-\frac{M}{r}\right)^2\right]^{\frac{1}{2}}$$

Now, note for $r \to M$, we have that $\frac{f}{m}\frac{q}{r-M} >> 1$. Thus, applying this approximation and factoring a $1/r^2$ (which then becomes $1/M^2$) out of the second term, we obtain:

$$\frac{dr}{d\tau} = \pm\left[E^2\left(\frac{m(r-M)}{fq}\right)^2 - \frac{1}{M^2}(r-M)^2\right]^{\frac{1}{2}}$$

Factoring out $M^{-2}(r-M)^2$, we obtain our final form:

$$\frac{dr}{d\tau} = \pm\left[E^2\left(\frac{mM}{fq}\right)^2 - 1\right]^{\frac{1}{2}} M^{-1}(r-M) \tag{159}$$

For the final step, let us define:

$$\frac{1}{K} = \pm\left[E^2\left(\frac{mM}{fq}\right)^2 - 1\right]^{\frac{1}{2}} M^{-1}$$

thus giving:

$$\frac{dr}{d\tau} = \frac{1}{K}(r-M)$$

which is an easy differential equation, yielding:

$$\tau = K\ln(r-M) \tag{160}$$

which shows that the particle's trajectory diverges at infinite proper time, showing that the singularity at $r = M$ is not a physical singularity.

6.2 Particles in the MTZ Metric Coupled to the MTZ Hair

Now, consider the MTZ metric given in Eq 111 and the MTZ scalar hair (with $r_0 = G\mu$), given as:

$$ds^2 = -\left(\frac{r^2}{l^2} - \left(1+\frac{r_0}{r}\right)^2\right)dt^2 + \left(\frac{r^2}{l^2} - \left(1+\frac{r_0}{r}\right)^2\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{161}$$

$$\phi(r) = \sqrt{\frac{3}{4\pi G}}\frac{r_0}{r+r_0} \tag{162}$$

Now, since the above calculation was in general until Eq 156, we can begin with the general form of the constant of motion, given as:

$$E = -g_{00}\frac{dt}{d\lambda}$$

Thus, substituting our value from Eq 161, we obtain our new constant of motion:

$$E = \left(\frac{r^2}{l^2} - \left(1+\frac{r_0}{r}\right)^2\right)\frac{dt}{d\lambda} \tag{163}$$

Once again, we will expand Eq 153 for orbits with zero angular momentum, and thus only time and radial terms survive:

$$\left(\frac{r^2}{l^2}-\left(1+\frac{r_0}{r}\right)^2\right)\left(\frac{dt}{d\lambda}\right)^2-\left(\frac{r^2}{l^2}-\left(1+\frac{r_0}{r}\right)^2\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2=\left(1+\frac{f}{m}\sqrt{\frac{3}{4\pi G}}\frac{r_0}{r+r_0}\right)^2$$

As above, we will substitute Eq 163 and solve for $\frac{dr}{d\lambda}$, which yields:

$$\frac{dr}{d\lambda}=\pm\left[E^2-\left(\frac{r^2}{l^2}-\left(1+\frac{r_0}{r}\right)^2\right)\left(1+\frac{f}{m}\sqrt{\frac{3}{4\pi G}}\frac{r_0}{r+r_0}\right)^2\right]^{\frac{1}{2}} \tag{164}$$

Now, note that we have:

$$\left(\frac{d\tau}{d\lambda}\right)^{-1}=\left[\left(1+\frac{f}{m}\sqrt{\frac{3}{4\pi G}}\frac{r_0}{r+r_0}\right)^{-2}\right]^{\frac{1}{2}}$$

which follows from Eq 153. Thus, we will once again manipulate the chain rule, giving:

$$\frac{dr}{d\tau}=\frac{dr}{d\lambda}\left(\frac{d\tau}{d\lambda}\right)^{-1}$$

which substituting the above two results yields:

$$\frac{dr}{d\tau}=\left[E^2\left(1+\frac{f}{m}\sqrt{\frac{3}{4\pi G}}\frac{r_0}{r+r_0}\right)^{-2}-\left(\frac{r^2}{l^2}-\left(1+\frac{r_0}{r}\right)^2\right)\right]^{\frac{1}{2}}$$

Now, consider $r\to -r_0$, where $\frac{f}{m}\sqrt{\frac{3}{4\pi G}}(\frac{r_0}{r+r_0}) >> 1$, in which case the above becomes:

$$\frac{dr}{d\tau}=\left[E^2\left(\frac{m}{f}\right)^2\frac{4\pi G}{3}\left(\frac{r+r_0}{r_0}\right)^2-\frac{r^2}{l^2}\right]^{\frac{1}{2}} \tag{165}$$

Note that the differential equation given in Eq 165 is a separable differential equation, allowing us to integrate and obtain the proper time, τ, as in Section 6.1. For ease of calculation, we will rename $K=E^2(\frac{m}{f})\frac{4\pi G}{3}$, and set $l^2=1$ and $r_0=-2$, we obtain the following result:

$$\tau=\frac{2i\ln\left[\frac{2i(-8+4(r-2)+K^2(r-2))}{\sqrt{4-K^2}}+2\sqrt{-16-16(r-2)-4(r-2)^2+K^2(r-2)^2}\right]}{\sqrt{4-K^2}} \tag{166}$$

Plotting the real and imaginary parts of this as a function of r with $K=10$, we obtain the following plot:

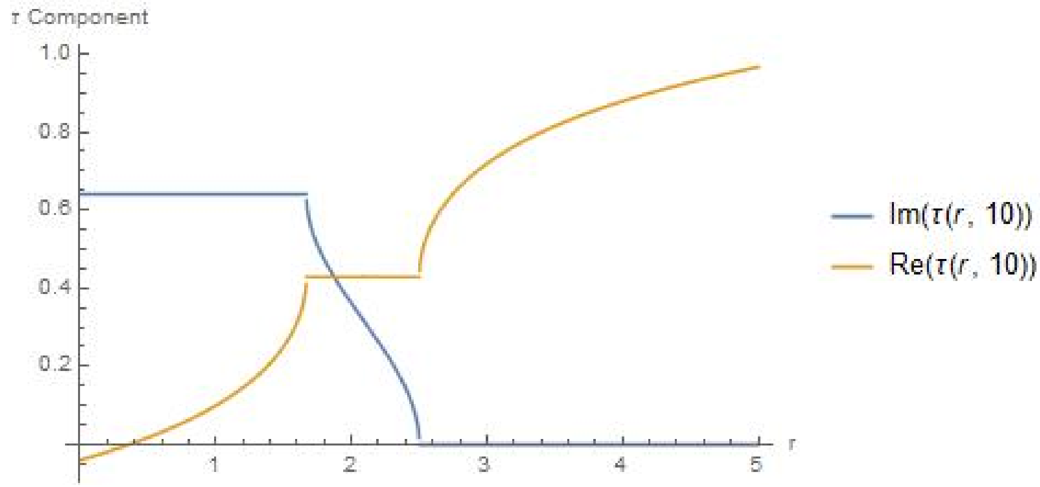

Figure 2: Real and Imaginary Components Proper Time of Particles Coupled to the MTZ Hair as a Function of r with $K = 10$

Here, we see that the proper time actually picks up an imaginary component before the singularity in the scalar hair, which here is at $r = 2$, but neither component diverges at the singularity as in the case studied in [16]. Additionally, we can look at $|\tau|^2$:

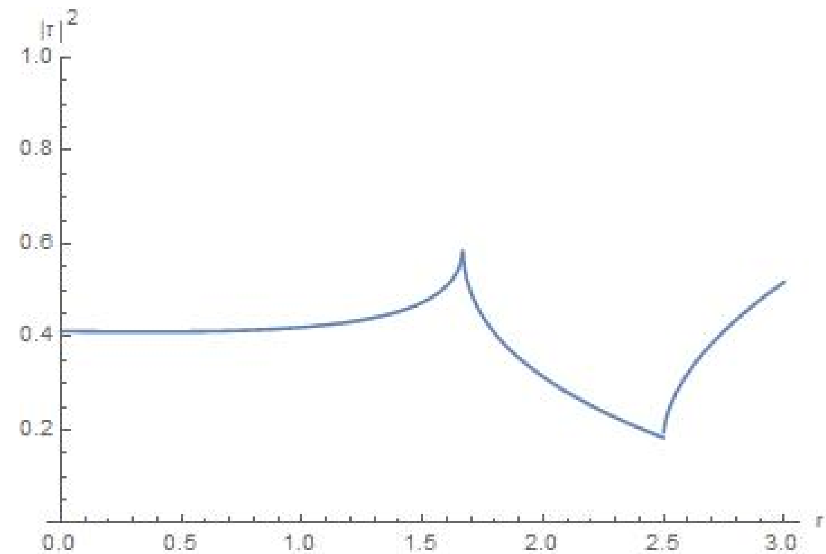

Figure 3: Squared Magnitude of Proper Time of Particles Coupled to the MTZ Hair as a Function of r with $K = 10$

Interestingly, in this plot of the mod of τ with respect to r, we see that this actually becomes non-differentiable at $r = 2.5$.

Due to the fact that the proper time is not divergent at the singularity in the scalar hair, we have that the point $r = -G\mu$ is singular [17]. Thus, we should expect to see some sort of instability of the black hole before the large negative mass regime that prevents the continuous variation of black hole mass below $\frac{G\mu}{l} = -\frac{1}{4}$.

7 A Method for Calculating Viscosity over Entropy in the MTZ System

Thus, we want to actually calculate η/s for this system. To do this, we will use the method given in [18], namely we will consider a perturbation of the metric, a gravitational wave (quantized by the graviton) of frequency ω, given as $\delta g_{\mu\nu} = e^{i\omega t}h(u)$ where the radial component of this perturbation, the shear mode, the satisfies the following equation of motion [18]:

$$h(u)\bigg(-\frac{2m^2V_X}{f(u)} - \frac{2i\omega}{uf(u)}\bigg) + h'(u)\bigg(\frac{f'(u)}{f(u)} + \frac{2i\omega}{f(u)} - \frac{2}{u}\bigg) + h''(u) = 0 \tag{167}$$

Above, V_X is the derivative of potential associated with the mass of the graviton (which we will assume, in general, to not be restricted to zero) with respect to $X = \frac{r^2}{l^2}$, m is the mass of the graviton, and f refers to the function of the metric as described in Eq 64, where by convention, we have taken $u = \frac{1}{u}$. Note that while in [18] the system considered was an AdS Schwarzschild black hole, Eq 167 still holds since both the Schwarzschild black hole and the MTZ black hole are of the same form, namely that given in Eq 64. Thus, Multiplying Eq 167 by $uf(u)$, we obtain our desired form of the equation of motion, given as:

$$h_{xy}(u)(-2um^2V_X) - 2i\omega) + h'_{xy}(u)(uf'(u) + 2iu\omega - 2f(u)) + uf(u)h''_{xy}(u) = 0 \tag{168}$$

Thus, substituting the the blackening factor (i.e. $f(u)$) from the MTZ solution, we obtain:

$$(-2i\omega)h(u)+\bigg(\frac{2}{u^2L^2}+2G\mu u(1+G\mu u)+2iu^{-1}\omega-\frac{2}{u^2L^2}+2(1+G\mu u)^2\bigg)h'(u)+u^{-1}\bigg(\frac{1}{u^2L^2}-(1+G\mu u)^2\bigg)h''(u) = 0$$

After expanding and simplifying, we obtain:

$$(-2i\omega)h(u) + (6G\mu u + 4G^2\mu^2u^2 + 2iu^{-1}\omega + 2)h'(u) + u^{-1}\bigg(\frac{1}{u^2L^2} - (1+G\mu u)^2\bigg)h''(r) = 0 \tag{169}$$

In general, fields in the bulk can be expanded as [18]:

$$\Phi(t,\vec{x}) = A(\vec{x})r^{-a}(1+\cdots) + B(\vec{x})r^{-b}(1+\cdots) \tag{170}$$

Where $A(\vec{x})$ and $B(\vec{x})$ are defined as the "leading" and "subleading" terms, respectively, and are integration constants obtained from the following procedure [18]: Eq 169 is integrated from the boundary (i.e. r at ∞) to some arbitrary point in the bulk, and again from the outermost event horizon to the same point in the bulk. In the case of the field $h(r)$, we have [18]:

$$h(r) = h_l(\omega)(1+\cdots) + h_s(\omega)(1+\cdots) \tag{171}$$

Where $h_l(\omega)$ and $h_s(\omega)$ are the leading and subleading terms described above. The viscosity can intuitively be understood as the response of the black hole to these perturbations in the metric, and will be related to the imaginary component of the Green's function as [18]:

$$\eta = \lim_{\omega \to 0} \left[-\frac{1}{\omega} \text{Im}\mathcal{G}(\omega) \right] \tag{172}$$

Where, from the AdS/CFT dictionary, we can write the Green's function as [18]:

$$\mathcal{G}(\omega) = \frac{3h_s(\omega)}{2h_l(\omega)} \tag{173}$$

Additionally, the entropy density of the system is given in terms of the AdS length, l, and the event horizon r_h [18]:

$$s = \frac{2\pi l^2}{r_h^2} \tag{174}$$

Thus, compiling Eq 172 and Eq 174, we obtain:

$$\frac{\eta}{s} = \lim_{\omega \to 0} \left[-\frac{r_h^2}{2\pi l^2 \omega} \text{Im}\mathcal{G}(\omega) \right] \tag{175}$$

The following section details our numerical results from the above outlined procedure.

8 Viscosity over Entropy: Numerical Results

Renaming $\mu = M$, we obtain the following plot for η/s as a function of re-normalized black hole mass M:

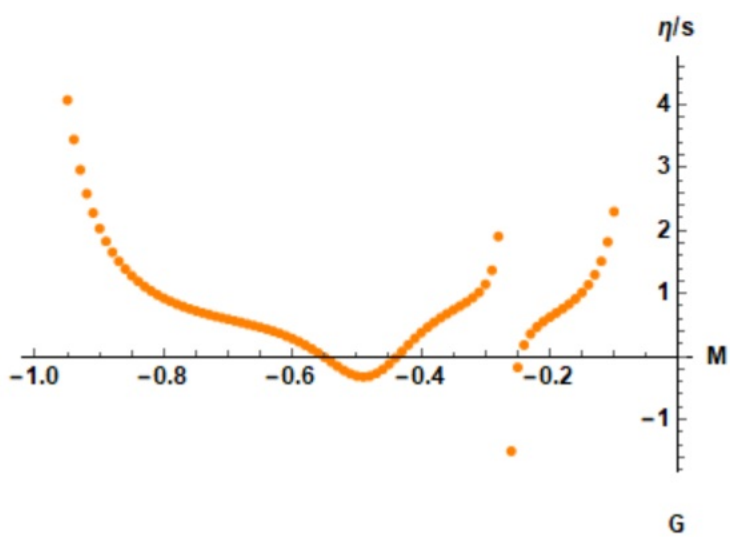

Figure 4: η/s in the MTZ Black Hole as a Function of Black Hole Mass

In the small negative mass regime, we can see arbitrary breaking of the bound before η/s becomes negative, and eventually diverges as it approaches $M = -\frac{1}{4}$. Negative viscosity has been shown to correspond to a superradiant instability of the black hole [11], thus indicating that mass cannot continuously varied past $M = -\frac{1}{4}$. This supports the claim from Section 6.2.

In the large negative mass regime, we can see a pathology in the model right around $M = -\frac{1}{2}$, where the ratio of η/s becomes negative, which is not physical.

So as to see if the pathology at $M = -\frac{1}{2}$ could be eliminated, the equation of motion given in Eq 167 was adjusted in the Mathematica code to include non-zero graviton mass. η/s was plotted again as a function of black hole mass, M, for graviton mass values of $m = 0.01$, $m = 0.1$, and $m = 1$. Plot behavior became strange for $m = 1$. For $m = 0.01$, plot looked similar to the case with massless graviton. At $m = 0.1$, a new plot was generated, which qualitatively looked similar to if the plot were inverted about the horizontal axis. At $M = -\frac{1}{2}$, η/s was positive instead of negative. Thus, between $m = 0.01$ and $m = 0.1$, η/s must have a zero (by the Intermediate Value Theorem). η/s was then plotted as a function of m between 0.01 and 0.1 at $M = -0.5$. This was used to approximate a value of m such that η/s was not negative at $M \approx -0.5$. Thus, η/s was plotted again as a function of M with $m = 0.046$. For this, the data showed that η/s was still negative at $M = -0.47$. Thus, repeating the above procedure, η/s as a function of m for $M = -0.47$ to approximate a value of m for which η/s had a zero to enter this value back into the plot of η/s as a function of M. This plot contained zero between $m = 0.052$ and $m = 0.053$. Thus, $m = 0.053$ was substituted into the plot of η/s as a function of M since η/s was positive at $m = 0.053$, as desired.

For $m = 0.052$, we have the following plot of η/s.

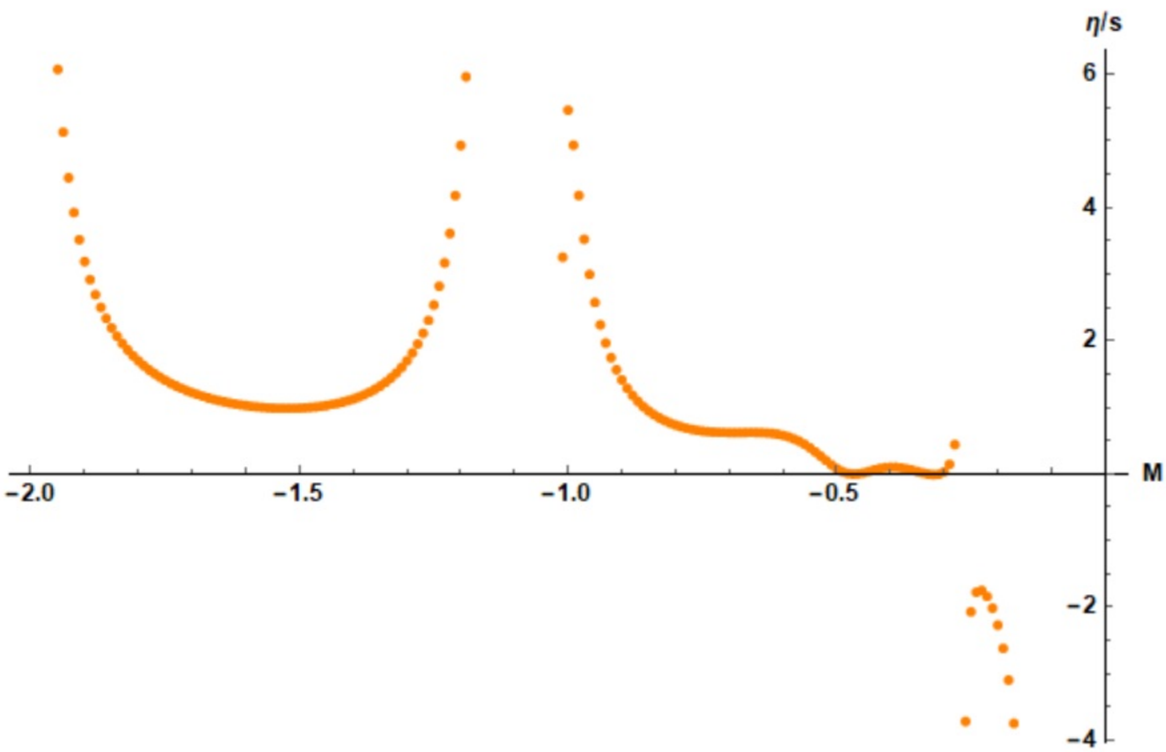

Figure 5: η/s in the MTZ Black Hole as a Function of Black Hole Mass for $m = 0.052$

Note that now that we have increased graviton mass to 0.052 from $m = 0$ as in Figure 4, we have a second local minimum that dips negative at $M \approx -0.33$. Thus, we can plot η/s as a function of graviton mass for $M = -0.47$ and $M = -0.33$, which are the two local minima depicted in the middle section of the plot in Figure 4.

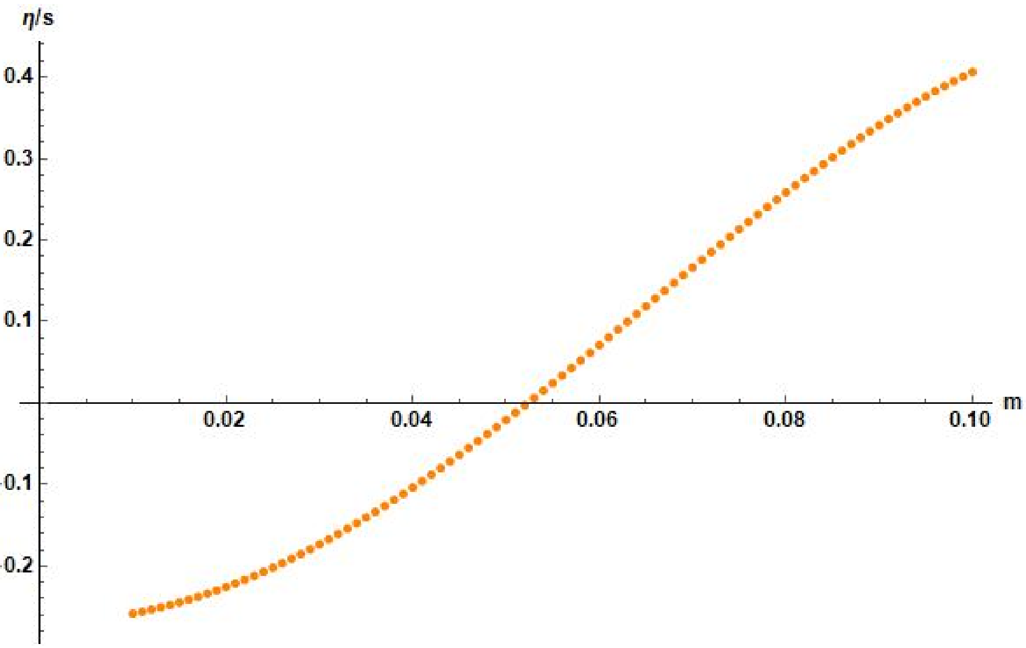

Figure 6: η/s in the MTZ Black Hole as a Function of Graviton Mass for $M = -0.47$

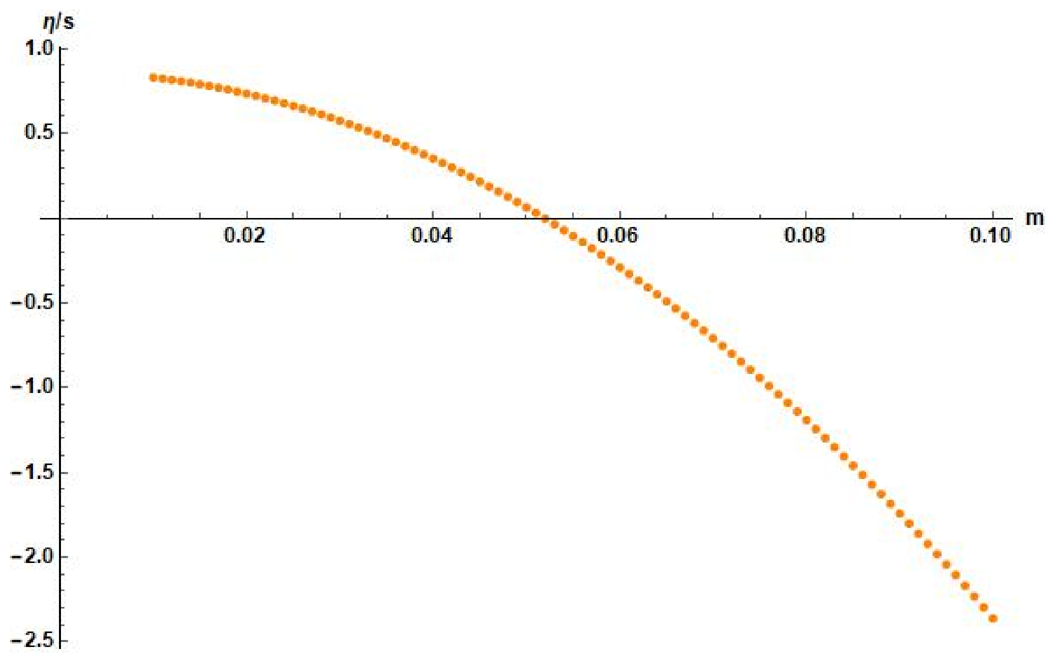

Figure 7: η/s in the MTZ Black Hole as a Function of Graviton Mass for $M = -0.33$

Upon examining the raw data for η/s as a function of graviton mass for $M = -0.33$ and $M = -0.47$, one can see that zero occurs for $M = -0.47$ in the interval $m \in [0.0523, 0.0524]$ and for $M = -0.33$ in the interval $m \in [0.0519, 0.0520]$. Thus, η/s is negative at both M values on the interval $m \in [0.0520, 0.0523]$, and are opposite signs elsewhere. Thus, this negative viscosity cannot be eliminated in the large negative mass regime. Furthermore, we can note that in Figure 5, η/s is negative for $-\frac{1}{4} < m < 0$, indicating that the small negative mass regime is unstable. Thus, we can conclude that the MTZ solution is unstable due to superradiant instabilities [11] for graviton mass $m = 0.052$ and larger (as is shown below).

Further investigations into behavior of η/s for larger graviton masses appear to show similarly unstable behavior, for example setting $m = 0.1$ and $m = 1$, as shown below.

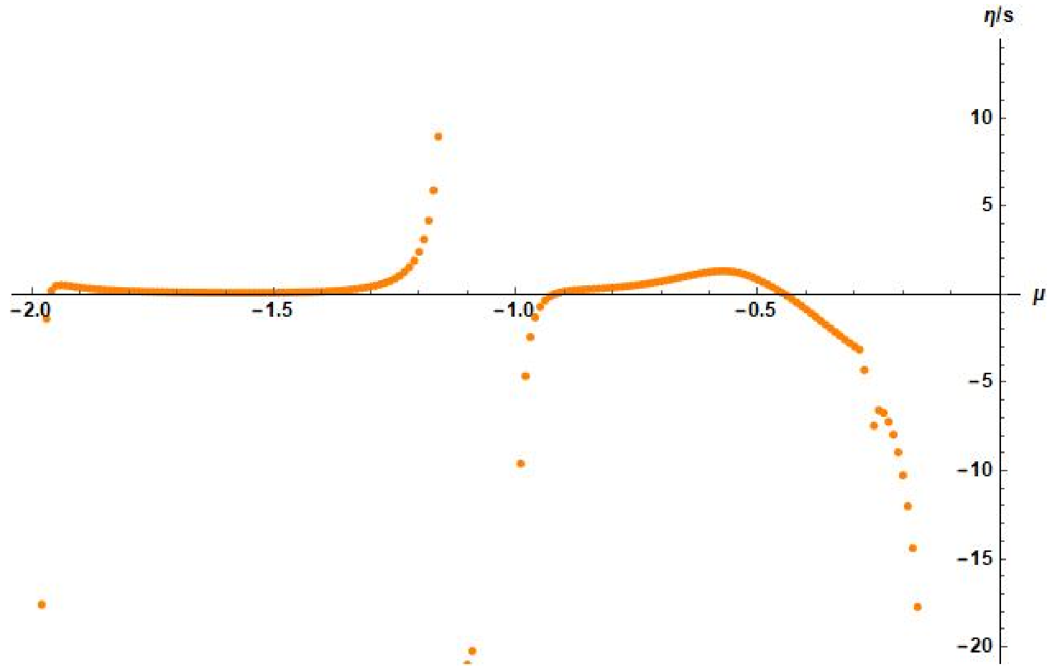

Figure 8: η/s in the MTZ Black Hole as a Function of BH Mass for $m = 0.1$

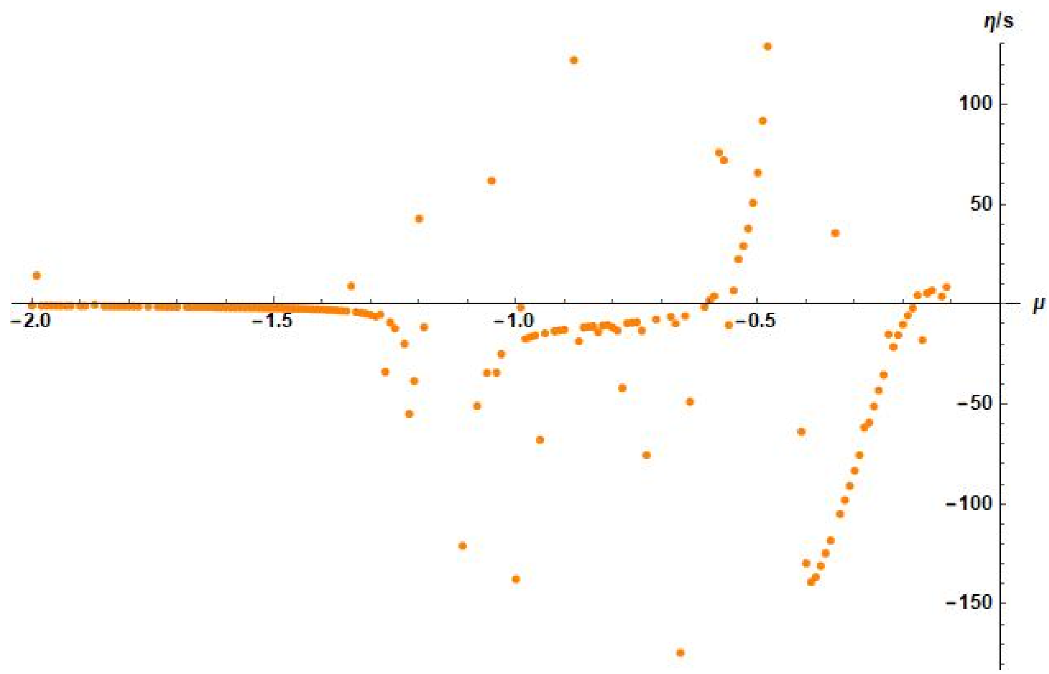

Figure 9: η/s in the MTZ Black Hole as a Function of BH Mass for $m = 0.1$

In addition to the above investigation of the pathology in the model for large negative mass, consider a different form of the potential associated with the massive graviton, i.e. $V = X^2 = (\frac{r}{L})^4$. This new form shows up in the equation of motion, and for graviton masses considered above (i.e.

less than 0.1), the form of η/s as a function of black hole mass, M, is qualitatively indistinguishable from that above for $m = 0$. However, when $m = 0$, we obtain the following behavior:

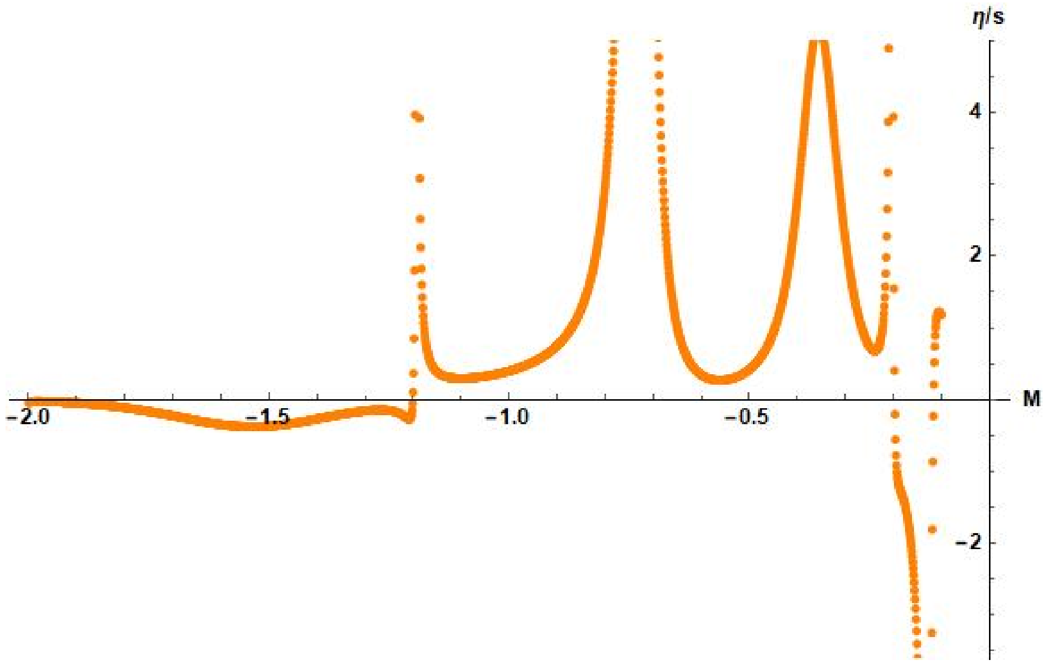

Figure 10: η/s in the MTZ Black Hole as a Function of Black Hole Mass for $m = 1$

We see that η/s remains positive on the interval $M \in [-1.2, -0.2]$, however η/s is once again negative for small negative mass, and large negative mass beyond $M = -1.2$.

9 Summary

While we have shown that the formation of hair in the large negative mass regime is given by a first order phase transition, we also showed that trajectories of particles coupled to this scalar hair diverge at the singularity in the scalar hair at finite proper time. This indicates that the naked singularity in the scalar hair for $\frac{G\mu}{l} < -\frac{1}{4}$ is pathological. Thus, mass values in the negative mass regime beyond $\frac{G\mu}{l} = -\frac{1}{4}$ would violate weak cosmic censorship [19].

In the small negative mass regime, namely $-\frac{1}{4} < \frac{G\mu}{l} < 0$, we have shown that for zero graviton mass, the KSS bound can be arbitrarily broken in a system holographically dual to the MTZ black hole system. At a certain value slightly above $\frac{G\mu}{l} = -\frac{1}{4}$, viscosity becomes negative and then diverges, indicating a superradiant instability of the black hole as the black hole mass approaches this value, and thus mass cannot be continuously varied to reach $-\frac{1}{4}$. In the negative mass regime beyond $-\frac{1}{4}$, there are unavoidable superradiant instabilities, namely negative viscosity, that cannot be removed by varying graviton mass in the set of graviton potentials considered above (namely $V(X) = X^2$ and $V(X) = X^3$). The emergent superradiant instability that arises as a result of continuously decreasing the black hole mass introduces a form of cosmic censorship, preventing the singularity in the scalar hair from becoming visible to the asymptotic observer for sufficiently negative black hole mass.

This work provides a new means of grossly violating the KSS bound in a holographic context that is not dependent on graviton mass. Future work could include studying the ratio of shear viscosity to entropy density in systems similar to the MTZ solution, or other non-trivial hairy black hole solutions in anti-de Sitter space.

10 Acknowledgments

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11 Appendices

11.1 Christoffel Symbols for the Schwarzschild Case

We can now derive the various Christoffel symbols below. It must be noted that the Greek sub/superscripts span dimensions 0-3 while latin sub/superscripts span only dimensions 1-3, where 0 is the time dimension by convention. Additionally, since $g_{\mu\nu}$ is diagonal $g_{\mu\nu}$ such that $\mu \neq \nu$ is vanishing. Also, when being differentiated in the given coordinate system (spherical), the only dimension in which a nonzero derivative is found is when $i, j = 1$ (or the radial dimension). It should also be mentioned that the Christoffel symbols are symmetric about the two subscripts due to the symmetry of the metric tensor, and that they are equal to zero $\forall \lambda \neq \mu$ due to the diagonal nature of the metric tensor. Below are all the possible Christoffel symbols that are relevant to the metric tensor"

$$\Gamma^0_{00} = \frac{1}{2} g^{00}\{g_{00,0} + g_{00,0} - g_{00,0}\} = 0$$

$$\Gamma^0_{0i} = \frac{1}{2} g^{00}\{g_{00,i} + g_{0i,0} - g_{0i,0}\} = \frac{1}{2U}\frac{dU}{dr}(i = 1, else = 0)$$

$$\Gamma^0_{ij} = \frac{1}{2} g^{00}\{g_{0i,j} + g_{0j,i} - g_{ij,0}\} = 0$$

$$\Gamma^1_{00} = \frac{1}{2} g^{11}\{g_{10,0} + g_{10,0} - g_{00,1}\} = \frac{1}{2V}\frac{dU}{dr}$$

$$\Gamma^1_{0i} = \frac{1}{2} g^{11}\{g_{10,i} + g_{1i,0} - g_{0i,1}\} = 0$$

$$\Gamma^1_{ij,i\neq j} = \frac{1}{2} g^{11}\{g_{1i,j} + g_{1j,i} - g_{ij,1}\} = 0$$

$$\Gamma^1_{11} = \frac{1}{2} g^{11}\{g_{11,1} + g_{11,1} - g_{11,1}\} = -\frac{1}{2V}\bigg(-\frac{dV}{dr}\bigg) = \frac{1}{2V}\frac{dV}{dr}$$

$$\Gamma^1_{22} = \frac{1}{2} g^{11}\{g_{12,2} + g_{12,2} - g_{22,1}\} = -\frac{1}{2V}\bigg(-\frac{d}{dr}(r^2)\bigg) = -\frac{r}{V}$$

$$\Gamma^1_{33} = \frac{1}{2} g^{11}\{g_{13,3} + g_{13,3} - g_{33,1}\} = -\frac{1}{2V}\bigg(-\frac{d}{dr}(r^2 \sin^2\theta)\bigg) = -\frac{r\sin^2\theta}{V}$$

$$\Gamma^2_{00} = \frac{1}{2} g^{22}\{g_{20,0} + g_{20,0} - g_{00,2}\} = 0$$

$$\Gamma^2_{0i} = \frac{1}{2} g^{22}\{g_{20,i} + g_{2i,0} - g_{0i,2}\} = 0$$

$$\Gamma^2_{ii,i=1,2} = \frac{1}{2} g^{22}\{g_{2i,i} + g_{2i,i} - g_{ii,2}\} = 0$$

$$\Gamma^2_{ii,i=3} = \frac{1}{2}g^{22}\{g_{2i,i} + g_{2i,i} - g_{ii,2}\} = -\frac{1}{2r^2}\left(\frac{d}{d\theta}(r^2\sin^2\theta)\right) = -\sin\theta\cos\theta$$

$$\Gamma^2_{12} = \frac{1}{2}g^{22}\{g_{21,2} + g_{22,1} - g_{12,2}\} = \frac{1}{r}$$

$$\Gamma^2_{13} = \frac{1}{2}g^{22}\{g_{21,3} + g_{23,1} - g_{13,2}\} = 0$$

$$\Gamma^2_{23} = \frac{1}{2}g^{22}\{g_{22,3} + g_{23,2} - g_{23,2}\} = 0$$

$$\Gamma^3_{00} = \frac{1}{2}g^{33}\{g_{30,0} + g_{30,0} - g_{00,3}\} = 0$$

$$\Gamma^3_{0i} = \frac{1}{2}g^{33}\{g_{30,i} + g_{3i,0} - g_{0i,3}\} = 0$$

$$\Gamma^3_{ii} = \frac{1}{2}g^{33}\{g_{3i,i} + g_{3i,i} - g_{ii,3}\} = 0$$

$$\Gamma^3_{12} = \frac{1}{2}g^{33}\{g_{31,2} + g_{32,1} - g_{12,3}\} = 0$$

$$\Gamma^3_{13} = \frac{1}{2}g^{33}\{g_{33,1} + g_{31,3} - g_{13,3}\} = -\frac{1}{2r^2\sin^2\theta}\left(\frac{d}{dr}(-r^2\sin^2\theta)\right) = \frac{1}{r}$$

$$\Gamma^3_{23} = \frac{1}{2}g^{33}\{g_{32,2} + g_{33,2} - g_{23,3}\} = -\frac{1}{2r^2\sin^2\theta}\left(\frac{d}{d\theta}(-r^2\sin^2\theta)\right) = \frac{\cos\theta}{\sin\theta} = \cot\theta$$

11.2 Explicit Calculation of Ricci Curvature Tensor Components and the Ricci Scalar for the Schwarzschild Case

Once again, let us begin by explicitly expanding Eq 5:

$$\begin{aligned} R_{\mu\nu} = {} & \Gamma^0_{\mu\nu,0} - \Gamma^0_{\mu0,\nu} + \Gamma^\alpha_{\mu\nu}\Gamma^0_{\alpha0} - \Gamma^\alpha_{\mu0}\Gamma^0_{\alpha\nu} \\ & +\Gamma^1_{\mu\nu,1} - \Gamma^1_{\mu1,\nu} + \Gamma^\alpha_{\mu\nu}\Gamma^1_{\alpha1} - \Gamma^\alpha_{\mu1}\Gamma^1_{\alpha\nu} \\ & +\Gamma^2_{\mu\nu,2} - \Gamma^2_{\mu2,\nu} + \Gamma^\alpha_{\mu\nu}\Gamma^2_{\alpha2} - \Gamma^\alpha_{\mu2}\Gamma^2_{\alpha\nu} \\ & +\Gamma^3_{\mu\nu,3} - \Gamma^3_{\mu3,\nu} + \Gamma^\alpha_{\mu\nu}\Gamma^3_{\alpha3} - \Gamma^\alpha_{\mu3}\Gamma^3_{\alpha\nu}. \end{aligned} \tag{176}$$

Thus, we can start with the case where $\mu \neq \nu$

$$\begin{aligned} R_{ij} = {} & \Gamma^0_{ij,0} - \Gamma^0_{i0,j} + \Gamma^\alpha_{ij}\Gamma^0_{\alpha0} - \Gamma^\alpha_{i0}\Gamma^0_{\alpha j} \\ & +\Gamma^1_{ij,1} - \Gamma^1_{i1,j} + \Gamma^\alpha_{ij}\Gamma^1_{\alpha1} - \Gamma^\alpha_{i1}\Gamma^1_{\alpha j} \\ & +\Gamma^2_{ij,2} - \Gamma^2_{i2,j} + \Gamma^\alpha_{ij}\Gamma^2_{\alpha2} - \Gamma^\alpha_{i2}\Gamma^2_{\alpha j} \\ & +\Gamma^3_{ij,3} - \Gamma^3_{i3,j} + \Gamma^\alpha_{ij}\Gamma^3_{\alpha3} - \Gamma^\alpha_{i3}\Gamma^3_{\alpha j}. \end{aligned} \tag{177}$$

Plugging in the Christoffel symbols into this, we obtain the following, though this will be explained out thoroughly in English since many of the cancellations and emergent values come about for logical reasons that are not quite trivial:

- Since the system does not evolve with time, and since the first term is differentiated with respect to time, the term is vanishing.
- The second term before being differentiated is $\frac{1}{2U}\frac{dU}{dr}$. However, this is then differentiated with respect to j, which is defined as not being equal to i, which in this case must be 1 for the term not to equal 0. This expression is a function of only r and when differentiated by anything other than r, it becomes zero, and thus the term disappears.

- The second symbol of the third term forces $\alpha = 1$ for the term to be non-vanishing. However, for the first symbol of the third term $\alpha = 1$ drives the term to 0, thus giving that the term is vanishing $\forall \alpha$.
- The first symbol in the fourth term is only non-vanishing for $\alpha = 0$, which forces $i = 1$. However, for the second symbol of the fourth term, since $i \neq j$, $j \neq i$, and thus the fourth term is vanishing.
- The fifth term is 0 for all $i \neq j$ (See Appendices).
- The sixth term is only non-vanishing when $i = 1$. However, since $i \neq j$, and when the term is vanishing when differentiated by anything other than r (which is represented by 1), the term is vanishing.
- By examining the second symbol of the seventh term, one can see that $\alpha = 1$ or else the term is vanishing. However, this drives the first symbol of the seventh term to 0, and thus the seventh term is vanishing.
- For the first symbol of the eighth term, by looking at the Appendices where the explicit Christoffel Symbols are given for this case, one can see that the term is only non-vanishing for $\alpha = i$. However, by doing the same for the second symbol of the eighth term, one finds that it is only non-vanishing for $\alpha = j$. These conditions are mutually exclusive since $i \neq j$, and thus the eighth term is vanishing.
- The ninth term is only non-vanishing before differentiation when $i, j = 1, 2$. However this returns a function of r, which when differentiated by θ is 0 and the term is thus vanishing.
- The tenth term before being differentiated is non-vanishing only when $i = 1$. However, this term is a function of only r, and since $j \neq i$, term is differentiated with respect to any variable other than r, and thus the tenth term is vanishing.
- The second symbol of the eleventh term forces $\alpha = 1$ in order to be non-zero. However, for the first symbol of the eleventh term, when $\alpha = 1$, the term equals 0, and thus the entire term is vanishing.
- The two symbol of the twelfth term are non-vanishing individually for three cases: $\alpha = 1$ and $i = 2$, $\alpha = 2$ and $i = 1$, and $\alpha = 3$ and $i = 3$. For the first case, the second symbol of the twelfth term is only non-vanishing for $j = 2$. But since $i = 2$, and $i \neq j$, the term is vanishing overall. For the second case, the second symbol of the twelfth term is only non-vanishing for $j = 1$. But similarly, since $i = 1$, and $i \neq j$, the term is vanishing overall. The final case is the exact same as the first two where j must equal 3 to be non-vanishing, but $i = 3$, which disallows this condition, and thus the twelfth term is completely vanishing.
- None of the Christoffel symbols are functions of ϕ. The thirteenth term is differentiated with respect to ϕ and therefore the term equals 0.
- Before differentiation, the fourteenth term is only non-vanishing for $i = 1, 2$. However, since when $i = 1$ and $i = 2$ the term is a function of r (1) and θ (2) respectively, and since the term is then differentiated by j where $i \neq j$, the term is vanishing.
- For the first symbol of the fifteenth term to be non-vanishing, α must equal 2 or 3. However, if $\alpha = 3$, then the second symbol is vanishing. Thus we are left with $\alpha = 2$, which gives $\Gamma^2_{12}\Gamma^3_{23} = \frac{1}{r}\cot\theta$.
- For the sixteenth term, we have a similar case to the twelfth term where for certain α, the only case in which both terms are non-vanishing are when $i = j$, which is a contradiction to the condition that $i \neq j$. The only case in which this does not occur is when $\alpha = 3$, which fores i and j to equal 1 and 2 in no particular order (due to symmetry of Christoffel symbols). This results in the sixteenth term simplifying to $-\Gamma^3_{13}\Gamma^3_{23} = -\frac{1}{r}\cot\theta$.Thus this cancels pairwise with the fifteenth term.

Thus, we have

$$R_{ij} = 0 \tag{178}$$

The next case that must be considered for $\mu \neq \nu$ is R_{0i}, which is given as the following:

$$\begin{aligned} R_{0i} = &\Gamma^0_{0i,0} - \Gamma^0_{00,i} + \Gamma^\alpha_{0i}\Gamma^0_{\alpha 0} - \Gamma^\alpha_{00}\Gamma^0_{\alpha i} \\ &+\Gamma^1_{0i,1} - \Gamma^1_{01,i} + \Gamma^\alpha_{0i}\Gamma^1_{\alpha 1} - \Gamma^\alpha_{01}\Gamma^1_{\alpha i} \\ &+\Gamma^2_{0i,2} - \Gamma^2_{02,i} + \Gamma^\alpha_{0i}\Gamma^2_{\alpha 2} - \Gamma^\alpha_{02}\Gamma^2_{\alpha i} \\ &+\Gamma^2_{0i,3} - \Gamma^3_{03,i} + \Gamma^\alpha_{0i}\Gamma^3_{\alpha 3} - \Gamma^\alpha_{03}\Gamma^3_{\alpha i} \end{aligned} \tag{179}$$

- The first term is differentiated with respect to time, and since this is a static body, this term is vanishing.
- The second term of this is zero since the Christoffel symbol prior to differentiation is zero, which when differentiated is also zero.
- The first symbol in the third term is only non-vanishing for $\alpha = 0$; however, this drives the second symbol to zero, and thus the term is vanishing.
- The first symbol in fourth term is only non-vanishing for $\alpha = 1$, but the second symbol is only non-vanishing for $\alpha = 0$, which is contradiction and thus the fourth term is also zero.
- The fifth and sixth terms are vanishing since the Christoffel symbols prior to differentiation are zero, which when differentiated is also zero.
- The first symbol in the seventh term is non-vanishing for $\alpha = 0$, which drives the second symbol to zero, and thus the whole term is vanishing.
- The eighth term is vanishing for the same reason as the seventh term.
- The ninth and tenth terms are vanishing for the same reason as the second term.
- The eleventh and twelfth terms are vanishing for the same reason as seventh term.
- The thirteenth and fourteenth terms are vanishing for the same reason as the second term.
- The fifteenth and sixteenth terms are vanishing for the same reason as the seventh term.

Thus we have that

$$R_{ij} = 0 \tag{180}$$

thus giving that $\forall \mu, \nu$ such that $\mu \neq \nu$, $R_{\mu\nu} = 0$.

Now we must compute the four cases of $R_{\mu\nu}$ such that $\mu = \nu$. Let us begin with R_{00}:

$$\begin{aligned} R_{00} = &\Gamma^0_{00,0} - \Gamma^0_{00,0} + \Gamma^\alpha_{00}\Gamma^0_{\alpha 0} - \Gamma^\alpha_{00}\Gamma^0_{\alpha 0} \\ &+\Gamma^1_{00,1} - \Gamma^1_{01,0} + \Gamma^\alpha_{00}\Gamma^1_{\alpha 1} - \Gamma^\alpha_{01}\Gamma^1_{\alpha 0} \\ &+\Gamma^2_{00,2} - \Gamma^2_{02,0} + \Gamma^\alpha_{00}\Gamma^2_{\alpha 2} - \Gamma^\alpha_{02}\Gamma^2_{\alpha 0} \\ &+\Gamma^3_{00,3} - \Gamma^3_{03,0} + \Gamma^\alpha_{00}\Gamma^3_{\alpha 3} - \Gamma^\alpha_{03}\Gamma^3_{\alpha 0} \end{aligned} \tag{181}$$

- The Christoffel symbols of the first and second terms are zero and thus the first and second terms are zero (also cancel pairwise).
- The third term is only non-vanishing for $\alpha = 1$, however this is also true for the fourth term, and the third and fourth terms cancel pairwise.

- The fifth term Christoffel symbol before differentiation yields $\Gamma^1_{00} = \frac{1}{2V}\frac{dU}{dr}$, which when differentiated yields $\Gamma^1_{00,1} = -\frac{1}{2V^2}\frac{dV}{dr}\frac{dU}{dr} + \frac{1}{2V}\frac{d^2U}{dr^2}$.
- The sixth term is differentiated with respect to time and since this is a static body, this term is vanishing.
- The first symbol of the seventh term is only non-vanishing for $\alpha = 1$. Thus, this yields $\Gamma^1_{00}\Gamma^1_{11} = \frac{1}{4V^2}\frac{dU}{dr}\frac{dV}{dr}$.
- The first symbol of the eighth term is only non-zero for $\alpha = 0$, which thus gives $-\Gamma^0_{01}\Gamma^1_{00} = -\frac{1}{4UV}(\frac{dU}{dr})^2$.
- The ninth and tenth term Christoffel symbols are zero before differentiation and are thus vanishing (see Appendices).
- The eleventh term is only non-vanishing for $\alpha = 1$, yielding $\Gamma^1_{00}\Gamma^2_{12} = \frac{1}{2Vr}\frac{dU}{dr}$.
- The first symbol of the twelfth term is vanishing $\forall\alpha$ and thus the term is vanishing.
- The Christoffel symbol in the thirteenth and fourteenth terms are zero before differentiation and are therefore vanishing.
- The fifteenth term is only non-vanishing for $\alpha = 1$, giving $\Gamma^1_{00}\Gamma^3_{13} = \frac{1}{2Vr}\frac{dU}{dr}$.
- The first symbol of the final term is vanishing $\forall\alpha$ and therefore the term is vanishing.

Thus, compiling our non-vanishing terms, we obtain

$$R_{00} = \Gamma^1_{00,1} - \Gamma^1_{01}\Gamma^1_{10} + \Gamma^1_{00}\Gamma^1_{11} + \Gamma^1_{00}\Gamma^2_{12} + \Gamma^1_{00}\Gamma^3_{13}$$

$$R_{00} = -\frac{1}{2V^2}\frac{dV}{dr}\frac{dU}{dr} + \frac{1}{2V}\frac{d^2U}{dr^2} - \frac{1}{4UV}\left(\frac{dU}{dr}\right)^2 + \frac{1}{4V^2}\frac{dU}{dr}\frac{dV}{dr} + \frac{1}{2Vr}\frac{dU}{dr} + \frac{1}{2Vr}\frac{dU}{dr}$$

$$R_{00} = -\frac{1}{4V^2}\frac{dV}{dr}\frac{dU}{dr} + \frac{1}{2V}\frac{d^2U}{dr^2} - \frac{1}{4UV}\left(\frac{dU}{dr}\right)^2 + \frac{1}{Vr}\frac{dU}{dr} \tag{182}$$

Next we can obtain R_{11}, given as:

$$\begin{aligned} R_{11} = {} & \Gamma^0_{11,0} - \Gamma^0_{10,1} + \Gamma^\alpha_{11}\Gamma^0_{\alpha 0} - \Gamma^\alpha_{10}\Gamma^0_{\alpha 1} \\ & + \Gamma^1_{11,1} - \Gamma^1_{11,1} + \Gamma^\alpha_{11}\Gamma^1_{\alpha 1} - \Gamma^\alpha_{11}\Gamma^1_{\alpha 1} \\ & + \Gamma^2_{11,2} - \Gamma^2_{12,1} + \Gamma^\alpha_{11}\Gamma^2_{\alpha 2} - \Gamma^\alpha_{12}\Gamma^2_{\alpha 1} \\ & + \Gamma^3_{11,3} - \Gamma^3_{13,1} + \Gamma^\alpha_{11}\Gamma^3_{\alpha 3} - \Gamma^\alpha_{13}\Gamma^3_{\alpha 1} \end{aligned} \tag{183}$$

- The Christoffel symbol of the first term before differentiation is zero, and thus the term is vanishing.
- The Christoffel symbol of the second term before differentiation is given as $-\frac{1}{2U}\frac{dU}{dr}$. Thus, when differentiated with respect to r, the term becomes $-\frac{1}{2U}\frac{d^2U}{dr^2} + \frac{1}{2U^2}(\frac{dU}{dr})^2$.
- The first symbol of the third term is only non-vanishing for $\alpha = 1$, therefore giving $\Gamma^1_{11}\Gamma^0_{10} = \frac{1}{4UV}\frac{dU}{dr}\frac{dV}{dr}$.
- Both symbols of the fourth term are only non-vanishing for $\alpha = 0$, which gives $-\Gamma^0_{10}\Gamma^0_{01} = -(\Gamma^0_{01})^2 = -\frac{1}{4U^2}(\frac{dU}{dr})^2$.
- The fifth and sixth terms are both nonzero, though since they cancel pairwise, there is no need to determine what exactly each one equals.
- The seventh and the eighth terms also cancel pairwise.

- The ninth term symbol before differentiation is zero and thus the term is vanishing.
- The tenth term symbol before differentiation is $-\Gamma^2_{12} = -\frac{1}{r}$, which when differentiated with respect to r gives $-\Gamma^2_{12,1} = \frac{1}{r^2}$.
- The eleventh term is only non-vanishing for $\alpha = 1$ which gives $\Gamma^1_{11}\Gamma^2_{12} = \frac{1}{2Vr}\frac{dV}{dr}$.
- The twelfth term is only non-vanishing for $\alpha = 2$. This thus gives $-\Gamma^2_{12}\Gamma^2_{21} = -\frac{1}{r^2}$, which cancels pairwise with the tenth term.
- The thirteenth term symbol before differentiation is zero and thus the term is vanishing.
- The fourteenth term symbol before differentiation is $-\Gamma^3_{13} = -\frac{1}{r}$, which when differentiated with respect to r gives $-\Gamma^3_{13,1} = \frac{1}{r^2}$.
- The fifteenth term is only non-vanishing for $\alpha = 1$ which gives $\Gamma^1_{11}\Gamma^3_{13} = \frac{1}{2Vr}\frac{dV}{dr}$.
- The final term is only non-vanishing for $\alpha = 3$. This thus gives $-\Gamma^3_{13}\Gamma^3_{31} = -\frac{1}{r^2}$,which cancels pairwise with the fourteenth term.

Compiling our non-vanishing terms for R_{11} we obtain

$$R_{11} = -\Gamma^0_{10,1} - \Gamma^0_{10}\Gamma^0_{01} + \Gamma^1_{11}\Gamma^0_{10} + \Gamma^1_{11}\Gamma^2_{12} + \Gamma^1_{11}\Gamma^3_{13}$$

$$R_{11} = -\frac{1}{2U}\frac{d^2U}{dr^2} + \frac{1}{2U^2}\left(\frac{dU}{dr}\right)^2 - \frac{1}{4U^2}\left(\frac{dU}{dr}\right)^2 + \frac{1}{4UV}\frac{dU}{dr}\frac{dV}{dr} + \frac{1}{2Vr}\frac{dV}{dr} + \frac{1}{2Vr}\frac{dV}{dr}$$

$$R_{11} = -\frac{1}{2U}\frac{d^2U}{dr^2} + \frac{1}{4U^2}\left(\frac{dU}{dr}\right)^2 + \frac{1}{4UV}\frac{dU}{dr}\frac{dV}{dr} + \frac{1}{Vr}\frac{dV}{dr} \tag{184}$$

Next we will obtain R_{22} which is given as:

$$\begin{aligned} R_{22} = {} & \Gamma^0_{22,0} - \Gamma^0_{20,2} + \Gamma^\alpha_{22}\Gamma^0_{\alpha 0} - \Gamma^\alpha_{20}\Gamma^0_{\alpha 2} \\ & + \Gamma^1_{22,1} - \Gamma^1_{21,2} + \Gamma^\alpha_{22}\Gamma^1_{\alpha 1} - \Gamma^\alpha_{21}\Gamma^1_{\alpha 2} \\ & + \Gamma^2_{22,2} - \Gamma^2_{22,2} + \Gamma^\alpha_{22}\Gamma^2_{\alpha 2} - \Gamma^\alpha_{22}\Gamma^2_{\alpha 2} \\ & + \Gamma^2_{22,3} - \Gamma^3_{23,2} + \Gamma^\alpha_{22}\Gamma^3_{\alpha 3} - \Gamma^\alpha_{23}\Gamma^3_{\alpha 2} \end{aligned} \tag{185}$$

- The first and second term symbols are zero before differentiation and thus the terms are vanishing.
- The third term is only non-vanishing for $\alpha = 1$, which gives $\Gamma^1_{22}\Gamma^0_{10} = -\frac{r}{2UV}\frac{dU}{dr}$.
- The fourth term is vanishing $\forall\alpha$.
- The fifth term symbol before differentiation is $\Gamma^1_{22} = -\frac{r}{V}$ which when differentiated gives $-\frac{1}{V} + \frac{r}{V^2}\frac{dV}{dr}$.
- The sixth term symbols is zero before differentiation and thus the term is vanishing.
- The seventh term is only non-vanishing for $\alpha = 1$, which gives $\Gamma^1_{22}\Gamma^1_{11} = -\frac{r}{2V^2}\frac{dV}{dr}$.
- The eighth term is only non-vanishing for $\alpha = 2$, which gives $-\Gamma^2_{21}\Gamma^1_{22} = \frac{1}{V}$, which cancels pairwise with the fifth symbol.
- The ninth and tenth terms cancel pairwise.
- The eleventh and twelfth terms cancel pairwise.
- The thirteenth term symbol is zero before differentiation and is thus vanishing.

- The fourteenth term symbol before differentiation is $-\Gamma^3_{23} = \cot\theta$, which when differentiated with respect to θ gives $-\Gamma^3_{23,2} = \csc^2\theta$.
- The fifteenth term is only non-vanishing for $\alpha = 1$, which gives $\Gamma^1_{22}\Gamma^3_{13} = -\frac{1}{V}$.
- The final term is only non-vanishing for $\alpha = 3$ which gives $-\Gamma^3_{23}\Gamma^3_{32} = -\cot^2\theta$.

Compiling these non-vanishing terms, we obtain

$$R_{22} = \Gamma^1_{22}\Gamma^0_{10} + \Gamma^1_{22,1} + \Gamma^1_{22}\Gamma^1_{11} - \Gamma^3_{23,2} - \Gamma^3_{23}\Gamma^3_{32} + \Gamma^1_{22}\Gamma^3_{13}$$

$$R_{22} = -\frac{r}{2UV}\frac{dU}{dr} - \frac{1}{V} + \frac{r}{V^2}\frac{dV}{dr} + \frac{1}{V} - \frac{r}{2V^2}\frac{dV}{dr} + \csc^2\theta - \cot^2\theta - \frac{1}{V}$$

$$R_{22} = -\frac{r}{2UV}\frac{dU}{dr} + \frac{r}{2V^2}\frac{dV}{dr} + 1 - \frac{1}{V} \tag{186}$$

Now we can obtain the final term, R_{33}:

$$\begin{aligned} R_{33} = {} & \Gamma^0_{33,0} - \Gamma^0_{30,3} + \Gamma^\alpha_{33}\Gamma^0_{\alpha 0} - \Gamma^\alpha_{30}\Gamma^0_{\alpha 3} \\ & + \Gamma^1_{33,1} - \Gamma^1_{31,3} + \Gamma^\alpha_{33}\Gamma^1_{\alpha 1} - \Gamma^\alpha_{31}\Gamma^1_{\alpha 3} \\ & + \Gamma^2_{33,2} - \Gamma^2_{32,3} + \Gamma^\alpha_{33}\Gamma^2_{\alpha 2} - \Gamma^\alpha_{32}\Gamma^2_{\alpha 3} \\ & + \Gamma^3_{33,3} - \Gamma^3_{33,3} + \Gamma^\alpha_{33}\Gamma^3_{\alpha 3} - \Gamma^\alpha_{33}\Gamma^3_{\alpha 3} \end{aligned} \tag{187}$$

- The first and second term symbols are zero prior to differentiation and therefore are vanishing.
- The third term is only non-vanishing for $\alpha = 1$, which gives $-\Gamma^1_{33}\Gamma^0_{10} = \frac{r\sin^2\theta}{2UV}\frac{dU}{dr}$.
- The fourth term is vanishing $\forall\alpha$.
- The fifth term symbol before being differentiated gives $-Gamma^1_{33} = -\frac{r\sin^2\theta}{V}$ which when differentiated with respect to r gives $\Gamma^1_{33,1} = -\frac{\sin^2\theta}{V} + \frac{r\sin^2\theta}{V^2}\frac{dV}{dr}$.
- Before differentiation, the sixth term gives $-\Gamma^3_{13} = -\frac{1}{r}$, which when differentiated with respect to ϕ is vanishing.
- The seventh term is only non-vanishing for $\alpha = 1$, which gives $\Gamma^1_{33}\Gamma^1_{11} = -\frac{r\sin^2\theta}{2V^2}\frac{dV}{dr}$.
- The eighth term is only non-vanishing for $\alpha = 3$ which gives $-\Gamma^3_{31}\Gamma^1_{33} = \frac{\sin^2\theta}{V}$.
- The ninth term prior to differentiation is $-\sin\theta\cos\theta$, which when differentiated with respect to θ gives $\Gamma^2_{33,2} = -\cos^2\theta + \sin^2\theta$.
- The tenth term is vanishing before differentiation, and thus remains 0.
- The second symbol of the eleventh term is only non-zero for $\alpha = 1$, thus making the eleventh term $\Gamma^1_{33}\Gamma^2_{12} = -\frac{\sin^2\theta}{V}$.
- The second symbol of the twelfth term is only non-vanishing for $\alpha = 3$, which forces the first symbol to zero, which causes the entire term to become $-\Gamma^3_{32}\Gamma^2_{33} = \sin\theta\cos\theta\cot\theta = \cos^2\theta$.
- The thirteenth and fourteenth terms are vanishing prior to differentiation and thus remain 0.
- The fifteenth and sixteenth terms cancel pairwise.

Compiling these terms, we obtain

$$R_{33} = \Gamma^1_{33}\Gamma^0_{10} + \Gamma^1_{33,1} - \Gamma^3_{31}\Gamma^1_{33} + \Gamma^1_{33}\Gamma^1_{11} + \Gamma^2_{33,2} + \Gamma^3_{32}\Gamma^2_{33} + \Gamma^1_{33}\Gamma^2_{12}$$

$$R_{33} = -\frac{r\sin^2\theta}{2UV} - \frac{\sin^2\theta}{V} + \frac{r\sin^2\theta}{V^2}\frac{dV}{dr} + \frac{\sin^2\theta}{V} - \frac{r\sin^2\theta}{2V^2}\frac{dV}{dr} - \cos^2\theta + \sin^2\theta + \cos^2\theta - \frac{\sin^2\theta}{V}$$

$$R_{33} = -\frac{r\sin^2\theta}{2UV} + \frac{r\sin^2\theta}{2V^2}\frac{dV}{dr} + \sin^2\theta - \frac{\sin^2\theta}{V}$$

$$R_{33} = \left(-\frac{r}{2UV}\frac{dU}{dr} + \frac{r}{2V^2}\frac{dV}{dr} + 1 - \frac{1}{V}\right)\sin^2\theta \tag{188}$$

We can rewrite this in terms of R_{22}

$$R_{33} = R_{22}\sin^2\theta \tag{189}$$

Important to solving Einstein's field equation is the Ricci scalar, given as:

$$R = g^{\mu\nu}R_{\mu\nu} \tag{190}$$

which using the Einstein summation convention tells us to sum over all $\mu\nu$. Plugging in the above results, we obtain:

$$\begin{aligned} R = &-\frac{1}{4UV^2}\frac{dV}{dr}\frac{dU}{dr} + \frac{1}{2UV}\frac{d^2U}{dr^2} - \frac{1}{4U^2V}\left(\frac{dU}{dr}\right)^2 + \frac{1}{UVr}\frac{dU}{dr} \\ &+\frac{1}{2UV}\frac{d^2U}{dr^2} - \frac{1}{4U^2V}\left(\frac{dU}{dr}\right)^2 - \frac{1}{4UV^2}\frac{dU}{dr}\frac{dV}{dr} - \frac{1}{V^2r}\frac{dV}{dr} \\ &+\frac{1}{2rUV}\frac{dU}{dr} - \frac{1}{2rV^2}\frac{dV}{dr} - \frac{1}{r^2} + \frac{1}{r^2V} \\ &+\frac{1}{2rUV}\frac{dU}{dr} - \frac{1}{2rV^2}\frac{dV}{dr} - \frac{1}{r^2} + \frac{1}{r^2V} \end{aligned} \tag{191}$$

This, this simplifies to:

$$R = -\frac{1}{2UV^2}\frac{dV}{dr}\frac{dU}{dr} + \frac{1}{UV}\frac{d^2U}{dr^2} - \frac{1}{2U^2V}\left(\frac{dU}{dr}\right)^2 - \frac{2}{V^2r}\frac{dV}{dr} + \frac{2}{rUV}\frac{dU}{dr} - \frac{2}{r^2} + \frac{2}{r^2V} \tag{192}$$

11.3 The Schwarzschild Metric Explicit Steps

Reiterating Eq 34:

$$R_{00} - \frac{1}{2}g_{00}R = 0 \tag{193}$$

$$R_{11} - \frac{1}{2}g_{11}R = 0 \tag{194}$$

$$R_{22} - \frac{1}{2}g_{22}R = 0 \tag{195}$$

$$R_{33} - \frac{1}{2}g_{33}R = 0 \tag{196}$$

Plugging in the results above Eq 193 and canceling minus signs gives:

$$\frac{1}{4V^2}\frac{dV}{dr}\frac{dU}{dr} - \frac{1}{2V}\frac{d^2U}{dr^2} + \frac{1}{4UV}\left(\frac{dU}{dr}\right)^2 - \frac{1}{Vr}\frac{dU}{dr}$$

$$-\frac{U}{2}\left(\frac{1}{2UV^2}\frac{dV}{dr}\frac{dU}{dr}-\frac{1}{UV}\frac{d^2U}{dr^2}+\frac{1}{2U^2V}\left(\frac{dU}{dr}\right)^2+\frac{2}{V^2r}\frac{dV}{dr}-\frac{2}{rUV}\frac{dU}{dr}+\frac{2}{r^2}-\frac{2}{r^2V}\right)=0$$

$$\frac{1}{4V^2}\frac{dV}{dr}\frac{dU}{dr}-\frac{1}{2V}\frac{d^2U}{dr^2}+\frac{1}{4UV}\left(\frac{dU}{dr}\right)^2-\frac{1}{Vr}\frac{dU}{dr}$$

$$\frac{1}{4V^2}\frac{dV}{dr}\frac{dU}{dr}-\frac{1}{2V}\frac{d^2U}{dr^2}+\frac{1}{4UV}\left(\frac{dU}{dr}\right)^2+\frac{U}{V^2r}\frac{dV}{dr}-\frac{1}{rV}\frac{dU}{dr}+\frac{2}{r^2}\left(1-\frac{1}{V}\right)=0$$

This simplifies to the following:

$$\frac{1}{V^2}\frac{dV}{dr}+\frac{1}{r}\left(1-\frac{1}{V}\right)=0 \tag{197}$$

Next, plugging in the results above Eq 194 and canceling minus signs gives:

$$\frac{1}{2U}\frac{d^2U}{dr^2}-\frac{1}{4U^2}\left(\frac{dU}{dr}\right)^2-\frac{1}{4UV}\frac{dU}{dr}\frac{dV}{dr}-\frac{1}{Vr}\frac{dV}{dr}$$

$$+\frac{V}{2}\left(\frac{1}{2UV^2}\frac{dV}{dr}\frac{dU}{dr}-\frac{1}{UV}\frac{d^2U}{dr^2}+\frac{1}{2U^2V}\left(\frac{dU}{dr}\right)^2+\frac{2}{V^2r}\frac{dV}{dr}-\frac{2}{rUV}\frac{dU}{dr}+\frac{2}{r^2}-\frac{2}{r^2V}\right)=0$$

$$\frac{1}{2U}\frac{d^2U}{dr^2}-\frac{1}{4U^2}\left(\frac{dU}{dr}\right)^2-\frac{1}{4UV}\frac{dU}{dr}\frac{dV}{dr}-\frac{1}{Vr}\frac{dV}{dr}$$

$$+\frac{1}{4UV}\frac{dV}{dr}\frac{dU}{dr}-\frac{1}{2U}\frac{d^2U}{dr^2}+\frac{1}{4U^2}\left(\frac{dU}{dr}\right)^2+\frac{1}{Vr}\frac{dV}{dr}-\frac{1}{rU}\frac{dU}{dr}+\frac{V}{r^2}\left(1-\frac{1}{V}\right)=0$$

This simplifies to the following:

$$-\frac{1}{U}\frac{dU}{dr}+\frac{1g}{r}(V-1)=0 \tag{198}$$

Next, plugging in the results above Eq 195 and canceling minus signs gives:

$$\frac{r}{2UV}\frac{dU}{dr}-\frac{r}{2V^2}\frac{dV}{dr}-1+\frac{1}{V}$$

$$+\frac{r^2}{2}\left(\frac{1}{2UV^2}\frac{dV}{dr}\frac{dU}{dr}-\frac{1}{UV}\frac{d^2U}{dr^2}+\frac{1}{2U^2V}\left(\frac{dU}{dr}\right)^2+\frac{2}{V^2r}\frac{dV}{dr}-\frac{2}{rUV}\frac{dU}{dr}+\frac{2}{r^2}-\frac{2}{r^2V}\right)=0$$

$$\frac{r}{2UV}\frac{dU}{dr}-\frac{r}{2V^2}\frac{dV}{dr}-1+\frac{1}{V}$$

$$+\frac{r^2}{4UV^2}\frac{dV}{dr}\frac{dU}{dr}-\frac{r^2}{2UV}\frac{d^2U}{dr^2}+\frac{r^2}{4U^2V}\left(\frac{dU}{dr}\right)^2+\frac{r}{V^2}\frac{dV}{dr}-\frac{r}{UV}\frac{dU}{dr}+1-\frac{1}{V}=0$$

This simplifies to the following:

$$-\frac{1}{U}\frac{dU}{dr}+\frac{1}{V}\frac{dV}{dr}+\frac{r}{2UV}\frac{dU}{dr}\frac{dV}{dr}-\frac{r}{U}\frac{d^2U}{dr^2}+\frac{r}{2U^2}\left(\frac{dU}{dr}\right)^2=0 \tag{199}$$

This is the same result as for $\mu\nu = 33$ and thus that equation will again be omitted here.

Now that we have a system of equations, we can use these to solve for the components of the metric. Taking the first equation's result,

$$\frac{1}{V^2}\frac{dV}{dr}+\frac{1}{r}\left(1-\frac{1}{V}\right)=0$$

Rearranging, we obtain:

$$-\frac{1}{r}\left(1-\frac{1}{V}\right)=\frac{1}{V^2}\frac{dV}{dr}$$

$$-\frac{dr}{r} = \frac{dV}{V^2(1-\frac{1}{V})} = \frac{dV}{V(V-1)} = \frac{dV}{V-1} - \frac{dV}{V}$$

Integrating both sides of this, we get

$$-\int \frac{dr}{r} = \int \frac{dV}{(V-1)} - \int \frac{dV}{V}$$

$$\ln\frac{1}{r} + K = \ln\frac{V-1}{V}$$

Letting $C = e^K$, we get

$$\frac{C}{r} = \frac{V-1}{V}$$

Which when solved for V becomes:

$$\frac{CV}{r} = V - 1$$

$$1 = V - \frac{CV}{r}$$

$$V = \frac{1}{1-\frac{C}{r}} \tag{200}$$

We can then plug this into Eq 198, giving:

$$-\frac{1}{U}\frac{dU}{dr} + \frac{1}{r}\left(\frac{1}{1-\frac{C}{r}} - \frac{1-\frac{C}{r}}{1-\frac{C}{r}}\right) = 0$$

$$-\frac{1}{U}\frac{dU}{dr} + \frac{1}{r}\left(\frac{\frac{C}{r}}{1-\frac{C}{r}}\right) = 0$$

Rearranging, we obtain:

$$\frac{dr}{r}\left(\frac{C}{r-C}\right) = \frac{dU}{U}$$

$$\left(\frac{1}{r-C} - \frac{1}{r}\right)dr = \frac{dU}{U}$$

Thus, integrating as above, we get:

$$\ln\frac{r-C}{r} = \ln U + A$$

Without loss of generality, here we take $A = 0$, giving:

$$U = \frac{r-C}{r}$$

Which when put in the same form as above gives

$$U = 1 - \frac{C}{r} \tag{201}$$

Plugging these results into Eq 2, we obtain:

$$ds^2 = (1-\frac{C}{r})dt^2 - \left(\frac{1}{1-\frac{C}{r}}\right)dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2 \tag{202}$$

Connecting this with Newtonian Gravity using c as the escape velocity, we find that $C = \frac{2GM}{c^2}$, giving the final form the Schwarzschild metric:

$$ds^2 = (1-\frac{2GM}{c^2 r})dt^2 - \left(\frac{1}{1-\frac{2GM}{c^2 r}}\right)dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2 \tag{203}$$

11.4 Christoffel Symbols for a Black Hole with Scalar Hair

The definition of a Christoffel symbol remains the same and is thus given as:

$$\Gamma^{\mu}_{\nu\sigma} = \frac{1}{2}g^{\mu\lambda}\{g_{\lambda\nu,\sigma} + g_{\lambda\sigma,\nu} - g_{\nu\sigma,\lambda}\}$$

Thus giving

$$\Gamma^{0}_{00} = \frac{1}{2}g^{00}\{g_{00,0} + g_{00,0} - g_{00,0}\} = 0$$

$$\Gamma^{0}_{0i} = \frac{1}{2}g^{00}\{g_{00,i} + g_{0i,0} - g_{0i,0}\} = \frac{1}{2}\frac{f'(r)}{f(r)}(i = 1, else = 0)$$

$$\Gamma^{0}_{ij} = \frac{1}{2}g^{00}\{g_{0i,j} + g_{0j,i} - g_{ij,0}\} = 0$$

$$\Gamma^{1}_{00} = \frac{1}{2}g^{11}\{g_{10,0} + g_{10,0} - g_{00,1}\} = \frac{1}{2}f'(r)f(r)$$

$$\Gamma^{1}_{0i} = \frac{1}{2}g^{11}\{g_{10,i} + g_{1i,0} - g_{0i,1}\} = 0$$

$$\Gamma^{1}_{ij,i\neq j} = \frac{1}{2}g^{11}\{g_{1i,j} + g_{1j,i} - g_{ij,1}\} = 0$$

$$\Gamma^{1}_{11} = \frac{1}{2}g^{11}\{g_{11,1} + g_{11,1} - g_{11,1}\} = -\frac{1}{2}\frac{f'(r)}{f(r)}$$

$$\Gamma^{1}_{22} = \frac{1}{2}g^{11}\{g_{12,2} + g_{12,2} - g_{22,1}\} = \frac{-f(r)a(r)a'(r)}{1 - k\rho^2}$$

$$\Gamma^{1}_{33} = \frac{1}{2}g^{11}\{g_{13,3} + g_{13,3} - g_{33,1}\} = -f(r)a(r)a'(r)\rho^2$$

$$\Gamma^{2}_{00} = \frac{1}{2}g^{22}\{g_{20,0} + g_{20,0} - g_{00,2}\} = 0$$

$$\Gamma^{2}_{0i} = \frac{1}{2}g^{22}\{g_{20,i} + g_{2i,0} - g_{0i,2}\} = 0$$

$$\Gamma^{2}_{11} = \frac{1}{2}g^{22}\{g_{21,1} + g_{21,1} - g_{11,2}\} = 0$$

$$\Gamma^{2}_{22} = \frac{1}{2}g^{22}\{g_{22,2} + g_{22,2} - g_{22,2}\} = \frac{1}{2}g^{22}g_{22,2} = \frac{k\rho}{1 - k\rho^2}$$

$$\Gamma^{2}_{33} = \frac{1}{2}g^{22}\{g_{23,3} + g_{23,3} - g_{33,2}\} = -\rho(1 - k\rho^2)$$

$$\Gamma^{2}_{12} = \frac{1}{2}g^{22}\{g_{21,2} + g_{22,1} - g_{12,2}\} = \frac{a'(r)}{a(r)}$$

$$\Gamma^{2}_{23} = \frac{1}{2}g^{22}\{g_{22,3} + g_{23,2} - g_{23,2}\} = 0$$

$$\Gamma^{2}_{13} = \frac{1}{2}g^{22}\{g_{21,3} + g_{23,1} - g_{13,2}\} = 0$$

$$\Gamma^{3}_{00} = \frac{1}{2}g^{33}\{g_{30,0} + g_{30,0} - g_{00,3}\} = 0$$

$$\Gamma^{3}_{0i} = \frac{1}{2}g^{33}\{g_{30,i} + g_{3i,0} - g_{0i,3}\} = \frac{1}{2}g^{33}g_{33,0} = 0$$

$$\Gamma^3_{ii} = \frac{1}{2}g^{33}\{g_{3i,i} + g_{3i,i} - g_{ii,3}\} = 0$$

$$\Gamma^3_{12} = \frac{1}{2}g^{33}\{g_{31,2} + g_{32,1} - g_{12,3}\} = 0$$

$$\Gamma^3_{23} = \frac{1}{2}g^{33}\{g_{32,3} + g_{33,2} - g_{23,3}\} = \frac{1}{\rho}$$

$$\Gamma^3_{13} = \frac{1}{2}g^{33}\{g_{31,3} + g_{33,1} - g_{13,3}\} = \frac{a'(r)}{a(r)}$$

11.5 Explicit Calculation of the Ricci Tensor Components for the Hairy Black Hole

As a reminder, the definition of the Ricci curvature tensor is given as:

$$\begin{aligned} R_{\mu\nu} = &\Gamma^0_{\mu\nu,0} - \Gamma^0_{\mu0,\nu} + \Gamma^\alpha_{\mu\nu}\Gamma^0_{\alpha0} - \Gamma^\alpha_{\mu0}\Gamma^0_{\alpha\nu} \\ &+\Gamma^1_{\mu\nu,1} - \Gamma^1_{\mu1,\nu} + \Gamma^\alpha_{\mu\nu}\Gamma^1_{\alpha1} - \Gamma^\alpha_{\mu1}\Gamma^1_{\alpha\nu} \\ &+\Gamma^2_{\mu\nu,2} - \Gamma^2_{\mu2,\nu} + \Gamma^\alpha_{\mu\nu}\Gamma^2_{\alpha2} - \Gamma^\alpha_{\mu2}\Gamma^2_{\alpha\nu} \\ &+\Gamma^3_{\mu\nu,3} - \Gamma^3_{\mu3,\nu} + \Gamma^\alpha_{\mu\nu}\Gamma^3_{\alpha3} - \Gamma^\alpha_{\mu3}\Gamma^3_{\alpha\nu}. \end{aligned} \tag{204}$$

First, let us consider R_{0i}, which is thus given as:

$$\begin{aligned} R_{0i} = &\Gamma^0_{0i,0} - \Gamma^0_{00,i} + \Gamma^\alpha_{0i}\Gamma^0_{\alpha0} - \Gamma^\alpha_{00}\Gamma^0_{\alpha i} \\ &+\Gamma^1_{0i,1} - \Gamma^1_{01,i} + \Gamma^\alpha_{0i}\Gamma^1_{\alpha1} - \Gamma^\alpha_{01}\Gamma^1_{\alpha i} \\ &+\Gamma^2_{0i,2} - \Gamma^2_{02,i} + \Gamma^\alpha_{0i}\Gamma^2_{\alpha2} - \Gamma^\alpha_{02}\Gamma^2_{\alpha i} \\ &+\Gamma^2_{0i,3} - \Gamma^3_{03,i} + \Gamma^\alpha_{0i}\Gamma^3_{\alpha3} - \Gamma^\alpha_{03}\Gamma^3_{\alpha i} \end{aligned} \tag{205}$$

Now for the sake of book-keeping, let us number each term, i.e. $\Gamma^0_{0i,0}$ is 1, $\Gamma^0_{00,i}$ is 2, and so on.

1. Vanishing since $\Gamma^0_{0i} = 0$ as given above.
2. Vanishing since $\Gamma^0_{00} = 0$ as given above.
3. The first symbol in this term is non-zero for $\alpha = 0$. However, as above, we have that $\Gamma^0_{00} = 0$ giving that the second symbol is vanishing for $\alpha = 0$ and thus the whole term is vanishing.
4. The first symbol in this term is non-zero for $\alpha = 1$. However, as above, we have that $\Gamma^0_{1i} = 0$ giving that the second symbol is vanishing for $\alpha = 1$ and thus the term is vanishing.
5. Vanishing since $\Gamma^1_{01} = 0$ as given above.
6. Vanishing since $\Gamma^0_{01} = 0$ as given above.
7. The first symbol in this term is non-zero for $\alpha = 0$. However, as above, we have that $\Gamma^1_{01} = 0$ giving that the second symbol is vanishing for $\alpha = 0$ and thus the term is vanishing.
8. The first symbol in this term is non-zero for $\alpha = 0$. However, as above, we have that $\Gamma^0_{0i} = 0$ giving that the second symbol is vanishing for $\alpha = 0$ and thus the term is vanishing.
9. Vanishing since $\Gamma^2_{0i} = 0$ as given above.
10. Vanishing since $\Gamma^2_{02} = 0$ as given above.

11. The first symbol in this term is non-zero for $\alpha = 0$. However, as above, we have that $\Gamma^2_{02} = 0$ giving that the second symbol is vanishing for $\alpha = 0$ and thus the term is vanishing.

12. Vanishing since the first symbol is vanishing $\forall\alpha$.

13. Vanishing since $\Gamma^3_{0i} = 0$ as given above.

14. Vanishing since $\Gamma^3_{03} = 0$ as given above.

15. The first symbol is non-zero for only $\alpha = 0$, for which the second symbol is zero, and thus the whole term is vanishing.

16. Vanishing since the first symbol is zero $\forall\alpha$.

Since every term is zero for R_{0i}, we have that

$$R_{0i} = 0 \tag{206}$$

as was for the Schwarzschild case.
Next, we will consider R_{ij} for $i \neq j$

$$\begin{aligned} R_{ij} = {} & \Gamma^0_{ij,0} - \Gamma^0_{i0,j} + \Gamma^\alpha_{ij}\Gamma^0_{\alpha 0} - \Gamma^\alpha_{i0}\Gamma^0_{\alpha j} \\ & +\Gamma^1_{ij,1} - \Gamma^1_{i1,j} + \Gamma^\alpha_{ij}\Gamma^1_{\alpha 1} - \Gamma^\alpha_{i1}\Gamma^1_{\alpha j} \\ & +\Gamma^2_{ij,2} - \Gamma^2_{i2,j} + \Gamma^\alpha_{ij}\Gamma^2_{\alpha 2} - \Gamma^\alpha_{i2}\Gamma^2_{\alpha j} \\ & +\Gamma^3_{ij,3} - \Gamma^3_{i3,j} + \Gamma^\alpha_{ij}\Gamma^3_{\alpha 3} - \Gamma^\alpha_{i3}\Gamma^3_{\alpha j}. \end{aligned} \tag{207}$$

Continuing as above, we have:

1. Vanishing since $\Gamma^0_{ij} = 0$.

2. Non-zero for $i = 1$ before differentiation. Since $i \neq j$, $j = 2$, but Γ^0_{01} is independent of Ω, and thus the term is vanishing.

3. The first symbol is non-zero for $\alpha = 2$, but since $\Gamma^0_{20} = 0$, the term is vanishing.

4. The first symbol is non-zero only for $\alpha = 0$ and $i = 1$. The second symbol is non-zero only for $\alpha = 0$ and $j = 1$. But $i \neq j$, and therefore the term is vanishing.

5. Vanishing since $\Gamma^1_{ij} = 0$.

6. Before differentiation, the term is non-zero for $i = 1$. Since $i \neq j$, we have that $j = 2$. But since Γ^1_{11} is independent of ρ, the term is vanishing.

7. The first symbol is non-zero for $\alpha = 2$. However, since $\Gamma^1_{21} = 0$ and thus the second symbol is zero for $\alpha = 2$ and thus the term is vanishing.

8. For $\alpha = 1$, the first symbol is non-zero for $i = 1$. However, the second symbol for $\alpha = 1$ is non-zero for $j = 1$, but $i \neq j$ so the term is vanishing for $\alpha = 1$. For $\alpha = 2$, the first symbol is non-zero for $i = 2$, and when $\alpha = 2$, the second symbol is non-zero for $j = 2$. But again, $i \neq j$ and thus the term is vanishing.

9. Since Γ^2_{12} is independent of ρ, the term is vanishing.

10. Before differentiation, the term is non-zero for $i = 1$, which implies that $j = 2$. However, since Γ^2_{12} is independent of ρ, we have that the term is vanishing.

11. The first symbol is non-zero for $\alpha = 2, 3$, and the second symbol is non-zero for $\alpha = 1, 2$. Thus, taking the intersection of these two sets, we can say that $\alpha = 2$, and thus the term becomes $\Gamma^2_{12}\Gamma^2_{22}$. However, note that this is equal to the previous term, and thus the two terms cancel pairwise.

12. For $\alpha = 1$, the first symbol is non-zero for $i = 2$ which implies that $j = 1$ since $i \neq j$. For this case, the second symbol becomes Γ^2_{11}, which as given above is zero and thus the term is vanishing for $\alpha = 1$. For $\alpha = 2$, the first term is non-zero for $i = 1$, which implies that $j = 2$. Thus,the second symbol becomes $\Gamma^2_{22} = \dfrac{k\rho}{1 - k\rho^2}$. Thus, the term becomes $\Gamma^2_{12}\Gamma^2_{22}$

13. All symbols are independent of ϕ and thus the term is vanishing.

14. Before differentiation, the symbol is non-zero for $i = 1, 2$. For $i = 1$, we have that the symbol is dependent on only r, and thus after differentiation, the symbol is zero for $j = 2, 3$, which it must since $i \neq j$ and thus the term is vanishing. Similarly, for $i = 2$, $j = 1, 3$ since $i \neq j$, and since for $i = 2$, the symbol is dependent on only ρ, we can say that after differentiation with respect to either r or ϕ (i.e. $j = 1, 3$), the symbol is vanishing.

15. The first symbol is non-zero for $\alpha = 2, 3$, and the second symbol is non-zero for $\alpha = 1, 2$. Thus, taking the intersection of these two sets, we have that $\alpha = 2$ and $i = 1$ an $j = 2$ (without loss of generality due to the symmetry of the Christoffel symbols). Thus, the term becomes $-\Gamma^2_{12}\Gamma^3_{23}$, which note is the additive inverse of the previous term, and thus the two terms cancel pairwise.

16. The first symbol is non-zero only for $\alpha = 3$ and $i = 1, 2$. The second symbol for $\alpha = 3$ is non-zero only for $j = 1, 2$. Thus, we have two cases where $i = 1$ and $j = 2$ or $i = 2$ and $j = 1$, which produce the same result, and the term becomes $\Gamma^3_{13}\Gamma^3_{23}$

Compiling these results, we see that we have

$$R_{ij} = 0 \tag{208}$$

Next we will consider R_{00}, given as:

$$\begin{aligned} R_{00} &= \Gamma^0_{00,0} - \Gamma^0_{00,0} + \Gamma^\alpha_{00}\Gamma^0_{\alpha 0} - \Gamma^\alpha_{00}\Gamma^0_{\alpha 0} \\ &+\Gamma^1_{00,1} - \Gamma^1_{01,0} + \Gamma^\alpha_{00}\Gamma^1_{\alpha 1} - \Gamma^\alpha_{01}\Gamma^1_{\alpha 0} \\ &+\Gamma^2_{00,2} - \Gamma^2_{02,0} + \Gamma^\alpha_{00}\Gamma^2_{\alpha 2} - \Gamma^\alpha_{02}\Gamma^2_{\alpha 0} \\ &+\Gamma^3_{00,3} - \Gamma^3_{03,0} + \Gamma^\alpha_{00}\Gamma^3_{\alpha 3} - \Gamma^\alpha_{03}\Gamma^3_{\alpha 0} \end{aligned} \tag{209}$$

Continuing, we have:

1. Vanishing since $\Gamma^0_{00} = 0$.
2. Vanishing since $\Gamma^0_{00} = 0$.
3. Cancels pairwise with the fourth term.
4. Cancels pairwise with the third term.
5. Before differentiation, we have $\Gamma^1_{00} = \dfrac{1}{2}f'(r)f(r)$. Thus, differentiating with respect to r and multiplying by -1, we obtain $-\dfrac{1}{2}f''(r)f(r) - \dfrac{1}{2}(f'(r))^2$
6. As above, $\Gamma^1_{01} = 0$ and thus the term is vanishing.
7. The first symbol is non-zero only for $\alpha = 1$ and thus the term becomes $-\Gamma^1_{00}\Gamma^1_{11} = -\dfrac{1}{2}f'(r)f(r)\left(-\dfrac{1}{2}\dfrac{f'(r)}{f(r)}\right) = \dfrac{1}{4}(f'(r))^2$.

8. The first symbol of this term is non-zero only for $\alpha = 0$, which makes the second symbol become Γ^1_{00}. Thus, the term becomes $\left(\frac{1}{2}\frac{f'(r)}{f(r)}\right)\left(\frac{1}{2}f'(r)f(r)\right) = \frac{1}{4}(f'(r))^2$

9. Vanishing since $\Gamma^2_{00} = 0$, as given above.

10. Vanishing since $\Gamma^2_{02} = 0$, as given above.

11. The first symbol is non-zero for only $\alpha = 1$, giving that the second term becomes Γ^2_{12}. Thus, the term becomes $-\Gamma^1_{00}\Gamma^2_{12} = -\left(\frac{1}{2}f'(r)f(r)\right)\left(\frac{a'(r)}{a(r)}\right) = -\frac{f'(r)f(r)a'(r)}{2a(r)}$.

12. The first symbol is vanishing $\forall\alpha$ and thus the term is vanishing.

13. Vanishing since $\Gamma^3_{00} = 0$ as given above.

14. Vanishing since $\Gamma^3_{0i} = 0$ as given above.

15. The first symbol is non-zero for only $\alpha = 1$. Thus, the term becomes $-\Gamma^1_{00}\Gamma^3_{13} = \frac{1}{2}f'(r)f(r)\left(\frac{a'(r)}{a(r)}\right)$

16. The first symbol is vanishing $\forall\alpha$ and thus the term is vanishing.

Thus, compiling these results, we obtain:

$$R_{00} = -\frac{1}{2}f''(r)f(r) - \frac{f'(r)f(r)a'(r)}{a(r)} \tag{210}$$

Next, consider R_{11}, given as:

$$\begin{aligned} R_{11} = {} & \Gamma^0_{11,0} - \Gamma^0_{10,1} + \Gamma^\alpha_{11}\Gamma^0_{\alpha 0} - \Gamma^\alpha_{10}\Gamma^0_{\alpha 1} \\ & +\Gamma^1_{11,1} - \Gamma^1_{11,1} + \Gamma^\alpha_{11}\Gamma^1_{\alpha 1} - \Gamma^\alpha_{11}\Gamma^1_{\alpha 1} \\ & +\Gamma^2_{11,2} - \Gamma^2_{12,1} + \Gamma^\alpha_{11}\Gamma^2_{\alpha 2} - \Gamma^\alpha_{12}\Gamma^2_{\alpha 1} \\ & +\Gamma^3_{11,3} - \Gamma^3_{13,1} + \Gamma^\alpha_{11}\Gamma^3_{\alpha 3} - \Gamma^\alpha_{13}\Gamma^3_{\alpha 1} \end{aligned} \tag{211}$$

1. Vanishing since $\Gamma^0_{11} = 0$ as given above.

2. Before differentiation, we have $\Gamma^0_{01} = \frac{1}{2}\frac{f'(r)}{f(r)}$. Thus, differentiating, we obtain $\frac{1}{2}(f''(r)f^{-1}(r) - (f'(r))^2 f^{-2}(r)) = \frac{1}{2}\frac{f''(r)}{f(r)} - \frac{1}{2}\left(\frac{f'(r)}{f(r)}\right)^2$.

3. The first symbol is non-zero only for $\alpha = 1$. Thus, the term becomes $-\left(-\frac{1}{2}\frac{f'(r)}{f(r)}\right)\left(\frac{1}{2}\frac{f'(r)}{f(r)}\right) = \frac{1}{4}\left(\frac{f'(r)}{f(r)}\right)^2$

4. The first symbol is non-zero for only $\alpha = 0$. Thus the term becomes $(\Gamma^0_{01})^2 = \frac{1}{4}\left(\frac{f'(r)}{f(r)}\right)^2$

5. Cancels pairwise with the sixth term.

6. Cancels pairwise with the fifth term.

7. Cancels pairwise with the eighth term.

8. Cancels pairwise with the seventh term.

9. Vanishing since $\Gamma^2_{11} = 0$ as given above.

10. Before differentiation, we have $\Gamma^2_{12} = \frac{a'(r)}{a(r)}$. Thus, differentiating, we obtain $\Gamma^2_{12,1} = \frac{d}{dr}\left(\frac{a'(r)}{a(r)}\right) = \frac{a''(r)}{a(r)} - \left(\frac{a'(r)}{a(r)}\right)^2$.

11. The first symbol is non-zero for only $\alpha = 1$, which thus makes the term become $-\left(-\frac{1}{2}\frac{f'(r)}{f(r)}\right)\left(\frac{a'(r)}{a(r)}\right) = \frac{1}{2}\frac{f'(r)a'(r)}{f(r)a(r)}$

12. The first symbol is non-zero for only $\alpha = 2$. Thus, the term becomes $(\Gamma^2_{12})^2 = \left(\frac{a'(r)}{a(r)}\right)^2$.

13. Vanishing since $\Gamma^3_{ii} = 0$, as given above.

14. Before differentiation, we have $\Gamma^3_{13} = \frac{a'(r)}{a(r)}$. Thus, differentiating, we have $\Gamma^3_{13,1} = \frac{d}{dr}\left(\frac{a'(r)}{a(r)}\right) = \frac{a''(r)}{a(r)} - \left(\frac{a'(r)}{a(r)}\right)^2$

15. The first symbol is non-zero for only $\alpha = 1$, thus making the term $-\Gamma^1_{11}\Gamma^3_{13} = \frac{1}{2}\frac{f'(r)}{f(r)}\left(\frac{a'(r)}{a(r)}\right)$

16. The first symbol is non-zero for only $\alpha = 3$, and thus the term becomes $(\Gamma^3_{13})^2 = \left(\frac{a'(r)}{a(r)}\right)^2$

Thus compiling these above terms, we obtain:

$$R_{11} = \frac{1}{2}\frac{f''(r)}{f(r)} - \frac{1}{2}\left(\frac{f'(r)}{f(r)}\right)^2 + \frac{1}{4}\left(\frac{f'(r)}{f(r)}\right)^2 + \frac{1}{4}\left(\frac{f'(r)}{f(r)}\right)^2 + \frac{a''(r)}{a(r)} - \left(\frac{a'(r)}{a(r)}\right)^2 + \left(\frac{a'(r)}{a(r)}\right)^2 + \frac{1}{2}\frac{f'(r)a'(r)}{f(r)a(r)}$$
$$+ \frac{a''(r)}{a(r)} - \left(\frac{a'(r)}{a(r)}\right)^2 + \left(\frac{a'(r)}{a(r)}\right)^2 + \frac{1}{2}\frac{f'(r)}{f(r)}\left(\frac{a'(r)}{a(r)}\right)$$

Thus, simplifying, we obtain:

$$R_{11} = \frac{1}{2}\frac{f''(r)}{f(r)} + 2\frac{a''(r)}{a(r)} + \frac{f'(r)a'(r)}{f(r)a(r)} \tag{212}$$

Next, consider R_{22}, which is give as

$$\begin{aligned} R_{22} = {} & \Gamma^0_{22,0} - \Gamma^0_{20,2} + \Gamma^\alpha_{22}\Gamma^0_{\alpha 0} - \Gamma^\alpha_{20}\Gamma^0_{\alpha 2} \\ & + \Gamma^1_{22,1} - \Gamma^1_{21,2} + \Gamma^\alpha_{22}\Gamma^1_{\alpha 1} - \Gamma^\alpha_{21}\Gamma^1_{\alpha 2} \\ & + \Gamma^2_{22,2} - \Gamma^2_{22,2} + \Gamma^\alpha_{22}\Gamma^2_{\alpha 2} - \Gamma^\alpha_{22}\Gamma^2_{\alpha 2} \\ & + \Gamma^2_{22,3} - \Gamma^3_{23,2} + \Gamma^\alpha_{22}\Gamma^3_{\alpha 3} - \Gamma^\alpha_{23}\Gamma^3_{\alpha 2} \end{aligned} \tag{213}$$

1. Vanishing since $\Gamma^0_{22} = 0$ as given above.

2. Vanishing since $\Gamma^0_{02} = 0$ as given above.

3. The second symbol is non-zero for only $\alpha = 1$, and thus the term becomes $-\frac{-f(r)a(r)a'(r)}{1 - k\rho^2}\left(\frac{1}{2}\frac{f'(r)}{f(r)}\right) = \frac{f'(r)a(r)a'(r)}{2(1 - k\rho^2)}$.

4. The first symbol is vanishing $\forall\alpha$ and thus the term is vanishing.

5. Before differentiation, we have $\dfrac{f(r)a(r)a'(r)}{1-k\rho^2}$. Thus, differentiating, we obtain $\dfrac{f'(r)a'(r)a(r)+f(r)(a'(r))^2+f(r)a(r)a''(r)}{1-k\rho^2}$.

6. Vanishing since $\Gamma^1_{12}=0$ as given above.

7. The second symbol is non-zero for only $\alpha=1$, and thus we obtain $-\left(-\dfrac{1}{2}\dfrac{f'(r)}{f(r)}\right)\left(\dfrac{-f(r)a(r)a'(r)}{1-k\rho^2}\right)=-\dfrac{f'(r)a(r)a'(r)}{2(1-k\rho^2)}$.

8. The first symbol is non-zero for only $\alpha=2$. Thus, we obtain $\left(\dfrac{a'(r)}{a(r)}\right)\left(\dfrac{-f(r)a(r)a'(r)}{1-k\rho^2}\right)=-\dfrac{f(r)(a'(r))^2}{1-k\rho^2}$

9. Cancels pairwise with the tenth term.

10. Cancels pairwise with the ninth term.

11. Cancels pairwise with the twelfth term.

12. Cancels pairwise with the eleventh term.

13. Vanishing since $\Gamma^3_{ii}=0$ as given above.

14. Before differentiation, the symbol is given as $\Gamma^3_{23}=\dfrac{1}{\rho}$. Thus, differentiating with respect to ρ, we obtain $\Gamma^3_{23,2}=-\dfrac{1}{\rho^2}$.

15. Both symbols are non-zero only for $\alpha=1,2$, in which case we sum over the two results, which are $-\Gamma^1_{22}\Gamma^3_{13}=\dfrac{f(r)a(r)a'(r)}{1-k\rho^2}\left(\dfrac{a'(r)}{a(r)}\right)=\dfrac{f(r)(a'(r))^2}{1-k\rho^2}$ and $-\Gamma^2_{22}\Gamma^3_{23}=-\dfrac{k}{1-k\rho^2}$ respectively.

16. The first term is only non-zero for $\alpha=3$, in which case the term becomes $(\Gamma^3_{23})^2=\dfrac{1}{\rho^2}$, which we see cancels pairwise with term number 13.

Thus, compiling, we obtain:

$$R_{22}=\frac{f'(r)a(r)a'(r)}{2(1-k\rho^2)}+\frac{f'(r)a'(r)a(r)+f(r)(a'(r))^2+f(r)a(r)a''(r)}{1-k\rho^2}-\frac{f(r)(a'(r))^2}{1-k\rho^2}-\frac{f'(r)a(r)a'(r)}{2(1-k\rho^2)}$$
$$+\frac{f(r)(a'(r))^2}{1-k\rho^2}-\frac{k}{1-k\rho^2}$$

which thus simplifies to:

$$R_{22}=\frac{f'(r)a'(r)a(r)+f(r)(a'(r))^2+f(r)a(r)a''(r)-k}{1-k\rho^2} \tag{214}$$

Finally, consider R_{33}, which is give as:

$$\begin{aligned}R_{33}&=\Gamma^0_{33,0}-\Gamma^0_{30,3}+\Gamma^\alpha_{33}\Gamma^0_{\alpha 0}-\Gamma^\alpha_{30}\Gamma^0_{\alpha 3}\\&+\Gamma^1_{33,1}-\Gamma^1_{31,3}+\Gamma^\alpha_{33}\Gamma^1_{\alpha 1}-\Gamma^\alpha_{31}\Gamma^1_{\alpha 3}\\&+\Gamma^2_{33,2}-\Gamma^2_{32,3}+\Gamma^\alpha_{33}\Gamma^2_{\alpha 2}-\Gamma^\alpha_{32}\Gamma^2_{\alpha 3}\\&+\Gamma^3_{33,3}-\Gamma^3_{33,3}+\Gamma^\alpha_{33}\Gamma^3_{\alpha 3}-\Gamma^\alpha_{33}\Gamma^3_{\alpha 3}\end{aligned} \tag{215}$$

1. Vanishing since $\Gamma^0_{ii} = 0$ as given above.
2. Vanishing since $\Gamma^0_{0i} = 0$ as given above.
3. The second symbol is non-zero for only $\alpha = 1$, and thus the term becomes $-\Gamma^1_{33}\Gamma^0_{01} = -f(r)a(r)a'(r)\rho^2\left(\dfrac{f'(r)}{2f(r)}\right) = -\dfrac{1}{2}f'(r)a(r)a'(r)\rho^2$.
4. Vanishing since the first symbol is zero $\forall\alpha$.
5. Before differentiation, we have $f(r)a(r)a'(r)\rho^2$. Thus, differentiating, we obtain $f'(r)a'(r)a(r)\rho^2 + f(r)(a'(r))^2\rho^2 + f(r)a(r)a''(r)\rho^2$.
6. Vanishing since $\Gamma^1_{ij} = 0$ as given above.
7. Both symbols are non-zero for only $\alpha = 1$, for which the term becomes $-\Gamma^1_{33}\Gamma^1_{11} = -f(r)a(r)a'(r)\rho^2\left(\dfrac{f'(r)}{2f(r)}\right) = -\dfrac{1}{2}f'(r)a(r)a'(r)\rho^2$.
8. Both symbols are non-zero for only $\alpha = 3$. Thus, the term becomes $\Gamma^3_{13}\Gamma^1_{33} = -f(r)a(r)a'(r)\rho^2\left(\dfrac{a'(r)}{a(r)}\right) = -f(r)(a'(r))^2\rho^2$.
9. Before differentiation, we have $-\Gamma^2_{33} = \rho(1-k\rho^2)$. Thus, differentiating with respect to ρ, we have $-\Gamma^2_{33,2} = 1 - 3k\rho^2$.
10. Vanishing since $\Gamma^2_{23} = 0$ as given above.
11. Both symbols are non-zero for $\alpha = 1, 2$, in which case we sum over the two results, which are $-\Gamma^1_{33}\Gamma^2_{12} = f(r)a(r)a'(r)\rho^2\left(\dfrac{a'(r)}{a(r)}\right) = f(r)(a'(r))^2\rho^2$ and $-\Gamma^2_{33}\Gamma^2_{22} = \rho(1-k\rho^2)\dfrac{k\rho}{1-k\rho^2} = k\rho^2$.
12. The first symbol is non-zero for only $\alpha = 3$, and thus the symbol becomes $\Gamma^3_{23}\Gamma^2_{33} = -(1-k\rho^2)$
13. Cancels pairwise with the fourteenth term.
14. Cancels pairwise with the thirteenth term.
15. Cancels pairwise with the sixteenth term.
16. Cancels pairwise with the fifteenth term.

Thus, compiling, we obtain:

$$R_{33} = -\frac{1}{2}f'(r)a(r)a'(r)\rho^2 + f'(r)a'(r)a(r)\rho^2 + f(r)(a'(r))^2\rho^2 + f(r)a(r)a''(r)\rho^2 - f(r)(a'(r))^2\rho^2$$
$$-\frac{1}{2}f'(r)a(r)a'(r)\rho^2 + 1 - 3k\rho^2 - (1-k\rho^2) + k\rho^2$$

which simplifies to:

$$R_{33} = f(r)a(r)a''(r)\rho^2 - k\rho^2 \tag{216}$$

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