Essays in Contest Theory:

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The majority of this work focuses on the theoretical analysis of collective action, group efficiency, and incentive mechanisms in team contests where individual outlays of heterogeneous agents are not observable. The reward allocation within the group is instead dependent on observable worker characteristics, modeled as individual abilities, as well as on the observable level of aggregate output. I study the incentives for free-riding and the group-size paradox under a very general set of intra-team allocation rules. I further derive the optimal allocation mechanism which rewards agents according to a general-logit specification based on their relative ability. I derive conditions under which a team's performance is most sensitive to the ability of its highest-skill members, while at the same time higher spread in the distribution of ability has a positive effect on group output.

In the final chapter I shift attention to the problem of optimal player order choice in dynamic pairwise team battles. I show that even if player order choice is conducted endogenously and sequentially after observing the outcomes of earlier rounds, then complete randomization over remaining agents is always a subgame perfect equilibrium. The zero-sum nature of these type of contests implies that expected payoffs for each team are independent of whether the contest matching pairs are determined endogenously and sequentially or announced before the start of the game. In both cases the ex-ante payoffs are equivalent to those when an independent contest organizer randomly draws the matches.

Contents

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To Hideo,
a role model, a guide, and a friend.

Chapter 1

The Evolution of the Theory of Collective Action and the Group-size Paradox

A vast subfield of the early Contest Theory literature during the late 1970s and 1980s focused on extending the seminal works of Olson (1965), Tullock (1967, 1980), and Krueger (1974). These were mostly articles aimed at studying the rent-seeking effects that typically occur in situations where multiple agents individually compete for private rewards. It was not until the early 1990s, however, that researchers started shifting their attention to group contests, reward allocation, and incentives. The first such work to exclusively focus on rent-seeking for pure public goods was that of Katz, Nitzan, and Rosenberg (1990). Most of the early motivation for the increased interest in this particular direction of research at the time had to do with the numerous cases where the rent sought was a public good. The rise of the dollar in the early 80s left many U.S. industries in a difficult position. Struggles for government support in the form of industry-wide subsidies and lobbying for relaxed regulation were a common theme at the time and eventually culminated with the Plaza Accord in 1985. A central question of interest that arose was: how did the total rent-seeking of those businesses and institutions compare to the public good benefit they would eventually receive?

This was precisely the question that Katz, Nitzan, and Rosenberg (1990) embarked to answer. The authors consider a pollution clean-up example in one of two distinct locations under the jurisdiction of a local authority. Identical clean-up costs and identical valuations of clean air by all individuals across both locations are assumed. The only heterogeniety between the two locations come from the different number of individuals living in each location. The two groups engage in a rent seeking contest, where the probability of success for location 1 is given, following Tullock (1980), as the ratio between the total amount spent on rent-seeking by location 1 and the total amount spent on rent-seeking by both locations 1 and 2. To start with, the authors assume risk neutrality of all agents. Under these conditions Katz, Nitzan, and Rosenberg (1990) show that the total rent-seeking done for the public good will equal only one half of the benefit to a single individual. This suggests that the total rent dissipation is likely to be very small compared to the cost of providing the public good in practice. Furthermore, they show that this result is completely independent of both the total number of individuals and the distribution of those individuals across the two locations. These results seem very robust as they continue to hold even in the case of more than two teams, different wealth levels across locations, and risk averse agents.

Another interesting facet of Katz, Nitzan, and Rosenberg's (1990) model is that in the unique symmetric equilibrium the free-rider problem seems to disappear on the team-level. The reason for this comes from the fact that the free-rider problem is exactly counterbalanced by the increase in the total size of the prize for the group - both increase equivalently with group size. Each extra individual added to one location decreases the rent-seeking by others. But the aggregate decrease he causes is exactly equal to his own rent-seeking. It should be noted that in the unique equilibrium all teams, regardless of size, end up with equal rent-seeking, thus allowing each equal probability of winning the contest. In a way, this exactly replicates Tullock's (1980) result for individual agents competing for private reward and suggests that the groups in this model really behave as if they were single contestants - a possible explanation for the irrelevance of the size and distribution of agents across locations. This naturally also implies that members of less populous locations will spend more per-capita than members of more populous ones. Thus, the free-rider effect is clearly observable on the individual level. On the group level, however, the aggregate prize increases at the same rate, so the free-rider effect dissipates.

A very similar work that almost complements Katz, Nitzan, and Rosenberg's (1990) is that of Nitzan (1991). Once again the analysis is centered around a number of groups which compete for a single rent; all group members voluntarily decide on their individual rent-seeking effort; and a single group wins the entire prize with probability following Tullock (1980). However, unlike Katz, Nitzan, and Rosenberg's (1990), the single reward is characterized as a private good that has to be divided among winning group members by some prespecified allocation rule. The specific rule considered by Nitzan (1991) is a mixed allocation method in which a fraction of the prize is divided under the egalitarian approach (equal distribution), while the remaining fraction is allocated according to relative effort.

In the extreme case when the entire reward is split equally among team members Nitzan (1991) shows that total rent-seeking is a small fraction of the total value of the prize, it increases with the number of competing groups and decreases with the total population. Furthermore, the free-riding effect is very clearly pronounced, leading larger groups to spend less on rent-seeking and leaving smaller groups members with higher expected payoffs and higher chance of winning the rent. The opposite extreme when the allocation mechanism is based purely on relative effort presents rather contrasting results. The number of teams no longer affects the total rent dissipation and all agents across all teams have equal expected payoff in equilibrium. Indeed, all agents exert the same level of individual rent seeking regardless of the size of their team or the distribution of individuals across teams. This result implies that the free-riding effect is completely cancelled out by the positive group-size effect. The rent seeking of each group is thus proportional to its size and larger groups have higher chances of winning the rent. Most of the rent is dissipated in this case. Finally, Nitzan (1991) shows that in the full version of his model, when the allocation rule is part-egalitarian and part based on relative effort, a few interesting observations transpire. First, the number of competing teams does still have a positive effect on the total rent seeking (this effect only disappears if the entire reward is split according to relative effort). Secondly and not surprisngly, the degree of egalitarianism negatively impacts the degree of total rent dissipation. Third, and perhaps most interesting, the effect of the total population size is ambiguous - it is positive if the level of egalitarianism is very low, but negative otherwise. The one factor that is completely neutral across all cases is that a change in the distribution of individuals across groups always leaves the degree of rent dissipation unchanged.

Even though Nitzan's (1991) mixed allocation rule approach provides a lot of intuitive understanding of both the rent dissipation and the significance of the free-rider problem, one big concern that remained unanswered was the determination of the level of egalitarianism in each group. Nitzan (1991) assumed for the sake of existence of Nash Equilibrium that all teams follow the same allocation mechanism. This certainly does not have to be the case if the groups or teams were allowed to be strategic in their choice of egalitarian versus performance-based prize sharing. This question was first addressed by Lee (1995). He took the exact model presented by Nitzan (1991), but transformed it into a two stage game. In stage one each team representative picks what fraction of the reward will be allocated based on the egalitarian rule and what fraction would be given based on relative effort in order to maximize the team's aggregate expected payoff. This choice is made conditional on the equilibrium allocation rules of all opposing groups, thus making the solution a subgame perfect equilibrium. In the second stage all members of all groups observe the realised levels of egalitarianism and pick individual outlays optimally. Lee (1995) shows that in the two team case, if both teams are equally populous, then the optimal allocation rule is to reward group members entirely based on relative contributions. If, however, the two teams are of unequal size, then the smaller team still should allocate the reward based on relative efforts, while the bigger team should employ the mixed rule in Nitzan (1991). The degree of egalitarianism in the larger group also grows with the degree of disparity in team sizes. In equilibrium both teams exert equal levels of rent-seeking, thus making both equally likely to win. Furthermore, this rent dissipation is once again equivalent to that of the single player case in Tullock (1980). Given this, the above result should make intuitive sense - as the bigger team grows in numbers, it needs less and less individual contribution from each member in order to compete agaisnt the smaller team. This allows it a choice of a more relaxed approach when incentivizing its agents, hence allowing for a more liberal use of the egalitarian rule despite the free-rider effect it may cause. Finally, Lee (1995) shows that it is not really the team's larger size that drives this effect. Even with more than two groups it is optimal for every team leader to chose the relative performance-based allocation rule and completely forego the egalitarian approach as long as all teams are smaller than half the total population. Thus, the only case in which a partially egalitarian mechanism could show up in equilibrium is when one of the teams has grown large enough to include more than half of all agents.

It should be noted that even thought both groups end up sharing equal rent, the ability of the smaller group in Lee (1995) to compete is severely limited by the restriction that the degree of egalitarianism be a positive fraction. Baik and Lee (1997) substantially extend the original two-stage game by removing this restriction. They allow the fraction of the rent distributed equally to be both negative or bigger than 1. The fraction can be negative if a group takes equal contributions from all members and then redistributes them according to relative outlays. Alternatively, a fraction bigger than 1 would imply that a group collects resources from its agents based on their relative effort and then redistributes those equally among everyone. The latter case could actually discourage members from sharing in the rent seeking effort and could even lead to a negative aggregate group outlays. The authors allow this to be the case as long as the total outlays of both teams remain positive. If this were to occur, then the group with negative outlays will lose the contest with probability 1. This is a very interesting possibility as the winning team effectively ”bribes” the loser by allowing them to borrow some of their outlays, thus exacerbating the free-rider problem on the losing team and securing its own victory.

In the subgame perfect equilibrium, however, this does not occur. The more populous team still chooses a degree of egalitarianism between 0 and 1, just as in the original model of Lee (1995). The difference in Baik and Lee (1997) is that the smaller team ends up choosing a negative degree of egalitarianism. It borrows equally from all its members and then redistributes the collected extra rent among them according to relative effort. This incentivizes the smaller group to share a larger portion of the total outlays and gives it higher probability of winning the contest. Just as in the previous few papers with private rents, the degree of rent dissipation remains independent of the distribution of individuals across teams. It is equal to $(N-1)/N$ where N is the total populatoin, the same as in the single player contest of Tullock (1980).

Baik and Lee (1997) also add an extra third stage to the contest which occurs before the allocation rule selection. In this extra stage they allow individual members to change teams. If the total number of members is even, then in equilibrium both teams end up with equal size and the expected payoff to each agent is the same. If the total number of players is odd, then in equilibrium one of the teams has an extra member. The sharing rules in stage two will then be very close to allocation based on relative effort. These results strongly rely on having only two competing groups and on the use of the relaxed parameter restriction framework for the allocation rules in Nitzan (1991).

One point that fails to be addressed in the works discussed thus far is the possibility that large groups lacking proper incentives sometimes fail to organize and lobby. A situation in which some contestants reitre from seeking a rent is commonly referred to as oligopolization. Different works in the late 80s try to characterize when oligopolization might occur based on mostly individual player characteristics. Hillman and Riley (1989) explain this as a consequence of disparities in the prize valuations of individual agents. Skaperdas (1991) attributes it to endowment differences, and Hirshleifer (1989) shows that it can be a result of the structure of contest success functions. None of these works, however, discuss the importance of group characteristics. Ueda (2002) is the first to properly address this issue, once again in the context of Nitzan's (1991) model. Ueda (2002) points out that both Nitzan (1991) and most of the following literature in the 90s focuses too strongly on the interior equilibria, and he argues that this is not equivalent to lack of oligopolization. Consider a two group contest for instance. According to the view of the earlier works, if one of the teams seizes to compete, then the other would need to show only minimal effort to win the prize, resulting in monopolization. But this is certainly not the case in the presence of internal team incentives. The players on the remaining team may still put in a lot of effort because of the within-group competition.

In this context, Ueda (2002) is able to find the degree of rent dissipation even in the most general case with many teams. He defines a measure γ_i which represents the stand-alone incentives for any group member of team i - the marginal benefit from effort that this individual would earn even if all her teammates decide to refrain from participation. He then considers the difference between γ_i and the relative effort of this individual compared to the total outlays by everyone else. In a way this weighs the difference between the individual marginal benefit from effort and the individual equilibrium effect on the probability to win from this same effort. Ueda (2002) shows that if we considered all possible subsets of competing groups, then in the unique Nash Equilibrium the set of active groups maximize this difference on average (it should be noted that this difference only depends negatively on the degree of egalitarianism and the size of the group, so it can be calculated exogenously for every team). In other words, in any Nash Equilibrium, each active individual achieves the highest degree of individual reward relative to her own effect on the probability to win on average. This result allows Ueda (2002) not only to find an explicit formula for the degree of rent dissipation, but also to show that in equilibrium rent dissipation is maximized across all possible subsets of possibly active groups. To interpret this differently, only groups whose members receive stand-alone incentives γ_i exceeding the degree of rent dissipation will choose to remain active in equilibrium. This throws a lot of light on the occurence of monopolization in the original work of Nitzan (1991). If a team can achieve high enough stand-alone incentives by employing allocation rules based sufficiently on performance, then it can potentially make the benefit of effort in competing groups too low, thus completely removing them from the contest. Perhaps ironically, this still maximizes the degree of rent dissipation. In a way, even though other groups retire from the contest, the remaining team still exerts a lot of effort to maintain this status. This bears a lot of resemblance to entry-deterrence games in Industrial Organization, but it also goes beyond that because there is still internal competition between the monopolizing team members for the division of the reward.

These oligopolization results only hold in the context of predetermined levels of egalitarianism. It would make sense that if team leaders are allowed to choose the allocation mechanisms endogenously and maximize group welfare, then they would not simply let their group drop out of the contest. Ueda (2002) confirms this intuition by showing that when applying the same formulation to Baik and Lee's (1997) two-stage game with endogenous choice of distribution rules oligopolization never occurs. Every member of every team will always put in positive effort in the unique equilibrium.

Despite the ubiquitous presence of the central theme of rent dissipation in Nitzan (1991), Lee (1995), and Baik and Lee (1997), these works inevitably throw light on another very closely related problem - that of the the group-size paradox. Dating back to Olson's (1965) thesis, the group-size paradox states that larger groups are less effective at attaining their goals than smaller ones. It should be noted that the concepts of group and effectiveness more often than not come hand-in-hand . After all, agents tend to pool efforts only if the group action is likely more effective than individual outlays in the first place. And yet the question of what group effectiveness is can be quite ambiguous. The most common interpretation, established in the early works of Olson (1965), Chamberlin (1974), and Oliver and Marlwell (1988) is that effectiveness is nothing else than the group's probability of earning a certain prize or rent. Olson's (1965) argument that larger groups are less likely to accomplish their goals raised a lot of questions at its time. Most of us are more likely accustomed to the polar opposite of divide and conquer. The early literature identified the natural culprit for the occurence of the group-size paradox as the free-rider problem. Larger groups tend to make individual deviations seem less impactful, while at the same time reducing the size of private rewards. A natural way to resolve this problem is to address the question of how to divide the rewards in ways to counter the free-rider problem. The results of Nitzan (1991), Lee (1995), and Baik and Lee (1997) clearly suggest that incentive schemes that are less egalitarian and more performance-based seem to avoid and reverse the group-size paradox. However, alternative individual incentives seem not to be the only way to resolve this phenomenon.

A very different approach, based on the early intuition of Chamberlin (1974), is offered by Esteban and Ray (2001). The authors show that even if the division of private rewards is performed in a completely egalitarian way, the group-size paradox is still very unlikely to occur as long as the rewards exhibit at least some public characterisitcs. The latter are quite likely in reality as even purely monetary rewards often come with a degree of recognition or reputation. To demonstrate this result, Esteban and Ray (2001) focus on one key element that was mostly overlooked in the literature up to that point - the convexity of the effort cost function. They argue that the standard case of linear cost of effort is very unrealistic: imperfect capital markets make the opportunity costs of borrowing large amounts steeper, while time-based effort should also come at increasing cost as it tends to make the less remaining time for alternate actions more valuable. In their model, a fixed fraction of the economic rent is provided as a public good, while the remaining private good is dividied equally among winning group members. This is to be distinguished from the perceived degree of publicness, which is given as the fraction of the individual reward that comes as a public good. The latter depends on group size - larger groups will divide the private component among more members, thus making the public good constitute a higher fraction of individual rewards, even though it is a predetermined fraction on the group level. It turns out that it is the difference between precisely this perceived degree of publicness of the good and the elasticity of the individual cost of effort that determines whether the group-size paradox occurs or not. Esteban and Ray (2001) first show a very strong result - if in the unique symmetric equilibrium the effort cost elasticities are larger than one (quadratic cost or steeper), then the level of collective action increases in group size. They further show that the group-size paradox also seizes to exist if those elasticities are smaller than one as long as the degree of publicness of the rent is high enough. The closer the effort cost elasticities are to 0 (linear cost), the higher the required threshold of publicness of the good in order to ensure this result. But even if these conditions are violated, it is still possible for the group-size paradox to disappear depending on the sizes of the participating teams. If the teams are populous enough, then the degree of perceived publicness of the good may be large enough relative to the cost elasticities, so that the actual level of publicness does not matter. This last result is particualry interesting because it can explain why it is typically large and diversified organizations and not narrower specialized groups that succeed in their lobbying efforts for government support. These conclusions also suggest that the group-size paradox is only certain to occur in the extreme case of linear costs and purely private rents. For any other parameter values it is more likely for the effect to be reversed. Additionally, these results hold not only when comparing different-sized teams in any given equilibrium, but also when possible growth in the size of any given group leads to an adjustment towards a new quilibrium. Finally, Esteban and Ray (2001) were able to classify the types of prizes for which their conclusions apply as equivalent to Chamberlin's (1974) original definition of normal goods - goods for which a unit increase in effort by team members leads to an equilibrium individual effort decrease of less than one unit. The convexity of the cost of effort is sufficient to establish this relationship in equilibrium.

The evidence in favor of the disappearance of the group-size paradox seems even stronger if one allows the private prize sharing rule to be endogenized. This was first established by Nitzan and Ueda (2011), whose work serves as a nice bridge between the group-size paradox literature and the rent dissipation literature discussed thus far. On one hand Nitzan and Ueda (2011) generalize Esteban-Ray's model by still focusing on a mixed public-private reward and convex costs of individual effort. But on the other hand, they allow for analysis of the further effects of incentives by using the endogenous allocation mechanism for the private component of the reward from Nitzan (1991). They consider a two-stage model in which each group leader first chooses what fraction of the private rewards will be allocated based on relative effort and what fraction based on the egalitarian rule. This is done in a way that maximizes the group's aggregate welfare. In the second stage each member chooses effort optimally. Most importantly, the first-stage choice of the allocation rule is done in private and not observable by outsiders. The rationale behind this assumption is that the sharing rule is not meant to be a strategic variable. For instance, the employment of very strong incentives within a group, if observable, may send a signal to opponents about the nature of the competition they will face, and thus affect both their expectations and equilibrium efforts. This may undermine the value of such rule in the first place, but even if we were to abstract from such equilibrium effects, what is to say that the the group will not change its rule later in secret? Such allocation mechanisms are naturally chosen internally and it would be challenging to enforce public revelation. Thus, the degree of egalitarianism remains private information only observable by team members. The implications of this assumption are crucial for the choice of equilibrium concept in such a model because individual agents never face a fully defined subgame. Each group member is not able to infer the payoff functions of opposing group players and will have to form believes about the other group's allocation rules. Hence the use of perfect Bayesian Equilibrium. The authors show that there is a unique such equilibrium in which every individual in the same group chooses symmetric effort. Moreover, the endogenous level of egalitarianism is positively related to a group's equilibrium probability of winning the prize. This closely supports the earlier intuition that only weaker (in this case smaller) teams resort to performance-based incentives in order to be able to compete with stronger teams. Finally, Nitzan and Ueda (2011) also show that the group-size paradox completely disappears as long as the prize is not completely private. This is a much stronger result than the one in Esteban and Ray (2001) as it does not depend on the elasticity of the cost function. Since effort is extracted optimally, the group leader is able to creative postive effort externalities from each agent. The latter effectively sets the marginal cost of individual effort equal to the added marginal benefit of all group members, thus significantly increasing her outlay. This positive externality naturally scales with group size - the larger the group, the higher the added benefit of using the optimal incentive scheme. Additionally, this clearly implies that groups with lower marginal costs will have higher probabilities of success and will have the freedom to use more egalitarian allocation rules.

The study of incentives discussed so far has centered exclusively on models in which the inherent symmetry amongst agents in each group has lead to similarly symmtrical treatment of those agents in equilibrium. This neglects one of the main aspects of Olson's (1965) thesis: giving incentives to a subgroup of agents might lead to an improvement in the group's welfare. The reason why this effect has escaped most of the contest theory literature in the 90s and early 2000s is to a big extent due to the fact that all these models treat agents' efforts as perfectly substitutable. Unfortunately, perfect substitutability is an extreme that enforces equal treatment of individuals in symmetric equilibria and in many ways fails to adequately address the core ideas of Olson (1965). Without adding explicit hetorogeneity among group members, Ray, Baland, and Dagnelie (2007) were among the first to provide early insight regarding this issue. They consider a single-team model in which group output is given by a constant elasticity of substitution production function of individual outlays. The agents are otherwise symmetric and the cost of effort is linear, but the share each agent receives of the total prize can be different. In this context, Ray, Baland, and Dagnelie (2007) study the dependancy of aggregate social surplus (the difference between total team output and the added cost of effort) on the structure of the share vector. The authors show that if the elasticity of substitution is no more than 2, then the marginal effects of the individual shares on aggregate welfare is decreasing in share size. In other words, transfering a small fraction of reward from individuals who are treated more favorably by the allocation rule to individuals who receive less, would provide positive incentives and increase aggregate output as long as personal efforts are strong enough complements. The only vector of shares that cannot be improved in this manner is the one that treats all members equally. It is very important to understand that this is not an optimality result in a mechanism design sense. Instead, it is an improvability result given a starting equilibrium. Ray, Baland, and Dagnelie (2007) also show that if the elasticity of substitution is greater than 2 then the equal treatment (fully egalitarian) outcome is improvable by equal-treatment minorities, i.e. outcomes in which the rewards are split evenly, but only among a subset of agents. It is argued via simulations that as the degree of substitutability among agents increases, the number of agents that receive positive shares must decrease. It had been commonly observed that Olson's (1965) arguments seem adequate when agents are perfect substitutes, but that they fall apart when agents are perfect complements. The significance of Ray, Baland, and Dagnelie (2007) is to provide a bridge that reconciles both sides of the argument while providing a complete characterization across the full spectrum of complementarity between individual efforts.

A different approach to introducing heterogneity in the collective action problem is presented in Epstein and Mealem (2009). In their model each individual receives a different benefit if her team wins the prize. This private valuation is not affected by the size of her team as the reward is considered a public good within the team. The authors consider two mathematically equivalent formulations. In the first the group's success chance is given by relative group outlays as in Tullock (1980) and the cost of effort is convex. The alternative formulation includes linear cost of effort but diminishing returns to individual effort. Similar to Esteban and Ray (2001), the convexity of the effort cost function limits the potential for free-riding. As the degree of convexity increases, the relative equilibrium efforts between any two group members become proportionally equal to the ratio between their private benefits from winning. Furthermore, the same increase in convexity (or decline in the marginal benefit of effort) in the limit leads to an equilibrium outcome in which only the number of individuals in each group matters. The individual valuations seize to have effect on the group's probability of success. This is simply a result of the declining personal outlays. In the limit each member's contribution is so small that the relative differences between players become irrelevant. Perhaps more intriguing in Epstein and Mealem (2009) are the results relating to group composition and interaction between teams. The authors show that an increase in a given individual valuation will increase the equilibrium effort of that player, while decreasing the efforts of all her teammates. Note that the group's total outlays will increase, thus eliminating the group-size paradox. Furthermore, this increment in valuation will stimulate the other team's members if their group is stronger (defined by the level of aggregate valuation added across members). The opposite effect will occur if the other team is weaker - then all its members will reduce their equilibrium effort, thus making their team even weaker. Finally, Epstein and Mealem (2009) consider the question of increasing group size by adding either one or two additional members (with equal aggregate valuation). There are two competing effects from increase in a group's aggregate benefit from the reward. On the one hand this would affect everyone's individual effort levels to increase team output, but on the other hand this benefit is affected by diminishing returns to individual effort. It is shown that if the convexity of effort is low enough, then it is optimal to add one single individual with higher valuation to the group. If, however, the marginal benefit of effort declines sharply, then it is always optimal to add two individuals with smaller valuations. Indeed, the best change in team composition would be to take full advantage of cost-sharing and to add as large a group as possible with as small individual benefits as possible, presenting a very strong argument against the group-size paradox.

The significance of inequality of individual valuations is further studied by Nitzan and Ueda (2014), who extend Epstein and Mealem (2009) in two ways. First, they allow for a prize with mixed public and private characteristics. Second, even thought the public component is equally valued by all group members, the private component is dividied according to predetermined and possibly unequal shares. It is the presence of these unequal shares that leads to heterogeneity of individual valuations. Nitzan and Ueda (2014) show that more unequal stakes lead to improvements in group performance as long as the group's equilibrium effort is low to begin with. Since the inter-group contest success probabilities are given by relative team efforts, this implies that this result is appicable to situations when either the team is a small part of a contest against a multitude of opponents, or if a very strong opponent exists. The nature of this result is most emphasized when the cost of effort is linear. In that case it is optimal to concentrate the private rewards in the hands of a single group member, leading to monopolization of the private component. However, Nitzan and Ueda (2014) point out that ironically this monopolization comes at the cost of welfare lost by the monopolizing agent. In the alternative case when the cost of effort is convex, the authors show that more equal share distribution always benefits the performance of the group and higher individual stakes always lead to higher expected utility within the group.

One way in which the analysis of Nitzan and Ueda (2014) can be extended is by addressing the possible existence of complementarity between agents in a group. Indeed, this has been an overarching problem in the literature on collective action up to this point. Multiple works noted above demonstrate the importance of convexities in the cost of effort, while others such as Ray, Baland, and Dagnelie (2007) focused on complementarities in production. But to my knowledge, it was not until recently that models started combining both features within the same framework. A recent work that adequately extends the findings of Esteban and Ray (2001) and Nitzan and Ueda (2014) in this respect is that of Kobayashi and Konishi (2020). The authors show that high degree of complementarity between team members' efforts accompanied by sufficient convexities in the cost of effort lead to outcomes in which the egalitarian rule is optimal. Contrarily, if effort costs are near linear or if the agents are closer to being perfect substitutes, then focusing all the incentives in the hands of a single team member maximizes team performance. Thus, Kobayashi and Konishi (2020) very nicely unify the results from the previously discussed literature, while at the same time providing a specific threshold separating the cases in which egalitarianism and monopolization occur optimally. Furthermore, they strengthen the previous results of Nitzan and Ueda (2014) and Esteban and Ray (2001), which were based on Lorentz domination-based arguments, by directly addressing the optimality of the chosen sharing rules in a full two-stage game. In stage one each group leader selects a sharing rule for her team with the objective to maximize the team's winning probability. In the second stage the team members observe the allocation rule and chose effort in a Nash Equilibrium across all members and all teams. A key to the analysis of Kobayashi and Konishi (2020) is the use of the constant elasticity of subsititution effort aggregator within each group. Together with a constant-elasticity of the marginal cost of effort formulation, this allows them to characterize equilibrium team output and the second stage equilibrium on the group level entirely, thus providing the above-mentioned results.

Finally, a very similar framework is employed by Crutzen, Flamand, and Sahuguet (2020). The main distinction is that the authors consider the case of multiple indivisible rewards. They compare two types of prize-sharing rules within a team: egalitarian versus a list rule. Under the list rule the rewards are allocated according to a predetermined list of members, while under the egalitarian rule a fair lottery is used. Crutzen, Flamand, and Sahuguet (2020) show that, once again, the degree of complementarity between individual efforts and the degree of convexity of the effort cost function are the key determinants for which kind of mechanism stimulates group perfromance more. They confirm the findings of Kobayashi and Konishi (2020) in that high degree of convexity or high degree of complementarity make the egalitarian rule preferable in terms of increasing the probability of the team's success. In the opposite case when the list rule performs better, Crutzen, Flamand, and Sahuguet (2020) aslo discuss the differences between open and closed lists. From the perspective of political science open lists are typically considered superior in the sense that they provide more incentives across the board. With closed lists individuals ranked near the top or bottom rarely tend to exert much effort regardless. It is shown that this is indeed the case only for intermediate values of the complementarity between individual efforts. As agents become close to perfect substitutes or as the cost function approaches linear, then the closed list would perform better.

Chapter 2

Efficiency and Incentives in Teams with Heterogenous Agents

2.1 Introduction

Compensation schemes designed to motivate team members have been a central issue in the economic analysis of labor provision (Irlenbusch, Ruchala, 2006), as well as in the theory of contests in general. Many businesses differentiate between merit-pay rewards such as salary and performance-based rewards such as bonuses and variable pay options. In many cases these performance-based rewards are not conditioned on individual performance, but rather on team performance (Hamilton, Nickerson, and Owan, 2003). Typically there are two reasons for that: (1) individual performance is not observable or not verifiable in court; and (2) these mechanisms are designed to encourage cooperative behavior in order to take advantage of existing complementarities between individual efforts in production.

Motivated by these observations from the empirical labor literature, this paper focuses on the theoretical analysis of collective action, group efficiency, and incentive mechanisms in situations when individual outlays of heterogeneous agents are not observable. Since the reward allocation within the group cannot be conditioned on effort, it is going to be dependent instead on observable worker characteristics, hereby modeled as "individual abilities", as well as on the observable level of aggregate output. The heterogeneity granted by the differences in agent productivity and the inability of the team leader to condition rewards on personal outlays are the two main distinguishing features of this work. A third distinguishing feature lies in the incentives of the group leader. Here she is not a benevolent social planner trying to maximize group welfare. Instead she tries to maximize team output directly. In this model it is perhaps reasonable to think of the team leader as a department chair trying to coordinate the efforts of her subordinates, or in the context of sports as the coach stimulating his team to compete. Similar to a mechanism designer, the objective of the team leader is to select an allocation rule that adequately incentivizes the agents to exert more effort. The model unfolds as a two-stage game. The team leader first selects an allocation mechanism - a complete contingency plan of how the total reward will be divided depending on the realized abilities of the agents and on the aggregate level of their output. Then the agents observe the allocation rule and everyone's abilities, and choose effort optimally in a Nash Equilibrium.

At this point, it is instructive to mention that I split the analysis into two distinct sections. First I study the properties of the second stage equiilbrium for a fairly general set of allocation rules. This allows me to discuss incentives for free-riding and the disappearance of the group-size paradox in a broader sense without having to narrow the focus on any one specific mechanism. The particular restrictions are that the allcoation rule is weakly monotonic, non-wasteful, and the elasticity of the marginal benefit of output for each agent is bounded by the elasticity of the marginal cost of effort. The last condition is required for the second-order condition for individual optimization. These allocation rules allow the team's leader to reward certain agents at the expense of others as the group becomes more productive. Having examined the general case, I only then move on to derive the optimal allocation rule and study its implications for optimal group structure.

Motivated by Olson's (1965) thesis, most of the previous works on collective action starting the early 90s have focused on providing better understanding of the group-size paradox, the idea that larger groups are less effective at achieving commong goals. The early literature identified the natural culprit for the occurence of the group-size paradox as the free-rider problem. Larger groups tend to make individual deviations seem less impactful, while at the same time reducing the size of private rewards. There are three main approaches commonly used to address this issue: (1) introduce pubilc component to the otherwise private rewards earned by the agents, as was originally done by Nitzan (1991); (2) introduce incentives that diminish the free-rider problem by conditioning individual payoffs on relative effort (as oppposed to equal shares), mostly emphasized in the extensions provided by Lee (1995) and Baik and Lee (1997); and (3) by introducing complementarities in individual efforts, as well as cost-sharing incentives generated by high elasticity of the marginal cost of effort, highlighted by the insightful work of Esteban and Ray (2001) and Ray, Baland, Dagnelie (2007). Finally it should be noted that just as in this paper, Ray, Baland, Dagnelie (2007) also consider a situation in which efforts are not observable. However, they use constant exogenous shares in their allocation rule. I will show that using linear shares eliminates one of the key incentives for free-riding, and thus limits the scope of the discussion.

The philosophy behind the model in this paper in many ways unifies the preceding literature by focusing on the common factor among those different approaches - the relationship between the marginal benefits of individual agents and the level of aggregate output. Observe the common theme that runs in many of those results. When we have fully private rewards, adding a new member to a group reduces the share of everyone else, thus reducing the marginal benefit they receive when output increases. By adding a public component to the rewards, we can limit the magnitude of this decrease and hence limit free-riding incentives. Similarly, when the individual payoffs are mostly private, the team representative can fight the free-riding incentives by directly condioning payoffs on relative efforts. This is once again just another way to prevent the marginal benefit of output from declining too sharply as the group increases in size. A contribution of this work is to show that at the core of how we address the free-rider incentives and the group-size paradox is the general problem of how fast marginal benefits decline with output. For this purpose, in the first part of this paper, I focus very broadly on any weakly monotonic allocation rules for which the elasticity of the merginal benefits of all agents are bounded by the elasticity of the cost of effort. The idea is that as output increases, all active agents are rewarded more, but some at increasing and others at decreasing rates. If the rate of decrease of the marginal benefit of any given agent is too sharp, this particular agent will fall victim to free-riding.

Following in the steps of Kobayashi and Konishi (2020) and Crutzen, Flamand, and Sahuguet (2020), I use a constant elasticity of marginal cost of effort and a CES aggregator of individuals outlays that controls for the elasticity of substitution. I show that if the elasticity of the marginal benefits of the agents with respect to both output and the productivity of their teammates are too negative relative to the degree of complementarity and the elasticity of output with respect to individual productivity, then free-riding occurs. This is a general result that unifies all of the preceding works by focusing on the core issue. In particular, suppose that the aggregate potency of a group increases, either because of the addition of a new member, or because of increase in the productivity (or human capital) of the existing agents. Then, if the allocation rule overrewards the new member at the expense of everyone else, or if the rate at which their marginal benefits decline with output are too steep (negative), then those agents will engage in free-riding and the organization will more likely suffer from a decrease in group effectiveness. The previous literature on collective action limits its attention to the first effect only (elasticity with respect to the potency of the group) and neglects the second (elasticity with respect to aggregate outlays). I show that even if the first effect is not strong enough to cause free-riding by itself, the addition of the second effect could be enough to do so. One way to minimize these free-riding incentives is by implementing allocation rules that are close to linear shares (eliminating the second effect). It should be noted that Ray, Baland, Dagnelie (2007) use such linear shares and the group-size paradox disappears. My work provides deeper understanding of why that occurs.

In the second half of this paper I move on to discuss the optimal allocation mechanism, in which each individual is rewarded based on their relative ability compared to that of everyone else (up to a power higher than unity). In the case of symmetric player abilities, this reduces to the egalitarian rule, confirming the unimprovability result of Ray, Baland, Dagnelie (2007) as a special case. This also conforms with the corresponding findings in Kobayashi and Konishi (2020) and Crutzen, Flamand, and Sahuguet (2020). Naturally, the group-size paradox is completely eliminated under the optimal rule. Moreover, I am able to obtain a characterization of the optimal group structure when ability is transferable between individuals. Consider a scenario, for instance, when an agent is replaced by someone with slightly higher ability, while another agent is replaced by someone with slightly lower ability - a tradeoff that could occur during the hiring process preceding the game. Would this type of effective transfer of ability be beneficial for the productivity of the group? I show that as long as convexity of the cost of effort and the complementarity in production are both not too hith, then the equilibrium effort choice of high ability players is more sensitive to changes in productivity of their teammates, and at the same time aggregate efficiency is maximized when investing addtional ability in those high ability players in the first place. This implies that transfering a small amount of productivity from a low ability to a high ability player can improve aggregate efficiency. Furthermore, if we want to transfer a fixed ϵ amount of ability, the most efficient way to do this is by taking it away from the lowest ability member and giving it to the highest ability one. Combined, these results imply that the distribution of abilities should be as spread out as possible, a phenomenon that I refer to as ”preferences for diversity”. In the limit it would be most effective to have one very high skill member accompanied by a ”competitive fringe” of low ability players.

Before proceeding with the model I provide a detailed perspective on the evolution of the collective action literature in the following section.

2.2 Single Team Model

Consider a group of n agents who jointly engage in collective action resulting in the production of certain output. Each agent $i \in \{1, ..., n\}$ is endowed with a different level of productivity, which will be denoted by $a_i \geq 0$, and exerts effort denoted by the continuous variable $e_i \geq 0$. Throughout the rest of this paper I will refer to this productivity as the ability of each member. By assumption, the abilities of all agents are common knowledge within the group. I am allowing for the abilities of some agents to be zero and I will refer to those agents as dormant, since they are unable to contribute to output. The remaining agents with positive ability will be considered active. The presence of dormant players is important because it will later allow room for increasing team size as a result of improvements in player productivity.

The joint level of output X is determined according to a CES production technology:

$$X = \left(\sum_{i=1}^{n} (a_i e_i)^{1-\sigma} \right)^{\frac{1}{1-\sigma}} \tag{2.1}$$

This formulation allows me to consider different degrees of effort complementarity measured by $\sigma \in (0,1)$. For values of $\sigma > 1$ a unique degenerate zero-effort equilibrium exists, so I ignore this case. Naturally, the reciprocal $\frac{1}{\sigma}$ is the elasticity of substitution between individual effort outlays.

Following Esteban and Ray (2001), I assume that effort becomes increasingly more costly with individual outlays. More specifically, I use the convex cost function

$$C(e_i) = \frac{e_i^{\beta+1}}{\beta+1} \tag{2.2}$$

The variable $\beta > 0$ is the constant elasticity of the marginal cost of effort ($\beta = \frac{e_i C''(e_i)}{C'(e_i)}$). This specification corresponds to situations in which increasing effort takes more time investment, thus increasing the opportunity cost of any remainng free time for alternative activities. Finally, the cost function is common among all team members as it is perceived that they partake in identical production activities within the group.

I assume that the group is represented by a team leader who collects a fixed fraction of total output $(1-\gamma)X$. The remaining fraction γX is fully transferable back to the agents. The team leader is not a benevolent social planner. His incentive is to maximize group output. This is a departure from the standard literature on collective action where joint welfare is typically the objective function. In this model it is perhaps reasonable to think of the team leader as a department chair trying to coordinate the efforts of her subordinates, or in the context of sports as the coach stimulating his team to compete. Similar to a mechanism designer, the objective of the team leader is to select an allocation rule that adequately incentivizes the agents to exert more effort. In this paper, however, payoff schemes based on individual outlays will not be achievable. Instead, I assume, as is often the case, that the agent's efforts are not contractible in advance (even though they can be inferred in equilibrium). The only available information on which the team leader can condition the individual compensations are the agents' abilities and the realized total team output (which is assumed observable).

The chosen allocation rule will be denoted by

$$Q(\mathbf{a}, X) = [q_1(\mathbf{a}, X), ..., q_i(\mathbf{a}, X), ..., q_n(\mathbf{a}, X)] \tag{2.3}$$

where $\mathbf{a}$ is the full vector of individual abilities and $q_i(\mathbf{a}, X) \geq 0$ is the share of the reward given to any player i. In what follows I will often suppress the argument $\mathbf{a}$ and denote these shares simply by $q_i(X)$ for brevity, but it should be emphasized that any allocation rule is a full plan of action - it specifices how the reward is to be divided for any vector of abilities. In particular, it conveys how the shares would be adjusted if the abilities of any subset of players were to change. I restrict the space of feasible sharing rules to all twice continuously differentialbe,weakly monotonic (with respect to X). That is, for any active player i: $\frac{\partial q_i}{\partial X} \geq 0$ An additional assumption sufficient for individual optimality is that $\epsilon_i^X < \beta$, where $\epsilon_i^X = \frac{X \frac{\partial^2 q_i}{\partial X^2}}{\frac{\partial q_i}{\partial X}}$ is the elasticity of agent i's marginal benefit with respect to aggregate output. Note that this elasticity could be even larger in any equilibrium. However, as the share of any given individual in total output converges to 1, the elasticity condition becomes necessary and sufficient. Given that β can be very large, the restriction $\epsilon_i^X < \beta$ should allow for enough convexity in the individual shares, even though it may be a tighter bound than necessary.

By assumption team output is fully transferable (after the team leader collects a fraction) back to the agents, hence I restrict attention to non-wasteful allocation rules:

$$\sum_{i=1}^{n} q_i(\mathbf{a}, X) = \gamma X \tag{2.4}$$

These restrictions imply that as the team becomes more productive the reward each active individual receives also increases weakly. However, this can occur at a decreasing rate. Intuitively, as group ouput increases all contributors are positively rewarded, but their overall share of output remains the same. This still allows a lot of freedom in the individual shares as they can be locally concave or convex (but not too convex as the elasticities of the marginal benefits are bounded by β).

The rest of the model unfolds as a two-stage game. The team leader first selects an allocation mechanism $Q(\mathbf{a}, X)$. The agents observe the allocation rule and everyone's abilities, and choose effort optimally in a Nash Equilibrium.

2.2.1 Equilibrium Effort Selection

In this, as well as the following two sections, I abstract away from the incentives provided by the team leader and the first stage of the game. Instead, I consider any exogenously given allocation rule $Q(\mathbf{a}, X)$ and more broadly study the properties of the resulting second-stage equilibrium.

The individual utility of agent i is given as the value of her share of the reward net of effort:

$$U_i(q_i(X), e_i) = q_i(X) - C(e_i) \tag{2.5}$$

It should be immediately obvious that for every dormant agent it is always a dominant strategy to exert zero effort - increasing effort by such an individual is costly and yet it brings no benefit since it does not affect output. On the other hand, each active member of the group maximizes her utility by selecting effort optimally, given the effort choice of everyone else:

$$\max_{e_i \geq 0} q_i(X) - \frac{e_i^{\beta+1}}{\beta+1}$$

The first order condition is:

$$\frac{\partial U}{\partial e_i} = \frac{\partial q_i}{\partial X}(X)\frac{\partial X}{\partial e_i} - e_i^{\beta} = 0$$

Using the fact that:

$$\frac{\partial X}{\partial e_i} = a_i^{1-\sigma} X^{\sigma} e_i^{-\sigma}$$

it is straightforward to show that equilibrium individual effort equals

$$e_i^{\star} = X^{\star\frac{\sigma}{\sigma+\beta}} \left(a_i^{1-\sigma} \frac{\partial q_i}{\partial X}(\mathbf{a}, X^{\star}) \right)^{\frac{1}{\sigma+\beta}} \tag{2.6}$$

Equation (2.6) implicitly solves for the equilibrium effort of agent i as a function of the equilibrium team ouput $X^{\star}$. The first term on the right-hand side is a scale effect suggesting that being on a more productive team may provide feedback incentives for higher individual contributions. Moreover, this aggregate feedback effect becomes stronger as the complementarity in production σ increases, thus matching the natural intuition that being on a more productive or competitive team can be stimulating especially when working as a team is essential. The presence of the second term (a_i) shows that in any equilibrium higher ability individuals tend to contribute more. Strategic effects aside, members with higher ability find it easier to compensate for the disutility of effort because their contribution is more valuable to team output per unit of outlays. Furthermore, the positive ability effect becomes stronger the weaker the complementarity between team members and the lower the elasticity of marginal cost. The former should be quite understandable as higher complementarities would limit the effectiveness of any one agent and hence constrain the importance of individual ability. Contrarily, higher substitutability would allow the higher ability of one member to compensate for the possibly low contributions by teammates, therefore stimulating that member to work even harder. This effect could easily be countered, however, by the choice of allocation rule. The presence of the last term suggests that if the marginal benefit of total output to agent i is too low, then it can disincentivize the agent. Additionally, it should be noted that equation (2.4) implies that

$$\sum_{i=1}^{n} \frac{\partial q_i}{\partial X}(\mathbf{a}, X^{\star}) = \gamma \tag{2.7}$$

We can observe that there will be a distinct trade-off from the perspective of the team leader when it comes to motivating team members. A more substantial increase in the marginal benefit from output for one of them must often come at the cost of disincentivizing someone else.

2.2.2 Equilibrium Group Output

By aggregating the individual equilibrium efforts from (2.6) it is straghtforward to show the following result:

Proposition 1 *Second stage equilibria exist for any $Q(\boldsymbol{a}, X)$ and any vector $\boldsymbol{a}$. Equilibrium team output is implicitly defined by the solution to:*

$$X^{\star\beta} = \left(\sum_{i=1}^{n} a_i^{\frac{(1-\sigma)(\beta+1)}{\sigma+\beta}} \left(\frac{\partial q_i}{\partial X}(\boldsymbol{a}, X^\star) \right)^{\frac{1-\sigma}{\sigma+\beta}} \right)^{\frac{\sigma+\beta}{1-\sigma}} \tag{2.8}$$

*Having solved for team output $X^\star$, the unique individual effort outlays of any active agent can be found from (*2.6*).*

The advantage of aggregating (2.6) across individuals is not only that it allows us to confirm existence and uniqueness of second-stage equilibrium, but also that it provides a useful equation (2.8) that allows us to characterize the second stage equilibrium level of team output. Also note that there may be multiple fixed points in equation (2.8), but for each of them the vector of equilibrium individual efforts is unique, thus leading to possibly multiple distinct equilibria.

2.2.3 Discussion

The contest theory literature on collective action historically has focused on one recurrent question: how does group size affect team output? The overarching consensus is that in cases when agents are perfect substitutes strong free-rider incentives arise that can reduce team output, i.e. the group size paradox. Many works starting with Nitzan (1991) and culminating with Esteban and Ray (2001) have shown that adding public component of the rewards can easily reverse the group size paradox. But how about settings when the goods are entirely private? The seminal work of Ray, Baland, and Dagnelie (2007) shows that complementarity between agents can have similar consequences in eliminating the group-size paradox. In all of the preceding literature, however, agents are treated symmetrically in terms of their contributions, and so the only way to increase team size is by adding more players. The hetorogenous nature of my model allows to address group size questions from a slightly different perspective: how does an increase in individual ability affect group output? To observe the similar nature of the two problems consider a situation in which a dormant member receives a positive boost in ability. This outcome is equivalent to that of adding a new member to the team (recall that the allocation rule specifies a full contingency plan depending on player abilities, so this type of change is well defined within the confines of the model). Alternatively, a member i may leave the group and be replaced by someone even more productive (interpreted once again as an increase in ability a_i). The key insight is that it is the aggregate potency of the group that should be the determining factor for free-rider incentives and aggregate output rather than the number of agents alone.

Proposition 2 *Suppose that agent j is active in any initial equilibrium. Let*

$$\mathcal{E}_j = \sum_{i=1}^{n} \epsilon_i^{a_j}(X^\star)\frac{(a_i e_i)^{\star 1-\sigma}}{(a_j e_j)^{\star 1-\sigma}}$$

where $\epsilon_i^{a_j}(X^\star) = \frac{a_j \frac{\partial^2 q_i(X^\star)}{\partial X \partial a_j}}{\frac{\partial q_i(X^\star)}{\partial X}}$ is the elasticity of the marginal benefit of output for agent i with respect to the ability of agent j in equilibrium. Then an increase in the ability of agent j increases group efficiency, $\frac{\partial X^\star}{\partial a_j} > 0$, if and only if $(\beta+1)+(1-\sigma)\mathcal{E}_j > 0$.

Proposition 2 extends the results of Kobayashi and Konishi (2020). It confirms that if we consider an increase in ability of any given individual as a form of indirectly increasing group size, then the group size paradox completely disappears when either the degree of complementarity σ or the elasticity of the marginal cost of effort β are high enough. This result accounts for the adjustment towards the new equilibrium. Observe that as $\sigma \to 1$, $\mathcal{E}_j \to \sum_{i=1}^{n} a_i \frac{\frac{\partial^2 q_i}{\partial X \partial a_j}}{\frac{\partial q_i}{\partial X}}$. As long as the elasticities are bounded the term $(1-\sigma)\mathcal{E}_j$ becomes arbitrarily small, ensuring that the condition is satisfied. The boundedness of elasticities is a rather mild requirement that prevents the team leader from dropping the marginal rewards to zero when a teammate's ability increases. Thus, we can conlude that high complementarity between individual efforts rules out the group-size paradox. The individual whose ability has increased only holds a limited impact on aggregate output and more effort from teammates is likely required in order to to take advantage of her productivity increase, thus increasing aggregate efficiency. Similarly, for any given σ, increasing the elasticity of the marginal cost of effort arbitrarily high limits the potential for large increases in effort by the chosen individual. Group members are likely to realize that cost-sharing is essential, thus limiting free-riding and eliminating the group-size paradox.

Next, we are going to turn attention to the special case of increasing the ability of a dormant agent. But before we do that I impose the following assumption:

Assumption 3 (Irrelevance of Dormant Agents) *Dormant agents have no effect on the distribution of marginal benefits from group output among everyone else:*

$$\lim_{a_j \to 0} \frac{\partial^2 q_i}{\partial X \partial a_j} = 0, \forall i \neq j$$

Assumption 3 ensures that as an individual becomes active, at the margin, the effect of his initial ability increase on the marginal benefits of everyone else vanishes in the limit. Intuitively, when an agent "activates" his ability is still essentially zero and he has no effective contribution to ouput. Any changes in output must originate from the active agents' abilities and efforts and thus the presence of the dormant player should have no relavance for the marginal benefits of everyone else.

Corollary 4 *If agent j is dormant in the initial equilibrium, then $\frac{\partial X^\star}{\partial a_j} \geq 0$ if and only if*

$$\sum_{i\neq j}(a_i e_i^\star)^{1-\sigma}\frac{\partial^2 q_i}{\partial X\partial a_j}\geq 0$$

If Assumption 3 *holds, then* $\frac{\partial X^\star}{\partial a_j}=0$.

The last result shows that Irrelevance of Dormant Agents is a sufficient condition for the group size paradox to disappear when adding a brand new member is considered. If Assumption 3 is violated, then the cross-partial effects in Corollary 4 are likely negative (in the most intuitive case adding a new player could mean lower shares for everyone else) and group-size paradox would arise. It should be noted that Independence of Dormant Agents ensures that the equilibrium efforts of all other agents are not affected when any given dormant player becomes marginally active (shown in the proof of Proposition 6. In light of this the result of Corollary 4 should not come as a surprise - if all other agents maintain the same effort and the ability of the new agent added is marginally zero, then clearly group output will not be initially affected. The results above can be unified in the following Corollary:

Corollary 5 *Suppose that Irrelevance of Dormant Agents holds. Let*

$$\bar{\mathcal{E}}=-\sup_{\{X^\star,j\}}|\mathcal{E}_j|$$

be finite, and suppose that $(\beta+1)+(1-\sigma)\bar{\mathcal{E}}>0$. *Then the group-size paradox disappears for any discrete increase in the ability of any agent:*

$$\triangle X^\star=\int_{a_i}^{a_i+\triangle a_i}\frac{\partial X^*}{\partial a}da>0$$

As long as the elasticities $\epsilon_i^{a_j}(X^\star)$ are bounded, then either high enough β or high enough σ can guarantee disappearance of the group-size paradox. This result unifies much of the preceding literature, in which particular bounds on β and σ typically ensure this result. In order to better understand how the incentives of individual players are affected, however, it would be instructive to study the behavior of their optimal effort not only in this marginal case, but also when the ability of already active agents increases.

Proposition 6 *The equilibrium effort of agent i responds to changes in the ability of agent j as follows:*

(i) $\frac{\partial e_i^\star}{\partial a_j} = 0$ if agent j is dormant under Independence of Irrelevant Alternatives

(ii) $sign\left(\frac{\partial e_i^\star}{\partial a_j}\right) = sign\left[(\sigma + \epsilon_i^X)\epsilon_X^{a_j} + \epsilon_i^{a_j}\right]$ when $a_j > 0$,

where $\epsilon_i^X = \frac{X^\star \frac{\partial^2 q_i}{\partial X^2}}{\frac{\partial q_i}{\partial X}}$ is the elasticity of the marignal benefit of group output to agent i in equilibrium, $\epsilon_X^{a_j} = \frac{\partial X^\star}{\partial a_j}\frac{a_j}{X^\star}$ is the elasticity of equilibrium output with respect to a_j, and $\epsilon_i^{a_j} = \frac{a_j \frac{\partial^2 q_i}{\partial X \partial a_j}}{\frac{\partial q_i}{\partial X}}$ is the elasticity of agent i's marginal benefit of output with respect to the ability of agent j.

The second part of Proposition 6 allows us to identify the source of possible free-rider incentives. To understand this, first note that $\epsilon_i^X \leq 0$ for at least some agents (it is not possible for all marginal benefits to be increasing or the agents' aggregate reward would increase faster than output). It is also instructive to consider the most "standard" case when $\epsilon_i^{a_j} \leq 0$. Intuitively, if agent j's marginal benefit of output increases in her own ability, then at least some of the other agents must have their marginal benefits reduced. Furthermore, consider the case when the elasticity of the marginal cost of effort is high enough, $\beta > -\mathcal{E}_j - 1$, so that the group-size paradox disappears and $\epsilon_X^{a_j} > 0$ (this assumed mostly for convenience as it will allow us to see the effects of changing σ without worrying about a sign change in $\epsilon_X^{a_j}$) . Under these assumptions, Proposition 6 says that agent i is affected by free-rider incentives if and only if:

$$\epsilon_i^X + \sigma < -\frac{\epsilon_i^{a_j}}{\epsilon_X^{a_j}} \tag{2.9}$$

Given the "standard" assumptions above, the right hand side is positive. Note that in the literature on collective action $\epsilon_i^{a_j}$ is commonly referred to as the negaitve free-rider effect, while $\epsilon_X^{a_j}$ as the positive group-size effect, i.e. free-riding is considered to originate from the negative sign of $\epsilon_i^{a_j}$. Typically, the relative size of these two effects is the key to eliminating the group-size paradox (making the ratio on the right-hand side as small as possible). The third effect, ϵ_i^X is new to this work. It has remained hidden in previous works either because of the use of constant-share allocation rules or because of the use of symmetric equilibria. In both of those cases $\epsilon_i^X = 0$. Thus, the presence of ϵ_i^X in Propostition 6 challenges the common definition of what free-riding is while also providing an explanation for why it appears so predominant in previous works.

There are two ways to think about this result. Firstly, given any σ, even if the group-size effect is positive, it is still possible that a free-rider effect occurs among some team members. In particular this happens to those individuals whose elasticity of marginal benefit either with respect to output (ϵ_i^X) or with respect to a_j,($\epsilon_i^{a_j}$), is too negative. This should make a lot of intuitive sense. If an agent's marginal reward declines too fast with aggregate output, then a boost in a teammate's productivity is more likely to encourage them to lower their effort. This occurs both because the marginal benefit of their effort declines with the equilibrium adjustment, but also because the output contribution from their teammate is likely to compensate for it. This disincentivizing effect is further exacerbated if the agent whose ability increased is overrewarded at their expense due to the redistribution of marginal benefits (a very negative $\epsilon_i^{a_j}$). The philosophical perspective presented here is that free-riding does not only depend on each agent becoming a smaller part of the group, but also on her being treated as such by the allocation rule. Even if $\sigma < -\frac{\epsilon_i^{a_j}}{\epsilon_X^{a_j}}$ (as in the majority of ealier works), it is still possible for the team leader to choose ϵ_i^X high enough for specific targeted agents and completely prevent them from free-riding.

A second way to consider the condition of Proposition 6 is to take any given allocation rule and consider what happens as σ changes. Suppose now that the agents become stronger substitutes. If σ becomes low enough then it is possible for all agents (other than j) to be afflicted by free-rider incentives.

Proposition 7 *Suppose that $\beta > -\mathcal{E}_j - 1$. Consider any allocation rule $Q(\boldsymbol{a}, X)$ such that*

$$\min_{\{i:a_i>0\}} \{\sup_X \epsilon_i^X\} = \bar{\epsilon} < 0$$

Then for any $\sigma < |\bar{\epsilon}|$ an increase in agent j's ability will lead to lower equilibrium effort by everyone else.

The condition on β is just meant to ensure that the condition of Proposition 2 is satisfied for consistency. This is the case in which the equilibrium increase in effort by agent j is sufficient to overcompensate for this discouraging effect on everyone else and the group-size paradox still vanishes.

This result has huge implications since it suggests that a lot of control over how agents respond to increases in teammate abilities lies in the hands of the team leader. For any given σ she can minimize the free-riding incentives by ensuring that these elasticities are close to zero. The potential for free riding will be minimized in the extreme case when all of these elasticities are zero, the case of linear shares. We can interpret this observation as a bridge between the general case discussed thus far and the optimal rule in the following section. Reducing free-riding incentives should have a significant impact on maximizing team efficiency. Thus, the result above hints that the optimal allocation rule should exhibit linear shares. This is indeed the case.

2.3 Optimal Allocation of Rewards

We can now turn to the problem of the team leader whose payoff is a fraction of total output. Therefore, she wants to choose the allocation rule $Q(\mathbf{a}, X)$ in a way that maximized aggregate efficiency $X^\star$ in the second-stage equilibrium. This is where equation (2.8) from Proposition (1) is going to be very useful.

Lemma 8 *Suppose that $\beta > 1 - 2\sigma$. Given any vector of abilities $\boldsymbol{a}$ and any fixed level of aggregate output X, the expression*

$$G\left(\boldsymbol{a}, X, \frac{\partial q_1}{\partial X}(\boldsymbol{a}, X), ..., \frac{\partial q_n}{\partial X}(\boldsymbol{a}, X)\right) = \left(\sum_{i=1}^{n} a_i^{\frac{(1-\sigma)(\beta+1)}{\sigma+\beta}} \left(\frac{\partial q_i}{\partial X}(\boldsymbol{a}, X)\right)^{\frac{1-\sigma}{\sigma+\beta}}\right)^{\frac{\sigma+\beta}{1-\sigma}}$$

obtains maximum when

$$\frac{\partial q_i}{\partial X}(\boldsymbol{a}, X) = \frac{\alpha_i}{\sum_{j=a}^{n} \alpha_j}\gamma \tag{2.10}$$

where $\alpha_i = a_i^{\frac{(1-\sigma)(\beta+1)}{\beta-(1-2\sigma)}}, \forall i = 1, ..., n.$

Proof. In any given state of the world defined by $\mathbf{a}$ and X, the problem of maximizing G reduces to that of choosing the marginal benefits to each member $\gamma_i = \frac{\partial q_i}{\partial X}(\mathbf{a}, X)$ optimally. We have the following constrained optimization problem:

$$\max_{(\gamma_i)_{i=1}^n} \sum_{i=1}^{n} a_i^{\frac{(1-\sigma)(\beta+1)}{\sigma+\beta}} \gamma_i^{\frac{1-\sigma}{\sigma+\beta}}$$

subject to:

$$\sum_{i=1}^{n} \gamma_i = \gamma$$

where the constraint is just a rewritten form of inequality 2.7.

The assumption $\beta > 1 - 2\sigma$ ensures that this is a covex optimization problem. Note that given the objective function, it is clear that the inequality constraint must bind at the optimum.

The first order condition for constrained optimization is:

$$\frac{1-\sigma}{\sigma+\beta} a_i^{\frac{(1-\sigma)(\beta+1)}{\sigma+\beta}} \gamma_i^{\frac{1-\sigma}{\sigma+\beta}-1} = \lambda, \forall i = 1, ..., n$$

or

$$\gamma_i = \left(\frac{a_i}{a_1}\right)^{\frac{(1-\sigma)(\beta+1)}{\beta-(1-2\sigma)}} \gamma_1$$

After plugging in the constraint it is straightforward to obtain:

$$\gamma_i^{\star} = \frac{a_i^{\frac{(1-\sigma)(\beta+1)}{\beta-(1-2\sigma)}}}{\sum_{j=1}^{n} a_j^{\frac{(1-\sigma)(\beta+1)}{\beta-(1-2\sigma)}}} \gamma$$

■

A key observation from Lemma 8 is that the marginal benefits that maximize Γ are independent of the state of the world X and only depend on the relative abilities of the agents. This ensures that for any vector of abilities $\mathbf{a}$ the expression $\sum_{i=1}^{n} a_i^{\frac{(1-\sigma)(\beta+1)}{\sigma+\beta}} \left(\frac{\partial q_i}{\partial X}(\mathbf{a}, X)\right)^{\frac{1-\sigma}{\sigma+\beta}}$ is uniformly bounded as a function of X. We summarize this observation in the following Lemma:

Lemma 9 *For any vector of individual abilities* $\boldsymbol{a}$ *and any allocation rule* $Q(\boldsymbol{a}, X)$*, the left-hand side of equation (*2.8*)*

$$G(X) = \left(\sum_{i=1}^{n} a_i^{\frac{(1-\sigma)(\beta+1)}{\sigma+\beta}} \left(\frac{\partial q_i}{\partial X}(\boldsymbol{a}, X) \right)^{\frac{1-\sigma}{\sigma+\beta}} \right)^{\frac{\sigma+\beta}{1-\sigma}}$$

is uniformly bouded by the constant function

$$A(X) = \gamma \left(\sum_{i=1}^{n} a_i^{\frac{(1-\sigma)(\beta+1)}{\beta-(1-2\sigma)}} \right)^{\frac{\beta-(1-2\sigma)}{1-\sigma}} \tag{2.11}$$

Proof. Plug in the optimal shares from Lemma 8 yields the results.

$$G(X) \leq \gamma \left(\sum_{i=1}^{n} a_i^{\frac{(1-\sigma)(\beta+1)}{\sigma+\beta}} \left(\frac{a_i^{\frac{(1-\sigma)(\beta+1)}{\beta-(1-2\sigma)}}}{\sum_{j=1}^{n} a_j^{\frac{(1-\sigma)(\beta+1)}{\beta-(1-2\sigma)}}} \right)^{\frac{1-\sigma}{\sigma+\beta}} \right)^{\frac{\sigma+\beta}{1-\sigma}} = \gamma \left(\sum_{i=1}^{n} a_i^{\frac{(1-\sigma)(\beta+1)}{\beta-(1-2\sigma)}} \right)^{\frac{\beta-(1-2\sigma)}{1-\sigma}}$$

■

Finally we are ready to state the optimal allocation mechanism.

Proposition 10 *Suppose that $\beta > 1 - 2\sigma$. Then, for any vector of abilities $\boldsymbol{a}$, the constant share allocation rule*

$$q_i(\boldsymbol{a}, X) = \frac{a_i^{\frac{(1-\sigma)(\beta+1)}{\beta-(1-2\sigma)}}}{\sum_{j=1}^{n} a_j^{\frac{(1-\sigma)(\beta+1)}{\beta-(1-2\sigma)}}} \gamma X \tag{2.12}$$

maximizes second-round equilibrium output. The value of the corresponding equilibrium output equals

$$X^\star = A^{\frac{1}{\beta}} = \gamma^{\frac{1}{\beta}} \left(\sum_{k=1}^{n} a_k^{\frac{(1-\sigma)(\beta+1)}{\beta-(1-2\sigma)}} \right)^{\frac{\beta-(1-2\sigma)}{\beta(1-\sigma)}}$$

and the individual effort outlays by

$$e_i^\star = \gamma^{\frac{1}{\beta}} a_i^{\frac{2(1-\sigma)}{\beta-(1-2\sigma)}} \left(\sum_{k=1}^{n} a_k^{\frac{(1-\sigma)(\beta+1)}{\beta-(1-2\sigma)}} \right)^{\frac{2\sigma-1}{\beta(1-\sigma)}}$$

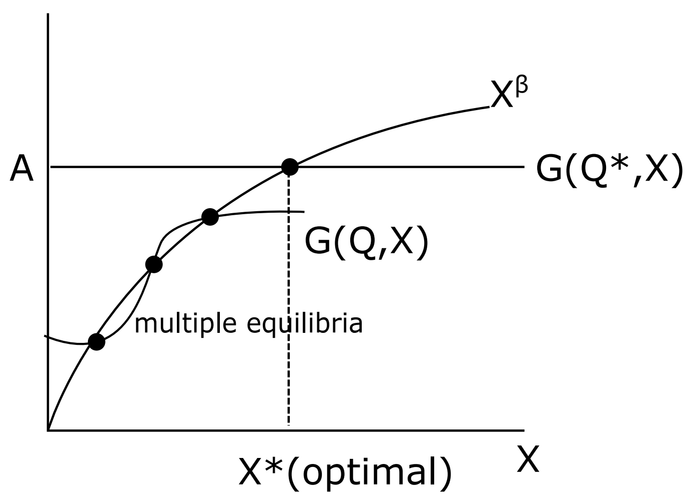
Proof. Plug in the optimal shares from Lemma 8 yields:

The choice of $q_i(\mathbf{a}, X) = \gamma_i^\star X$ ensures that the left-hand side of equation (2.8) obtains its smallest uniform upper bound. Given that the left-hand side of equation (2.8) is the exponential function X^β this ensures that the unique fixed point in equation (2.8) is maximized.

■

Figure 1 below shows visually the idea behind the optimality of the allocation rule in Proposition 10. For any given allocation rule Q, the right-hand side $G(Q, X)$ of Equation (2.8) is bounded by the constant A. When the optimal allocation rule $Q^\star$ is used the right-hand side achieves this smallest uniform bound: $G(Q^\star, X) = A$. In this case the equilibrium is unique and dominates every other equilibrium under any other allocation rule.

Figure 2.1: Fixed Points in Aggregate Output



This result as well as the discussion in the following sections strongly depend on the assumption that $\beta > 1-2\sigma$. It should be noted that in the opposite case when $\beta < 1-2\sigma$, i.e. when the cost of effort is near linear and/or when agents' inputs are near perfect substitutes, then it is optimal to give the entire reward to the highest-ability player. Focusing all incenctives in the hands of a single individual corresponds closely to Olson's idea that the efficiency of collective action diminishes with group size when the rewards are entirely private and individuals are close substitutes. In this model this effect is further emphasized by the fact that when agents are perfect substitutes it is socially wasteful to incentivize anyone but the highest ability individual as he provides the highest contribution per unit of effort (assuming near linear effort cost). In this paper we focus on the more interesting outcome when $\beta > 1-2\sigma$.

2.3.1 Discussion

Proposition 10 implies that the optimal (team payoff-maximizing) allocation rule assigns each team member a fixed share of the team reward, equal to her relative ability compared to the ability of her teammates (up to a power $\frac{\beta+1}{\beta-(1-2\sigma)}$). This holds if either the cost of effort is sufficiently convex ($\beta_j > 1$ - quadratic cost or higher), or if the degree of complementarity between individual member efforts is sufficiently high ($\sigma > \frac{1}{2}$). It is easiest to relate this result to the existing literature by looking at a symmetric ability case.

Corollary 11 *Suppose that $\beta > 1-2\sigma$. If all agents are endowed with identical ability a, then the optimal allocation rule is the egalitarian rule $q_i^\star(X) = \frac{\gamma X}{n}$.*

Ray, Baland, Dagnelie (2007) show that the egalitarian rule is not improvable by Lorenz domination as long as $\sigma > \frac{1}{2}$. The result provided here both incorporates their finding as a special case, but also stengthens it by using optimality instead of improvability. If there is low degree of substitutability between team members' efforts, then each member's contribution is relatively more essential for her team's output, and providing equal incentives is most cost efficient. Similarly, if the costs of effort exhibit high degree of convexity, then it would be very expensive (in terms of the share rewarded) for a team to stimulate one individual to work very hard. Instead, it would be much less costly to extract moderate effort from multiple individuals, hence the egalitarian rule.

Next, I state a few results regarding team composition that are jointly discussed below.

Corollary 12 (Disappearance of the group-size paradox) *Suppose that $\beta > 1-2\sigma$. Then aggregate efficiency is increasing in individual ability, $\frac{\partial X^*}{\partial a_i} > 0$ if $a_i > 0$, and $\frac{\partial X^*}{\partial a_i} \to 0$ when $a_i \to 0$.*

Corollary 13 (Individual Effort Response) *Suppose that $\beta > 1-2\sigma$ and suppose that the optimal allocation rule is used. Then:*

(i) $\frac{\partial e_i^\star}{\partial a_j} > 0$ if and only if $\sigma > \frac{1}{2}$ for any active agent i. Free-riding occurs if and only if $\sigma < \frac{1}{2}$

(ii) $\frac{\partial e_i^\star}{\partial a_j} = 0$ for any dormant agent i. Dormant agents are not responsive to changes in group potency.

(iii) $\lim_{a_j \to 0} \frac{\partial e_i^\star}{\partial a_j} = 0$

(iv) Higher ability individuals are always more sensitive to changes in a teammate's ability:

$$\frac{\partial e_i^\star}{\partial a_j} = \left(\frac{a_i}{a_k}\right)^{\frac{2(1-\sigma)}{\beta-(1-2\sigma)}} \frac{\partial e_k^\star}{\partial a_j}$$

(v) Changes in the ability of high skill individuals have stronger impact on the equilibrium effort of everyone else if and only if $\sigma < \frac{2}{\beta+3}$.

$$\frac{\partial e_i^\star}{\partial a_j} = \left(\frac{a_j}{a_k}\right)^{\frac{2(1-\sigma)-\sigma(\beta+1)}{\beta-(1-2\sigma)}} \frac{\partial e_i^\star}{\partial a_k}$$

Corollary 14 (Preferences for Diversity) *Suppose that $\beta > 1-2\sigma$ and $\sigma < \frac{2}{\beta+3}$. Also suppose that ability is freely transferable among individuals. Then under the optimal allocation rule:*

$$\frac{\partial X^\star}{\partial a_j} = \left(\frac{a_j}{a_i}\right)^{\frac{2(1-\sigma)-\sigma(\beta+1)}{\beta-(1-2\sigma)}} \frac{\partial X^\star}{\partial a_i}$$

(i) Consider two members i and j with with $a_i < a_j$. Then taking away $\epsilon > 0$ ability from i and giving it to j will result in higher group efficiency. The benefit of this transfer decreases in magnitude for higher σ, and disappears as $\sigma \to 1$.

(ii) If $\epsilon > 0$ ability were to be transferred between any two individuals, then the aggregate increase in productivity is maximized when the transfer occurs from the lowest ability to the highest ability agent.

Corollary 12 ascertains that the group-size paradox is completely absent as long as the optimal allocation rule is used. Note that the fact that $\frac{\partial X^*}{\partial a_i} = 0$ when $a_i = 0$ does not pose a contradiction to this claim. The group leader will always be indifferent between hiring a dormant agent or not given the relative ability distribution rule. However, as long as the added player has any positive ability $a_i > 0$, no matter how small, her total effect on team efficiency will always be positive: $\triangle X^\star = \int_0^{a_i} \frac{\partial X^*}{\partial a} da > 0$. Thus, "activating" a dormant individual always has a positive effect on team output and the group-size paradox vanishes.

At the same time, however, there could still be free-rider incentives as Corollary 13 shows. If the elasticity of substitution is higher than 2, then adding an agent with ability a_j to the group will lower the equilibrium effort of everyone else: $\triangle e_i^\star =$

$\int_{a=0}^{a_j} \frac{\partial e_i^\star}{\partial a} da < 0, \forall i \neq j$. Given Corollary 12, it must be the case that individual j's added effort overcompensates in the aggregate.

Corollary 12 also allows us to observe that the highest ability individuals have the strongest incentive effects on the rest of the group as long as it is not the case that both β and σ are very high. Since both of these variables negatively impact the importance of ability, it makes should make intuitive sense that they cannot both be arbitrarily large ($\sigma < \frac{2}{\beta+3}$) for this result to hold. The equilibrium share of any agent i is more severely impacted by an ϵ change in the ability of another agent j than from change in the ability of agent k as long as $a_j > a_k$. It is very important to understand how this works throught the interplay of parts (i) and (v) of Corollary 12. Consider first the case when the complementarity between individual outlays is high enough so that no free-riding occurs $\sigma > \frac{1}{2}$. Then the more we decrease the equilibrium share of any agent i while increasing the ability (and hence share) of agent j, the more i is stimulated to work harder in order to compensate for his share loss, while taking advantage of the complementarity of efforts. Given (v), this is achieved by choosing j to be the higest ability teammate. Now consider the opposite case when $\sigma < \frac{1}{2}$ and free-riding occurs. Then increasing the ability of agent j makes her increase her outlays, while reducing those of everyone else. Even though the rest of the agents reduce their equilibrium effort the most when j is the highest ability agent, the decrease in their shares is also the biggest, so the two effects counter each other. Thus, agent j's (the only agent whose effort increases in equilibrium) share increase is maximized, giving her the strongest incentives to increase her outlays. The combined effect of her higher ability and higher incentivization overcompensates for the effort loss in everyone else as given by Corollary 12. Additionally, we can also note that part (iv) of Corollary 13 indicates that the highest ability individuals are also the ones stre sensitive to changes in other player's productivity. This further reinforces the intuition above - if the $\epsilon > 0$ extra ability were allocated to someone else, then the highest ability agent would generate the most sizeable free-rider effect since she is most sensitive. By choosing her as the recipient of ϵ we ensure that the least sensitive subset of $n-1$ agents will engage in free-riding.

Corollary14 presents an additional unique result that implies that widening the spread of the ability distribution within any given subset of agents will always increase the group's efficiency. We should be careful not to interpret this result as implying that the incentives should be focused in the hands of one agent while removing everyone else from the group. The transfer is conditional on both agents being active and thus Corollary14 merely addresses the distribution of ability and its effect on aggregate potency once the set of active agents is determined. It is still beneficial to add more agents to the team by Corollary 12, but once they join, the spread between individual abilities should be as wide as possible. Combining both results suggests that both bigger and more diverse teams are more productive. Additionally, it also transipres from the proof of Corollary14 that if an $\epsilon > 0$ ability were to be added to the team, it is best invested in the higher productivity members as increases in their marginal effectiveness are most impactful. Corollary 13 shows why this is the case when we consider the strategic effect of this ability transfer. Equilibrium individual effort outlays are more sensitive to changes in the ability of high-skill individuals than to changes in the ability of low-skill individuals. Thus, investing any fixed $\epsilon > 0$ increment in ability to the most skilled agent is most efficient because it provides the strongest strategic response from the rest of the team.

2.4 Appendix A

2.4.1 Second Order Condition for Utility Maximization

$$\frac{\partial^2 U_i}{\partial e_i^2} = \frac{\partial^2 q_i}{\partial X^2}\left(\frac{\partial X}{\partial e_i}\right)^2 + \frac{\partial q_i}{\partial X}\frac{\partial^2 X}{\partial e_i^2} - \beta e_i^{\beta-1} < 0$$

Note that because the effort aggregator exhibits constant returns to scale in individual effort it follows that:

$$\frac{\partial^2 X}{\partial e_i^2} = \sigma X^{\sigma-1} a_i^{1-\sigma} e_i^{-\sigma-1}\left(e_i \frac{\partial X}{\partial e_i} - X\right) < 0$$

Plugging in from the first order condition for e_i^{β} and rearranging terms yields:

$$\frac{\partial^2 U_i}{\partial e_i^2} < 0 \iff \frac{a_i^{1-\sigma} e_i^{\star 1-\sigma}}{X^{\star 1-\sigma}}(\epsilon_i^X + \sigma) < \sigma + \beta$$

The first term is the individual equilibrium output of player i relative to that of the entire team ($X^{\star 1-\sigma} = \sum_{j=1}^{n} a_j e_j^{\star 1-\sigma}$) before aggregating and so it is less than or equal to unity. The inequality provides an upper bound on the elasticity ϵ_i^X of the marginal benefit to player i with respect to output X. It is always satisfied if $\epsilon_i^X < \beta$.

2.4.2 Proposition 1 Proof:

Plug in from (2.6) into the equation from team output (2.1):

$$X^\star = \left(\sum_{i=1}^{n} a_i^{1-\sigma}\left(X^{\star \frac{\sigma}{\sigma+\beta}}\left(a_i^{1-\sigma}\frac{\partial q_i}{\partial X}(X^\star)\right)^{\frac{1}{\sigma+\beta}}\right)^{1-\sigma}\right)^{\frac{1}{1-\sigma}}$$

Redistributing terms yields (2.8). Note that the left-hand side of (2.8) is a strictly increasing (exponential) function of $X^\star$, while the right-hand side is bounded, and continuous as a function of $X^\star$. Continuity follows from C^2 and boundedness from

(2.7). Thus, we obtain existence of $X^\star$. Plugging back into (2.6) yields the unique Nash Equilibrium effort levels for any given $X^\star$.

2.4.3 Proposition 2 Proof:

Implicitly differentiate equation (2.8):

$$\left[\frac{\beta(1-\sigma)}{\sigma+\beta} X^{\star\frac{\beta(1-\sigma)}{\sigma+\beta}-1} - \sum_{i=1}^{n} a_i^{\frac{(1-\sigma)(\beta+1)}{\sigma+\beta}} \frac{1-\sigma}{\sigma+\beta} \left(\frac{\partial q_i}{\partial X}\right)^{\frac{1-\sigma}{\sigma+\beta}-1} \frac{\partial^2 q_i}{\partial X^2}\right] \frac{\partial X^\star}{\partial a_j} =$$

$$= \frac{(1-\sigma)(\beta+1)}{\sigma+\beta} a_j^{\frac{(1-\sigma)(\beta+1)}{\sigma+\beta}-1} \left(\frac{\partial q_j}{\partial X}\right)^{\frac{1-\sigma}{\sigma+\beta}} + \frac{1-\sigma}{\sigma+\beta} \sum_{i=1}^{n} a_i^{\frac{(1-\sigma)(\beta+1)}{\sigma+\beta}} \left(\frac{\partial q_i}{\partial X}\right)^{\frac{1-\sigma}{\sigma+\beta}-1} \frac{\partial^2 q_i}{\partial X \partial a_j}$$

The term in the square brackets of the left-hand side can be rewritten as follows:

$$\frac{1}{X^\star}\frac{(1-\sigma)}{\sigma+\beta}\left[\beta X^{\star\frac{\beta(1-\sigma)}{\sigma+\beta}} - \sum_{i=1}^{n} a_i^{\frac{(1-\sigma)(\beta+1)}{\sigma+\beta}} \left(\frac{\partial q_i}{\partial X}\right)^{\frac{1-\sigma}{\sigma+\beta}} \epsilon_i^X\right] >$$

$$\frac{1}{X^\star}\beta\frac{(1-\sigma)}{\sigma+\beta}\left[X^{\star\frac{\beta(1-\sigma)}{\sigma+\beta}} - \sum_{i=1}^{n} a_i^{\frac{(1-\sigma)(\beta+1)}{\sigma+\beta}} \left(\frac{\partial q_i}{\partial X}\right)^{\frac{1-\sigma}{\sigma+\beta}}\right] = 0$$

We have used the assumption that $\epsilon_i^X < \beta$. Thus, the sign of $\frac{\partial X^\star}{\partial a_j}$ is the same as the sign of the right-hand side. Rewrite the right-hand side as follows:

$$\frac{1-\sigma}{(\sigma+\beta)X^{\star\frac{(1-\sigma)\sigma}{\sigma+\beta}}}[(\beta+1)a_j^{\frac{(1-\sigma)(\beta+1)}{\sigma+\beta}-1} \left(\frac{\partial q_j}{\partial X}\right)^{\frac{1-\sigma}{\sigma+\beta}} X^{\star\frac{(1-\sigma)\sigma}{\sigma+\beta}} +$$

$$\sum_{i=1}^{n} \frac{a_i}{a_j} a_i^{\frac{(1-\sigma)(\beta+1)}{\sigma+\beta}-1} \left(\frac{\partial q_i}{\partial X}\right)^{\frac{1-\sigma}{\sigma+\beta}} X^{\star\frac{(1-\sigma)\sigma}{\sigma+\beta}} \frac{a_j \frac{\partial^2 q_i}{\partial X \partial a_j}}{\frac{\partial q_i}{\partial X}}] =$$

$$= \frac{1-\sigma}{(\sigma+\beta)X^{\star\frac{(1-\sigma)\sigma}{\sigma+\beta}}}\left[(\beta+1)\frac{e_j^{\star 1-\sigma}}{a_j^{\sigma}} + (1-\sigma)\sum_{i=1}^{n}\frac{a_i}{a_j}\frac{e_i^{\star 1-\sigma}}{a_i^{\sigma}}\epsilon_i^{a_j}\right]$$

$$= \frac{(1-\sigma)e_j^{\star 1-\sigma}}{(\sigma+\beta)X^{\star\frac{(1-\sigma)\sigma}{\sigma+\beta}}}\left[(\beta+1) + (1-\sigma)\sum_{i=1}^{n}\frac{(a_i e_i)^{\star 1-\sigma}}{(a_j e_j)^{\star 1-\sigma}}\epsilon_i^{a_j}\right]$$

which gives the desired result.

2.4.4 Proposition 3 Proof:

$$\frac{\partial e_i^\star}{\partial a_j} = a_i^{\frac{1-\sigma}{\sigma+\beta}}\frac{1}{\sigma+\beta}\left(X^{\star\sigma}\frac{\partial q_i}{\partial X}\right)^{\frac{1}{\sigma+\beta}-1}\left[\left(\sigma X^{\star\sigma-1}\frac{\partial q_i}{\partial X} + X^{\star\sigma}\frac{\partial^2 q_i}{\partial X^2}\right)\frac{\partial X^\star}{\partial a_j} + X^{\star\sigma}\frac{\partial^2 q_i}{\partial X \partial a_j}\right]$$

The sign depends only on the term in the square brackets above. Note that when agent j is dormant this term becomes zero, hence $\frac{\partial e_i^\star}{\partial a_j} = 0$. Suppose instead that $a_j > 0$. The term in the square brackets can be rewritten as:

$$\frac{X^{\star\sigma}\frac{\partial q_i}{\partial X}}{a_j}\left[(\sigma+\epsilon_i^X)\epsilon_X^{a_j} + \epsilon_i^{a_j}\right]$$

where $\epsilon_i^X = \frac{X^\star \frac{\partial^2 q_i}{\partial X^2}}{\frac{\partial q_i}{\partial X}}$ is the elasticity of the marignal benefit of group output to agent i in equilibrium, $\epsilon_X^{a_j} = \frac{\partial X^\star}{\partial a_j}\frac{a_j}{X^\star}$ is the elasticity of equilibrium output with respect to a_j, and $\epsilon_i^{a_j} = \frac{a_j \frac{\partial^2 q_i}{\partial X \partial a_j}}{\frac{\partial q_i}{\partial X}}$ is the elasticity of the marginal benefit of output to agent i with respect to the ability of agent j.

2.4.5 Corollary (12) Proof:

$$\frac{\partial X^\star}{\partial a_i} = \frac{\beta+1}{\beta}a_i^{\frac{(1-\sigma)(\beta+1)}{\beta-(1-2\sigma)}-1}\left(\sum_{k=1}^{n}a_k^{\frac{(1-\sigma)(\beta+1)}{\beta-(1-2\sigma)}}\right)^{\frac{\beta-(1-2\sigma)}{\beta(1-\sigma)}-1} \geq 0$$

with strict inequality for $a_i > 0$.

2.4.6 Corollary (13) Proof:

$$\frac{\partial e_i^\star}{\partial a_j} = a_i^{\frac{2(1-\sigma)}{\beta-(1-2\sigma)}} \frac{(1-\sigma)(\beta+1)}{\beta-(1-2\sigma)} \frac{2\sigma-1}{\beta(1-\sigma)} \left(\sum_{k=1}^{n} a_k^{\frac{(1-\sigma)(\beta+1)}{\beta-(1-2\sigma)}}\right)^{\frac{2\sigma-1}{\beta(1-\sigma)}-1} a_j^{\frac{2(1-\sigma)-\sigma(\beta+1)}{\beta-(1-2\sigma)}}$$

2.4.7 Corollary (14) Proof:

$$\frac{\frac{\partial X^\star}{\partial a_j}}{\frac{\partial X^\star}{\partial a_i}} = \left(\frac{a_j}{a_i}\right)^{\frac{2(1-\sigma)-\sigma(\beta+1)}{\beta-(1-2\sigma)}}$$

Since the power $\frac{2(1-\sigma)-\sigma(\beta+1)}{\beta-(1-2\sigma)}$ is positive only if $\sigma < \frac{2}{\beta+3}$, it implies that higher ability members bring higher marginal benefit from individual productivity in equilibrium.

Chapter 3

Optimal Intra-team Allocation in Group Contests with Heterogeneous Agents

In this chapter I consider a multi-team extension of the basic model which should allow better understanding of how the fundamental results can adapt when interactions between teams are introduced.

The basic assumptions on the individual team level are essentially the same as the single-team model, but I include them for completeness. Section 1.1 describes the characteristics of the participants in the contest: individuals and teams. I then move on to discuss the general structure of the team contests in section 1.2. Once this is done, section 1.3 sets up the within-team reward allocation problem. Finally, section 1.4 defines equilibrium in this extended model.

3.0.1 Team-Specific Characteristics

There are J teams and each team $j = 1, ..., J$ consists of n_j members, $N = \sum_{i=1}^{J} n_i$. Each member i of team j is endowed with ability $a_{ij} \geq 0$ and makes effort denoted by the continuous variable $e_{ij} \geq 0$. It is assumed that the individual abilities remain private information within each team - each player knows the ability of her teammates, but not the abilities of agents on other teams. I assume that all individuals on the same team face the same team-specific cost function $C_j(e_{ij}) = \frac{e_{ij}^{\beta_j+1}}{\beta_j+1}$. Note that the teams and players are completely heterogeneous - both the cost parameters and number of players differ across teams, and so does individual player ability.

Team output X_j is, once again, given by a constant elasticity of substitution aggregator:

$$X_j = \left(\sum_{i=1}^{n_j} (a_{ij} e_{ij})^{1-\sigma_j} \right)^{\frac{1}{1-\sigma_j}}$$

The elasticity of substitution $\frac{1}{\sigma_j}$ also differs across teams. The same restriction $0 < \sigma_j < 1$ applies since the $\sigma_j > 1$ case leads to a unique zero-effort equilibrium.

3.0.2 Inter-Team Prize Allocation Mechanism

I consider the most general type of contest in which the overall prize pool is allocated among the competing teams based on their team outputs. Let X denote the vector of realized team outputs: $X = (X_1, ..., X_J)$. Each X_j is the aggregated effort of all team j members. The vector X is important in two different ways: 1) it determines the size of the overall prize pool through a general production function $F(X)$, and 2) it is used to determine how the prize pool is allocated among teams through an inter-team allocation mechanism $p(X) = (p_1(X), ..., p_J(X))$.

If we consider each team as a different department within a firm, then the function $F(X_1, ..., X_J)$ summarizes the firm-level production technology and shows how the efforts of individual departments affect the overall size of aggregate production (each team can be interpreted as a factor of production). In this case, it could make sense to impose standard classical assumptions on the production technology. Alternatively, the overall prize pool may be fixed at a given value V, so that team effort has no effect on it: $F(X) = V$, just as would be the case in a standard Tullock contest. To keep this as general as possible I only impose non-negative and non-increasing marginal products: $\frac{\partial F}{\partial X_j} \geq 0$ and $\frac{\partial^2 F}{\partial X_j^2} \leq 0$. This allows to endogenize size of the reward, without ruling out fixed prize contests.

The prize is divided among the teams according to a predetermined allocation mechanism denoted by $p(X) = (p_1(X), ..., p_J(X))$. The share won by team j, $p_j(X_1, ..., X_j)$ is twice continuously differentiable and monotonic in own-team effort $\frac{\partial p_j}{\partial X_j} > 0$. I assume that the reward allocation mechanism $p(X)$ is common knowledge among all competing teams and that for any given realization of team outputs $X = (X_1, ..., X_J)$ the shares $p_j(X)$ are assigned to satisfy the resource constraint $\sum_{j=1}^{J} p_j(X) \leq F(X)$. It should be noted that the divisibility of the reward is immaterial in the context of this paper. The shares $p_j(X)$ could easily be interpreted as winning probabilities of the (then) indivisible reward $F(X)$.

To understand the scope of these assumptions note two familiar special cases: (1) If $F(X) = V$ and $p_j(X) = \frac{X_j}{X_1+...+X_J}$, then this is a standard Tullock contest with fixed prize pool - a very common case in the contest theory literature; (2) If $F(X)$ is a constant returns to scale technology and $p_j(X) = \frac{\partial F}{\partial X_j}$, then we have a standard classical production function in which the factors of production receive their marginal product.

3.0.3 Intra-Team Reward Alocation

Each team $j = 1, ..., J$ is represented by a leader who keeps a fraction $(1 - \gamma_j)$ of the team's payoff $p_j(X)$. The team leader's goal is once again to maximize team output. The allocation rule within each each is going to be denoted by $Q_j(\mathbf{a}_j, p_j(X)) = (q_{1j}(\mathbf{a}_j, p_j(X)), q_{2j}(\mathbf{a}_j, p_j(X)), ..., q_{n_j j}(\mathbf{a}_j, p_j(X)))$. Each $q_{mj}(\mathbf{a}_j, p_j(X))$ is the value awarded to member m of team j and is conditional on the realized outcome X only via the team reward $p_j(X)$. The key assumption here is that each team considers only the size of its own prize when deciding on how to solve this allocation problem. The performance of the opposing teams only affects this allocation indirectly, through the team contest mechanism $p(X)$. This rules out ”grudges” against opposing teams in which the team leader is willing to punish his team members if an opposing team does well. Also note that $\mathbf{a}_j$ is the vector of abilities on team j only. The shares $\{q_{mj}(\mathbf{a}_j, p_j(X))\}_{m=1}^{k_j}$ are assumed to be continuous, twice continuously differentiable, weakly monotonic in $p_j(X)$: $\frac{\partial q_{mj}}{\partial p_j(X)} \geq 0$, and once again that $\epsilon_{ij}^{X_j} < \beta_j$ (for a discussion of these assumptions visit the single-team problem. Just as in the single-team case, after the leader of team j takes his share of the team reward $(1 - \gamma_j)p_j(X)$, the remainder is fully transferable back to the team members $\sum_{m=1}^{n_j} q_{mj}(\mathbf{a}_j, p_j(X)) = \gamma_j p_j(X)$ for any $p_j(X)$.

3.1 Equilibrium

The problem is modelled as a two-stage game. In stage one all team leaders select an allocation mechanism for their team and in stage two all agents select effort optimally. Note that the abilities of all agents and the allocation rules chosen remain private information within each team.

Consider first the extended effort-choice game played among all individuals across all teams. The solution concept used in the second round departs from a standard Nash Equilibrium. In particular, the equilibrium efforts of individual agents will be selected as a best reponse to their teammates' actions, but more importantly as a best response only to the aggregated output of other groups. Each agent is very limited in her understanding of the institutional characteristics of the other teams. She does not know their reward allocation mechanism, she does not know their abilities, and she may not even know how many members they have. When choosing her best response she is only able to act strategically against their aggregate output choice. The latter affects her and the output of her team via the inter-team contest rule only.

Before defining the equilibrium formally, it is instructive to discuss the incentives for both agents and leaders in slightly more detail first. Consider the optimization problem for each individual player. Given the effort of all other agents on her team, given her team's allocation rule, and given the team output levels of all other teams, each member m of group j chooses effort e_{mj} to maximize her utility:

$$\max_{e_{mj}\geq 0} U_{mj}(e_{mj}) = q_{mj}(\mathbf{a}_j, p_j(X)) - C_j(e_{mj})$$

Observe that the utility of agent m on team j is only affected by the collective action of other teams $X_1, ..., X_{j-1}, X_{j+1}, ...X_j$. If two or more agents on another team change their efforts in a way that does not affect their team's output, then she will remain unaffected. Similarly, if both the allocation rule and efforts within any other team change in a way that leaves their output constant, she is once again unaffected. Indeed, she may not even be able to detect that such changes ocurred. By assuming that she plays best response to those aggregate actions only, and not to the individual play of the agents generating them, I am able to go around the incomplete information structure of the problem.

Using this observation, I define the equilibrium in individual effort choice as follows:

Definition 15 (Equilibrium among the agents) *A vector of efforts* $\{\{e^{\star}_{mj}\}_{m=1}^{n_j}\}_{j=1}^{J}$ *constitutes an equilibrium among the agents if for all agents* i *on all teams* j*:*

$$e_{ij}^{\star} \in \arg\max_{e_{ij}\geq 0} q_{ij}(\boldsymbol{a}_j, p_j(e_{-ij}^{\star}, e_{ij}, X_{-j}^{\star})) - C_j(e_{ij})$$

i.e. given her team's allocation rule Q_j, each agent selects her effort optimally as a best response to her teammates' efforts $\{e_{mj}^{\star}\}_{m\neq i}$,and as a best response to the equilibrium ouputs of all other teams $X_{-j}^{\star}$.

Next, consider the first-round problem for the team-leaders. Recall that each team leader's payoff is $(1-\gamma_j)p_j(X)$. The equilibrium of the game among the agents $X^{\star}$ will satisfy the utitlity maximization for the team leaders if and only if:

$$Q_j^{\star} \in \arg\max_{Q_j} p_j(X_{-j}^{\star}, X_j(Q_j)), \forall j = 1, ..., J$$

Given the equilibrium output choice of other teams, the aggregate output of team j must maximize leader j's utility. This is achieved by selecting the allocation rule optimally.

3.2 Individual Effort Choice

Consider the optimization problem for each individual player. Given the effort of all other agents on her team, given her team's allocation rule, and given the team output levels of all other teams, each member m of group j chooses effort e_{mj} to maximize her utility:

$$\max_{e_{mj}\geq 0} U_{mj}(e_{mj}) = q_{mj}(\mathbf{a}_j, p_j(X)) - C_j(e_{mj})$$

Note that for brevity I will supress the argument $\mathbf{a}_j$ and denote $\frac{\partial q_{mj}}{\partial p_j(X)} = q'_{mj}(p_j(X))$. It is a dominant strategy for every dormant team member to put in zero effort. The general form of the First Order Condition is for any active agent is:

$$q'_{mj}(p_j(X))\frac{\partial p_j}{\partial X_j}(X_1,...,X_J)\frac{\partial X_j}{\partial e_{mj}} = C'_j(e_{mj})$$

Note that with the CES effort aggregator:

$$\frac{\partial X_j}{\partial e_{mj}} = a_{mj}^{1-\sigma_j} X_j^{\sigma_j} e_{mj}^{-\sigma_j}$$

The First Order Condition for e_{mj} then simplifies to:

$$e^*_{mj} = \left(a_{mj}^{1-\sigma_j} q'_{mj}(p_j(X))\right)^{\frac{1}{\sigma_j+\beta_j}} \left(\frac{\partial p_j}{\partial X_j}\right)^{\frac{1}{\sigma_j+\beta_j}} X_j^{\frac{\sigma_j}{\sigma_j+\beta_j}} \tag{3.1}$$

Equation (3.1) implicitly shows that when each individual chooses her effort contribution optimally, she does not need to take into account the individual effort of opposing players, but it is sufficient to consider only the aggregated effort of opposing teams. Given the aggregated output of other teams, and given the effort of her teammates, her effort choice is affected by her ability a_{mj}, the rate at which she gets rewarded for increasing team output $q'_{mj}(p_j(E))$, the rate at which her team gets rewarded by the contest allocation mechanism $\frac{\partial p_j}{\partial E_j}$, the degree of complementarity of team members' effort σ_j, and the degree of convexity of the team-specific cost funtion β_j. Comparing this to the single team case, note the only difference is the presence of the $\frac{\partial p_j}{\partial X_j}$ term which accounts for the fact that the team reward is not simply the given team's output, but is instead given by the intra-team contest rule.

3.2.1 Equilibrium Group Ouput

Just as in the single-team model we can aggregate the first-order conditions to obtain the following equation that defines the best response $X_j^\star$.

Proposition 16 *Suppose that* $\frac{\partial}{\partial X_j}\left(\frac{X_j^{\beta_j}}{\frac{\partial p_j}{\partial X_j}}\right) > 0$. *Then for any vector of team outputs by other teams* $(X_1, ..., X_{j-1}, X_{j+1}, ..., X_J)$, *and any allocation rule* $Q_j(\boldsymbol{a}_j, p_j(X))$, *there exist a unique best response* $e^\star_{mj}$ *by each member of team* j.

The aggregated "group best reponse" $X_j^\star$ *for team* j *can be uniquely obtained by solving:*

$$(X_j^\star)^{\beta_j} = \left(\frac{\partial p_j}{\partial X_j}(X^\star)\right)\left(\sum_{m=1}^{n_j} a_{mj}^{\frac{(1-\sigma_j)(\beta_j+1)}{\sigma_j+\beta_j}} q'_{mj}(p_j(X^\star))^{\frac{1-\sigma_j}{\sigma_j+\beta_j}}\right)^{\frac{\sigma_j+\beta_j}{1-\sigma_j}} \tag{3.2}$$

and the individual best reponses $e^\star_{mj}$ *can be found by plugging into Equation* 3.1

The proof follows exactly in the steps of the single-team case. One thing that has to be taken care of is the presence of the $\frac{\partial p_j}{\partial X_j}(X^\star)$. It imposes a constraint on the types of team contests for which Proposition 16 holds. It states that as a given team increases its aggregate effort, the rate at which its resulting reward grows cannot increase faster than the elasticity of the marginal cost of effort $\beta_j > 0$, i.e. we rule out contests in which team rewards can grow exponentially with effort. This assumption is very mild and should be trivially satisfied in most fixed-reward contests, as well as in firm production models exhibiting constant or decreasing returns. Consider a classical model of a production funcion in which each factor(team) is paid is marginal product, for example. The condition is then equivalent to assuming that the marginal product of each factor cannot increase faster than the exponantial rate β_j.

3.3 Optimal Allocation Rule

Recall that the objective for each team leader was to maximize $p_j(X)$ with respect to X_j. Note that by assumption $\frac{\partial p_j}{\partial X_j} > 0$. Hence, the following Lemma:

Lemma 17 *Given any vector of output choices by other teams, $\boldsymbol{X}_{-j}$, the utility of the leader of team j is maximized if and only if her team's output X_j is also maximized conditional on $\boldsymbol{X}_{-j}$.*

Lemma (17) is an intermediate step showing that utility maximization and output maximization conditional on other teams' choices are equivalent from the perspective of the group leader.

Proposition 18 (Optimal Intra-Team Allocation Rule:) *Suppose that the following conditions hold:*

(i) $\frac{\partial}{\partial X_j}\left(\frac{X_j^{\beta_j}}{\frac{\partial p_j}{\partial X_j}}\right) > 0$

(ii) $\beta_j > 1 - 2\sigma_j$

Then selecting the intra-team allocation rule

$$Q_j^{\star}(\boldsymbol{a}_j, p_j(X)) = (q_{1j}^{\star}(\boldsymbol{a}_j, p_j(X)), ...q_{k_j j}^{\star}(\boldsymbol{a}_j, p_j(X)))$$

such that

$$q_{mj}^{\star}(\boldsymbol{a}_j, p_j(X)) = \frac{a_{mj}^{\frac{(1-\sigma_j)(\beta_j+1)}{\beta_j-(1-2\sigma_j)}}}{\sum_{h=1}^{k_j} a_{hj}^{\frac{(1-\sigma_j)(\beta_j+1)}{\beta_j-2(1-\sigma_j)}}}\gamma_j p_j(X),$$

for all $m = 1, ..., n_j$ is a dominant strategy for every team leader in the first stage.

Proof.

By Lemma 17, the objective of the team leader is to maximize $X_j^{\star}$ for any vector $\mathbf{X}_{-j}$. But for the same vector $\mathbf{X}_{-j}$ the aggregated best response of team j's agents must satisfy equation 3.2. This problem is equivalent to that of maximizing group output in the individual effort choice game conditional on the equilibrium behavior of other teams. This is the same optimization problem as in the single-team case, hence the result. ■

Imposing the optimal allocation rule incentivizes each team to select the best response $X_j^\star$ collectively in agreement with the team leader's agenda.

Proposition 19 (Second Stage Equilibria under the Optimal Rule:) *Under the conditions of Proposition* 18*, an equilibrium exists and must satisfy the following system of equations:*

(i) Aggregate equilibrium output levels are determined by the system of equations:

$$X_j^{\star\beta_j} = A_j \frac{\partial p_j}{\partial X_j}(X^\star) \tag{3.3}$$

where $A_j = \gamma_j \left(\sum_{m=1}^{n_j} a_{mj}^{\frac{(1-\sigma_j)(\beta_j+1)}{\beta_j-(1-2\sigma_j)}} \right)^{\frac{\beta_j-(1-2\sigma_j)}{1-\sigma_j}}$.

(ii) For any vector of equilibrium outputs $X^\star$*, the individual equilibrium efforts can be obtained from Equation* 3.1*.*

Proof. We are going to show existence of equilibrium for the aggregated ouput X. Note that the set of all possible aggregate output vectors is closed and convex. The equilibrium aggregate output of team j is also bounded as long as $\frac{\partial}{\partial X_j}\left(\frac{X_j^{\beta_j}}{\frac{\partial p_j}{\partial X_j}}\right) > 0$ is satisified. The mapping given by the system of equations (3.3) is continuous. By Brouwer's Fixed Point Theorem there exists a fixed point $X^\star$. Individual equilibrium efforts are always uniquely determined for any $X^\star$ by point (ii) of Proposition 19, hence we obtain existence.

Plugging in the consistent allocation rule into Equation (3.2) gives the desired result. ■

Proposition 19 shows that discussing the equilibrium on the team level as described by the system 3.3 is sufficient since the individual equilibrium efforts can always be derived uniquely afterwards.

3.4 Special Case: Tullock Contest with a Fixed Reward

It would be interesting to gain more understanding about the interaction between teams in any equilibrium. In order to do this, in this section I turn my attention on a type of contest rule that has been very prevalent in the literature on collective action - a fixed reward contest where the probability of success for each team is given by Tulock's (1980) relative effort formulation.

Consider a contest with symmetric costs of effort $\beta_j = \beta, \forall j$, a fixed prize pool of value V, and a reward mechanism in which the share of the prize p_j earned by each team j is proportional to its relative effort: $p_j = \frac{X_j}{X_1+...+X_J}$. In the context of the general model from the previous section, the total prize pool is given by the production function $F(X_1, ..., X_J) = V$ where agregate output is fixed and independent of team inputs, and the total prize earned by each team j is equal to $p_j(X) = p_j V = \frac{X_j}{X_1+...+X_J} V$. Each p_j can be interpreted as the probability of winning the reward V.

Denote total ouput t by $X = X_1 + ... + X_J$ and let $X - X_j = X_{-j}$. Then,

$$\frac{\partial p_j}{\partial X_j} = \frac{(X_1 + ... + X_J) - X_j}{(X_1 + ... + X_J)^2} V = \frac{X_{-j}}{X^2} V$$

The condition of Proposition (18) is satisfied:

$$(i)\frac{\partial}{\partial X_j}\left(\frac{X_j^{\beta}}{\frac{\partial p_j}{\partial X_j}}\right)=\frac{\partial}{\partial X_j}\left(\frac{X_j^{\beta}}{\frac{X_{-j}}{X^2}V}\right)=\frac{\beta X_j^{\beta-1}X^2+2X_j^{\beta}X}{X_{-j}V}>0$$

Assuming that $\beta_j > 1-2\sigma_j, \forall j \leq J$,we can apply the result of Proposition (19). Equilibrium team efforts must satisfy:

$$X_j^{\star\beta}=A_j\frac{X_{-j}^{\star}}{X^{\star 2}}V \tag{3.4}$$

where recall that $A_j=\gamma_j\left(\sum_{m=1}^{k_j}a_{mj}^{\frac{\beta}{\beta-2(1-\sigma_j)}}\right)^{\frac{\beta-2(1-\sigma_j)}{1-\sigma_j}}$. In what follows I am going to refer to A_j as the aggregate ability of team j.

Rewrite equation (3.4) as follows:

$$X_j^{\star\beta}=\frac{VA_j}{X_j^{\star}}\frac{X_{-j}^{\star}}{X^{\star}}\frac{X_j^{\star}}{X^{\star}}$$

which simplifies to

$$X_j^{\star}=\left(p_j^{\star}(1-p_j^{\star})A_jV\right)^{\frac{1}{\beta+1}} \tag{3.5}$$

where $p_j^{\star}=\frac{X_j^{\star}}{X_1^{\star}+...+X_J^{\star}}$ Equation (3.5) implicitly solves for the aggregated best response effort of team j as a function of its equilibrium share $p_j^{\star}$, aggregate team ability A_j, and the value of the reward V.

Proposition 20 *Equilibrium team effort for any team j with aggregate ability A_j is maximized when $p_j^{\star}=\frac{1}{2}$ with $\frac{\partial X_j^{\star}}{\partial p_j^{\star}}>0$ if $p_j^{\star}<\frac{1}{2}$ and $\frac{\partial X_j^{*}}{\partial p_j^{\star}}<0$ if $p_j^{\star}>\frac{1}{2}$.*

Proof.

$$\frac{\partial X_j^{\star}}{\partial p_j^{\star}}=\frac{1}{\beta+1}\left(p_j^{\star}(1-p_j^{\star})A_jV\right)^{\frac{-\beta}{\beta+1}}A_jV(1-2p_j^{\star})\lesseqgtr 0\Leftrightarrow p_j^{\star}\gtreqless\frac{1}{2}$$

■

Corollary 21 *Equilibrium team effort for any team j is increasing in other teams' aggregate outlays if $p_j^\star > \frac{1}{2}$ and decreasing in other teams' aggregate outlays if $p_j^\star < \frac{1}{2}$.*

Proof.

$$p_j = \frac{X_j}{X_j + X_{-j}} \Rightarrow X_j = \frac{p_j}{1 - p_j} X_{-j}$$

Equation (3.5) is thus equivalent to:

$$\left(\frac{p_j^\star}{1 - p_j^\star}\right)^{\beta+1} \frac{1}{p_j^\star(1 - p_j^\star)} = \frac{VA_j}{X_{-j}^{\star\beta+1}}$$

or

$$\frac{p_j^{\star\beta}}{(1 - p_j^\star)^{\beta+2}} = \frac{VA_j}{X_{-j}^{\star\beta+1}} \tag{3.6}$$

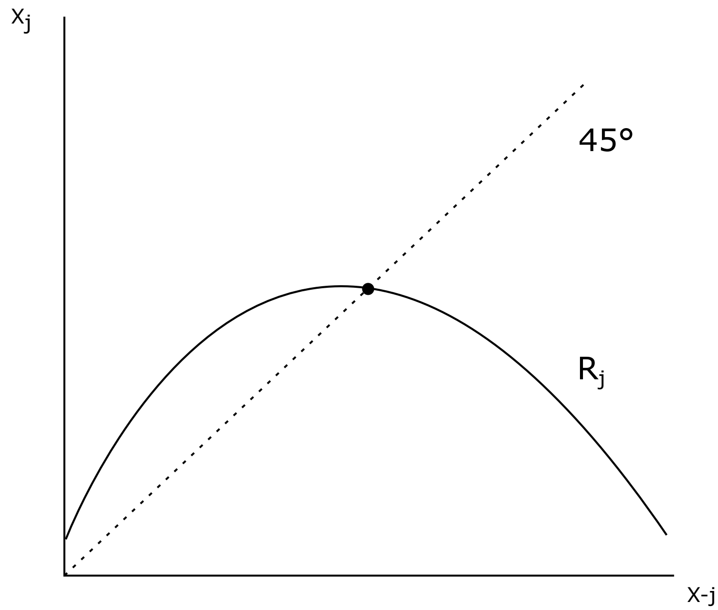
The left-hand side is increasing in $p_j^\star$, hence $\frac{\partial p_j^\star}{\partial X_{-j}} < 0$. The result follows from Proposition 20.

■

Corollary 21 shows that competition incentivizes a team to exert more effort only as long as it wins half of the total reward V or more.

A "winning" team with $p_j^\star > \frac{1}{2}$ is positively stimulated by extra competition, leading it to increase team effort, while "losing" teams with $p_j^\star < \frac{1}{2}$ are discouraged by higher competition in equilibrium. Figure 1 shows the reaction function for team j. The peak always lies on the 45° line.

Figure 3.1: Best Response for team j



3.4.1 Equilibrium in the Two Team Case

Next we explicitly solve for the second stage Nash Equilibrium in the case of two competing teams only.

Theorem 22 *The unique equilibrium in the two team case is given by:.*

$$X_1^\star = \left[\frac{V A_1^{\frac{2+\beta}{1+\beta}} A_2^{\frac{1}{1+\beta}}}{\left(A_1^{\frac{1}{1+\beta}} + A_2^{\frac{1}{1+\beta}} \right)^2} \right]^{\frac{1}{1+\beta}} ; X_2^\star = \left[\frac{V A_2^{\frac{2+\beta}{1+\beta}} A_1^{\frac{1}{1+\beta}}}{\left(A_1^{\frac{1}{1+\beta}} + A_2^{\frac{1}{1+\beta}} \right)^2} \right]^{\frac{1}{1+\beta}}$$

Proof.

Using $p_2^\star = 1 - p_1^\star$ and the equilibrium equations:

$$X_1^\star = [p_1^\star (1 - p_1^\star) V A_1]^{\frac{1}{1+\beta}}$$

$$X_2^\star = [p_2^\star (1 - p_2^\star) V A_2]^{\frac{1}{1+\beta}}$$

we obtain:

$$\frac{p_1^\star}{p_2^\star} = \frac{X_1^\star}{X_2^\star} = \left(\frac{A_1}{A_2}\right)^{\frac{1}{\beta+1}}$$

or:

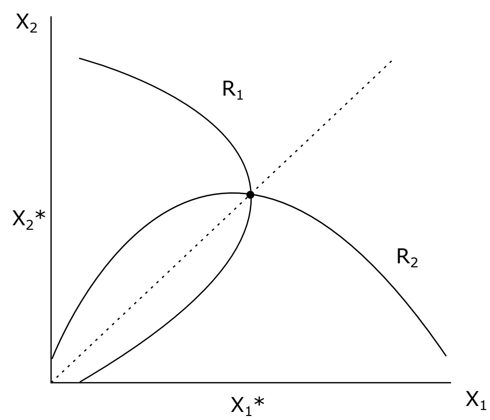
$$p_1^\star = \frac{A_1^{\frac{1}{1+\beta}}}{A_1^{\frac{1}{1+\beta}} + A_2^{\frac{1}{1+\beta}}}; p_2^\star = \frac{A_2^{\frac{1}{1+\beta}}}{A_1^{\frac{1}{1+\beta}} + A_2^{\frac{1}{1+\beta}}}$$

Plugging back into the equilibrium conditions gives the desired result.

■

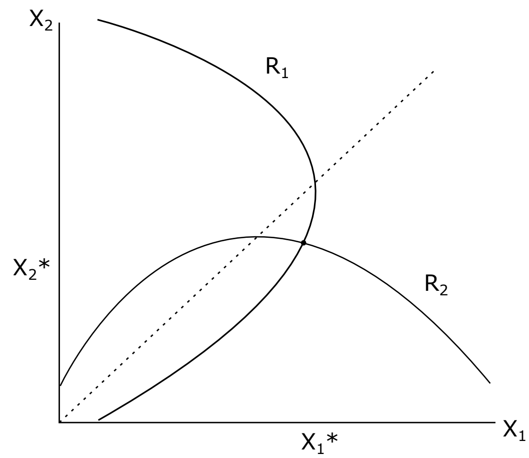
In the symmetric case when $A_1 = A_2 = A$, the equilibrium shares are $p_1^\star = p_2^\star = \frac{1}{2}$ and the best response finctions intersect at their peaks on the 45° line ($X_1^\star = X_2^\star = \left(\frac{VA}{4}\right)^{\frac{1}{\beta+1}}$) . Figure 2 depicts this case. Figure 3 shows the equilibrium when $A_1 > A_2$ (higher team skill shifts the best response peak higher along the 45° line).

Figure 3.2: Equilibrium in the Symmetric Case $A_1 = A_2$



These results suggest that in order to maximize total team outlays $X_1^\star + X_2^\star$, the two teams should be as equally matched as possible ($A_1 = A_2$). This turns out to be the case and the result is presented in the following section.

Figure 3.3: Equilibrium in the case $A_1 > A_2$



3.4.2 Effort-Maximizing Inter-Team Skill Distribution

So far we have treated the composition of each team, represented by the aggregate team ability A_j, as exogenous. However, it is often the case in the real world that individual skills are transferable and any given employee could be assigned to different departments or teams. If that is the case, then an interesting question might arise for the firm manager: how should different eployees be assigned across teams in order to maximize total equilibrium output (effort)? Alternatively, a common point of interest in the collective action literature is the size of rent dissipation. In this context, it would be interesting to see how the distribution of aggregate ability across teams maximizes the extent of rent dissipation.

Proposition 23 *If there are only two teams with aggregate abilities A_1 and A_2, and aggregate team ability is freely transferable between teams, then total rent dissipation is maximized when $A_1 = A_2$.*

Proof.

It can be shown that total outlays in the unique equilibrium are given by:

$$X_1^* + X_2^* = V^{\frac{1}{\beta+1}} A_1^{\frac{1}{(\beta+1)^2}} A_2^{\frac{1}{(\beta+1)^2}} \left(A_1^{\frac{1}{\beta+1}} + A_2^{\frac{1}{\beta+1}} \right)^{\frac{\beta-1}{\beta+1}}$$

The problem for the contest designer is:

$$\max_{\{A_1,A_2\}} X_1^* + X_2^*, s.t. A_1 + A_2 \leq \bar{A}$$

The first order condition for A_1 is:

$$V^{\frac{1}{\beta+1}} \frac{1}{(1+\beta)^2} A_1^{\frac{1}{(1+\beta)^2}-1} A_2^{\frac{1}{(1+\beta)^2}} \left(A_1^{\frac{1}{1+\beta}} + A_2^{\frac{1}{1+\beta}} \right)^{\frac{\beta-1}{1+\beta}}$$

$$+V^{\frac{1}{1+\beta}} A_1^{\frac{1}{(1+\beta)^2}} A_2^{\frac{1}{(1+\beta)^2}} \frac{\beta-1}{1+\beta} \left(A_1^{\frac{1}{1+\beta}} + A_2^{\frac{1}{1+\beta}} \right)^{\frac{\beta-1}{\beta}} \frac{1}{1+\beta} A_1^{\frac{1}{\beta}} = \lambda$$

Combining this with the first order condition for A_2 we obtain:

$$A_2 \left[\beta A_1^{\frac{1}{1+\beta}} + A_2^{\frac{1}{1+\beta}} \right] = A_1 \left[\beta A_2^{\frac{1}{1+\beta}} + A_1^{\frac{1}{1+\beta}} \right]$$

Let $x = \left(\frac{A_1}{A_2}\right)^{\frac{1}{\beta+1}} > 0$. Then the last equation becomes:

$$x^{\beta+1} + (\beta-1)x^{\beta} - (\beta-1)x - 1 = 0$$

Rewrite this as:

$$\beta x(x^{\beta} - 1) + (x^{\beta+2} - 1) = 0$$

Note that since $\beta > 0$, the left hand side is always negative for $0 < x < 1$ and always positive for $x > 1$. Thus the obvious solution $x^* = 1$ is the only point that satisifes the equality, i.e. $A_1 = A_2$.

■

Proposition 23 shows that in order to maximize total rent dissipation, the contest designer should try to assign agents to teams in a way that makes the resulting two teams as evenly balanced in terms of skill as possible. The intuition behind this result can be observed from Figure 2. When the teams are equally matched, the equilibrium is at the intersection of the peaks of the two best-response functions, hence the maximum achievable effort is exerted by both teams.

3.4.3 Equilibrium With J Teams

We begin the equilibrium analysis in the general case of J teams with an observation: a team's equilibrium share is always positive as long as its aggregate ability is nonzero. This follows directly from the equilibrium condition:

$$\frac{p_j^{\star\beta}}{(1-p_j^\star)^{\beta+2}} = \frac{VA_j}{X_{-j}^{\star\beta+1}}$$

Theorem 24 *Monotonicity of Equilibrium: For any two teams j and k: $X_j^\star \geq X_k^\star$ if and only if $A_j \geq A_k$.*

Proof. As long as we exclude the degenerate case of $A_j = 0$, all teams will exert positive effort in equilibrium. Using $p_j = \frac{X_j}{X}$ and $1 - p_j = \frac{X-X_j}{X}$ we can plug into $X_j^\star = [p_j^\star(1-p_j^\star)VA_j]^{\frac{1}{\beta+1}}$ to obtain:

$$X_j^{\star\beta+1} = \frac{X_j^\star}{X^\star}\frac{X^\star - X_j^\star}{X^\star}VA_j$$

For any two given teams j and k:

$$\frac{X_j^{\star\beta}}{X_k^{\star\beta}} = \frac{X^\star - X_j^\star}{X^\star - X_k^\star}\frac{A_j}{A_k}$$

Denote by $X^\star_{-jk}$ the equilibrium effort exerted by all teams other than j and k: $X^\star_{-jk} = X^\star - X_j^\star - X_k^\star$. Then the equation above can be rewritten as:

$$\frac{X_j^{\star\beta}(X^\star_{-jk} + X_j^\star)}{X_k^{\star\beta}(X^\star_{-jk} + X_k^\star)} = \frac{A_j}{A_k}$$

For any given equilibrium effort by the rest of the teams $X^\star_{-jk}$, the expression $X_j^{*\beta}(X^\star_{-jk} + X_j^\star)$ is an increasing function of $X_j^\star$, hence the result.

■

Corollary 25 *Equilibrium in the Symmetric Case: If $A_j = A_k = \bar{A}, \forall j, k \leq J$, then $p_j^* = \frac{1}{J}$ and $X_j^* = [\frac{J-1}{J^2}\bar{A}V]^{\frac{1}{\beta+1}}, \forall j \leq J$.*

Proof. In the symmetric case $A_j = A_k, \forall j, k \leq J$: for any X^*_{-jk} and any X_k^* the unique solution to:

$$X_j^{\star\beta}(X^\star_{-jk} + X_j^\star) = X_k^{\star\beta}(X^\star_{-jk} + X_k^\star)$$

is

$$X_j^\star = X_k^\star$$

■

Proposition 26 *The Equilibrium with any number of teams J is unique.*

Proof. Consider once again the equation:

$$\frac{X_j^{\star\beta}}{X_k^{\star\beta}} = \frac{X^\star - X_j^\star}{X^\star - X_k^\star}\frac{A_j}{A_k}$$

Using $\frac{p_j^\star}{p_k^\star} = \frac{X_j^\star}{X_k^\star}$ we can rewrite this as:

$$\frac{p_j^{\star\beta}}{p_k^{\star\beta}} = \frac{1-p_j^\star}{1-p_k^\star}\frac{A_j}{A_k}$$

or expressed relative for team 1:

$$\frac{p_j^{\star\beta}}{1-p_j^\star} = \frac{A_j}{A_1}\frac{p_1^{\star\beta}}{1-p_1^\star}$$

The last equation implies that for any $p_1^\star$, there is a unique solution $p_j^\star = g_j(p_1^\star)$, an increasing function. Then we can recover the value of $p_1^\star$ from $\sum_{j=1}^{k} g_j(p_1^\star) = 1$.

■

3.5 Appendix B

3.5.1 Second Order Condition

I show the conditions under which the solution above is indeed a maximum. There are two key assumptions needed here. The first one is that $\sigma_j < 1$ and the second is the following technical condition: $\frac{\partial}{\partial X_j}\left(\frac{X_j^{\beta_j}}{\frac{\partial p_j}{\partial X_j}}\right) > 0$ (for a discussion of the latter see Proposition 1):

$$\frac{\partial^2 U_{mj}(e_{mj})}{\partial e_{mj}^2} = a_{mj}q''_{mj}(p_j(X))\left(\frac{\partial p_j}{\partial X_j}\right)^2 \frac{\partial X_j}{\partial e_{mj}} X_j^{\sigma_j} e_{mj}^{-\sigma_j} + a_{mj}q'_{mj}(p_j(X))\frac{\partial^2 p_j}{\partial X_j^2}\frac{\partial X_j}{\partial e_{mj}} X_j^{\sigma_j} e_{mj}^{-\sigma_j}$$

$$+a_{mj}q'_{mj}(p_j(X))\frac{\partial p_j}{\partial E_j}\frac{\partial X_j}{\partial e_{mj}}\sigma_j X_j^{\sigma_j-1} e_{mj}^{-\sigma_j} + a_{mj}q'_{mj}(p_j(X))\frac{\partial p_j}{\partial X_j} X_j^{\sigma_j}(-\sigma_j)e_{mj}^{-\sigma_j-1} - (\beta_j)e_{mj}^{\beta_j-1}$$

$$= a_{mj} q'_{mj}(p_j(X)) \frac{\partial p_j}{\partial X_j} X_j^{\sigma_j} \frac{\partial X_j}{\partial e_{mj}} e_{mj}^{-\sigma_j} \left(\frac{\partial p_j}{\partial X_j} \frac{q''_{mj}(p_j(X))}{q'_{mj}(p_j(X))} + \frac{\frac{\partial^2 p_j}{\partial X_j^2}}{\frac{\partial p_j}{\partial X_j}} + \frac{\sigma_j}{X_j} - \frac{\sigma_j}{e_{mj} \frac{\partial X_j}{\partial e_{mj}}} \right) - (\beta_j) e_{mj}^{\beta_j - 1}$$

$$= e_{mj}^{\beta_j + \sigma_j} a_{mj} X_j^{\sigma_j} e_{mj}^{-\sigma_j} e_{mj}^{-\sigma_j} \left(\frac{\partial p_j}{\partial X_j} \frac{q''_{mj}(p_j(X))}{q'_{mj}(p_j(X))} + \frac{\frac{\partial^2 p_j}{\partial X_j^2}}{\frac{\partial p_j}{\partial X_j}} + \frac{\sigma_j}{X_j} - \frac{\sigma_j}{e_{mj} \frac{\partial X_j}{\partial e_{mj}}} \right) - (\beta_j) e_{mj}^{\beta_j - 1}$$

$$= e_{mj}^{\beta_j - \sigma_j} a_{mj} X_j^{\sigma_j} \left(\frac{\partial p_j}{\partial X_j} \frac{q''_{mj}(p_j(X))}{q'_{mj}(p_j(X))} + \frac{\frac{\partial^2 p_j}{\partial X_j^2}}{\frac{\partial p_j}{\partial X_j}} + \frac{\sigma_j}{X_j} - \frac{\sigma_j}{e_{mj} \frac{\partial X_j}{\partial e_{mj}}} \right) - (\beta_j) e_{mj}^{\beta_j - 1}$$

$$= e_{mj}^{\beta_j - \sigma_j} a_{mj} X_j^{\sigma_j} \left(\frac{\partial p_j}{\partial X_j} \frac{q''_{mj}(p_j(X))}{q'_{mj}(p_j(X))} + \frac{\frac{\partial^2 p_j}{\partial X_j^2}}{\frac{\partial p_j}{\partial X_j}} + \frac{\sigma_j X_j^{\sigma_j}}{X_j e_j \frac{\partial X_j}{\partial e_{mj}}} \left(a_{mj} e_{mj}^{1-\sigma_j} - X_j^{1-\sigma_j} \right) \right) - (\beta_j) e_{mj}^{\beta_j - 1}$$

$$= e_{mj}^{\beta_j - \sigma_j} a_{mj} X_j^{\sigma_j} \left(\frac{\partial p_j}{\partial X_j} \frac{q''_{mj}(p_j(X))}{q'_{mj}(p_j(X))} + \frac{\sigma_j X_j^{\sigma_j}}{X_j e_j \frac{\partial X_j}{\partial e_{mj}}} \left(a_{mj} e_{mj}^{1-\sigma_j} - X_j^{1-\sigma_j} \right) \right)$$

$$+ e_{mj}^{\beta_j - \sigma_j - 1} a_{mj} X_j^{\sigma_j} \frac{\frac{\partial^2 p_j}{\partial X_j^2}}{\frac{\partial p_j}{\partial X_j}} - (\beta_j) e_{mj}^{\beta_j - 1}$$

By assumption $q''_{mj}(p_j(X)) \leq 0$.

$$\frac{\partial}{\partial X_j} \left(\frac{X_j^{\beta_j}}{\frac{\partial p_j}{\partial X_j}} \right) > 0 \Rightarrow (\beta_j) X_j^{\beta_j} \frac{\partial p_j}{\partial X_j} - X_j^{\beta_j} \frac{\partial^2 p_j}{\partial X_j^2} > 0$$

or

$$X_j^{\beta_j} \left(\frac{(\beta_j)}{X_j} \frac{\partial p_j}{\partial X_j} - \frac{\partial^2 p_j}{\partial X_j^2} \right) > 0 \Rightarrow \frac{\frac{\partial^2 p_j}{\partial X_j^2}}{\frac{\partial p_j}{\partial X_j}} < \frac{(\beta_j)}{X_j}$$

After plugging these results into the second order condition to obtain:

$$\frac{\partial^2 U_{mj}(e_{mj})}{\partial e_{mj}^2} < e_{mj}^{\beta_j-\sigma_j} a_{mj} X_j^{\sigma_j} \left(\frac{\sigma_j X_j^{\sigma_j}}{X_j e_{mj} \frac{\partial E_j}{\partial e_{mj}}} \left(a_{mj} e_{mj}^{1-\sigma_j} - X_j^{1-\sigma_j} \right) \right)$$

$$+ e_{mj}^{\beta_j-\sigma_j} a_{mj} X_j^{\sigma_j} \frac{(\beta_j - 1)}{X_j} - (\beta_j - 1) e_{mj}^{\beta_j-1}$$

$$= \sigma_j e_{mj}^{\beta_j-1} X_j^{\sigma_j-1} \left(a_{mj} e_{mj}^{1-\sigma_j} - X_j^{1-\sigma_j} \right) + \frac{(\beta_j) e_{mj}^{\beta_j-1}}{X_j^{1-\sigma_j}} \left(a_{mj} e_{mj}^{1-\sigma_j} - X_j^{1-\sigma_j} \right)$$

Recall from the CES effort aggregator and assuming $\sigma_j < 1$, we have: $E_j = \left(\sum_{h=1}^{k_j} a_{hj} e_{hj}^{1-\sigma_j} \right)^{\frac{1}{1-\sigma_j}} \Rightarrow a_{mj} e_{mj}^{1-\sigma_j} - E_j^{1-\sigma_j} \leq 0$. Note that the last expression can equal 0 in the case when there is only one person on team j, hence the weak inequality.

Thus it follows that as long as $e_{mj} > 0$ (interior solution), then the second order condition for individual utility maximization is satisfied:

$$\frac{\partial^2 U_{mj}(e_{mj})}{\partial e_{mj}^2} < \sigma_j e_{mj}^{\beta_j-1} X_j^{\sigma_j-1} \left(a_{mj} e_{mj}^{1-\sigma_j} - X_j^{1-\sigma_j} \right) + \frac{(\beta_j) e_{mj}^{\beta_j-1}}{X_j^{1-\sigma_j}} \left(a_{mj} e_{mj}^{1-\sigma_j} - X_j^{1-\sigma_j} \right) \leq 0.$$

Chapter 4

Equilibrium Player Choices in Team Contests with Multiple Pairwise Battles

4.1 Introduction

In their influential paper in group contests, Fu, Lu, and Pan (2015) analyze a multi-battle team contest in which players from two rival teams form pairwise matches to compete in distinct component battles—each player fights exactly one battle in the whole contest. They naturally assume that the winning probability of battles is depicted by a function that is homogeneous of degree zero in players' efforts,[1] they show that the outcomes of past battles do not distort the outcomes of future battles, as long as the pairwise matches between the players from the two teams stay the same. That is, (i) the winning probability in each battle (match) is independent of the history of that battle, (ii) the winning probability of a team is independent of the sequence of battles, and (iii) the winning probability of a team is independent of temporal structure of the component battles (i.e., one-shot or sequential). Moreover, they also show that neither the total expected effort nor the overall outcome of the

[1] This genre of technology includes many well-accepted models, such as the general Tullock contest and first-price all-pay auction.

contest depends on (i) the battle sequence or (ii) the temporal structure. These are quite striking results, which have interesting implications for team competitions in sports and other areas.[2] In a different line of the literature, Hamilton and Romano (1998) consider the situation in which the two team leaders choose the order of players in a multi-battle contest strategically. They assume that each individual match has an exogenously fixed winning probability and the number of players in each team is exactly the same as the number of battles. Employing a one-shot simultaneous ordering choice game, they show that there is a mixed strategy equilibrium in which both teams assign the same probability to every ordering of the players, and that the winning outcome of the team contest (expected winning probability) is shown to be unique by von Neumann's minimax theorem in a two-person zero-sum game. This is also an intriguing observation.

In this study, we combine these two papers—we will consider the team leaders' strategic assignment problem of players to component battles as in the multi-battle team contest of Fu, Lu, and Pan (2015). In doing so, we will analyze not only Hamilton and Romano's (1998) one-shot ordering choice problem but also a sequential battle-by-battle player choice problem following the spirit of Fu, Lu, and Pan (2015). A one-shot simultaneous ordering choice before the first battle starts may not actually be the most common practice—for example, in the Davis Cup in men's tennis, the team captains announce which players are called to compete in the next match only after the revelation of the results of the previous matches. In the MLB World Series, team managers announce starting pitchers on each game day. In addition, by introducing a sequential player choice of the next battle, we may reveal intriguing insights regarding the following questions: Is it important to have a lead in the early

[2]Despite of all of these neutrality results by Fu, Lu, and Pan (2015), Barbieri and Serena (2019) show that the expected winners' efforts are higher if battles are held simultaneously than if they are held sequentially under a natural condition.

stage (a momentum/discouraging effect)? Do players fight more fiercely in the late stage? Do the results from previous rounds affect leaders' or players' decisions in later stages?

We first reproduce the result by Hamilton and Romano (1998) on one-shot ordering choice games when pairwise battle winning probabilities are exogenous by directly using the resulting matchings instead of strategy profiles. In this way, we can also show that the expected winning probability of a team is the same as the one when the contest organizer chooses a matching of players totally randomly (Proposition 1). Using this result, we show that the totally mixed strategy Nash equilibrium in Hamilton and Romano (1998) extends to a one-shot order choice game in the Fu, Lu, and Pan multi-battle contest environment in which each player's effort level is endogenously determined, and that the expected winning probability of a team is the same when the contest organizer chooses a matching of players totally randomly (Theorem 1). Although Fu, Lu, and Pan (2015) assume that the pairwise player matching in their multi-battle contests is fixed, we show that their invariance result regarding the outcome (winning probability) of each pairwise battle is more general than that—as long as a pair of players are matched in one of the multiple battles in a team contest, the expected outcome (winning probability) stays the same, irrespective of the rest of the matches. Thus, for any realization of a matching as a result of (mixed strategy) equilibrium, the history independence result for the wining probability of each pairwise match in Fu, Lu, and, Pan (2015) still follows, resulting in the Hamilton-Romano totally random equilibrium. In the sequential player-choice game, however, the argument is more involved. For any history, each of the matched players in a battle needs to foresee what the winning probability of her team if the current battle is won or lost, and they make their effort decisions based on this information. We will demonstrate Fu, Lu, and Pan's (2015) induction arguments still works and show that the team's ex ante winning probability is again the same as the ones under the Hamilton-Romano totally random Nash equilibrium and the totally random matching of players by the contest organizer (Theorem 2). As a corollary, we can say that the ex ante expected equilibrium effort of each player is invariant of the type of player choice game—one-shot or sequential. Thus, we can add another invariance result to Fu, Lu, and Pan (2015).

In the next subsection, we provide a brief literature review. In Section 2, we will start with a three-battle contest example with exogenously fixed winning probabilities for each pairwise match between players from the two teams. This illustrates the equivalence between the outcome (ex ante team winning probability) of the one-shot game and the one of the sequential move game. In Section 3, we introduce the general model using matching language and replicate Hamilton and Romano's (1998) result by using matching theory (Proposition 1). Then, in Section 4, we endogenize the winning probability of each race and show that the same results hold (Theorems 1 and 2, and Corollary 1). In Section 5, we conclude by providing examples of illustrating the importance of each of our assumptions: namely, the number of players in each team and the number of battles need to be the same for our equivalence results.

4.1.1 Related Literature

Our paper contributes the burgeoning literature on multi-battle contests.[3] Harris and Vickers (1987) model a two-firm R&D competition as series of individual stages; in each stage the success probability depends on the firms' efforts in that stage. The first firm to win N or more stages than its opponent wins the whole competition—a *tug-of-war* game. They show that the trailing firm makes less efforts, and the effort decreases as the deficit increases—the momentum effect. Klumpp and Polborn (2006) consider

[3]For a complete review of this literature, please see Kovenock and Roberson (2012).

a multi-district campaign spending game, e.g., US presidential primaries, in which district competitions are modeled as Tullock competitions and hold sequentially. The candidate who gets a majority of the districts wins the game. They show that the momentum effect exists and candidates tend to spend more in early voting districts. In Konrad and Kovenock (2009), two players compete in a race comprised of a sequence of battles, and each component battle is modeled as a first-price all-pay auction. They characterize the unique subgame perfect equilibrium and show that the expected effort in each battle can be non-monotonic as the competition gets tighter. All in all, both outcomes and strategies in each battle depend on the previous results. These history dependence results in this line of the literature are rooted in the presumption that the participants in each battle are the same.

There are papers other than Fu, Lu, and Pan (2015) that consider multi-battle group contests in which each battle is played by different players. Häfner (2017) investigates a tug-of-war game played by a (potentially) infinite number of different players, and shows that there exists a unique Markov-perfect equilibrium. Barbieri and Serena (2019) show that the expected winners' efforts are higher if battles are held simultaneously than sequentially in the Fu-Lu-Pan model under a natural condition. These two papers assume that the matching of players is prefixed, but Barbieri and Serena (2019) also consider a contest-design problem and show that the sequential game in which players are ordered from less efficient to more efficient is the setup that minimizes winners' efforts. Fu and Lu (2018) consider a strategic player assignment game in a two-team, two-stage, all-pay contest, in which each team has one stronger and one weaker player. This model has the closest motivation to ours, but there is a fundamental difference between the two. They assume that the team with the higher aggregate effort wins the prize, whereas we assume that the team with the majority of individual battle victories wins. They show that in equilibrium, both teams assign the stronger players in the second stage as long as the intra-team heterogeneity of player ability is not excessive. Thus, it is easy to see that our neutrality result crucially depends on this difference. Klumpp, Konrad, and Solomon (2019) consider a sequential multi-battle Blotto game that respects the majoritarian rule, but where the resource is not reusable.[4] They show that the player should split the resource evenly across all battles in the unique equilibrium, and thus the winning chance in each battle is independent of how many games were won/lost before that battle.

4.2 A Three-Player Example with Exogenous Winning Probabilities

Here, we present Example 1. Teams A and B each have three players labeled 1, 2, and 3. Suppose for simplicity that the winning probability in each pairwise battle is exogenously given. Since each match is a zero-sum game, we summarize these winning probabilities in a single matrix (Q) from the perspective of team A only:

$$Q = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix}$$

where $q_{ij} \in [0, 1]$ is the winning probability of team A's player i when i is matched with team B's player j for all $i, j = 1, 2$, and 3.

We first analyze the Nash equilibrium strategy profiles of the one-shot game in which both team leaders simultaneously choose the order in which their players com-

[4]Konard (2018) also shows this even split result in a best-of-three contest. Konrad (2018) studies the best-of-three sequential Blotto game and shows that if the resource can be reused in future battles, there are discouragement effects for the lagging player and a showdown effect when the battle is decisive.

pete. Note that each team leader's strategies are player orderings. For example, if leader A plays 123 and leader B plays 123, then the resulting pairwise battles are $(1,1),(2,2)$, and $(3,3)$. Also, a strategy profile of 123 and 123 is the same as that of 132 and 132, and a strategy profile of 123 and 321 is the same as that of 132 and 312, etc. The winning probability of team A for any strategy combination is, in principle, calculable. Therefore, we have the following payoff (winning probability) matrix for leader A:

Leader B

Leader A		123	132	213	231	312	321
	123	α	β	γ	δ	ϵ	ϕ
	132	β	α	δ	γ	ϕ	ϵ
	213	γ	ϵ	α	ϕ	β	δ
	231	ϵ	γ	ϕ	α	δ	β
	312	δ	ϕ	β	ϵ	α	γ
	321	ϕ	δ	ϵ	β	γ	α

where, for example, $\alpha = q_{11}q_{22}q_{33} + q_{11}q_{22}(1-q_{33}) + (1-q_{11})q_{22}q_{33} + q_{11}(1-q_{22})q_{33}$ and $\beta,\gamma,\delta,\epsilon,\phi$ are similarly defined. Notice that $\alpha,\beta,\gamma,\delta,\epsilon,\phi$ show up exactly once for each row and column (though some of them may take the same values).

Now, assume that leader B plays all pure strategies with probability $\frac{1}{6}$ each. Clearly, leader A is indifferent between all pure strategies. Let leader A play all pure strategies with probability $\frac{1}{6}$ each. Then, leader B is also indifferent between all pure strategies. Thus, this is a mixed strategy equilibrium. Since this is a two-person zero-sum game, the Nash equilibrium payoff and the minimax value are the same. Moreover, by von Neumann's theorem, the minimax value is unique. Thus, we have unique Nash equilibrium winning probability $\bar{P}_A$, which is supported by a complete randomization with equal probabilities. This is the same result as in Hamilton and Romano (1998).

Notice that leader A's expected payoff is

$$\bar{P}_A = \frac{\alpha + \beta + \gamma + \delta + \epsilon + \phi}{6}$$

We now turn to a sequential choice game. That is, leader A and leader B simultaneously choose the first player, observe the outcome of the resulting match, and then again choose their second player simultaneously (the players for the third battle are automatically determined using the leftover players). The question is what is a subgame perfect equilibrium of this sequential game. We start with an analysis of each subgame. Suppose that in the first battle, team A's player 1 and team B's player 1 were matched and one of them won the first match. Whoever won, the rest of the game reduces to the order choice of the remaining two players on each team only. The resulting payoff matrix is as follows:

		$\frac{1}{2}$	$\frac{1}{2}$
		23	32
$\frac{1}{2}$	23	a	b
$\frac{1}{2}$	32	b	a

In this matrix, $a, b \in [0, 1]$ are team A's winning probabilities (strategy profiles $(23, 23)$ and $(32, 32)$ achieve the same winning probability, since players 2 and 2 and 3 and 3 are matched anyway). Notice that the unique Nash equilibrium in this zero-sum game is that both teams play 23 with probability $\frac{1}{2}$. This does not depend on which team won in the first match. Furthermore, the battle $(1, 1)$ in the first round was chosen completely arbitrarily and the equal-probability continuation equilibrium is not affected. This means that in *every* subgame, both teams play $\frac{1}{2}$ and $\frac{1}{2}$ for the rest of the ordering no matter who was paired in the first battle and regardless of who wins it.

Now, we consider the first round battle. Suppose that leader B selects players 1, 2, and 3 with probability $\frac{1}{3}$ for each. If leader A chooses player 1 and leader B happens to choose player 1, leader A knows that the subsequent battles $\{(2,2),(3,3)\}$ and $\{(2,3),(3,2)\}$ happen with an equal probability of $\frac{1}{2}$ for each. That is, the sets of pairwise battles $\{(1,1),(2,3),(3,2)\}$ and $\{(1,1),(2,2),(3,3)\}$ end up played with probability $\frac{1}{3}\times\frac{1}{2}=\frac{1}{6}$ for each. The same argument applies to the case when leader B happens to play 2 and 3. In the end, each possible matching is played with probability $\frac{1}{6}$. Thus, leader A is indifferent between choosing players 1, 2, or 3 in the first round, and in the second round he chooses the rest of the orderings with probability $\frac{1}{2}$ for each (this is equivalent to choosing a player from the two remaining players with probability $\frac{1}{2}$). Clearly, leader A will place probability $\frac{1}{3}$ for each of his three players in the first round. His equilibrium payoff is again $\bar{P}_A$. This discussion shows that the sequential game outcome is the same as the simultaneous game outcome. By induction, we can see that the argument works for any (odd) number of players.□

4.3 The Basic Model—Exogenous Winning Probabilities in Battles

There are two teams, A and B. Each team has $2n+1$ players where $n \in N$. The whole competition consists of $2n+1$ sequential (or simultaneous) head-to-head battles. The winning team is the one which wins $n+1$ battles. There is a team leader in charge of deciding the order in which players on each team will enjoy a payoff of 1 if his team wins. Let $N^A = \{i_1, ..., i_{2n+1}\}$ and $N^B = \{j_1, ..., j_{2n+1}\}$ be the sets of players of teams A and B, and let i and j be the representative elements in N^A and N^B, respectively. Team ν's leader can choose the ordering of the players—$\pi^A : \{1, ..., 2n+1\} \to N^A$ and $\pi^B : \{1, ..., 2n+1\} \to N^B$ are one-to-one mappings. The two leaders announce the ordering of their players simultaneously at the beginning of the competition. Let Π^ν be the set of all orderings, and then a strategy combination is denoted by $(\pi^A, \pi^B) \in \Pi^A \times \Pi^B$.

Let a *matching* $\mu : N^A \to N^B$ be a one-to-one function such that $\mu^{-1}(\mu(i)) = i$ for all $i \in N^A$. Since $|N^A| = |N^B|$, $\mu(N^A) = N^B$. Let $M(N^A, N^B)$ denote the set of all matchings. Note that there are $(2n+1)!$ possible matchings, and for each π^A, there is exactly one π^B that generates a particular matching μ. Moreover, given a matching μ, there are $(2n+1)!$ combinations of $(\pi^A, \pi^B) \in \Pi^A \times \Pi^B$ that yield the same μ.

We assume that the winning probability of each match of players from teams A and B is independent of how other players are matched and which player wins. Team A's players' winning probabilities when they are matched with each of the players on team B are exogenously given by[5]

$$Q = \begin{pmatrix} q_{i_1 j_1} & \cdots & q_{i_1 j_{2n+1}} \\ \vdots & \ddots & \vdots \\ q_{i_{2n+1} j_1} & \cdots & q_{i_{2n+1} j_{2n+1}} \end{pmatrix}$$

where a generic match is denoted by (i, j) with team A's (i's) winning probability being q_{ij}. This Q matrix is perfectly general. We allow for the cases in which player i_1 does well against most of the players on team B, but i_1 somehow always loses against

[5]In the next section, we endogenize winning probabilities in battles by considering multi-battle contest game following Fu, Lu, and Pan (2015).

j_{2n+1}.

The static nature of the winning probability matrix Q implies that the payoffs of this game depend only on the resulting matching, i.e. two strategy profiles that lead to the same matching will result in identical payoffs for both teams. Denote the expected payoffs from a given matching for each team by $\tilde{P}^A(\mu)$ (and $\tilde{P}^B(\mu) = 1 - \tilde{P}^A(\mu)$) accordingly. Let $W = \left\{S \in 2^{\{1,2,\ldots,2n+1\}} : |S| \geq n+1\right\}$.

$$\tilde{P}^A(\mu) \equiv \sum_{S \in W} \left(\prod_{r \in S} \left(q_{i_r \mu(i_r)} \right) \times \prod_{r \notin S} \left(1 - q_{i_r \mu(i_r)} \right) \right).$$

There are $(2n+1)!$ strategy profiles $\left(\pi^A, \pi^B\right) \in \Pi^A \times \Pi^B$ that achieve the same matching $\mu \in M(N^A, N^B)$, where $M(N^A, N^B)$ denotes the set of all possible matchings. Also note that there are $(2n+1)!$ elements in M and $((2n+1)!)^2$ elements in $\Pi^A \times \Pi^B$. We now consider team A's winning probability when there exists a contest organizer who picks a matching *totally randomly* to be

$$\bar{P}^A \equiv \frac{1}{(2n+1)!} \sum_{\mu \in M(N^A, N^B)} \tilde{P}^A(\mu).$$

Since the corresponding matching for any given combination of (π^A, π^B) is unique, we can slightly abuse the notation to let $\mu : \Pi^A \times \Pi^B \to M(N^A, N^B)$ be the matching generated from permutations $\left(\pi^A, \pi^B\right)$, such that $\mu(i) = \pi^B(\left(\pi^A\right)^{-1}(i))$ for all $i \in N^A$. Then, A's ex ante winning probability given (π^A, π^B) can be written as

$$P^A(\pi^A, \pi^B) \equiv \tilde{P}^A(\mu(\pi^A, \pi^B)).$$

Similarly, define $P^B(\pi^A, \pi^B)$. It is clear that $P^A(\pi^A, \pi^B) + P^B(\pi^A, \pi^B) = 1$.

Thus, the game with two team leaders who maximize their teams' winning probability is a zero-sum game with strategy sets Π^A and Π^B, and with a $\Pi^A \times \Pi^B$ payoff matrix $P \equiv \left(P^A(\pi^A, \pi^B)\right)_{\pi^A \in \Pi^A, \pi^B \in \Pi^B}$. In this case, a mixed strategy is $m^v : \Pi^v \to [0, 1]$ with $\sum_{\pi^v \in \Pi^v} m^v(\pi^v) = 1$ for $v = A, B$. Let $\bar{m}^v(\pi^v) = \frac{1}{|\Pi^v|} = \frac{1}{(2n+1)!}$ for all $\pi^v \in \Pi^v$ and $k = A, B$ be the mixed strategy that assigns equal probability to all strategies. Notice that for each A's pure strategy $\pi^A \in \Pi^A$, each $\mu \in M(N^A, N^B)$ realizes once and only once for some $\pi^B \in \Pi^B$. With some abuse of notation, we have

$$P^A(\pi^A, \bar{m}^B) = \frac{1}{(2n+1)!} \sum_{\mu \in M(N^A, N^B)} \tilde{P}^A(\mu) = \bar{P}^A$$

for any $\pi^A \in \Pi^A$**,** and team A is indifferent between all possible orderings if team B employs $\bar{m}^B$. For the same reason, team B obtains payoff

$$P^B(\bar{m}^A, \pi^B) = 1 - \bar{P}^A$$

for any $\pi^B \in \Pi^B$. Therefore, we obtain the result by Hamilton and Romano (1998).

Proposition 1 (Hamilton and Romano 1998) *Suppose that the winning probabilities of all pairwise battles are described by a static matrix Q. A total randomization over all orderings of players with equal probability $(\bar{m}^A, \bar{m}^B)$ is a Nash equilibrium of the one-shot ordering-choice game. Moreover, in every Nash equilibrium of the game, team A's winning probability, $\bar{P}^A$, is exactly the same as the one when the contest organizer picks a matching of players totally randomly .*

Note that there are many other Nash equilibria in our static game, although the equilibrium payoffs are unique, as is shown in von Neumann (1928). For example, consider the following $2n + 1$ strategies: $\pi_1^v = (i_1, ..., i_{2n+1})$, $\pi_2^v = (i_{2n+1}, i_1, ..., i_{2n})$, $\pi_2^v = (i_{2n}, i_{2n+1}, i_1, ..., i_{2n-1})$,..., and $\pi_{2n+1}^v = (i_2, ..., i_{2n+1}, i_1)$. Let $\hat{m}^v$ be $\hat{m}^v(\pi_\ell^v) =$

$\frac{1}{2n+1}$ for all $\ell = 1, ..., 2n+1$ and $\hat{m}^v(\pi^v) = 0$ for any other π^v. If team B uses strategy $\hat{m}^B$, then each player on team A is matched with all of the team B players with equal probability $\frac{1}{2n+1}$. Thus, team A is indifferent between all strategies in Π^A. Therefore, $\hat{m}^A$ is one of the best responses to $\hat{m}^B$, and $\left(\hat{m}^A, \hat{m}^B\right)$ is a Nash equilibrium, too. There are many other ways to select $2n+1$ pure strategies that do this same thing. Hence, we have a continuum of Nash equilibria with the same expected payoffs.

4.4 Endogenous Winning Probabilities in Battles

So far, we have assumed that the winning probabilities for team A's players against team B's players are exogenously determined for all possible pairs of player matches, i.e., players' behavior is exogenous. In this section, we relax this assumption following the arguments of the invariance results in Fu, Lu, and Pan (2015). We again assume that $\left|N^A\right| = \left|N^B\right| = 2n+1$ and that the leaders of teams A and B simultaneously choose the player ordering at the beginning of the contest. Consider a battle between players $i \in N^A$ and $j \in N^B$. Although the same result applies to any of the examples listed in their paper, we will focus on a variation of a complete-information generalized Tullock contest (Model 6 in Fu, Lu, and Pan 2015). To apply their invariance result, assume that (*ij-pair-specific*) contest success function $q_{ij}(x_i, x_j)$ is (i) homogenous of degree zero in x_i and x_j, (ii) $\frac{\partial q_{ij}}{\partial x_i} > 0$ and $\frac{\partial^2 q_{ij}}{\partial x_i^2} < 0$, and (iii) $\frac{\partial q_{ij}}{\partial x_j} < 0$ and $\frac{\partial^2 q_{ij}}{\partial x_j^2} > 0$, where x_i and x_j are effort levels by players i and j, respectively. Players i and j have constant marginal costs of effort $c_i, c_j > 0$ and benefits $V_i, V_j > 0$ from their team's winning the majority of battles. If this is just a single battle played by i and j, then players i and j solve the following problems, respectively:

$$\max_{x_i} q_{ij}(x_i, x_j)V_i - c_i x_i$$

and

$$\max_{x_j} \left(1 - q_{ij}(x_i, x_j)\right) V_j - c_j x_j.$$

The following result is first shown by Malueg and Yates (2005). For completeness, we include a concise proof.[6]

Lemma 1 (Malueg and Yates 2005) *In a complete information general Tullock contest played by (i, j), team A member i's equilibrium winning probability* is $\bar{q}_{ij} = q_{ij}(\frac{c_j}{V_j}, \frac{c_i}{V_i})$. *Moreover, if the equilibrium effort vector given the full prize is $(x_i^*(i,j), x_j^*(i,j))$, then the prize is multiplied by p, and thus the equilibrium effort vector is $(px_i^*(i,j), px_j^*(i,j))$;* i.e., *the equilibrium efforts are homogeneous of degree one in the value of the prize.*

Proof. The first order conditions are

$$\frac{\partial q_{ij}(x_i, x_j)}{\partial x_i} V_i - c_i = 0 \tag{4.1}$$

and

$$-\frac{\partial q_{ij}(x_i, x_j)}{\partial x_j} V_j - c_j = 0 \tag{4.2}$$

Since $q_{ij}(x_i, x_j)$ is homogenous of degree zero, we have a Euler equation

$$\frac{\partial q_{ij}(x_i, x_j)}{\partial x_i} x_i + \frac{\partial q_{ij}(x_i, x_j)}{\partial x_j} x_j = 0.$$

These three equations imply

$$\frac{x_i}{x_j} = \frac{V_i c_j}{V_j c_i}.$$

[6] This is a variation on Observations 1 and 2 in Fu, Lu, and Pan (2015) in our context. In their Extensions and Caveats section, they show that asymmetric valuations can be allowed as long as there is no personal battle-specific payoff.

Thus, team A's equilibrium winning probability is written as

$$\bar{q}_{ij} = q_{ij}(\frac{V_i}{c_i}, \frac{V_j}{c_j}).$$

Since $q_{ij}(x_i, x_j)$ is homogenous of degree zero, $\frac{\partial q_{ij}(x_i,x_j)}{\partial x_i}$ and $\frac{\partial q_{ij}(x_i,x_j)}{\partial x_j}$ are homogeneous of degree -1. Thus, we have

$$\frac{\partial q_{ij}(px_i, px_j)}{\partial (px_i)} = \frac{1}{p}\frac{\partial q_{ij}(x_i, x_j)}{\partial x_i}$$

for all $p > 0$ (the same result holds for x_j). This implies

$$\frac{\partial q_{ij}(px_i, px_j)}{\partial (px_i)} pV_i - c_i = \frac{\partial q_{ij}(x_i, x_j)}{\partial x_i} V_i - c_i = 0.$$

That is, if $(x_i, x_j) = (x_i^*(i, j), x_j^*(i, j))$ solves the system of equations (4.1) and (4.2), then $(x_i, x_j) = (px_i^*(i, j), px_j^*(i, j))$ solves the system of equations

$$\frac{\partial q_{ij}(x_i, x_j)}{\partial x_i} pV_i - c_i = 0$$

and

$$-\frac{\partial q_{ij}(x_i, x_j)}{\partial x_j} pV_j - c_j = 0.$$

We have completed the proof.□

Thus, as long as conditions (i), (ii), and (iii) are satisfied, the winning probability of player i in a battle with player j is intact at $\bar{q}_{ij}$ since players i and j face the same probability of their battle to be pivotal p in every contest with multiple pairwise battles. This is the Observation 2 in Fu, Lu, and Pan (2015). Denote $\bar{Q}(N^A, N^B) =$

$(\bar{q}_{ij})_{i\in N^A, j\in N^B}$ to be the pairwise winning probability of player i on team A against j on team B. Thus, the winning probability of team A in multi-battle contest under fixed matching μ is always described by

$$\tilde{P}^A(\mu) \equiv \sum_{S\in W}\left(\prod_{r\in S}\left(\bar{q}_{i_r\mu(i_r)}\right) \times \prod_{r\notin S}\left(1-\bar{q}_{i_r\mu(i_r)}\right)\right).$$

Using this, we immediately get the following result.

Theorem 1. *In a multi-battle generalized Tullock contest, suppose that the two teams simultaneously choose the order which their players will fight in the battles. We have the following: (i) for any realized matching μ and any pair (i, j) with $\mu(i) = j$, player i's winning probability is invariant at $\bar{q}_{ij}$, and (ii) a total-randomization strategy profile—both players' placing probability $\frac{1}{(2n+1)!}$ in all orderings—is a mixed-strategy Nash equilibrium; and (iii) team A's expected winning probability is $\bar{P}^A = \frac{1}{(2n+1)!}\sum_{\mu\in M}\tilde{P}^A(\mu)$.*

Proof. By Observations 1 and 2 in Fu, Lu, and Pan (2015) and Lemma 1, we know that for any *realized* matching $\mu \in M(N^A, N^B)$, in any battle by matched players (i, j) with $\mu(i) = j$, team A wins with probability $\bar{q}_{ij}$. Thus, team A's winning probability matrix is $\bar{Q}(N^A, N^B)$. This implies that by Proposition 1, (ii) and (iii) must hold.□

Now, we will consider sequential battle-by-battle player-choice games. Consider a state $s \in S$ with $s = \left(k, \ell, h; T^A, T^B\right)$, where k is number of battles left, and ℓ and h denote the numbers of wins that teams A and B need to become the winning team at state s, respectively. Moreover, T^A and T^B denote the set of remaining players for team A and B, respectively, and S is the set of all states. Note that $k = |T^A| = |T^B|$ and $\ell + h = k + 1$. We use the functions $k(s) = k$, $\ell(s) = \ell$, $h(s) = s$, $T^A(s) = T^A$, and $T^B(s) = T^B$ to indicate the relevant information at state $s = \left(k, \ell, h; T^A, T^B\right)$. We start with the following definition. In state s, let

$$\bar{P}^A(s) \equiv \frac{1}{k(s)!} \sum_{\mu \in M(T^A(s), T^B(s))} \tilde{P}(\mu; k(s), \ell(s))$$

where

$$\tilde{P}(\mu; k, \ell) \equiv \sum_{S \in W(k,\ell)} \left(\prod_{r \in S} \left(\bar{q}_{i_r \mu(i_r)} \right) \times \prod_{r \notin S} \left(1 - \bar{q}_{i_r \mu(i_r)} \right) \right)$$

and

$$W(k, \ell) \equiv \left\{ S \in 2^{\{1,\dots,k\}} : |S| \geq \ell \right\}.$$

Note that $W(k, \ell)$ is the set of winning coalitions when a team needs to win ℓ out of k battles. Similar to the previous section, $\bar{P}^A(s)$ is A's winning probability when there is a contest organizer who totally randomly assigns players to battles after the state s. We let $\triangle(T^A(s))$ and $\triangle(T^B(s))$ be the sets of mixed actions for leader A and B, respectively, and define $\sigma^\nu : S \to \triangle(N^\nu)$ such that $\sigma^\nu(s) \in \triangle(T^\nu(s))$ as the mixed strategy of the leader ν. One possible subgame perfect equilibrium strategy is $\bar{\sigma}^\nu(s) = \frac{1}{T^\nu(s)}(1, 1, ..., 1) \in \triangle T^\nu(s)$ for $\nu = A, B$.

In each state s, we need to consider every possible pair of players in the next battle. For each pair, $i \in T^A(s)$ and $j \in T^B(s)$, depending on the the winner of the battle, the next state will be either $s^i_{-ij} = (k-1, \ell-1, h; T^A \backslash \{i\}, T^B \backslash \{j\})$ or $s^j_{-ij} = (k-1, \ell, h-1; T^A \backslash \{i\}, T^B \backslash \{j\})$. The former s^i_{-ij} denotes the state that succeeds s after a battle between i and j with i being the winner. Furthermore, we will prove the following result using induction arguments starting from the last battle.

Theorem 2. *In a multi-battle generalized Tullock contest, suppose that the two teams simultaneously choose their players battle by battle sequentially. Then, we have the following: (i) in any battle in any stage, if players $i \in T^A$ and $j \in T^B$ are matched,* *i's winning probability is invariant at* $\bar{q}_{ij}$*; (ii) the total-randomization strategy profile* $(\bar{\sigma}^A, \bar{\sigma}^B)$ *is a subgame-perfect equilibrium; and (iii) team A's expected winning probability in the beginning of each state* $s \in S$ *is* $\bar{P}^A(s)$*. In particular, for the initial state* s_0*, we have* $\bar{P}^A(s_0) = \bar{P}^A = \frac{1}{(2n+1)!} \sum_{\mu \in M} \tilde{P}^A(\mu)$*.*

Proof. By induction, we will show that for any state s with $k(s) \leq \hat{k}$ with (i)-(iii) satisfied, then for a state s' with $k(s) = \hat{k} + 1$, (i)-(iii) are again satisfied.

Suppose that $\hat{k} = 1$. For any state s with $k(s) = 1$, the only meaningful case is $k = \ell = h = 1$ (otherwise, the game is over). Clearly, the last players i and j make the best effort to obtain the award V_i and V_j, respectively, so the winning probability of team A is $\bar{q}_{ij}$. In any other case, the game is over. Moreover, (ii) and (iii) in this case are trivial.

Now, suppose that $\hat{k} = 2$. There are two meaningful cases: $(k, \ell, h) = (2, 2, 1)$ or $(2, 1, 2)$. Consider $(k, \ell, h) = (2, 2, 1)$. Let $T^A = \{i, i'\}$ and $T^B = \{j, j'\}$. We know that if the game is not over after this round (team A player wins), then team A's winning probability is $q_{i'j'}$ if $\{i', j'\}$ is selected in the subgame. What, then, about the second last stage played by players i and j? The payoff functions of players i and j are given as

$$[q_{ij}(x_i, x_j)\bar{q}_{i'j'}]\, V_i - c_i x_i$$

and

$$[1 - q_{ij}(x_i, x_j)\bar{q}_{i'j'}]\, V_j - c_j x_j,$$

respectively. The first order conditions are

$$\frac{\partial q_{ij}(x_i, x_j)}{\partial x_i} \bar{q}_{i'j'} V_i - c_i = 0$$

and

$$-\frac{\partial q_{ij}(x_i,x_j)}{\partial x_j}\bar{q}_{i'j'}V_j - c_j = 0.$$

Thus, $\frac{x_i}{x_j} = \frac{V_i c_j}{V_j c_i}$ and $q_{ij}(\frac{c_j}{V_j}, \frac{c_i}{V_i}) = \bar{q}_{ij}$. The matrix game of this subgame is described by

		$\frac{1}{2}$	$\frac{1}{2}$
	$\ell = 2$	jj'	$j'j$
$\frac{1}{2}$	ii'	$\bar{q}_{ij}\bar{q}_{i'j'}$	$\bar{q}_{ij'}\bar{q}_{i'j}$
$\frac{1}{2}$	$i'i$	$\bar{q}_{ij'}\bar{q}_{i'j}$	$\bar{q}_{ij}\bar{q}_{i'j'}$

Clearly, a mixed strategy profile with equal probability, $(\bar{\sigma}^A(s), \bar{\sigma}^B(s))$, is an equilibrium and is unique unless $\bar{q}_{ij}\bar{q}_{i'j'} = \bar{q}_{ij}\bar{q}_{i'j'}$. Team A's winning probability (expected payoff) is $\frac{1}{2}\left(\bar{q}_{ij}\bar{q}_{i'j'} + \bar{q}_{ij'}\bar{q}_{i'j}\right) = \bar{P}^A\left(T^A, T^B, 2, 2, 1\right)$. Case $(k,\ell,h) = (2,1,2)$ can be treated symmetrically by swapping teams A and B. This proves that the induction hypothesis holds for $k = 2$.

Consider any subgame starting at state $s = \left(k,\ell,h; T^A, T^B\right)$ with $|T^A| = |T^B| = k$ and suppose that the induction hypothesis is correct for all states $\tilde{s} = \left(\tilde{k},\tilde{\ell},\tilde{h}; \tilde{T}^A, \tilde{T}^B\right)$ with $|\tilde{T}^A| = |\tilde{T}^B| = \tilde{k}$ where $\tilde{k} < k$. Denote the set of all possible matchings between the members of T^A and T^B by $M(T^A, T^B)$. Similarly, denote the set of all possible matchings between the members of T^A and T^B in which player $i \in T^A$ is matched to player $j \in T^B$ by $M(T^A, T^B; (i,j))$. Then, the continuation state when player i wins is $s^i_{-ij} = (k-1, \ell-1, h; T^A\backslash\{i\}, T^B\backslash\{j\})$ and when j wins the state is $s^j_{-ij} = (k-1, \ell, h-1; T^A\backslash\{i\}, T^B\backslash\{j\})$. We first show that (i) holds for any s with $k(s) = k$. The payoff functions of players i and j after being matched in state s are

$$\begin{aligned} u_i &= q_{ij}(x_i,x_j)\bar{P}^A\left(s^i_{-ij}\right)V_i + \left(1 - q_{ij}(x_i,x_j)\right)\bar{P}^A\left(s^j_{-ij}\right)V_i - c_i x_i \\ &= q_{ij}(x_i,x_j)\left(\bar{P}^A\left(s^i_{-ij}\right) - \bar{P}^A\left(s^j_{-ij}\right)\right)V_i - c_i x_i + \bar{P}^A\left(s^j_{-ij}\right)V_i \end{aligned}$$

and

$$
\begin{aligned}
u_j &= (1 - q_{ij}(x_i, x_j)) \left(1 - \bar{P}^A \left(s^j_{-ij}\right)\right) V_j + q_{ij}(x_i, x_j) \left(1 - \bar{P}^A \left(s^i_{-ij}\right)\right) V_j - c_j x_j \\
&= (1 - q_{ij}(x_i, x_j)) \left(\bar{P}^A \left(s^i_{-ij}\right) - \bar{P}^A \left(s^j_{-ij}\right)\right) V_j - c_j x_j - \left(1 - \bar{P}^A \left(s^i_{-ij}\right)\right) V_j,
\end{aligned}
$$

respectively. Thus, by our Lemma 1, equilibrium efforts (x_i, x_j) satisfy $\frac{x_i}{x_j} = \frac{c_j}{c_i}$ and team A's winning probability is invariant at $q_{ij}(x_i, x_j) = \bar{q}_{ij}$.

Now, let team leader B randomize his choice of player at state s with equal probability, i.e., he uses $\bar{\sigma}^B(s) = \frac{1}{k(s)}$. The resulting winning probability for team A from selecting player i at state s is:

$$
\begin{aligned}
& \frac{1}{k} \sum_{j \in T^B} \left[\bar{q}_{ij} \bar{P}^A(s^i_{-ij}) + (1 - \bar{q}_{ij})\, \bar{P}^A(s^j_{-ij})\right] \\
= & \frac{1}{k} \sum_{j \in T^B} \left[\bar{q}_{ij} \frac{1}{(k-1)!} \sum_{\mu \in M(T^A(s^i_{-ij}), T^B(s^i_{-ij}))} \tilde{P}(\mu, k-1, l-1)\right] \\
& + \frac{1}{k} \sum_{j \in T^B} \left[(1 - \bar{q}_{ij}) \frac{1}{(k-1)!} \sum_{\mu \in M(T^A(s^j_{-ij}), T^B(s^j_{-ij}))} \tilde{P}(\mu, k-1, l)\right]
\end{aligned}
$$

$$
\begin{aligned}
= & \frac{1}{k!} \sum_{j \in T^B} \sum_{\mu \in M(T^A(s^i_{-ij}), T^B(s^i_{-ij}))} \bar{q}_{ij} \tilde{P}(\mu, k-1, l-1) \\
& + \frac{1}{k!} \sum_{j \in T^B} \sum_{\mu \in M(T^A(s^j_{-ij}), T^B(s^j_{-ij}))} (1 - \bar{q}_{ij}) \tilde{P}(\mu, k-1, l) \\
= & \frac{1}{k!} \sum_{j \in T^B} \sum_{\mu \in M(T^A - \{i\}, T^B - \{j\})} \left[\bar{q}_{ij} \tilde{P}(\mu, k-1, l-1) + (1 - \bar{q}_{ij}) \tilde{P}(\mu, k-1, l)\right] \\
= & \sum_{j \in T^B} \frac{1}{k} \left[\frac{1}{(k-1)!} \sum_{\mu \in M(T^A, T^B; (i,j)))} P(\mu, k, l)\right] = \bar{P}^A(s),
\end{aligned}
$$

where $M(T^A, T^B; (i,j))$ is a collection of all matchings $\mu : T^A \to T^B$ with $\mu(i) = j$.

Each term inside the brackets in the last line equation is the ex ante probability that team A wins the tournament when player i faces player j, but since the sum is over all $j \in T^B$, the sum equals the overall probability of winning the tournament under the assumption that the leader of team B mixes equally among all players. We know this since each battle's winning probability is independent of other battles' outcomes from Lemma 1. Note that from the inductive assumption that mixing equally is a subgame perfect equilibrium in every subsequent state of the world, it follows that each subgame is weighted by the same probability $\frac{1}{(k-1)!}$.

Clearly, the winning probability of team A is $\bar{P}^A(s)$, regardless of which player $i \in T^A(s)$ is chosen by team A at state s. Thus, team A is indifferent between all available players. Team A can place equal probability on each player, which makes team B indifferent between all available players. This concludes that team A's equilibrium winning probability at state s is $\bar{P}^A(s)$, which is team A's winning probability when every possible match occurs with equal probability. Our induction argument is complete.

Note that at the initial state $s_0 = (2n+1, n+1, n+1; N^A, N^B)$, we have $\bar{P}^A(s_0) = \bar{P}^A = \frac{1}{(2n+1)!} \sum_{\mu \in M} \tilde{P}^A(\mu)$ by definition.$\square$

Next, we turn to each player's ex ante expected effort. By Theorem 2, we know that every matching μ occurs with probability $\frac{1}{(2n+1)!}$ ex ante. Since the winning probability matrix $\bar{Q}$ is independent according to Theorems 1 and 2, we can calculate the probability that i and j are matched at state $s = (k, \ell, h; T^A, T^B)$, with this battle being pivotal. First, state s occurs with many possible matchings prior to it—in all elements $\tilde{\mu} \in M(N^A \backslash T^A(s), N^B \backslash T^B(s))$. Second, player i is matched with player j with probability $\frac{1}{k(s)}$. Third, after the battle (i,j) is over, there are many possible realization matchings $\mu|_{(T^A\backslash\{i\},T^B\backslash\{j\})} \in M\left(T^A(s)\backslash\{i\}, T^B(s)\backslash\{j\}\right)$ with probability $\frac{1}{(k(s)-1)!}$ for each. Fourth, for each possible realization matching $\hat{\mu} \in M\left(T^A\backslash\{i\}, T^B\backslash\{j\}\right)$, the probability that this (i,j) battle is pivotal is

$$p(s,(i,j)) = \sum_{\hat{\mu}\in M(T^A(s)\backslash\{i\},T^B(s)\backslash\{j\})} \sum_{S\in D(k(s)-1,\ell(s)-1)} \prod_{r\in S}\left(\bar{q}_{i_r\hat{\mu}(i^r)}\right) \times \prod_{r\notin S}\left(1-\bar{q}_{i_r\hat{\mu}(i_r)}\right),$$

where $D(k,\ell) \equiv \{S \subseteq \{1,...,k\} : |S| = \ell\}$. Since this probability is common to players i and j, player i's expected effort when i and j are matched is $p(s,(i,j))x_i^*(i,j)$ by Lemma 1. And state s occurs with probability

$$P(s) = \sum_{\tilde{\mu}\in M(N^A\backslash T^A(s),N^B\backslash T^B(s))} \sum_{S\in D(2n+1-k(s),n+1-\ell(s))} \prod_{r\in S}\left(\bar{q}_{i_r\tilde{\mu}(i^r)}\right) \times \prod_{r\notin S}\left(1-\bar{q}_{i_r\tilde{\mu}(i_r)}\right).$$

Therefore, player i's expected effort when i is matched with j is

$$\begin{aligned} E(x_i|(i,j)) &= \sum_{s\in\mathcal{S}|(i,j)\in T^A(s)\times T^B(s)} P(s)p(s,(i,j))x_i^*(i,j) \\ &= \sum_{\tilde{\mu}\in M(N^A\backslash\{i\},N^B\backslash\{j\})} \sum_{S\in D(2n,n)} \prod_{r\in S}\left(\bar{q}_{i_r\tilde{\mu}(i^r)}\right) \times \prod_{r\notin S}\left(1-\bar{q}_{i_r\tilde{\mu}(i_r)}\right) x_i^*(i,j). \end{aligned}$$

Thus, the coefficient of $x_i^*(i,j)$ is nothing but the probability that this battle becomes pivotal. This implies that neither a sequential choice nor a one-shot choice makes a difference. Hence, player i's ex ante expected effort in both cases is

$$\begin{aligned} E(x_i) &= \frac{1}{2n+1}\sum_{j\in N^B} E(x_i|(i,j)) \\ &= \frac{1}{2n+1}\sum_{j\in N^B} \sum_{\tilde{\mu}\in M(N^A\backslash\{i\},N^B\backslash\{j\})} \sum_{S\in\{S'\in\{1,...,2n\}:|S'|=n\}} \prod_{r\in S}\left(\bar{q}_{i_r\tilde{\mu}(i^r)}\right) \times \prod_{r\notin S}\left(1-\bar{q}_{i_r\tilde{\mu}(i_r)}\right) x_i^*(i,j), \end{aligned}$$

and Fu, Lu, and Pan's (2015) total effort equivalence result extends to our case, too.

Corollary 1. *The expected effort level of each player in a one-shot ordering choice game is equal to the level in battle-by-battle sequential choice game.*

Although we only considered a fully sequential player-choice game in Theorem 2, Fu, Lu, and Pan's (2015) invariance results hold even if the game involves battles with a more general temporal structure. Their Theorem 3's logic would hold, although the argument gets messier by that.

4.5 Concluding Remarks

In this paper, we show that Fu, Lu, and Pan's (2015) invariance results extend even if the team leaders strategically choose the order in which players are sent to the battleground. Somewhat surprisingly, the total randomization of player choice at any level is the equilibrium strategy irrespective of whether team leaders' choices are made as one-shot or battle-by-battle decisions. The independence of each battle's winning probability is quite robust as long as the zero homogeneity of the contest success function of each battle is satisfied. For the invariance results on the expected winning probability of the whole contest and ex ante effort levels, the choices of player orderings add more subtleties. First, the number of players who participate in the $2n + 1$ battles from each team needs to be exactly $2n + 1$. The following example illustrates the importance of this assumption. For simplicity, we consider a game with an exogenous winning probability matrix.

Example 2. Suppose that there are three battles and teams A and B have four and three players, respectively. We assume the following exogenous probability matrix:

$$Q = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \\ q_{41} & q_{42} & q_{43} \end{pmatrix} = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0 \end{pmatrix}$$

That is, player 1 in team B is a dominant player, but players 1, 2, 3 on team A and players 2 and 3 on team B are exactly in the same league. Player 4 in team A is a weak player, but is good at dealing with the dominant player 1 on team B (an assassin). In this case, if team A selects $\{1, 2, 3\}$, team A can win only when both players that are not matched with team B's dominant player win. Thus, team A's winning probability is $0.5 \times 0.5 = 0.25$. If team A includes the assassin player 4, then it has a positive winning probability only when assassin player is matched with the dominant player. This implies that team A's winning probability is $\frac{1}{3} \times 0.5 = 0.1333 < 0.25$. Thus, in a one-shot static ordering choice game, team A does not use player 4.

In contrast, in a battle-by-battle player choice game, in race 3, if team B still has the dominant player 1, team A will certainly use player 4 if it still has her. If so, does team B keep player 1 till race 3? Since the situation is similar to matching pennies, randomization is needed, so player 1 may be kept. Consider the following case in race 2. Team A won the first round, and still has players 2, 3, and 4, while team B has players 1 and 2. Team B must win the next two races to win the team contest.

second race		$\frac{3}{4}$	$\frac{1}{4}$
	$\ell = 1$	1	2
$\frac{3}{4}$	2, 3	0.5	0.75
$\frac{1}{4}$	4	0.75	0

Thus, with one additional player, our equivalence result no longer holds.□

Next, unlike in Fu, Lu, and Pan (2015), our player-order choice game does not preserve the invariance in a team's winning probability if battles are weighted unevenly. In the last section of Fu, Lu, and Pan (2015), they demonstrate the robustness of invariance results that allows for component battles to carry different weights. This result follows in their model, since each battle and the players who play in them are tied up there. However, in our game, team leaders assign players to each battle. If a certain battle is weighted heavily, team leaders' strategy would be affected. We conclude the paper with the following simple example (fixed Q again).

Example 3. Suppose that there are three battles with potentially different weights, and teams A and B have three players each. The team that wins with a total weight more than $\frac{1}{2}$ wins the contest. We assume the following exogenous probability matrix:

$$Q = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix} = \begin{pmatrix} 0.5 & 0.7 & 0.9 \\ 0.3 & 0.5 & 0.7 \\ 0.1 & 0.3 & 0.5 \end{pmatrix}$$

Player 1 is the dominant player on each team. If the weight of each battle is $\frac{1}{3}$ each as before, then we know that the total randomization is used in any setup. But now, suppose that the first battle's weight is more than $\frac{1}{2}$. In this case, only the first battle matters for the contest outcome. Obviously, both team leaders assign their best player to the first battle. Thus, our results no longer hold.□

Finally, even though this chapter addresses the invariance results when each team member is selected to play a single match, it should be noted similar results can be extended for cases when this condition is partially relaxed. One such environment is studied in Anbarci, Sun, and Ünver (2020). The authors consider sequentially fair mechanisms in penalty shootouts - mechanisms designed to resolve ties in high level sports competitions such as soccer and hockey. They show that the fixed order in which the penalties are taken can affect the fairness of such mechanisms, and with the exception of one specific such order under sudden-death, all other mechanisms are sequentially unfair. However, Anbarci, Sun, and Ünver (2020) demonstrate that taking any such mechanism consisting of a sudden-death element, and extending its continuation with a sequentially-fair mechanism with sudden-death rounds can lead to a sequentially-fair mechanism. In any such mechanism the player-order choice once again becomes irrlevant.

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