Bounded Powers Extend:

Author: Cristina Mullican

Boston College Electronic Thesis or Dissertation, 2020

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Cristina Mullican

A dissertation

submitted to the Faculty of

the department of Mathematics

in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

Boston College
Morrissey College of Arts and Sciences
Graduate School

March 2020

Bounded Powers Extend

Cristina Mullican

Advisor: Assoc. Prof. Ian Biringer, PhD

We are interested in proving the following statement: Given a 3-manifold M with boundary and a homeomorphism of the boundary $f : \partial M \to \partial M$ such that there is some power that extends to M, there is some k depending only on the genus $g(\partial M)$ and some $\ell < k$ such that f^ℓ extends to M. We will prove that the power needed to extend is not uniformly bounded with some examples, we will prove the statement is true if M is boundary incompressible and we will show that the general statement reduces to effectivising some technical results about pure homeomorphisms extending to compression bodies.

Acknowledgements

I would like to thank my advisor Ian Biringer for his thoughtful guidance, encouragement, and support during my graduate career at BC and especially throughout this project. Thanks for instilling in me a deeper wonder of the story that mathematics has to tell. Thanks to my committee members Kathryn Lindsey and Martin Bridgeman for reviewing my project. Many thanks as well to Ellen Goldstein and Juliana Belding for your mentorship.

To my fellow classmates who have become wonderful friends. Siddhi, Mia, Melissa, Tom, and Mustafa, I wouldn't have survived this without each of you and you all made the struggle a joy!

Also, I would like to thank my CoaH family for your love, friendship, and spiritual care for me and my family over the last 6 years. And finally, my deepest love and gratitude to Gerhard for your love, support, and care for me and Max. You have served our family so well in the midst of my pursuing my dreams. I love you!

And thanks to the initiative "A Room of One?s Own" for focused time to write.

This dissertation is dedicated to Maxwell.

Contents

1 Introduction

In order to understand 3-manifolds with boundary, we might study homeomorphisms of the manifold to itself. This would induce a homeomorphism on the boundary, a surface homeomorphism. Since surface homeomorphisms are well understood, it is productive to reverse the process: For M a 3-manifold with boundary component S, we can ask when a homeomorphism of S extends to M. Precisely, for a manifold M with a boundary component S, we say that a homeomorphism $f : S \to S$ *extends* to M if there exists some homeomorphism $\phi : M \to M$ such that $\phi|_S = f$.

Surface homeomorphism extension is broadly studied in 3-manifold topology and geometry. One interesting instance is the cobordism of surface automorphisms. Let $f : S \to S$ and $f' : S' \to S'$ be surface homeomorphisms. We say f and f' are *cobordant* if there exists some 3-manifold M with boundary components S and S' and an automorphism F of M such that $F|_S = f$ and $F|_{S'} = f'$. Both Bonahon [13] and independently Edmonds and Ewing [14], show that the cobordism classes of surface automorphisms form a group isomorphic to $\mathbb{Z}^\infty \oplus (\mathbb{Z}/2)^\infty$. Furthermore, in [6], Casson and Long give an algorithm to determine if a surface homeomorphism extends to a *compression body* (see below) which is a key step in determining if a homeomorphism is nullcobordant.

1.1 Main Conjecture

We are interested in proving the following conjecture:

Conjecture 1. Let M be a 3 manifold with boundary and let $B = S_{g_1} \sqcup \cdots \sqcup S_{g_\ell}$ be a disjoint union of a subset of the components of ∂M, closed surfaces S_{g_i} of genus g_i with $G(B) := \sum_{i=1}^{\ell} g_i$. If $f : B \to B$ is a homeomorphism such that f raised to some power extends to M, then there is a bounded power $\ell \leq k(G(B))$ such that f^ℓ also extends to M.

One might wonder if any homeomorphisms require a power to extend. Section 3 gives examples of some 3 manifolds M with boundary S and homeomorphisms $f : S \to S$ that require a power to extend to M, including pseudo-Anosov homeomorphisms.

1.2 Partial Extension

Given these examples, one might wonder if a lesser power might extend to a submanifold of M.

Let M be a compact, orientable, and irreducible 3-manifold with some compressible boundary component S. Then for a homeomorphism $f : S \to S$, we say that f *partially extends* to M if there is some nontrivial compression body $C \subset M$ with $\partial_+ C = S$ and a homeomorphism $\phi : C \to C$ such that $\phi|_S = f$.

Biringer, Johnson, and Minsky [1] prove the following:

Theorem (BJM, 2013)**.** *Let $f : \Sigma \to \Sigma$ be a pseudo-Anosov homeomorphism of some compressible boundary component Σ of a compact, orientable and irreducible 3-manifold M. Then the (un)-stable lamination of f is an $\mathbb{R}$-projective limit of meridians if and only if f has a power that partially extends to M.*

Ackermann gives an alternate proof in [5] using earlier machinery of Casson and Long (see [6] and [7]). Maher and Schleimer give an alternate proof using train tracks and subsurface projections (see [8]).

Partial extension comes up naturally when hyperbolizing 3-manifolds created as gluings. For instance, in [2], Lackenby studied the hyperbolization of 3-manifolds obtained by 'generalized Dehn surgery', i.e. manifolds obtained by attaching a handlebody H to a compact 3-manifold M along some boundary component. He showed (modulo the Geometrization Theorem) that if M is 'simple' and we choose any homeomorphism $\phi : \partial M \to \partial H$ and a homeomorphism $f : \partial H \to \partial H$ such that no power partially extends to H, then for infinitely many integers n, the manifold $M \cup_{f^n \circ \phi} H$ is hyperbolic.

Inspired by both of these theorems, it is natural to ask if there is some bound on the power of f required to partially extend as this would imply there are only a finite number of powers of f to check for partial extension and subsequent obstruction to hyperbolization. We will show that there is no universal bound via construction in the proof of the following:

Theorem 32. *For all $i \in \mathbb{N}$ there is a compact, orientable and irreducible 3-manifold M_i with compressible boundary component Σ_i and a pseudo-Anosov $f_i : \Sigma_i \to \Sigma_i$ such that f_i^i partially extends to M_i and f_i^j does not for $j < i$.*

This illustrates why in Conjecture 1, we bound our power by a function of the genus of the boundary of M.

1.3 Outline

In Section 2, we give background on surface homeomorphisms, and compression bodies. In Section 3 we present examples of boundary homeomorphisms that require a power to extend including pseudo-Anosov maps. In Section 4 we prove Theorem 32.

In Section 5, we use the Prime Decomposition Theorem to prove the following theorem.

Theorem 37. *If Conjecture 1 is true for pure homeomorphisms on the boundary of orientable irreducible 3-manifolds, it is true for homeomorphisms on the boundary of orientable 3-manifolds in general.*

This along with following work of Bonahon allows us to reduce proving Conjecture 1 in general to proving it for compression bodies and irreducible manifolds that are boundary incompressible.

Theorem 41 ([13], Theorem 2.1)**.** *Let M be an irreducible 3-manifold. There exists $V \subset M$ a disjoint union of compression bodies for ∂M, unique up to isotopy, called the characteristic compression body, such that $M - V$ is boundary incompressible and irreducible.*

In Section 6 we prove Theorem 42 which says that Conjecture 1 is true for pure boundary homeomorphisms of irreducible boundary-incompressible manifolds. A key tool we use is Theorem 43 ([18], Theorem 3.8 Characteristic torus/annulus decomposition) which is the analogue of the JSJ decomposition theorem for manifolds with boundary.

As a corollary to this, we prove the following:

Corollary 62. *Let M be an irreducible, orientable boundary-incompressible manifold. If B is a union of a subset of the boundary components of M with homeomorphism $f : B \to B$ such that a power of f extends to M, then f^ℓ extends to M such that*

$$\ell \leq 12G(B)!(3G(B)-3)![210(B(G)-1)]^{2G(B)-2}.$$

In Section 7, we reduce proving Conjecture 1 to proving the following:

Conjecture 2. If $h : S_g \to S_g$ is a homeomorphism and C is an S_g-compression body such that some power of h extends to C, then there exists S_g-compression body $C' \subset C$ such that for some $j \leq m(g)$ (with $m(g)$ an increasing function) we have that h^j extends to C'.

Assuming this conjecture, in Proposition 63 we get a bound for extension on a compression body. With Theorem 41 in mind we generalize to the case where M is a disjoint union of compression bodies in the following corollary:

Corollary 64. Let $V = S_{g_1}[K_1] \sqcup \cdots \sqcup S_{g_n}[K_n]$ be a disjoint union of compression bodies and suppose that B is a union of a subset of the S_{g_i}'s. If homeomorphism $f : B \to B$ has a power that extends to V and if Conjecture 2 is true then there is an $i \leq m(G(B))^{[1+G(B)(2G(B)-2)]G(B)}$ such that f^i extends to V.

Then in Proposition 65 we put the compression body case and the incompressible boundary case together to get a bound on extension for an irreducible manifold. As a corollary, we get the following for a general orientable 3-manifold:

Corollary 66. Assuming Conjecture 2, if M is an orientable 3-manifold with boundary and if B is a union of a subset of components of ∂M such that $f : B \to B$ has a power that extends to M, then there is some

$$i \leq \left[12G(B)!(3G(B)-3)!m(G(B))^{[1+G(B)(2G(B)-2)]G(B)}[210(B(G)-1)]^{2G(B)-2}\right]^{G(B)}$$

such that f^i extends to M.

In Section 8, we give ideas toward proving Conjecture 2 including a proof of the following:

Theorem 71. Let $h : S_g \to S_g$ be a homeomorphism and $\gamma \subset S_g$ an essential simple closed curve. Set

$$\mathcal{C}_{h,\gamma} = \{S_g\text{-compression body } C : h \text{ extends to } C \text{ and } \gamma \text{ compresses in } C\}.$$

Then there are at most

$$(6g-6)^{2g-2}(6g-6)^{(2g-2)^2(2g-3)}$$

minimal elements of $\mathcal{C}_{h,\gamma}$.

This along with an effectivization of a theorem of Casson and Long (see Theorem 1.2 in [6]) would prove a weaker version of Conjecture 2 that applies to only pseudo-Anosovs instead of homeomorphisms in general.

2 Background

2.1 Surface homeomorphisms

First, let us consider the homeomorphisms of a torus. The orientation preserving automorphisms of the torus up to isotopy are isomorphic to $SL(2,\mathbb{Z})$.

Fact 1. *If an orientation preserving homeomorphism $f : T^2 \to T^2$ fixes a curve up to isotopy, then f is isotopic to a power of a Dehn twist.*

Proof. Let orientation preserving homeomorphism $\phi : T^2 \to T^2$ correspond to $A_\phi = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{Z})$. Then there is some simple closed curve α represented by vector $v \in \mathbb{Z}\times\mathbb{Z}$ such that $(A_\phi)v = v$. Thus $(A_\phi - I)v = 0$. Since such a v exists this implies $\det(A_\phi - I) = 0$. Thus $(a-1)(d-1) - cb = 0$ $(*)$.

Since $A_\phi \in SL(2,\mathbb{Z})$, we have $\det(A_\phi) = 1$ and so $-cb = 1 - ad$. Plugging this into $(*)$, we get $(a-1)(d-1) + 1 - ad = 0$ which gives $a + d = 2$. Thus $|\mathrm{tr}(A_\phi)| = 2$. By the classification of $\mathrm{Mod}^+(T^2)$, every $A \in SL(2,\mathbb{Z})$ with $|\mathrm{tr}(A)| = 2$ corresponds to a homeomorphism of T^2 that is a power of a Dehn Twist. Thus up to isotopy, ϕ is a power of a Dehn Twist. $\square$

Recall the following classification of surface homeomorphisms of surfaces $S_{g,b,n}$ of genus g with b boundary components and n punctures. See [22], Chapter 13 for details.

Theorem 2 (Nielsen-Thurston classification)**.** *Let $\chi(S_{g,b,n}) < 0$. Each homeomorphism $f : S_{g,b,n} \to S_{g,b,n}$ is isotopic to a homeomorphism that is periodic, reducible, or pseudo-Anosov. Furthermore, the pseudo-Anosov mapping classes are neither periodic nor reducible.*

A map is *reducible* if it preserves a nonempty set of disjoint isotopy classes of essential simple closed curves $\{c_1, ..., c_n\}$ in $S_{g,b,n}$. This set is called a *reduction system* for f. The *canonical reduction system* for f is the intersection of all maximal reduction systems with respect to inclusion of reduction systems of f. The following corollary to Theorem 2 proved in [23] allows us to cut $S_{g,b,n}$ along the c_i and look at the classification of f restricted to the resulting pieces.

Corollary 3 ([22], Cor 13.3)**.** *Let $\chi(S_{g,b,n}) < 0$ and $S = S_{g,b,n}$. Let $f \in Mod(S)$ and let $\{c_1, ..., c_m\}$ be its canonical reduction system. Choose representatives of the c_i with pairwise disjoint closed neighborhoods $R_1, ..., R_m$ which are annuli. Let $R_{m+1}, ..., R_{m+p}$ denote the closures of the connected components of $S - \bigcup_{i=1}^{m} R_i$. Let $\eta_i : Mod(R_i) \to Mod(S)$ denote the homomorphism induced by the inclusion $R_i \to S$. Then there is a representative ϕ of f that permutes the R_i so that some power of ϕ leaves each R_i invariant. What is more, there exists a $k \geq 0$ so that $\phi^k(R_i) = R_i$ for all i and $f^k = \Pi_{i=1}^{m+p} \eta(f_i)$ where $f_i \in Mod(R_i)$ is a power of a Dehn twist for $1 \leq i \leq m$ and $f_i \in Mod(R_i)$ is either pseudo-Anosov or the identity for $m+1 \leq i \leq m+p$.*

Definition 4. We say that map $f^k : S_g \to S_g$ as described above is *pure*.

Note that if $n = b = 0$, the order $|\{c_1, ..., c_n\}| \leq 3g - 3$, the order of a pants decomposition in a closed genus g surface and thus for any homeomorphism $f : S_g \to S_g$, there is some $k \leq (3g-3)!$ such that f^k is pure.

Let us generalize to the case that $f : B \to B$ is a homeomorphism of B a disjoint union of closed surfaces. We denote a closed surface of genus g by S_g throughout.

Definition 5. We say $f : B \to B$ for $B = S_{g_1} \sqcup \cdots \sqcup S_{g_n}$ is *pure* if f satisfies the following:

1. f fixes each component of B
2. on each component S_{g_i} with $g_i > 1$, f is pure (as defined in Definition 4)
3. on each component S_{g_i} with $g_i = 1$, $f = id$ or f is not finite order.

Definition 6. Let $B = S_{g_1} \sqcup \cdots \sqcup S_{g_n}$ be a disjoint union of closed surfaces S_{g_i} of genus g_i. We say that $G(B) := \sum_{i=1}^{n} g_i$ is the *genus sum* of B.

Corollary 7. *If $B = S_{g_1} \sqcup \cdots \sqcup S_{g_n}$ is a disjoint union of closed surfaces S_{g_i} of genus $g_i > 0$ for all i with $f : B \to B$ a homeomorphism then there is some $k \leq 12G(B)!(3G(B)-3)!$ such that f^k is pure.*

Proof. For a homeomorphism $f : B \to B$, the map f may permute the components of B. Then $G(B)$ is an upper bound on the number components in B under the assumption that each component is at least genus one. Hence there is some $\ell \leq G(B)!$ such that f^ℓ maps each component of B to itself. And for a homeomorphism that maps each component of B to itself, one can check that the reduction system of the genus 2 or greater components of B contains at most $3G(B)-3$ curves. Thus there is some $k \leq G(B)!(3G(B)-3)!$ such that f^k is pure on each component component of B with genus greater than 1.

Recall that $\mathrm{Mod}(T^2) \cong SL(2,\mathbb{Z})$. Since 4 is an upper bound on the order of every finite cyclic subgroup of $SL(2,\mathbb{Z})$, then f^{12} restricted to the tori components of B is either the identity or has infinite order. $\square$

Now, we look at the well known result that the isometry group of a hyperbolic surface is finite. Following the Primer on Mapping Class Group's treatment [22] of Hurwitz's famous $84(g-1)$ theorem, we generalize to surfaces with punctures.

Proposition 8. *If X is a hyperbolic surface genus g with n punctures then $|Isom(X)| \leq 42(2g+n-2)$.*

Proof. We begin by proving that if X is a hyperbolic surface homeomorphic to $S_{g,n}$, a genus g surface with n punctures, then $\mathrm{Isom}(X)$ is finite in any hyperbolic metric.

By the thick-thin decomposition and the Margulis constant, for every $x \in X$ and all $f \in \mathrm{Isom}(X)$, $d(x, f(x))$ is uniformly bounded. Let $\mathcal{F}$ be an infinite sequence of $f_n \in \mathrm{Isom}(X)$. Then $\mathcal{F}$ is uniformly equicontinuous. So by Arzela-Ascoli, two functions in $\mathcal{F}$ are isotopic.

It is sufficient to show that if $f \in \mathrm{Isom}(X)$ is isotopic to the identity, it is the identity. Consider $\tilde{f}$ an isometry on $\mathbb{H}^2$. Either $\tilde{f}$ has 1 fixed point in $\mathbb{H}^2$, 1 fixed point in $\partial\mathbb{H}^2$, 2 fixed points in $\partial\mathbb{H}^2$, or fixes $\partial\mathbb{H}^2$. Since $\tilde{f}$ is isotopic to the identity, $\tilde{f}$ conjugates to the identity and so $\tilde{f}$ must fix $\partial\mathbb{H}^2$. Hence $\tilde{f}$ is the identity.

Now, set $Y = X/\mathrm{Isom}^+(X)$. Since $|\mathrm{Isom}^+(X)|$ is finite, $\mathrm{Area}(Y) = \mathrm{Area}(X)/|\mathrm{Isom}^+(X)|$.

Suppose that there is some lower bound $M < \mathrm{Area}(Y)$ for all orbifolds $Y = S_{g,n}/\mathrm{Isom}^+(X)$. It would follow that $|\mathrm{Isom}^+(X)| < \mathrm{Area}(X)/M$.

Recall that the Euler Characteristic of orbifold Y with m cone points P_i with degree p_i is $\chi(Y) = 2-2g-n-m+\sum_{i=1}^{m}\frac{1}{p_i}$. By the Gauss Bonnet Theorem,

$$\begin{aligned}\mathrm{Area}(Y) &= -2\pi\chi(Y)\\ &= -2\pi\left(2-2g-n-\sum_{i=1}^{m}\left(1-\frac{1}{p_i}\right)\right)\end{aligned}$$

To find a lower bound for Area(Y), we need to find an upper bound for $\chi(Y)$.

Note that for orbifold Y' with $g = 0 = n$ with 3 cone points of degree 2, 3, and 7, $\chi(Y) = -\frac{1}{42}$.

Suppose to the contrary that there is a Y with $\chi(Y) > -\frac{1}{42}$. Let's consider the cases.

1. First, note that if Y has no cone points, then $\chi(Y)$ is a negative integer.
2. Suppose $g > 1$. Then $\chi(g) \leq -2$
3. If $g = 1$ then either $n > 0$ or $m > 0$ in order for Y to be hyperbolic.
    (a) If $n > 0$ then $\chi(Y) \leq 2 - 2 - n \leq -1$
    (b) If $m > 0$ then note that each cone point subtracts off at least $\frac{1}{2}$ in its contribution to Euler characteristic. Thus $\chi(Y) \leq -\frac{1}{2}$.
4. Suppose $g = 0$. Then $\chi(Y) = 2 - n - \left(\sum_{i=1}^{m} 1 - \frac{1}{p_i}\right)$
    (a) If $n > 2$, $\chi(Y) \leq -1$.
    (b) If $n = 2$ and $m \geq 1$, then $\chi(Y) \leq -\frac{1}{2}$.
    (c) If $n = 2$ and $m = 0$ then Y is not hyperbolic.
    (d) If $n = 1$ and $m \geq 3$ then $\chi(Y) \leq -\frac{1}{2}$.
    (e) If $n = 1$ and $m = 2$ with both degree 2, $\chi(Y) = 0$ which means Y is not hyperbolic.
    (f) if $n = 1$ and $m = 2$ with degree 2 and 3, $\chi(Y) = -\frac{1}{6}$. Any other degrees will result in a lower Euler characteristic.
    (g) if $n = 1$ and $m = 1$ with degree k, $\chi(Y) = \frac{1}{k}$ which means Y is not hyperbolic.
    (h) if $n = 0$, then X is a closed surface. The proof of Theorem 7.10 in [22] examines these cases to show that for all $Y = S_{g,0}/\text{Isom}(X)$, there is no hyperbolic orbifold with Euler characteristic greater than $-\frac{1}{42}$.

Therefore, orbifold Y' from above has the greatest upper bound for hyperbolic Y. Thus for all hyperbolic orbifolds Y, $\chi(Y) \leq -\frac{1}{42}$. It follows that $\text{Area}(Y) \geq \frac{\pi}{21}$. Thus

$$\begin{aligned}
|\text{Isom}(X)| &\leq \text{Area}(X)/\frac{\pi}{21} \\
&= -2\pi(2-2g-n)/\frac{\pi}{21} \\
&= 42(2g+n-2)
\end{aligned}$$

□

Definition 9. Let M be a 3-manifold with nonempty boundary. Then the components of ∂M are closed surfaces. We say that M is *boundary-incompressible* if no essential simple closed curve on ∂M bounds an embedded disk in M.

2.2 Compression bodies

A *compression body* C is an orientable, compact, irreducible 3-manifold with a preferred boundary component $\partial_+ C$ that π_1-surjects. We say $\partial_+ C$ is the *exterior boundary component* of C.

Definition 10. If we fix a closed surface S to be the exterior boundary, an S-*compression body* is a pair (C, m) where C is a compression body with a homeomorphism $m : S \to \partial_+ C$. Here, m is the *marking* of C which is dropped when non-ambiguous.

Any S-compression body C can be constructed in the following way: Set $\{\alpha_1, ..., \alpha_n\}$ to be a maximal disjoint set of simple closed curves on S that bound disks in C. First consider $S \times [0,1]$ and attach 2-handles along annuli in $S \times \{0\}$ whose core curves are $\{\alpha_i\} \times \{0\}$. Then we attach 3-balls along any resulting boundary components that are homeomorphic to S^2. In this construction, $\partial_+ C = S \times \{1\}$. For details see [3].

When an S-compression body can be constructed as above by attaching 2-handles along simple closed curves $\{\alpha_1, ..., \alpha_n\}$ and subsequent 3-balls, we denote it as $S[\alpha_1, ..., \alpha_n]$ and call $\{\alpha_1, ..., \alpha_n\}$ a *compressing system*.

The *trivial* S-compression body is homeomorphic to $S \times I$. We call $\partial C \backslash \partial_+ C$ the *interior boundary* of C.

For S-compression bodies (C, m) and (D, n) we write $(C, m) \subset (D, n)$ if there exists an embedding $H : C \to D$ such that $n = H|_{\partial_+ C} \circ m$.

Two S-compression bodies (C, m) and (C', m') are equivalent if there exists a homeomorphism $\varphi : C \to C'$ such that the diagram below commutes.

$$\begin{tikzcd}
 & \partial_+ C' \\
S \arrow[ur, "m'"] \arrow[dr, "m"'] & \\
 & \partial_+ C \arrow[uu, "\varphi|_{\partial_+ C}"']
\end{tikzcd}$$

A homeomorphism $f : S \to S$ sends (C, m) to (C, mf^{-1}). This action respects the equivalence relation. From here forward, we will drop all markings and will abusively refer to compression bodies when we mean equivalence classes of marked compression bodies. In particular, we denote this action as $f(C)$. Notice that $f(S[\alpha_1, ..., \alpha_n]) = S[f(\alpha_1), ..., f(\alpha_n)]$.

Definition 11. Let C be an S-compression body and $f : S \to S$ a homeomorphism. We say that f *extends* to C if there is a homeomorphism $\phi : C \to C$ such that $\phi|_{\partial_+ C} = f$. Here we can write $f(C) = C$. We say f *partially extends* to C if there is an S-compression body $D \subset C$ with a homeomorphism $\psi : D \to D$ such that $\psi|_{\partial_+ D} = f$.

Definition 12. Let α be a simple closed curve in $\partial_+ C$. If α bounds a disk in C then we say α is a *meridian* of C and α *compresses* in C. The *disk set* of an S-compression body denoted $\mathcal{D}(C)$ is the set of all simple closed curves in S that compress in C, viewed as a subset of the vertices of the curve graph $\mathcal{C}(S)$.

In fact, the disk set completely determines a compression body.

Proposition 13 ([3], Cor. 2.2)**.** *Let C and D be S-compression bodies. Then C and D are isomorphic if and only if $\mathcal{D}(C) = \mathcal{D}(D)$, and $C \subseteq D$ if and only if $\mathcal{D}(C) \subseteq \mathcal{D}(D)$.*

Definition 14. Let α be an essential simple closed curve in the exterior boundary of compression body C. If there is some simple closed curve α' in the interior boundary of C such that α and α' together bound an embedded annulus in C we say α *bounds an annulus*

Recall the following well known fact which is proved in detail in [16]:

Fact 15. *Let C be an S-compression body. A Dehn twist $T_\alpha : S \to S$ extends to C if α compresses in C or bounds an annulus in C.*

The idea of the proof is to define a Dehn twist of C by twisting in a neighborhood of the disk or annulus bounded by α.

In the compression body $S[\alpha_1, ..., \alpha_n]$ there are likely many curves besides the α_i that compress.

Fact 16. *If two boundary components of an embedded pair of pants in $\partial_+ C$ bound disks then the third boundary component also bounds a disk.*

Proof. Let P be a pair of pants embedded in $\partial_+ C$ with boundary components c_1, c_2 and c_3 such that c_1, c_2 bounding disks d_1 and d_2 respectively. Since $P \cup d_1 \cup d_2$ is homeomorphic to a disk then c_3 compresses. $\square$

Definition 17. An S-compression body C is *small* if it can be written as $S[a]$ for some simple closed curve $a \in S = \partial_+ C$. A compression body is *minimal* if it does not contain any (non-trivial) sub-compression bodies. Let $\mathcal{C}$ be a set of S-compression bodies. Then $C \in \mathcal{C}$ is *minimal* among the elements of $\mathcal{C}$ if the following holds: if there is some $D \in \mathcal{C}$ such that $D \subseteq C$, then $D = C$.

Minimality and smallness are related in the following work of Biringer and Vlamis.

Proposition 18 ([3], Cor. 2.7)**.** *An S-compression body is minimal if and only if it is a solid torus or a small compression body obtained by compressing a separating curve.*

A compression body can be built out of minimal compression bodies in the following way:

Definition 19. A *sequence of minimal compressions* of an S-compression body C is a chain $S \times [0,1] = C_0 \subset C_1 \subset \cdots \subset C_k = C$ of S-compression bodies where C_{i+1} is obtained from C_i by gluing in a minimal F_i-compression body to F_i, an interior boundary component of C_i. (See [3] section 2.2 for details.)

In this sequence, each compression body is obtained by gluing in either a solid torus or obtained by compressing a single separating curve. Biringer and Vlamis give a formula for the number of steps required to obtain a compression body from minimal compressions:

Proposition 20 ([3], Prop. 2.10)**.** *If C is any S-compression body with interior boundary $F_1 \sqcup \cdots \sqcup F_n$, then the length k of any sequence of minimal compressions $S \times [0,1] = C_0 \subset C_1 \subset \cdots \subset C_k = C$ is $\mathfrak{h}(C) := 2g(S) - 1 - \sum_{i=1}^{n}(2g(F_i) - 1)$, the* height *of C.*

Proposition 21 ([3], Cor. 2.4)**.** *Let $C \subset E$ be S-compression bodies and let $\partial\_C = F_1 \sqcup \cdots \sqcup F_n$ where $n \leq g$. Then E is isomorphic to an S-compression body obtained by gluing to C a collection of (possibly trivial) F_i-compression bodies D_i, one for each i.*

2.3 Curve Surgery in Compression bodies

Lemma 22 ([3], Lemma 2.8)**.** *Suppose that α and β are both meridians of compression body C in minimal position. Then the intersections with α divide β into a collection of arcs, one of which, say b has the following properties:*

1. *both intersections of b with α happen on the same side of α*
2. *the union of b with either of the two arcs of α with the same endpoints is a meridian which is disjoint from (after isotopy) but not isotopic to a*

Definition 23. In the case of Lemma 22, let a and a'' be the two arcs of α. We say $\alpha' := a' \cup b$ and $\alpha'' := a'' \cup b$ are *obtained by b-surgery* on α. In general, given an S-compression body C with meridian α and arc b in S with $b \cap \alpha = \partial b$ where b satisfies (1) of Lemma 22, the intersections of b with α divide α into two arcs a' and a''. We say $\alpha' := a' \cup b$ and $\alpha'' := a'' \cup b$ are obtained by b*-surgery* on α.

Note that as long as both α and β compress in some compression body, we can perform b-surgery on α, even if we are considering a context where α and or β is not a meridian. If both are meridians, then b-surgery yields curves that are meridians as well by Lemma 22.

Definition 24. Let δ be a meridian in S-compression body C. Let b be an arc in S with $\partial b = b \cap \delta$ satisfying (1) of Lemma 22. Then we say arc b is a δ*-wave* if b is homotopic in C rel endpoints to an arc of δ. (See definition 3.2 in [25])

Fact 25. *Let α be a meridian in S-compression body C. Let b be an arc in S with $\partial b = b \cap \delta$ satisfying (1) of Lemma 22. Then b is an α wave if and only if the curves α' and α'' obtained by b-surgery on α are meridians which are disjoint from α and b.*

Proof. Suppose b is an α-wave. Then b is homotopic to in C rel endpoints to an arc a' of α. This homotopy sweeps out a disk in C with boundary $b \cup a'$ and so $\alpha' := a' \cup b$ obtained from b-surgery of α is a meridian of C. After isotopy α' is disjoint from b and α (since $\alpha \cap b = \partial b$) and cobounds a pair of pants with $\alpha'' = b \cup (\alpha - a')$. So by Fact 16, α'' is also a meridian of C.

Conversely, if α' is a meridian, then we can homotope b rel its endpoints through this disk to a'. So b is an α-wave. □

2.4 Compressing Systems

Definition 26. Let $K = \{\delta_1, ..., \delta_n\}$ be a set of disjoint meridians in C. We say that γ is in *tight position* with respect to K if γ contains no arcs that are δ_i-waves for $1 \leq i \leq n$.

Fact 27 ([3], Lemma 3.5)**.** *For C an S_g-compression body and α any simple closed curve on S_g, there is a compressing system with respect to which α is in tight position.*

Proof. Let $K_1 = \{\kappa_1, ..., \kappa_m\}$ be a compressing system for S-compression body C and let α be an essential simple closed curve of S.

Note that the set of points $\left(\bigcup_{\kappa \in K_1} \kappa\right) \cap \alpha$ divides α into arcs. Suppose that arc $a \subset \alpha$ is a κ_i-wave for some i. Then we obtain κ_i' and κ_i'' via a-surgery on κ_i which are both meridians and disjoint from κ_i and a by Fact 25. Set $C' := [\kappa_1, \cdots \kappa_{i-1}, \kappa_i', \kappa_i'', \kappa_{i+1}, ..., \kappa_m]$. Since both κ_i' and κ_i'' compress in C, then $C' \subset C$. Notice that κ_i cobounds a pair of pants with κ_i' and κ_i'' hence by Fact 16, κ_i compresses in C' implying $C \subset C'$. Thus $C = C'$ and so $K_2 := \{\kappa_1, \cdots \kappa_{i-1}, \kappa_i', \kappa_i'', \kappa_{i+1}, ..., \kappa_m\}$ is a compressing system for C. And by replacing κ_i from K_1 with κ_1' and κ_i'' in K_2, we have removed ∂a from among the points of intersection of α with the compressing system for C. Thus

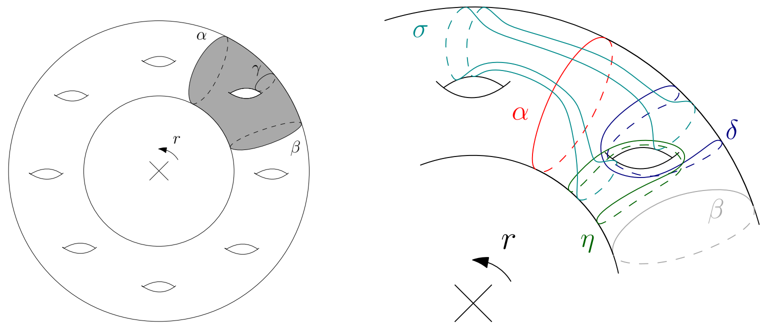
$$\left|\left(\bigcup_{\kappa\in K_1}\kappa_i\right)\cap\alpha\right| \geq \left|\left(\bigcup_{\kappa\in K_2}\kappa_i\right)\cap\alpha\right| + 2.$$

Continue the process starting with a κ_i wave of K_j. Since $\left|\left(\bigcup_{\kappa\in K_j}\kappa\right)\cap\alpha\right|$ is finite and strictly decreasing as j increases, the process will terminate with K_ℓ a compressing system for C with which α is in tight position either when there are no κ_i waves for all $\kappa_i \in K_\ell$ or when $\left|\left(\bigcup_{\kappa\in K_\ell}\kappa_i\right)\cap\alpha\right| = 0$ in which case α is disjoint from the curves of K_ℓ and hence there are certainly no κ_i waves in α. $\square$

3 Examples of Extension

We will examine several examples of surface homeomorphisms that do not extend to the whole 3-manifold but where powers do extend. Let S_g be a closed surface of genus g.

Example 28. As pictured in Figure 1 (a), consider the 3-manifold $S_g[\alpha, \beta, \gamma]$. Let $r : S_g \to S_g$ rotate S_g one "click" counterclockwise. That is, let r be the rotation of angle $2\pi/(g-1)$ to the left along the axis pictured and so r has order $g-1$. Homeomorphisms of 3-manifolds map meridians to meridians. Notice for example that $r(\alpha)$ does not bound a disk in $S_g[\alpha, \beta, \alpha]$. Thus r cannot extend to a homeomorphism of $S_g[\alpha, \beta, \gamma]$ and in fact r^j for $1 \leq j < g-1$ does not extend to $S_g[\alpha, \beta, \gamma]$ but $r^{g-1} = id$ does extend to $S_g[\alpha, \beta, \gamma]$.



(a) $S_g[\alpha, \beta, \gamma]$ (b) Set $\mathcal{K} = \{\alpha, \gamma, \delta, \eta, \sigma\}$

Figure 1: Examples of extending maps

A classical theorem of Wiman states that any periodic homeomorphism of a surface of genus g has order at most $4g+2$ up to isotopy [11]. Therefore, Conjecture 1 holds when f is periodic with $k(g) = 4g+2$.

By the Nielsen-Thurston classification of homeomorphisms (see Theorem 2), every surface homeomorphism is isotopic to either a periodic, reducible, or pseudo-Anosov homeomorphism. Pseudo-Anosovs are the most interesting case of the three and should be thought of as highly mixing-up the entire surface. Consider the following example which is similar to Example 28 but gives a pseudo-Anosov map instead of a periodic one.

Example 29. Considering again the 3-manifold $S_g[\alpha, \beta, \gamma]$ from Figure 1 (a). We will construct a pseudo-Anosov of S_g as a composition of Dehn twists using the following theorem of Fathi.

Theorem 30 ([9], Theorem 0.2)**.** *Let h be a mapping class of surface a S and $\alpha_1, ..., \alpha_k$ be simple closed curves on S. Suppose that the orbits under h of the α_i are distinct and fill S. Then there exists an $n \in \mathbb{N}$ such that for every $(n_1, ..., n_k) \in \mathbb{Z}^k$ with $|n_i| \geq n$, the class $T_{\alpha_k}^{n_k} ... T_{\alpha_1}^{n_1} h$ is pseudo-Anosov.*

In order to apply Fact 15, we add an additional condition to the curves about which we will twist: they must bound disks or annuli in $r^j(S_g)$ for all $j \in \{1, 2, ..., g-1\}$ so that the Dehn twists extend to the orbit of S_g under r.

Each curve in $\mathcal{K}$ as pictured in Figure 1 (b) bounds a disk in $S_g[\alpha, \beta, \gamma]$ and moreover, each curve in $\mathcal{K}$ bounds a disk or annulus in $r^j(S_g[\alpha, \beta, \gamma])$. Note that the orbit of $\mathcal{K}$ under r fills S and so applying Theorem 0.2 of Fathi [9], there are $n_1, ..., n_5 \in \mathbb{Z}$ such that $f := T_\alpha^{n_1} \circ T_\gamma^{n_2} \circ T_\delta^{n_3} \circ T_\eta^{n_4} \circ T_\sigma^{n_5} \circ r$ is pseudo-Anosov. By our construction we have

$$\begin{aligned} f(S_g[\alpha, \beta, \gamma]) &= T_\alpha^{n_1} T_\gamma^{n_2} T_\delta^{n_3} T_\eta^{n_4} T_\sigma^{n_5} r(S_g[\alpha, \beta, \alpha]) \\ &= r(S_g[\alpha, \beta, \gamma]). \end{aligned}$$

As seen in the last example, $r^j(S_g[\alpha, \beta, \gamma]) \neq S_g[\alpha, \beta, \gamma]$ for $j < g-1$. In other words, r^j does not extend to $S_g[\alpha, \beta, \gamma]$ for $j < g-1$ which implies that for $j < g-1$, the pseudo-Anosov homeomorphism f^j does not extend to $S_g[\alpha, \beta, \gamma]$. But since $f^{g-1}(S_g[\alpha, \beta, \gamma]) = r^{g-1}(S_g[\alpha, \beta, \gamma]) = S_g[\alpha, \beta, \gamma]$, our pseudo-Anosov does extend at power $g-1$.

In our next example, we will construct a pseudo-Anosov map that requires a power to extend to a handlebody, which is interesting because we can't exploit a lack of compression disks to construct the homeomorphism.

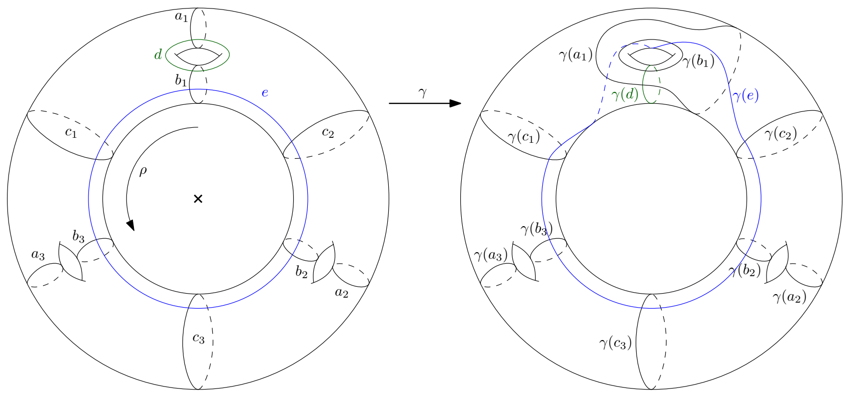


Figure 2: Handlebody $H := S_4[a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3]$, homeomorphism $\rho : S_4 \to S_4$ a $\frac{2\pi}{3}$ counterclockwise rotation about the axis, and homeomorphism $\gamma : S_4 \to S_4$.

Example 31. Consider the handlebody $H := S_4[a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3]$ as pictured in Figure 2. Let $\rho : S_4 \to S_4$ be the $\frac{2\pi}{3}$ counterclockwise rotation about the axis pictured. Let γ be the reducible homeomorphism as follows: the identity on the genus 2 component of $S_4 \backslash \{c_1, c_2\}$. On the genus 1 component of $S_4 \backslash \{c_1, c_2\}$ considered as the flat unit square torus with boundary components c_1 and c_2, γ is the transformation $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ which results in the image depicted in Figure 2. Note that ρ has order 3 and γ has order 4.

Moreover, note that $\gamma, \gamma^2, \gamma^3$ don't extend to H since for example, meridian b_1 is mapped to $\gamma(b_1)$ which does not compress in H. Composing our maps, we have homeomorphism $\rho \circ \gamma : S_4 \to S_4$ that requires power 12 to extend to H.

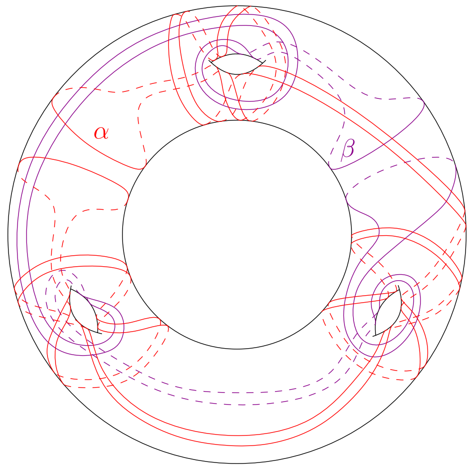


Figure 3: The homeomorphism $f := T_\alpha^{n_1} \circ T_\beta^{n_2} \circ \gamma\rho$ is pseudo-Anosov

Consider the simple closed curves α and β which fill S_4. Clearly this implies that the orbits of α and β under $\gamma\rho$ also fill S_4. Therefore, we can apply Fathi's Theorem 30, giving integers n_1 and n_2 such that $f := T_\alpha^{n_1} \circ T_\beta^{n_2} \circ \gamma\rho$ is pseudo-Anosov.

Notice that this pair of curves are band-sums of meridians of H and thus compress in H. Moreover, α and β are band-sums of meridians of $(\gamma\rho)^i(H)$ for all integers i since $\gamma\rho$ preserves the disks bounded by the c_i. Therefore, T_α and T_β extend to the handlebodies $(\gamma\rho)^i(H)$ for all integers i by Fact 15.

Therefore, $f^i(H) = (\gamma\rho)^i(H)$. Hence, f^i doesn't extend to H for $i < 12$ but f^{12} does extend to H.

Note that this example can be expended to a family of examples of genus g handlebodies H_g with a pseudo-Anosov map f_g such that f_g^i doesn't extend to H_g for $i < (g-1)4$ but $f^{(g-1)4}$ does.

4 Partial Extension

Recall from Definition 11 that we say $f : S \to S$ *partially extends* to M, a compact, orientable, and irreducible 3-manifold with some compressible boundary component

S if there is some nontrivial compression body $C \subset M$ with $\partial_+ C = S$ and a homeomorphism $\phi : C \to C$ such that $\phi|_S = f$. Let S_g be a closed surface of genus g.

We will prove Theorem 32 by constructing a family of manifolds and corresponding pseudo-Anosovs. In fact, the manifolds we construct below are compression bodies. We restate our Theorem 32 to reflect this:

Theorem 32. *For $g \in \mathbb{N}$ there is an S_{2g}-compression body C_g and a pseudo-Anosov $f_g : S_{2g} \to S_{2g}$ such that f_g^g extends to C_g and f_g^j does not partially extend to C_g for $j < g$.*

Proof. Fix $2g$, the genus of the exterior boundary component, and consider the compression body $K_1 := S_{2g}[\gamma, \alpha]$ as shown in Figure 4. We will construct a pseudo-Anosov homeomorphism $f_g : S_{2g} \to S_{2g}$ such that f_g^g extends to K_1 but f_g^j does not extend to any sub-compression body of K_1 for $j \leq g$.

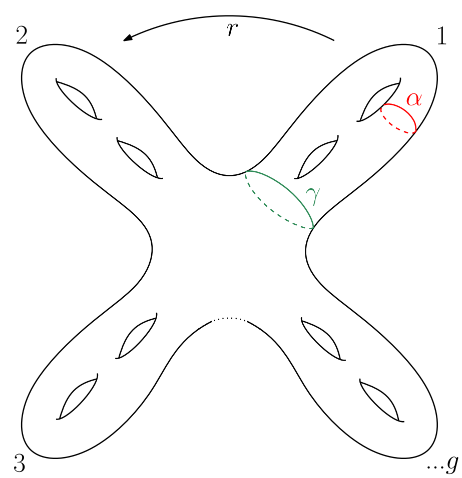


Figure 4: Compression body $K_1 := S_{2g}[\gamma, \alpha]$

Let r be the rotation of S_{2g} by $\frac{2\pi}{g}$. Define $K_j := r^{j-1}(K_1)$ for $1 < j \leq g$. Since r^g is the identity then r^g extends to K_1. Thus $r(K_i) = K_{(i+1)\text{mod}(g)}$.

Lemma 33. *Each curve in the set $\mathcal{K} = \{\sigma, \varphi, \gamma, \beta, \delta\}$ (see Figure 5) bounds a disk or annulus in $K_1, ..., K_g$.*

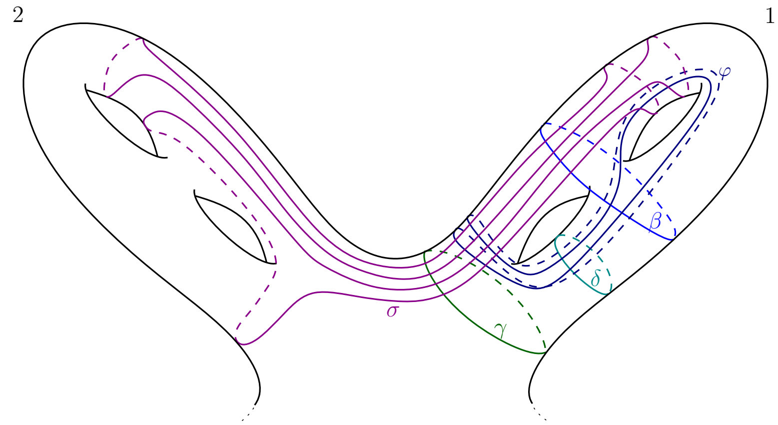


Figure 5: Set $\mathcal{K} = \{\sigma, \varphi, \gamma, \beta, \delta\}$.

Proof. First, note that every curve in $\mathcal{K}$ bounds an annulus in K_i for $i > 2$ and every curve in $\mathcal{K} \backslash \{\sigma\}$ bounds an annulus in K_2.

Clearly γ bounds a disk and δ bounds annulus in K_1. Since meridian α and curve β co-bound a pair of pants, by Fact 16, β also bounds a disk in K_1. Also by Fact 16, curve φ' bounds a disk in K_1 as seen in Figure 6. Notice that φ is a band-sum of this disk bounded by φ'. Thus φ also bounds a disk in K_1.

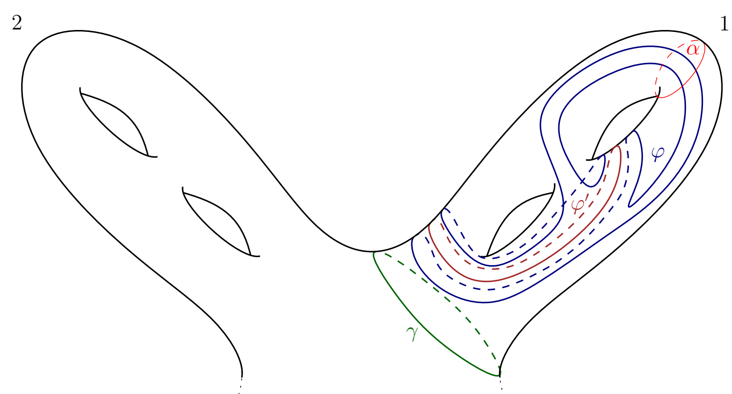


Figure 6: Curve φ, a bandsum of φ'

In Figure 7 in steps (a) through (d) we see that σ homotopes to $\tilde{\sigma_1}$ in K_1 which bounds an annulus. Hence σ bounds an annulus in K_1. In Figure 8 we see that in K_2, σ is homotopic to $\tilde{\sigma_2}$ which co-bounds a pair of pants with $r(\gamma)$ and $r(\alpha)$ which both compress in K_2. Thus σ bounds a disk in K_2.

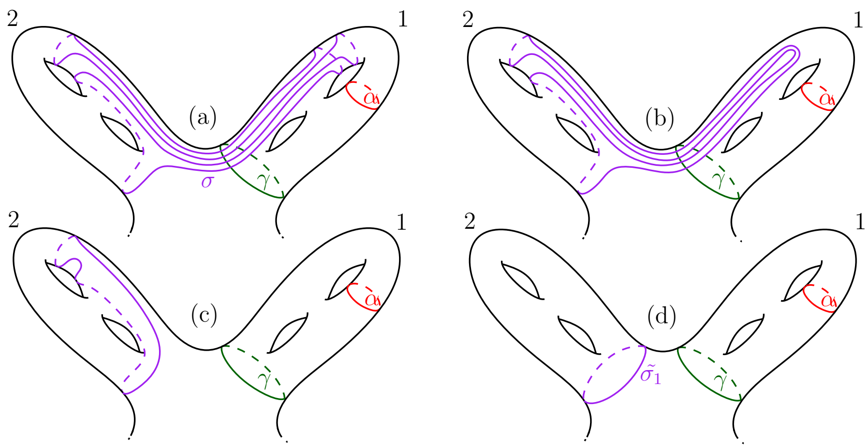


Figure 7: Curve σ bounds an annulus in K_1

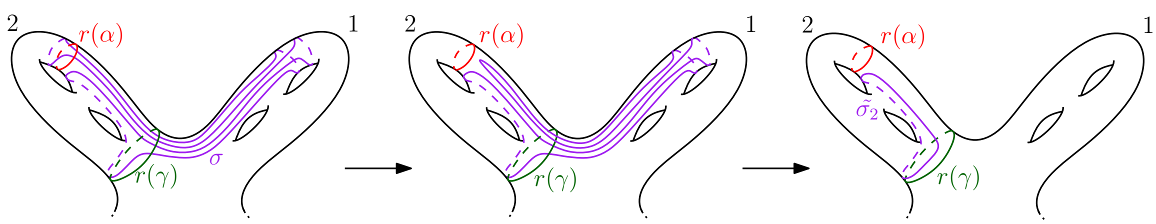


Figure 8: Curve σ bounds a disk in K_2

$\square$

We will apply the following theorem of Fathi:

Theorem ([9], Theorem 0.2)**.** *Let h be a mapping class of surface a S and $\gamma_1, ..., \gamma_k$ be simple closed curves on S. Suppose that the orbits under h of the γ_i are distinct and fill S. Then there exists an $n \in \mathbb{N}$ such that for every $(n_1, ..., n_k) \in \mathbb{Z}^k$ with $|n_i| \geq n$, the class $T_{\gamma_k}^{n_k} ... T_{\gamma_1}^{n_1} h$ is pseudo-Anosov.*

The curves $\bigcup_{i=1}^{g} r^i(\mathcal{K})$ fill S_{2g} as follows. Looking at Figure 2, we see that φ, σ, b_1, and δ fill the genus 2 component of $S_{2g} \backslash \gamma$. Also, note that the genus 0 component of $S_{2g} \backslash \bigcup_{i=0}^{g-1} r^i(\gamma)$ is filled by $\bigcup_{i=0}^{g-1} r^i(\sigma)$. Hence the orbit of $\mathcal{K}$ under r fills S_{2g}.

By Fathi's theorem there exists $n_{g_1}, \cdots, n_{g_5} \in \mathbb{Z}$ such that

$$T_{\gamma}^{n_{g1}} \circ T_{\sigma}^{n_{g2}} \circ T_{\varphi}^{n_{g3}} \circ T_{\beta}^{n_{g4}} \circ T_{\delta}^{n_{g5}} \circ r := f_g$$

is pseudo-Anosov.

Lemma 34. *The homeomorphism $f_g^g : S_{2g} \to S_{2g}$ extends to K_1 but f_g^j does not extend to K_1 for $j < g$.*

Proof. Recall that $r(K_i) = K_{(i+1)\mathrm{mod}(g)}$. By Lemma 33, every curve in $\mathcal{K}$ bounds a disk or annulus so Fact 15 implies that T_c extends to K_i for all $c \in \mathcal{K}$ and $1 \leq i \leq g$. Therefore, $f_g(K_i) = K_{(i+1)\mathrm{mod}(g)}$. Hence, $f_g^g(K_1) = K_1$ and so f_g^g extends to K_1 but f_g^j does not for $j < g$. $\square$

We must now show that f^j for $j < g$ does not *partially* extend to K_1. First, we prove two lemmas.

Lemma 35. *Every common meridian of K_i and K_j for $i \neq j$ is separating.*

Proof. Let's first consider the homology of K_i. Set $\alpha_i := r^{i-1}(\alpha)$ and $\gamma_i := r^{i-1}(\gamma)$. Hence both α_i and γ_i compress in K_i. Let D_{α_i} and D_{γ_i} denote the disks in K_i bounded by α_i and γ_i respectively. Consider the following portion of the the Mayer-Vietoris long exact sequence:

$$H_1(S_{2g} \cap (D_{\alpha_i} \sqcup D_{\gamma_i})) \xrightarrow{\Phi_i} H_1(S_{2g}) \oplus H_1(D_{\alpha_i} \sqcup D_{\gamma_i}) \xrightarrow{\Psi_i} H_1(K_i).$$

Note that $H_1(D_{\alpha_i} \sqcup D_{\gamma_i}) = 0$ giving $H_1(S_{2g}) \xrightarrow{\Psi_i} H_1(K_i)$. Also, $S_{2g} \cap (D_{\alpha_i} \sqcup D_{\gamma_i}) =$

$\{\alpha_i, \gamma_i\}$. By exactness of the Mayer-Vietoris sequence, $\text{Im}(\Phi_i) = \Phi_i(\{\alpha_i, \gamma_i\}) = \ker(\Psi_i)$. Moreover, γ_i is separating and hence $\Phi_i([\gamma_i]) = 0$ in $H_1(S_{2g})$. Therefore, $\ker(\Psi_i)$ is generated by α_i.

Note that the $[\alpha_i]$ for $i \in \{1, 2, ..., g\}$ form a subset of a basis for $H_1(S_{2g})$. If ω is a simple closed curve in S_{2g} and a meridian of both K_i and K_j then $[\omega] \in \ker(\Psi_i) \cap \ker(\Psi_j) \subset H_1(S_{2g})$. So $[\omega] \in \langle [\alpha_i] \rangle \cap \langle [\alpha_j] \rangle$ implying $[\omega] = 0$ in $H_1(S_{2g})$. Therefore, ω is a separating curve of S_{2g}. □

Lemma 36. *Every S-compression body $D \subset K_i \cap K_j$ is small.*

Proof. For all i, the interior boundary components of K_i are homeomorphic to $S_1 \sqcup S_{2g-2}$. Applying Proposition 20 for all i, the height of K_i is

$$\begin{aligned} \mathfrak{h}(K_i) &= (2(2g) - 1) - [2(1) - 1] - [2(2g-2) - 1] \\ &= 4g - 1 - 1 - (4g - 5) \\ &= 3. \end{aligned}$$

Let D be an S-compression body with $D \subset K_i \cap K_j$. Then $\mathfrak{h}(D) < 3$.

If $\mathfrak{h}(D) = 1$ then $S_{2g} \times [0, 1] = C_0 \subsetneq C_1 = D$ is a sequence of minimal compressions for D. By Proposition 18, since D is not a solid torus, D is small.

If $\mathfrak{h}(D) = 2$ then $S_{2g} \times [0, 1] = C_0 \subsetneq C_1 \subsetneq C_2 = D$ is a sequence of minimal compressions for D. By Lemma 35, both compressions are along separating curves. Therefore, the interior boundary of D is $F_1 \sqcup F_2 \sqcup F_3$ with F_i a closed surface where $g(F_1) + g(F_2) + g(F_3) = 2g$ and $g(F_\ell) > 0$ for $\ell \in \{1, 2, 3\}$. Recall that the interior boundary components of K_i are $S_{2g-2} \sqcup S_1$. Then $D \subset K_i$ implies that $g(F_\ell) \geq 2g - 2$ for some $\ell \in \{1, 2, 3\}$. Without loss of generality, say $g(F_1) \geq 2g - 2$. This forces

$g(F_1) = 2g - 2$, $g(F_2) = 1$ and $g(F_3) = 1$.

There is some sequence of minimal compressions with $S_{2g} \times [0,1] = C_0 \subsetneq C_1 \subsetneq C_2 = D \subsetneq C_3 = K_i$. Thus we must compress a curve in F_2 or F_3 to obtain K_i (to preserve the genus $2g-2$ interior boundary component). Likewise, we must compress a curve in F_2 or F_3 to obtain K_j. Therefore, K_i and K_j share F_1, the same genus $2g-2$ interior boundary component. But this is false by construction of K_i and K_j. Therefore, $\mathfrak{h}(D) \neq 2$.

Thus, D is a small compression body. $\square$

To prove Theorem 32 recall we must show that for all $j < g$, f_g^j does not *partially* extend to K_1. Assume for sake of contradiction that there is some S_{2g}-compression body $K' \subsetneq K_1$ with $\ell < g$ such that f_g^ℓ extends to K'. Then $f_g^\ell(K') = K'$. Since $f_g^\ell(K_1) = K_{\ell+1}$, this implies that $K' \subset K_{\ell+1}$. Thus $K' \subset K_1 \cap K_{\ell+1}$. By Lemma 36, K' must be a small S_{2g}-compression body and by Lemma 35, K' has exactly one meridian, a, a separating curve. Then f_g^ℓ must map the disk bounded by a to itself. This implies that $f_g^\ell(a) = a$ which contradicts the fact that f_g^ℓ is pseudo-Anosov.

Therefore, f^g extends to K_1 and no lesser power partially extends to K_1. As the genus to goes to infinity, the power required for f to extend to the corresponding compression body K_1 also goes to infinity. $\square$

5 Reducing to irreducibility and pure homeos

We will reduce Conjecture 1 to the case that M is an irreducible manifold. Recall that a 3-manifold M is *irreducible* if every embedded S^2 bounds a 3-ball.

Theorem 37. *If Conjecture 1 is true for pure homeomorphisms on the boundary of orientable irreducible 3-manifolds, it is true for homeomorphisms on the boundary of orientable 3-manifolds in general.*

Proof. Let M be a 3 manifold with boundary and let $B = S_{g_1} \sqcup \cdots \sqcup S_{g_\ell}$ be a disjoint union of a subset of the components of ∂M, closed surfaces S_{g_i} of genus g_i.

By the Prime Decomposition Theorem [16], there is a unique decomposition of M into a connected sum of prime manifolds, that is a connected sum of irreducible manifolds and S^2 bundles over S^1. Here, we take all 3-cells to be interior to M and therefore preserve the components of B. Since S^2 bundles over S^1 are closed manifolds, it follows that a prime manifold with a boundary component S_{g_i} will be irreducible and in particular, for $g_j = 0$, we have that S_{g_j} is isotopic to S^2, in the decomposition giving that S_{g_j} will be the surface of a 3-ball. Let $M = M_{\mathrm{irr}_1} \# \cdots \# M_{\mathrm{irr}_\ell} \# M_{\mathrm{irr}_{\ell+1}} \# \cdots \# M_{\mathrm{irr}_{\ell+m}} \# M_{\mathrm{bun}_1} \# \cdots \# M_{\mathrm{bun}_n}$ where the M_{irr_j} for $1 \leq j \leq \ell$ are 3-balls, the M_{irr_j} for $\ell + 1 \leq j \leq \ell + m$ are the irreducible manifolds that are not 3-balls, and the M_{bun_j} are the S^2 bundles over S_1.

Proposition 38. *For prime decomposition*

$$M = M_{irr_1} \# \cdots \# M_{irr_\ell} \# M_{irr_{\ell+1}} \# \cdots \# M_{irr_{\ell+m}} \# M_{bun_1} \# \cdots \# M_{bun_n}$$

as above and an orientation preserving homeomorphism $h : B \to B$ which fixes all non-spherical components of B with the property that h that extends to each M_{irr_j} adjacent to a component of B, we have that h extends to M.

Proof. We will define a homeomorphism $H : M \to M$ such that $H|_B = h$.

Set $B_h \subset B$ to be the disjoint union of non-spherical components of B (which are all fixed by h by assumption). Since h is orientation preserving, we can isotope h so that h fixes a disk on each $S_{g_i} \subset B_h$. By assumption, for M_{irr_j} adjacent to some S_{g_i}, there exists a homeomorphism $H_j : M_{\mathrm{irr}_j} \to M_{\mathrm{irr}_j}$ such that $H_j|_{S_{g_i}} = h$. Since M_{irr_j} is orientable, there is a collar neighborhood $S_{g_i} \times I$ embedded in M_{irr_j} where we can isotope H_j to be h on the fibers of the neighborhood. So for each M_{irr_j} adjacent to a component of B_h, we have a homeomorphism H_j that fixes a 3-ball b_{irr_j}.

Now, consider the M_{irr_j} that are not adjacent to a component of B. For such j, define $H_j : M_{\mathrm{irr}_j} \to M_{\mathrm{irr}_j}$ to be the identity. Likewise, for the M_{bun_j} which are closed and thus not adjacent to any component of B, define $H'_j : M_{\mathrm{bun}_j} \to M_{\mathrm{bun}_j}$ to be the identity.

Now consider the components of $B \backslash B_h = \{P_1, ..., P_d\}$ which are S^2 boundary components of M. In the prime decomposition each P_i is a boundary component of some 3-ball M_{irr_j} for $1 \leq j \leq \ell$. Isotope h so that it is the identity on each S^2 composed with the permutation σ where $h(P_1, ..., P_d) = (P_{\sigma(1)}, ..., P_{\sigma(d)})$.

Let D^3 be a 3-ball with $\{p_1, ..., p_d\}$ disjoint marked points. There is an orientation preserving self-homeomorphism $g : D^3 \to D^3$ such that $g|_{\partial D^3} = id$ and $g(p_i) = p_{\sigma(i)}$. Isotope g so that it permutes $\{b_1, ..., b_d\}$, small disjoint embedded 3-balls with b_i centered at p_i.

Now, connect sum the $M_{\mathrm{irr}_1} \# \cdots \# M_{\mathrm{irr}_\ell} \# M_{\mathrm{irr}_{\ell+1}} \# \cdots \# M_{\mathrm{irr}_{\ell+m}} \# M_{\mathrm{bun}_1} \# \cdots \# M_{\mathrm{bun}_n}$ in the following way. For the M_{irr_j} with $\ell + 1 \leq j \leq \ell + m$ adjacent to a boundary component of B_h, we take the deleted 3-ball to be embedded in b_{irr_j}. Next, for each P_i with corresponding M_{irr_j} for $1 \leq j \leq \ell$, connect sum the M_{irr_j} to D^3 along a deleted 3-ball contained in b_i. Then connect sum the D^3 via some embedded 3-ball in b_{irr_j} for some j. Finally, connect sum the rest inside some b_{irr_j}.

Notice that the induced map $H : M \to M$ from the H_j, identity map and g on the appropriate components of the decomposition restricts to h on the boundary of M. $\square$

In the above set-up, set $B_i := B \cap \partial M_{\mathrm{irr}_i}$ for $\ell + 1 \leq i \leq \ell + m$.

Corollary 39. *If Conjecture 1 holds for a boundary homeomorphism that preserves the positive genus boundary components for each irreducible component of M adjacent* *to B in the prime decomposition of M, then the conjecture holds for M. Precisely, let $f : B \to B$ be an orientation preserving homeomorphism that preserves all positive genus components of B and has a power that extends to M. If for each M_{irr_i} adjacent to a component of B, there is a bounded power $\ell_i \leq k(G(B_i))$ such that $(f|_{B_i})^{\ell_i}$ extends to M_{irr_i} then there is some $\ell \leq [k(G(B))]^{G(B)}$ such that f^ℓ extends to M.*

Proof. By assumption, for each M_{irr_i} adjacent to a component of B, there is a bounded power $\ell_i \leq k(G(B_i))$ such that $(f|_{B_i})^{\ell_i}$ extends to M_{irr_i}. Let ℓ' be the product of all the ℓ_i. Then $f^{\ell'}$ extends to all the M_{irr_i} and is orientation preserving. And $\ell' \leq \prod_{i=\ell+1}^{\ell+m} k(G(B_i)) \leq k(G(B))^m \leq [k(G(B))]^{G(B)}$. □

Now, we use the fact that a bounded power of any homeomorphism will preserve the positive genus components of B and in fact a bounded power f is a pure homeomorphism (see Corollary 3 and Definitions 4 and 5 and Corollary 7).

Fact 40. *Let M a 3-manifold with boundary and let $B \subset \partial M$ be a disjoint union of components of ∂M. Let $f : B \to B$ be a homeomorphism. If the pure map f^n has a bounded power $\ell \leq k(G(B))$ that extends to M then f has a bounded power that extends to M.*

Proof. From Corollary 39, it suffices to prove for irreducible M. If M is a 3-ball, then the mapping class group of ∂M is trivial and hence f extends to M. Otherwise, the genus of all boundary components of M is positive and so by Corollary 7, there is a bounded $n \leq 12G(B)!(3G(B)-3)!$ such that f^n fixes each component of B and is pure on each component (and hence orientation preserving). So if $(f^n)^\ell$ extends to M for $\ell \leq k(G(B))$ then for some bounded $m \leq 12G(B)!(3G(B)-3)!k(G(B))$, f^m extends to M. □

Thus we have reduced to the case where we need to prove that Conjecture 1 holds when M is irreducible and f is pure. $\square$

6 Incompressible case

With the goal of proving Conjecture 1, we can employ the following theorem of Bonahon now that we have reduced to the case of irreducible manifolds.

Theorem 41 ([13], Theorem 2.1)**.** *Let M be an irreducible 3-manifold. There exists $V \subset M$ a disjoint union of compression bodies for ∂M, unique up to isotopy, called the characteristic compression body, such that $M - V$ is boundary incompressible and irreducible.*

If Conjecture 1 is true for both compression bodies and for boundary-incompressible manifolds, then the conjecture holds for irreducible manifolds in general. We will address this in detail in Section 7. In this section we show that the conjecture is true if M is *boundary-incompressible*, that is if any simple closed curve in ∂M does bound an embedded disk in M.

Theorem 42. *Let M be an irreducible, orientable boundary-incompressible manifold. If B is a union of a subset of the boundary components of M with pure homeomorphism $f : B \to B$ such that f^k extends to M, then there is some $\ell \leq [210(B(G)-1)]^{2G(B)-2}$ such that f^ℓ extends to M.*

We will use the Characteristic torus/annulus decomposition as formulated by Bonahon which is attributed to K. Johannson [15], W. Jaco and P. Shalen [20] [21], and Bonahon and Siebenmann [19].

Theorem 43 ([18], Theorem 3.8 Characteristic torus/annulus decomposition)**.** *Let M be an orientable 3-manifold of finite type with boundary, which contains no essential*

2-sphere or compression disk for its boundary. Then, up to isotopy, there is a unique compact 2-dimensional submanifold F of M such that :

1. *Every component F_1 of F is 2-sided (i.e. its normal bundle is trivial in M) and is either an essential 2-torus or an annulus with $F_1 \cap \partial M = \partial F_1$.*
2. *For every component W of $M - F$, either*
   (a) *W contains no essential embedded 2-torus or annulus,*
   (b) *W admits a Seifert fibration for which $W \cap \partial M$ is a union of fibers, or*
   (c) *W admits the structure of a $\mathbb{D}^1$-bundle over a surface of finite type such that the corresponding $\partial\mathbb{D}^1$-bundle is equal to $W \cap \partial M$.*
3. *Property (2) fails when any component of F is removed.*

In addition, note that the ends of a Seifert fibered component W of $M - F$ all are of the toric type, and can be delimited by 2-tori in W;

Remark 44. Recall in general, a map $f : (S, \partial S) \to (M, \partial M)$ is essential if (1) it cannot be homotoped (rel ∂S) to $f' : S \to \partial M$ and (2) the induced homomorphism $f_* : \pi_1(S) \to \pi_1(M)$ is injective. (see [28]) In regards to (a) above, W contains no essential annuli in the sense that $W - F$ is pared (see below). An annulus $(A, \partial A) \hookrightarrow (W, \partial W)$ that can be isotoped into ∂W such that A must intersect F it is considered essential.

Our strategy is to show that Theorem 42 holds when M is replaced with a component of $M - F$ and then see that this gives us the theorem for M as well.

Proof. Let M be an irreducible, orientable boundary-incompressible manifold and B a disjoint union of a subset of the boundary components of M with a pure homeomorphism $f : B \to B$ which preserves the components of B such that there is some k such that f^k extends to M.

Fact 45. *A pure homeomorphism f preserves the components of ∂F.*

Proof. Since F is unique up to isotopy, f^k preserves $\partial M \cap F$. Since f is also pure, then f must preserve $\partial M \cap F$. Suppose that c is a curve of ∂F. Then $f(c) \subset \partial F$. If c is contained in some R_i such that $f|_{R_i} = id$ then $f(c) = c$. (see Corollary 3). If c is a core curve of an annulus R_j such that $f|_{R_j} = T_c^n$ then $f(c) = c$ since f is pure. Suppose that γ is a curve in the canonical reducing system of f and let R_γ be a regular neighborhood of γ. Note that no finite set of arcs transverse to γ in R_γ with geometric intersection 1 with γ are preserved by T_γ^i. Hence, f preserves ∂F only if $\partial F \cap \gamma = \emptyset$. Suppose R_j is a region on which f is pseudo-Anosov. Since $f|_{R_j}$ preserves no finite set of arcs then it follows that $\partial F \cap R_j = \emptyset$. Thus, f preserves each component of ∂F. □

Now we will consider the different components of $M - F$. We start with components that are adjacent to B.

6.1 Interval bundles

First, suppose W a component of $M - F$ satisfies (c) above. That is W admits the structure of a $\mathbb{D}^1$-bundle over a surface of finite type $S = S_{g,b}$ such that the corresponding $\partial\mathbb{D}^1$-bundle is equal to $W \cap \partial M$.

Proposition 46. *Let W be a component of $M - F$ satisfying (c) in Theorem 43, and let $f : B \to B$ be a pure homeomorphism such that at least one component of $W \cap \partial M$ is contained in B. Suppose that f^k extends to M and so $f^k|_{B\cap W}$ extends to W. Then $f|_{B\cap W}$ extends to W.*

Proof. Suppose W a component of $M - F$ admits the structure of a $\mathbb{D}^1$-bundle over a surface of finite type $S = S_{g,b}$ such that the corresponding $\partial\mathbb{D}^1$-bundle is equal to

$W \cap \partial M$. If exactly one of the $S \times \{0\}$, $S \times \{1\}$ is contained in B, say $S \times \{0\}$, then f extends to W by defining $F : W \to W$ to be f on each of the fibers of W.

Suppose $(S \times \{0\} \cup S \times \{1\}) \subset B$. Since f is pure and fixes $F \cap B$, we know that f sends $S \times \{0\}$ to itself and likewise $S \times \{1\}$. For every simple closed curve $\alpha \subset S$, we have $\alpha \times \{0\}$ is isotopic to $\alpha \times \{1\}$ in W and moreover, since f^k extends, $f^k(\alpha \times \{0\})$ is isotopic to $f^k(\alpha \times \{1\})$. Thus, $f^k|_{S \times \{0\}} = f^k|_{S \times \{1\}}$ up to isotopy. Since f is pure, f^k is pure. In the notation of Corollary 3, suppose R_i is a component of S where f^k is the identity on $R_i \times \{0\}$ and $R_i \times \{1\}$. Then f is also the identity on R_i. Thus f extends to $R_i \times [0, 1]$. Now suppose R_i is a component where f^k is pseudo-Anosov on $R_i \times \{0\}$ and $R_i \times \{1\}$ with dilatation λ and stable lamination L_s and unstable lamination L_u. Then f is pseudo-Anosov with dilatation $\lambda^{1/k}$ and stable lamination L_s and unstable lamination L_u for both $R_i \times \{0\}$ and $R_i \times \{1\}$ and so $f|_{R_i \times \{0\}}$ is isotopic to $f|_{R_i \times \{1\}}$. Hence f extends to $R_i \times [0, 1]$. Finally suppose R_i is an annular component of S with core curve a where f^k is a power of a Dehn Twist, $T^p_{a \times \{0\}}$ on $R_i \times \{0\}$ and $T^p_{a \times \{1\}}$ on $R_i \times \{1\}$. Since f is also pure, then f must be a power of a Dehn twist about a, T^j_a such that $jk = p$. It follows that f extends to $R_i \times [0, 1]$. It follows that f extends to W. □

6.2 Seifert Fibered Spaces

Next, we consider the case when W a component of $M - F$ is a Seifert fibered space. Recall that all boundary components of an orientable Seifert Fibered Space are tori.

Proposition 47. *Let W is a component of $M - F$ satisfying (b) in Theorem 43 that is an orientable Seifert Fibered Space, and let $f : B \to B$ be a pure homeomorphism such that at least one component of $W \cap \partial M$ is contained in B. Suppose that f^k extends to M and so $f^k|_{B \cap W}$ extends to W. Then $f|_{B \cap W}$ extends to W.*

Proof. Suppose that W is a Seifert fibered space. We break this up in to 4 cases: W is the solid torus, $T^2 \times I$, a twisted I bundle over a Klein bottle, or a Seifert fibered space that is none of the former cases.

Claim 48. *The proposition holds when W is not homeomorphic to a solid torus, $T^2 \times I$ nor a twisted I bundle over a Klein bottle.*

Proof. Assume that W is not homeomorphic to a solid torus, $T^2 \times I$ nor a twisted I bundle over a Klein bottle. We start with a theorem.

Theorem 49. *Suppose that M is an oriented Seifert fibered space with boundary and that M is not homeomorphic to a solid torus, $S^1 \times S^1 \times I$ or a twisted I-bundle over the Klein bottle. Let $\pi : M \longrightarrow X$ be a Seifert fibration, where X is a 2-orbifold with boundary components $b_1, \ldots, b_m$. Set $B_i := \pi^{-1}(b_i)$, fix $p_i \in b_i$ and set $h_i = \pi^{-1}(p_i) \subset B_i$.*

If $\phi : M \longrightarrow M$ is an orientation preserving homeomorphism that leaves each component of ∂M invariant, then for each i, the restriction $\phi|_{B_i}$ is isotopic to $T_{h_i}^{n_i}$, the n_i^{th} power of a Dehn twist of B_i around the curve h_i, and $\sum_i n_i = 0$.

Here, when defining the Dehn twists, we are regarding $B_i \subset \partial M$ with the boundary orientation coming from the specified orientation of M. If we switch the orientation on M, the powers n_i all negate, but their sum is still zero.

Proof. By Theorem VI.18 in [27], we may assume after isotopy that ϕ is fiber preserving. Hence, each restriction $\phi|_{B_i}$ is a power $T_{h_i}^{n_i}$ of a Dehn twist around the fiber h_i, and the goal is to prove $\sum_i n_i = 0$.

Let's first assume that X is oriented, and let $x_j,\ j \in J$ be the singular points. Pick a collection of disjoint disks D_j, where $x_j \in D_j$, and and let

$$X_0 = X \setminus \cup_j D_j, \quad M_0 = \pi^{-1}(X_0).$$

Since M and X are both orientable, the bundle $\pi : M_0 \longrightarrow X_0$ restricts to a trivial bundle over every loop in X_0. And since X_0 is homotopy equivalent to a wedge of circles, it follows that $\pi : M_0 \longrightarrow X_0$ is trivial. So, we can fix a section

$$X_0 \hookrightarrow M_0, \quad x \longmapsto x^* \tag{1}$$

of π. Let $q_j := \partial D_j$, equipped with the boundary orientation. Then

$$H_1(M, \mathbb{Z}) = \left\langle a_k^*, b_k^*, q_j^*, b_i^*, h \;\middle|\; \alpha_j q_j^* + \beta_j h = 0 \;\forall j,\; \sum_j q_j^* + \sum_i b_i^* = 0 \right\rangle \tag{2}$$

Here, a_k, b_k are pairs of simple closed curves on X_0, where the number of pairs is the genus of X, the α_j, β_j are pairs of relatively prime integers determining the types of the singular fibers of M, and h is the common homology class of all the regular fibers, e.g. the h_i. In each case, $(\cdot)^*$ denotes the image of $(\cdot)$ under the map from Equation (1). See Corollary 6.2 of [26] for details in the case that X has no boundary; the same arguments apply in our case.

Since ϕ is fiber preserving, it induces a homeomorphism $\bar{\phi} : X \longrightarrow X$ with $\pi \circ \phi = \bar{\phi} \circ \pi$. Note that $\bar{\phi}$ permutes the singular set of X, which is finite. To prove the theorem, it suffices to work with a nontrivial power of ϕ: after all, passing to the k^{th} power just scales all the n_i by k, and $\sum_i k n_i = 0$ if and only if $\sum_i n_i = 0$. So, after passing to a power of ϕ, we may assume $\bar{\phi}$ fixes all the singular points of X. After isotopy, we can also assume that $\bar{\phi}$ leaves invariant each disk D_j. So if $Q_j := \pi^{-1}(q_j) \subset M$, then ϕ restricts to a homeomorphism of each Q_j. The map $\phi|_{Q_j}$ fixes the isotopy class in Q_j of the π-fibers, but it also fixes the isotopy class of curves on Q_j that bound disks in $\pi^{-1}(D_j)$. These curves are linearly independent in $H_1(Q_j, \mathbb{Z})$, since the fibers of a fibered solid torus are never meridians. Hence, $\phi|_{Q_j}$

is isotopic to the identity. Since $q_j^* \subset Q_j$, it follows that

$$[\phi(q_j^*)] = [q_j^*] \in H_1(M, \mathbb{Z}), \quad \forall j.$$

So, using the last relation in Equation (2), we get that in homology,

$$\sum_j [q_j^*] + \sum_i [b_i] = 0 = (\phi_*) \left[\sum_j q_j^* + \sum_i b_i^* \right] = \sum_j [q_j^*] + \sum_i [\phi \circ b_i]. \tag{3}$$

But since ϕ restricts to $T_{h_i}^{n_i}$ on the boundary component B_i, Equation (3) implies

$$\sum_i [b_i] = \sum_i [\phi \circ b_i] = \sum_i n_i[h_i] + [b_i] = \sum_i n_i[h] + [b_i],$$

so as $[h]$ has infinite order in $H_1(M, \mathbb{Z})$, we must have $\sum_i n_i = 0$.

Assume now that X is nonorientable. Again, we may assume after isotopy that ϕ is fiber preserving, and hence that Consider the degree 2 cover

$$\hat{M} \longrightarrow M$$

corresponding to the subgroup of $\pi_1 M$ consisting of loops γ such that $\pi \circ \gamma$ is an orientable loop in X. Then $\hat{M}$ is also a Seifert fibered space and there is a diagram

$$\begin{array}{ccc} \hat{M} & \longrightarrow & M \\ \downarrow{\scriptstyle \hat{\pi}} & & \downarrow{\scriptstyle \pi} \\ \hat{X} & \longrightarrow & X \end{array}$$

where the vertical arrows are Seifert fibrations and the bottom arrow is the orientation double cover of X. Since ϕ descends to a homeomorphism $\bar{\phi}$ of X, the subgroup described above is clearly preserved by ϕ, and hence ϕ lifts to a fiber preserving homeomorphism

$$\hat{\phi} : \hat{M} \longrightarrow \hat{M}.$$

Since the boundary components of X are orientable curves, each component $B_i \subset \partial M$ is covered homeomorphically by a pair of components of $\partial\hat{M}$. On each of these components, the map $\hat{\phi}$ acts as $T_{\hat{h}_i}^{n_i}$, where here $\hat{h}_i$ covers h_i homeomorphically. So, by the orientable case above, we have

$$\sum_i 2n_i = 0, \implies \sum_i n_i = 0. \quad \square$$

Let $\pi : W \longrightarrow X$ be a Seifert fibration, where X is a 2-orbifold with boundary components $b_1, \ldots, b_m$. Set $B_i := \pi^{-1}(b_i)$, fix $p_i \in b_i$ and set $h_i = \pi^{-1}(p_i) \subset B_i$.

Since $f^k|_{B \cap W}$ extends to W, we have that f^k restricted to a boundary component B_i of W must be a power of a Dehn twist $T_{h_i}^{n_i}$ with $\sum_{i=1}^m n_i = 0$.

Suppose W has only one boundary component. Then by the proposition, $n_1 = 0$ and so f^k is the identity on the boundary. Hence f has finite order. Since f is pure then $f = id$ and so $f|_{B \cap W}$ extends to W.

Suppose W has more than one boundary component. In general, because f^k is a power of a Dehn twist, $f|_{B_i} = T_{h_i}^{\ell_i}$ such that $k\ell_i = n_i$ for $B_i \subset B$ and so $\sum_i \ell_i = 0$. Thus it is possible that f might be a restriction of some map on W. In fact we will construct a map $F : M \to M$ such that $F|_{B_i} = f_{B_i}$ for all $B_i \subset B$.

We can isotope the base orbifold X so that it has the form pictured in Figure 9 where all the handles, möbius bands and singular points are outside a regular neighborhood of the b_i. Therefore, $Y \subset X$ which is shaded red in the figure is orientable.

Fact 50. *The bundle over Y in M is trivial. That is $\tilde{Y} := \pi^{-1}(Y) \cong Y \times S^1$.*

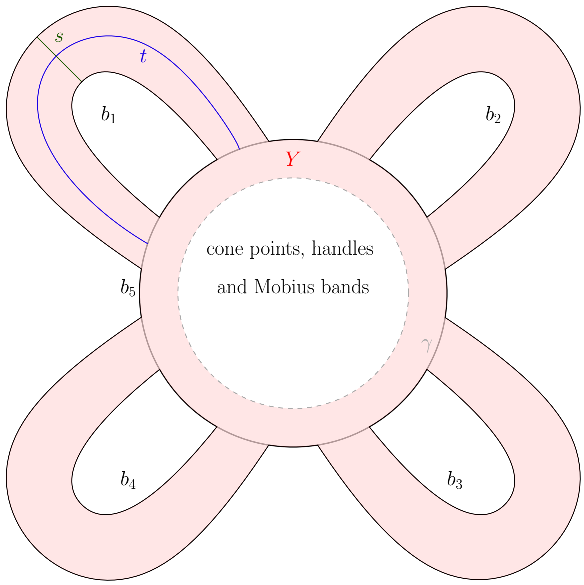


Figure 9: Orbifold X with 5 boundary components and $Y \subset X$ orientable.

Proof. Note that any S^1 bundle over S^1 is homeomorphic to either the torus or the Klein bottle. Since both Y and M are orientable, then $\tilde{Y} := \pi^{-1}(Y)$ must be orientable. Thus for any curve γ in Y, $\pi^{-1}(\gamma)$ must be homeomorphic to the torus. Note that Y deformation retracts to a wedge of S^1's. Therefore, $\tilde{Y} \cong Y \times S^1$. $\square$

Fix $1 \leq i < m$. Consider the strip $R_i \cong I \times I$ bounded by an arc of b_i and an arc of b_m as shown in Figure 9 where $\{0\} \times I \subset b_m$ and $\{1\} \times I \subset b_i$. Set $M' := \pi^{-1}(X - (\cup_i R_i))$. Then for our desired $F : M \to M$, set $F|_{M'} = id$.

On the strip R_i, define $F|_{\pi^{-1}(R_i)} : I \times I \times S^1 \to I \times I \times S^1$ such that $(s, t, p) \mapsto (s, t, r_{2\pi\ell_i t}(p))$ where r_t is rotation by p so that $F|_{\pi^{-1}(b_i)=B_i} = T_{h_i}^{\ell_i}$.

Note that $\pi^{-1}(\{0\} \times I)$ is an annulus contained in B_m and $F|_{\pi^{-1}(\{0\}\times I)} = T_{h_i}^{-\ell_i}$ for all i. It follows that $F|_{B_m} = T_{h_m}^{\left(\sum_{i=1}^{m-1} -\ell_i\right)} = f|_{B_m}$. Hence we have proven the claim and $f|_{W \cap B}$ extends to W.

$\square$

Claim 51. *If W is a solid torus with $f : \partial W \to \partial W$ such that f^k extends to W then*

f does as well.

Proof. Now suppose that W is a solid torus. By assumption, $\partial W \subset B \subset \partial M$. Therefore, M is a solid torus. (We will address the case that $\partial W \subset F$ below).

Recall that $H_1(T^2, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ and that the orientation preserving homeomorphisms of T^2 up to isotopy as a group are isomorphic to $SL(2, \mathbb{Z})$ where a class $[g] \in \mathrm{Mod}^+(T^2)$ corresponds to $g_* : H_1(T^2, \mathbb{Z}) \to H_1(T^2, \mathbb{Z})$. Let $H_1(T^2, \mathbb{Z}) = \langle [\ell], [m] \rangle$ where we make the following identifications: $[\ell] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $[m] = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Let h be a framing $h : S^1 \times D^2 \to M$ of the solid torus with $h(\{1\} \times \partial D^2) = [m]$ and $h(S^1 \times \{1\}) = [\ell]$. Notice here that $[m]$ is the boundary of a disk.

Since f^k extends to M then $f^k([m])$ also bounds a disk and so $f^k([m]) = [m]$. So for $(f^k)_* = \begin{bmatrix} a & b \\ c & f \end{bmatrix} \in SL(2, \mathbb{Z})$ we have

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \pm \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and so $b = 0$ and $d = \pm 1$ giving $(f^k)_* = \begin{bmatrix} a & 0 \\ c & \pm 1 \end{bmatrix}$

Since $(f^k)_* \in SL(2, \mathbb{Z})$, then $\det((f^k)_*) = 1$. Hence $a = \pm 1$. Thus $(f^k)_* = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$ or $(f^k)_* = \begin{bmatrix} -1 & 0 \\ c & -1 \end{bmatrix}$ for some $c \in \mathbb{Z}$. Recall the following fact.

Fact 52. *If $[\sigma] \in Mod^+(T^2)$ with corresponding matrix $A_\sigma \in SL(2, \mathbb{Z})$ with $|tr(A_\sigma)| = 2$ then $[\sigma]$ is a power of a Dehn twist. If $|tr(A_\sigma)| < 2$ then $[\sigma]$ is periodic. If $|tr(A_\sigma)| > 2$, $[\sigma]$ is Anosov.*

Since $f_* \in SL(2, \mathbb{Z})$, if $(f^k)_* = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$ then $f_* = \begin{bmatrix} 1 & 0 \\ \ell & 1 \end{bmatrix}$ with $\ell k = c$ and if $(f^k)_* = \begin{bmatrix} -1 & 0 \\ c & -1 \end{bmatrix}$ then f must be odd and $f_* = \begin{bmatrix} -1 & 0 \\ \ell & -1 \end{bmatrix}$ with $-k\ell = c$.

Either way, f is a power of a Dehn twist that fixes meridian m and so f extends to M. $\square$

Claim 53. *If $W = T^2 \times I$ and $f : B \to B$ is a pure homeomorphism such that at least one component of ∂W is contained in B with $f^k|_{B\cap W}$ extending to W, then $f|_{B\cap W}$ extends to W.*

Proof. If exactly one of the $T^2 \times \{0\}$, $T^2 \times \{1\}$ is contained in B, say $T^2 \times \{0\}$, then f extends to W by defining $F : W \to W$ to be f on each of the fibers of W. When we consider the boundary orientation coming from the specified orientation of W, then $F|_{T^2\times\{0\}} = f$ and $F|_{T^2\times\{1\}} = -f$.

Suppose $(T^1 \times \{0\} \cup T^1 \times \{1\}) \subset B$. By assumption, f fixes the components of B and so f sends $T^2 \times \{0\}$ to itself and likewise $T^2 \times \{1\}$.

Let α be an essential simple closed curve in $T^2 \times \{0\}$. Then $\alpha \times \{0\}$ is isotopic to $\alpha \times \{1\}$ in W. Since f^k extends, $f^k(\alpha \times \{0\})$ is isotopic to $f^k(\alpha \times \{1\})$. Thus, $f^k|_{T^2\times\{0\}} = f^k|_{T^2\times\{1\}}$ up to isotopy. Note the orientation of each of the fibers is the same, we are not considering orientation on $T^2 \times \{1\}$ as the boundary orientation from W.

If f^k is periodic, then f is periodic with order $j \leq 4$. So $f^j = id$ extends to W.

If f^k is a power of a Dehn twist, then as we saw above, f must also be a power of a Dehn twist–completely determined by f^k.

If f^k is Anosov then A_{f^k} has two real positive eigenvalues λ and $1/\lambda$ and corresponding eigenspaces in $\mathbb{R}^2$ where one is stretched and one is contracted by a factor of λ. It follows that A_f must have the same eigenspaces but stretch and contract by a factor of $\lambda^{1/k}$.

In the above to cases, we can conclude that there is a unique homeomorphism f' up to isotopy such that $f'^k = f^k$. Thus $f|_{T^2\times\{0\}}$ is isotopic to $f|_{T^2\times\{1\}}$ and thus extends to W. $\square$

Claim 54. *Suppose W is a twisted I-bundle over the Klein bottle and we have a pure homeomorphism $f : \partial W \to \partial W$ such that f^k extends to W. Then f extends to W.*

Proof. Finally, we consider the case where W is a twisted I-bundle over the Klein bottle which can be described in the following way. Set $W = T^2 \times [0,1]/(x,1) \sim (\sigma(x),1)$ such that $\sigma : T^2 \to T^2$ is an orientation reversing fixed point free involution. Note that $\partial W = T^2 \times \{0\}$ which by assumption is contained in B. In this case, $W = M$ since there is only one boundary component of W.

Note that $K^2 = T^2/\sigma$ with the induced covering map $\rho : T^2 \to K^2$ with the bundle map $\pi : T^2 \times [0,1]/\sim \to K^2$ with $(p,t) \mapsto \rho(p)$.

First, we consider what kind of homeomorphisms $g : T^2 \to T^2$ extend to W.

Fact 55. *If $g : T^2 \to T^2$ extends to $M = T^2 \times [0,1]/(x,1) \sim (\sigma(x),1)$ and $\sigma : T^2 \to T^2$ is an orientation reversing fixed point free involution then g commutes with σ up to homotopy.*

Proof. Let $c \in \partial M = T^2 \times \{0\}$ be an essential simple closed curve. Then in M, the curve $c \times \{0\}$ is isotopic to $c \times \{1\}$ in M which is identified with $\sigma(c) \times \{1\}$ which is isotopic through M to $\sigma(c) \times \{0\}$. So there is an essential annulus A embedded in M with $\partial A = c \cup \sigma(c) \subset \partial M$.

Replacing c with $g(c)$ above, we get that $g(c) \times \{0\}$ is isotopic to $\sigma(g(c)) \times \{0\}$.

Let $G : M \to M$ be the extension of g to M. Then $G(A)$ is also an essential annulus with $\partial G(A) = g(c) \cup g(\sigma(c)) \subset \partial M$. So $g(c) \times \{0\}$ is isotopic to $g(\sigma(c)) \times \{0\}$.

Thus for any essential simple closed curve $c \in T^2$, $\sigma(g(c))$ is isotopic to $g(\sigma(c))$. $\square$

Set $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $t : \mathbb{R}^2 \to \mathbb{R}^2$ such that $(x,y) \mapsto (x, y + 1/2)$. Define

$\sigma = t \circ A$. Taking $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, it follows that σ is an orientation reversing fixed point free involution of T^2.

Then $[\sigma]_A = A$ acts on $H_1(T^2, \mathbb{Z})$. If $g : T^2 \to T^2$ extends to W, then $g_* : H_1(T^2, \mathbb{Z}) \to H_1(T^2, \mathbb{Z})$ is an element $g_* = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{Z})$ such that on homology we have $g_* A_\sigma = A_\sigma g_*$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$\begin{bmatrix} a & -b \\ c & -d \end{bmatrix} = \begin{bmatrix} a & b \\ -c & -d \end{bmatrix}.$$

Therefore, $b = 0$ and $c = 0$. Moreover, since $\det(g_*) = 1$, $ad = 1$ which implies $a = d = 1$ or $a = d = -1$. Thus, $g = id$ or $g^2 = id$.

Suppose that f^k extends to W. Then $f^k = \pm id$. Hence f^k has finite order and thus f does as well. Since f is pure, this implies that $f = id$ and thus extends to W. $\square$

So we have addressed all the cases of the orientable Seifert fibered spaces and have proved the proposition. $\square$

6.3 Hyperbolic manifolds

Now we look at the case where W is neither Seifert fibered nor a surface cross an interval such that $\partial W \cap B \neq \emptyset$ which we will prove are hyperbolic.

Proposition 56. *Let $\hat{W}$ be the disjoint union of all the components of $M - F$ satisfying (a) in Theorem 43 that is neither a SFS nor an I-bundle, and let $f : B \to B$ be a pure homeomorphism such that at least one component of $\partial\hat{W}$ is contained in B. Suppose that $f^k|_{\partial\hat{W}\cap B}$ extends to $\hat{W}$. Then $(f|_{\partial\hat{W}\cap B})^i$ extends to $\hat{W}$ for $i \leq [210(B(G) - 1)]^{2G(B)-2}$.*

Proof. First, consider W a component of $\hat{W}$.

Fact 57. *In the set up above, if $f^k|_{B\cap W}$ extends to W then $f|_{W\cap B}$ is periodic.*

Proof. Suppose W contains no essential embedded 2-torus or annulus.

Any component of $M-F$ that is neither a Seifert fibered nor admits a the structure of a $\mathbb{D}^1$-bundle over a surface is a pared manifold $(W, F\cap\partial W)$ as defined by Thurston.

Definition 58. A *pared manifold* is a pair (M, P) such that the following hold

1. M is a compact orientable irreducible 3-manifold
2. $P \subset \partial M$ is a disjoint union of incompressible tori and annuli P_j
3. no two components of P are isotopic in ∂M
4. every abelian noncyclic subgroup of $\pi_1(M)$ is conjugate to a subgroup of $\pi_1(P_j)$ for some j
5. there are no essential cylinders $(A, \partial A) \hookrightarrow (M, P)$

Definition 59. In an *acylindrical pared manifold* (M, P), every essential embedded annulus $(A, \partial A) \hookrightarrow (M, \partial M)$ has non-empty intersection $\partial A \cap P$.

A *pared manifold* (M, P) *has incompressible boundary* if every meridian α has nontrivial intersection with P.

A *hyperbolic metric with totally geodesic boundary on a pared manifold* (M, P) is a complete hyperbolic metric on $M - P$ such that $\partial(M - P)$ is totally geodesic. Here, the components of P correspond to cusps.

Notice that $(W, F \cap \partial W)$ has incompressible boundary. Let $\alpha \in \partial W$ bound a disk in W. Since by assumption, M is boundary incompressible, then $\alpha \not\subset \partial M$ and so $F \cap \alpha \neq \emptyset$. Thus $\alpha \cap P_i \neq \emptyset$ for some P_i component of $F \cap \partial W$.

Furthermore, $(W, F \cap \partial W)$ is acylindrical by Theorem 43. The following theorem of Thurston's tells us about the structure of $(W, F \cap \partial W)$.

Theorem 60 ([24], p. 14)**.** *Any acylindrical pared manifold (M, P) with incompressible boundary admits a hyperbolic metric with totally geodesic boundary.*

Thus $(W, F \cap \partial W)$ admits a hyperbolic metric with totally geodesic boundary where the $F \cap \partial W$ correspond to cusps. By assumption $f^k|_{W \cap B}$ has an extension, say $F_k : W \to W$.

Claim 61. *The metric with geodesic boundary on W is unique up to isotopy.*

Proof. Glue two copies of W, say W_1 and W_2 along their boundary to get the double of W denoted DW which is a closed manifold without boundary. Then by Mostow's Rigidity Theorem, there is a unique hyperbolic metric on DW. Let τ be the natural involution of DW that interchanges W_1 and W_2 and fixes the boundary of W. Then the quotient of DW by τ gives back W with the corresponding unique hyperbolic structure on W. Since W has totally geodesic boundary, it follows that this unique hyperbolic structure on W induces a hyperbolic metric on ∂W which is unique up to isotopy. (See [18] Section 2.5 for details). $\square$

Therefore, F_k is isotopic to an isometry of W and $f^k|_{W \cap B}$ is isotopic to an isometry. Hence $f^k|_{W \cap B}$ is periodic and thus $f|_{W \cap B}$ is as well. $\square$

Note that $F \cap \partial \hat{W} = P_1 \sqcup P_k \sqcup T^2 \sqcup \cdots \sqcup T^2$ where each P_i is an essential annulus contained in some component of $\partial \hat{W}$. Then $\partial \hat{W} - F = S_{g_1, b_1} \sqcup \cdots \sqcup S_{g_j, b_j}$ such that S_{g_i, b_i} is a genus g_i surface with b_i boundary components (and no punctures). When

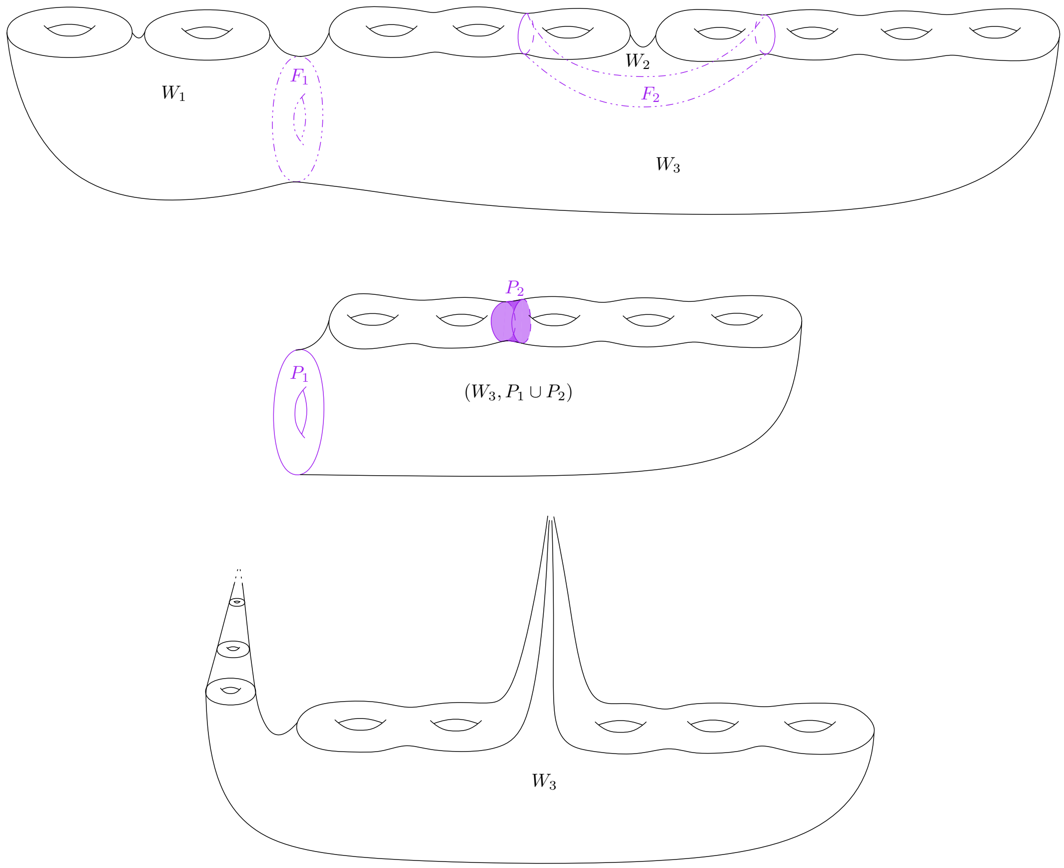


Figure 10: An example of M with Characteristic Submanifold $F = F_1 \cup F_2$. $M - F$ has 3 components. W_1 admits the structure $S_{1,1} \times I$. W_2 is a SFS. $(W_3, F_1 \cup F_2)$ is a pared manifold which admits a hyperbolic metric with totally geodesic boundary. Note how F_1 and F_2 correspond to cusps in the metric.

we endow $\hat{W}$ with a hyperbolic metric from the above theorem, the P_i correspond to cusps, and so the totally geodesic boundary of $\hat{W}$ is $S_{g_1,n_1} \sqcup \cdots \sqcup S_{g_j,n_j}$ such that $b_i = n_i$ and S_{g_i,n_i} is the genus g_i surface with n_i punctures (and no boundary components) since these boundary components b_i correspond to punctures in the induced metric on $\partial\hat{W}$.

Recall that $f|_{\hat{W}\cap B}$ fixes ∂F and thus $f|_{\hat{W}\cap B}$ preserves each S_{g_i,n_i}. By The Nielsen realization theorem ([22], Thm 7.2) and the fact that $|\mathrm{Isom}(X)| \leq 42(2g+n-2)$ (see Proposition 8), the order of $f|_{S_{g_i,n_i}}$ is bounded by $42(2g_i + n_i - 2)$. Moreover, for all S_{g_i,n_i}, we have $g_i \leq G(B)$ and $n_i \leq 3G(B) - 3$. Hence the order of $f|_{S_{g_i,n_i}}$ is bounded by $42(2G(B) + (3G(B) - 3) - 2) = 210(G(B) - 1)$.

Let j be the number of components of $\partial W - F$. Then $j \leq 2G(B) - 2$ (an upper bound on the number of pants in a pants decomposition of B). Hence we can say that the order of $f|_{\hat{W} \cap B}$ is bounded above by $[210(B(G) - 1)]^{2G(B)-2}$.

Note that this bound is not sharp because as the number of punctures and components increase, the genus decreases.

Thus there is some $i \leq [210(B(G) - 1)]^{2G(B)-2}$ such that $f|^i_{\partial W} = id$. Hence f^i extends to $\hat{W}$.

$\square$

6.4 Putting it all together: boundary-incompressible manifolds

Now we must prove that Theorem 42 holds for M as a whole. Let $A \subset F$ be an essential annulus with $\partial A \subset B$. If A is adjacent to an $S \times I$ component and a hyperbolic component, isotope f so that it fixes ∂A point-wise and that any twisting occurs in the $S \times I$ component. By assumption, f^k extends to a homeomorphism $F_k : M \to M$.

Consider W a hyperbolic component of $M - F$ with a torus boundary component T. As in the proof of Claim 61, let DW be the double of W which is a finite volume hyperbolic 3-manifold without boundary. Let $H : W \to W$ be a homeomorphism. Then H lifts to $\tilde{H} : DW \to DW$. By Mostow's Rigidity Theorem, $\tilde{H}$ is isotopic to an isometry of DW and moreover, the mapping class group of DW is finite. Thus the mapping class group of W is also finite. Therefore, $F_k|_T$ must be of finite order. Suppose that T is a boundary component of W' a Seifert fibered component of $M - F$. We know that $F_k|_T$ is a power of a Dehn twist which has infinite order if it's not the identity. Thus $F_k|_T = id$. If W' has boundary components in B, then $f|_{T_i \in B \cap \partial W'} = T_h^{n_i}$ such that $\sum_{T_i \subset \partial W'} n_i = 0$. It follows that f^i extends to W' and $f^i|_T = id$ for all i.

Hence, for some $i \leq [210(B(G)-1)]^{2G(B)-2}$, f^i is the identity on all the hyperbolic W and on all $S \times I$ and SFS components, f^i extends and induces the identity on any boundary components that are adjacent to the hyperbolic components. Thus, f^i extends to M. $\square$

Corollary 62. *Let M be an irreducible, orientable boundary-incompressible manifold. If B is a union of a subset of the boundary components of M with homeomorphism $f : B \to B$ such that f^k extends to M, then f^ℓ extends to M such that*

$$\ell \leq 12G(B)!(3G(B)-3)![210(B(G)-1)]^{2G(B)-2}.$$

Proof. This follows directly from Fact 40 and Theorem 42. $\square$

7 General Case

In this section we reduce proving Conjecture 1 to proving Conjecture 2 (see below). In the following section, we will present ideas toward proving Conjecture 2.

By Theorem 37 we first reduce to the case where M is an irreducible manifold. At the end of this section, we present a bound for a general orientable manifold in terms Conjecture 2.

Recall the following Theorem of Bonahon.

Theorem 41 ([13], Theorem 2.1)**.** Let M be an irreducible 3-manifold. There exists $V \subset M$ a disjoint union of compression bodies for ∂M, unique up to isotopy, called the *characteristic compression body*, such that $M - V$ is boundary incompressible and irreducible.

The characteristic compression body V is simple to construct: Let K be a maximal set of essential simple closed curves on ∂M that bound disks in M where $S_{g_1}, ..., S_{g_n}$ are the components of ∂M that contain curves of K. Set $K_i = K \cap S_{g_i}$. Then $V = S_{g_1}[K_1] \sqcup \cdots \sqcup S_{g_n}[K_n]$. Notice that V is a disjoint union of compression bodies as described in Section 2.2.

First, we would like to prove Conjecture 1 when M is a single compression body and then we can address the case where M is a disjoint union of compression bodies such as the Characteristic Compression body. We will use the following conjecture:

Conjecture 2. If $h : S_g \to S_g$ is a homeomorphism and C is an S_g-compression body such that some power of h extends to C, then there exists S_g-compression body $C' \subset C$ such that for some $j \leq m(g)$ (with $m(g)$ an increasing function) we have that h^j extends to C'.

Proposition 63. *Let $f : S_g \to S_g$ be a homeomorphism and C an S_g-compression body such that some power of f extends to C. If Conjecture 2 is true, then there exists some $i \leq m(g)^{1+g(2g-2)}$ such that f^i extends to C.*

Proof. Let $f : S_g \to S_g$ be a homeomorphism and C an S_g-compression body such that f^k extends to C. By Conjecture 2, there exists a $j_1 \leq m(g)$ and a nontrivial $C_1 \subset C$ such that f^{j_1} extends to C_1. If $C_1 = C$ then f^{j_1} extends to C.

Suppose not. The interior boundary $\partial_- C_1 = F_1^1 \sqcup \cdots \sqcup F_n^1$. Biringer and Vlamis' Proposition 21 lets us decompose C in terms of C_1 in the following way: There exist F_i^1-compression bodies D_i^1 such that $C = C_1 \bigcup_{\partial_- C_1} D_1^1 \sqcup \cdots \sqcup D_n^1$.

Fix i. We have homeomorphism $f^{j_1}|_{F_i^1} : S_{g(F_i^1)} \to S_{g(F_i^1)}$. Since $f^{j_1 k}$ extends to C then $(f^{j_1}|_{F_i^1})^k$ extends to D_i^1. Again by Conjecture 2, there exist some $C_i^2 \subset D_i^1$ such that $(f^{j_1}|_{F_i^1})^{j_{2_i}}$ extends to C_i^2 with $j_{2_i} \leq m(g(F_i^1)) \leq m(g)$.

Set $j_2 := \Pi_{i=1}^n j_{2_i}$ and $C_2 = C_1 \bigcup_{\partial_- C_1} C_1^2 \sqcup \cdots \sqcup C_n^2$. Note that $j_2 \leq m(g)^g$. Then $f^{j_1 j_2}$ extends to C_2. Moreover, since $C_1 \neq C$, there is some i such that D_i^1 is nontrivial and hence C_i^2 is also nontrivial implying $C_1 \neq C_2$. We have a sequence of compressions, $S_g \times [0,1] \subsetneq C_1 \subsetneq C_2$.

In general, if $C_\ell \neq C$, we have The interior boundary $\partial_- C_\ell = F_1^\ell \sqcup \cdots \sqcup F_n^\ell$ for $n < g$. There exist F_i^ℓ-compression bodies D_i^ℓ such that $C = C_\ell \bigcup_{\partial_- C_\ell} D_1^\ell \sqcup \cdots \sqcup D_n^\ell$.

Fix i. We have homeomorphism $f^{\Pi_{i=1}^\ell j_i}|_{F_i^\ell} : S_{g(F_i^\ell)} \to S_{g(F_i^\ell)}$. Since $f^{k\Pi_{i=1}^\ell j_i}$ extends to C then $(f^{\Pi_{i=1}^\ell j_i}|_{F_i^\ell})^k$ extends to D_i^ℓ. Again by Conjecture 2, there exist some $C_i^{\ell+1} \subset D_i^\ell$ such that $(f^{\Pi_{i=1}^\ell j_i}|_{F_i^\ell})^j_{\ell+1}$ extends to $C_i^{\ell+1}$ with $j_{\ell+1_i} \leq m(g(F_i^\ell)) \leq m(g)$.

Set $j_{\ell+1} := \Pi_{i=1}^n j_{\ell+1_i}$ and $C_{\ell+1} = C_\ell \bigcup_{\partial_- C_\ell} C_1^{\ell+1} \sqcup \cdots \sqcup C_n^{\ell+1}$. Note that $j_\ell \leq m(g)^g$. Then $f^{\Pi_{i=1}^{\ell+1} j_i}$ extends to $C_{\ell+1}$. Moreover, since $C_\ell \neq C$, there is some i such that D_i^ℓ is nontrivial and hence $C_i^{\ell+1}$ is also nontrivial implying $C_\ell \neq C_{\ell+1}$. We have a sequence of compressions, $S_g \times [0,1] \subsetneq C_1 \subsetneq \cdots \subsetneq C_{\ell+1}$.

By Proposition 20, there is some $a \leq 2g - 1$ such that $C_a = C$ with $f^{\Pi_{i=1}^a j_i}$ extending to c. Since $j_1 \leq m(g)$ and for $i > 1$, $j_i \leq m(g)^g$, we have that $\Pi_{i=1}^a j_i \leq m(g)(m(g)^g)^{a-1} = m(g)^{1+g(a-1)} \leq m(g)^{1+g(2g-2)}$.

$\square$

Now we show that if M is a disjoint union of compression bodies that Conjecture 1 is true assuming Conjecture 2 is true.

Corollary 64. *Let $V = S_{g_1}[K_1] \sqcup \cdots \sqcup S_{g_n}[K_n]$ be a disjoint union of compression bodies and suppose that B is a union of a subset of the S_{g_i}'s. If homeomorphism $f : B \to B$ has a power that extends to V and if Conjecture 2 is true then there is an $i \leq m(G(B))^{[1+G(B)(2G(B)-2)]G(B)}$ such that f^i extends to V.*

Proof. Assuming Conjecture 2, for each $S_{g_i}[K_i]$, there is some i such that $f|_{S_{g_i}}^i$ extends for $i \le m(G(B))^{[1+G(B)(2G(B)-2)]}$. Therefore, $f^{\prod_{i=1}^{n} i}$ extends to V. There are at most $G(B)$ compression bodies in the disjoint union V hence $n \le G(B)$. □

Now we consider the case where M is some irreducible manifold. By Theorem 41, any homeomorphism of M preserves V. And any homeomorphism of V induces a homeomorphism on $\partial(M - V)$. So we can put together the incompressible case with compression bodies case to get the following.

Proposition 65. *Assuming Conjecture 2, if M is an irreducible orientable 3-manifold with boundary and if B is a union of a subset of components of ∂M such that $f : B \to B$ has a power that extends to M, then there is some*

$$i \le m(G(B))^{[1+G(B)(2G(B)-2)]G(B)} 12G(B)!(3G(B)-3)![210(B(G)-1)]^{2G(B)-2}$$

such that f^i extends to M.

Proof. Let M be an irreducible orientable 3-manifold with boundary with homeomorphism $f : B \to B$ such that f^k extends to M. First, by Theorem 41, we have characteristic compression body V. Assuming Conjecture 2, by Corollary 64, there is some $i \le m(G(B))^{[1+G(B)(2G(B)-2)]G(B)}$ such that f^i extends to V.

Let $\partial_+ V$ denote the disjoint union of the exterior boundary components of the compression bodies of V and $\partial_- V$ denote the disjoint union of interior boundary components of the compression bodies of V. Then f^i induces a map $f_i : \partial_- V \to \partial_- V$.

Consider $M - V$ which is an orientable irreducible boundary incompressible 3-manifold with boundary $(\partial M - \partial_+ V) \cup \partial_- V$. Set $B' = (B - \partial_+ V) \cup \partial_- V$. Then we have $f' : B' \to B'$ such that $f'|_{\partial_- V} = f_i$ as defined above and $f'|_{B - \partial_+ V} = f^i$. Since f^k extends to M then f^{ki} does as well. Hence, f'^k extends to $M - V$.

Note that $G(B') \le G(B)$ since the genus of the interior boundary of $S_{g_i}[K_i]$ is less than the genus of the exterior boundary. By Corollary 62, there is a

$$
\begin{aligned}
j &\leq 12G(B')!(3G(B')-3)![210(G(B')-1)]^{2G(B')-2} \\
&\leq 12G(B)!(3G(B)-3)![210(G(B)-1)]^{2G(B)-2}
\end{aligned}
$$

such that f'^j extends to $M-V$. It follows that f^{ij} extends to M. $\square$

Assuming Conjecture 2 is true, we can obtain a bound for any orientable 3-manifold by applying Corollary 39.

Corollary 66. *Assuming Conjecture 2, if M is an orientable 3-manifold with boundary and if B is a union of a subset of components of ∂M such that $f: B \to B$ has a power that extends to M, then there is some*

$$
i \leq \left[12G(B)!(3G(B)-3)!m(G(B))^{[1+G(B)(2G(B)-2)]G(B)}[210(B(G)-1)]^{2G(B)-2}\right]^{G(B)}
$$

such that f^i extends to M.

Note that for each irreducible M' in the prime decomposition of M, there is an $i' \leq 12G(B)!(3G(B)-3)!m(G(B))^{[1+G(B)(2G(B)-2)]G(B)}[210(B(G)-1)]^{2G(B)-2}$ such that $f|_{\partial M' \cap B}^{i'}$ is orientation preserving since i' can be chosen to be even (we can do so in the step where we take a power of f such that f is pure on $M'-V'$). Thus we can apply Corollary 39 to get the bound above.

8 Extension in Compression bodies

In this section we present ideas toward a proof of Conjecture 2.

We first effectivize Casson and Long's Corollary 2.5 in [6] that a simple closed curve of S compresses in only finitely many minimal S-compression bodies for which homeomorphism $f : S \to S$ extends (see Theorem 71).

Definition 67. Let C be S_g-compression body and α an essential simple closed curve of S_g. If B is an S_g-compression body satisfying both $C \subset B$ and α compresses in B then we say B is $C * \alpha$.

Let C and D be S_g-compression bodies. If B is an S_g-compression body satisfying both $C \subset B$ and $D \subset B$ then we say B is $C * D$.

Theorem 68. *Let S be a closed genus g surface. Set*

$$\mathcal{C}_{D*E} := \{S - \textit{compression bodies } C : C \textit{ is } D * E\}.$$

There are at most $(6g-6)^{(2g-2)^2}$ minimal elements of $\mathcal{C}_{D*E}$.

First we prove two lemmas:

Lemma 69. *Let C be S_g-compression body and α an essential simple closed curve of S_g that does not compress in C. Then there exists a set of compression bodies $\mathcal{D}$ with $|\mathcal{D}| \leq 6g-6$ with the property that $C \subsetneq D$ for all $D \in \mathcal{D}$ such that for any $C * \alpha$ compression body E, there exists a $D' \in \mathcal{D}$ with $D' \subset E$.*

Proof. Fix S_g-compression body C and let α be an essential closed curve of S_g that does not compress in C. By Fact 27, there exists compressing system $\{\kappa_i, ..., \kappa_k\}$ for C for which α is in tight position.

If α is disjoint from the $\kappa_i, ..., \kappa_k$, then $\{\kappa_i, ..., \kappa_k, \alpha\}$ is a compressing system. Denote $D := S_g[\kappa_i, ..., \kappa_k, \alpha]$. Then $C \subsetneq D$ and also $D \subset E$.

Now suppose α is not disjoint from $\kappa_i, ..., \kappa_k$. Set $K = \kappa_1 \cup \cdots \cup \kappa_k$ and hence $K \cap \alpha \neq \emptyset$. Fix E to be a compression body that is $C * \alpha$.

Let D_α be an embedded disk in E with boundary α and for each κ_i, define D_{κ_i} similarly. We may assume the D_{κ_i} are all disjoint and that D_α intersects them transversely. This intersection is then a set of disjoint arcs $A \subset D_\alpha$. Let α be an *outermost arc* in A, that is an arc that bounds a component of $D_\alpha - A$ with an arc of α. Denote this arc of α as a. Thus α is homotopic rel $\partial\alpha$ to arc a through D_α. Also note that α is homotopic rel $\partial\alpha$ to an arc of κ_j through D_{κ_j} for some j. Denote this arc k. Thus a is homotopic to k in E, thus a is a κ_j-wave. It follows that a-surgery on κ_j gives ${\kappa_j}' = k \cup a$ which is a meridian in E disjoint from κ_j by Fact 25. Moreover, κ_j' and κ_j'' are disjoint from κ_i for $i \neq j$ since the κ_i form a compressing system. Set $D_a = S_g[\kappa_1, ..., \kappa_k, \kappa_j']$. Then $D_a \subset E$. Recall that by assumption, α is tight position with respect to $\{\kappa_1, ..., \kappa_k\}$ in C. Therefore, a is not a κ_j-wave in C. So by Fact 25, κ_j' and κ_j'' are not meridians of C. Hence $C \subsetneq D$. Therefore, $D_a \in \mathcal{D}$.

Note that for for each E satisfying $C * \alpha$, an ourtermost arc of $K \cap D_\alpha$ may be homotopic to a different arc of $\alpha - K$. However, any arc $a \subset \alpha - K$ that is homotopic to an outermost arc of $K \cap D_\alpha$ in some $C * \alpha$ compression body E will satisfy both of the following properties.

(1) $\partial a \subset \kappa_i$ for some $\kappa_i \in K$. That is, a is incident to a single component of K.

(2) both intersections of a with κ_i happen on the same side of κ_i by Lemma 22

Set $A = \{\text{arc } a \subset \alpha : a \text{ satisfies (1) and (2)}\}$. Each $a \in A$ determines a compression body $D_a \in \mathcal{D}$. However, these may not be distinct.

Suppose that arcs $a_i, a_j \in A$ are parallel in $\overline{S_g \backslash K}$. Parallelism gives that both a_i and a_j have boundaries on the same curve $\kappa_\ell \in K$ and property (2) gives that $\partial(a_i \cup a_j)$ is contained in a single boundary component b of $\overline{S_g \backslash K}$. Thus a_i, a_j and b bound an embedded rectangle in $\overline{S_g \backslash K}$. Thus we can homotope κ_ℓ' obtained by a_1-surgery through the embedded rectangle to κ_ℓ'', the curve obtained by a_j-surgery as illustrated in Figure 11. Thus, $D_{a_i} = D_{a_j}$.

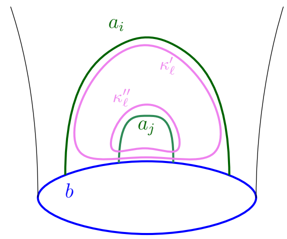


Figure 11: Because a_i and a_j are parallel, κ'_ℓ is homotopic to κ''_ℓ

This implies that $|\mathcal{D}|$ is bounded by the number of homotopy classes of properly embedded arcs that can be realized disjointly in $\overline{S_g \backslash K}$. An Euler Characteristic argument shows that in fact, $|\mathcal{D}| \leq 6g - 6$. $\square$

Lemma 70. *Let C be a non-trivial S_g-compression body and γ an essential simple closed curve of S_g. Set $\mathcal{C}_{C*\gamma} = \{S_g\text{-compression bodies } D : D \text{ is } C * \gamma\}$. The number of minimal elements of $\mathcal{C}_{C*\gamma}$ is bounded by $(6g-6)^{2g-2}$.*

Proof. Let C be an S_g-compression body C and let γ be an essential closed curve of S_g.

If γ compresses in C, then $C \in \mathcal{C}_{C*\gamma}$ and is the sole minimal element.

Suppose that γ does not compress in C. Let $E \in \mathcal{C}_{C*\gamma}$ and assume that E is minimal. By Lemma 69, there exist $D_{1_1}, ..., D_{\ell_1}$ for $\ell_1 \leq 6g - 6$ such that there is some j with $C \subsetneq D_{j_1} \subset E$. If γ compresses in D_{j_1}, then by minimality of E, $E = D_{j_1}$. In this case, there are at most $6g - 6$ minimal elements of $\mathcal{C}_{C*\gamma}$.

If not, set $D_{j_1} = D_1$. Since γ does not compress in D_1, again by Lemma 69, there exist $D_{1_2}, ..., D_{\ell_2}$ with $\ell_2 \leq 6g-6$ such that for some j, $D_1 \subsetneq D_{j_2} \subset E$. Set $D_{j_2} = D_2$. By minimality of E, if γ compresses in D_2, then $E = D_2$. In this case, there were at most $6g-6$ possibilities for D_1 and then at most $6g-6$ possibilities for D_2, dependent on D_1. Thus there are at most $(6g-g)^2$ minimal elements of $\mathcal{C}_{C*\gamma}$.

If not, continue the process.

In general, if γ does not compress in D_i for all $i \leq n-1$, then by Lemma 69, there exist $D_{1_n}, ..., D_{\ell_n}$ with $\ell_n \leq 6g-6$ such that for some j, $D_1 \subsetneq \cdots \subsetneq D_{n-1} \subsetneq D_{j_n} \subset E$. Set $D_{j_n} = D_n$. By minimality of E, if γ compresses in D_n, then $E = D_n$. In this case, there were at most $6g-6$ possibilities for D_1 through D_n and so there are at most $(6g-6)^n$ possibilities for D_n, dependent on previous D_i. Thus there are at most $(6g-g)^n$ minimal elements of $\mathcal{C}_{C*\gamma}$.

By Proposition 20, in any sequence of compressions $S_g \times [0,1] \subsetneq C_1 \subsetneq \cdots \subsetneq C_k$ we have the bound $k \leq 2g-1$.

We have $S_g \times [0,1] \subsetneq C \subsetneq D_1 \subsetneq \cdots \subsetneq D_n$ which is a length $n+1$ sequence of compressions. Thus $n+1 \leq 2g-1$ and so for some $n \leq 2g-2$, $D_n = E$.

Thus there are at most $(6g-6)^{2g-2}$ minimal elements of $\mathcal{C}_{C*\gamma}$. $\square$

Now we prove Theorem 68.

Proof. If $E \subseteq D$, then $D \in \mathcal{C}_{D*E}$ is minimal and moreover the only minimal element.

Now suppose that $E \not\subset D$ (as well as $D \not\subset E$) and let $C \in \mathcal{C}_{D*E}$ be minimal. This implies that $D \subsetneq C$ (since $D = C$ contradicts $E \not\subset D$). It also implies that D is a nontrivial compression body. We look to bound the length of a chain of compressions from D to C.

There is some simple closed curve β_1 that compresses in E but not in D. Then by Lemma 70 there is a bounded list of $D * \beta_1$ minimal compression bodies $C_{1_1}, ..., C_{\ell_1}$ with $\ell_1 \leq (6g-6)^{2g-2}$. Then for some i, $C_{i_1} \subseteq C$. Set $C_1 := C_{i_1}$. If $C = C_1$ then we have the chain $D \subsetneq C_1 = C$ and there are at most $\ell \leq (6g-g)^{2g-2}$ minimal elements of $\mathcal{C}_{D*E}$.

If $C_1 \subsetneq C$, by minimality of C, there is some meridian β_2 of E that doesn't compress in C_1. Then by Lemma 70 there is a bounded list of $C_1 * \beta_2$ minimal compression bodies $C_{1_2}, ..., C_{\ell_2}$ with $\ell_2 \leq (6g-6)^{2g-2}$. Then for some i, $C_{i_2} \subseteq C$. Set

$C_2 := C_{i_2}$. Then we have the chain $D \subsetneq C_1 \subsetneq C_2$. There were at most $(6g-6)^{2g-2}$ possibilities for C_1 and then at most $(6g-6)^{2g-2}$ possibilities for C_2, dependent on C_1. If $C = C_2$ then there are at most $(6g-6)^{2(2g-2)}$ different chains of compressions to give some minimal element C and so there are at most $(6g-6)^{2(2g-2)}$ minimal elements of $\mathcal{C}_{D*E}$.

In general, if $C_i \subsetneq C$ for all $i < j$, then by minimality of C there is some meridian β_j of E that doesn't compress in C_{j-1}. Then by Lemma 70 there is a bounded list of $C_{j-1} * \beta_j$ minimal compression bodies $C_{1_j}, ..., C_{\ell_j}$ with $\ell_j \leq (6g-6)^{2g-2}$. Then for some i, $C_{i_j} \subseteq C$. Set $C_j := C_{i_j}$. Then we have the chain $D \subsetneq C_1 \subsetneq \cdots \subsetneq C_j$. For this chain, there are at most $(6g-6)^{2g-2}$ for each C_i dependent on the previous selection. So there are at most $(6g-6)^{j(2g-2)}$ possibilities for C_j. If $C = C_j$ then there are at most $(6g-6)^{j(2g-2)}$ minimal elements of $\mathcal{C}_{D*E}$.

Thus we have some sequence of compressions $S_g \times [0,1] \subsetneq D \subsetneq C_1 \subsetneq \cdots \subsetneq C_j$ of length $j+1$. By Proposition 20, $j+1 \leq 2g-1$ and so for some $j \leq 2g-2$, $C_j = C$. Thus there are at most $(6g-6)^{(2g-2)^2}$ minimal elements of $\mathcal{C}_{D*E}$. $\square$

Theorem 71. *Let $h : S_g \to S_g$ be a homeomorphism and $\gamma \subset S_g$ an essential simple closed curve. Set*

$$\mathcal{C}_{h,\gamma} = \{S_g\text{-compression body } C : h \text{ extends to } C \text{ and } \gamma \text{ compresses in } C\}.$$

Then there are at most

$$(6g-6)^{2g-2}(6g-6)^{(2g-2)^2(2g-3)}$$

minimal elements of $\mathcal{C}_{h,\gamma}$.

Proof. Let $C \in \mathcal{C}_{h,\gamma}$ be minimal. By Lemma 70, there are at most $(6g-g)^{2g-2}$ minimal

$S[\gamma] * h(\gamma)$ compression bodies $D_{1_1}, ..., D_{\ell_1}$. Since $h(\gamma)$ compresses in C, then one of these minimal $S[\gamma]*h(\gamma)$ compression bodies, say D_{i_1} is contained in C. Set $D_1 = D_{i_1}$. If $D_1 = C$ then we have a sequence of compressions $S_g \times [0,1] \subsetneq D_1 = C$.

Suppose $D_1 \subsetneq C$. By minimality of C, we have $D_1 \notin \mathcal{C}_{h,\gamma}$. Since γ compresses in D_1 by construction, it follows that h doesn't extend to D_1. In other words, $h(D_1) \not\subseteq D_1$. Applying Theorem 68, there are at most $(6g-6)^{(2g-2)^2}$ minimal $D_1 * h(D_1)$ compression bodies $D_{1_2}, ..., D_{\ell_2}$. Note that for all i, $D_1 \neq D_{i_2}$ because $h(D_1) \subset D_{i_2}$ and $h(D_1) \not\subset D_1$. Since h extends to C and $D_1 \subset C$, it follows that for some i, $D_{i_2} \subset C$. Set $D_2 := D_{i_2}$. We have a sequence of compressions $S_g \times [0,1] \subsetneq D_1 \subsetneq D_2$.

Suppose $D_{n-1} \subsetneq C$. By minimality of C, we have $D_{n-1} \notin \mathcal{C}_{h,\gamma}$. Since γ compresses in D_{n-1} by construction, it follows that h doesn't extend to D_{n-1}. In other words, $h(D_{n-1}) \not\subset D_{n-1}$. Applying Theorem 68, there are at most $(6g-6)^{(2g-2)^2}$ minimal $D_{n-1} * h(D_{n-1})$ compression bodies $D_{1_n}, ..., D_{\ell_n}$ all satisfying $D_{i_n} \neq D_{n-1}$ as above. Since h extends to C and $D_{n-1} \subset C$, it follows that for some i, $D_{i_n} \subset C$. Set $D_n := D_{i_n}$.

We have a sequence of compressions $S_g \times [0,1] \subsetneq S[\gamma] \subsetneq D_1 \subsetneq \cdots \subsetneq D_n$ which has length $n+1$. By Proposition 20, $n \leq 2g-2$. Hence this process must terminate with some $D_n = C$ for $n \leq 2g-2$.

Therefore, any minimal element $C \in \mathcal{C}_{h,\gamma}$ has a sequence of compressions $S_g \times [0,1] \subsetneq S[\gamma] \subsetneq D_1 \subsetneq \cdots \subsetneq D_n$ for $n \leq 2g-2$ where D_1 is one of the at most $(6g-g)^{2g-2}$ minimal $S[\gamma] * h(\gamma)$ compression bodies, and D_i for $2 \leq i \leq n$ is one of the at most $(6g-6)^{(2g-2)^2}$ minimal $D_{i-1} * h(D_{i-1})$ compression bodies. So there are at most $(6g-g)^{2g-2}(6g-6)^{(2g-2)^2(2g-3)}$ minimal elements of $\mathcal{C}_{h,\gamma}$. □

Casson and Long prove that for a pseudo-Anosov homeomorphism $f : S_g \to S_g$, there is a finite set $\mathcal{M}_f$ of essential simple closed curves of S_g, such that if f extends to compression body C, then some $c \in \mathcal{M}_f$ is a meridian of C. (see Theorem 1.2 in

[6])

Some coarse geometry arguments from Maher and Schleimer's paper [8] may help to effectivize Casson and Long's Theorem 1.2. We will state this as a conjecture.

Conjecture 3. Suppose that $f : S_g \to S_g$ is a pseudo-Anosov homeomorphism. Then there exists set $\mathcal{M}_f = \{\gamma_1, ..., \gamma_\ell\}$ for $\ell \leq k(g)$ such that if f extends to an S_g-compression body C, then some γ_i is a meridian of C.

The following proposition is similar to Conjecture 2 however it is weaker: in Conjecture 2, f is any homeomorphism but in the proposition below, f must be pseudo-Anosov.

Proposition 72. *Assuming Conjecture 3, if $f : S_g \to S_g$ is pseudo-Anosov and there is some S_g-compression body C such that some power of f extends to C, then there exists an S_g-compression body $C' \subset C$ such that for some $i \leq [k(g)(6g-g)^{2g-2}(6g-6)^{(2g-2)^2(2g-3)}]!$, we have that f^i extends to C'.*

Proof. First we prove the following lemma.

Lemma 73. *Let $f : S_g \to S_g$ be a pseudo-Anosov and set*

$$\mathcal{C}_f = \{S_g\text{-compression bodies } C : f \text{ extends to } C\}.$$

If Conjecture 3 is true, then there are at most $k(g)(6g-g)^{2g-2}(6g-6)^{(2g-2)^2(2g-3)}$ minimal elements of $\mathcal{C}_h$.

Proof. Let $C \in \mathcal{C}_f$ be minimal. Then by Conjecture 3, there is some $\gamma_i \in \mathcal{M}_f$ which compresses in C. There are at most $k(g)$ curves γ_i and at most $(6g-g)^{2g-2}(6g-6)^{(2g-2)^2(2g-3)}$ minimal compression bodies to which f extends where γ_i compresses. Thus we multiply to get the bound. □

Let $f : S_g \to S_g$ be a pseudo-Anosov and assume that f^k extends to S_g-compression body C.

Set $\mathcal{C}^M_{f^k} = \{D \in \mathcal{C}_f : D \text{ is minimal}\}$, the set of minimal S_g-compression bodies to which f^k extends.

Note that f permutes the elements of $\mathcal{C}^M_{f^k}$. To see this, let $D \in \mathcal{C}^M_{f^k}$. Then $f^k(f(D)) = f(f^k(D)) = f(D)$. Hence $f(D) \in \mathcal{C}^M_{f^k}$.

From Lemma 73, $|\mathcal{C}^M_{f^k}| \leq k(g)(6g-g)^{2g-2}(6g-6)^{(2g-2)^2(2g-3)}$. Thus, the order i of f as a permutation of $\mathcal{C}^M_{f^k}$ is at most $[k(g)(6g-g)^{2g-2}(6g-6)^{(2g-2)^2(2g-3)}]!$. By minimality of the compression bodies of $\mathcal{C}^M_{f^k}$, there is some $C' \in \mathcal{C}^M_{f^k}$ such that $C' \subset C$ with $f^i(C') = C'$. $\square$

Note that we can't use the above proposition to prove a pseudo-Anosov analogue to Proposition 63 because of the following: Let $h : S_g \to S_g$ be a pseudo-Anosov that has a power that extends to some S_g-compression body C. Then by the above proposition, there exists a bounded i such that h^i has an extension $H_i : C' \to C'$ for some $C' \subset C$. Recall that $\partial_- C'$ is the interior boundary of C', that is $\partial C' - \partial_+ C'$. Note that $H_i|_{\partial_- C'}$ is a surface homeomorphism but is likely not a pseudo-Anosov homeomorphism. Thus we cannot induct like we did in the proof of Proposition 63 to get a bounded power of extension to C. This illustrates why we need the stronger Conjecture 2.

Thus we would like to prove the following generalized form of Conjecture 3.

Conjecture 4. Suppose that homeomorphism $h : S_g \to S_g$ is pure and is pseudo-Anosov on subsurface X of S. Then there exists a set of non-peripheral, essential simple closed curves of X, $\mathcal{M}_h = \{\gamma_1, ..., \gamma_i\}$ for $i \leq \ell(g)$ such that if C is any proper X-compression body to which h extends, then some γ_i compresses in C.

If Conjecture 4 is true, one might use tools of forthcoming work of Biringer and

Lecuire [25] to prove Conjecture 2 which would in turn prove Conjecture 1 for all orientable 3-manifolds as described in Section 7.

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