HEEGAARD SPLITTINGS AND COMPLEXITY OF FIBERED KNOTS

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This dissertation explores a relationship between fibered knots and Heegaard splittings in closed, connected, orientable three-manifolds. We show that a fibered knot, which has a sufficiently complicated monodromy, induces a minimal genus Heegaard splitting that is unique up to isotopy. Moreover, we show that fibered knots in the three-sphere has complexity at most 3.

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Dedicated to my father.

1 Introduction

The study of three-dimensional manifolds has been a mainstream subfield of topology since the beginning of the 20^{th} century with the ultimate goal of classifying three-manifolds.

One way to understand the properties of a three-manifold M is to analyze surfaces and knots embedded in it. This approach proved to be useful by contributing many beautiful theorems to the three-manifolds literature, some of which will be presented below. The most important examples of this phenomenon are Heegaard surfaces, introduced by Heegaard [12] in 1898, and fibered knots, introduced by Alexander [1] in 1923. A Heegaard surface in a three-manifold M cuts M into two handlebodies, where a fibered knot in M has a complement that fibers over S^1. Both notions provide a decomposition of M into simpler pieces, which in certain cases tells a lot about the properties of M.

An obvious relationship between fibered knots and Heegaard surfaces is that every fibered knot induces a Heegaard surface. This naturally follows from the definitions of these objects, and the Heegaard surface is obtained as a union of the knot with two distinct fibers that come from the fibration of the knot complement. However, the induced Heegaard surface is, in general, not the minimal genus Heegaard surface. So, the classification of the Heegaard surfaces of a three-manifold in terms of the fibered knots is an interesting problem to discover.

In this dissertation, we study the interaction between Heegaard surfaces and fibered knots in three-manifolds to give a sufficient condition on the complexity of a fibered knot for it to induce a minimal genus Heegaard surface. In particular, we prove that a sufficiently complicated fibered knot induces a minimal Heegaard surface. A classification of Heegaard surfaces in terms of complicated fibered knots follows.

2 Background

2.1 Essential Surfaces in Three-Manifolds

A surface S embedded in a compact orientable three-manifold M is called *properly embedded* when S intersects ∂M transversely and $S \cap \partial M = \partial S$. In particular, when M is a closed three-manifold, we require S to be a closed surface. Historically, surfaces properly embedded in three-manifolds have been strong tools to study three-manifold topology, and we will exhibit examples of this phenomenon in this chapter.

The existence of an embedded sphere $S \subset M$ that does not bound a three-ball in M is quite informative about M. Therefore, we provide a special name for such spheres. If a sphere S embedded in M does not bound a three-ball, then it is called an *essential* sphere. If S is not essential, we say it is *inessential*.

Definition 2.1. If a three-manifold M contains no essential spheres, then M is called *irreducible*. If there exists an essential sphere in M, we call M *reducible*.

The celebrated Theorem of Alexander [2], as a generalization of Jordan Curve Theorem [14], states that every two-sphere properly embedded in S^3 bounds a three-ball on each side. It essentially follows that S^3, $\mathbb{R}^3$, and the three-ball are irreducible three-manifolds. On the other hand, $S^1 \times S^2$ is a reducible three-manifold since an S^2 fiber does not bound a three-ball, as it is non-separating, i.e., its complement is connected. More generally, if M is not a three-ball, then every non-separating sphere in M is essential.

The notion of essentialness can be generalized for two-sided surfaces with positive genus as follows. Before the definition, note that a surface S properly embedded in a three-manifold M is called *two-sided* if a tubular neighborhood $N(S)$ of S in M is homeomorphic to $S \times [-1, 1]$.

Definition 2.2. A two-sided surface $S \subset M$, which is neither a disk nor a sphere, is called *incompressible* if for every embedded disk $D \subset M$ with $\partial D = D \cap S$, the simple closed curve ∂D also bounds a disk in S. If S is not incompressible, we say that it is *compressible*. Moreover, a disk $D \subset M$ with $\partial D = D \cap S$ is called a *compressing disk* for S if ∂D does not bound a disk in S.

The notion of compressing disks can also be generalized as *∂-compressing* disks for surfaces with boundary as follows.

Definition 2.3. A two-sided surface S, which is neither a disk nor a sphere, is called *boundary-incompressible* (or *∂-incompressible*) if for any embedded disk $D \subset M$ such that $\partial D = \alpha \cup \beta$ for some connected arcs $\alpha = D \cap S$ and $\beta = D \cap \partial M$, the arc α cuts off a disk D' from S, i.e., there exists a disk D' in S such that $\partial D' = \alpha \cup \beta'$ for some connected arc $\beta' \subset \partial S$. Such a disk D is called a *∂-compressing disk* for S if $\alpha = D \cap S$ does not cut off a disk from S. If S is not ∂-incompressible, then we say it is *∂-compressible*.

Finally, we define the notion of essentialness for positive genus surfaces as follows.

Definition 2.4. An incompressible and ∂-incompressible surface $S \subset M$ is called *essential* in M.

Example 2.5. Just as an S^2 fiber of $S^1 \times S^2$ is essential in $S^1 \times S^2$, one can show that an S_g fiber in $S^1 \times S_g$ is incompressible/essential in $S^1 \times S_g$ for any positive genus surface S_g.

We have excluded spheres in the last definition because we have already defined a notion of essentialness for them. We have also excluded disks because every simple closed curve in a disk bounds a subdisk, by Jordan Curve Theorem [14], and therefore no compression, as defined above, is possible for disks. We define essentialness for disks as follows.

Definition 2.6. A disk D properly embedded in a three-manifold M is called *essential* if ∂D does not bound a disk in ∂M.

Notice that the existence of an essential disk in M is equivalent to that ∂M is compressible, by definitions.

Now let us discuss how we can "surger" a compressible or ∂-compressible surface S to obtain *simpler* surfaces embedded in M, using compressing or ∂-compressing disks. The operations we define here are going to be quite useful in the following chapters.

Definition 2.7 (Compression)**.** If D is a compressing disk for a surface $S \subset M$, take a closed product neighborhood $N(D) \cong D \times [-1, 1]$ in M such that $\partial D \times (-1, 1)$ is an open annulus in S. Remove $\partial D \times (-1, 1)$ from S and cap off the two created boundary components with parallel copies of D, namely $D^{\pm} = D \times \{\pm 1\} \subset N(D)$. The resulting surface, say S', is simpler than S in the sense that it has larger Euler charcteristic: $\chi(S') = \chi(S) + 2$. The operation of creating S' from S, using the compressing disk D, is called the *compression* of S along D (or along ∂D).

Definition 2.8 (∂-Compression)**.** Similar to compression defined above, if D is a ∂-compressing disk for a properly embedded surface $S \subset M$ with $D \cap S = \alpha$, then by removing $\mathring{N}(\alpha)$ from S and then by adding parallel copies of D, we get a surface S' of Euler characteristic $\chi(S') = \chi(S) + 1$. This operation is called the *∂-compression* of S along D (or along $\alpha = \partial D \cap S$).

Compression and ∂-compression operations simplify a surface S by yielding a surface S' that has larger Euler characteristic. Another possible way to simplify a surface is to isotope it into ∂M. This scenario requires a separate definition.

Definition 2.9. A properly embedded surface $S \subset M$ is called *∂-parallel* in M if there is an isotopy (rel ∂S) of M taking the surface S into a subsurface of ∂M.

The following lemma proves that being essential and being ∂-parallel are complimentary cases for a properly embedded incompressible surface $S \subset M$ that has non-empty boundary.

Lemma 2.10 ([11], Lemma 1.10)**.** *Let S be a connected incompressible surface in an irreducible three-manifold M such that ∂S is in a torus component T of ∂M. Then S is either ∂-incompressible (hence essential) or a ∂-parallel annulus.*

This lemma (or a variation of it) is going to be quite useful in the subsequent chapters when we work with irreducible three-manifolds with torus boundary (such as knot exteriors).

2.2 Heegaard Surfaces

Most of the time, it is convenient to regard a three-manifold as a union of simpler pieces such as handlebodies. Therefore, we will first introduce handlebodies in this section and then we will discuss how they are used to build and to study three-manifolds.

2.2.1 Handlebodies

A *handlebody* is a compact orientable three-manifold obtained by gluing three-dimensional one-handles $I \times D^2$ via homeomorphisms of $\{0,1\} \times D^2$ into the boundary of a three-ball. Alternatively, a handlebody is a compact orientable three-manifold H with boundary such that there are essential disks $D_1, \ldots, D_g$ in H for which $H \setminus \mathring{N}\big(\cup_i D_i\big)$ is a three-ball. Such a collection of disks is called a *disk system* for H. The number g is called the *genus* of the handlebody H, denoted by $g(H)$.

Example 2.11. (1) The three-ball and the solid torus are handlebodies.

(2) For any compact orientable surface Σ with boundary, $\Sigma \times I$ is a handlebody of genus $1-\chi(\Sigma)$. To see that, one can easily exhibit $k = (1-\chi(\Sigma))$ properly embedded arcs, say $\gamma_1, \ldots, \gamma_k \subset \Sigma$, which cut Σ into a disk D. Then the disks $D_i = \gamma_i \times I$ cut $\Sigma \times I$ into the three-ball $D \times I$.

One convenience of working with handlebodies is that they do not contain essential spheres.

Proposition 2.12. *Any handlebody is irreducible.*

Proof. Let H be a genus g handlebody with a disk system $D_1, \ldots, D_g$. Let S be a sphere in H. Isotope S to intersect the disk system minimally. The minimality assumption implies that S is disjoint from the disk system. Otherwise, if S intersects the disk sytem, we can further isotope S to eliminate curves of intersection that are innermost in S. Since S is disjoint form the disk system, it lies in the three-ball $B = H \setminus (\cup_i D_i)$. It follows from Alexander's Theorem [2] that S bounds a ball in B, and hence in H. □

The boundary ∂H of a genus g handlebody H is a closed orientable genus g surface. One could imagine H as a solid orientable genus g surface standardly embedded in $\mathbb{R}^3$, or a thickening of a wedge of g circles in $\mathbb{R}^3$. The last characterization of H implies that $\pi_1(H) \cong \pi_1(\vee_g S^1) \cong \mathbb{Z}^{*g}$. Handlebodies are among the simplest three-manifolds, and they are classified up to homeomorphism by their genus.

One convenience of working with handlebodies is that they contain essential disks. When a pair of disjoint essential disks is given we can create a new one, via *band sum*, which is a very useful operation. So, let us define it.

Definition 2.13 (Band sum)**.** Let D_1, D_2 be disjoint essential disks in a handlebody H, and $\alpha \subset \partial H$ an arc connecting D_1 to D_2 with interior disjoint from D_1 and D_2. A closed neighborhood N of $(D_1 \cup \alpha \cup D_2)$ in H is a solid pair of pants with three disks on the boundary. Two of these three disks are isotopic to D_1 and D_2, by construction. The third disk on the boundary of N is called the *band sum* of D_1 and D_2 along α, denoted by $D_1 *_\alpha D_2$.

In the above definition, notice that the disk $D_1 *_\alpha D_2$ is not essential if and only if D_1 and D_2 cut off a one-handle h from H such that α is contained h. Therefore, when a pair of essential disks is given, the bad sum operation is a convenient way to create a new essential disk. This will turn out to be useful when we prove some statements regarding Heegaard splittings below.

2.2.2 Heegaard Splittings

Let U, V be genus g handlebodies and $f : \partial U \to \partial V$ a homeomorphism. When we identify U and V along their boundaries via f, we obtain a closed, connected, orientable three-manifold $M = U \sqcup_f V$. Notice that the identified surface $\partial U = \partial V$ represents a properly embedded separating surface $P \subset M$ which bounds handlebodies U and V in M. This motivates the following definition.

Definition 2.14. A properly embedded two-sided genus g surface P in a closed three-manifold M is called a *Heegaard surface* if it cuts M into two handlebodies U and V. The triple (P, U, V) is said to be a *genus g Heegaard splitting* of M. We sometimes denote the Heegaard splitting by (U, V) or just P.

The following celebrated theorem made Heegaard splittings crucial tools in the study of three-manifolds.

Theorem 2.15 (Moise [16])**.** *Every closed, connected, orientable three-manifold has a Heegaard splitting.*

In [16], Moise, in fact, proved that any three-manifold M has a triangulation, i.e., M is homeomorphic to a simplicial complex. Therefore, for a given three-manifold M, the thickening of the one-skeleton of a simplicial complex homeomorphic to M is a handlebody $U \subset M$ such that the complement of the interior of U is another handlebody $V \subset M$. The existence of Heegaard splittings yields the following important invariant for three-manifolds.

Definition 2.16 (Heegaard genus)**.** The *Heegaard genus* of a three-manifold M, denoted by $g(M)$, is the minimum genus among all Heegaard splittings of M.

Example 2.17. (1) A three-manifold has Heegaard genus 0 if and only if it is homeomorphic to S^3, since any gluing of two three-balls always gives S^3.

(2) $S^1 \times S^2$ has Heegaard genus 1, since it can be obtained by gluing two copies of $S^1 \times D^2$ via the identity map on their boundaries.

(3) Lens spaces have genus 1 Heegaard splittings (see e.g. [19]). This fact follows from the characterization of lens spaces as unions of solid tori. A closed three-manifold M is called a *lens space* if it is homeomorphic to neither S^3 nor $S^1 \times S^2$, and it has Heegaard genus 1. If M is defined by the gluing map $f : \partial(S^1 \times D^2) \to \partial(S^1 \times D^2)$

mapping the $(0,1)$ element to (p,q) element in $H_1(S^1 \times S^1; \mathbb{Z})$ with the ordered base $\big([S^1 \times \{*\}], [\{*\} \times \partial D^2]\big)$, then the resulting lens space is $L(p,q)$.

(4) The Poincare homology sphere has a genus 2 Heegaard splitting. Besides, the three-torus $S^1 \times S^1 \times S^1$ has a genus 3 Heegaard splitting (see e.g. [19]).

(5) When we glue two genus g handlebodies via id map along their boundaries, we obtain the connected sum of g copies of $S^1 \times S^2$.

Comparing Heegaard splittings of a three-manifold, up to a certain equivalence relation, is a meaningful thing to do. Two Heegaard splittings (P, U, V) and (P', U', V') of a three-manifold M are called *homoemorphic* if there is a homoemorphism $f : M \to M$ which maps P to P', U to U', and V to V'. They are called *isotopic* if they are homoeomorphic via a homeomorphism f that is isotopic to $\mathrm{id} : M \to M$. When P and P' have distinct genera this definition does not make sense as they cannot even be homeomorphic. In this case, one might ask whether one of the given Heegaard splittings can be obtained from the other by some operation. To this end, we introduce a way of creating new Heegaard splittings from a given one, which is called *stabilization.*

Definition 2.18 (Stabilization)**.** Let (P, U, V) be a genus g Heegaard splitting of a three-manifold M. Choose a properly embedded arc $\gamma \subset U$ for which there exists an arc $\beta \subset P$ such that $\partial\beta = \partial\gamma$ and $\gamma \cup \beta$ bounds an embedded disk D in U (such an arc γ is called *unknotted* in U). Take a closed neighborhood $N(\gamma)$ of γ in U. Define $U' = U - \mathring{N}(\gamma)$ and $V' = V \cup N(\gamma)$ in M. Observe that (1) V' is a handlebody since it is obtained by attaching the one-handle $N(\gamma)$ to V, and (2) U' is a handlebody since the remnant of the disk D in U' completes a disk system of U to a disk system of U'. Hence, both U' and V' are handleboies of genus $(g+1)$ in M with common boundary, say P'. The Heegaard splitting (P', U', V') is called a *stabilization* of (P, U, V).

Note that different choices for γ define isotopic Heegaard splittings, so stabilization is well-defined, up to isotopy. Stabilization is a tool to compare any pair of Heegaard splittings (possibly with the same genus). The first regularity theorem about Heegaard splittings is proven in the 30's by Reidemeister and Singer independently, using the stabilization operation.

Theorem 2.19 (Reidemeister [18], Singer [26]). *Any pair of Heegaard splittings (P, U, V) and (A, X, Y) of a closed orientable three-manifold M has isotopic stabilizations.*

Later on, in the 60's, Waldhausen proved the following theorem which states that the Heegaard splittings of S^3 are fairly simple. For an exposition of Waldhausen's proof in English see [25].

Theorem 2.20 (Waldhausen [29], Theorem 3.1). *Any positive genus Heegaard splitting of S^3 is stabilized from the genus 0 Heegaard splitting. Therefore, any two Heegaard splittings of S^3 with the same genus are isotopic.*

The convenience of working with Heegaard splittings is that when we have a Heegaard splitting (P, U, V) of a three-manifold M, the Heegaard surface P can be compressed both in U and in V. So, to understand some properties of M, we can analyze how essential disks in U and essential disks in V interact with each other. We have the following definitions involving the intersections of essential disks from each side of a Heegaard surface.

Definition 2.21. We say that a Heegaard splitting (P, U, V)

- is *stabilized* if there exist properly embedded essential disks $D \subset U$ and $E \subset V$ such that $E \cap V = \partial E \cap \partial V$ is a single point.

- is *reducible* if there are essential disks $D \subset U$ and $E \subset V$ such that $\partial D = \partial E$.

- is *weakly reducible* if there are essential disks $D \subset U$ and $E \subset V$ such that $\partial D \cap \partial E = \emptyset$.
- is *irreducible* if it is not reducible, i.e., if there are no essential disks $D \subset U$ and $E \subset V$ such that $\partial D = \partial E$.
- is *strongly irreducible* if it is not weakly reducible, i.e., any pair of essential disks $D \subset U$ and $E \subset V$ intersect.

Note that being stabilized and being a stabilization of a Heegaard splitting are, in fact, equivalent. Moreover, reducibility is stronger than weakly reducibility since two essential disks $D \subset U$ and $E \subset V$ with the same boundary in P can be isotoped slightly to have non-intersecting boundaries. Hence, by contrapositive, strong irreducibility is stronger than irreducibility. In general, strong irreducibility and irreducibility are not equivalent. For example, the genus 3 minimal Heegaard splitting of the three-torus $S^1 \times S^1 \times S^1$ is irreducible but not strongly irreducible (see [6]). However, when P has genus 2, strong irreducibility and irreducibility are equivalent.

Proposition 2.22. *If a genus 2 Heegaard splitting (P, U, V) is weakly reducible, then it is reducible.*

Proof. In this proof, we will use the fact that two disjoint essential separating curves in the closed orientable genus 2 surface are isotopic.

Let $D \subset U$ and $E \subset V$ be essential disks with disjoint boundary. If ∂D and ∂E are both separating in P, then they are isotopic by the fact. Hence, we can isotope D and E to have common boundary, which proves reducibility of P. So assume that one of the disks, say D, is non-separating. We can choose an arc γ in P such that (1) γ is disjoint from ∂E, and (2) γ travels from one side of D to the other side of D. Let D' be the band sum of two isotopic copies, D^+ and D^-, of D along γ.

Then D' is a separating essential disk in U disjoint from E. Now we have two cases. If E is separating, then ∂E and $\partial D'$ should be isotopic, and we are done. If E is non-separating, then similarly we can construct an essential band sum, say E', of E^+ and E^- along an arc disjoint from $\partial D'$. Hence, $\partial D'$ and $\partial E'$ are isotopic, and the result follows similarly. □

2.2.3 Heegaard Surfaces vs. Essential Surfaces

In this subsection, we will discuss how Heegaard surfaces interact with essential surfaces. Surprisingly, the notions of irreducibility and strong irreducibility show up effectively in this discussion.

Reduciblity of a Heegaard splitting (P, U, V) is interesting by its own because for disks $D \subset U$ and $E \subset V$ with the same boundary in the Heegaard surface P, the union $S = D \cup E$ is a sphere intersecting P in a single simple closed curve $\partial D = \partial E$. Such a sphere yields the following definition.

Definition 2.23. Let (P, U, V) be a Heegaard splitting of M. A sphere $S \subset M$, which intersects P in a single loop that bounds an essential disk in each handlebody, is called a *Haken sphere.*

We have the following remarkable result of Wolfgang Haken which gives a quite strong sufficient condition for existence of Haken spheres.

Lemma 2.24 (Haken's Lemma [10])**.** *If a closed three-manifold M is reducible, then for every Heegaard splitting (P, U, V) of M, there exists an essential sphere $S \subset M$ such that $S \cap P$ is a single loop, equivalently, S intersects each handlebody in a single essential disk. In other words, every Heegaard splitting of a reducible three-manifold is reducible.*

Next we discuss how being reducible and being stabilized are related for a Heegaard splitting. In fact, they are closely related.

Proposition 2.25. *Assume that (P, U, V) is not a genus 1 Heegaard splitting of S^3. If P is stabilized, then it is reducible.*

Proof. Assume that (P, U, V) is stabilized with disks $D \subset U$ and $E \subset V$ that intersect at a single point in P. Consider the wedge of two circles $\partial D \cup \partial E$ on P. A small closed neighborhood of this set has a circle boundary, say C, which bounds (1) a band sum, say D', of two parallel copies of D along the arc $\partial E \setminus \mathring{N}(D)$ in U, and (2) a band sum, say E', of two parallel copies of E along the arc $\partial D \setminus \mathring{N}(E)$ in V.

To complete the proof, we show that the disks D' and E' are essential in U and V. Assume for a contradiction that they are not essential. Then $C = \partial D' = \partial E'$ bounds a disk F in P. It follows that P is the union of a thickening of a wedge of two circles, namely $\partial D \cup \partial E$, with the disk F. Therefore, P is a torus. Moreover, $S = D' \cup E'$ bounds three-balls on each side, and hence M is S^3. But this contradicts to the assumption. □

To sum up, being stabilized is stronger than being reducible except for a special case. Note also that they are not equivalent, i.e., being reducible does not necessarily imply being stabilized. For example, if there exists an essential sphere in M, then a minimal genus Heegaard splitting would be unstabilized (by definition) and reducible (by Haken's Lemma). However, if there is no essential sphere in a three-manifold M, we can guarantee that being reducible implies being stabilized for any Heegaard splitting of M.

Proposition 2.26. *Let M be an irreducible three-manifold. If a Heegaard splitting (P, U, V) of M is reducible, then it is stabilized.*

Proof. Let $D \subset U$ and $E \subset V$ be essential disks with the same boundary, say C, in P. The sphere $S = D \cup E$ bounds a three-ball B in M, by irreducibility. As D and E are essential disks, $C = \partial D = \partial E$ cannot bound a disk in P. Therefore, the subsurface $P' = P \cap B$ of P with boundary C cannot be a disk. We can obtain S^3 from B by identifying the complementary disks D and E in $S = \partial B$ via a homeomorphism. Let F be the image of D and E in S^3. Then $P' \cup F$ is a closed positive genus surface which defines a Heegaard splitting for S^3. By Waldhausen's Theorem (Theorem 2.20), this Heegaard splitting is stabilized. Moreover, we can isotope the stabilizing disks away from the disk F. So these stabilizing disks, in fact, exist in M with boundaries on P. Hence, (P, U, V) is stabilized. □

One might also ask how irreducibility and stong irreducibility are related for Heegaard splittings. As we stated above, when P has genus 2, stong irreducibility and irreducibility are equivalent. However, strong irreducibility is generally strictly stronger than irreducibility. Surprisingly, the two notions are equivalent when there is no essential surface in M.

Theorem 2.27 (Casson-Gordon [6], Theorem 3.1)**.** *Let (P, U, V) be an irreducible Heegaard splitting of a closed orientable three-manifold M. If (P, U, V) is not strongly irreducible, then M contains a positive genus essential surface.*

Another remarkable result regarding the interaction of Heegaard splittings with essential surfaces is the following lemma introduced by Scharlemann.

Lemma 2.28 (Scharlemann's no-nesting Lemma [23])**.** *Let (P, U, V) be a strongly irreducible Heegaard surface in a three-manifold M. If $\alpha \subset P$ is a simple closed curve that bounds a disk in M, then α bounds a properly embedded disk in U or V.*

The convenience of this lemma is that it help us manipulate the compressions of a strongly irreducible Heegaard splitting P, as we will see in the following chapters.

2.2.4 Primitive and Disk-Busting Curves on Heegaard Surfaces

Let $C \subset M$ be a simple closed curve that lies on a Heegaard surface P that bounds handlebodies U and V in M. The interaction of C with the essential disks in U or V is generally something useful to study.

Definition 2.29. Let (P, U, V) be a Heegaard splitting of M. A simple closed curve $C \subset P$ is called *disk-busting* in U (respectively in V) if $C \cap D$ is non-empty for any essential disk $D \subset U$ (respectively $D \subset V$).

Definition 2.30. Let (P, U, V) be a Heegaard splitting of M. A simple closed curve $C \subset P$ is called *primitive* in U (respectively in V) if $|C \cap D| = 1$ for some essential disk $D \subset U$ (respectively $D \subset V$). Alternatively, a primitive curve in U is called a *core* in U.

When a curve $C \subset P$ is primitive in U, we can isotope P into U in a way that, when we remove $\mathring{N}(C)$ from M, the surface P is a Heegaard splitting of the resulting space. For this construction to make sense, we will introduce a brief generalization of Heegaard splittings for compact three-manifolds with boundary.

In the construction of compact three-manifolds as unions simpler pieces, we replace handlebodies by compression-bodies. What is a compression-body? Let F be a closed orientable genus g surface and consider $X = F \times I$. Choose a colletion of non-isotopic essential simple closed curves, say $\alpha_1, \ldots, \alpha_n$, in $F \times \{0\}$, and glue three-dimensional two-handles $D^2 \times I$ to X by identifiying $\partial D^2 \times I$ with annuli neighborhoods $N(\alpha_i) \cong (\alpha_i \times I)$ in $F \times \{0\}$. Finally, if this process creates any sphere boundaries, attach a three-ball to each of them. The resulting three-manifold, say W, is called a genus g *compression-body*. We denote $F \times \{1\} \subset \partial W$ by $\partial W_+ \cong F$, and $(\partial W \setminus \partial W_+)$ by

∂W_-. A handlebody is a compression-body W such that ∂W_- is empty.

Similar to handlebodies, compression-bodies also have collections of disks cutting them into meaningful pieces, and we can characterize compression-bodies with use of such disks. More precisely, the two-handles we attach along the curves $\alpha_i \subset F \times \{0\}$ can be completed to properly embedded disks in W (by adding the annuli $\alpha_i \times I \subset F \times I$) such that the complement of the disks is a union of three-balls with $\partial W_- \times I$. In particular, when ∂W_- is connected, W can be regarded as a union of $\partial W_- \times I$ and a handlebody glued along a disk.

Definition 2.31. Let M be a compact orientable three-manifold with boundary. A closed, orientable surface P in M is called a *Heegaard surface* in M if there are compression-bodies W_1, W_2 in M such that $P = \partial(W_1)_+ \cap \partial(W_2)_+$ and $\partial M = \partial(W_1)_- \cup \partial(W_2)_-$. In other words, we require P to cut M into two compression-bodies.

Most of the definitions we have provided for Heegaard splittings in the closed setting generalize similarly in the compact setting. Moreover, it is possible to extend (or modify) some of the results given in the closed setting to the compact setting. For example, the Haken's Lemma can be modified for Heegaard splittings of three-manifolds with boundary in the following way.

Lemma 2.32 ([6])**.** *If there exists an essential disk D in M, then there is one which intersects a given Heegaard splitting in a single loop.*

We finalize this section by discussing Heegaard splittings of the complements of primitive curves. Let $C \subset P$ be a primitive curve in U. Choose an essential disk $D \subset U$ that intersects C in a single point. Take two parallel copies, say D^+ and D^-, of D in U so that $D^\pm$ are the "ends" of a tubular neighborhood $N(D) \subset U$. Let D' be the band sum of D^+ and D^- along the arc $C \setminus \mathring{N}(D)$. Then D' is an essential separating disk cutting U into a solid torus, say T, and a genus $g-1$ handlebody, say H. By construction, D is a meridian and C is a longitude for T. Now, if we push C into the solid torus T as a core, then P lies in the complement of C. Moreover, $U' = U \setminus \mathring{N}(C)$ is a compression-body. Hence, P cuts $M \setminus \mathring{N}(C)$ into a pair (U', V), where U' compression-body and V is a handlebody. Thus, P naturally defines a Heegaard splitting for $M \setminus \mathring{N}(C)$.

In a subsequent chapter, we will study with knots $K \subset M$ that are primitive on a Heegaard surface P. Using the construction above, we will be able to regard P as a Heegaard surface of the knot exterior $M \setminus \mathring{N}(K)$, which will be a very useful assumption.

2.3 Fibered Links and Complexity

A collection of circles that is properly embedded in a three-manifold M is called a *link*. When L has a single component, it is called a *knot*. We generally denote knots by K, and links with multiple components by L. The *exterior* of a link $L \subset M$, denoted by X_L, is the complement of an open tubular neighborhood $\mathring{N}(L)$ in M.

2.3.1 Fibered Link Exteriors

A link $L \subset M$ is called *fibered* if there is a fibration $p : (M \setminus L) \to S^1$ with fibers, called *pages*, homeomorphic to the interior of a compact surface Σ. Identifying S^1 with $[0, 2\pi]/\sim$, we denote the page $p^{-1}(\theta)$ by Σ_θ, and each page has L as its boundary. When L is a fibered link, the restriction of the fibration map p to X_L is still a fibration with fibers homeomorphic to the compact surface Σ. When we cut X_L open along a fiber, we get an interval bundle homeomorphic to $\Sigma \times [0, 2\pi]$. Hence, X_L is homeomorphic to the mapping torus $M_\phi = \Sigma \times [0, 2\pi]/(x, 0) \sim (\phi(x), 2\pi)$ for some homeomorphism $\phi : \Sigma \to \Sigma$ such that $\phi|_{\partial\Sigma} = \mathrm{id}$. The homeomorphism ϕ is called a *monodromy* of the fibered link L.

The simplest example of a fibered knot is the unknot in S^3. If $K \subset S^3$ is the unknot, then its exterior X_K is homeomorphic to the solid torus $S^1 \times D^2$ such that the essential disks $\{*\} \times D^2$ are indeed disks bounded by K in S^3. Hence, the exterior X_K is fibered, and K is a fibered knot.

Let $K \subset S^3$ be a knot, fibered or non-fibered, with exteriror X_K. As an immediate corollary of the Alexander's theorem [2], X_K contains no essential spheres. Because if S is a sphere in X_K, then regarding S in S^3, it bounds three-balls on both sides in S^3, and one of the balls completely lies in X_K. This proves the following.

Theorem 2.33. *For any knot $K \subset S^3$, the exterior X_K is irreducible.*

This a very nice property that comes in handy when studying knots in the three-sphere. However, it is not necessarily true for knots in other three-manifolds. For instance, if M contains an essential sphere S, pick any knot $K \subset M$ that does not intersect S. It follows that S is still an essential sphere in X_K, i.e., X_K is reducible. Nevertheless, we will work with fibered knots and links in this dissertation, and one can easily show that the exterior of a fibered link contains no essential spheres, which will be helpful in our work with fibered links.

Theorem 2.34. *If $L \subset M$ is a fibered link, then the exterior X_L is irreducible.*

Proof. Let S be a sphere in X_L. We will show that S is inessential. Isotope S so that $S \cap \Sigma_0$ is minimal. Since Σ_0 is incompressible in X_L, the minimality assumption implies that $|S \cap \Sigma_0| = 0$. Therefore, S lies in the handlebody $H = X_L \setminus \mathring{N}(\Sigma_0)$. Since handlebodies are irreducible (see Proposition 2.12), it follows that S bounds a three-ball B in H. The ball B also lies in X_L. Therefore, S is not essential in X_L. □

2.3.2 Heegaard Splittings Induced by Fibered Links

If L is an fibered link in M with fibration $p : (M \setminus L) \to S^1$, then $U = (p^{-1}([0,\pi]) \cup L)$ and $V = (p^{-1}([\pi, 2\pi]) \cup L)$ are handlebodies homeomorphic to $\Sigma \times I$ embedded in M. Therefore, the triple (P, U, V) defines a Heegaard splitting of genus $g = g(\Sigma \times I) = 1 - \chi(\Sigma)$ for M, where $P = \Sigma_0 \cup L \cup \Sigma_\pi$.

An interesting question is whether minimal genus Heegaard splittings are, in general, defined in this way or not. There are many examples of three-manifolds M such that no minimal genus Heegaard splitting is induced by a fibered link. For example, the following statement implies that most lens spaces do not have minimal genus Heegaard splittings induced by a fibered link.

Proposition 2.35 ([17]). *Let M be a lens space. The minimal genus Heegaard splitting of M is induced by a fibered link if and only if $M \cong L(p, 1)$ for some integer $p \neq 0, \pm 1$.*

Moreover, Rubinstein [20] proves that "most" three-manifolds of Heegaard genus 2 do not have a minimal genus Heegaard splitting induced by a fibered link. In [20], Rubinstein also proposes the question whether there is a sufficient condition for a minimal genus Heegaard splitting to be induced by a fibered link (or knot). In this dissertation, we tackle this question. We give a sufficient condition in terms of complexity of fibered knots/links.

2.3.3 Complexity of Fibered Links

Let $L \subset M$ be a fibered link with monodromy ϕ. The complexity of ϕ is measured using the arc-and-curve complex of the page Σ, which is defined in the following way.

Definition 2.36 (Arc-and-curve complex). A properly embedded arc in Σ is called *inessential* if it cuts off a disk from Σ. A simple closed curve embedded in Σ is called *inessential* if either it is trivial (it bounds a disk in Σ) or it is peripheral (it cuts off an annulus from Σ). An arc or a curve properly embedded in Σ is called *essential* if it is not inessential. Let $Z \subset \partial\Sigma$ be a collection of points, one in each boundary component of Σ. The *arc-and-curve complex* of Σ, denoted by $\mathcal{A}C(\Sigma)$, is the abstract simplicial complex of which vertices are isotopy (rel Z) classes of essential arcs and curves in Σ, and k-simplices are k-tuples of pairwise disjoint (up to isotopy rel Z) essential arcs and curves in Σ. In particular, if two non-isotopic (rel Z) essential arcs or curves are disjoint up to isotopy, then they bound an edge in $\mathcal{A}C(\Sigma)$.

For simplicity, we do not distinguish an arc or a curve from its isotopy class in notation. The *distance*, denoted by $d_{\mathcal{A}C}(\gamma_1, \gamma_2)$, between two isotopy classes γ_1, γ_2 of essential arcs or curves in Σ is then the minimum number of edges between corresponding vertices in the arc-and-curve complex. For a fibered link L with the monodromy $\phi : \Sigma \to \Sigma$, we define the *complexity* of ϕ (or L) by

$$d_{\mathcal{A}C}(\phi) = \min\{d_{\mathcal{A}C}(\gamma, \phi(\gamma)) \mid \gamma \text{ is a vertex in } \mathcal{A}C(\Sigma)\}.$$

One defines the arc complex, denoted by $\mathcal{A}(\Sigma)$, and the curve complex, denoted by $\mathcal{C}(\Sigma)$, similarly. The corresponding complexities $d_{\mathcal{A}}(\phi)$ and $d_{\mathcal{C}}(\phi)$ are also defined in a similar fashion. It immediately follows that $d_{\mathcal{A}C}(\phi) \leq d_{\mathcal{A}}(\phi)$ and $d_{\mathcal{A}C}(\phi) \leq d_{\mathcal{C}}(\phi)$.

In further chapters, we will analyze fibered knots that have high complexity. Before going further on our analysis, note that there are indeed highly complicated monodromies. One way to construct such a monodromy is to take the sufficiently high power of a pseudo-Anasov map. For details, see [21].

Theorem 2.37 ([21] Theorem 3.5)**.** *For any compact, orientable surface Σ with boundary and any integer n, there is a monodromy $\phi : \Sigma \to \Sigma$ such that $d_{\mathcal{A}C}(\phi) > n$.*

3 The Main Results

3.1 Motivation and the Main Theorems

This study was motivated by the following conjecture, which we will refer to as Schleimer's Conjecture.

Conjecture 3.1 (Schleimer [24], Thompson [27])**.** *For any three-manifold M, there is a constant $t(M)$ with the following property: if $K \subset M$ is a fibered knot, then the monodromy of K has complexity at most $t(M)$. Moreover, $t(S^3) = 1$.*

Remark 3.2. Note that the assertion $t(S^3) = 1$ implies that if $K \subset S^3$ is a non-trivial fibered knot with monodromy $\phi : \Sigma \to \Sigma$, then $d(\phi) \leq 1$, i.e., there exists an essential arc or curve α in Σ such that $\phi(\alpha)$ and α are disjoint. This a very strong and challenging conjecture. However, in this dissertation, we will show that $t(S^3) \leq 3$.

The ultimate goal here is to provide a complexity bound for fibered knots/links in a given three-manifold M. One way to do that is to look at the properties of the Heegaard splitting induced by a given fibered knot/link. Indeed, our first theorem is about the complexity of fibered links that induce weakly reducible Heegaard splittings.

Theorem A. *Let $L \subset M$ be a fibered link with monodromy ϕ. If L induces a weakly reducible Heegaard splitting, then $d_{\mathcal{A}}(\phi) \leq 4$.*

The celebrated theorem of Waldhausen (Theorem 2.20) states that every genus $g \geq 2$ Heegaard splitting of S^3 is stabilized, hence weakly reducible. Therefore, Theorem A has the following immediate corollary.

Corollary 3.3. *If $K \subset S^3$ is a non-trivial fibered knot with monodromy ϕ, then $d_{\mathcal{A}}(\phi) \leq 4$.*

The corollary implies that Schleimer's Conjecture holds for the three-sphere with $t(S^3) \leq 4$. Theorem A also has the following corollary, which provides a sufficient condition for a genus one fibered knot to induce a minimal genus Heegaard splitting.

Theorem B. *Let K be a genus one fibered knot with monodromy ϕ in a three-manifold M. If $d_{\mathcal{A}}(\phi) > 4$, then K induces a minimal genus Heegaard splitting.*

Notice that a once-punctured torus Σ contains pairs of disjoint non-isotopic essential arcs, although there are no pairs of disjoint non-isotopic essential curves in it. Therefore, $\mathcal{A}(\Sigma)$ is well-defined, with respect to the definition given above, and speaking about $d_{\mathcal{A}}(\phi)$ makes sense for genus one fibered knots. Note that we do not claim any uniqueness up to isotopy in the statement of Theorem B. At this point, it is not known to us whether a sufficiently complicated genus one fibered knot must induce a unique Heegaard splitting up to isotopy or not.

Below we will present straightforward proofs for Theorems A and B. However, using advanced tools from three-manifold topology, we can (a) provide a smaller complexity bound for fibered knots in the three-sphere and (b) generalize Theorem B for fibered knots of any genus. For a given three-manifold M, one could achieve this by analyzing how minimal genus Heegaard surfaces in M interact with the pages of a given fibered knot. Similar work was previously done by David Bachman and Saul Schleimer [4], where they compare Heegaard splittings and fibrations of closed, connected, orientable three-manifolds. To motivate our work, we shortly introduce their results. Let M be a closed, connected, orientable three-manifold that fibers over S^1. Let $\varphi : \Sigma \to \Sigma$ be the monodromy of M with complexity $d_{\mathcal{C}}(\varphi)$, defined by the action of φ on the curve complex of the closed fiber Σ.

Theorem 3.4 (Bachman-Schleimer [4], Theorem 3.1)**.** *If there exists a properly embedded, closed, essential surface $S \subset M$ with genus $g \geq 2$, then either S is isotopic to a fiber or $d_{\mathcal{C}}(\varphi) \leq 2g - 2$.*

Theorem 3.5 (Bachman-Schleimer [4], Theorem 6.1)**.** *If there exists a strongly irreducible Heegaard surface $P \subset M$ with genus $g \geq 2$, then $d_{\mathcal{C}}(\varphi) \leq 2g - 2$.*

Theorem 3.4 is straightforward and it follows from Thurston's argument that an essential surface $S \subset M$ can be isotoped into a nice position with respect to fibers of M (see [28], Theorem 4). Theorem 3.5, on the other hand, is more challenging part of Bachman-Schleimer's work, and it shows that strongly irreducible Heegaard surfaces behave like essential surfaces when it comes to bounding the complexity of a fibration of M. It is now a well-known phenomenon in the three-manifold topology that strongly irreducible Heegaard surfaces, due to their complicated nature, sometimes behave like essential surfaces. The main technique that is used to generalize a result from essential surfaces to strongly irreducible Heegaard surfaces is the double sweepout technique, which we review in the upcoming chapters.

Note that if a three-manifold M fibers over S^1 with fibers of genus g, then M has a *standard* Heegaard splitting with genus $2g+1$, which is induced by the fibration (see [4] for details). Bachman-Schleimer's theorems stated above implies a classification of small genus Heegaard splittings of a fibered three-manifold M that has a sufficiently complicated monodromy φ.

Corollary 3.6. *Let M be a fibered three-manifold with monodromy φ. Any Heegaard splitting $P \subset M$ satisfying $-\chi(P) < d_{\mathcal{C}}(\varphi)$ is a stabilization of the standard splitting.*

The last corollary can be restated as follows: If a fibered three-manifold M has a sufficiently complicated monodromy, then the standard Heegaard splitting is the minimal one, and small genus Heegaard splittings are the stabilizations of the standard splitting. In this dissertation, following Bachman-Schleimer, we use double sweepout techniques to prove a similar result for fibered knots. In particular, we show that a sufficiently complicated fibered knot in M induces the minimal genus Heegaard splitting, and small genus Heegaard splittings are stabilizations of the minimal one. We state these results and more in Theorems C and D below.

Theorem C. *Let $K \subset M$ be a non-trivial fibered knot with monodromy ϕ and pages of genus greater than one.*

1. *If $M \cong S^3$, then $d_{\mathcal{A}}(\phi) \leq 3$.*
2. *If $M \cong S^1 \times S^2$, then $d_{\mathcal{A}}(\phi) \leq 3$.*
3. *If M is a lens space, then $d_{\mathcal{A}}(\phi) \leq 4$.*
4. *Let $P \subset M$ be a minimal Heegaard surface with genus $g \geq 2$. If $d_{\mathcal{AC}}(\phi) > 2g+2$, then P is induced by K and it is unique up to isotopy.*

Notice that Theorem C directly implies that Schleimer's conjecture holds for certain fibered knots.

Corollary 3.7. *Schleimer's conjecture holds for fibered knots which do not induce minimal genus Heegaard splittings.*

A slightly different version of Part (4) of Theorem C was previously announced in the unpublished preprint [13] (see Theorem 1) by Jesse Johnson, where the proof was given using an axiomatic thin position argument and Bachman's index theory [3]. Here we provide a more direct proof based on standard thin position and double sweepout arguments. The techniques we use in the proof of Theorem C are strong enough to provide the following result.

Theorem D. *Let $K \subset M$ be a fibered knot with monodromy ϕ and $P \subset M$ a minimal Heegaard splitting with genus $g \geq 2$. If $d_{\mathcal{AC}}(\phi) > 2h+2$ for some integer $h > g$, then any non-minimal Heegaard surface $P' \subset M$ with genus $g' \leq h$ is a stabilization of P.*

Theorems C and D together implies that a higly complicated fibered knot in a three-manifold M classifies small genus Heegaard splittings of M, which is a quite strong result for the Heegaard splittings literature.

3.2 Proofs of Theorems A and B

In this section, we present proofs of Theorems A and B after introducing some preliminary lemmas.

3.2.1 The Main Lemma for Theorem A

Say L is a fibered link that induce a weakly reducible Heegaard splitting. Let Σ_θ, $\theta \in [0, 2\pi]$, be the pages of L and (P, U, V) the weakly reducible Heegaard splitting induced by L. With an abuse of notation, we may regard $U = \Sigma \times [0, \pi]$ with $\partial U = \Sigma_0 \cup (\partial\Sigma \times [0, \pi]) \cup \Sigma_\pi$, and $V = \Sigma \times [\pi, 2\pi]$ with $\partial V = \Sigma_\pi \cup (\partial\Sigma \times [\pi, 2\pi]) \cup \Sigma_{2\pi}$. We denote the *vertical boundaries* $\partial\Sigma \times [0, \pi]$ by $\partial_v U$ and $\partial\Sigma \times [\pi, 2\pi]$ by $\partial_v V$.

Since (P, U, V) is weakly reducible, there exist essential disks $D \subset U$ and $E \subset V$ such that $\partial D \cap \partial E = \emptyset$ in P. Isotope D and E such that they are transverse to $\partial\Sigma_0 = \partial\Sigma_{2\pi}$ and $\partial\Sigma_\pi$, and $|D \cap \partial_v U|$ and $|E \cap \partial_v V|$ are minimal simultaneously. Theorem A follows from a straightforward analysis of how the essential disks D and E interact with the pages Σ_θ and the vertical disks in U and V. The minimality assumption on D and E provides the following.

Claim 3.8. *Each arc of intersection in $D \cap \Sigma_0$, $D \cap \Sigma_\pi$, $E \cap \Sigma_\pi$, or $E \cap \Sigma_{2\pi}$ is essential in the corresponding page.*

Proof. Assume that such an arc of intersection, say $\alpha \subset D \cap \Sigma_0$, is inessential in Σ_0. Then we can isotope D in U to eliminate α from the intersection. However, such an isotopy eliminates at least two vertical arcs from $|D \cap \partial_v U|$, which contradicts the minimality assumption. $\square$

The proof of Theorem A will follow from the following lemma.

Lemma 3.9. *Let I be the interval $[0, 1]$. If an essential disk $D \subset \Sigma \times I$ intersects Σ_0 and Σ_1 in essential arcs, then there exists a component α of $\partial D \cap \Sigma_0$ and a component β of $\partial D \cap \Sigma_1$ such that α and β are disjoint up to isotopy when projected to Σ, i.e., $d_{\mathcal{A}}(\alpha, \beta) \leq 1$ in $\mathcal{A}(\Sigma)$.*

Proof. Let α be a component of $\partial D \cap \Sigma_0$ that intersects the collection of arcs $\partial D \cap \Sigma_1$ minimally up to isotopy when projected to Σ. Take the vertical disk $D_\alpha = \alpha \times I$ in $\Sigma \times I$. Isotope the disks D and D_α so that

1. the disk D still intersects Σ_0 and Σ_1 in essential arcs,

2. the intersection $\partial D \cap D_\alpha$ lies Σ_1, and

3. the disks D and D_α intersect minimally in Σ_1.

We can achieve this by picking an isotopic copy of α in Σ_0 that is disjoint from $\partial D \cap \Sigma_0$ to span D_α, and then isotoping the arcs of $\partial D \cap \Sigma_1$ to intersect the isotopic copy of α in Σ_1 minimally. By irreducibility of handlebodies, we can further isotope the interior of D in U to eliminate simple closed curves from $D \cap D_\alpha$. Now the intersection $D \cap D_\alpha$ only consists of arcs. Let δ be an arc in $D \cap D_\alpha$ that is outermost in D, i.e., δ cuts off a disk Δ from D that does not intersect D_α. Let δ' be the arc $\Delta \cap \partial D$ so that δ' does not intersect D_α.

Claim 3.10. *The endpoints of δ' are in Σ_1. Moreover, $|\delta' \cap \partial\Sigma_1|$ is an even number.*

Proof. The first claim follows from the fact that ∂D and ∂D_α intersects in Σ_1. So, $\partial\delta = \partial\delta'$ should be in Σ_1. The second claim then follows from the fact that $\partial\Sigma_1$ is separating in ∂U. Since δ' has both endpoints in Σ_1, everytime it leaves Σ_1, it must enter back in Σ_1. Hence, δ' hits $\partial\Sigma_1$ an even number of times. □

Claim 3.11. $|\delta' \cap \partial\Sigma_1| \neq 0$.

Proof. For a contradiction, assume that $|\delta' \cap \partial\Sigma_1| = 0$, i.e., δ' lies in Σ_1. Then δ' and the segment, say δ'', of $D_\alpha \cap \Sigma_1$ between the endpoints of δ' bounds a disk $\Delta'' \subset U$ which has boundary $\delta' \cup \delta''$ in Σ_1. Since Σ_1 is incompressible in U, the simple closed curve $\delta' \cup \delta''$ bounds a disk Δ'' in Σ_1. It follows that we can isotope D, by pushing δ' away from δ'' in Σ_1 through Δ'', to reduce the intersection $D \cap D_\alpha$. This contradicts the minimality of the intersection $D \cap D_\alpha$ in Σ_1. □

Claim 3.12. $|\delta' \cap \partial\Sigma_1| \neq 2$.

Proof. For a contradiction, assume that $|\delta' \cap \Sigma_0| = 2$. Then, δ' and the segment of $D_\alpha \cap \Sigma_1$ between the endpoints of δ' forms a disk which is isotopic to the vertical disk $D_{\alpha'}$, where $\alpha' = \delta' \cap \Sigma_0 \subset D \cap \Sigma_0$. It follows that $D_{\alpha'}$ intersects D in less number of points than D_α. In particular, α' intersects $D \cap \Sigma_1$ is less number of points than α up to isotopy when projected to Σ, which is impossible by the choice of α. □

Claim 3.13. *If $|\delta' \cap \partial\Sigma_1| \geq 4$, then the lemma is true.*

Proof. If $|\delta' \cap \Sigma_0| \geq 4$, then $\delta' \cap \Sigma_1$ contains an arc of intersection, say $\beta \subset D \cap \Sigma_1$, that is essential in Σ_1. Moreover, β is disjoint from an isotopic copy of α in Σ_1, since δ' does not intersect D_α away from its endpoints. Notice that the arcs α and β satisfy the lemma, i.e., the lemma is true in this case. □

By Claims 3.10, 3.11, and 3.12, $|\delta' \cap \partial\Sigma_1|$ is even and cannot be less than 4. Thus, the lemma follows from Claim 3.13. □

3.2.2 Proof of Theorem A

Proof. Let (P, U, V) be the Heegaard splitting induced by a fibered link L with pages Σ_θ, $\theta \in [0, 2\pi]$, where $U = \Sigma \times [0, \pi]$, and $V = \Sigma \times [\pi, 2\pi]$ with an abuse of notation. If (P, U, V) is weakly reducible, then there are properly embedded essential disks $D \subset U$ and $E \subset V$ so that $\partial D \cap \partial E = \emptyset$. Choose D and E transverse to $\partial\Sigma_0 = \partial\Sigma_{2\pi}$ and $\partial\Sigma_\pi$ such that $|D \cap \partial_v U|$ and $|E \cap \partial_v V|$ are minimal simultaneously. By Claim 3.8, D intersects Σ_0 and Σ_π in essential arcs, and E intersects Σ_π and $\Sigma_{2\pi}$ in essential arcs.

Now, by Lemma 3.9, there are arcs $\alpha \subset D \cap \Sigma_0$ and $\beta \subset D \cap \Sigma_\pi$ such that $d_{\mathcal{A}}(\alpha, \beta) \leq 1$ in the arc complex $\mathcal{A}(\Sigma)$. Similarly, there are arcs $\beta' \subset E \cap \Sigma_\pi$ and $\alpha' \subset E \cap \Sigma_{2\pi}$ such that $d_{\mathcal{A}}(\beta', \alpha') \leq 1$. Since ∂D and ∂E are disjoint, β and β' are disjoint essential arcs in Σ_π, i.e., $d_{\mathcal{A}}(\beta, \beta') \leq 1$. Similarly, $\phi(\alpha)$ and α' are disjoint essential arcs in $\Sigma_{2\pi}$, i.e., $d_{\mathcal{A}}(\alpha', \phi(\alpha)) \leq 1$ as well. Combining these inequalities under the triangle inequality, we get

$$d_{\mathcal{A}}(\phi) \leq d_{\mathcal{A}}(\alpha, \phi(\alpha)) \leq d_{\mathcal{A}}(\alpha, \beta) + d_{\mathcal{A}}(\beta, \beta') + d_{\mathcal{A}}(\beta', \alpha') + d_{\mathcal{A}}(\alpha', \phi(\alpha)) \leq 4,$$

as desired. □

3.2.3 Proof of Theorem B

Proof. We prove the contrapositive. Let $K \subset M$ be a genus one fibered knot which does not induce a minimal genus Heegaard splitting. Note that K induces a genus 2 Heegaard surface, say P, in M. If P is not a minimal genus Heegaard surface, then M is either S^3, $S^1 \times S^2$, or a lens space. It follows from [29], [10], or [5] respectively that P

is a weakly reducible Heegaard surface. Thus, Theorem A implies that $d_{\mathcal{A}}(\phi) \leq 4$. □

3.3 Outline of the Proofs of Theorems C and D

In this section, we will list the key results that are provided in the upcoming chapters and used in the proofs of Theorems C and D. At the end of the section, we will prove Theorems C and D, using the listed results.

3.3.1 Key Results

In Chapter 4, we prove the following proposition by analyzing the interaction between essential surfaces embedded in M and the pages of fibered links.

Proposition 4.1. *Let $L \subset M$ be a fibered link with monodromy ϕ. If M contains an essential sphere, then $d_{\mathcal{A}}(\phi) \leq 3$. If M contains an incompressible surface of genus $g > 0$, then $d_{\mathcal{AC}}(\phi) \leq 2g + 2$.*

The proposition, when combined with some classical theorems about Heegaard splittings, implies the following.

Theorem 4.2. *Let $L \subset M$ be a fibered link with monodromy ϕ. If M contains a genus $g \geq 2$ Heegaard surface P, which is weakly reducible but not stabilized, then*

$$d_{\mathcal{AC}}(\phi) \leq \begin{cases} 3 & , \text{ if } g = 2, \\ -\chi(P), & \text{if } g \geq 3. \end{cases}$$

In particular, if a minimal genus Heegaard surface $P \subset M$ is weakly reducible, then the given complexity bound holds.

In Chapter 5, we introduce the thin position and double sweepout techniques.

These will be useful to achieve a complexity bound for the monodromy of a fibered knot K that cannot be isotoped into a Heegaard surface P. The main result of Chapter 5 is the following.

Theorem 5.1. *Let $K \subset M$ be a fibered knot with monodromy ϕ. If $P \subset M$ is a Heegaard surface of genus g such that K cannot be isotoped into P, then*

$$d_{\mathcal{A}}(\phi) \leq \begin{cases} 3 \quad , & \text{if } g = 0, \\ 2g+2\,, & \text{if } g \geq 1. \end{cases}$$

Proposition 4.1 and Theorem 5.1 suffice to prove parts (1), (2), and (3) of Theorem C. Moreover, the last two theorems suffice to prove part (4) of Theorem C when a minimal genus Heegaard surface P is weakly reducible or K cannot be isotoped into P. So, in Chapter 6, we analyze the case that a fibered knot $K \subset M$ lies in a strongly irreducible Heegaard surface P and prove the following.

Proposition 6.1. *Let $K \subset M$ be a fibered knot with monodromy ϕ. If (P, U, V) is a strongly irreducible Heegaard splitting in M such that $K \subset P$, then one of the following holds:*

1. *P is isotopic to the Heegaard surface induced by K.*
2. $d_{\mathcal{AC}}(\phi) \leq 2g - 2$.
3. *K is isotopic to a core in U or V.*

In Chapter 7, the final chapter, we resolve the only case left by the last three statements for a complete proof Theorem C. Namely, we prove the following theorem, by using double sweepout arguments.

Theorem 7.1. *Let $K \subset M$ be a fibered knot with monodromy ϕ and (P, U, V) a Heegaard splitting of genus $g \geq 2$ in M such that K is a core in U or V. If P is strongly irreducible in M, then one of the following holds:*

1. *P is isotopic to the Heegaard surface induced by K.*

2. *$d_{\mathcal{C}}(\phi) \leq 2g - 2$.*

We finish this section by proving Theorems C and D, which readily follow from the results stated above.

3.3.2 Proof of Theorem C

Proof. We prove each statement separately.

1. The bound for S^3 follows from Theorem 5.1 because a non-trivial fibered knot cannot be isotoped into a Heegaard sphere in S^3.

2. The bound for $S^1 \times S^2$ follows from Proposition 4.1 because $S^1 \times S^2$ contains an essential sphere.

3. Let K be a fibered knot in a lens space. If K can be isotoped into a Heegaard torus T, then $A = T \setminus \mathring{N}(K)$ is an incompressible annulus that is not ∂-parallel in X_K. It follows from Lemma 6.2 (see below) that $d_{\mathcal{A}}(\phi) \leq 1 \leq 4$. On the other hand, if K cannot be isotoped into T, then by Theorem 5.1, we obtain $d_{\mathcal{A}}(\phi) \leq 4$ since T has genus 1.

4. Let K be a fibered knot with monodromy ϕ and (P, U, V) a minimal Heegaard splitting with genus $g \geq 2$. Assume that $d_{\mathcal{AC}}(\phi) > 2g+2$. By Theorem 4.2, P is strongly irreducible since it is minimal. By Theorem 5.1, K can be isotoped into P. By Proposition 6.1, K is isotopic to a core in, say, U. Finally, by Theorem

7.1, P is isotopic to the Heegaard surface induced by K. Since P is arbitarily chosen, we deduce that P is the unique minimal genus Heegaard surface in M up to isotopy. □

3.3.3 Proof of Theorem D

Proof. Let $K \subset M$ be a fibered knot with monodromy ϕ such that $d_{\mathcal{AC}}(\phi) > 2h+2$ for some integer $h > g$. Assume that (P', U', V') is a Heegaard splitting in M with genus g' such that $g < g' \leq h$. By Theorem C, K can be isotoped into P'. Moreover, since P' is not a minimal genus splitting, it follows from Theorem C that K does not induce P'. Therefore, by Proposition 6.1, K is isotopic to a core in U'. It follows that none of the necessary conditions of Theorem 7.1 holds for P' and we deduce that P' is weakly reducible. Finally, by Theorem 4.2, P' is stabilized. Since this holds for any genus $g' \leq h$, we deduce that P' can be destabilized into the minimal genus Heegaard surface. □

3.4 The Main Lemmas for Theorems C and D

In the proof of Theorems C and D, surfaces properly embedded in the fibered knot exterior X_K (or in the link exterior X_L) will play an important role. In this section, we will discuss how to position a properly embedded surface $F \subset X_L$ nicely with respect to the pages of L. When F has boundary, the isotopy class of ∂F in ∂X_L will be important. Therefore, let us first distinguish the isotopy classes of simple closed curves in ∂X_L.

Definition 3.14 (Boundary Slopes)**.** If $K_1, \ldots, K_a \subset M$ are the components of L, then $\partial X_L = \partial X_{K_1} \cup \ldots \cup \partial X_{K_a}$ in M. On a torus component ∂X_{K_i}, we can regard the isotopy types of essential simple closed curves as *slopes* in $\mathbb{Q} \cup \{\infty\}$. The *zero slope* is then the isotopy type of the simple closed curve $\partial\Sigma_\theta \cap \partial X_{K_i}$, and any other isotopy type is called a *non-zero slope*. Moreover, the isotopy type of a simple closed curve that bounds a disk in $N(K_i)$ is called the *infinity slope* or the *meridional slope* in ∂X_{K_i}, and any other isotopy type is called a *non-meridional slope*. A surface that has non-empty meridional boundary components in ∂X_L is called a *meridional* surface.

3.4.1 Regular Surfaces and Perfect Surfaces

In this part, we define special positions for surfaces properly embedded in a fibered link exterior.

Definition 3.15. A properly emdedded surface $F \subset X_L$ with (possibly empty) boundary components of non-zero slopes in the components of ∂X_L is said to be *regular* in X_L if

1. The components of ∂F and $\partial\Sigma_\theta$ are transverse in ∂X_L for each θ;
2. F is transverse to each Σ_θ except for finitely many $\theta_1, \ldots, \theta_m \in [0, 2\pi]$;
3. F is transverse to each $\Sigma_{\theta_i}, i = 1, \ldots, m$, except for a single saddle or center tangency.

When F is regular, the pages that are not transverse to F are called *critical* with respect to F. A page that is not critical is called *non-critical*.

Note that the definition of a regular surface can be extended for surfaces with boundary components of zero slopes, however, we will mostly be dealing with meridional surfaces in this dissertation. Clearly, every properly embedded surface $F \subset X_L$ with (possibly empty) boundary components of non-zero slopes can be isotoped to be regular in X_L. To prove Proposition 4.1, we will analyze the tangencies of a regular surface, which intersects pages in essential arcs and curves.

Definition 3.16. Let $F \subset X_L$ be a regular surface with (possibly empty) boundary components of non-zero slopes in ∂X_L. A saddle tangency of F to a page Σ_θ is called *inessential* if for any $\epsilon > 0$ sufficiently small, the component of $F \cap (\Sigma \times [\theta - \epsilon, \theta + \epsilon])$ that contains the tangency has a boundary component that is a trivial simple closed curve in $\Sigma_{\theta \pm \epsilon}$. If a saddle tangency is not inessential, then it is called *essential.*

In the proof of Proposition 4.1, inessential saddle tangencies will be negligible, and the number of essential saddles will be important for complexity calculations. We prove the following two lemmas to provide an upper bound for the number of essential saddle tangencies of a regular surface $F \subset X_L$.

Lemma 3.17. *Let Σ_θ be a page of L. If $\alpha \subset \Sigma_\theta$ is a simple closed curve that bounds a disk in X_L, then α bounds a disk in Σ_θ.*

Proof. Let $D \subset X_L$ be an embedded disk bounded by α such that D intersects Σ_θ transversely, and D intersects Σ_θ minimally among all such disks in X_L. It suffices to show that the interior of D is disjoint from Σ_θ, as Σ_θ is incompressible. Assume for a contradiction that $\mathring{D}$ is not disjoint from Σ_θ and pick a simple closed curve $\beta \subset D \cap \Sigma_\theta$ that is innermost in D so that β bounds a subdisk Δ in D, which is disjoint from Σ_θ. Since Σ_θ is incompressible, β bounds a disk Δ' in Σ_θ. If α lies in Δ', then α is trivial in Σ_θ, so we can assume that α does not lie in Δ'. It follows that $\Delta \cup \Delta'$ forms a sphere that bounds a ball in X_L and we can isotope D through this ball to eliminate β (and other curves in Δ') from the intersection of D with Σ_θ, while maintaining $\alpha = \partial D$. This contradicts the minimality assumption on D. □

Lemma 3.18. *If $F \subset X_L$ is a regular surface with $\chi(F) \leq 0$, then the number of essential saddle tangencies of F to the pages is at most $|\chi(F)| = -\chi(F)$.*

Proof. Let c denote the number of center tangencies, s the number of inessential saddle tangencies, and s' the number of essential saddle tangencies of F to the pages.

Since each center tangency contributes 1, and each saddle tangency contributes -1 to the Euler characteristic of F, we have

$$\chi(F) = c - (s + s') \implies s' = -\chi(F) + (c - s).$$

To prove that $s' \leq -\chi(F)$, we will show that $(c - s)$ is non-positive, equivalently $s \geq c$. We will do this by analyzing the singular foliation, say $\mathcal{F}$, of F defined by its intersections with the pages. Note that we regard every arc or curve α that is in the intersection of F with a page Σ_θ as a leaf of $\mathcal{F}$, while α is a subset of F. So, we write $\alpha \in \mathcal{F}$ and $\alpha \subset F$.

If $\mathcal{F}$ has no leaf that is a trivial simple closed curve in F, then there is no center tangency of F to the pages, i.e., $c = 0$, and $s \geq c$ trivially holds. So, we can assume that $\mathcal{F}$ has leaves that are trivial simple closed curves in F. Then there exists a collection $\mathcal{C} = \{C_1, \ldots, C_k\} \subset \mathcal{F}$ of leaves such that

- for each $j \in \{1, \ldots, k\}$, C_j bounds a disk D_j in F, and D_j's are pairwise disjoint.
- the collection $\mathcal{C}$ is outermost and maximal in F, i.e., if there exists another curve $C' \in \mathcal{F}$ that is trivial in F, then there exists a $j \in \{1, \ldots, k\}$ such that either C' is in D_j or C' and C_j cobound an annulus transverse to the pages.

For each $j = 1, \ldots, k$, let a_j (resp. b_j) be the number of center (resp. saddle) tangencies in D_j. It follows that for each $j \in \{1, \ldots, k\}$,

$$\chi(D_j) = a_j - b_j = 1 \implies b_j = 1 - a_j.$$

The lemma will follow from the following observations.

1. Outside $\cup_{j=1}^{k} D_j$ there are no center tangencies: This is because the collection $\mathcal{C}$ is maximal.

2. For $j = 1, \ldots, k$, each saddle tangency in D_j is inessential: Let p be a saddle tangency of F to a page Σ_θ that lies in some D_j. For $\epsilon > 0$ sufficiently small, the component of $F \cap (\Sigma \times [\theta - \epsilon, \theta + \epsilon])$ that contains the tangency has boundary components that lie in D_j. By the previous lemma, those boundary components are trivial in $\Sigma_{\theta \pm \epsilon}$ since they bound disks in D_j. Hence, by definition, p is an inessential saddle tangency.

3. For $j = 1, \ldots, k$, each C_j is trivial in the corresponding page (by the previous lemma) because C_j bounds a disk D_j.

4. For $j = 1, \ldots, k$, each C_j meets a different saddle tangency outside $\cup_{j=1}^k D_j$: Otherwise, two C_j's merge at the same inessential saddle tangency, which yields a third curve C' that is trivial in F and not contained in $\cup_{j=1}^k D_j$. Notice that C' cannot cobound an annulus with any C_j, which contradicts the maximality assumption on the collection $\mathcal{C}$.

By (1), we have $c = a_1 + \ldots + a_k$. By (2), the number of inessential saddle tangencies inside $\cup_{j=1}^k D_j$ is $b_1 + \ldots + b_k$. By (3) and (4), the number of inessential saddle tangencies outside $\cup_{j=1}^k D_j$ is at least k (one for each C_j). Therefore, we obtain

$$s \geq k + b_1 + \ldots + b_k = k + (a_1 - 1) + \ldots + (a_k - 1) = a_1 + \ldots + a_k = c,$$

as desired. □

So far, we argued that every properly embedded surface $F \subset X_L$ with (possibly empty) boundary components of non-zero slopes can be isotoped to be regular in X_L with certain tangency properties. However, to prove Theorems C and D, we will need a regular surface in X_L with an additional nice property, called *perfectness*. Now, we define that property.

Definition 3.19 (Perfectness)**.** Let $F \subset X_L$ be a properly embedded regular surface.

- If F has non-empty boundary of non-zero slopes, then it is called *perfect* if for any non-critical page Σ_θ, every arc in $F \cap \Sigma_\theta$ is essential in Σ_θ.
- If F is a closed surface, then it is called *perfect* if for any non-critical page Σ_θ, there exists a simple closed curve in $F \cap \Sigma_\theta$ that is essential in Σ_θ.

In the proof of Theorem C, we present a perfect surface in the knot exterior in various cases. In each case, the existence of a perfect surface provides the complexity bound stated in Theorem C. In the following subsection, we will analyze how the existence of a perfect surface provides a complexity bound.

3.4.2 Complexity Bounds

In this subsection, we observe how a perfect surface in X_L imposes a complexity bound for the monodromy ϕ of a fibered link L. The following lemma is adapted from Lemma 17 in [13] with an improvement on the upper bound.

Lemma 3.20. *Let $F \subset X_L$ be a genus g perfect surface with non-empty meridional boundary components in ∂X_L such that $|\partial F| = 2n$. Then*

$$d_{\mathcal{A}}(\phi) \leq \begin{cases} 0 \ , & \text{if } g = 0 \text{ and } n = 1, \\ 3 \ , & \text{if } g = 0 \text{ and } n \geq 2, \\ 2g \ , & \text{if } g \geq 1 \text{ and } n = 1, \\ 2g+2, & \text{if } g \geq 1 \text{ and } n \geq 2. \end{cases}$$

Proof. Let S be the preimage of F under the quotient map $q : \Sigma \times [0, 2\pi] \to X_L$, which maps $\Sigma \times \{\theta\}$ to Σ_θ in the natural way. For simplicity, we will not distinguish

$\Sigma \times \{\theta\}$ and Σ_θ. We can assume that Σ_0 is a regular page with respect to F (after slightly rotating the pages, if necessary). Since $|\partial F| = 2n$, there exist n arcs in $S \cap \Sigma_0$, and $S \cap (\partial\Sigma \times [0, 2\pi])$ consists of vertical arcs $\{x_1, \ldots, x_{2n}\} \times [0, 2\pi]$ as the boundary components are meridional.

By Lemma 3.18, the number, say m, of essential saddle tangencies of F to the pages is at most $-\chi(F) = 2g + 2n - 2$. Let $0 < \theta_1 < \ldots < \theta_m < 2\pi$ be the angles such that Σ_{θ_i}'s contain the essential saddle tangencies. For $i = 1, \ldots, m-1$, pick $\theta_i < t_i < \theta_{i+1}$ such that Σ_{t_i} is transverse to F. Furthermore, set $t_0 = 0$ and $t_m = 2\pi$.

Now we will argue how many different isotopy classes of arcs in Σ can be observed in the intersection of F with the pages. To begin with, there are n essential arcs in $F \cap \Sigma_0$. Moreover, for any pair $\theta < \theta'$ in $[0, 2\pi]$, we have the following observations:

(a) If $\Sigma \times [\theta, \theta']$ contains no essential saddle tangencies of F, then the arcs in $F \cap \Sigma_\theta$ and $F \cap \Sigma_{\theta'}$ represent the same isotopy types in Σ.

(b) If $\Sigma \times [\theta, \theta']$ contains a single tangency of F that is an essential saddle, then there are at most two isotopy classes of arcs in $F \cap \Sigma_{\theta'}$ that are different from the isotopy classes of arcs in $F \cap \Sigma_\theta$. Different isotopy classes are introduced as an arc and a simple closed curve (or two arcs) in Σ_θ merge at the essential saddle tangency.

These observations imply that as θ increases from 0 to 2π, $F \cap \Sigma_\theta$ realizes at most two new arc types in Σ_θ exactly when θ passes through one of $\theta_i, i = 1, \ldots, m$. It follows that the number, say N, of isotopy classes of arcs that can be observed in $F \cap \Sigma_\theta$, for $\theta \in [0, 2\pi]$, is at most

$$n + 2m \leq n + 2[-\chi(F)] = n + 2(2g + 2n - 2) = 4g + 5n - 4.$$

Note also that each of these arcs is essential in its respective page because F is assumed to be perfect. Moreover, every arc type that is realized in $F \cap \Sigma_\theta$ has its endpoints in $\{x_1, \ldots, x_{2n}\} \subset \Sigma_\theta$ by abusing the notation. For $j = 1, \ldots, 2n$, let k_j be the number of isotopy classes of arcs in $F \cap \Sigma_\theta$ that have x_j as an endpoint. It follows that

$$k_1 + \ldots + k_{2n} = 2N \leq 10n + 8g - 8$$

because each of the N isotopy classes is counted twice (once for each endpoint) in the sum. We deduce that for some x_j, the number k_j is at most $(10n + 8g - 8)/2n = 5 + (4g - 4)/n$. Without loss of generality, let x_1 be the endpoint realized by

$$k \leq 5 + (4g - 4)/n$$

isotopy classes of arcs. Let $\alpha_1, \ldots, \alpha_k$ be those isotopy classes. For $i = 1, \ldots, k-1$, up to relabeling, we can assume that α_{i+1} is introduced as α_i merges into a saddle tangency of F. Therefore, α_i and α_{i+1} represent disjoint isotopy classes in Σ, i.e., $d_{\mathcal{A}}(\alpha_i, \alpha_{i+1}) \leq 1$. Moreover, $\phi(\alpha_1) = \alpha_k$ because $\phi(x_1) = x_1$. Thus, we obtain

$$d_{\mathcal{A}}(\phi) \leq d_{\mathcal{A}}(\alpha_1, \phi(\alpha_1)) = d_{\mathcal{A}}(\alpha_1, \alpha_k) \leq \sum_{i=1}^{k-1} d_{\mathcal{A}}(\alpha_i, \alpha_{i+1}) = k - 1.$$

Now we will run a case analysis depending on the values of g and n to provide the complexity bounds stated in the lemma.

Case 1. $g = 0$ and $n = 1$: In this case, F is an annulus and has no essential saddle tangencies to the pages since $\chi(F) = 0$, and we get $k = 1$. Hence, we obtain $d_{\mathcal{A}}(\phi) \leq k - 1 = 0$, which implies $d_{\mathcal{A}}(\phi) = 0$.

Case 2. $g = 0$ and $n \geq 2$: In this case, we have $k \leq 5 + (4g - 4)/n = 5 - 4/n \leq 4$. Thus, we obtain $d_{\mathcal{A}}(\phi) \leq k - 1 \leq 3$.

Case 3. $g \geq 1$ and $n = 1$: In this case, $k \leq 5 + 4g - 4 = 4g + 1$. However, $4g + 1$ is an unnecessarily large upper bound. Because when $n = 1$, we have a single isotopy class of arc observed by $F \cap \Sigma_\theta$ in between each pair of consecutive essential saddle tangencies. Hence, $k \leq 1 + m \leq 1 + 2 + 2g - 2 = 2g + 1$. Thus, we obtain $d_{\mathcal{A}}(\phi) \leq k - 1 \leq 2g + 1 - 1 = 2g$.

Case 4. $g \geq 1$ and $n \geq 2$: In this case, $k \leq 5 + (4g - 4)/n \leq 2g + 3$. Thus, we obtain $d_{\mathcal{A}}(\phi) \leq k - 1 \leq 2g + 3 - 1 = 2g + 2$. □

We will now prove a similar lemma for closed perfect surfaces in X_L.

Lemma 3.21. *Let $F \subset X_L$ be a properly embedded, closed, perfect surface. If F is a torus, then $d_{\mathcal{C}}(\phi) \leq 1$. If F has genus $g \geq 2$, then $d_{\mathcal{C}}(\phi) \leq -\chi(F)$.*

Proof. Let S be the preimage of F under the quotient map $q : \Sigma \times [0, 2\pi] \to X_L$ mapping $\Sigma \times \{\theta\}$ to Σ_θ in the natural way. For simplicity, we will not distinguish $\Sigma \times \{\theta\}$ from Σ_θ.

Assume F is a torus. Since F is perfect, there exists a curve component α of $F \cap \Sigma_0$ that is essential in Σ_0. Notice that there are no essential saddle tangencies in F because $\chi(F) = 0$. Therefore, the isotopy type of α does not change at all from Σ_0 to $\Sigma_{2\pi}$. Hence, there exists a curve $\beta \subset F \cap \Sigma_{2\pi}$ that is isotopic to α in $\Sigma_{2\pi}$. It follows that either $\phi(\alpha) = \beta$ or $\phi(\alpha) \cap \beta = \emptyset$. In both cases, we obtain $d_{\mathcal{C}}(\beta, \phi(\alpha)) \leq 1$, and hence $d_{\mathcal{C}}(\alpha, \phi(\alpha)) \leq 1$. It follows that $d_{\mathcal{C}}(\phi) \leq 1$.

Now assume that F has genus $g \geq 2$. Let $\theta_1 < \ldots < \theta_m$ be the angles such that Σ_{θ_i}'s contain the essential saddle tangencies of F. By Lemma 3.18, we have $m \leq -\chi(F)$. We can assume that $0 < \theta_i < 2\pi$ by slightly rotating the pages, if necessary. For $i = 1, \ldots, m - 1$, pick $\theta_i < t_i < \theta_{i+1}$ such that Σ_{t_i} is transverse to F. Since F is perfect, each transversal intersection $S \cap \Sigma_{t_i}$ contains a simple closed curve, say α_i, that is essential in Σ_{t_i}. Furthermore, pick a simple closed curve $\alpha_0 \subset S \cap \Sigma_0$

that is essential in Σ_0 and set $\alpha_m = \phi(\alpha_0)$ in $S \cap \Sigma_{2\pi}$. For $i = 0, \ldots, m$, we have the following observations:

(a) If α_i and α_{i+1} are in the boundary of the component of $F \cap \Sigma \times [t_i, t_{i+1}]$ that contains the essential saddle tangency of $F \cap \Sigma_{\theta_i}$, then α_{i+1} is introduced as α_i merges into the saddle tangency, and they represent disjoint isotopy classes.

(b) If one of α_i and α_{i+1} is not in the boundary of the component of $F \cap \Sigma \times [t_i, t_{i+1}]$ that contains the essential saddle tangency of $F \cap \Sigma_{\theta_i}$, then it is observed in both Σ_{t_i} and $\Sigma_{t_{i+1}}$ since it is not affected by the essential saddle tangency. Therefore, α_i and α_{i+1} represent disjoint isotopy classes.

The observations imply that $d_{\mathcal{C}}(\alpha_i, \alpha_{i+1}) \leq 1$ for $i = 0, \ldots, m$, when they are regarded as essential curves in Σ. It immediately follows that

$$d_{\mathcal{C}}(\phi) \leq d_{\mathcal{C}}(\alpha_0, \phi(\alpha_0)) = d_{\mathcal{C}}(\alpha_0, \alpha_m) \leq \sum_{i=0}^{m-1} d_{\mathcal{C}}(\alpha_i, \alpha_{i+1}) \leq m = -\chi(F),$$

as desired. □

Lemmas 3.20 and 3.21 will be the tools that will provide us with the complexity bound in many cases. Equipped with these lemmas, now we are ready to prove the key results that go in to the proofs of Theorems C and D.

4 Fibered Links and Essential Surfaces

In this chapter, we argue that the complexity of a monodromy is bounded when M contains an essential surface. Namely, we will provide the following.

Proposition 4.1. *Let $L \subset M$ be a fibered link with monodromy ϕ. If M contains an essential sphere, then $d_{\mathcal{A}}(\phi) \leq 3$. If M contains an incompressible surface of genus $g > 0$, then $d_{\mathcal{AC}}(\phi) \leq 2g + 2$.*

Here is an immediate corollary of the proposition combined with Casson-Gordon's theorem (Theorem 2.27) and Waldhausen's theorem (Theorem 2.20).

Theorem 4.2. *Let $L \subset M$ be a fibered link with monodromy ϕ. If M contains a genus $g \geq 2$ Heegaard surface P, which is weakly reducible but not stabilized, then*

$$d_{\mathcal{AC}}(\phi) \leq \begin{cases} 3, & \text{if } g = 2, \\ -\chi(P), & \text{if } g \geq 3. \end{cases}$$

In particular, if a minimal genus Heegaard surface $P \subset M$ is weakly reducible, then the given complexity bound holds.

We will prove the proposition and its corollary at the end of this chapter after introducing some preliminary lemmas.

4.1 Finding a Perfect Surface in the Link Exterior

Notice that Proposition 4.1 is stated not only for fibered knots but also for fibered links. For the rest of this chapter, fix a fibered link $L \subset M$ with a monodromy ϕ and pages Σ_θ, for $\theta \in [0, 2\pi]$. In the proof of the proposition, we take an essential/incompressible surface S that intersects the fibered link L minimally. It follows that $F = S \cap X_L$ is a meridional incompressible and ∂-incompressible surface. Following Theorem 4 in [28], such a surface can be isotoped to only have saddle tangencies to the pages, and it intersects the pages of L in essential arcs and curves. One then can obtain the desired complexity bound by analyzing how the isotopy types of the arcs and curves in $\Sigma_\theta \cap F$ change as we travel from $\Sigma_0 \cap F$ to $\Sigma_{2\pi} \cap F$, similar to the proof of Theorem 3.1 in [4]. Alternatively, we will show that such a surface F is perfect in X_L and the complexity bound will follow from Lemmas 3.20 and 3.21.

Remark 4.3. An incompressible and ∂-incompressible surface, such as F mentioned above, is called *index-zero* in the terminology of Bachman [3]. A complexity bound in this case follows from Lemmas 4 and 17 in [13], which uses the counting argument mentioned above.

Even though every surface in X_L is regular up to isotopy, there are surfaces that are not perfect in X_L. A trivial example is a ∂-parallel annulus. However, we can show that incompressible surfaces that are not boundary-parallel are perfect. This will be useful to prove Proposition 4.1 for essential/incompressible surfaces that cannot be isotoped into X_L.

Lemma 4.4. *Let $F \subset X_L$ be a regular surface with non-empty boundary of non-zero slopes in ∂X_L. If F is incompressible, then it is either perfect or a ∂-parallel annulus.*

Proof. Assume F is not perfect. We will show that it is a ∂-parallel annulus in X_L.

Let Σ_θ be a page such that $F \cap \Sigma_\theta$ contains an arc that is inessential in Σ_θ. Then there exists an arc $\alpha \subset F \cap \Sigma_\theta$ that cuts off a disk $\Delta \subset \Sigma_\theta$, which does not intersect F. Since ∂F and $\partial \Sigma_\theta$ are transverse, $\partial\Delta$ meets two distinct boundary components of ∂F, which cobound an annulus A in a component of ∂X_L. Let $N(\Delta) \cong \Delta \times [-1, +1]$ represent a neighborhood of Δ in X_L, and $\Delta^\pm$ denote $\Delta \times \{\pm 1\}$. Joining Δ^+ and Δ^- with the band $B = A \setminus N(D)$, we obtain a disk $D = \Delta^+ \cup B \cup \Delta^-$ in X_L such that $D \cap F = \partial D$. Since F is incompressible, we deduce that ∂D bounds a disk in F, and thus, F is an annulus. By construction, F is also ∂-parallel. $\square$

A similar lemma holds for incompressible surfaces in M that lie in X_L. However, to provide the lemma, we will need the following operation.

Definition 4.5 (Annulus Surgery)**.** Let F be any closed surface in the fibered link exterior X_L. Assume that F is transverse to a page Σ_θ such that a curve $\gamma \subset F \cap \Sigma_\theta$ cuts off an annulus $A \subset \Sigma_\theta$, which is disjoint from F. Take a neighborhood $N(A) \cong A \times [-1, 1]$ of A in X_L so that $N(A) \cap F = N(\gamma) \cong \gamma \times [-1, 1] \subset F$. Now let T be the boundary component of ∂X_L which meets A, and A' the annulus $T \setminus N(A)$. It follows that $[N(\gamma) \cup A^- \cup A' \cup A^+]$ is a peripheral torus isotopic to T in X_L, where $A^\pm$ represents $A \times \{\pm 1\}$ in $N(A)$. Isotoping F in M through the solid torus bounded by T in M, we can replace the annulus $N(\gamma)$ by the annulus $[A^- \cup A' \cup A^+]$. This operation is called the *annulus surgery* of F along γ since replacing an annulus in F is the essential part of it.

In Figure 4.1, we describe the annulus surgery in a schematic picture, where the binding L and the curve of intersection γ are represented as dots. Notice that this isotopy of F eliminates γ from $F \cap \Sigma_\theta$.

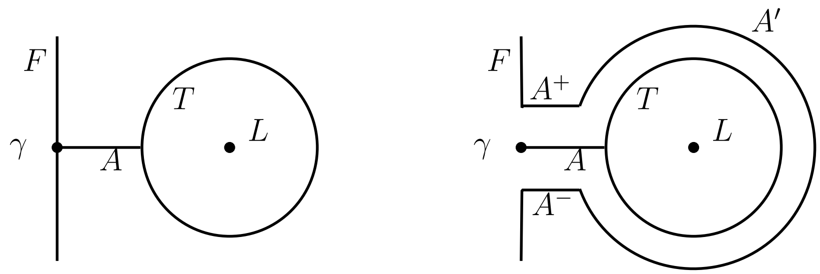


Figure 4.1: The result of the annulus surgery is on the right.

The annulus surgery is useful to establish the following lemma.

Lemma 4.6. *Let $F \subset M$ be an incompressible surface disjoint from the fibered link L. Then F is perfect in X_L.*

Proof. We prove the contrapositive. Assume that F is not perfect. Let Σ_θ be a page transverse to F such that every curve in $F \cap \Sigma_\theta$ is inessential in Σ_θ. If there are trivial curves in the intersection, then by applying the standard innermost curve argument, we can isotope F to eliminate those trivial curves from the intersection. So, we can assume that every curve in $F \cap \Sigma_\theta$ is peripheral in Σ_θ. Applying repeated annulus surgeries to F along the peripheral curves, starting with outermost ones, we can further isotope F in M to eliminate all peripheral curves from $F \cap \Sigma_\theta$. At the end, we get an isotopic copy of F in M that is disjoint from the page Σ_θ. In other words, F can be isotoped into the handlebody $X_L \setminus N(\Sigma_\theta)$, which implies that F is compressible in M. □

4.2 Proof of Proposition 4.1

Proof. Let $S \subset M$ be an essential/incompressible surface of genus g that intersects the fibered link L transversely and minimally among all genus g essential surfaces embedded in M. We have two cases.

Case 1. $S \cap L$ is non-empty: In this case, $F = S \cap X_L$ is a properly embedded genus g surface with meridional boundary components in ∂X_L since S intersects L transversely. It follows from standard arguments that F is incompressible in X_L and it is not a ∂-parallel annulus. By Lemma 4.4, F is perfect in X_L. Therefore, by Lemma 3.20, we obtain $d_{\mathcal{A}}(\phi) \leq 3$ when $g(F) = 0$ (i.e., when S is a sphere), and $d_{\mathcal{A}}(\phi) \leq 2g + 2$ when $g(F) = g(S)$ is positive.

Case 2. $S \cap L$ is empty: In this case, S cannot be a sphere since there exists no essential sphere in a fibered link exterior. By Lemma 4.6, S is a perfect surface in X_L. It follows from Lemma 3.21 that $d_{\mathcal{C}}(\phi) \leq \max\{1, 2g - 2\} \leq 2g + 2$. □

4.3 Proof of Theorem 4.2

Proof. Let (P, U, V) be a genus $g \geq 2$ Heegaard splitting in M, which is weakly reducible but not stabilized. We have two cases.

Case 1. P is an irreducible splitting: When P is weakly reducible but irreducible, it follows from Theorem 3.1 in [6] that M contains an essential surface S of positive genus, which is obtained by compressing the Heegaard surface P at least once in both U and V. Hence, $S \subset M$ is an essential surface such that $0 < g(S) \leq g - 2$. By Proposition 4.1, we obtain $d_{\mathcal{A}C}(\phi) \leq 2g(S) + 2 \leq 2(g - 2) + 2 = 2g - 2 = -\chi(P)$.

Case 2. P is a reducible splitting: When P is reducible but not stabilized, it is a corollary of Theorem 3.1 in [29] that M contains an essential sphere. Hence, by Proposition 4.1, we obtain $d_{\mathcal{A}C}(\phi) \leq d_{\mathcal{A}}(\phi) \leq 3$. When $g \geq 3$, this particularly implies the desired bound $d_{\mathcal{A}C}(\phi) \leq 3 < 2g - 2 = -\chi(P)$.

Finally, the last statement in the theorem then follows from the fact that a minimal genus Heegaard surface is never stabilized. □

5 Fibered Knots in Thin Position

In this chapter, we will prove the following theorem which provides the complexity bound stated in Theorem C, when K does not lie on a minimal genus Heegaard splitting up to isotopy. Namely, we will prove the following.

Theorem 5.1. *Let $K \subset M$ be a fibered knot with monodromy ϕ. If $P \subset M$ is a Heegaard surface of genus g such that K cannot be isotoped into P, then*

$$d_{\mathcal{A}}(\phi) \leq \begin{cases} 3 \quad , & \text{if } g = 0, \\ 2g+2\,, & \text{if } g \geq 1. \end{cases}$$

In the previous chapter, our assumptions were strong enough to provide a meridional incompressible and perfect surface in the fibered knot exterior, which helped us execute a combinatorial argument that gives a complexity bound on the monodromy. However, there exist fibered knots which contain no incompressible surfaces in their exterior (namely, the small knots). In this chapter, we use thin position and double sweepout arguments to provide a meridional surface that behaves similarly to perfect surfaces. Such a surface will reveal itself as a level surface for a thin position of K with respect to a sweepout of the Heegaard surface P (see below for definitions). The techniques we use here are similar to those in [4], [9], and [15]. Before proving the theorem, we will introduce literature, notation, and some useful lemmas. The proof of the theorem will be presented at the end of this chapter.

Assumption. For the rest of this chapter, assume that $K \subset M$ is a fibered knot with monodromy ϕ, which cannot be isotoped into the given Heegaard surface P.

5.1 Sweepouts of Heegaard Splittings

A *spine* of a handlebody U is a connected graph G in U such that $U \setminus G$ is homeomorphic to $\partial U \times (0,1]$. Let P be a Heegaard surface bounding a pair of hendlebodies (U, V) in M. Let G_U and G_V be spines of U and V, respectively. Then $M \setminus (G_U \cup G_V)$ is homeomorphic to $P \times (0,1)$. A *sweepout* of the Heegaard surface P is a smooth function $H : P \times I \to M$ such that $H(P \times \{0\}) = G_U$, $H(P \times \{1\}) = G_V$, and $H(P \times \{t\})$ is isotopic to P for any $t \neq 0, 1$. For simplicity, we will denote $H(P \times \{t\})$ by P_t, and we will not distinguish $H(P \times (0,1))$ from $P \times (0,1)$. On the other hand, a *height function* of P is the map from $h : P \times (0,1) \to (0,1)$, which maps P_t to t.

5.2 Thin Position

Thin position for knots was invented by Gabai [8] and applied in many places in the three-manifolds literature. For convenience, we recall the definition of a thin position. Fix a sweepout of P in M with height function h. By an isotopy of K, we may assume that $K \cap (G_U \cup G_V) = \emptyset$ and that $h|_K$ is a Morse function, i.e., $h|_K$ has only finitely many non-degenerate critical values $a_1, \ldots, a_n$ such that K has a unique tangency to each P_{a_i}. Given such a Morse position of K, let $t_1, \ldots, t_{n-1} \in (0,1)$ be non-critical values of $h|_K$ such that $a_i < t_i < a_{i+1}$ for each $i = 1, \ldots, n-1$. We call the number $\Sigma_{i=1}^{n-1} |P_{t_i} \cap K|$ the *width* of the Morse position. A *thin position* of K is then a Morse position of the minimal width. In a thin position of K with respect to a Heegaard surface P, for each non-critical value t of $h|_K$, the Heegaard surface P_t intersects a tubular neighborhood $N(K)$ of K in meridional disks. In other words, $F_t = P_t \setminus N(K)$ is a meridional surface in X_K and we call it a *level surface*.

Now let $a < b$ in $(0, 1)$ such that a is a local minimum of $h|_K$, b is a local maximum of $h|_K$, and (a, b) contains no critical values of $h|_K$. The family $\{F_t \,|\, t \in (a, b)\}$ is called a *middle slab*. We will analyze the intersection of the pages Σ_θ with the levels F_t in a middle slab to introduce a useful level surface F_s in the middle slab.

Assumption. For the rest of this chapter, assume that K is in thin position with respect to a fixed sweepout $\{P_t \,|\, t \in [0, 1]\}$ of the given Heegaard surface P and fix a middle slab $\{F_t \,|\, t \in (a, b)\}$.

5.3 Intersection Graphics of Surface Families

One can isotope the pages Σ_θ so that they are standard with respect to level surfaces F_t near ∂X_K. Moreover, by Cerf theory [7], the pages Σ_θ can be further isotoped so that the pages Σ_θ and the level surfaces F_t of the middle slab are in *Cerf position*, that is, the set

$$\Lambda = \{(\theta, t) \in S^1 \times (a, b) \,|\, \Sigma_\theta \text{ is not transverse to } F_t\}$$

is a one-dimensional graph in the open annulus $A = S^1 \times (a, b)$ satisfying the following properties:

1. If (θ, t) is in the complement of Λ, then Σ_θ and F_t intersect transversely in a collection of arcs and simple closed curves (by definition of Λ).

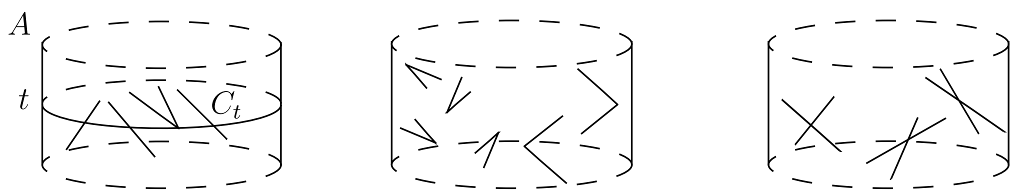
2. If (θ, t) and (θ', t') are in the same connected component of $A \setminus \Lambda$, then $\Sigma_\theta \cap F_t$ and $\Sigma_{\theta'} \cap F_{t'}$ have the *same* intersection pattern.

3. If (θ, t) is on an edge of Λ, then Σ_θ and F_t are transverse except for a single center or saddle tangency. Moreover, the tangency type does not alter along an edge of Λ. In other words, every edge represents a center or saddle tangency.

4. For any number $t \in (a, b)$, the horizontal circle $C_t := S^1 \times \{t\} \subset A$ contains at most one vertex of Λ. Similarly, for any angle $\theta \in [0, 2\pi]$, the vertical interval $I_\theta := \{\theta\} \times (0, 1)$ contains at most one vertex of Λ (see Figure 5.1a).

5. A vertex of Λ is either a *birth-and-death* vertex with valence 2 as in Figure 5.1b, or a *crossing* vertex with valence 4 as in Figure 5.1c.

6. The edges of Λ are not tangent to any horizontal circle C_s or vertical interval I_θ (see Figure 5.1a).



(a) Horizontal circles. (b) Birth-and-death. (c) Crossing vertices.

Figure 5.1: Local pictures of the intersection graphic Λ.

Definition 5.2. The graph Λ is called an *intersection graphic* of the families Σ_θ and F_t. A connected component of the $A \setminus \Lambda$ is called a *region* of $A \setminus \Lambda$.

Assumption. For the rest of this chapter, assume that the pages Σ_θ and the level surfaces F_t of the middle slab are in a Cerf position providing an intersection graphic Λ in the annulus A, satisfying the properties listed above.

5.4 Labeling the Levels of the Middle Slab

Following Section 1 of [9], we label a level surface F_t with L (respectively with H) if there exists a page Σ_θ such that $\Sigma_\theta \cap F_t$ contains a properly embedded arc $\alpha \subset \Sigma_\theta$ that cuts off a half disk Δ^- (respectively Δ^+) from Σ_θ such that the arc $\beta^- = \Delta^- \cap \partial\Sigma_\theta$ (respectively $\beta^+ = \Delta^+ \cap \partial\Sigma_\theta$) lies completely below (respectively above) F_t. We say that Δ^-/Δ^+ is a *low/high* disk for F_t. Notice that, in the definition, $\Delta^\pm$ are allowed to include circles from the intersection of $\Sigma_\theta \cap F_t$.

One can also define the labeling for the regions of $A \setminus \Lambda$ in the following way: A region R of $A \setminus \Lambda$ is labeled with L (respectively with H) if there exists a point (θ, t) in R such that an arc in $\Sigma_\theta \cap F_t$ cuts off a low disk (respectively a high disk) for F_t from Σ_θ.

Observation 5.3. *By properties of the Cerf position, if a region R receives a label, then for every t, for which the horizontal circle C_t meets R, the level F_t receives the same label. Moreover, by definition of the labeling, a level F_t is labeled with L or H if and only if there exists a region R that receives the label L or H, respectively, and meets the horizontal circle C_t.*

Remark 5.4. We will see below that a level surface F_t that intersects a page Σ_θ in arcs that are inessential in Σ_θ receives a label. Using thin position arguments, we will introduce a level surface F_s, which is not labeled, and therefore, has no inessential arcs of intersection with the pages. As in the proof of Lemma 3.20, such a surface will be an essential tool to execute a combinatorial argument which provides the complexity bound stated in Theorem 5.1.

Thin position arguments and the intersection properties of the pages Σ_θ and the levels F_t will provide the following two lemmas which will be useful in detecting a surface F_s that is not labeled.

Claim 5.5. *For $\delta > 0$ small, $F_{a+\delta}$ is labeled with L and $F_{b-\delta}$ is labeled with H.*

Proof. Since the pages are standard near ∂X_K with respect to the level surfaces F_t, there exists a page Σ_θ, which hangs down near the local maximum b. Therefore, for t values sufficiently close to b, $\Sigma_\theta \cap F_t$ contains an arc that cuts off a high disk for F_t from Σ_θ, i.e., F_t is labeled with H. Similarly, for t values sufficiently close to a, the level surface F_t is labeled with L. □

Since we essentially use the same labeling with Section 1 of [9], we immediately obtain the following.

Claim 5.6. *There exists no $t \in (a, b)$ such that F_t is labeled with both L and H. In other words, every level surface F_t receives at most one label. Hence, every region of $A \setminus \Lambda$ receives at most one label.*

Proof. The first statement follows from Lemma 1.1 in [9]. It immediately follows that a region R receives at most one label. Otherwise, if there is a region receiving both labels, then every level F_t, for which the horizontal circle C_t meets R, receives both labels. □

The last two claims imply that the label of F_t change from L to H as t increases from 0 to 1. Next we show that there must be a level surface F_s, which receives no label. We introduce this surface and analyze its properties in the following section.

5.5 A Special Level

Now we are ready to introduce a special level in the middle slab. Let

$$s := \sup\{t \in (a, b) \mid F_t \text{ is labeled with } L\}.$$

The level surface F_s will be the surface in the knot exterior which provides the complexity bound stated in Theorem 5.1. In this section, we will show that F_s can be assumed to satisfy certain conditions towards the proof of the theorem, by proving the following.

Lemma 5.7. *For any θ, any transversal arc in the intersection $\Sigma_\theta \cap F_s$ is essential in Σ_θ. Moreover, either Theorem 5.1 holds or F_s satisfies the following properties:*

1. *The horizontal circle C_s contains a crossing vertex of Λ.*

2. *For any $\epsilon > 0$, there exist numbers $s_- \in (s-\epsilon, s)$ and $s_+ \in (s, s+\epsilon)$ such that F_{s_-} is labeled with L and F_{s_+} is labeled with H.*

We will prove Lemma 5.7 at the end of this section. First, we present a discussion that provides a sequence of claims that are used in the proof of the lemma.

Claim 5.8. *The number s equals neither a nor b.*

Proof. We immediately obtain $a < s$ from Claim 5.5 because for sufficiently small δ values, $F_{a+\delta}$ is labeled with L. On the other hand, assume for a contradiction that $s \geq b$, hence $s = b$. Then there exist t values arbitrarily close to b such that F_t are labeled with L. By Claim 5.5, such levels are labeled with H as well, which is impossible by Claim 5.6. □

Claim 5.9. *The level surface F_s is not labeled.*

Proof. Assume that F_s is labeled with either L or H. We show that both cases lead to a contradiction.

Case 1. F_s is labeled with L: In this case, a transversal arc of intersection in $\Sigma_\theta \cap F_s$ that bounds a low disk for F_s in Σ_θ persists in the intersection $\Sigma_\theta \cap F_t$ for any

$t \in (s-\epsilon, s+\epsilon)$, for $\epsilon > 0$ sufficiently small. Therefore, for every number $t \in (s, s+\epsilon)$, the level F_t receives the label L. But this contradicts that s is the supremum.

Case 2. F_s is labeled with H: In this case, a transversal arc of intersection in $\Sigma_\theta \cap F_s$ that bounds a high disk for F_s in Σ_θ persists in $\Sigma_\theta \cap F_t$ for any $t \in (s-\epsilon, s+\epsilon)$, for $\epsilon > 0$ sufficiently small. Therefore, for any $t \in (s-\epsilon, s)$, the level F_t receives the label H. Since F_s is not labeled with L (by the previous case), there exists a number $t \in (s-\epsilon, s)$ such that F_t receives the label L as well, which contradicts Claim 5.6. □

Claim 5.10. *For any $\epsilon > 0$, there exists $t \in (s-\epsilon, s)$ such that F_t is labeled with L.*

Proof. Since s is the supremum of L-labeled levels and F_s is not labeled (by Claim 5.9), parameters of the L-labeled levels must be arbitrarily close to s. □

Claim 5.11. *For any angle θ, transversal arcs in $\Sigma_\theta \cap F_s$ are essential in Σ_θ.*

Proof. Assume for a contradiction that $\Sigma_\theta \cap F_s$ contains a transversal arc of intersection α that is inessential in Σ_θ. We will show that F_s is labeled, which contradicts Claim 5.9.

Case 1. Σ_θ and F_s intersect transversely: In this case, α cuts off a half disk Δ from Σ_θ such that Δ intersects F_s transversely in embedded arcs and simple closed curves. Then an arc of intersection $\alpha' \subset \Delta \cap F_s \subset \Sigma_\theta \cap F_s$, that is outermost in Δ, cuts off a half disk $\Delta' \subset \Delta$ which is a low or high disk for F_s in Σ_θ. This implies F_s is labeled.

Case 2. Σ_θ and F_s do not intersect transversely: In this case, (θ, s) is in the intersection graphic Λ and we can find an angle θ' sufficiently close to θ so that

(i) The point (θ', s) lies in a region of $A \backslash \Lambda$, that is, $\Sigma_{\theta'}$ and F_s intersect transversely.

(ii) The intersection arc α persists in $\Sigma_{\theta'} \cap F_s$ as an inessential arc in $\Sigma_{\theta'}$.

In other words, $\Sigma_{\theta'}$ and F_s are as in the previous case. An identical argument yields a label for F_s. □

Notice that the last claim establishes the first statement in Lemma 5.7. Now we will introduce other claims of a different flavor to analyze the intersection graphic Λ.

Claim 5.12. *If the horizontal circle $C_s \subset A$ contains no vertex of the intersection graphic Λ, then Theorem 5.1 holds.*

Proof. If there exists no vertex of Λ in C_s, then the level surface F_s is in regular position with respect to pages. Moreover, by Claim 5.11, every transversal arc of intersection in $\Sigma_\theta \cap F_s$ is essential in Σ_θ for any angle θ. In other words, F_s is a meridional perfect surface in X_K (see Definition 3.19), where $g(F_s)$ equals the Heegaard genus g of M. By Lemma 3.20, we get $d_{\mathcal{A}}(\phi) \leq 3$ when $g = 0$, and $d_{\mathcal{A}}(\phi) \leq 2g + 2$ when $g \geq 1$. Thus, Theorem 5.1 holds. □

Claim 5.13. *If there exists an $\epsilon > 0$ such that F_t is not labeled for any $t \in (s, s + \epsilon)$, then Theorem 5.1 holds.*

Proof. Assume that there exists an $\epsilon > 0$ such that for any $t \in (s, s + \epsilon)$, the level F_t is not labeled. Then we can choose a number $s' \in (s, s + \epsilon)$ such that $F_{s'}$ receives no label and $C_{s'}$ contains no vertex of Λ. In other words, $F_{s'}$ satisfies the hypotheses of Claims 5.11 and 5.12. Applying identical arguments to $F_{s'}$, we deduce Theorem 5.1 holds. □

Claim 5.14. *If there exists a birth-and-death vertex on C_s, then Theorem 5.1 holds.*

Proof. Assume that C_s contains a birth-and-death vertex. Recall that every horizontal circle in A contains at most one vertex of Λ. So, away from the birth-and-death vertex, C_s intersects edges of Λ transversely. We introduce a case analysis depending on the location of the edges adjacent to the vertex on C_s, and we either reach at a contradiction or show that Theorem 5.1 holds.

Case 1. One edge is above C_s, and the other is below: In this case, there exists an $\epsilon > 0$ such that for any $t \in (s-\epsilon, s)$, the horizontal circle C_t meets the same regions as C_s. Since F_s is unlabeled by Claim 5.9, all regions intersecting C_s are unlabeled. In other words, for any $t \in (s-\epsilon, s)$, all regions intersecting C_t are unlabeled. This implies that the level F_t is unlabeled for any $t \in (s-\epsilon, s)$, which is impossible by Claim 5.10.

Case 2. Both edges are above C_s: In this case, again there exists an $\epsilon > 0$ such that for any $t \in (s-\epsilon, s)$, the horizontal circle C_t meets the same regions as C_s. Similarly, this implies that for any $t \in (s-\epsilon, s)$, F_t is unlabeled, which is impossible by Claim 5.10.

Case 3. Both edges are below C_s: In this case, there exists an $\epsilon > 0$ such that for any $t \in (s, s+\epsilon)$, the horizontal circles C_t and C_s meet the same regions. Similarly, F_t is unlabeled for $t \in (s, s+\epsilon)$. Thus, by Claim 5.13, Theorem 5.1 holds. □

Now we are ready to prove Lemma 5.7 and finish this section.

Proof of Lemma 5.7. By Claim 5.11, any transversal arc of intersection in $\Sigma_\theta \cap F_s$ is essential in Σ_θ.

Now assume that (1) does not hold. Then either (a) C_s contains no vertex or (b) C_s contains a birth-and-death vertex. In case (a), Theorem 5.1 holds by Claim 5.12. In case (b), Theorem 5.1 holds by Claim 5.14.

Finally, assume (2) does not hold. By Claim 5.10, for any $\epsilon > 0$, there exists $s_- \in (s-\epsilon, s)$ such that F_{s_-} is labeled with L. Therefore, if (2) does not hold, then there exists an $\epsilon > 0$ such that for any $t \in (s, s+\epsilon)$, F_t is not labeled. Thus, by Claim

5.13, Theorem 5.1 holds. □

5.6 Analyzing the Crossing Vertex

In the previous section we showed that for $s = \sup\{t \in (a,b) \,|\, F_t \text{ is labeled with } L\}$, any transversal intersection arc in $\Sigma_\theta \cap F_s$ is essential in Σ_θ. Moreover, in Lemma 5.7, we showed that if F_s does not satisfy one of the following properties, then Theorem 5.1 holds:

1. The horizontal circle C_s contains a crossing vertex of Λ.

2. For any $\epsilon > 0$, there exist numbers $s_- \in (s-\epsilon, s)$ and $s_+ \in (s, s+\epsilon)$ such that F_{s_-} is labeled with L and F_{s_+} is labeled with H.

Since, our ultimate goal is to prove Theorem 5.1, in this section, we assume that F_s satisfies (1) and (2), and we analyze F_s further to prove some claims that will be used in the proof of the theorem.

Let (ψ, s) be the crossing vertex of Λ that is in C_s. By rotating the open book, if necessary, we can assume that ψ is a non-zero angle, and Σ_0 is transverse to F_s. Let R^+ be the region that is adjacent to the edges above C_s at (ψ, s), R^- the region that is adjacent to the edges below C_s at (ψ, s). Moreover, let R^w (respectively R^e) be the region to the west (respectively to the east) of the vertex (ψ, s). Let the four edges adjacent to the vertex (ψ, s) be e_1, e_2, e_3, e_4, as in Figure 5.2.

Claim 5.15. *The region R^+ is labeled with H and R^- is labeled with L. The surfaces Σ_ψ and F_s intersect transversely except for two saddle tangencies. Moreover, the two saddle tangencies are entangled, i.e., $\Sigma_\psi \cap F_s$ has a connected singular component containing both saddle tangencies.*

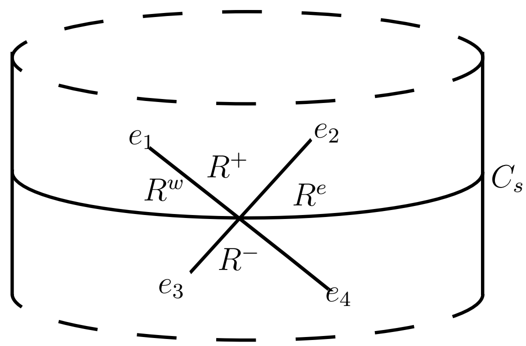


Figure 5.2: The local picture of Λ near the crossing vertex (ψ, s).

Proof. Assume for a contradiction that R^- is not labeled. Let $R_1, \ldots, R_n$ be the regions that meet C_s. By properties of the intersection graphic Λ, the horizontal circle C_s intersects edges of Λ transversely away from the vertex (ψ, s). Then there exists an $\epsilon > 0$ such that for any $t \in (s - \epsilon, s)$, C_t intersects the regions $R_1, \ldots, R_n$, and R^-. Since C_s is not labeled, none of the regions R_i are labeled. Since R^- is not labeled either, it follows that C_t meets no labeled regions. This implies that F_t is not labeled for $t \in (s - \epsilon, s)$, which contradicts the assumption (2) above. On the other hand, if we assume that R^+ is not labeled, it follows from the same argument that there exists an $\epsilon > 0$ such that F_t is not labeled for $t \in (s, s + \epsilon)$, which again contradicts the assumption (2) above.

To prove the second claim, choose $\epsilon > 0$ sufficiently small so that $(\psi - \epsilon, s) \in R^w$ and $(\psi, s + \epsilon) \in R^+$. If we travel from $(\psi - \epsilon, s)$ to $(\psi, s + \epsilon)$ along the straight line between them, we cross Λ once at the edge e_1. Since R^w is not labeled and R^+ is labeled, this implies that the tangency represented by e_1 changes arc types in the intersections. Thus, e_1 must represent a saddle tangency rather than a center tangency. A similar argument implies that e_2 must represent a saddle tangency as well, because R^+ is labeled and R^e is not labeled. Thus, the edges e_1 and e_2 represent two saddle tangencies between Σ_ψ and F_s. Finally, observe that the saddle tangency represented by e_1 introduces inessential arcs of intersection and the saddle tangency represented by e_2 eliminates the same inessential arcs of intersection. Thus, the two saddle tangencies meet the same singular component of $\Sigma_\psi \cap F_s$, which implies that they are entangled. □

Claim 5.16. *Let G be the singular component of $\Sigma_\psi \cap F_s$ containing the two entangled saddles. Then G meets $\partial\Sigma_\psi$.*

Proof. Assume for a contradiction that $\partial\Sigma_\psi \cap G = \emptyset$. This implies that no arcs in $\Sigma_\psi \cap F_{s-\epsilon}$ interact with the entangled saddles. In particular, if we travel from $(\psi-\epsilon, s)$ to $(\psi, s+\epsilon)$ along the edge e_1 between them, the entangled saddle represented by e_1 does not alter the arc types in $\Sigma_{\psi-\epsilon} \cap F_s$. Thus, every arc in $\Sigma_\psi \cap F_{s+\epsilon}$ is essential in Σ_ψ, and so R^+ is not labeled, which is impossible by Claim 5.15. □

Now we will analyze how the entangled saddles of F_s to the page Σ_ψ affect the type of intersection arcs from $\Sigma_{\psi-\epsilon} \cap F_s$ to $\Sigma_{\psi+\epsilon} \cap F_s$. Fix $\epsilon > 0$ small enough so that Σ_ψ is the only critical page in $\Sigma \times [\psi - \epsilon, \psi + \epsilon]$, and let $\widehat{F}$ be the component of $(\Sigma \times [\psi - \epsilon, \psi + \epsilon]) \cap F_s$ that contains the singular component of $\Sigma_\psi \cap F_s$.

Claim 5.17. *For any pair of arcs $\alpha^\pm \subset \Sigma_{\theta\pm\epsilon} \cap \widehat{F}$, we have $d_{\mathcal{A}}(\alpha^+, \alpha^-) \leq 2$.*

Proof. Let $N(G)$ be a neighborhood of the singular component G of $\Sigma_\psi \cap F_s$ in Σ_ψ. Notice that $G \subset \Sigma_\psi$ is a graph with two vertices of valence 4 away from $\partial\Sigma_\psi$, where $g(\Sigma_\psi) \geq 2$. Therefore, $N(G)$ does not fill the surface Σ_ψ, i.e., there exists an essential arc, say β, in Σ_ψ disjoint from $N(G)$.

Let $\pi : \Sigma \times [\psi - \epsilon, \psi + \epsilon] \to \Sigma_\psi$ be the projection map. It follows that $\pi(\alpha^\pm) \subset N(G)$ up to isotopy. Therefore, the arc β is disjoint from $\pi(\alpha^+)$ and $\pi(\alpha^-)$ up to isotopy. Thus, we get $d_{\mathcal{A}}(\alpha^+, \alpha^-) \leq 2$. □

Claim 5.18. *Any simple closed curve in $\Sigma_{\psi\pm\epsilon} \cap \widehat{F}$ is non-trivial $\Sigma_{\psi\pm\epsilon}$.*

Proof. Without loss of generality, assume for a contradiction that $\Sigma_{\psi-\epsilon} \cap \widehat{F}$ contains a simple closed curve α that is trivial in $\Sigma_{\psi-\epsilon}$. We can also assume that this curve interacts with the saddle tangency represented by the edge $e_1 \subset \Lambda$ (see Figure 5.2). So, if we travel from $(\psi - \epsilon, s)$ to $(\psi, s + \epsilon)$ along the straight line between them, an essential arc enters into the saddle tangency with the trivial curve α, and the arc types in the intersection do not change. This, in particular, implies that every arc in $\Sigma_\psi \cap F_{s+\epsilon}$ is essential in Σ_ψ, and so R^+ is not labeled, which is impossible by Claim 5.15. □

Lemma 5.19. *The number of essential saddles of* $\Sigma \times ([0, \psi - \epsilon] \cup [\psi + \epsilon, 2\pi]) \cap F_s$ *is at most* $-\chi(F_s) - 2$.

Proof. Let c be the number of center tangencies, s the number of saddle tangencies, and s_i (respectively s_e) the number of inessential (respectively essential) saddles of F_s to the pages in $\Sigma \times ([0, \psi - \epsilon] \cup [\psi + \epsilon, 2\pi])$ for $\epsilon > 0$ sufficiently small. Since there are exactly two saddle tangencies of F_s in $\Sigma \times [\psi - \epsilon, \psi + \epsilon]$, a standard Euler characteristic calculation provides

$$\chi(F_s) + 2 = c - s = (c - s_i) - s_e \implies s_i = -\chi(F_s) - 2 + (c - s_i).$$

So, it suffices to show that $c - s_i \leq 0$, or equivalently, $c \leq s_i$.

By the last two claims, the saddle tangencies of $\Sigma_\psi \cap F_s$ are neither contained in a subdisk of F_s nor they interact with any inessential curve of intersection in $\Sigma_{\psi\pm\epsilon} \cap F_s$. Hence, it follows from the arguments of Lemma 3.18 that away from the entangled saddles we have $c \leq s_i$, as desired. □

5.7 Proof of Theorem 5.1

As in the proof of Lemma 3.20, the result essentially follows from a counting argument that measures how much the arc types change as we travel from Σ_0 to $\Sigma_{2\pi}$ along a level surface in X_K through the saddle tangencies. The counting arguments slightly differ between the cases $g = 0$ and $g \geq 1$. Therefore, at the end, we will provide different proofs for the three-sphere and higher genus three-manifolds. However, first let us provide the arguments that are common to both cases.

Consider the meridional surface F_s, where $s := \sup\{t \in (a,b) \mid F_t \text{ is labeled with } L\}$ with the labeling defined in Section 5.4 above. By Lemma 5.7, any transversal arc of intersection in $\Sigma_\theta \cap F_s$ is essential in Σ_θ. We can assume that F_s is transverse to the page $\Sigma_0 = \Sigma_{2\pi}$ by slightly rotating the open book, if necessary. Moreover, by Claim 5.15, we can assume that there exists an angle $\psi \neq 0$ such that the page Σ_ψ is transverse to F_s except for two entangled saddle tangencies. For any angle $\theta \neq \psi$, the level F_s is transverse to Σ_θ except for possibly a single center or saddle tangency.

Let there be $2n$ boundary components of F_s. Note that $n \geq 2$ because $n = 1$ would imply that K is isotopic on to the Heegaard surface P since K is in thin position. We can denote ∂F_s as $\{x_1, \ldots, x_{2n}\} \times S^1 \subset \partial\Sigma \times S^1 \cong \partial X_K$, where x_i are distinct points in $\partial\Sigma$. By Claim 5.16, the singular component G of $\Sigma_\psi \cap F_s$ meets $\partial\Sigma_\psi \subset \partial X_K$. It follows that G meets ∂X_K at either 2, 4, or 6 points. For simplicity, let us say an endpoint x_i is *singular* if $\{x_i\} \times S^1$ meets the singular component G. Otherwise, say x_i is *non-singular*. Hence, among $x_1, \ldots, x_{2n}$ there are either 2, 4, or 6 singular endpoints. We denote the number of singular endpoints by r.

Let S be the preimage of F_s under the quotient map $q : \Sigma \times [0, 2\pi] \to X_K$, which maps $\Sigma \times \{\theta\}$ to Σ_θ in the natural way. For simplicity, we will not distinguish $\Sigma \times \{\theta\}$ and Σ_θ. Observe that $S \cap (\partial\Sigma \times [0, 2\pi])$ consists of vertical arcs $\{x_1, \ldots, x_{2n}\} \times [0, 2\pi]$.

Moreover, since Σ_0 is transverse to F_s, the intersection $S \cap \Sigma_0$ consists of some simple closed curves and exactly n essential arcs so that $S \cap \Sigma_{2\pi}$ consists of images of those curves and arcs.

By Lemma 5.19, there are at most $-\chi(F_s) - 2 = (2g + 2n - 4)$ essential saddle tangencies of F_s to the pages in $\Sigma \times ([0, \psi - \epsilon] \cup [\psi + \epsilon, 2\pi])$ for $\epsilon > 0$ sufficiently small. As θ increases from 0 to 2π, the arc types in $\Sigma_\theta \cap F_s$ can change only if a page contains an essential saddle of F_s. Moreover, as we pass through each essential saddle tangency away from Σ_ψ, at most two new arcs can be introduced. As we pass through the entangled saddles in Σ_ψ, at most $r/2$ arc types are introduced. Therefore, the total number of essential arc types that are introduced by essential and entangled saddle tangencies is $2(2g + 2n - 4) + r/2 = 4g + 4n - 8 + r/2$. With the n arcs in $\Sigma_0 \cap F_s$, we deduce that the preimage $S = q^{-1}(F_s)$ intersects the pages of $\Sigma \times [0, 2\pi]$ in at most $4g + 5n - 8 + r/2$ distinct essential arc types. Now, for $i = 1, \ldots, 2n$, let k_i be the number of essential arcs that have endpoints in x_i. Since each arc has two endpoints, when we add k_i's, we get

$$k_1 + k_2 + \ldots + k_{2n} = 8g + 10n - 16 + r,$$

which is the equality that will allow us to apply combinatorial arguments. Now let us prove the theorem.

Proof of Theorem 5.1 for the three-sphere. For $M = S^3$, the Heegaard genus is $g = 0$ and we obtain

$$k_1 + k_2 + \ldots + k_{2n} = 10n - 16 + r.$$

Case 1. $2 \leq n \leq 4$: In this case, there is an endpoint, say x_1, in $\partial\Sigma$ realizing at most 3 arc types, say $\alpha_1, \alpha_2, \alpha_3$. Since $\alpha_1 \subset \Sigma_0$ and $\alpha_3 \subset \Sigma_{2\pi}$ have the same endpoint x_1, we deduce that $\phi(\alpha_1) = \alpha_3$.

If x_1 is a non-singular endpoint, then $d_{\mathcal{A}}(\alpha_1, \alpha_2) \leq 1$ and $d_{\mathcal{A}}(\alpha_2, \alpha_3) \leq 1$ by since the arc types are introduced by essential saddle tangencies away from Σ_ψ. Thus,

$$d_{\mathcal{A}}(\phi) \leq d_{\mathcal{A}}(\alpha_1, \phi(\alpha_1)) = d_{\mathcal{A}}(\alpha_1, \alpha_3) \leq 2 \leq 3.$$

On the other hand, if x_1 is a singular endpoint, then assume without loss of generality that α_2 is introduced as α_1 interacts with the entangled saddles at Σ_ψ. It follows from Claim 5.17 that $d_{\mathcal{A}}(\alpha_1, \alpha_2) \leq 2$. On the other hand, we have $d_{\mathcal{A}}(\alpha_2, \alpha_3) \leq 1$, which provides

$$d_{\mathcal{A}}(\phi) \leq d_{\mathcal{A}}(\alpha_1, \phi(\alpha_1)) = d_{\mathcal{A}}(\alpha_1, \alpha_3) \leq 3.$$

Case 2. $n \geq 5$: In this case, the sum is $k_1 + k_2 + \ldots + k_{2n} = 10n - 16 + r \leq 10n - 10$ since the number r of singular endpoints is at most 6. It follows that either there is an endpoint realizing 3 distinct arc types, or there are at least ten endpoints realizing 4 distinct arc types. If there is an endpoint realizing 3 distinct arc types, then the discussion in Case 1 implies $d_{\mathcal{A}}(\phi) \leq 3$. So, assume that there are at least ten endpoints realizing 4 distinct arc types. In particular, there exists a non-singular endpoint, say x_7, realizing 4 distinct arc types. Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the arc types that are realized by x_7. It follows that $d(\alpha_j, \alpha_{j+1}) \leq 1$, for $j = 1, 2, 3$, since no α_j is involved with the entangled saddles. Since $\phi(\alpha_1) = \alpha_4$, we obtain

$$d_{\mathcal{A}}(\phi) \leq d_{\mathcal{A}}(\alpha_1, \phi(\alpha_1)) = d_{\mathcal{A}}(\alpha_1, \alpha_4) \leq 3.$$

This completes the proof of Theorem 5.1 for the three-sphere. □

Now let us present a proof for three-manifolds with higher Heegaard genus.

Proof of Theorem 5.1 for $g \geq 1$. If M has Heegaard genus $g = g(F_s) \geq 1$, since r is at most 6, we get $k_1 + k_2 + \ldots + k_{2n} = 8g + 10n - 16 + r \leq 8g + 10n - 10$.

Case 1. $n = 2$: In this case, the inequailty turns into $k_1 + k_2 + k_3 + k_4 \leq 8g + 10$. Therefore, there exists an endpoint, say x_1, realizing at most $2g + 2$ arc types, say $\alpha_1, \alpha_2, \ldots, \alpha_{2g+2}$. Since $\alpha_1 \subset \Sigma_0$ and $\alpha_{2g+2} \subset \Sigma_{2\pi}$ have the same endpoint x_1, we deduce that $\phi(\alpha_1) = \alpha_{2g+2}$.

If x_1 is not a singular endpoint, then $d_{\mathcal{A}}(\alpha_j, \alpha_{j+1}) \leq 1$ for each $j = 1, \ldots, 2g+1$ since the arc types are introduced by essential saddle tangencies away from Σ_ψ. Hence, we obtain

$$d_{\mathcal{A}}(\phi) \leq d_{\mathcal{A}}(\alpha_1, \phi(\alpha_1)) = d_{\mathcal{A}}(\alpha_1, \alpha_{2g+2}) \leq 2g + 1 \leq 2g + 2.$$

On the other hand, if x_1 is a singular endpoint, then assume without loss of generality that α_2 is introduced as α_1 interacts with the entangled saddles at Σ_ψ. It follows from Claim 5.17 that $d_{\mathcal{A}}(\alpha_1, \alpha_2) \leq 2$. On the other hand, we have $d_{\mathcal{A}}(\alpha_j, \alpha_{j+1}) \leq 1$ for $j = 2, \ldots, 2g+1$, which provides

$$d_{\mathcal{A}}(\phi) \leq d_{\mathcal{A}}(\alpha_1, \phi(\alpha_1)) = d_{\mathcal{A}}(\alpha_1, \alpha_{2g+2}) \leq 2g + 2.$$

Case 2. $n \geq 3$: In this case,

$$k_1 + k_2 + \ldots + k_{2n} = 8g + 10n - 16 + r \leq 8g + 10n - 10,$$

and hence there is an edpoint realizing at most $5 + (8g - 10)/6 \leq 2g + 2$ edpoints (which can be seen by a case analysis on values of g). The discussion in Case 1 above works equally in this case. Thus, we get $d_{\mathcal{A}}(\phi) \leq 2g + 2$, as desired. □

6 Fibered Knots in Strongly Irreducible Heegaard Surfaces

In the last two chapters, we showed that the complexity bound stated in Theorem C holds when a minimal genus Heegaard splitting $P \subset M$ is weakly reducible, or K cannot be isotoped into P. The remaining case is that the fibered knot K lies on a strongly irreducible minimal genus Heegaard surface P. Therefore, in this chapter, we will prove the following proposition. Notice that we state the proposition for any strongly irreducible Heegaard splitting rather than minimal genus ones, to prove Theorem D as well.

Proposition 6.1. *Let $K \subset M$ be a fibered knot with monodromy ϕ. If (P, U, V) is a strongly irreducible Heegaard splitting in M such that $K \subset P$, then one of the following holds:*

1. *P is isotopic to the Heegaard surface induced by K.*
2. *$d_{\mathcal{AC}}(\phi) \leq 2g - 2$.*
3. *K is isotopic to a core in U or V.*

We will prove Proposition 6.1 at the end of the chapter. In the proof of the proposition, the surface $P \setminus N(K)$ embedded in X_K will play an essential role. Since

K is assumed to lie in P, this surface will have non-meridional boundary components of an integral (possibly zero) slope in ∂X_K. Therefore, before proving Proposition 6.1, we will provide some complexity bounds when X_K contains a non-meridional essential surface.

6.1 Preliminary Lemmas

Lemma 6.2. *Let $F \subset X_K$ be an essential surface with non-empty boundary components of a non-zero slope in ∂X_K. Assume that F is not boundary parallel in X_K. If F is an annulus, then $d_{\mathcal{A}}(\phi) \leq 1$. If $\chi(F) \leq -1$, then $d_{\mathcal{A}}(\phi) \leq -\chi(F)$.*

Proof. By Theorem 4 in [28], we can isotope F in X_K so that F only has saddle tangencies to $m = -\chi(F)$ pages. Moreover, since F is an essential surface that is not boundary parallel, every arc of intersection in $F \cap \Sigma_\theta$ is essential in Σ_θ for any θ (see Lemma 4.4). Let S be the preimage of F under the quotient map $q : \Sigma \times [0, 2\pi] \to X_K$, which maps $\Sigma \times \{\theta\}$ to Σ_θ in the natural way.

If F is an annulus, then there are no tangencies. Fix an arc $\alpha \subset F \cap \Sigma_0$. Since there are no saddle tangencies, there exists an arc $\beta \subset F \cap \Sigma_{2\pi}$, which is isotopic to α. Since F is properly embedded in X_K, either $\beta = \phi(\alpha)$, or β and $\phi(\alpha)$ are disjoint. Thus, we obtain $d_{\mathcal{A}}(\alpha, \phi(\alpha)) = d_{\mathcal{A}}(\beta, \phi(\alpha)) \leq 1$, which implies that $d_{\mathcal{A}}(\phi) \leq 1$.

Now assume that $\chi(F) \leq -1$, so there exist $m = -\chi(F) \geq 1$ saddle tangencies. Let $\Sigma_{\theta_1}, \ldots, \Sigma_{\theta_m}$ be the pages that are transversal to F except for a single saddle tangency, where $0 < \theta_1 < \ldots < \theta_m < 2\pi$. For each $i = 1, \ldots, m-1$, fix an angle t_i in (θ_i, θ_{i+1}) and choose an arc $\alpha_i \subset F \cap \Sigma_{t_i}$. Furthermore, choose an arc $\alpha_0 \subset F \cap \Sigma_0$ and set $\alpha_m = \phi(\alpha_0) \subset F \cap \Sigma_{2\pi}$. Since, for each $i = 0, \ldots, m-1$, there is only a single saddle tangency of F in $\Sigma \times [t_i, t_{i+1}]$, we can isotope α_{i+1} into Σ_{t_i} so that it is disjoint from α_i. In other words, for each $i = 0, \ldots, m-1$, we have $d_{\mathcal{A}}(\alpha_i, \alpha_{i+1}) \leq 1$. Thus, by the triangle inequality, we obtain

$$d_{\mathcal{A}}(\phi) \leq d_{\mathcal{A}}(\alpha_0, \phi(\alpha_0)) = d_{\mathcal{A}}(\alpha_0, \alpha_m) \leq \sum_{i=0}^{m-1} d_{\mathcal{A}}(\alpha_i, \alpha_{i+1}) \leq m = -\chi(F),$$

as desired. □

Remark 6.3. We believe that the complexity bound in the last lemma could be given in terms of the genus rather than the Euler characteristic of F, by a careful application of the combinatorial arguments introduced in the proof of Lemma 3.20. This would be more convenient especially when the number of boundary components of F is large. However, in the proof of Proposition 6.1, we will be dealing with surfaces that have small number of boundary components. Therefore, a complexity bound in terms of Euler characteristic is fine for our purposes.

Next, we provide three lemmas that will be useful in the proof of Proposition 6.1 when we have an incompressible surface in X_K with boundary of the zero slope.

Lemma 6.4 ([30], Proposition 3.1)**.** *Let $F \subset X_K$ be a properly embedded incompressible surface. If F is disjoint from a page Σ_θ, then each component of F is either a ∂-parallel annulus or isotopic to a page.*

Lemma 6.5. *Let $F \subset X_K$ be a properly embedded incompressible surface that has no ∂-parallel annulus component. Assume that F has non-empty boundary components of the zero slope. If there exists a page Σ_θ such that $F \cap \Sigma_\theta$ consists of peripheral curves in Σ_θ, then F is isotopic to a union of pages.*

Proof. Isotope F to intersect Σ_θ minimally. By the previous lemma, it suffices to show that F is disjoint from Σ_θ. Assume for a contradiction that F is not disjoint from Σ_θ. Let us define $N = X_K \setminus N(\Sigma_\theta)$ and $S = F \cap N = F \setminus N(\Sigma_\theta)$.

Claim. S is incompressible in N.

Proof. Assume for a contradiction that S is compressible in N. Choose a compressing disk D for S and let $\gamma = \partial D = D \cap S$. Since F is an incompressible surface, γ bounds a disk $E \subset F$ which does not lie in S. Therefore, E intersects Σ_θ, and a component δ of $E \cap \Sigma_\theta \subset F \cap \Sigma_\theta$ is peripheral in Σ_θ by assumption. Since the peripheral curve $\delta \subset \Sigma_\theta$ bounds a disk in E, we deduce that $\partial\Sigma_\theta$ bounds a disk in X_K. This implies that K is the unknot in $M = S^3$, which contradicts the assumption that M has a strongly irreducible Heegaard splitting. □

By assumption, ∂S is peripheral in the horizontal boundary $\Sigma \times \{0,1\}$ of $N \cong \Sigma \times I$. By Lemma 6.4, we deduce that each component S is either a page or a ∂-parallel annulus in N. It follows that the intersection of F with the page Σ_θ consists of peripheral curves in F. Let $\gamma \subset F \cap \Sigma_\theta$ be an outermost curve of intersection, which cuts off an annulus A from F. We can isotope F to eliminate γ from the intersection $F \cap \Sigma_\theta$, which contardicts the minimality assumption. □

Lemma 6.6. *Let $F \subset X_K$ be a properly embedded incompressible surface that is not a collection of ∂-parallel annuli. Assume that F has non-empty boundary components of the zero slope. If F is not isotopic to a union of pages, then $d_{\mathcal{C}}(\phi) \leq -\chi(F)$.*

Proof. If F is not isotopic to a union of pages and ∂-parallel annuli, by Theorem 4 in [28], we can isotope F in X_K so that F is transverse to all but $m = -\chi(F)$ pages, say $\Sigma_{\theta_1}, \ldots, \Sigma_{\theta_m}$, where $0 < \theta_1 < \ldots < \theta_m < 2\pi$, and F is transverse to each Σ_{θ_i} except for a single saddle tangency. Choose numbers $0 = t_0 < t_1 < \ldots < t_{m-1} < t_m = 2\pi$ such that each t_i is in (θ_i, θ_{i+1}). It follows that

1. Each simple closed curve in $F \cap \Sigma_{t_i}$ is non-trivial in Σ_θ, since a trivial curve would yield a center tangency.

2. For any $i = 0, \dots, m$, at least one curve of intersection in $F \cap \Sigma_{t_i}$ is non-peripheral in Σ_θ, for otherwise F would be isotopic to a union of pages by Lemma 6.5.

Now choose a curve α_i in each $F \cap \Sigma_{t_i}$ that is essential in Σ_{t_i} while ensuring that $\phi(\alpha_0) = \alpha_m$. Observe that for each $i = 0, \dots, m-1$, there is only a single saddle tangency of F in $\Sigma \times [t_i, t_{i+1}]$. Therefore, we can isotope α_{i+1} into Σ_{t_i} so that it is disjoint from α_i. In other words, for each $i = 0, \dots, m-1$, we have $d_{\mathcal{C}}(\alpha_i, \alpha_{i+1}) \leq 1$. Thus, by the triangle inequality, we get

$$d_{\mathcal{C}}(\phi) \leq d_{\mathcal{C}}(\alpha_0, \phi(\alpha_0)) = d_{\mathcal{C}}(\alpha_0, \alpha_m) \leq \sum_{i=0}^{m-1} d_{\mathcal{C}}(\alpha_i, \alpha_{i+1}) \leq m = -\chi(F),$$

as desired. □

Before the proof of Proposition 6.1 we will introduce one more lemma.

Lemma 6.7. *Let (P, U, V) be a strongly irreducible Heegaard splitting in M and K a knot that lies in P. If the surface $F = P \setminus N(K)$ can be compressed in U or V to an annulus A that is ∂-parallel in X_K, then K is a core in U or V, respectively.*

Proof. The proof is symmetric with respect to U and V. Therefore, we will give a proof only for U. Since A is ∂-parallel in X_K, it is ∂-compressible. Let $\Delta \subset X_K$ be a ∂-compressing disk of A, where $\partial\Delta$ is union of arcs α and β such that $\alpha = \partial\Delta \cap A$ and $\beta = \partial\Delta \cap \partial X_K$. Isotope α away from the disks in A introduced by the compressions of F so that α lies in P.

Let B be the annulus component of $\partial X_K \setminus \partial A$ that contains β. If B is the annulus $\partial X_K \cap U$ (resp. $\partial X_K \cap V$), we can find a half disk Δ' in $N(K) \cap U$ (resp. in $N(K) \cap V$) such that $\partial\Delta' = \beta \cup \beta'$, where β' is a spanning arc for the annulus $B' = N(K) \cap P$. (This is because the core of B is an intergral slope in ∂X_K.) Concatenating Δ and

Δ' along β, we obtain a disk $D \subset M$ such that $\partial D = \alpha \cup \beta'$ is a simple closed curve in P that intersects K (which is the core of B') exactly once. Since P is strongly irreducible, it follows from 2.28 that $\alpha \cup \beta'$ bounds a disk $D' \subset U$. Finally, since $\partial D'$ intersects K exactly once, we can push K into U as a core. $\square$

Now we are ready to prove Proposition 6.1.

6.2 Proof of Proposition 6.1

Proof. Consider the surface $F = P \setminus N(K)$ properly embedded in X_K. Notice that F has two boundary components of a non-meridional integral slope in ∂X_K. Since F is obtained by removing an annulus from P, we have $\chi(F) = \chi(P) = 2 - 2g$. Now we provide the proof of the proposition by a case analysis depending on the compressibility of F in X_K, and in each case we show that one of the conclusions asserted in the proposition holds.

Case 1. F is incompressible in X_K: In this case, we have two subcases depending on the boundary slopes of F.

Subcase 1. F realizes the zero slope: In this case, first note that F cannot be an annulus beacuse that would imply P is a torus, which is outruled by the assumprion that $g(P) \geq 2$. Then, by Lemma 6.6, either $d_{\mathcal{C}}(\phi) \leq -\chi(F) = 2g - 2$, i.e., (2) holds, or F is isotopic to a union of two pages. If F is isotopic to a union of two pages, then P is isotopic to the Heegaard surface induced by K, i.e., (1) holds.

Subcase 2. F realizes a non-zero slope: In this case, it directly follows from Lemma 6.2 that $d_{\mathcal{A}}(\phi) \leq -\chi(F) = 2g - 2$, and hence (2) holds.

Case 2. F is compressible in X_K: In this case, by Lemma 2.28, there exists a compressing disk of F lies in one of the handlebodies U or V. With no loss of generality, assume that there exists a compressing disk for F in U. Let $G \subset X_K$ be the surface obtained by "maximally" compressing F in U.

Claim 6.8. *Every non-sphere component of G is incompressible in X_K.*

Proof. Assume for a contradiction that there is a non-sphere component S of G that is compressible in X_K. Let $\gamma \subset S$ be a curve that bounds a compressing disk D for S in X_K. We can isotope γ into $F \cap S$ because $S \setminus F$ is a union of disks in S (which are introduced by the compressions of F in U). Hence, γ is an essential curve on the strongly irreducible Heegaard splitting P that bounds a disk in X_K. By Lemma 2.28, D can be assumed to lie in either U or V completely. If $D \subset U$, then F is not maximally compressed in U, which is a contradiction. If $D \subset V$, then D is an essential disk in V that is disjoint from the compressing disks of F in U, which contradicts the strong irreducibility of P. □

Notice that $\chi(G) > \chi(F) = 2 - 2g$ and G has two boundary components since it is obtained from F by compressions. Let S be the union of the components of G that contains the boundary. Since G is incompressible, so is S. Notice that S cannot be a union of two pages, for otherwise the strongly irreducible Heegaard surface P would be compressed into the Heegaard surface induced by K, which is impossible by Theorem 2.1 in [6]. Hence, the following two cases complete the proof.

Subcase 1. S is a ∂-parallel annulus in X_K: In this case, by Lemma 6.7, K is a core in U. In other words, (3) holds.

Subcase 2. S is not a ∂-parallel annulus in X_K: In this case, Lemmas 6.2 and 6.6 (depending on the boundary slope of S) imply that we have $d_{\mathcal{AC}}(\phi) \leq -\chi(S) \leq -\chi(G) < 2g - 2$, i.e., (2) holds. □

7 Primitive Fibered Knots in Strongly Irreducible Heegaard Surfaces

The discussion so far leaves behind one case to discuss for a complete proof of Theorem C: (P, U, V) is a strongly irreducible Heegaard splitting and K is a fibered knot in M such that K is a core in U or V. Therefore, in this chapter, we will prove the following.

Theorem 7.1. *Let $K \subset M$ be a fibered knot with monodromy ϕ and (P, U, V) a Heegaard splitting of genus $g \geq 2$ in M such that K is a core in U or V. If P is strongly irreducible in M, then one of the following holds:*

1. *P is isotopic to the Heegaard surface induced by K.*

2. *$d_{\mathcal{C}}(\phi) \leq 2g - 2$.*

In Chapter 4, we showed that if there exists a closed surface $S \subset X_K$ that is incompressible in M, then $d_{\mathcal{C}}(\phi) \leq 2g(S) - 2$. In this chapter, we will achieve a similar complexity bound when there is a Heegaard splitting $P \subset X_K$ that is strongly irreducible in M. We will generalize the result of Chapter 4 from closed incompressible surfaces to strongly irreducible Heegaard splittings, by using the double sweepout technique along with a labeling, similar to [4] and [15]. Some of the arguments will be very similar to Chapter 4 and we will give short explanations for such arguments.

We will also refer to the figures of Chapter 4. Before proving the theorem, we will introduce literature, notation and some useful lemmas. The proof of the theorem will be presented at the end of this chapter.

7.1 Intersection Graphics of Surface Families

Assume that (P, U, V) is a strongly Heegaard splitting of M and K is a core in U. Let (P, U', V) denote the Heegaard splitting of X_K determined by P, where U' is the compression-body obtained by removing $\mathring{N}(K)$ from U. A *spine* of U', denoted by $G_{U'}$, is a wedge of $\partial_- U' = \partial X_K$ with a spine of a genus $g-1$ handlebody embedded in U' such that $U' \setminus G_{U'}$ is homeomorphic to $P \times (0, 1]$.

For fixed spines $G_{U'}$ of U' and G_V of V, a *sweepout* of the Heegaard splitting (P, U', V) is a smooth function $H : P \times I \to X_K$ such that $H(P \times \{0\}) = G_{U'}$, $H(P \times \{1\}) = G_V$, and $H(P \times \{t\})$ is isotopic to P for any $t \neq 0, 1$. For simplicity, we will denote $H(P \times \{t\})$ by P_t, and we will not distinguish $H(P \times (0, 1))$ from $P \times (0, 1)$. For any $t \in (0, 1)$, let

(a) U'_t denote the compression-body $P \times [0, t]$ bounded by P_t in X_K,

(b) U_t denote the handlebody $U'_t \cup N(K)$ bounded by P_t in M,

(c) V_t denote the handlebody $P \times [t, 1]$ bounded by P_t in M.

One can isotope the pages Σ_θ in X_K so that they are standard with respect to P_t near the spines $G_{U'}$ and G_V. Moreover, by Cerf theory [7], the pages Σ_θ can be further isotoped so that the families Σ_θ and P_t are in *Cerf position*, that is, the set

$$\Lambda = \{(\theta, t) \in S^1 \times (a, b) \,|\, \Sigma_\theta \text{ is not transverse to } P_t\}$$

is a one-dimensional graph in the open annulus $A = S^1 \times (0,1)$ satisfying the properties (1)-(6) provided in Subsection 5.3. Similar to Chapter 4, the graph Λ is called an *intersection graphic* of the families Σ_θ and P_t, and a connected component of the $A \setminus \Lambda$ is called a *region* of $A \setminus \Lambda$.

Assumption. For the rest of this chapter, assume that K is a fibered knot in M with pages Σ_θ, for $\theta \in S^1 = [0, 2\pi]/\sim$, and (P, U, V) is a strongly irreducible Heegaard splitting of M, and K is a core in U. Let P_t, $t \in (0,1)$, be a sweepout of P in X_K such that the families Σ_θ and P_t of are in a Cerf position providing an intersection graphic Λ in the annulus $A = S^1 \times (0,1)$, satisfying the properties mentioned above.

7.2 Labeling

We label a level surface P_t with U (resp. with V) if there exists a page Σ_θ, which is transverse to P_t, such that every component of $\Sigma_\theta \cap P_t$ is an inessential curve in Σ_θ that is not disk-busting in the handlebody U_t (resp. in V_t). Alternatively, we label a region R of $(A \setminus \Lambda)$ with U (resp. with V) if there exists a point $(\theta, t) \in R$ such that every component of $\Sigma_\theta \cap P_t$ is an inessential curve in Σ_θ that is not disk-busting in the handlebody U_t (resp. in V_t).

In the proof of Theorem 7.1, we will eventually show that if P is not isotopic in M to the Heegaard surface induced by K, then there exists a level surface P_s, which is not labeled. Such a surface will behave similarly to a perfect surface in X_K and will help us achieve a complexity bound, similar to the proof of Theorem 5.1. In this section, we will prove the following lemma which serves to that purpose.

Lemma 7.2. *If there exists a level surface P_t that is labeled with both U and V, then P_t, and therefore P, is isotopic to the Heegaard surface induced by K.*

Before proving the lemma, we will introduce a few claims that will be useful. Since we have already fixed a Heegaard splitting (P, U, V) for M, we denote the Heegaard splittings by (H, X, Y) instead of (P, U, V) in the statements, for the sake of no confusion.

Claim 7.3. *If H is a Heegaard surface of X_K, then any page of K intersects H.*

Proof. Assume for a contradiction that there exists a page Σ_θ of K such that $\Sigma_\theta \cap H = \emptyset$. By Dehn filling X_K along the boundary of a page, we obtain a fibered three-manifold $\widehat{M}$. Moreover, H persists in $\widehat{M}$ as a Heegaard surface that is disjoint form the positive genus fiber $\widehat{\Sigma}_\theta = \Sigma_\theta \cup$(a filling disk). This implies that the fiber $\widehat{\Sigma}_\theta$ lies in a handlebody bounded by H in $\widehat{M}$, which contradicts the incompressibility of $\widehat{\Sigma}_\theta$. □

Claim 7.4. *Assume that (H, X, Y) is a strongly irreducible Heegaard splitting of M, and K is a core in X (or Y). Let Σ_θ be a page of K such that $\Sigma_\theta \cap H$ is a collection of simple closed curves that are peripheral in Σ_θ. Then at least one component $\Sigma_\theta \cap H$ is disk-busting in either X or Y.*

Proof. Assume for a contradiction that no curve in $\Sigma_\theta \cap H$ is disk-busting in X or Y. First note that $\Sigma_\theta \cap H$ is non-empty by Claim 7.3. Since all curves in $\Sigma_\theta \cap H$ are peripheral in Σ_θ, there exists a component J of $\Sigma_\theta \cap H$ which cuts off an annulus A from Σ_θ that contains all other curves of intersection. By assumption, J is not disk-busting in both X and Y. Isotope K to J along the annulus A to position K in H so that the page Σ_θ completely lies in one of the handlebodies, say X, and K is not disk-busting in X. It follows that $F = H \setminus N(K)$ is a surface in X_K that is disjoint from Σ_θ and compressible in the handlebody X. Let $G \subset X$ be the surface obtained by maximally compressing F in X. By Claim 6.8, G is incompressible in X_K. Since Σ_θ is incompressible in X, before compressing F in X we can isotope Σ_θ away from the compressing disks that yield G. Therefore, we can assume that G and Σ_θ are disjoint. Now let S be the union of the components of G which contain $\partial G = \partial F$, so $S \subset X_K$ is an incompressible surface disjoint from Σ_θ with two boundary components of the zero slope in ∂X_K. By Lemma 6.4, we have the following two possibilities for S both of which yield a contradiction.

1. S is a ∂-parallel annulus in X_K: In this case, K is a core in X by Lemma 6.7. By pushing K into X, H becomes a Heegaard splitting of X_K that is disjoint from the page Σ_θ, which is impossible by Claim 7.3.

2. S is isotopic to a union of two pages: In this case, the union of S with the annulus $B = H \cap N(K)$ yields a Heegaard surface H' induced by K. In other words, the strongly irreducible Heegaard surface $H \subset M$ can be compressed in X to another Heegaard surface H', which is impossible by Theorem 2.1 in [6].

This completes the proof. □

Claim 7.5. *Assume that (H, X, Y) is a strongly irreducible Heegaard splitting of M, and K is a core in X (or Y). Let Σ_θ be a page of K such that no component of $\Sigma_\theta \cap H$ bounds an essential disk in X or Y. Then we can isotope H so that every component of $\Sigma_\theta \cap H$ is non-trivial in both Σ_θ and H.*

Proof. First note that any curve $\gamma \subset \Sigma_\theta \cap H$ that is trivial in Σ_θ is also trivial in H. Otherwise, by Lemma 2.28, γ bounds an essential disk D in X or Y, which contradicts to the assumption. On the other hand, any curve $\gamma \subset \Sigma_\theta \cap H$ that is trivial in H is also trivial in Σ_θ by Lemma 3.17 (basically because Σ_θ is incompressible). Therefore, we can isotope H to eliminate trivial curves from the intersection by applying the standard "innermost intersection curve" argument. □

Now we are ready to prove the main lemma of this section.

Proof of Lemma 7.2. Let P_t be labeled with both U and V, i.e., there exist pages Σ_U and Σ_V such that

(a) every curve in $\Sigma_U \cap P_t$ is inessential in Σ_U and not disk-busting in U_t;

(b) every curve in $\Sigma_V \cap P_t$ is inessential in Σ_V and not disk-busting in V_t.

Claim. Both $\Sigma_U \cap P_t$ and $\Sigma_V \cap P_t$ have no component that bounds an essential disk in U_t or V_t.

Proof of the claim. Assume for a contradiction that $\Sigma_U \cap P_t$ has a component γ_U that bounds an essential disk in U_t or V_t. By labeling, γ_U is not disk-busting in U_t. Since P_t is strongly irreducible, we deduce that γ_U cannot bound a disk in V_t. So, γ_U bounds an essential disk $D_U \subset U_t$. Now we have two cases depending on $\Sigma_V \cap P_t$, and both yield a contradiction.

Case 1. $\Sigma_V \cap P_t$ has a component γ_V that bounds an essential disk in U_t or V_t: In this case, since γ_V is not disk-busting in V_t and P_t is strongly irreducible, we deduce that the curve γ_V bounds an essential disk D_V in V_t. However, this implies that $D_U \subset U_t$ and $D_V \subset V_t$ do not intersect, which contradicts the strong irreducibility of P_t.

Case 2. $\Sigma_V \cap P_t$ has no component that bounds an essential disk in U_t or V_t: In this case, by Claim 7.5, we can isotope P_t so that $\Sigma_V \cap P_t$ contains no trivial curves. After the isotopy, $\Sigma_V \cap P_t$ is a collection of peripheral curves that are not disk-busting in V_t. On the other hand, since $\gamma_U \subset \Sigma_U \cap P_t$ bounds an essential disk that is disjoint from $\Sigma_V \cap P_t$, we deduce that $\Sigma_V \cap P_t$ is not disk-busting in U_t either. However, this is impossible by Claim 7.4. □

It follows from Claim 7.5 that we can isotope P_t to eliminate all simple closed curves of $\Sigma_U \cap P_t$ and $\Sigma_V \cap P_t$ that are trivial in Σ_U and Σ_V, respectively. After the isotopy, $\Sigma_U \cap P_t$ (resp. $\Sigma_V \cap P_t$) is a collection of peripheral cuves in Σ_U (resp. in Σ_V). Since $\Sigma_U \cap P_t$ is not disk-busting in U_t, by Claim 7.4, we deduce that it has a component γ_V that is disk-busting in V_t. Similarly, $\Sigma_V \cap P_t$ has a component γ_U that is disk-busting in U_t.

Now we will show that P_t is isotopic to the Heegaard surface induced by K. Let N be the complement of an open tubular neighborhood $\mathring{N}(\Sigma_U \cup \Sigma_V)$ in X_K and $F = P_t \cap N$. Notice that each component of N is homeomorphic to $\Sigma \times I$.

First we prove that F is incompressible in N. Assume for a contradiction that F is compressible with a compressing disk D. Then $\alpha = \partial D$ can be regarded as an essential curve in P_t that bounds a disk in M. By Lemma 2.28, α bounds an essential disk in U_t or V_t, which is impossible because γ_U and γ_V are disk-busting in U and V.

Note that each component of ∂F is peripehral in the horizontal boundary components of N. Therefore, F can be isotoped in N so that ∂F lies in the vertical boundary components $\partial\Sigma \times I$. It follows from Lemma 6.4 that $F = P_t \cap N$ is isotopic to a union of pages and ∂-parallel annuli in N. We deduce that P_t is isotopic in M to a union of a collection pages and annuli. Since P_t is not a torus, it contains a subsurface that is homeomorphic to a page. Since the only closed connected surface that can be constructed as a union of pages and annuli is the Heegaard surface induced by K, it follows that P_t is isotopic to the Heegaard surface induced by K. □

7.3 A Special Level

In the previous section, we showed that if there exists a level surface P_t that receives both labels U and V, then P is induced by the fibered knot K, which is one of the possible conclusions in Thorem 7.1. In this section, we will show that if P is not induced by K, then there exists a level P_s that does not receive a label and this surface will provide the complexity bound stated in Theorem 7.1.

Claim 7.6. *For $\delta > 0$ small, P_δ is labeled with U, and $P_{1-\delta}$ is labeled with V.*

Proof. This is basically because Σ_θ and P_t have standard intersection near the spines.

For t values near 0, every curve $\gamma \subset \Sigma_\theta \cap P_t$ is inessential in Σ_θ. If γ is trivial in Σ_θ, then it bounds a disk in U_t. If γ is peripheral in Σ_θ, then it is primitive in U_t. In both cases, γ is not disk-busting. So, P_t is labeled with U.

For t values near 1, every curve $\gamma \subset \Sigma_\theta \cap P_t$ is inessential in Σ_θ and bounds a disk in V_t. So, P_t is labeled with V. □

Lemma 7.7. *If P is not induced by K, there exists a level surface P_s that is not labeled.*

Proof. Let $s := \sup\{t \in (0,1) | P_t \text{ is labeled with } U\}$. The lemma follows from the following observations, which follow from arguments that are in Chapter 4:

1. $0 < s$: This is because P_δ is labeled with U for $\delta > 0$ sufficiently small.

2. $s < 1$: If $s = 1$, then there are t values arbitrarily close to 1 such that P_t receives both labels. Hence, by Lemma 7.2, P is induced by K up to isotopy.

3. P_s is not labeled with U: If P_s is labeled with U, then for small $\delta > 0$, $P_{s+\delta}$ is labeled U, which contradicts the definition of s.

4. For any $\epsilon > 0$, there exists a $t \in (s - \epsilon, s)$ such that P_t is labeled with U: If this does not hold, s cannot be the supremum of the parameters of U-labeled levels.

5. P_s is not labeled with V: If P_s is labeled with V, then for small $\delta > 0$, $P_{s-\delta}$ is labeled V. Hence, there are t values arbitrarily close to s which are labeled with both U and V. Therefore, by Lemma 7.2, P is induced by K up to isotopy.

Thus, P_s is an unlabeled level as stated in observations (3) and (5). $\square$

Now let us fix a level surface P_s that is unlabeled. Unlike Chapter 4, we do not necessarily specify s to be $\sup\{t \in (0,1) \,|\, P_t \text{ is labeled with } U\}$.

Lemma 7.8. *If P is not induced by K, then for any angle θ such that $\Sigma_\theta \cap P_s$ is transversal, there exists a component of $\Sigma_\theta \cap P_s$ that is essential in Σ_θ.*

Proof. We prove the contrapositive. Assume that there exists a page Σ_θ such that $\Sigma_\theta \cap P_s$ is transversal and inessential in Σ_θ. Since P_s is not labeled, it follows that there exist components γ_U and γ_V in $\Sigma_\theta \cap P_s$ such that γ_U is disk-busting in U_s and γ_V is disk busting in V_s. This implies that no component of $\Sigma_\theta \cap P_s$ bounds an essential disk in U_s or V_s. Thus, by Lemma 7.5, we can isotope P_s to eliminate trivial curves of intersection so that $\Sigma_\theta \cap P_s$ consists of curves that are peripheral in Σ_θ. After the isotopy, $\Sigma_\theta \cap P_s$ still contains curves γ_U and γ_V that are disk-busting in U_s and V_s, respectively. Similar to the argument at the end of the proof of Lemma 7.2, this implies that P_s, and therefore P, is induced by K. We shortly explain it.

Let $N \cong \Sigma \times I$ be the complement of an open tubular neighborhood $\mathring{N}(\Sigma_\theta)$ in X_K and $F = P_s \cap N$. Similarly, F is incompressible in N and it is isotopic to a union of pages and ∂-parallel annuli in N (by Lemma 6.4). We deduce that P_s is isotopic in M to a union of a collection pages and annuli. Since P_s is not a torus, it contains a subsurface that is homeomorphic to a page. Since the only closed connected surface that is a union of pages and annuli is the Heegaard surface induced by K, it follows that P_s is isotopic to the Heegaard surface induced by K. $\square$

The discussion so far points out that if Conclusion (1) of Theorem 7.1 does not hold, then there exists a level surface P_s that is unlabeled and the intersection of this surface with any transverse page Σ_θ contains a simple closed curve that is essential in that page. Before the proof of Theorem 7.1, we will state and prove two more lemmas, which will be helpful to prove that such a surface imposes a complexity bound.

Lemma 7.9. *Assume that P is not induced by K. If P_s is an unlabeled level surface such that there is no vertex of Λ on the horizontal circle C_s, then $d_{\mathcal{C}}(\phi) \leq 2g - 2$.*

Proof. If C_s contains no vertex, then P_s is a regular surface. Moreover, since P is not induced by K, Lemma 7.8 implies that for any angle θ, if $\Sigma_\theta \cap P_s$ is transversal, then it contains a curve that is essential in Σ_θ. In other words, P_s is a perfect surface in X_K (see Definition 3.19). Hence, by Lemma 3.21, $d_{\mathcal{C}}(\phi) \leq -\chi(P_s) = 2g - 2$. □

Lemma 7.10. *Assume that P is not induced by K. If P_s is an unlabeled level surface such that there is a birth-and-death vertex of Λ on C_s, then $d_{\mathcal{C}}(\phi) \leq 2g - 2$.*

Proof. If there exists a birth-and-death vertex on C_s, then we can find a sufficiently small $\epsilon > 0$ such that $P_{s-\epsilon}$ (or $P_{s+\epsilon}$) is unlabeled, and $C_{s-\epsilon}$ contains no vertex. Hence, $P_{s-\epsilon}$ satisfies the hypothesis of the previous lemma, and the bound follows. □

7.4 Proof of Theorem 7.1

Proof. Assume that (P, U, V) and K are as in the statement of Theorem 7.1. We will assume that Conclusion (1) does not hold and show that Conclusion (2) holds. So, assume P is not isotopic to the Heegaard surface induced by K. It follows from Lemma 7.8 that there is an unlabeled level P_s such that for any page Σ_θ that is transversal to P_s, there exists a curve $\alpha \subset \Sigma_\theta \cap P_s$ that is essential in Σ_θ. Moreover, by Lemmas 7.9 and 7.10, we can assume that the horizontal circle C_s contains a crossing vertex (ψ, s) for otherwise we obtain $d_{\mathcal{C}}(\phi) \leq 2g - 2$, i.e., (2) holds. Under these assumptions, the following facts follow from the arguments of Chapter 4:

1. Σ_θ and P_s intersect transversely except for two entangled saddle tangencies.

2. If $\widehat{F}$ is the component of $P_s \cap \Sigma \times [\psi - \epsilon, \psi + \epsilon]$ that contains the saddle tangencies (for $\epsilon > 0$ small), then every component of $\Sigma_{\psi \pm \epsilon} \cap \widehat{F}$ is non-trivial in $\Sigma_{\psi \pm \epsilon}$.

3. P_s has at most $m = -\chi(P_s) - 2$ essential saddle tangencies to distinct pages in $\Sigma \times ([0, \psi - \epsilon] \cup [\psi + \epsilon, 2\pi])$.

By rotating the pages of K and reparametrizing θ, if necessary, we can assume that Σ_0 and P_s intersect transversely, and $\Sigma \times (\psi, 2\pi)$ contains no tangencies of P_s.

Now let S be the preimage of P_s under the quotient map $q : \Sigma \times [0, 2\pi] \to X_K$, which maps $\Sigma \times \{\theta\}$ to Σ_θ in the natural way. For simplicity, we will not distinguish $\Sigma \times \{\theta\}$ and Σ_θ. Let $0 = t_0 < t_1 < \ldots < t_m < \psi < t_{m+1} = 2\pi$ be angles such that, for $i = 1, \ldots, m-1$, Σ_{t_i} and P_s are transverse, and $\Sigma \times [t_i, t_{i+1}]$ contains a single essential saddle tangency of P_s. Furthermore, for $i = 0, 1, \ldots, m+1$, fix simple closed curves $\alpha_i \subset \Sigma_{t_i} \cap P_s$ that are essential in Σ_{t_i}, while ensuring $\phi(\alpha_0) = \alpha_{m+1}$. The following claims will complete the proof. Recall that $m = -\chi(P_s) - 2$ in the statements.

Claim. For $i = 0, \ldots, m-1$, we have $d_{\mathcal{C}}(\alpha_i, \alpha_{i+1}) \leq 1$.

Proof. This is basically because there exists a single essential saddle tangency of P_s in $\Sigma \times [t_i, t_{i+1}]$. If one of the curves, say α_i, does not interact with the saddle tangency, then $F \cap \Sigma_{t_{i+1}}$ contains a curve that is isotopic to α_i. Therefore, either $\alpha_i = \alpha_{i+1}$ or they are disjoint, and we get $d_{\mathcal{C}}(\alpha_i, \alpha_{i+1}) \leq 1$. So, we can assume that both curves interact with the essential saddle tangency of P_s in $\Sigma \times [t_i, t_{i+1}]$. In this case, the saddle tangency guides an isotopy of α_i into $\Sigma_{t_{i+1}}$ such that α_i and α_{i+1} are disjoint, and we get $d_{\mathcal{C}}(\alpha_i, \alpha_{i+1}) \leq 1$. □

Claim. We have $d_{\mathcal{C}}(\alpha_m, \alpha_{m+1}) \leq 2$.

Proof. Similar to the previous claim, we can assume that α_m and α_{m+1} interact with the entangled saddle of P_s to Σ_ψ. Otherwise, we get $d_{\mathcal{C}}(\alpha_m, \alpha_{m+1}) \leq 1$. Let G be the singular component of $\Sigma_\psi \cap P_s$, so G is a graph embedded in Σ_ψ with two vertices of valence 4. Since, $g(\Sigma_\psi) \geq 2$, we deduce that G does not fill Σ_ψ. On the other hand, α_m and α_{m+1} have isotopic copies that lie in a negihborhood $N(G) \subset \Sigma_\psi$. Since, G does not fill Σ_ψ, there exists an essential curve β outside $N(G)$. Therefore, β is disjoint from the isotopic copies of α_m and α_{m+1} in Σ_ψ, and we obtain $d_{\mathcal{C}}(\alpha_m, \alpha_{m+1}) \leq 2$. □

Finally, it follows from the last two claims that

$$d_{\mathcal{C}}(\phi) \leq d_{\mathcal{C}}(\alpha_0, \phi(\alpha_0)) = d_{\mathcal{C}}(\alpha_0, \alpha_{m+1}) \leq \sum_{i=0}^{m} d_{\mathcal{C}}(\alpha_i, \alpha_{i+1}) \leq m+2 = -\chi(P_s) = 2g-2,$$

as desired. □

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