Fermi Liquid Properties of Dirac Materials:

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Fermi Liquid Properties of Dirac Materials

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Boston College Morrissey College of Arts and Sciences Graduate School

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Fermi Liquid Properties of Dirac Materials

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One of the many achievements of renowned physicist L.D. Landau was the formulation of Fermi Liquid Theory (**FLT**). Originally debuted in the 1950s, **FLT** has seen abundant success in understanding degenerate Fermi systems and is still used today when trying to understand the physics of a new interacting Fermi system. Of its many advantages, **FLT** excels in explaining why interacting Fermi systems behave like their non-interacting counterparts, and understanding transport phenomena without cumbersome and confusing mathematics.

In this work, **FLT** is applied to systems whose low energy excitations obey the massless Dirac equation; i.e. the energy dispersion is linear in momentum, $\varepsilon \propto p$, as opposed to the normal quadratic, $\varepsilon \propto p^2$. Such behavior is seen in numerous, seemingly unrelated, materials including graphene, high T_c superconductors, Weyl semimetals, etc. While each of these materials possesses its own unique properties, it is their low energy behavior that provides the justification for their grouping into one family of materials called Dirac materials (**DM**). As will be shown, the linear spectrum and massless behavior leads to profound differences from the normal Fermi liquid behavior in both equilibrium and transport phenomena. For example, with mass having no meaning, we see the usual effective mass relation from **FLT** being replaced by an effective velocity ratio. Additionally, as **FLT** in d = 2 has been poorly studied in the past, and since the most famous **DM** in graphene is a d = 2 system, a thorough analysis of **FLT** in d = 2 is presented. This reduced dimensionality leads to substantial differences including undamped collective modes and altered quasiparticle lifetime.

In chapter 3, we apply the Virial theorem to **DM** and obtain an expression for the total average ground state energy $E = \frac{B}{r_s}$ where B is a constant independent of density and r_s is a dimensionless parameter related to the density of the system: the interparticle spacing r is related to r_s through $r = ar_s$ where a is a characteristic length of the system (for example, in graphene, a = 1.42 Å). The expression derived for E is unusual in that it's typically impossible to obtain a closed form for the energy with all interactions included. Additionally, the result allows for easy calculation of various thermodynamic quantities such as the compressibility and chemical potential. From there, we use the Fermi liquid results from the previous chapter and obtain an expression for B in terms of constants and Fermi liquid parameters F_0^s and F_1^s . When combined with experimental results for the compressibility, we find that the Fermi liquid parameters are density independent implying a unitary like behavior for **DM**. In chapter 4, we discuss the alleged universal KSS lower bound in **DM**. The bound, $\frac{\eta}{s} \ge \frac{\hbar}{4\pi k_B}$, was derived from high energy/string theory considerations and was conjectured to be obeyed by all quantum liquids regardless of density. The bound provides information on the interactions in the quantum liquid being studied and equality indicates a nearly perfect quantum fluid. Since its birth, the bound has been highly studied in various systems, mathematically broken, and poorly experimented on due to the difficult nature of measuring viscosity. First, we provide the first physical example of violation by showing $\frac{\eta}{s} \to 0$ as $T \to T_c$ in a unitary Fermi gas. Next, we determine the bound in **DM** in d = 2, 3 and show unusual behavior that isn't seen when the bound is calculated for normal Fermi systems.

Finally we conclude in chapter 5 and discuss the outlook and other avenues to explore in **DM**. Specifically, it must be pointed out that the physics of what happens near charge neutrality in **DM** is still poorly understood. Our work in understanding the Fermi liquid state in **DM** is necessary in understanding **DM** as a whole. Such a task is crucial when we consider the potential in **DM**, experimentally, technologically, and purely for our understanding.

"There may be more people who have more than talent than you, but there's no excuse for anyone to work harder than you."

Derek Sanderson Jeter



"What you lack in talent can be made up with desire, hustle, and giving 110% all the time."



Donald William Zimmer, 1951-2014

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Throughout my time as a graduate student, there was only one semester where I wasn't participating in some teaching role. My passion in physics started with wanting to become a teacher of some sort and that has never wavered. To all the students I've had over the years in all the various classes I've taught/TA-ed for, thank you for everything. Thank you for teaching me more than I ever thought I could learn, and thank you for letting me share my passion. Throughout these seven years you've helped me achieve my dream and love my time as a graduate student and I cannot thank you enough.

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Abbreviations

\mathbf{DoS}	Density of States
IS	Interacting \mathbf{S} ystem
NIS	$\mathbf{N} \mathrm{on}\text{-}\mathbf{I} \mathrm{n} \mathrm{teracting}~\mathbf{S} \mathrm{ystem}$
$\mathbf{QP} \text{ or } \mathbf{qp}$	\mathbf{Q} uasi \mathbf{P} article
\mathbf{FLT}	${\bf F}{\rm ermi}$ ${\bf L}{\rm iquid}$ ${\bf T}{\rm heory}$
\mathbf{FL}	${f F}{ m ermi}$ Liquid
DM	\mathbf{D} irac \mathbf{M} aterial
\mathbf{TLL}	Tomanaga Luttinger Liquid
LKE	\mathbf{L} andau \mathbf{K} inetic \mathbf{E} quation

This thesis is dedicated to my high school physics teacher, Mr. Mac, who introduced me to the world of physics and opened my eyes to what I was meant to do.

Chapter 1

Introduction

The physics of a single particle, e.g. an electron, can be understood within quantum mechanics. As any student of physics knows, all one has to do is go through the mundane task of solving the Schrödinger equation for the electron in the system of interest, determine the wave function and energies, and that's it ¹. But what happens when we consider more than one electron? For a realistic system, such as a metal, we must consider all electrons and tackle the Schrödinger equation with the many body Hamiltonian²

$$H = \sum_{i} \frac{p_{i}^{2}}{2m_{e}} + \sum_{I} \frac{p_{I}^{2}}{2m_{I}} + \frac{1}{2} \sum_{i} \sum_{i \neq j} \frac{e^{2}}{|\mathbf{r}_{i} - \mathbf{r}_{j}|} + \frac{1}{2} \sum_{I} \sum_{I \neq J} \frac{Z_{I} Z_{J} e^{2}}{|\mathbf{R}_{I} - \mathbf{R}_{J}|} - \sum_{i} \sum_{I} \frac{Z_{I} e^{2}}{|\mathbf{R}_{I} - \mathbf{r}_{i}|}$$
(1.1)

where the first two terms represent the kinetic energies of the electrons and the ions respectively, and the remaining three are the electron-electron interaction, nuclei-nuclei interaction, and the electron-nuclei interaction respectively. While eqn.(1.1) is a straightforward extension of the single electron Hamiltonian, obtaining an exact solution is an impossible task. Even within the Born Oppenheimer approximation³, eqn.(1.1) would still need to be solved for each electron (roughly 10^{23} electrons in a condensed matter system).

In spite of the daunting and seemingly impossible task of studying systems with Hamiltonians like/similar to eqn.(1.1), the work of many brilliant physicists lead to the progress in many body physics resulting in our current level of understanding ⁴ in the physics of metals, semiconductors, insulators, magnetic materials, etc. One such physicist, L.D. Landau, was responsible for numerous achievements and advances in physics such that his work has laid the foundation of our

 $^{^{1}}$ Additionally, as any student of physics knows, even for single electron problems, solving the Schrödinger equation is exact for a limited number of potentials and anything else must be done numerically.

 $^{^{2}}$ Of course this Hamiltonian will vary depending on the system being studied, the particles being studied, the various interactions present, etc. As this thesis is devoted to a condensed matter study, we present the Hamiltonian typically used for such cases (note that relativistic effects have been ignored).

 $^{^{3}}$ This is the assumption that the kinetic energy of the ions is negligible since the mass of the ions is orders of magnitude larger than that of the electrons. In terms of timescale, the massive ions appear stationary on the timescale of electron relaxation.

⁴Our understanding is by no stretch of the imagination complete.

current knowledge in condensed matter physics. Landau Fermi Liquid theory (**FLT**), arguably his most profound contribution⁵, provided an elegant and powerful tool for studying the physics of interacting Fermi systems. Present day, **FLT** is still used to study interacting Fermi systems and it's the standard by which all theories are measured.

This thesis is organized as follows: within the remaining parts of chapter 1, we introduce Landau Fermi Liquid theory (a powerful theoretical tool for studying fermionic many body systems), Dirac materials (the systems that are the main focus of this work), review the properties of a one dimensional Dirac metal, discuss the validity of Fermi liquid theory. In chapter 2, we derive the Fermi liquid properties (both equilibrium and non-equilibrium) of Dirac materials in dimensions d = 2, 3. While some results remain unchanged from that of a normal Fermi liquid, we see numerous profound differences that result from the linear energy dispersion and massless behavior of electrons in Dirac materials. In chapter 3, we apply the Virial theorem to Dirac materials and derive a simple expression for the total ground state energy. This surprising result is a consequence of the linear spectrum in Dirac materials and allows us to make several remarks related to the density dependence of various quantities, including Fermi liquid parameters F_0^s and F_1^s . In chapter 4 we discuss how perfect the "fluid" of electrons in Dirac materials is by deriving the alleged universal KSS bound on η/s for quantum fluids in Dirac materials. We conclude in chapter 5 with a discussion of the numerous avenues for future works.

1.1 Basics of Fermi Liquid Theory

A question that plagued the early days of many body physics was the following:

How can certain many body systems, metals for example, be qualitatively explained by their non-interacting properties in spite of the presence of complex, and often strong, interactions?

In an effort to understand the physics of many body systems, numerous theoretical methods were developed and have all seen varying degrees of success. One such theory was the Landau Fermi Liquid Theory developed by physicist L.D. Landau [8–11]. Originally a phenomenological theory used to study ³He when Landau presented it, **FLT** was able to explain why the properties of interacting Fermi systems are very similar to those of the ideal (non-interacting) Fermi gas. Having now seen success in describing the physics of other interacting Fermi systems at low temperatures, such as heavy fermions in nuclear matter and the interacting electrons in a metal, **FLT** has become an important tool in understanding any degenerate Fermi system. In this section we briefly introduce and discuss **FLT** to get the reader acquainted with the theory before applying it to Dirac materials⁶ in chapter 2. For a much more thorough and detailed account, we refer the reader to standard texts on the subject written by Baym and Pethick [2], Pines and Nozières [12], Kevin Bedell [13], and Abrikosov, Gorkov, and Dzyaloshinskiĭ [14].

⁵An argument could be made for the Landau theory of phase transitions being his crowning achievement.

 $^{^6\}mathrm{Dirac}$ materials will be discussed in sections 1.2 and 1.3

At its core, **FLT** is the application of Landau's elementary excitation concept to a normal interacting Fermi system [2]. In general, there exist two types of elementary excitations: fermion quasiparticles (single particle like entities typically referred to as quasiparticles (**QP**) and sometimes referred to as Landau quasiparticles to avoid confusion [15]) and boson quasiparticles (known as collective modes, which will be discussed later). With the **QP** concept in mind, the main idea behind **FLT** can stated in the following way:

A system of **strongly interacting particles** can be described in terms of **"weakly" interacting quasiparticles**

To develop the **QP** picture, we start with a gas of free fermions. In characterizing the system, it is sufficient to specify a distribution function⁷ given by

$$n_{\mathbf{p}\sigma}\left(T\right) = \frac{1}{1 + \exp\left[\beta\left(\varepsilon_{\mathbf{p}\sigma} - \mu\right)\right]}, \qquad \beta = \frac{1}{k_B T}$$
(1.2)

where $n_{\mathbf{p}\sigma}$ tells us the occupation of each state $|\mathbf{p}, \sigma\rangle$ at a given temperature *T*. The ground state of the system is formed by filling all states with momentum $p \leq p_F$; this defines the Fermi sea/radius of the Fermi surface; states beyond the Fermi surface are empty. In the ground state, eqn.(1.2) takes the form of a step function

$$n_{\mathbf{p}\sigma} \left(T = 0 \right) = \Theta \left(\varepsilon_F - \varepsilon_{\mathbf{p}\sigma} \right)$$

depicted in fig.(1.1) and allows us to relate the Fermi momentum to the particle density

$$n = \sum_{\mathbf{p}\sigma} n_{\mathbf{p}\sigma}$$
$$= \frac{N}{V} = \frac{1}{3\pi^2} \left(\frac{p_F}{\hbar}\right)^2$$

At this point, Landau invoked the principle of adiabatic continuity i.e. slowly "turn on" the interactions thus turning the well established ground state into an interacting state (turning the Fermi gas into a Fermi liquid). This adiabatic "switching on" of the interactions is of paramount important for a number of reasons:

- 1. The quantum numbers used to label the non-interacting states are still good quantum numbers for the interacting states (namely **p** and σ)
- 2. This establishes a one-to-one correspondence between the non-interacting and interacting states
- 3. The number of non-interacting states (particles) is the same as the number of interacting states (**QP**)

⁷A derivation of $n_{\mathbf{p}\sigma}$ is given in **Appendix A**.



FIGURE 1.1: Distribution function for a non-interacting Fermi system at T = 0. Note the sharp discontinuity at $|\mathbf{p}| = p_F$.

Item 3 in the above list is the main reason why expressions for equilibrium properties of a free Fermi gas still apply with good agreement for an interacting Fermi liquid. Many equilibrium properties follow from the entropy. Since the entropy is derived purely on counting arguments, the entropy for a system of interacting QP is the same as that of a free Fermi gas.⁸We should note that although similarities between particles and QP have been established, these two objects are very different; the main difference being their respective lifetimes. Physical particles (e.g. electrons) have essentially infinite lifetime whereas QP have a finite lifetime ⁹.

Since the distribution function (eq.(1.2)) follows from the entropy, it should come as no surprise that $n_{\mathbf{p}\sigma}$ for an interacting system has the same form, albeit with a subtle, yet profound difference. For a free Fermi gas, $\varepsilon_{\mathbf{p}\sigma}$ was the single particle energy, which in most literature is $\varepsilon_{\mathbf{p}\sigma} = |\mathbf{p}|^2 / 2m$. For a system of \mathbf{QP} , $\varepsilon_{\mathbf{p}\sigma}$ represents the \mathbf{QP} energy, which must include the effect of other \mathbf{QP} in the surrounding area. This distinction can be summed up as follows:

non-interacting system :
$$\epsilon = \sum_{\mathbf{p}\sigma} \varepsilon_{\mathbf{p}\sigma} n_{\mathbf{p}\sigma}$$

interacting system : $\epsilon \neq \sum_{\mathbf{p}\sigma} \varepsilon_{\mathbf{p}\sigma} n_{\mathbf{p}\sigma}$

where ϵ is the total energy density. The form for $\varepsilon_{\mathbf{p}\sigma}$ in the interacting case was postulated by Landau to be a functional of $n_{\mathbf{p}\sigma}$, which was then expanded in terms of small deviations of the distribution function:

$$\varepsilon_{\mathbf{p}\sigma} = \varepsilon_{\mathbf{p}\sigma}^{(0)} + \frac{1}{V} \sum_{\mathbf{p}'\sigma'} f_{\mathbf{p}\mathbf{p}'}^{\sigma\sigma'} \delta n_{\mathbf{p}'\sigma'} + \dots$$
(1.3)

 $^{^{8}}$ A detailed derivation and corresponding counting argument for the entropy can be found in **Appendix A**

⁹An in-depth discussion on lifetime can be found in [14] and in Appendix D of this thesis

The first term, $\varepsilon_{\mathbf{p}\sigma}^{(0)}$, is the ground state \mathbf{QP} energy. The second term, $f_{\mathbf{pp}'}^{\sigma\sigma'}$, is the first functional derivative of $\varepsilon_{\mathbf{p}\sigma}$ with respect to $n_{\mathbf{p}'\sigma'}$ and is formally defined as the energy change of a \mathbf{QP} in state $|\mathbf{p},\sigma\rangle$ due to the presence of other \mathbf{QP} with momentum \mathbf{p}' and spin σ' [2]. These $f_{\mathbf{pp}'}^{\sigma\sigma'}$ terms, known as the \mathbf{qp} interactions, describe the interaction between two \mathbf{QP} and are related to the scattering amplitude of the interacting \mathbf{QP} . The total energy of the system, also a functional of $n_{\mathbf{p}\sigma}$, is given as¹⁰

$$\delta E = \frac{1}{V} \sum_{\mathbf{p}\sigma} \varepsilon_{\mathbf{p}\sigma}^{(0)} \delta n_{\mathbf{p}\sigma} + \frac{1}{2} \frac{1}{V^2} \sum_{\mathbf{p}\sigma, \mathbf{p}'\sigma'} f_{\mathbf{p}\mathbf{p}'}^{\sigma\sigma'} \delta n_{\mathbf{p}\sigma} \delta n_{\mathbf{p}'\sigma'} + \dots$$
(1.4)

where $\delta E[n_{\mathbf{p}\sigma}] = E[n_{\mathbf{p}\sigma}] - E^{(0)}$ with $E^{(0)}$ being the ground state energy and the **QP** interaction can also be viewed as the second functional derivative of the total energy E.

In general, the **qp** interactions are a function of four variables $f_{\mathbf{pp}'}^{\sigma\sigma'} \equiv f(\mathbf{p}, \mathbf{p}', \sigma, \sigma')$. However, in spin rotation invariant systems, we rewrite the **qp** interactions as

$$f_{\mathbf{p}\mathbf{p}'}^{\sigma\sigma'} = f_{\mathbf{p}\mathbf{p}'}^s + f_{\mathbf{p}\mathbf{p}'}^a \boldsymbol{\sigma} \cdot \boldsymbol{\sigma'}$$
(1.5)

where the superscripts s and a refer to symmetric and antisymmetric spin of the interacting **QP** respectively. Close to the Fermi surface, we're allowed to make the following approximations:

- 1. $|\mathbf{p}| = |\mathbf{p}'| \simeq p_F$
- 2. We only concern ourselves with the angle between \mathbf{p} and \mathbf{p}'

This allows us to expand the interaction terms in an orthogonal basis set (which will be different for d = 2 and d = 3)

$$f_{\mathbf{pp}'}^{s,a} = \begin{cases} \sum_{l} f_{l}^{s,a} T_{l} \left(\cos \theta \right) & d = 2\\ \sum_{l} f_{l}^{s,a} P_{l} \left(\cos \theta \right) & d = 3 \end{cases}$$
(1.6)

where $T_l(\cos\theta)$ are the Chebyshev polynomials, $P_l(\cos\theta)$ are the Legendre polynomials, and θ is defined as the angle between **p** and **p'**. In practice it is convenient to deal with dimensionless Landau parameters¹¹

$$F_l^{s,a} = g(0)f_l^{s,a} (1.7)$$

where g(0) is the density of states at the Fermi surface. From eqn.(1.6) we can see there are an infinite amount of Landau parameters. Luckily, in practice, only the first few moments (typically only l = 0, 1, 2) are necessary since higher order moments involve higher order distortions of the Fermi surface and those effects are small enough to be neglected.

¹⁰One of Landau's many brilliant insights was only going to second order with eqn.(1.3) and eqn.(1.4). It wasn't until years after **FLT** that such a truncation was valid by noticing that higher order terms lead to non-analytic and non-physical terms in the equilibrium properties derived from eqn.(1.3) and eqn.(1.4).

 $^{^{11}}$ For the remainder of this thesis, when we refer to Landau parameters, we will refer to eqn.(1.7)

With the interactions present, the distribution function, still at T = 0, is slightly modified as in fig.(1.2). The difference between the figures (other than the color) is the discontinuity at $p = p_F$. In fig.(1.2), the discontinuity is shortened and represents the quasiparticle weight Z which influences the lifetime of our quasiparticle [9, 14]. The one-to-one correspondence developed earlier explains why the distribution function has the same form for non-interacting and interacting systems but combining eqn.(1.2) and eqn.(1.3), we see a profound difference between the non-interacting and interacting case. Earlier it was stated that the energy of **QP** depends on all other **QP** around it. When looking at eqn.(1.2) and eqn.(1.3), we see this mathematically as $\varepsilon_{\mathbf{p}\sigma}$ depends on $n_{\mathbf{p}\sigma}$ and vice-versa. This is not the case in a non-interacting system as $\varepsilon_{\mathbf{p}\sigma}$ is given independent of $n_{\mathbf{p}\sigma}$.



FIGURE 1.2: Distribution function for an interacting Fermi system at T = 0. Unlike the non-interacting distribution function shown above, the discontinuity at p_F is shortened and denotes the quasiparticle weight, traditionally labeled Z where Z < 1.

Regardless of the phenomenology presented, a physical picture of a \mathbf{QP} is still lacking. In order to present one, we turn to standard screening arguments that can be found in any electromagnetism book and in the standard text on the electron liquid by Giuliani and Vignale [7] for screening specific to a Fermi liquid. When an electron is inserted into a metal, what happens? It pushes away other electrons and is attracted to whatever positive charge is available to itself. This entity, the electron plus its screening cloud, is the \mathbf{QP} . As one would imagine, the screening cloud screens the interactions which results in nerfing a strong long range interaction thus leaving a relatively weak short range interaction making **FLT** possible. The cartoon in fig.(1.3) illustrates this idea. The \mathbf{QP} , electron plus screening cloud, behaves similarly to the bare particle albeit with an enhanced effective mass¹² given as

$$\frac{m^*}{m} = \begin{cases} 1 + \frac{F_1^s}{2} & d = 2\\ 1 + \frac{F_1^s}{3} & d = 3 \end{cases}$$
(1.8)



FIGURE 1.3: Popular quasiparticle cartoon from A Guide to Feynman Diagrams in the Many-Body Problem by R.D. Mattuck

With eqn.(1.3), a calculation of all equilibrium properties is possible¹³ and one can see that the form for many expressions is slightly modified by the inclusion of certain Landau parameters. For example, the low temperature entropy calculated within **FLT** is

$$s = \frac{\pi^2}{3} k_B^2 g(0) T \tag{1.9}$$

the low temperature specific heat is

$$c_V = T\left(\frac{\partial s}{\partial T}\right)_V = \frac{\pi^2}{3}k_B^2 g(0)T \tag{1.10}$$

where both expressions retain the linear in temperature behavior that is seen in their noninteracting counterparts but carry the interacting behavior in the density of states term:

$$g(0) = \begin{cases} \frac{m^*}{\pi\hbar^2} & d = 2\\ \frac{m^* p_F}{\pi^2\hbar^3} & d = 3 \end{cases}$$
(1.11)

The zero temperature compressibility is given as

$$\kappa = \frac{1}{n^2} \frac{g(0)}{1 + F_0^s} \tag{1.12}$$

where again we revert back to the non-interacting expression if $F_0^s = 0$.

¹²A derivation of eqn.(1.8) can be found in **Appendix B**

¹³Such calculations are done in [2, 12, 13] as well as in Appendix B

In addition to the equilibrium properties of a Fermi liquid, **FLT** provides a way to calculate non-equilibrium (transport) properties as well [10, 11]; such properties are the Fermi liquids response to an external field/perturbation. In investigating the transport properties of an inhomogeneous Fermi liquid, we concern ourselves with the space and time dependence of the distribution function: $n_{\mathbf{p}\sigma} \rightarrow n_{\mathbf{p}\sigma}(\mathbf{r}, t)$. The space/time evolution is governed by a Boltzmannlike equation, known as the Landau kinetic equation (**LKE**):

$$\frac{\partial n_{\mathbf{p}\sigma}\left(\mathbf{r},t\right)}{\partial t} - \left\{\varepsilon_{\mathbf{p}\sigma}, n_{\mathbf{p}\sigma}\right\} = I\left[n_{\mathbf{p}'\sigma'}\right] \tag{1.13}$$

which is derived by taking the full time derivative of the distribution function $n_{\mathbf{p}\sigma}(\mathbf{r},t)$ where $\{,\}$ is the Poisson bracket [16]. A natural concern in obtaining eqn.(1.13) is that the quantum behavior of $n_{\mathbf{p}\sigma}(\mathbf{r},t)$ is ignored. In cases where quantum behavior needs to be considered, $n_{\mathbf{p}\sigma}(\mathbf{r},t)$ must be replaced with the Wigner function [2] and eqn.(1.13) is replaced by a quantum mechanical equation of motion¹⁴. However, if we restrict our study to external fields whose characteristic length, $\lambda = \hbar/q$, is much larger than the interparticle spacing, a, in the Fermi liquid¹⁵, then the distribution may be treated classically and eqn.(1.13) is valid [13]. On the right hand side of eqn.(1.13) is the collision integral¹⁶

$$I[n_{\mathbf{p}'\sigma'}] = \sum_{\mathbf{p}_{2}\sigma_{2}} \sum_{(\mathbf{p}_{3}\sigma_{3},\mathbf{p}_{4}\sigma_{4})} W(\mathbf{p}_{1}\sigma_{1}\mathbf{p}_{2}\sigma_{2};\mathbf{p}_{3}\sigma_{3}\mathbf{p}_{4}\sigma_{4}) \,\delta_{\mathbf{p}_{1}+\mathbf{p}_{2},\mathbf{p}_{3}+\mathbf{p}_{4}} \delta_{\sigma_{1}+\sigma_{2},\sigma_{3}+\sigma_{4}}$$
$$\times \,\delta\left(\varepsilon_{1}+\varepsilon_{2}-\varepsilon_{3}-\varepsilon_{4}\right) \left[n_{3}n_{4}\left(1-n_{1}\right)\left(1-n_{2}\right)-n_{1}n_{2}\left(1-n_{3}\right)\left(1-n_{4}\right)\right] \qquad (1.14)$$

which quantifies the net rate at which **QP** collisions increase the occupation of state $|\mathbf{p}\sigma\rangle$ [2]. In practice, we're typically concerned with situations where the distribution function and **QP** energy differ by a small amount from their equilibrium value, $n_{\mathbf{p}\sigma}(\mathbf{r},t) = n_{\mathbf{p}\sigma}^{(0)} + \delta n_{\mathbf{p}\sigma}(\mathbf{r},t)$ and $\varepsilon_{\mathbf{p}\sigma}(\mathbf{r},t) = \varepsilon_{\mathbf{p}\sigma}^{(0)} + \delta \varepsilon_{\mathbf{p}\sigma}(\mathbf{r},t)$, which leads to the linearized Landau kinetic equation (**LLKE**)

$$\frac{\partial}{\partial t}\delta n_{\mathbf{p}\sigma}\left(\mathbf{r},t\right) - \mathbf{v}_{\mathbf{p}\sigma} \cdot \nabla_{r}\left(\delta\varepsilon_{\mathbf{p}\sigma}\left(\mathbf{r},t\right)\frac{\partial n_{\mathbf{p}\sigma}^{(0)}}{\partial\varepsilon_{\mathbf{p}\sigma}^{(0)}} - \delta n_{\mathbf{p}\sigma}\left(\mathbf{r},t\right)\right) = I\left[n_{\mathbf{p}'\sigma'}\right]$$
(1.15)

where all terms higher than linear order in the deviations $\delta n_{\mathbf{p}\sigma}$ and $\delta \varepsilon_{\mathbf{p}\sigma}$ are neglected. Additionally, the Fourier transform of eqn.(1.15) is useful in analysis of collective modes and is given by

$$\left(\omega - \mathbf{q} \cdot \mathbf{v}_{\mathbf{p}\sigma}\right) \delta n_{\mathbf{p}\sigma}\left(\mathbf{q},\omega\right) + \left(\mathbf{q} \cdot \mathbf{v}_{\mathbf{p}\sigma}\right) \frac{\partial n_{\mathbf{p}\sigma}^{(0)}}{\partial \varepsilon_{\mathbf{p}\sigma}^{(0)}} \delta \varepsilon_{\mathbf{p}\sigma}\left(\mathbf{q},\omega\right) = iI\left[n_{\mathbf{p}'\sigma'}\right]$$
(1.16)

With eqns.(1.13 - 1.16), conservation laws for particle number, momentum, and energy are attainable. Additionally, collective modes, such as zero sound (c_0) and plasmons (ω_p), and transport

¹⁴The quantum mechanical equation of motion is very similar to eqn.(1.13). The main difference is the Poisson brackets are replaced by $(i\hbar)^{-1}$ times the commutator

¹⁵This condition can also be stated as $q \ll k_F$ where k_F is the Fermi wavevector. In terms of energy, the energy transfer due to the external field ω must be comparable to the excitation energy qv_F which must be less than the Fermi temperature i.e. $\omega \sim qv_F \ll T_F$

¹⁶The second summation, over $\mathbf{p}_3\sigma_3$ and $\mathbf{p}_4\sigma_4$, is over distinguishable final states only

coefficients spin diffusion (D), viscosity (η) , and thermal conductivity (K) may be derived allowing us to understand the transport properties of a Fermi liquid. A detailed derivation for spin diffusion D can be found in the work of Hua Li [17] while a study of viscosity η will be discussed in subsequent chapters of this thesis.

The treatment of Fermi liquid thus far has been phenomenological. It wasn't until a later paper by Landau [9] and the work (which then became a textbook) of Abrikosov, Gorkov, and Dzyaloshinskii [14] that the microscopic basis of Fermi liquid theory was established. Starting from the full **QP** Green's function at T = 0

$$G_{\sigma}(\mathbf{p},\omega) = \frac{1}{\hbar\omega - \varepsilon_{\mathbf{p}\sigma} + \mu - \Sigma_{\sigma}(\mathbf{p},\omega) + i\eta}$$
(1.17)

the connection to Fermi liquid quantities is contained in the, generally complex, self energy term $\Sigma_{\sigma}(\mathbf{p},\omega) = \Sigma_1 + i\Sigma_2$ where Σ_1 is the real part and Σ_2 is the imaginary part. First, using the real part of the self energy, one can obtain an expression for the effective mass[18, 19]:

$$\frac{m^*}{m} = \frac{1 - \frac{\partial \Sigma_1}{\partial \omega}}{1 + v_F \frac{\partial \Sigma_1}{\partial p}} \bigg|_{|\mathbf{p}| = p_F, \omega = 0}$$
(1.18)

which allows for a microscopic calculation of m^* and F_1^s . The numerator of eqn.(1.18) is related to the **QP** weight Z, as depicted in fig.(1.2). If we consider the single particle spectral function, a **QP** can be interpreted as a delta function peak in the spectral function where Z is the strength of this peak. Second, using the imaginary part Σ_2 , we get an expression for the **QP** lifetime (also the inelastic scattering rate)[19, 20]:

$$\frac{1}{\tau} = -2\Sigma_2 \tag{1.19}$$

where the temperature dependencies¹⁷ are [2, 12, 23-25]

$$\frac{1}{\tau} \sim \begin{cases} T^2 \ln\left(\frac{T^*}{T}\right) & d = 2\\ T^2 & d = 3 \end{cases}$$
(1.20)

where T^* is a cutoff/adjustable parameter indicating when high temperature corrections need to be considered. The lifetime is imperative for a proper Fermi liquid description as it allows us to confidently say when/where the **QP** picture is valid. Eqn.(1.20) is a consequence of the **QP** not being true eigenstates and therefore must exhibit decay. Additionally, these lifetime factors appear in transport coefficients and as we'll see in **chapter 2** and **chapter 4**, their influence can lead to profound physical results.

¹⁷These lifetimes are specific to a Fermi liquid. This distinction is important for Dirac materials when the chemical potential μ is at the charge neutrality point. Here, the lifetime is linear in temperature $\tau^{-1} \sim T$ as explained in [21, 22]

In spite of its birth in the late 1950s, **FLT** remains a powerful theoretical tool in condensed matter physics to this day. Originally designed to describe ³He (the "true" Fermi liquid¹⁸), the theory has proven useful in describing many degenerate interacting Fermi systems (of which multiple texts/papers exist). The relative ease in calculating both equilibrium and transport properties, a "simple" physical picture/interpretation, and a firm microscopic basis have made **FLT** a standard, and as such, will be used to study Dirac materials in **chapter 2** of this thesis. Although powerful, and wildly used, the theory does have its shortcomings¹⁹, which will be discussed in detail in the final section of this chapter. Nonetheless, we find that **FLT** results seem to hold in instances where it shouldn't be applicable thus hinting at Landau's work being more robust than originally thought. While some of our usage of **FLT** in this work might seem ill-placed, we encourage the reader to take a deep breath, have a little faith, and allow us to explain our case²⁰. Before moving onto the next section, it must be stressed that the discussion of **FLT** given here is surface level at best. Throughout this section, the standard literature on **FLT** has been mentioned and cited for the reader who wishes for a more in-depth treatment.

1.2 Dirac Materials

As discussed in the previous section, **FLT** is a powerful tool used to study interacting Fermi systems. Often the benchmark by which all other theoretical techniques are compared to, **FLT** is frequently the first method applied to a new system of interacting fermions. With this in mind, the theme of this thesis is the application of **FLT** to a unique class of interacting Fermi systems that exhibit quasi-relativistic nature. Such unique behavior is realized across a wide set of materials including graphene [22, 27, 28], *d*-wave superconductors, certain topological insulators, Dirac Semimetals (often dubbed "3D graphene")[29], and many more as shown in fig.(1.4) from [30].

Although fig.(1.4) shows a seemingly diverse list, with materials ranging from crystalline structures to quantum fluids, they all share the property that their low energy excitations²¹ behave as massless Dirac particles obeying the Dirac equation and thus having a linear energy dispersion. The existence of such Dirac points (or lines)²², regardless of their origin, separates the family of Dirac materials (**DM**) from normal metals and semiconductors [30]. Such a powerful organizing principle implies that the properties arising as a consequence of the Dirac spectrum are universal. For example, the power-law temperature dependence of the fermionic specific heat, response to impurities and magnetic fields, suppressed backscattering, and various transport

¹⁸We'll put this name to the test

¹⁹Alleged shortcomings. This will be discussed later on and was the theme of the following paper [26]

 $^{^{20}}$ If that doesn't work and you still don't believe us, well nature is on our side so we win

 $^{^{21}}$ This is typically how condensed matter systems are described as this usually determines responses to external probes.

 $^{^{22}}$ ARPES and STS are the main experimental techniques used to verify the linear behavior of the energy dispersion. Please refer to section 4 in [30] for summaries on such experiments done on single layer graphene, topological insulators (Bi₂Se₃ and Bi₂Te₃), high temperature superconductors (Bi₂Sr₂CaCu₂O_{8+x}), and Dirac semimetals (Na₃Bi).

Material	Pseudo-spin	Energy scale
Graphene, silicene, germanene	Sublattice	$1 - 3 \mathrm{eV}$
Artificial graphenes	Sublattice	$10^{-8} - 0.1 \mathrm{eV}$
Hexagonal layered heterostructures	Emergent	$0.01 - 0.1 \mathrm{eV}$
Hofstadter butterfly systems	Emergent	0.01 eV
Graphene-hBN heterostructures in high magnetic fields		
Band inversion interfaces: SnTe/PbTe, CdTe/HgTe, PbTe	Spin–orbit ang. mom.	0.3 eV
2D topological insulators: HgTe/CdTe, InAs/GaSb, Bi bilayer,	Spin–orbit ang. mom.	<0.1 eV
3D topological insulators: $Bi_{1-x}Sb_x$, Bi_2Se_3 , strained HgTe, Heusler alloys,	Spin–orbit ang. mom.	$\lesssim 0.3 \text{eV}$
Topological crystalline insulators: SnTe, $Pb_{1-x}Sn_xSe$	Orbital	$\lesssim 0.3 \text{eV}$
<i>d</i> -wave cuprate superconductors	Nambu pseudo-spin	$\lesssim 0.05 \mathrm{eV}$
³ He	Nambu pseudo-spin	0.3 µeV
3D Weyl and Dirac SM	Energy bands	Unclear
Cd ₃ As ₂ , Na ₃ Bi		

FIGURE 1.4: A table from Wehling *et. al.* depicting various Dirac materials, corresponding pseudo-spin, and energy range where the Dirac dispersion is present. The pseudo-spin is commonly referred to a "valley degeneracy" and typically contributes an additional factor of 2 in calculations thus leading to a total 4 fold degeneracy.

properties is expected to be seen in neutral superfluids, graphene, and d-wave superconductors [30].

In typical Fermi systems, the **qp** are governed by a Schrödinger equation with a Hamiltonian:

$$H = \frac{p^2}{2m^*}$$

where m^* is the effective mass. In contrast, the **qp** in Dirac materials obey a Dirac-like²³ Hamiltonian

$$H_D = v_g \mathbf{p} \cdot \boldsymbol{\sigma} + m v_g^2 \sigma_z \qquad \text{in 2D}$$
$$H_D = v_g \mathbf{p} \cdot \boldsymbol{\sigma} \qquad \text{in 3D}$$

Fig.(1.5) in the following section compares the dispersion of a normal Fermi system and a Dirac system. Aside from the obvious difference in the shape of the energy dispersion, another profound difference is in the mass term. We see in a normal Fermi system that mass directly plays as role whereas in the Dirac Hamiltonians, mass is generally missing. Even if one considers non-zero mass, which has to be considered if the experimentally probed energies are on the order of mv_g^2 , the results are still qualitatively different. For example, particles and holes in **DM** share the same effective mass which is related to the spectral gap $2v_g^2$ while in normal Fermi systems, particles and holes obey separate Schrödinger equations and have different effective masses[30].

The importance in studying these materials cannot be understated. Aside from the obvious benefits to material science and advancements in technology, Dirac materials act as a parent

 $^{^{23}}$ We say Dirac-like since the speed of light c is replaced by the Fermi velocity in Dirac materials v_g . Unfortunately, since this velocity is tied to the rest frame of the material, the Dirac equation in Dirac materials is not Lorentz covariant.

compound for the realization of other novel states [29], they're a platform to observe relativistic effects (such as half integer QHE, Klein paradox, and enhanced sensitivity to magnetic fields), and they provide an experimental playground for low dimensional physics²⁴ in addition to their high tunability. While each material has been extensively studied on its own, and microscopic origins for their respective Dirac dispersions have been established, the study of their universal behavior that stems from the linear dispersion has not been discussed. With this in mind that we apply **FLT** methods of the previous section to general **DM** in 2D and 3D with the claim that our results can be applied to a diverse class of materials within appropriate energy ranges.

1.3 A Discussion of the 1D Dirac Material

Although **FLT** has experienced abundant success in describing interacting Fermi systems, there are a few cases in which Landau's theory does not apply. One such case is a 1D interacting Fermi system where the decreased phase space results in the lowest energy excitations being collective boson modes that resemble sound modes²⁵[31, 32] and thus no good **qp** picture. This problem was solved by Tomanaga and Luttinger and led to Tomanaga Luttinger Liquid theory (**TLL**) which produced results that indicated non-Fermi Liquid behavior consistent with experimental observation ²⁶. A key difference in the 1D interacting Fermi system is the momentum distribution function²⁷ ($\hbar = 1$ for the remainder of this section)

$$n(k) \simeq \frac{1}{2} - C_1 \left| k - rk_F \right|^{\alpha} \operatorname{sgn} \left(rk - k_F \right) - C_2 \left(k - rk_F \right)$$
(1.21)

which clearly lacks the usual discontinuity at $k = k_F$ that's seen in higher dimensions. The exponent α , referred to as the anomalous exponent, is a trademark of **TLL** behavior as they appear in the power law correlation functions [33]. An additional feature unique to the 1D interacting Fermi system is the existence gapless bosonic modes involving separate charge and spin degrees of freedom; in general, these modes propagate with different speeds [34].

While the main work of this thesis is the application of **FLT** to **DM**, we start with a discussion of 1D **DM** for the purposes of providing a complete description of **DM** in all dimensions, as well as supplementing the relatively recent work by Ishii *et. al.* that observed **TLL** behavior in a 1D Dirac material [35]. For simplicity, let's consider a 1D Dirac system at T = 0 within the framework of **TLL** theory. Furthermore, we avoid gapped states and possible Wigner crystallization by avoiding other chiralities [1]. The **TLL** Hamiltonian is

$$H = H_0 + H_{\rm int} \tag{1.22}$$

 $^{^{24}}$ For years, the idea of 2D systems and systems with linear energy dispersion were thought to be purely theoretical. Dirac materials allow for past/future predictions to be tested.

 $^{^{25}}$ Throughout this thesis, the term quasiparticle refers to Landau quasiparticles, not collective modes

 $^{^{26}{\}rm The}$ problem of the 1D interacting Fermi system has been extensively studied and is typically solved via bosonization or renormalization group techniques

 $^{^{27}}r = -, +$ denotes left and right moving fermions respectively; C_1 and C_2 are constants whose explicit form can be found in [1, 32]

where H_0 is the Hamiltonian for a non-interacting Dirac gas

$$H_0 = v_g \sum_{rk\sigma} \left(rk - k_F \right) c_{k\sigma}^{r\dagger} c_{k\sigma}^r \tag{1.23}$$

where the Fermi velocity $v_g \sim 10^6$ m/s, $\sigma = \uparrow \downarrow$ is the single fermion spin, and r = +, - indicates what branch we are on $(p_F \text{ or } -p_F)$. H_{int} is the Tomanaga-Luttinger interaction Hamiltonian, commonly known as the g-ology model, given by

$$H_{\text{int}} = \frac{1}{L} \sum_{k_1, k_2, p, \sigma_1, \sigma_2} \left[\Gamma^{(2)}_{\sigma_1, \sigma_2} c^{\dagger\dagger}_{k_1, \sigma_1} c^{-\dagger}_{k_2, \sigma_2} c^{-}_{k_2 + p, \sigma_2} c^{\dagger}_{k_1 - p, \sigma_1} + \frac{1}{2} \Gamma^{(4)}_{\sigma_1, \sigma_2} \left(c^{\dagger\dagger}_{k_1, \sigma_1} c^{\dagger\dagger}_{k_2, \sigma_2} c^{+}_{k_2 + p, \sigma_2} c^{\dagger}_{k_1 - p, \sigma_1} c^{-\dagger}_{k_1, \sigma_1} c^{-\dagger}_{k_2, \sigma_2} c^{-}_{k_2 + p, \sigma_2} c^{-}_{k_1 - p, \sigma_1} \right) \right]$$
(1.24)

and the interaction is characterized by

$$\Gamma^i_{\sigma_1,\sigma_2} = g^s_i \delta_{\sigma_1,\sigma_2} + g^a_i \delta_{\sigma_1,-\sigma_2} \tag{1.25}$$

where k_1/k_2 and σ_1/σ_2 are the momenta and spin of the interacting fermions. In both eqn.(1.23) and eqn.(1.24), c^{\dagger} and c are the creation and annihilation operators for fermions on a given branch with momentum and spin as indicated. In the interaction, eqn.(1.24), we restrict $\Gamma_{\sigma_1,\sigma_2}^i$ in eqn.(1.25) to i = 2, 4 which indicates forward scattering on the separate branches and forward scattering on the same branch respectively²⁸. The superscripts s and a denote symmetric and anti-symmetric spin respectively. The coupling constants, $g_i^{s,a}$, are a measure of the interaction strength for given scattering processes i = 2, 4 and are in general momentum dependent; we neglect this dependence by restricting the momentum to within certain values of cutoff Λ , i.e. $k_F - \Lambda < k < k_F + \Lambda$ [31].

At this point, we can readily identify a subtle yet substantial difference between a 1D DM and a 1D normal metal in eqn.(1.23). Originally, the selling point of **TLL** theory was the results being "exact." With the Hamiltonian, similar to eqn.(1.22), exact forms for the Green's function can obtained G(x,t), as calculated by Dzyaloshinskiĭ and Larkin [33], and all meaningful properties follow²⁹. What needs to be stressed is the misleading nature of calling these results exact. Although the Green's function G(x,t) presented in [31–33] is exact, the calculation depends on the linearization of the energy dispersion around the Fermi points $\pm p_F$; it follows that linearization about different Fermi points will produce different results³⁰. This linearization is important in ordinary metals since the energy dispersion is generally quadratic $\varepsilon \propto p^2$. In

 $^{^{28}}$ It is important to note that the **TLL** model is not complete but rather it is the simplest model that shows deviation from Fermi liquid behavior. One can easily see that back scattering and Umklapp scattering processes have been ignored in eqn.(1.25). This exclusion is allowed as long as we're considering the low-energy properties of metals where interactions that do not commute with charge and spin are irrelevant (in semiconducting systems, these need to be included)[1]

²⁹This is one of the few times where a solution in (x,t) is possible while the Fourier transformed solution $G(k,\omega)$ isn't. Additionally, a number of thermodynamic properties are easily obtained through Bosonization instead. However, the Green's function approach provides microscopic analysis of the system, and leads to expressions for the number density n(k) and density of states $g(\omega)$.

³⁰The linearization will directly change the Fermi velocity v_F as well as the allowed momentum values determined by the cutoff Λ around $\pm p_F$.

Dirac metals, such as the single walled carbon nanotube (**SWNT**), a linear dispersion exists without any approximation³¹. The differences are shown in fig.(1.5).



FIGURE 1.5: Single particle energy for a system with parabolic dispersion (left) and linear dispersion (right). Left: Linearization around p_F leads to a different v_F (note the changing slopes represented by black lines). Right: Linearization is unnecessary here as the dispersion is already linear. This implies a constant v_F regardless of where p_F is (note the constant slope of the black lines).

We see in the left plot that as the Fermi energy ε_F is changed (different locations of horizontal, dashed, red line), the slope of the linearized black line changes. In the right plot, the spectrum is linear leading to the same slope of the black line regardless of where ε_F is (notice how all the black lines have the same slope). This distinction in the curvature of the dispersion between an ordinary metal and a Dirac metal leads to results that are valid for all momentum, not just those bounded by the cutoff.

Using the Hamiltonian (1.22), we can derive an expression for G(x,t) following similar procedure as that done by Dzyaloshinskii and Larkin [33]. In the absence of interactions, the free Green's function for fermions in a 1D Dirac metal is given as

$$G_r^{(0)}(k,\omega) = \frac{1}{\omega - v_g \left(rk - k_F\right) + i\delta \operatorname{sgn}\left(rk - k_F\right)} \qquad r = +, -$$

The general machinery for solving an interacting Fermi system using Green's functions is well known and typically leads to intractable integral equations. In general, we determine the vertex function Γ through an infinite series of Feynman diagrams and resort to approximations [14]. The benefit of the linear spectrum in 1D enters here as we can write an exact relation between the vertex function Γ_r and the full Green's function³² $G_r(k, \omega)$ as

$$\Gamma_r = \frac{G_r^{-1}(p,\varepsilon) - G_r^{-1}(p-k,\varepsilon-\omega)}{\omega - rv_q k}$$
(1.26)

 $^{^{31}}$ Of course the linearity of the dispersion will start to curve outside a certain energy range, but seeing as this is a low energy description, we need not concern ourselves with that.

 $^{^{32}}$ Conservation of particle number on each branch was also imposed in addition to the linear dispersion

from which we get the following Dyson equation³³ (for the positive branch)

$$\left[\varepsilon - v_g \left(p - k_F\right)\right] G_+\left(p,\varepsilon\right) = 1 + \frac{i}{4\pi^2} \int dk d\omega \frac{D_{4\parallel}}{\omega - v_g k} G_+\left(p - k,\varepsilon - \omega\right)$$
(1.27)

where $D_{4\parallel}$ is the effective coupling³⁴. Upon Fourier transforming eqn.(1.27), we obtain the following solution

$$G_{r}(x,t) = \frac{1}{2\pi} \frac{rx - v_{g}t + i/\Lambda}{rx - v_{g}t + i\delta(t)} \prod_{j=\sigma,\rho} \frac{1}{(rx - u_{j}t + i/\Lambda)^{1/2}} \frac{1}{\left[\Lambda^{2} \left(x^{2} - u_{j}^{2}t^{2} + \frac{i}{\Lambda} \left(2u_{j}t - \frac{i}{\Lambda}\right)\right)\right]^{\alpha_{j}}}$$
(1.28)

where $\delta(t) = \delta \operatorname{sgn}(t)$ is a bandwidth cutoff and Λ is the momentum cutoff discussed earlier. The exponents $\alpha_{\sigma,\rho}$ are of special interest and are of the following form for a 1D Dirac metal

$$\alpha_{\sigma,\rho} = \frac{1}{4u_{\sigma,\rho}} \left[v_g + \frac{1}{2\pi} \left(g_4^s \pm g_4^a \right) - u_{\sigma,\rho} \right]$$
(1.29)

where $u_{\sigma,\rho}$ can be interpreted as the velocities of two boson modes, a spin density mode and a charge density mode, given as

$$u_{\sigma,\rho}^{2} = \left[v_{g} + \frac{1}{2\pi} \left(g_{4}^{s} \pm g_{4}^{a} \right) \right]^{2} - \left[\frac{1}{2\pi} \left(g_{2}^{s} \pm g_{2}^{a} \right) \right]^{2}$$
(1.30)

These exponents, sometimes referred to as anomalous exponents, are a measure of interaction strength and appear in equilibrium and non-equilibrium properties. When compared to the exponents a normal 1D metal, which can be found in [31, 32, 34], we see that although the forms for the exponents are similar, the exponents in a 1D Dirac metal provide two pieces of crucial information: (1) The equilibrium/non-equilibrium properties will have the same exponent since linearization is superfluous and (2) the interaction strength in 1D Dirac systems is independent of particle density. The first of these conclusions follows directly from the earlier discussion when clarifying the exactness of the original **TLL** solutions. The second conclusion is more surprising, albeit easily seen if we compare eqn.(1.29) and eqn.(1.30) to those for an ordinary metal. Within the g-ology model, the coupling constants $g_i^{s,a}$ remain constant and this is true for both a normal metal and a Dirac metal. The difference is in the velocity terms. In a 1D normal metal, v_q is replaced by $v_F \propto n$ where n is the electron density. In a Dirac metal, the velocity term is a constant independent of electron density. Clearly, the exponents in eqn.(1.29)are independent of electron density which therefore leads to the conclusion that, at least in 1D, the interaction strength is independent of the density of electrons in the system. This implies that the interaction strength in a Dirac system is at some saturated value, perhaps even in a constant state of unitarity³⁵

 $^{^{33}}$ Feynman diagrams for the vertex function and the Dyson equation can be found in [31, 33].

 $^{^{34}}$ The effective couplings have simple algebraic forms containing the coupling constants $g_{2,4}^{s,a}$ and the polarization bubbles for respective branches. These effective interactions are a consequence of the linear dispersion which causes all diagrams containing loops with more than two interaction vertices to cancel out leaving self energy diagrams containing bubbles. Detailed forms for these constants can be found in [31]

 $^{^{35}}$ This idea of Dirac systems being in the unitary limit will show up later in this thesis.

Using the Green's function in eqn.(1.28), we can determine the momentum distribution function $n(k)^{36}$ through standard methods found in [14]:

$$n(k) = -i\sum_{r} \int_{-\infty}^{\infty} G_r(x, 0^-) e^{-ikx} dx$$

At $t = 0^{37}$, the Green's function behaves as $G_r(x) \simeq x^{-1-\alpha}$, where $\alpha = 2(\alpha_{\sigma} + \alpha_{\rho})$ and we therefore expect n(k) to have the same form. Evaluation of the integral is done using [36] and has the following solution:

$$n(k) = \frac{1}{2\pi} \operatorname{sgn}\left(k_F - rk\right) \left[\beta \left(\frac{1}{2}, \frac{1}{2}\left(\alpha - 1\right)\right) {}_1F_2\left(\frac{1}{2}; \frac{3}{2} - \frac{\alpha}{2}, \frac{3}{2}; \frac{1}{4} \frac{|rk - k_F|^2}{\Lambda}\right) \frac{|rk - k_F|}{\Lambda} + \frac{\sqrt{\pi}\Gamma\left(\frac{1}{2} - \frac{\alpha}{2}\right)}{\Gamma\left(1 + \frac{\alpha}{2}\right)} {}_1F_2\left(\frac{\alpha}{2}; 1 + \frac{\alpha}{2}, \frac{1}{2} + \frac{\alpha}{2}; \frac{1}{4} \frac{|rk - k_F|^2}{\Lambda^2}\right) \left(\frac{|rk - k_F|}{2\Lambda}\right)^{\alpha} \right]$$

where $\Gamma(a)$ and $\beta(a, b)$ are the standard gamma and beta functions respectively, and ${}_{1}F_{2}(a; b, c; z)$ is the generalized hypergeometric function. To cast n(k) in a more appealing form, we restrict ourselves to a small area around k_{F} and impose that $|rk - k_{F}| \rightarrow 0$ faster than $\Lambda \rightarrow 0$; this ensures that the momentum cutoff Λ is irrelevant [1] and leaves our results independent of the cutoff. This leads to the following final form

$$n(k) \simeq \frac{1}{2} - C_1 \left| k - rk_F \right|^{\alpha} \operatorname{sgn} \left(rk - k_F \right) - C_2 \left(k - rk_F \right)$$
(1.31)

where the constants C_1 and C_2 are³⁸

$$C_1 = \frac{1}{2\sqrt{\pi}} \frac{\Gamma\left(\frac{1}{2} - \frac{\alpha}{2}\right)}{\Gamma\left(1 + \frac{\alpha}{2}\right)} \left(\frac{1}{2\Lambda}\right)^{\alpha} \qquad C_2 = \frac{\beta\left(\frac{1}{2}, \frac{1}{2}\left(\alpha - 1\right)\right)}{2\pi} \frac{1}{\Lambda}$$

We see that the form of n(k) is the same as that of a normal metal given in eqn.(1.21); the difference is hidden in the exponent α which is independent of n for a Dirac metal but has complicated n dependence for a normal metal. Also note that both differ from n(k) at T = 0 for an interacting system in higher dimensions, shown in fig.(1.2).

To conclude our discussion of the 1D Dirac metal, we use eqn.(1.31) to derive the density of states (**DoS**) which is given by

$$g(E) = \frac{1}{L^d} \frac{dN}{dE} \rightarrow g\left(k\right) = \frac{1}{L^d} \frac{1}{v_g \hbar} \frac{dN}{dk}$$

in this case, d = 1 and L is the length of the **SWNT**. N is the number of electrons and is given by

$$N = \frac{L}{2\pi} \int n(k) dk$$

 $^{^{36}}$ A detailed derivation for n(k) in a 1D normal metal was derived by Voit [32]

 $^{^{37}\}text{Notice}$ how this also sets the bandwidth cutoff $\delta(t)$ to zero

³⁸Ignoring the cutoff, and using a value of $\alpha = 0.46 \pm 0.1$ as determined through experiment (see ref.[13] in [1]), we get values of $C_1 = 0.742$ and $C_2 = -0.319$



FIGURE 1.6: Plot of the momentum distribution n(k) for a 1D Dirac metal given by eqn.(1.31). Notice how this curve is fundamentally different from that given in fig.(1.1) and fig.(1.2). Specifically, n(k) goes smoothly through $k = k_F$ as opposed to having a discontinuity. This signals a breakdown of Fermi liquid theory due to the lack of well defined **QPs**

If we limit our integration over a small range³⁹ $k_F \pm \delta k$,

$$\delta N = \frac{L}{2\pi} \int_{k_F - \delta k}^{k_F} \frac{1}{2} - C_1 |k - rk_F|^{\alpha} \operatorname{sgn} (rk - k_F) - C_2 (k - rk_F) dk$$

$$\therefore \qquad \frac{\delta N}{\delta k} = \frac{L}{2\pi} \left(1 - \frac{C_1}{\alpha + 1} (\delta k)^{\alpha} + \frac{C_2}{2} \delta k \right)$$

Upon taking $\delta k \to 0$, we arrive at the desired derivative needed for the density of states g(E):

$$\lim_{\delta k \to 0} \frac{\delta N}{\delta k} = \frac{dN}{dk} = \frac{L}{2\pi} \left(1 - \frac{C_1}{\alpha + 1} \left(\delta k \right)^{\alpha} \right)$$

where we drop the C_2 term linear in δk since $0 < \alpha < 1$ which leads to δk going to zero faster than $(\delta k)^{\alpha}$. After putting g(k) in terms of ω^{40} , we arrive at the final expression (with a 4-fold degeneracy factor taken into account)

$$g(\omega) = g^{(0)}(\omega) \left(1 - \frac{C_1}{v_g^{\alpha}(\alpha+1)} |\omega|^{\alpha} \right); \qquad g^{(0)}(\omega) = \frac{2}{\pi v_g}$$
(1.32)

where $g^{(0)}(\omega)$ is the **DoS** for a non-interacting 1D Dirac metal. The **DoS** is of interest since it's directly measurable in ARPES experiments as was discussed in [1]. Plots for the **DoS** in eqn.(1.32) are shown in fig.(1.7) and fig.(1.8). In fig.(1.7) we see very different behavior in the **DoS** between a 1D normal metal and a 1D Dirac metal which is a direct consequence of the linear energy dispersion.

³⁹the preceding integral is done from $k_F - \delta k$ to k_F . Additionally, we focus on the positive branch, i.e. r = +. The results are the same if we consider the range k_F to $k_F + \delta k$ and/or the negative branch r = -

 $^{{}^{40}}E$ and ω are measured from the Fermi point, i.e. $E \to 0, \omega \to 0$ and $|k| \to k_F$ are synonymous.



FIGURE 1.7: The static ($\omega = 0$) density of states $g(\omega = 0)$ normalized to $2/\pi v_g$ as a function of electron number N = nL (*n* is the electron density and *L* is the length of the **SWNT**) for a 1D normal metal with parabolic energy dispersion (blue dashed curve) compared with that of a 1D Dirac metal with linear energy dispersion (pink curve). Although simple in that the Dirac metal is just a constant, we see how a seemingly simple change of energy dispersion can lead to stark differences.

In addition to **TLL** theory, one may use bosonization techniques to analyze 1D interacting Fermi systems. The Hamiltonian in eqn.(1.22) can be transformed into the following form⁴¹

$$H = u_{\rho} \sum_{p>0} \left(\alpha_1^{\dagger} \alpha_1 + \alpha_2^{\dagger} \alpha_2 \right) + u_{\sigma} \sum_{p>0} \left(\beta_1^{\dagger} \beta_1 + \beta_2^{\dagger} \beta_2 \right)$$
(1.33)

which clearly shows the separation of charge and spin that is unique to 1D interacting Fermi systems. These two independent boson modes give rise to two dispersions

$$\varepsilon_i = u_i k \qquad i = \rho, \sigma \tag{1.34}$$

where $u_{\rho,\sigma}$, given in eqn.(1.30), are the velocities of the two boson modes as stated earlier. If we impose the absence of a magnetic perturbation and the existence of a charge perturbation, we see a charge density mode⁴² that propagates and gives rise to a speed of sound

$$v_1 = v_g \sqrt{1 + \frac{2g_4}{v_g \pi} + \frac{1}{\left(v_g \pi\right)^2} \left[\left(g_4\right)^2 - \left(g_2\right)^2 \right]}$$
(1.35)

 $^{^{41}}$ Detailed bosonization for a 1D normal metal can be found in [32, 34]

 $^{^{42}\}mathrm{Not}$ to be confused with charge density wave



FIGURE 1.8: The density of states normalized to the non-interacting density of states $g(\omega)/g^{(0)}(\omega)$ as a function of $x = \frac{\omega}{\Lambda v_g}$. The curve is plotted for different values of α ; experimentally, a value of $\alpha = 0.46 \pm 0.1$ was obtained by Ishii *et al* (see ref [13] in [1]). We see that in the presence of interactions, the density of states behaves very differently from the non-interacting expression. Additionally, our curves show the suppression of $g(\omega)$ as ω increases, which was predicted in 1D normal metals.

where $g_i = g_i^s = g_i^a$ based on anti-symmetry arguments. In the weak coupling limit, we neglect terms quadratic and higher in the coupling constant to obtain

$$v_1 = v_g \sqrt{1 + \frac{2g_4}{v_g \pi}}$$
(1.36)

which is the same expression for the speed of sound derived in [33] using the Green's function method. Although both eqn.(1.35) and eqn.(1.36) have the density independent behavior that's seen through this section, it's particularly odd as this is a sound mode. A sound wave is a density fluctuation; typically understood as a wave of compression and rarefaction of particles [37]. Naturally, one would expect the speed of sound to vary as particle density varies⁴³. On the contrary, the speed of sound in a 1D Dirac metal is independent of particle density and only depends on the interaction parameter g_4 . This is reminiscent of Landau's predicted zero sound mode that was discussed in the previous section. Using the dispersions given in eqn.(1.34), a number of equilibrium properties are easily calculated. For example, we find a specific heat of the following form

$$c_V(T) = \frac{\pi k_B^2}{3} \left(\frac{1}{u_\sigma} + \frac{1}{u_\rho}\right) T$$

⁴³For example, it is well known that the speed of sound in air at room temperature is 340 m/s while in the much more dense water, the speed of sound is 1500 m/s. Additionally, when calculating the speed of (first) sound within **FLT**, we see $c_1 \propto v_F$ clearly showing dependence on particle density.
which has the usual linear T dependence characteristic of both Fermi and Bose systems at low temperature. Following suit with eqn.(1.36), the specific heat, as well as other quantities such as the compressibility, susceptibility, and conductivity, are all density independent.

Although they have their similarities, mainly in the forms of most expressions, the behavior of the 1D Dirac metal is strangely different from that of the 1D normal metal. A seemingly subtle change from a quadratic dispersion to a linear dispersion has had profound changes. The inherent density independence of the Fermi velocity v_g forces us to understand the behavior of electrons in these systems in a different way than we would in a normal metal. While other systems may coincidentally show density independent behavior in some properties, we conclude that, at least in 1D, <u>all</u> the properties of a Dirac material are independent of density. As we will see in the following chapters, when Dirac materials in 2D and 3D are explored, this density independence returns and has consequences that illuminate the underlying behavior of these exotic systems.

1.4 Is Fermi Liquid Theory Valid?

While **FLT** is powerful and has been extensively used to understand interacting Fermi systems, it does have its limitations. For example, in the previous section, we discussed why **FLT** can't be used to describe one dimensional Fermi systems. An alternative approach was introduced, TLL, and distinct non-FL behavior resulted (namely, the power law momentum distribution which is profoundly different from the **FL** result of a step function). Additionally, "near" the onset of a phase transition, \mathbf{FL} breaks down as well⁴⁴. Simply put, a phase transition "breaks" the notion of a one-to-one correspondence which was paramount to the formation of FLT. This is most easily understood in terms of superfluidity and BCS superconductivity: at the transition temperature, the **FL** state undergoes an instability and the low-energy states experience a re-ordering [20]. This re-ordering leads to the formation of bound states, i.e. Cooper pairs, which clearly number less than the number of non-interacting electrons. Thus the counting argument (discussed in detail in **appendix A**), which allowed us to use the same form of the entropy, is invalid. While we don't run into the issues of one dimension or phase transitions in the remainder of this thesis, the application of **FLT** to **DM** must be done so carefully due to two main factors: the **non-FL** behavior of electrons close to the Dirac point (charge neutrality point) and the two dimensional behavior of some **DM** (like graphene).

First let's discuss the application of **FLT** to **DM**. For systems in d = 2, 3, the low energy dispersion is linear as shown in Fig.(1.9) and Fig.(1.10). The difference in these figures is where the Fermi energy is placed. If the Fermi energy is placed at the **CNP**, we no longer have a Fermi liquid and instead have the so-called Dirac fluid [21, 38]. The reason for the breakdown of **FLT** when the Fermi energy is at the **CNP** can be understood in the following ways:

 $^{^{44}}$ As we'll see in **chapter 4**, the results of [26] imply that **FL** is more robust than it seems. Nevertheless, in this introductory chapter we stick to what is generally known/agreed upon



FIGURE 1.9: Energy dispersion of a **DM** with the chemical potential at the charge neutrality point (i.e. $\mu = 0$). In the vicinity near the charge neutrality point, the **FS** shrinks to a point resulting in no well defined **QPs**. **FLT** cannot be used in this region and other methods/models need to be used.

- At the **CNP**, the Fermi surface reduces to a point. This voids the argument that the Fermi surface constrains the possible scattering pathways [21] which is crucial for the weakly interacting nature of a Fermi liquid.
- At the **CNP**, there is an absence of screening, which is necessary for the formation of **QP**.
- At the **CNP**, photoemission experiments show a **QP** decay rate that is linear in energy (temperature) which clearly deviates from **FLT** predictions [39]

Additional arguments exist that all share the conclusion: **FLT** is not applicable when the Fermi energy (chemical potential) is at the **CNP**. Then the question we must answer is whether or not **FLT** is applicable at all for **DM**. Luckily, we may safely apply **FLT** if we can restore the Fermi surface/screening argument. The way to do so, is via doping or gating which leads to dispersion given in Fig.(1.10). By moving the Fermi energy away from the **CNP**, the Fermi surface regains its circular shape (spherical in three dimensions), and **FLT** may be applied without issue. Clearly, the reduction of the Fermi surface has drastic consequences on the Fermi liquid state. We will see that going from three to two dimensions, although **FLT** is applicable, this reduction of phase space has profound consequences primarily in transport phenomena. The subject of the applicability of **FLT** in **DM** has been discussed in-depth by Das Sarma [40], Kotov [22], and Castro Neto [28]. [40]

Second, we discuss any restrictions **FLT** may experience in d = 2; for d = 3, **FLT** can generally be used and any restrictions have been thoroughly discussed in the literature [2, 7–13]. Past work by Freedman [41] and Sacco [42] applied **FLT** in two-dimensional systems as it was universally accepted that perturbation theory in d = 2 would hold without issue. The work of Anderson [43–46] discussing possible issues regarding impurities and singular forward scattering terms that <u>could</u> kill the Fermi liquid state put their claims to the test and generated a movement



FIGURE 1.10: Energy dispersion of **DM** with the chemical potential moved away from the charge neutrality point (i.e. $\mu \neq 0$). This can be achieved through doping or gating the sample. Left: n-type doped/gated **DM** where we have a **FL** of electrons. Right: p-type doped/gated **DM** where we have a **FL** of holes.

to find failures of \mathbf{FLT}^{45} . These issues were expanded and discussed in more clarity by Stamp [47, 48]. More recent work by Das Sarma [49, 50] on two-dimensional Fermi liquids show effective mass divergences in dilute systems and it is unknown if **FLT** theory applies.

The breakdown of **FLT** and the search/discovery of non-**FL** systems is not new and continues to this day. For example, with the assimilation of topology in condensed matter physics, new ways of understanding **FLT** breakdown are possible and discussed in-depth by Su and Lu [51]. We stress that in this thesis, at least up until **chapter 4**, we apply **FLT** where its validity isn't questioned⁴⁶, i.e. well within the **FL** regime, sufficiently far away from the **CNP**, no phase transitions, where there is no singular behavior in $f_{\mathbf{pp}'}^{\sigma\sigma'}$ etc. In d = 2, 3, for a Dirac (linear) spectrum, the results obtained are fundamentally different from their normal metal counterparts further cementing **FLT** as a valuable tool in studying interacting Fermi systems, even ~ 60 years later. In chapter 2 we apply FLT to DM in d = 2, 3. The equilibrium properties are generally unchanged, but the non-equilibrium (transport) properties show exotic behavior that is a direct consequence of the linear dispersion and the reduced phase space. In chapter 3, we apply the Virial theorem to **DM** and show that the linear spectrum leads to an exact closed form solution for the total ground state energy. In chapter 4 we question the alleged universality of the KSS bound, $\eta/s \ge \hbar/4\pi k_B$, and discuss how "perfect" the quantum fluid in **DM** is. To conclude this work, in chapter 5, we summarize⁴⁷ our results and discuss future work stemming from this thesis.

 $^{^{45}}$ With the emergence of superconductivity in the mid to late 1900s, much work was done in trying to kill the Fermi liquid state in metals

 $^{^{46}}$ Additionally, a large number of papers present work that refutes/restricts Anderson's claims (See reference [2] in the work done by Stamp [48])

 $^{^{47}}$ With what we've done in this thesis, and what we've learned from applying **FLT** to **DM**, the most important lesson is "There is no school like the old school"

Chapter 2

Fermi Liquid Properties of 2D & **3D** Dirac Materials

Introduction & The Ideal Dirac Gas 2.1

Before applying **FLT** to **DM** in both d = 2, 3, let's briefly examine a free Dirac gas whose constituents are fermions. Similar to what is done in undergraduate statistical mechanics courses for a gas of "normal" electrons, we consider a gas of non-interacting Dirac fermions (i.e. electrons whose single particle spectrum is linear rather than parabolic). Since we're dealing with a system of fermions, the entropy density derived in **Appendix A** is valid for a free Dirac gas^1

$$s = -k_B \sum_{\mathbf{p}\sigma} n_{\mathbf{p}\sigma} \ln n_{\mathbf{p}\sigma} + (1 - n_{\mathbf{p}\sigma}) \ln (1 - n_{\mathbf{p}\sigma})$$
(2.1)

which leads to the usual Fermi-Dirac distribution

$$n_{\mathbf{p}\sigma} = \frac{1}{1 + \exp\left[\beta\left(\varepsilon_{\mathbf{p}\sigma} - \mu\right)\right]}, \qquad \beta = \frac{1}{k_B T}$$
(2.2)

where the single particle spectrum $\varepsilon_{\mathbf{p}\sigma} = \mathbf{v} \cdot \mathbf{p} = v_g p$ is linear rather than quadratic². At T = 0, $n_{\mathbf{p}\sigma}$ reduces to the usual step function as depicted in Fig. 1.1, allowing us to calculate numerous of properties for **DM** such as the number density:

$$n = \begin{cases} \frac{p_F^2}{\pi\hbar^2} & d = 2\\ \frac{2}{3}\frac{p_F^3}{\pi^2\hbar^3} & d = 3 \end{cases}$$
(2.3)

¹Recall that the entropy density was derived purely from a counting argument so the specific form of the single particle dispersion of the fermions should not change this expression 2v_g is a constant of order $\sim 10^6~{\rm m/s}$

and ground state energy density:

$$\epsilon = \begin{cases} \frac{2}{3}n\varepsilon_F & d = 2\\ \frac{3}{4}n\varepsilon_F & d = 3 \end{cases}$$
(2.4)

where details of both derivations can be found in **Appendix A** upon making the appropriate substitutions and changes :

- $\sum_{\sigma} = (\text{spin}) \times (\text{pseudospin}) = 4$; The degeneracy factor is now 4 in **DM** due to the addition of a pseudospin, or spin-like, degeneracy. The pseudo spin degeneracy is different for different **DM**. For example, in graphene, the pseudospin degeneracy is the degeneracy arising from the two identical triangular sublattices that make up the two-dimensional hexagonal lattice. While this degeneracy is important in preserving and ensuring the existence of the linear spectrum and the Dirac points [30], at the end of the day in our calculations, the pseudospin degeneracy merely contributes an additional factor of 2 making the total degeneracy 4.
- $\varepsilon_{\mathbf{p}\sigma} = \mathbf{v} \cdot \mathbf{p} = v_g p$ which leads to a Fermi energy of $\varepsilon_F = p_F v_g$; In the literature, **DM** are commonly referred to as zero density of states systems. This is true if the Fermi energy is set at the Dirac point. In this work, as was discussed in the previous chapter, we move the Fermi energy from the Dirac point (i.e. $\varepsilon_F \neq 0$) where **FLT** is applicable.

For the low temperature properties obtained in **Appendix A**, we find most expressions retain the same form (e.g. entropy linear in temperature, $s \propto T$) with a change in the **DoS**:

$$g^{(0)}(0) = \begin{cases} \frac{2p_F}{\pi\hbar^2 v_g} & d = 2\\ \frac{2}{3} \frac{p_F^2}{\pi^2\hbar^3 v_g} & d = 3 \end{cases}$$
(2.5)

where the superscript (0) denotes non-interacting. Barring minor changes to constants, the general behavior of these expressions is (so far) the same as those for a "normal" Fermi system³. At this point, we (perhaps naively) straightforwardly apply Fermi liquid theory in the same fashion as Landau did: adiabatically turning on interactions, establishing one-to-one correspondence, etc. But before discussing any of the equilibrium and transport properties, we need to discuss what happens to the effective mass.

 $^{{}^{3}}$ For the duration of this thesis, we use the term "normal" to mean a system in which the single electron spectrum is quadratic in momentum

2.2 Interpreting effective mass m^* & the introduction of effective velocity v_q^*

One of the benefits of **FLT** is its ease in explaining interacting phenomena: many expressions that describe non-interacting systems are valid when interactions are considered albeit with the addition of a few interaction parameters that simply renormalize the non-interacting expressions. For example, in the compressibility defined in **appendix B**, setting the interaction parameter $F_0^s = 0$ reduces to the expression for compressibility derived for a free Fermi gas. An elegant way to describe this was by the introduction of effective mass m^* which allowed us to picture a strongly interacting Fermi system as weakly interacting **qp**, i.e. the physics of strongly interacting particles was effectively the same as that of weakly interacting **qp** where the **qp** are essentially "heavier" particles where their mass is modified by eqn.(1.8):

$$\frac{m^*}{m}=1+\frac{1}{d}F_1^s$$

where d = 2, 3 represents the spatial dimension and m is the bare mass. The challenge with **DM** is that the electrons, and therefore the **qp**, exhibit massless behavior and so m and m^* have no meaning. A natural question to ask is if a similar relation exists for **DM** and if so, what ratio of quantities instead is renormalized by the F_1^s parameter?

One of the first unusual properties of **DM** was the inherent massless behavior of the particles and **qp**. Therefore, one might think the qualitative/quantitative issues involving m and m^* would reconciled first. Historically, well before the discovery of graphene, the work of Baym and Chin on the relativistic extension of **FLT** [52] seems like a good starting point to describe **DM**. Unfortunately, their work discusses general relativistic properties whereas **DM** are inherently in the ultrarelativistic limit (m = 0)⁴ and therefore only a small portion of their paper is applicable; that small portion will be discussed in **chapter 3**. After the discovery of graphene, numerous review articles [22, 28, 53] have been published, yet still discuss "mass of electrons" in graphene, in spite of the agreed upon massless behavior. In fact, some of the articles mentioned discuss measurements of the (cyclotron) mass in graphene which seems meaningless. It is our goal here to provide a reasonable extension of the effective mass idea that is appropriate and suitable for the massless electrons and **qp** in graphene.

To begin, recall the single particle dispersion for a normal metal $\varepsilon_{\mathbf{p}\sigma} = \frac{p^2}{2m}$ and for a **DM** $\varepsilon_{\mathbf{p}\sigma} = v_g p$. In the presence of interactions, we know the single particle spectrum has to get altered. First looking at the single particle dispersion for a normal metal, we can deduce that the interactions have to alter the mass term. This is due to Luttinger's theorem [54] which states that the volume enclosed by the Fermi surface is an interaction strength invariant. Since $p_F \propto$ (volume of the Fermi surface), the only parameter that can be altered by interaction is m. Applying the same line of reasoning to the linear dispersion for **DM**, the parameter that has to get altered by interactions is the velocity v_g .

 $^{^{4}}$ Baym & Chin's work still considers mass and in fact has a modified effective mass relation in their work

The effective mass relation eqn.(1.8) is derived from imposing Galilean invariance⁵ to the Fermi system [2, 7, 12, 13, 55, 56] and analyzing changes in the **qp** current density when the Fermi surface is "shifted":

$$\mathbf{j}_{\mathbf{p}\sigma} = \sum_{\mathbf{p}\sigma} \nabla_{\mathbf{p}} \varepsilon_{\mathbf{p}\sigma} n_{\mathbf{p}\sigma}$$
(2.6)

where $\nabla_{\mathbf{p}} \varepsilon_{\mathbf{p}\sigma}$ is the **qp** velocity. A key observation is **j** for particles (electrons) is equal to **j** for **qp** as a consequence of number conservation. The current density is proportional to the total momentum:

$$\mathbf{P} = \sum_{\mathbf{p}\sigma} \mathbf{p} n_{\mathbf{p}\sigma}$$
$$\sum_{\mathbf{p}\sigma} \mathbf{p} n_{\mathbf{p}\sigma} = \alpha \sum_{\mathbf{p}\sigma} \nabla_{\mathbf{p}} \varepsilon_{\mathbf{p}\sigma} n_{\mathbf{p}\sigma}$$

where α is a constant. Shifting the Fermi surface is the equivalent of varying $n_{\mathbf{p}\sigma}$ and after equating the two expressions, for total momentum and current density, we arrive at

$$v_g = v_g^* + \frac{p_F^{d-2}}{\left(2\pi\hbar\right)^d} \sum_{\sigma'} f_{\mathbf{pp'}}^{\sigma\sigma'} \cos\theta$$
(2.7)

where we notice, from unit analysis, that p/α must have units of velocity. Focusing on the interaction and matching terms, we arrive at the following relation:

$$\frac{v_g}{v_g^*} = \begin{cases} 1 + \frac{1}{2}F_1^s & d = 2\\ 1 + \frac{1}{3}F_1^s & d = 3 \end{cases}$$
(2.8)

Therefore we see that the **qp** in **DM** have an **effective velocity** relation instead of an effective mass relation, both mitigated by the **FL** parameter F_1^s .

Physically and intuitively, this makes sense. We see from eqn.(2.8) that as the interaction strength increases, the **qp** velocity v_g^* decrease. This is expected; the interactions act in such a way to "slow down" the **qp**. With eqn.(2.8), we write down an expression for the density of states in the presence of interactions:

$$g(0) = \begin{cases} \frac{2p_F}{\pi\hbar^2 v_g^*} & d = 2\\ \frac{2}{3} \frac{p_F^2}{\pi^2\hbar^3 v_g^*} & d = 3 \end{cases}$$
(2.9)

which is of course related to the non-interacting density of states:

$$g(0) = \frac{v_g}{v_g^*} g^{(0)}(0) \tag{2.10}$$

⁵Work on the relativistic Fermi liquid by Baym & Chin [52] apply Lorentz invariance and obtain the same effective mass relation (to first order). This makes sense if one recalls that the value of the Lorentz factor $\gamma = (1 - \beta^2)^{-1/2} \text{ w}/\beta = \frac{v}{c}$ dictates if relativistic effects need to be considered. Since $v_g \sim 10^6 \text{ m/s}$, relativistic corrections don't matter

Our result, eqn.(2.8) shows velocity reduction as a function of interaction strength. This may seem in contradiction to current research which predicts enhancement [22, 53] and experiments which claim to show enhancement by up to a factor of 2 [57]. Although this will be talked about more in **chapter 3**, at this point, we'd like to point out that these results and eqn.(2.8) are not necessarily contradictory. Careful examination of the velocity enhancement expressions are analyzed close to the **CNP** whereas our result is safely in the Fermi liquid regime. Additionally, it is interesting to note that eqn.(2.8) can show enhancement provided $F_1^s < 0$, perhaps indicating that as one gets closer to the **CNP**, the **qp** interaction starts to become attractive. A phenomena similar to what happens at the onset of cooper pair formation where as temperature is lowered, electrons start to "feel" an attractive interaction.

To conclude, we briefly discuss the possibility of marginal Fermi liquid behavior. Numerous papers by Varma et. al. and Kakehashi et. al. [58, 59] thoroughly discuss the similarities and differences between a marginal and normal Fermi liquid, with the most important difference being the behavior $\text{Im}\Sigma(\omega)$ where $\Sigma(\omega)$ is the self energy. Typically, for normal Fermi liquid behavior, $\text{Im}\Sigma(\omega) \sim \omega^2 \ln \omega$ while $\text{Im}\Sigma(\omega) \sim \omega$ for a marginal Fermi liquid. As discussed in **chapter 1**, these relate directly to the **qp** lifetime. Das Sarma et. al. [60] took up the task of determining if graphene was a Fermi liquid or not. Their results show marginal Fermi liquid behavior for undoped graphene and usual Fermi liquid behavior for doped graphene. Furthermore, using their Green's function methods, they calculate velocity renormalizations for both undoped and doped graphene. The velocity renormalization in doped graphene shows a velocity reduction (as ours does) or enhancement depending on the cut-off momentum, and also shows non-uniformity as a function of momentum further supporting the idea that our expression eqn.(2.8) and the expressions showing enhancement are not contradictory, but rather two ends of a single story.

2.3 Equilibrium Properties

A natural start point when applying **FLT** to a system is to determine the equilibrium properties: the specific heat, compressibility, magnetic susceptibility, and effective mass/velocity. Although these properties have been extensively discussed [2, 7, 12, 13] and experimentally measured, we must exercise caution before claiming validity of these expressions for a Fermi liquid in a **DM**. Luckily, as discussed in the previous sections, the new additions and features of a **DM**, i.e. the massless behavior/linear dispersion, don't make appreciable changes to the equilibrium properties.

Despite the quasirelativistic behavior of the electrons in **DM**, as long as we're in a regime where $\mu \neq 0$, as in Fig.(1.10), the equilibrium properties are generally the same as in an ordinary **FL** with changes to the density of states:

$$c_V = \frac{\pi^2}{3} k_B^2 g(0) T$$

$$\kappa = \frac{1}{n^2} \frac{g(0)}{1 + F_0^s}$$
$$\chi = \frac{\mu_B^2 g(0)}{1 + F_0^a}$$

where g(0) is given by eqn.(2.9) as opposed to eqn.(1.11). This doesn't come a complete shock as Baym & Chin [52] pointed out that much of the non-relativistic Fermi liquid results are applicable for the relativistic case. This can be seen for a **DM** if one views the derivation for compressibility in **appendix B**. Any changes/substitutions made in the derivation don't change the form, just the density of states as stated above. The care we must take alluded to in the beginning of this section is in regards to the effective mass/velocity. This was discussed in the previous section. To reiterate, we must adopt a new relation, an effective velocity as opposed to effective mass, when dealing with **DM**.

An interesting consequence, that will be discussed further in **chapter 3**, of applying **FLT** to **DM** is in the number of free parameters. As stated in the work of Hwang et. al. [61], a measurement of the compressibility cannot provide any information about velocity renormalization in graphene. In general, for a normal metal for example, this is true. In the compressibility, there is F_0^s but also F_1^s in the density of states term. This forces one to first measure the specific heat, determine F_1^s , then measure the compressibility to determine F_0^s . However, as we'll see after applying the Virial theorem to **DM**, the conclusions drawn by Hwang et. al. could be incomplete since the linearity of the single particle dispersion leads to a possible constraint on the Fermi liquid parameters allowing us to determine more from fewer experiments (i.e. The only free parameter is F_0^s).

2.4 Fermi Liquid transport in Two-Dimensional Dirac Materials

Discussed in **chapter 1**, the **LKE** eqn.(1.13) is the primary tool used to study transport phenomena in a Fermi liquid. To do so, we examine the time evolution of $n_{\mathbf{p}\sigma}$, considering small deviations from equilibrium, as stated above eqn.(1.15). A common concern, as suggested in Baym and Pethick [2] is if a semi-classical treatment of $n_{\mathbf{p}\sigma}$ is appropriate. This can also be interpreted as asking if the effects uncertainty principle (inherently quantum effect) are important. As stated by Bedell [13], the semi-classical condition is

$$\lambda \gg \frac{\hbar}{p}$$

where λ is the wavelength of the external perturbation and p is a characteristic momentum of the system. Unique to **DM** is the constant nature of v_q . We can thus put a number to λ :

$$\lambda \gg \frac{10^{-17}}{T}$$

where T is the temperature. Alternatively, if (system size) $\gg \lambda$, then the **qp** are localized and a semi-classical approach to $n_{\mathbf{p}\sigma}$ is fine. In terms of energy, a semi-classical approach is valid if (typical excitation energy $\ll \varepsilon_F$). From this viewpoint, it's easy to see why transport studies near/at the **CNP** do not typically use the **LKE**. Since $\varepsilon_F = 0$ at the **CNP**, excitation energies would have to be significantly less than zero which isn't physically possible.

In the next section, we discuss the atypical behavior of zero sound and plasmons in a <u>normal</u> d = 2 Fermi liquid. The reason for such detail is due to the lack of literature, both theoretical and experimental, on Fermi liquid transport in lower dimensions. Many such discussions are surface level and merely adopt the d = 3 transport equations and claim, without proof, that they're sufficient for d = 2. As we'll show in the next section, this is not the case.

2.4.1 Zero Sound in an ordinary two-dimensional Fermi Liquid

Before discussing the Fermi liquid transport properties in a two-dimensional Dirac material, we need to discuss the transport modes in a normal two-dimensional Fermi liquid. Originally thought to be a purely theoretical model, recent work on the sounds modes in a trapped Fermi gas, making an effective d = 2 system, has provided a means to compare our work with experiment [62]. As we'll see, the reduced phase space alone leads to profound differences⁶. We begin with the LKE as stated earlier in eqn.(1.13)

$$\frac{\partial n_{\mathbf{p}\sigma}}{\partial t} - \{\varepsilon_{\mathbf{p}\sigma}, n_{\mathbf{p}\sigma}\} = I \tag{2.11}$$

where $\{\dots, \dots\}$ are Poisson brackets, $n_{\mathbf{p}\sigma}$ is the quasiparticle distribution function, $\varepsilon_{\mathbf{p}\sigma}$ is the quasiparticle Hamiltonian, and I is the collision integral. As per usual, we linearize and Fourier transform eqn.(1.13) and are left with the **LLKE** given in eqn.(1.15)

$$\left(\omega - \mathbf{q} \cdot \mathbf{v}_{\mathbf{p}\sigma}\right) \delta n_{\mathbf{p}\sigma} \left(\mathbf{q}, \omega\right) + \left(\mathbf{q} \cdot \mathbf{v}_{\mathbf{p}\sigma}\right) \frac{\partial n_{\mathbf{p}\sigma}^0}{\partial \varepsilon_{\mathbf{p}\sigma}^0} \delta \varepsilon_{\mathbf{p}\sigma} \left(\mathbf{q}, \omega\right) = iI$$
(2.12)

By definition, the zero sound mode is a collision-less sound mode (a sound mode at zero temperature) which allows us to set I = 0. Writing $\delta n_{\mathbf{p}\sigma}$ in terms of Fermi surface distortions $\nu_{\mathbf{p}\sigma}$, we can cast eqn.(1.15) in a familiar form as derived by Baym and Pethick [2]

$$\nu_{\mathbf{p}\sigma} + \frac{\mathbf{q} \cdot \mathbf{v}_{\mathbf{p}\sigma}}{\omega - \mathbf{q} \cdot \mathbf{v}_{\mathbf{p}\sigma}} \sum_{\mathbf{p}'\sigma'} f_{\mathbf{p}\mathbf{p}'}^{\sigma\sigma'} \left(\frac{\partial n_{\mathbf{p}'\sigma'}^0}{\partial \varepsilon_{\mathbf{p}'\sigma'}^0} \right) \nu_{\mathbf{p}'\sigma'} = \frac{\mathbf{q} \cdot \mathbf{v}_{\mathbf{p}\sigma}}{\omega - \mathbf{q} \cdot \mathbf{v}_{\mathbf{p}\sigma}} U$$
(2.13)

It is at this point that we begin to see behavior unique to two-dimensional systems.

⁶For more details, one should consult the work found in [63, 64]

As was done in the previous section, we use the Chebyshev polynomials of first kind where the angle between \mathbf{p} and \mathbf{p}' , θ , is planar restricted. The orthogonality relation:

$$\int_{-1}^{1} \frac{dx}{\pi} \frac{T_l(x)T_{l'}(x)}{\sqrt{1-x^2}} = \begin{cases} \frac{1}{2}\delta_{ll'}, & l \neq 0, l' \neq 0\\ \\ 1, & l = l' = 0 \end{cases}$$

is what leads to the unusual behavior of the collective mode. Evaluating the second term on the l.h.s. of eqn.(2.13) leaves us with

$$\sum_{l} T_{l}(x)\nu_{l} + \frac{x}{s-x} \sum_{l'} \left(-\frac{1}{2}F_{l'}^{s}\right) (1+\delta_{l',0}) T_{l'}(x)\nu_{l'} = \frac{x}{s-x}U$$
(2.14)

where only the spin symmetric Landau parameters matter. The goal is to determine the poles of the response functions ν_l/U as these are the frequencies of oscillation of the propagating mode.

So far, with the exception of introducing Chebyshev polynomials as our orthogonal function basis, this analysis is what's done in three-dimensional Fermi liquids. A profound difference is in solving for the l = 0 response function. If we multiply eqn.(2.14) by $T_0(x)/\sqrt{1-x^2}$ and exploit the orthogonality relation, we arrive at

$$\nu_0 - F_0^s \nu_0 \int_{-1}^1 \frac{dx}{\pi} \frac{x}{s-x} \frac{1}{\sqrt{1-x^2}} = U \int_{-1}^1 \frac{dx}{\pi} \frac{x}{s-x} \frac{1}{\sqrt{1-x^2}}$$
(2.15)

where we've imposed $F_{l'}^s = 0$ for $l' \ge 1$. The integrals in eqn.(2.15) are highly non-trivial and need to be evaluated carefully. Details of the calculation can be found in [64]. The difficulty arises from the integration bounds also being singularities of the integrand. To circumvent this issue, a careful change of variables must be made which turns the above integrals into

$$\int_{-1}^{1} \frac{x}{s-x} \frac{1}{\sqrt{1-x^2}} dx = \pm \frac{1}{2i} \oint_{|z|=1} \frac{(z-i)(z+i)}{z(z-s-\sqrt{s^2-1})(z-s+\sqrt{s^2-1})}$$

where we assume a general complex form for $s: s = s_1 + is_2$ and the contour is the unit circle. The integral is then evaluated for |s| > 1 and |s| < 1 leading to

$$\int_{-1}^{1} \frac{dx}{\pi} \frac{x}{s-x} \frac{1}{\sqrt{1-x^2}} = -\left(1 - \frac{|s|}{\sqrt{s^2 - 1}}F(s)\right)$$
(2.16)

where the function F(s) is defined as

$$F(s) = 1 - [\delta(s_1)\Theta(|s| - 1) + \delta(s_2)\Theta(1 - |s|)]$$

and captures the behavior of |s| > 1 and |s| < 1 which determines whether the poles of the integral are inside/outside the contour. Finally, the response function ν_0/U , and subsequently the zero sound mode, can then be determined depending on the value of |s|.

• Response function for |s| > 1

For values of |s| > 1, the response function

$$\frac{\nu_0}{U} = \begin{cases} -\left(\left(1 - \frac{s}{\sqrt{s^2 - 1}}\right)^{-1} + F_0^s\right)^{-1} & s_1 \neq 0 \end{cases}$$
(2.17a)

$$\left(-(1+F_0^s)^{-1}\right) \qquad \qquad s_1 = 0 \qquad (2.17b)$$

First focusing on the response function when the real part of s is non-zero, given in eqn.(2.17a), and solving for the poles, we get the following two equations

$$\frac{s_2\sqrt{s^2-1}}{2s^2-2s_1\sqrt{s^2-1}-1} = 0 \tag{2.18a}$$

$$\frac{s^2 - s_1\sqrt{s^2 - 1} - 1}{2s^2 - 2s_1\sqrt{s^2 - 1} - 1} = -F_0^s$$
(2.18b)

Eqn.(2.18a) immediately sets the imaginary part of s to be zero (i.e. $s_2 = 0$) and subsequently sets s to be purely real (i.e. $s = s_1$). Solving eqn.(2.18b) for s yields the following for the propagating zero sound mode in a two-dimensional Fermi liquid

$$s = \frac{\omega}{qv_F} = \frac{1 + F_0^s}{\sqrt{1 + 2F_0^s}} \tag{2.19}$$

which is plotted in fig. $(2.1)^7$.

If we consider $F_0^s \gg 1$, we see

$$s \approx \sqrt{\frac{F_0^s}{2}} \to c_0 \approx v_F \sqrt{\frac{F_0^s}{2}}$$

while for $F_0^s \ll 1$,

$$s\approx1+\frac{1}{2}\left(F_{0}^{s}\right)^{2}\rightarrow c_{0}\approx v_{F}\left(1+\frac{1}{2}\left(F_{0}^{s}\right)^{2}\right)$$

both of which can be seen in the behavior of the blue curve in fig.(2.1). The interesting behavior of the two-dimensional zero sound mode is seen if we consider attractive interactions (i.e. $F_0^s < 0$). As per Baym & Pethick [2], for arbitrarily small attractive interactions, Landau damping strongly affects the zero sound mode. However, as discussed in previous works [12, 65, 66], a non-zero real and imaginary part of the response function is crucial to the existence of Landau damping. This is easily seen in the response function for a three-dimensional Fermi liquid as shown in fig.(2.2). As we've shown earlier, for |s| > 1and non-zero real part, the imaginary part must be zero. Thus the mode can propagate freely for a certain range of attractive interactions $-0.5 < F_0^s < 0$. Although experimental realization of a two-dimensional Fermi liquid with attractive interactions is due to the work of Fröhlich et. al. [67], to our knowledge, a theoretical description of the two-dimensional Fermi liquid with attractive interaction has not been carried out.

⁷The negative solution of eqn.(2.18b) is disregarded as this mode can't be excited at T = 0



FIGURE 2.1: A plot of s as a function of F_0^s for |s| > 1 (blue curve) given by eqn.(2.19). The dashed red line at $F_0^s = -0.5$ indicates a dynamic instability, different from the instability discussed in the previous section, of the zero sound mode. The unusual behavior of this mode can be seen for attractive interactions $-0.5 < F_0^s < 0$ where undamped propagation occurs. This is in stark contrast to the behavior found in three-dimensional Fermi liquids where, for arbitrarily small attractive interactions, the presence of Landau damping strongly inhibits the mode [2].

Turning our attention to the response function given by eqn.(2.17b), we see that ν_0/U is independent of s. This response function violates causality, established through the wellknown Kramers-Kronig relations [12]. As such, we disregard this solution and conclude that s must be purely real for |s| > 1.

• Response function for |s| < 1

For values of |s| < 1, we have

$$\frac{\nu_0}{U} = \begin{cases} -\left(\left(1 - \frac{s}{\sqrt{s^2 - 1}}\right)^{-1} + F_0^s\right)^{-1} & s_2 \neq 0 \\ (1 + F_0^s)^{-1} & s_2 \neq 0 \end{cases}$$
(2.20a)

$$\left(-(1+F_0^s)^{-1}\right) \qquad s_2 = 0 \qquad (2.20b)$$

First focusing on the situation given in eqn.(2.20a), and solving for the poles, we get the following two equations

$$\frac{-s_1\sqrt{1-s^2}}{1-2s_2\sqrt{1-s^2}} = 0 \tag{2.21a}$$

$$\frac{1 - s^2 - s_2\sqrt{1 - s^2}}{1 - 2s_2\sqrt{1 - s^2}} = -F_0^s \tag{2.21b}$$

Eqn.(2.21a) immediately sets the real part of s to be zero (i.e. $s_1 = 0$) and subsequently sets s to be purely imaginary (i.e. $s = is_2$). Solving eqn.(2.21b) for s yields the following



FIGURE 2.2: A plot of Ω_{00} as a function of s given by eqn.(1.3.16) from Baym and Pethick [2]. In solving for the zero sound mode in three dimensions, one solves $1 + F_0^s \Omega_{00} = 0$ for s after imposing different limits on F_0^s . If $F_0^s > 0$, we see an undamped zero sound mode and if $F_0^s < 0$, we see a Landau damped mode. This is very different from the two dimensional case where an undamped propagating mode can exist for an attractive interaction, and Landau damping is seemingly absent as s is either purely real or purely imaginary.

for zero sound mode in a two-dimensional Fermi liquid if |s| < 1

$$s = \pm i \left(\frac{1 + F_0^s}{\sqrt{1 + 2F_0^s + 2\left(F_0^s\right)^2}} \right)$$
(2.22)

which is plotted in fig.(2.3).

Discussion of the results plotted in fig.(2.3) requires more care than the discussion of the results in fig.(2.1). First note that in eqn.(2.22), we have kept both positive and negative solutions as the earlier argument in regards to negative solutions and T = 0 only holds assuming we're dealing with real frequencies. Analyzing the positive pole (green curve) first, we see an immediate violation of causality as this pole is in the upper complex plane. The remaining negative pole is purely imaginary and indicates a severely damped mode (i.e. no propagation allowed) that is distinctly different from the usual Landau damping due to the absence of a non-zero real part. Focusing on the alternative solution for the response function given by eqn.(2.20b), we see another response function constant in s. As per before, this result violates causality and is disregarded. Therefore we conclude that no mode can propagate for values of |s| < 1.

As we've seen from the above analysis, behavior of the zero sound mode in a two-dimensional Fermi liquid is drastically different from that in its three-dimensional counterpart. Previous work on zero sound in three-dimensions is numerous and all shows the same usual behavior: undamped propagation for $F_0^s > 0$ and/or |s| > 1 (outside the particle-hole



FIGURE 2.3: A plot of s_2 as a function of F_0^s for |s| < 1 given by eqn.(2.22). The dashed red line at $F_0^s = -1$ indicates the Pomeranchuk instability as derived in the previous section. While both poles seem to present possible modes for |s| < 1, they're both invalid for different reasons. The positive pole (green curve) is located in the upper half of the complex plane and therefore violates the well-known Kramer's Kronig relations. The negative pole (purple curve) is purely imaginary (as is s in general for |s| < 1) indicating a severely damped mode that cannot propagate. This is different from Landau damping, which requires both nonzero real and imaginary parts to s.

continuum), and Landau damping when $F_0^s < 0$ and/or |s| < 1 (inside the particle-hole continuum). While our work does show an undamped mode for repulsive interactions and |s| > 1, we see propagation for a small region of attractive interaction which is absent in three-dimensions; a sound mode may propagate in the presence of attractive interactions but this is first sound, not zero sound. Additionally, it appears that Landau damping is absent and a Landau damped mode is completely damped out in a two-dimensional Fermi liquid.

• Including higher order Landau parameters

We stress that the above calculation was performed for l = 0 distortions of the Fermi surface and assuming all Landau parameters essentially zero except F_0^s . A natural question to ask is if including higher order Landau parameters affects the behavior of the zero sound mode. Specifically, we want to ask whether or not Landau damping can be restored upon inclusion of the F_1^s Landau parameter.

Going back to eqn.(2.14), we multiply by $T_1(x)/\sqrt{1-x^2}$ and exploit the orthogonality relation, arriving at an equation similar to eqn.(2.15) with integrals of the following form

$$\int_{-1}^{1} \frac{x^n}{s - x} \frac{1}{\sqrt{1 - x^2}} dx$$

Evaluation of the integral above leads to a generalization of the integral in eqn.(2.16). Their result allows us to express eqn.(2.15) in the following general form, similar to what is done in [2] for a three-dimensional Fermi liquid

$$\frac{\pi}{2} \left(1 + \delta_{l0} \right) \nu_l + \sum_{l'} Q_{ll'} \left(1 + \delta_{l'0} \right) F_{l'}^s \nu_{l'} = -2Q_{l0}U \tag{2.23}$$

where $Q_{ll'}$, a function of s, is defined as

$$Q_{ll'} = Q_{l'l} = \frac{1}{2} \int_{-1}^{1} \frac{T_l(x)T_{l'}(x)}{\sqrt{1-x^2}} \frac{x}{x-s} dx$$

If we wish to include the F_1^s term, we take the l = 0 and l = 1 terms of eqn.(2.23)

$$\pi\nu_0 + 2Q_{00}F_0^s\nu_0 + Q_{01}F_1^s\nu_1 = -2Q_{00}U$$
(2.24a)

$$\frac{\pi}{2}\nu_1 + 2Q_{10}F_0^s\nu_0 + Q_{11}F_1^s\nu_1 = -2Q_{10}U$$
(2.24b)

where we impose $F_l^s = 0$ for $l \ge 2$. Solving for the response function we get

$$\frac{\nu_0}{U} = \frac{-2Q_{00} - 2F_1^s \frac{(Q_{01})^2}{\frac{\pi}{2} + Q_{11}F_1^s}}{\pi + 2F_0^s Q_{00} - 2F_0^s F_1^s \frac{(Q_{01})^2}{\frac{\pi}{2} + Q_{11}F_1^s}}$$
(2.25)

and subsequently solve for s such that the denominator of eqn.(2.25) is zero. As before, we obtain modes that freely propagate (s purely real), modes that violate causality (response function approaching a constant value as $s \to \infty$), and modes that are severely damped (s purely imaginary). However, a new feature that arises from retaining up to the F_1^s Landau parameter is the possibility of a Landau damped collective mode. For |s| > 1, we have

$$s = s_1 + is_2$$
 (2.26)

where

$$s_1 = \frac{1}{2}\sqrt{s^2 - 1} \left(\frac{F_1^s + 1}{F_1^s - 2}\right)$$
(2.27a)

$$s_2 = \pm \frac{1}{2} \sqrt{\frac{(2+F_1^s)\left(-6+F_0^s\left(F_1^s-2\right)+5F_1^s-\left(F_1^s+2\right)s^2\right)}{F_1^s-2}}$$
(2.27b)

and for |s| < 1, we have

$$s = s_3 + is_4$$
 (2.28)

where

$$s_3 = \frac{(F_1^s + 2)\sqrt{1 - s^2}}{2(F_1^s - 2)}$$
(2.29a)

$$s_4 = \frac{F_1^s + 2}{2(F_1^s - 2)} \left(\sqrt{1 - s^2} \pm \sqrt{\frac{6F_1^s + 4F_0^s(F_1^s - 2) - 4}{F_1^s + 2} - 2s^2} \right)$$
(2.29b)

Expressions for s_1 , s_2 , s_3 , and s_4 solely as function of F_0^s and F_1^s are straightforward to obtain, however they are cumbersome. The importance of eqns.(2.26)-(2.29b) is their complex nature which allows for the possibility of Landau damping subject to the following constraints on F_1^s :

$$F_1^s > \begin{cases} \frac{-1 + \sqrt{53 + F_0^s \left(29 + 4F_0^s\right)}}{4 + F_0^s} & |s| > 1 \end{cases}$$
(2.30a)

$$\begin{cases}
\frac{2+4F_0^s}{3+2F_0^s} & |s| < 1
\end{cases}$$
(2.30b)

and subsequently on |s|

$$s| < \begin{cases} \sqrt{\frac{4F_0^s F_1^s + 5F_1^s - 8F_0^s - 6}{F_1^s + 2}} & |s| > 1 \end{cases}$$
(2.31a)

$$\left\{ \sqrt{\frac{2F_0^s F_1^s + 3F_1^s - 4F_0^s - 2}{F_1^s + 2}} \quad |s| < 1 \right.$$
(2.31b)

which are derived from imposing that |s| must be real.



FIGURE 2.4: Plot of eqn.(2.30a) (light curve) and eqn.(2.30b) (dark curve). The curves show the lower bound on F_1^s (as a function of F_0^s). For large values of F_0^s , we see that the lower bound condition saturates at $F_1^s = 2$. For small values of F_0^s , the situation is different depending on |s|. For |s| > 1, the smallest value for F_1^s above which Landau damping can occur is $F_1^s = 1.5$ while for |s| < 1, as we approach the dynamic instability at $F_0^s = -0.5$, F_1^s needs to simply be larger than zero for Landau damping to be present. With a large lower bound of $F_1^s \ge 2$ for large values of F_0^s , it appears that the phenomena of Landau damping is still absent for |s| > 1in spite of including the F_1^s Fermi liquid parameter. For |s| < 1, Landau damping seems more likely and this agrees with what is seen in the three-dimensional case (The |s| < 1 mode is inside the particle-hole continuum indicating a Landau damped mode) with the exception of needing the addition of F_1^s .

The above analysis, assuming the next order Landau parameter F_1^s is non-zero, is summarized in fig.(2.4). As we can see, the inclusion of a non-zero F_1^s term is crucial for the possibility of Landau damping in the system albeit doesn't guarantee it. If |s| > 1, $F_1^s > 1.5$ for small values of F_0^s and $F_1^s > 2$ for large values of F_0^s . Both scenarios require

a rather large value of F_1^s therefore making the presence of Landau damping unlikely. If |s| < 1, Landau damping is more likely for small values of F_0^s since the lower bound on F_1^s tends to zero as $F_0^s \to -0.5$. To fully understand Landau damping in a two-dimensional Fermi liquid, one would need values for F_0^s and F_1^s obtained via experiments, for example on the specific heat and compressibility, on two-dimensional interacting Fermi systems. To our knowledge, such data isn't currently available and therefore we're left to make predictions based on the results from above. As such, we conclude that in spite of including an additional Fermi liquid parameter, F_1^s , Landau damping is still not present at least for large values of F_0^s . For small F_0^s , Landau damping remains unlikely for |s| > 1. If |s| < 1, the situation resembles that of a three-dimensional Fermi liquid and the mode is immediately inside the particle-hole continuum indicating its damped behavior.

2.4.2 Plasmons in a two-dimensional Fermi Liquid

In practice, most systems under investigation have non-zero charge and as such the zero sound mode discussed throughout this paper needs to be replaced with a plasmon mode. Emergence of plasmon modes within the Landau theory requires inclusion of the Coulomb interaction in eqn.(2.13). Discussion of the process, including the subtleties when dealing with the singularity at q = 0 and establishing screening, for a three-dimensional Fermi liquid can be found in Pines & Nozieres [12]. Carrying out the calculation, and including up to the F_1^s term, we arrive at the following for the frequency of a plasmon mode in a three-dimensional Fermi liquid

$$\omega^2 = \left[\omega_p^2 + \frac{1}{3}q^2 v_F^2 F_0^s\right] \left(\frac{m^*}{m}\right)$$
(2.32)

where ω_p is the RPA plasmon frequency⁸ in a three-dimensional electron gas [40]:

$$\omega_p^2 = \frac{4\pi n e^2}{m}$$

Eqn.(2.32) highlights two new features that arise from using the LKE to analyze a charged Fermi liquid: an effective mass (F_1^s) correction and a crossover to the zero sound mode as function of q.

The general analysis conducted earlier for zero sound doesn't appreciably change when looking at the plasmon mode. The biggest difference comes in altering the l = 0 **qp** interaction to incorporate the Coulomb interaction:

$$f_0^s \to \frac{2\pi e^2}{q} + f_0^s$$

⁸Note that there is a change from $m \to m^*$ in the particle density term n, however, Luttinger's theorem ensures that particle density remains constant under interaction

Continuing as before, assuming only F_0^s is non-zero and looking at $s \gg 1$ and |s| > 1 since the plasmon is a high frequency mode, we arrive at

$$\omega^2 = \omega_p^2 + \frac{1}{2}q^2 v_F^2 F_0^s \tag{2.33}$$

where now ω_p is the RPA plasmon frequency in a two-dimensional electron gas:

$$\omega_p^2 = \frac{2\pi n e^2 q}{m}$$

and $n = \frac{1}{2\pi} \left(\frac{p_F}{\hbar}\right)^2$ is the electron density in two-dimensions. Eqn.(2.33), only keeping the l = 0Landau parameter resembles that of eqn.(B.24) where the RPA plasmon frequency is shifted by a term that can be interpreted as zero sound provided $F_0^s \gg 1$. Keeping up to the F_1^s parameter in eqn.(2.14), performing the same integrals discussed earlier, and making an expansion due to large s, we have

$$2s^2\nu_0 - \tilde{F}_0^s\nu_0 - \frac{1}{2}F_1^s\nu_1 s = U$$

where $\tilde{F}_0^s = g(0)\frac{2\pi e^2}{q} + F_0^s$ and g(0) is the density of states of the two-dimensional electron gas at the Fermi energy. As in the three-dimensional case discussed in **appendix B**, we use the l = 0 moment of the kinetic equation, eqn.(B.5), to substitute for ν_1 allowing us to solve for the response function ν_0/U . Solving for the pole leaves us with a peculiar expression for frequency of the propagating mode:

$$\omega^2 = \frac{1}{1 - \frac{1}{4}F_1^s} \left[\omega_p^2 + \frac{1}{2}q^2 v_F^2 F_0^s \left(\frac{m^*}{m}\right) \right]$$
(2.34)

We can see a number of interesting differences between eqn.(2.32) and eqn.(2.33)/eqn.(2.34). First, in the q dependence, we see from eqn.(2.32) that the three-dimensional mode has robust plasmon behavior as it avoids the particle-hole continuum and remains undamped until q gets very large; notice the q dependence only being on the additional additive term with F_0^s . However, in a two-dimensional Fermi liquid, the plasmon mode has a linear q dependence and could theoretically be damped from all values of q; it appears the undamped propagation of plasmon modes in a two-dimensional Fermi liquid is sensitive to factors such as mass, particle density, and **qp** interactions F_0^s and F_1^s to avoid the particle-hole continuum. Comparing eqn.(2.32) and eqn.(2.34) where F_1^s was considered, we see an unusual difference. In eqn.(2.32) for the mode in three-dimensions, we see an effective mass term that scales/renormalizes the entire term. This is expected as by including mass renormalization effects, one would expect the behavior of the entire mode to be altered the same way. In two-dimensions, we don't see this. In eqn.(2.34), we see two features absent from eqn.(2.32). First, the effective mass correction only alters the additive F_0^s term. The RPA plasmon part remains unchanged. Second, we see an unusual factor of $\left(1-\frac{1}{4}F_1^s\right)^{-1}$ multiplying the whole expression. If $F_1^s > 0$, the terms acts to enhance the mode possibly helping to avoid the particle-hole spectrum and possible damping (that is, until $F_1^s = 4$ in which case the mode goes to infinity). If $F_1^s < 0$, the mode gets weakened and possibly dips into

the particle-hole spectrum and experiences Landau damping sooner than expected. This would agree with the general notion from Baym and Pethick who claimed that in a three-dimensional Fermi liquid, the slightest attractive interaction will damp the collective mode [2]. Nevertheless, it's clear that the experimental determination of F_0^s and F_1^s is imperative to understanding the underlying behavior of the mode. Additionally, as we've seen in this section and throughout this work, the dimensionality of the system plays a vital role in the electron behavior.

2.4.3 Zero Sound & Plasmons in a two-dimensional DM

As we saw in the previous subsection, a simple reduction of phase space leads to new and interesting phenomena, even in the "simplest" transport properties⁹. Therefore, it is reasonable to assume that a d = 2 **DM** would posses at least similar atypical collective mode behavior. Such an interest in the collision-less modes of d = 2 **DM**, such as graphene, is well-documented and thoroughly explored [68–71]. Unfortunately, as has been and will be the theme of this thesis, much of the investigation omits any study of the Fermi liquid regime as it's assumed the behavior of the collective modes is essentially that of a d = 3 Fermi liquid. Hopefully you've been convinced from the previous subsection that this isn't the case.

Following the analysis conducted for zero sound in a d = 2 Fermi liquid we notice no appreciable change in the expressions for s given by eqn.(2.19); for this discussion, we focus on the regimes where a mode can propagate freely |s| > 1. This is expected since, as said before, the main changes when considering the Fermi liquid properties of **DM** are in the density of states g(0) and the velocity v_g/v_g^* . Therefore, all the results from the previous discussion should all hold for a d = 2 **DM** like graphene.

The plasmon is slightly more interesting. Returning to eqn.(2.14) and making the appropriate change in the density of states, we get slightly modified versions of eqn.(2.33) and eqn.(2.34). First considering the case when only F_0^s is non-zero:

$$\omega^2 = \omega_p^2 + \frac{1}{2}q^2 v_g^2 F_0^s \tag{2.35}$$

and now with F_1^s :

$$\omega^{2} = \frac{1}{1 - \frac{1}{4}F_{1}^{s}} \left[\omega_{p}^{2} + \frac{1}{2} \left(qv_{g}^{*} \right)^{2} F_{0}^{s} \right] \frac{v_{g}}{v_{g}^{*}}$$
(2.36)

where ω_p is the RPA plasmon frequency for a d = 2 Dirac system defined as

$$\omega_p = \sqrt{\alpha} \left(g\pi n_2\right)^{1/4} v_g \sqrt{q}$$

 $^{^{9}}$ By no stretch of the imagination are zero sound and plasmon modes simple. We merely say simple to indicate the collision integral, which is typically the bane of everyone's existence, being set to zero

where α is the fine structure constant in graphene [21, 22, 28]¹⁰, g = 4 for both spin and valley degeneracy, and n_2 is the particle density defined in eqn.(2.3). For details on the derivation of ω_p using the RPA, please see the work by Das Sarma and Hwang [69].

Once again, we see the plasmon frequency is shifted from the usual RPA ω_p by F_0^s and possibly renormalized by F_1^s , provided it's non-zero. The interest in plasmons for **DM** is due to a few things: First, there are many papers using various methods to calculate the plasmon frequency allowing us to easily check for agreement or differences between our work and theirs. For example, when comparing with the results of [69], we see our result contains additional interaction parameters. We also see the **LKE** recover the RPA ω_p with all the well-known exotic behavior: same q dependence as in a "normal" Fermi system, quantum behavior (no classical analog) in the first order term arising primarily from the linear spectrum. Second, since all **DM**, like graphene, have net charge, the collective density oscillation is a plasmon and not zero sound meaning any experimental work done will have measured the plasmon frequency.

Perhaps the greatest interest in collective modes in **DM** is due to the possible realization of a long sought after crossover between zero sound and plasmons. Originally discussed by Pines and Nozières [12], the simple fact that the only difference between zero sound and a plasmon is the presence of charge implies a crossover from plasmon to zero sound behavior as a function of some parameter. Unfortunately, very few systems (if any) can experimentally verify this phenomena and thus this was viewed as more of a theoretical happenstance than anything. Fortunately, with the boom of graphene and **DM**, there is a possibility of experimentally probing/driving a system to such that we can view this crossover. Armed with the expressions for plasmons (eqn.(2.35)) and eqn.(2.36)) and zero sound (same as before), it is possible to measure the collective mode behavior and see which mode propagates. The question therefore becomes: where and when in a DM can we have charge-less behavior to see a zero sound mode? The answer: at the CNP! So, as a function of momentum, we should be able to crossover from the plasmon to zero sound. This is also seen in eqn.(2.35) and eqn.(2.36). The terms containing F_0^s closely resemble zero sound behavior. Furthermore, the depend on q^2 as opposed to q so for shorter wavelengths, zero sound behavior should be dominant. This conclusion agrees with the work of Svintsov [68] who develops a kinetic model describing $nonlocal^{11}$ transport in ballistic and hydrodynamic regimes. His work shows weakly damped sound modes and strongly damped plasmons near the CNP and opposite behavior in doped samples (plasmons well defined and sound modes strongly damped).

¹⁰In the literature, there is typically a κ in the denominator of the graphene fine structure constant. This is to capture the dielectric behavior of the substrate the graphene sheet is grown one. Obviously throughout these discussions, $\kappa = 1$

¹¹Recall earlier in section 4 when determining if the semi-classical approach was valid, we related the perturbation size λ to the locality of our system

Chapter 3

Constraints from the Virial Theorem

3.1 Introduction

The Virial Theorem, originally introduced by Clausius in 1870 and applied to problems of thermodynamics, has been widely studied and applied to many systems in both classical mechanics, special relativity, and quantum mechanics^[16]. By considering the following quantity, known as the virial,

$$G = \sum_{i} \mathbf{r}_{i} \cdot \mathbf{p}_{i} \tag{3.1}$$

a number of interesting properties may be derived depending on the system in question. For example, when considering systems under the influence of central forces only, the virial theorem leads to

$$2\overline{T} = (n+1)\overline{V} \tag{3.2}$$

where $\overline{T} \& \overline{V}$ are the average kinetic and potential energy respectively, and n is related to the potential that generates the forces $V = ar^{n+11}$. Other areas where the virial theorem has been applied include deducing the existence of dark matter and deriving the Chandrasekhar limit for the stability of white dwarf stars.

As discussed in chapter 1, the systems/materials of interest in this thesis are **DM** due to their unique properties [22, 27–30, 53, 72], specifically their massless behavior and constant Fermi velocity which we denote $v_g \sim 10^6$ m/s. In this chapter, we apply the Virial theorem to **DM** and discuss the implications. Motivation for the endeavor grew from the desire to understand the role of electron-electron interactions in **DM**. One may anticipate unconventional physics based on the linear dispersion alone. For example, the work of Dahal et. al. showed that Wigner

¹n = -2 is Coulomb interaction and leads to the well known result $\overline{K} = -\frac{1}{2}\overline{V}$

crystallization is absent in **DM** regardless of the low density close to the charge neutrality point [73]. This is in stark contrast to what happens in a conventional electron system, one with parabolic dispersion, where low particle density can lead to Wigner crystallization. Application of the Virial theorem to **DM** was first done by Stokes et. al. [74] with an unusual result. As a consequence of the linear dispersion, an expression for the <u>full</u>² average ground state energy is obtained:

$$\overline{E} = \frac{\mathcal{B}}{r_s} \tag{3.3}$$

where \mathcal{B} is a constant independent of particle density and r_s is a dimensionless parameter characterizing the density of the system ³. Sometime later, the work of Sokolik et. al. [76] introduced cutoffs to take into account the finite size of **DM** such as graphene. Their results agree with that of [74] when the cutoffs are taken to infinity.

In the next section, we outline and discuss the elegance of applying the Virial theorem to **DM** first performed by Stokes et. al. [74]. Although their result was originally obtained from applying the Virial theorem to an arbitrary two-dimensional system where the single electron dispersion is linear in momentum (meant to model graphene), the results are valid for **DM** in all dimensions. Although the result seems simple, its impact cannot be understated. Additionally, finding a closed form solution for any quantity when taking into account interactions is a rarity and doing so for the total average ground state energy is even more rare. In the final section, we discuss the profound consequences of eqn.(3.3) in regards to the compressibility and within the context of Fermi Liquid theory. We will see that the simple result obtained through the Virial theorem provides deep insight and predicts unusual behavior in **DM**.

3.2 The Virial Theorem in Graphene & other Dirac Materials

To begin, we consider a system of Dirac electrons in two-dimensions with standard looking Hamiltonian

$$H = T + V \tag{3.4}$$

where V is the Coulomb potential containing three separate contributions: the interaction between electrons:

$$V_{ee} = \frac{1}{2} \sum_{i \neq j} \frac{e^2}{|\mathbf{r}_i - \mathbf{r}_j|}$$

the interaction between electrons and the positive background:

$$V_{eb} = -e \sum_{i} \int_{A} d\mathbf{A} \frac{\rho}{|\mathbf{r}_{i} - \mathbf{x}|}$$

 $^{^{2}}$ Here we say "full" to indicate this expression takes into account interactions

³In the usual literature, r_s is defined as the Wigner-Seitz radius [75]

and the potential energy of the positive background itself:

$$V_b = \frac{1}{2} \int_A d\mathbf{A} \int_A d\mathbf{A}' \frac{\rho^2}{|\mathbf{x} - \mathbf{x}'|}$$

The homogeneous nature⁴ of these terms becomes crucial when Euler's homogeneous function theorem is employed, however, this is no different than what is done in the standard twodimensional electron gas. This should come as no surprise as the interaction terms are the same regardless of the single electron energy dispersion. As one might expect, the difference comes when considering the kinetic energy. Applying the virial in eqn.(3.1) to the Heisenberg equation of motion⁵ (i.e. take the time derivative), we get

$$\dot{G}=T+V+r_s\frac{\partial V}{\partial r_s}$$

where $r_s = r/a$ characterizes the density of electrons⁶, r is the interparticle spacing, and a is the lattice constant of graphene⁷.

As outlined in Goldstein [16], numerous arguments lead to the time average of \dot{G} going to zero⁸. The $r_s \frac{\partial V}{\partial r_s}$ term can be recast as $r_s \frac{\partial H}{\partial r_s}$ by noting that the kinetic energy is independent of r_s leading to

$$\overline{T} + \overline{V} = -r_s \frac{\partial \overline{H}}{\partial r_s}$$

Furthermore, H as defined in eqn.(3.4) in this case is simply the total energy E. Additionally, the LHS of the above equation is average total energy \overline{E} . This is in stark contrast to what is seen in the ordinary electron gas, i.e. with parabolic single particle spectrum, where the LHS doesn't completely reduce to the average total energy. So we have a simple first order differential equation in \overline{E} :

$$\overline{E} = -r_s \frac{\partial \overline{E}}{\partial r_s} \tag{3.5}$$

whose solutions are given by eqn.(3.3), where again, the constant \mathcal{B} is independent of r_s .

The results above, both the differential equation eqn.(3.5) and the solution eqn.(3.3), may seem trivial but nevertheless have a few remarkable features:

1. The ground state energy of the system, with full interactions, has an explicit closed form. Additionally, as Stokes et. al. have shown [74], the explicit form of the interaction doesn't matter. As long as the interaction is long range, the result holds. The specific form of the

 $^{{}^{4}}V_{e}$ is a homogeneous function of degree -1 in \mathbf{r}_{i} , V_{b} is a homogeneous function of degree -1 in r_{s} , and V_{eb} is a homogeneous function of degree -1 in $\{\mathbf{r}_{i}, r_{s}\}$

⁵The Heisenberg equation of motion is given in numerous quantum mechanics textbooks. Within the Heisenberg representation, where observables carry the time dependence and state vectors are stationary, we have an equation of motion for an observable A(t) given as $\frac{dA}{dt} = \frac{i}{\hbar}[H, A(t)] + \left(\frac{\partial A}{\partial t}\right)_{H}$

⁶No, Prof. Kalman, it is NOT the Wigner Seitz radius

⁷We specifically mention graphene as the result was originally derived for graphene. In general, a would represent the lattice constant for whatever crystal **DM** you have. If there is no crystalline structure, a would simply represent a characteristic length of your system

⁸The time average of \dot{G} will <u>not</u> be zero if frictional forces are present and the motion of the particles dies out

interaction influences the constant \mathcal{B} . The work done by Gochan & Bedell [1] calculated the constant \mathcal{B} for a one-dimensional **DM**

- 2. Thermodynamic quantities, such as the compressibility and chemical potential, can be calculated using standard statistical mechanics methods while including interaction effects. While this may seem somewhat trivial, we must remember that this can rarely be accomplished with the same ease as will be shown here.
- 3. The result, while derived for a 2D system, holds for all dimensions. This allows to determine a number of properties related to exotic materials⁹ that can be classified as **DM**

3.3 Thermodynamics

As stated in the previous section, eqn.(3.3) provides a unique opportunity to calculate various thermodynamic quantities with relative ease while taking into account the effects of interactions. First, we start with differential form of the first law of thermodynamics¹⁰:

$$dE = TdS - PdV + \mu dN \tag{3.6}$$

where at T = 0, we can express the pressure as a partial derivative of the energy with respect to volume (while holding particle number constant). This leads to the following relation

$$PV = n \frac{\partial E}{\partial n} \tag{3.7}$$

where n is the particle density. What we wish to do now is incorporate the parameter r_s . Recall from the previous chapter, eqn.(2.3) which gave values for the particle density in **DM**. From a standard solid state physics analysis[75], we can relate the Fermi momentum to the parameter r_s via $p_F = \hbar/r_s a$ allowing us to express particle density as

$$n = \begin{cases} \frac{1}{\pi a^2} r_s^{-2} & d = 2\\ \frac{2}{3} \frac{1}{\pi^2 a^3} r_s^{-3} & d = 3 \end{cases}$$
(3.8)

Simple chain rule on eqn.(3.7) leads to the following result:

$$PV = \frac{1}{d}E\tag{3.9}$$

where d = 1, 2, 3 is the dimension of your system/**DM**. We now have a clear path to the pressure and energy which will allow us to find other thermodynamic properties. Most notably, we'll find the compressibility, and discuss in the next section.

 $^{^{9}}$ For an extensive list, one should look at the review article by Wehling et. al. [30]

 $^{^{10}}$ In the remainder of this section, we will drop the overline notation to indicate average. As this section is centered around thermodynamics, it is statistical in nature and the quantities discussed are averages

Before concluding this brief digression on thermodynamics using eqn.(3.3), we can easily derive one more T = 0 quantity of interest: the chemical potential. First note that the Gibbs free energy at T = 0 can be written as

$$G = E + PV = E + \frac{1}{d}E$$

where in the last equality, we used eqn.(3.9). Further noting that the Gibbs free energy, in general, is equal to μN , we can write the chemical potential at $T = 0^{11}$ as

$$\mu = \left(1 + \frac{1}{d}\right) \frac{\mathcal{B}}{N} \frac{1}{r_s} \tag{3.10}$$

where \mathcal{B} is defined as before. Again the surprise here is the ease at which these macroscopic quantities, containing effects of interactions, can be obtained.

3.4 What are the consequences?

Richard P. Feynman once said, "It doesn't matter how beautiful your theory is. If it doesn't agree with experiment, it's wrong." As such, regardless of the elegance in eqn.(3.3), it's meaningless if it doesn't agree the results of experiment. Given the availability of experimental data for (isothermal) compressibility [77], it's logical to see what effect, if any, our result has. Recall that the compressibility is defined as

$$\kappa^{-1} = n \frac{\partial P}{\partial n} \tag{3.11}$$

where n is the electron density. Using eqn.(3.8) and eqn.(3.7), we arrive at the following expressions

$$\overline{\kappa}^{-1} = \begin{cases} \frac{3}{4} \frac{\mathcal{B}}{N} \left(\pi a^2\right)^{1/2} n^{-1/2} & d = 2\\ \frac{4}{9} \frac{\mathcal{B}}{N} \left(\frac{3\pi^2 a^3}{2}\right)^{1/3} n^{-2/3} & d = 3 \end{cases}$$
(3.12)

where $\overline{\kappa} = n^2 \kappa$ is what's typically measured. Since \mathcal{B} is a constant independent of density, we see that (in d = 2), $\overline{\kappa} \propto n^{1/2}$ which is in agreement with experiment, such as the work done by Martin et. al. [77]. This startling consequence is profound in that it contradicts past theoretical work which used more traditional methods, Hartree-Fock and RPA, to determine the energy and subsequently the compressibility [61, 78]. The work of Hwang et. al. [61] and Barlas et. al. [78], show logarithmic corrections in r_s to compressibility which is not seen experimentally. A possible reason for the logarithmic terms is the failure of any perturbative approach in r_s . In a standard d = 2 electron gas, the ratio of potential to kinetic energy depends on r_s thus allowing for a small parameter for a given density regime. For example, Hartree-Fock calculations are valid in low-density regimes ($r_s \gg 1$) whereas RPA calculations are valid in high-density regimes ($r_s \ll 1$). In graphene, or any **DM**, the ratio of these energies

¹¹Note that in the presence of interactions, $\mu \neq \varepsilon_F$

is independent of r_s and ~ 1 leading to a failed expansion. While this realization points to using different non-perturbative/computational methods, it should be noted that the Virial theorem approach outlined here is valid (as it didn't depend on a perturbative approach) and produced an expression, eqn.(3.3) and eqn.(3.12), consistent with experiment.

Our first application of the virial theorem result is to the d = 1 DM results discussed in chapter 1. We start with finding the compressibility

$$\kappa^{-1} = n^2 \frac{\partial^2 \mathcal{E}}{\partial n^2}$$

At this point, we must be careful since we don't know what dependence κ has on particle density; As we saw with eqn.(3.12), the dependence can vary wildly depending on the dimensionality. To determine the compressibility, we use eqn.(3.11) where the expression for pressure comes from the Green's function analysis discussed in **chapter 1** and [31, 33] arriving at

$$\kappa = \frac{4v_g}{\pi n^2 \left(v_1\right)^2}$$

where v_1 defined earlier eqn.(1.35), and in the absence of a magnetic field or assuming a spin rotation invariant system, we have eqn.(1.36) for v_1 . From this expression for κ , we see that $n^2 \kappa$ is independent of density allowing us to easily integrate for the energy density \mathcal{E}

$$\mathcal{E} = \frac{1}{2}\kappa^{-1}$$

where we have made the assumption that v_g , and g_4 are independent of density, which is valid within the cutoffs discussed in **chapter 1**. Combining

$$E = \frac{N\pi v_g}{8a} \left(1 + \frac{2g_4}{\pi v_g}\right) \frac{1}{r_s}$$
(3.13)

and comparing with eqn.(3.3), we obtain an expression for \mathcal{B} within the **TLL** model (remember that while the general form of eqn.(3.3) remains the same, the specific form and value of \mathcal{B} is model dependent) of

$$\frac{\mathcal{B}}{N} = \frac{\pi v_g}{8a} \left(1 + \frac{2g_4}{\pi v_g} \right) \tag{3.14}$$

where everything on the RHS of eqn.(3.14) must be density independent, further supporting our assertion that the Fermi velocity v_g and the coupling constants are density independent. Using a = 1.44 from [79], assuming our d = 1 **DM** are in the metallic state, and obtaining a rough estimate for g_4 from the work done by Ishii [35], we predict a value of \mathcal{B} of

$$\frac{\mathcal{B}}{N} \approx 2.8 \pm .2 \text{ eV}$$

where \hbar has been restored. It's interesting to note two things: (1) Although the constant \mathcal{B} does depend on the specific interaction, a calculation for \mathcal{B} using the Coulomb interaction was

performed and the values are close and within the error range above. This supports our idea that the linear single particle spectrum is the primary driving force for any new phenomena and the specific form of the interaction isn't important, what's important is that the interaction be long range. (2) The numerical value of \mathcal{B} is approximately the energy range where the linear spectrum is valid, right before any band curvature, as per tight-binding calculations (see fig.(1.4)). This hints at the possibility of \mathcal{B} being able to experimentally tell us the range in which certain materials behave as **DM**.

In d = 2, 3, we have the benefit of combining Fermi liquid theory with the virial theorem results. As such, we naturally combine eqn.(3.12) with the Fermi liquid result for compressibility derived in **appendix B**:

$$\kappa = \frac{1}{n^2} \frac{g(0)}{1 + F_0^s}$$

where g(0) is the interacting density of states at the Fermi energy defined by eqn.(2.9) and eqn.(2.10). Combining the Fermi liquid result above with eqn.(3.12), we obtain the following for \mathcal{B}/N

$$\frac{\mathcal{B}}{N} = \begin{cases} \frac{2}{3} \frac{v_g^* \hbar}{a} \left(1 + F_0^s\right) & d = 2\\ \frac{3}{4} \frac{v_g^* \hbar}{a} \left(1 + F_0^s\right) & d = 3 \end{cases}$$
(3.15)

At first glance, the expressions above provide an avenue for experimentally measuring F_0^s (whereas a value for v_g^* is obtained through measurements of the specific heat). Taking this a step further, we use the modified effective mass relation for **DM** eqn.(2.8) to get

$$\frac{\mathcal{B}}{N} = \begin{cases} \frac{2}{3} \frac{v_g \hbar}{a} \left(\frac{1+F_0^s}{1+\frac{1}{2}F_1^s} \right) & d=2\\ \frac{3}{4} \frac{v_g \hbar}{a} \left(\frac{1+F_0^s}{1+\frac{1}{3}F_1^s} \right) & d=3 \end{cases}$$
(3.16)

where we clearly have both F_0^s and F_1^s present. Recall that the original statement of the virial theorem result claimed the constant \mathcal{B} is density independent. Eqn.(3.16) therefore implies the following conclusions:

- 1. F_0^s and F_1^s are separately density independent
- 2. The terms $1 + F_0^s$ and $1 + \frac{1}{3}F_1^s$ have the same density dependence

This statement, following solely from the virial theorem is of importance for a few reasons. First, with the exception of very few results, a statement about Fermi liquid parameters is typically not possible. Few exceptions exist, for example when examining the ultracold Fermi gas in the unitary limit using Fermi liquid theory, it is possible to obtain values for the Fermi liquid parameters [17], provided we're using the induced interaction model. Otherwise, comparison with experimental data for compressibility and specific heat is needed to determine F_0^s and F_1^s .

While we don't have an explicit expression for these parameters¹², being able to make a statement about the density dependence from a seemingly simple calculation is a profound result.

The interpretation of this density dependence/independence is a little challenging. While one can interpret different Fermi liquid parameters as changes to the Fermi surface, it isn't apparent why two separate distortions¹³ should depend on **qp** density in the same way. If we were to assume that both F_0^s and F_1^s carry the same density dependence, we must be careful of the implications to conclusion 2 above. Let $F_0^s = a_0 f(n)$ and $F_1^s = a_1 f(n)$ where a_0 , a_1 are constants and f(n) is some function of density. For eqn.(3.16) to remain density independent, $a_1 = da_0$ where d = 2, 3. This result, however, implies that $F_1^s > F_0^s$, which is an unusual result, albeit not completely foreign; dilute solutions of ³He in ⁴He [2] as well as nuclear matter have been shown to have $F_1^s > F_0^s$. If we are to maintain that $F_0^s > F_1^s$, for example as in ³He under various pressures [2], then each Fermi liquid parameter must have different density dependencies.

Conclusion 1 leads to a more fruitful discussion, if we consider the consequences of imposing relativistic effects. First investigated by Baym & Chin [52], application of **FLT** to a relativistic Fermi liquid was originally done to study high density matter and relativistic electron gases where the main differences from "normal" Fermi liquids were due to Lorentz invariance and massless behavior, similar to the difference between **DM** and normal Fermi liquids! As can be seen in their paper, most of the results from non-relativistic **FLT** apply in the relativistic case with small, albeit interesting, modifications. The result of [52] pertinent to our discussion is eqn.(52), where in the ultrarelativistic limit (i.e. $M \to 0$), $f_1^s = 0$. If this is combined with our results from the Virial theorem, we can conclude a number of things:

- 1. F_0^s is independent of density
- 2. $F_1^s = 0$ which implies no velocity renormalization (mass renormalization in the case of a normal **FL**)

In this interpretation, we may regard the Fermi velocity in **DM**, v_g , to play the role of the speed of light, providing an upper bound for velocity electrons can travel at. Furthermore, without imposing anything related to the induced interaction model, we have arrived at a profound conclusion:

Ultrarelativistic Fermi liquids are inherently local with the Fermi velocity playing the role of the speed of light. Since electrons in **DM** behave as if massless, we can conclude that **DM**, when in their **FL** state, are local.

Operating under the premise that **DM** are local Fermi liquids, which assumes only the l = 0Landau parameters are nonzero [17, 80–82], the velocity renormalization defined by eqn.(2.8) is

 $^{^{12}}$ A microscopic calculation is needed to make any in-depth statement about Fermi liquid parameters. For more details, the reader shoulder consult the iconic text of Abrikosov, Gorkov, and Dzyaloshinskii [14]

 $^{^{13}}$ For a clear discussion on these distortions, please look at the book Introduction to Many-Body Physics by Piers Coleman [56]

no longer valid. At first glance, this seems to contradict well received review articles by Kotov et. al. [22], Castro Neto et. al. [28] and experimental data "claiming" velocity enhancement [57]. The velocity renormalization discussed in review articles [22, 28] is

$$v(k) = v_g \left(1 + \frac{\alpha}{4} \ln\left(\frac{\Lambda}{k}\right) \right)$$

where Λ is a momentum cutoff after which the single particle spectrum starts to show curvature and $\alpha = e^2/(\epsilon_0 v)$ is the fine structure constant in graphene. This expression, derived using Green's function techniques, shows not a only a velocity renormalization but an increase. If we're to accept the seeming inherent relativistic behavior, a velocity increase violates one of the postulates of special relativity, assuming v_g plays the role of the speed of light. To reconcile these differences, we simply point out that the Green's function method applied in [22, 28] apply to a region close to the **CNP**. Recall that in this region, as discussed in **chapter 1**, our Fermi liquid description breaks down and therefore the conclusion that this is a relativistic Fermi liquid is no longer valid. Experimental results in the work of Elias et. al. [57] claiming velocity enhancement measured cyclotron mass near the Dirac point over a wide range of densities. Although they did find velocity enhancement by more than a factor of 2, their results are for electrons close to the Dirac point and therefore don't contradict our local relativistic claim.

The inherent local behavior of the Fermi liquid in **DM** has an additional consequence if one considers combining these results with the properties of the unitary Fermi gas. Recall first the previous steps in the following order:

- 1. From the virial theorem, we arrive at the $(1 + F_0^s) / (1 + \frac{1}{d}F_1^s)$ being independent of density
- 2. Imposing ultra-relativistic behavior, we find $F_1^s = 0$ [52] which is indicative of a local Fermi liquid [80]
- 3. F_0^s must therefore be density independent

The conclusion that F_0^s is density independent is odd. Intuitively, it isn't clear why this should happen. In fact, it makes more sense for the **qp** interaction, especially the one related to compressibility, to be heavily dependent on the density of particles. To find further physical significance in this result, we turn to the induced interaction model, developed by Babu and Brown [83], and the more recent work using the model by Bedell and Quader¹⁴.

In short, the induced interaction model is an attempt to calculate the Landau parameters $F_l^{s,a}$ from first principles¹⁵ by writing the **qp** interaction function as the sum of a direct term and an induced term. The work of Quader is computational and, upon setting the direct term to infinity, he finds the Fermi liquid parameters to be independent of density (as we have determined) and furthermore finds the parameters to saturate at a constant value. This leads us to a startling

¹⁴The work discussed is still in progress and therefore will remain brief

 $^{^{15}\}mathrm{Recall}$ that these Landau parameters, when the \mathbf{FLT} was first introduced, were purely phenomenological parameters

conclusion. A model/system that has constant Fermi liquid parameters is a system in the unitary limit.

A system in the unitary limit, for example the unitary Fermi gas [84], is frequently known as a system with max interaction strength, max scattering amplitude, etc. In the unitary limit, higher order Fermi liquid parameters are zero and $F_0^s = -0.5^{16}$ and $F_0^a = \infty$. Obviously being saturated at constant values, and therefore independent of density, is reminiscent of the behavior we initially saw from our Virial theorem result. Our final conclusion is that the electrons in a **DM** are ultrarelativistic, inherently local, and intrinsically at the unitary limit.

Taking the expression above for expression \mathcal{B}/N and using eqn.(3.3), we can obtain expressions for the (average) energy

$$E = \begin{cases} \frac{2}{3}N\left(\frac{1+F_0^s}{1+\frac{1}{2}F_1^s}\right)\varepsilon_F & d=2\\ \frac{3}{4}N\left(\frac{1+F_0^s}{1+\frac{1}{3}F_1^s}\right)\varepsilon_F & d=3 \end{cases}$$
(3.17)

where the density dependence is tucked away inside ε_F , since we established earlier that at the very least the ratio of Fermi liquid parameters is density independence. The chemical potential follows and is

$$\mu = \begin{cases} \left(\frac{1+F_0^s}{1+\frac{1}{2}F_1^s}\right)\varepsilon_F & d=2\\ \left(\frac{1+F_0^s}{1+\frac{1}{3}F_1^s}\right)\varepsilon_F & d=3 \end{cases}$$
(3.18)

The chemical potential allows us to calculate the speed of first sound, defined in the non-relativistic limit as

$$c_1^2 = \frac{N}{m} \frac{\partial \mu}{\partial N} = \frac{1}{\sqrt{\kappa n m}}$$

an obvious complication is in how to deal with the mass term but as discussed in **chapter 2**, we can re-write as a ratio of momentum to velocity. Using eqn.(3.18), we get a speed of (first) sound

$$c_1^2 = \frac{v_g^2}{2} \left(\frac{1 + F_0^s}{1 + \frac{1}{2} F_1^s} \right)$$
(3.19)

where v_g is the renormalized velocity¹⁷. Eqn.(3.19) is in agreement with the expressions for speed of (first) sound obtained via the l = 0 and l = 1 **LKE**. Additionally, this result agrees with the analysis conducted in relativistic case by Baym and Chin [52]. Returning for a moment to the idea that **qp** in a **DM**, in the ultrarelativistic limit, are local, then

$$c_1^2 = \frac{v_g^2}{2} \left(1 + F_0^s\right)$$

 $^{^{16}}$ Recall that the work on zero sound in **chapter 2** shows how attractive Fermi liquid parameters (i.e. $F_0^s < 0)$ are valid

¹⁷Unfortunately, many texts on this subject cast the speed of first sound c_1 in two forms without differentiating until now: in eqn.(B.8) the velocity v_F is the renormalized Fermi velocity and in eqn.(3.19) v_g is the bare Fermi velocity

By the principles of special relativity, no velocity parameter can be greater than the speed of light, which is played by the role of v_g in this case. Therefore, we have an additional constraint on F_0^s :

$$\frac{1+F_0^s}{2} \le 1 \to F_0^s \le 1$$

Combining this with the Pomeranchuk instability condition discussed in **appendix B**, we see conclude that the F_0^s Landau parameter is bounded from below by -1 and bound from above by 1: $-1 \le F_0^s \le 1$.

The simple application of the Virial theorem has led to numerous profound results. Right away, we arrived at a closed form for the energy. At first glance, an expression of such simplicity perhaps couldn't be right, but using said expression to obtain a form for the compressibility showed agreement with experiment whereas Hartree-Fock and RPA calculations lead logarithmic terms that don't show up anywhere. The energy from the Virial theorem was then combined with Fermi liquid theory, non-relativistic and relativistic, and allowed us to make claims about the Fermi liquid parameters, eventually concluding that **DM** in their Fermi liquid phase behave as relativistic local Fermi liquids in the unitary limit. With this conclusion, future work may apply the local model within the induced interaction picture without being approximate. The work done shows that importance of starting from the beginning, with known theories and methods, to analyze new systems. The effects and conclusions drawn from this analysis would otherwise have been lost. Furthermore, we once again see the importance of the linear dispersion when determining the physical phenomena in **DM**.

Chapter 4

The KSS Bound in Dirac Materials

4.1 Introduction

The peculiar behavior for low viscous fluids has been a subject of study for years. Originally discussed in an informal talk by Purcell [85], the subject has gained tremendous interest in recent years after a makeover using field theory techniques. Using the anti-de Sitter/conformal field theory (AdS/CFT) correspondence, strongly interacting quantum field theories can be described in terms of weakly interacting gravitational systems. This has led to the conjecture that there exists a lower bound —the KSS bound— for η/s in a strongly coupled field theory given by [4, 86–91]

$$\frac{\eta}{s} \ge \frac{\hbar}{4\pi k_{\rm B}} \tag{4.1}$$

where η is the viscosity and s is the entropy density. Quantum fluids, such as the quark gluon plasma and the unitary Fermi gas, that obey eqn. (4.1), are called nearly perfect quantum liquids where equality denotes a "perfect" quantum liquid [92, 93]. Eqn.(4.1) has been thoroughly studied, for example by Baggioli [94] and Amoretti [95], to provide a possible description of unique phenomena such as strange metals, and what happens at the **CNP** in **DM**. Additionally, the AdS/CFT correspondence creates a bridge between gravitational physics and condensed matter physics and allows one to be studied in terms of the other [96, 97].

The alleged "universality" [98] of the bound has drawn much interest due to the rarity of such a statement/relation that holds across multiple subfields of physics. While many quantum liquids obey eqn.(4.1), it is the examples where the bound is violated that draw interest. Although violation of the bound is nothing new, most results are purely theoretical/mathematical [96, 99–102] and tangible experimentally verifiable violations are lacking. Our study of η/s in the unitary Fermi gas, discussed in the next section, tackles this issue while providing insight into how phase

transitions and viscoelastic behavior can affect η/s . In the final section, we discuss looking at η/s in **DM**. The work, although preliminary, hints at unusual behavior and can potentially shed light on how electrons in quasirelativistic systems behave.

4.2 The Unitary Fermi Gas

The first system in which we study η/s is the unitary Fermi gas, a system characterized by maximal/infinite scattering length and subsequently a regime of maximum strength interaction. Having been heavily studied [17, 84], both theoretically and experimentally, the appeal of the unitary Fermi gas is similar to what was obtained by the Virial theorem analysis in **chapter 3**: the "ease" at which a closed equation of state can be obtained and then readily compared to experiment. Our interest is in the possible superfluid phase transition, where superfluid fluctuations above the transition temperature, T_c , have significant effects on the spin transport [3, 103]. This result is the motivation for our work. We sought to determine if such superfluid fluctuations could have a similar impact in viscosity and subsequently the KSS bound. Recent experiments on the unitary Fermi gas ⁶Li show a normal/superfluid phase transition at a transition temperature, $T_c \approx 0.167T_F$ [6], where T_F stands for the Fermi temperature. As for the viscosity, recent advances in experiments have allowed for its measurement [104] and subsequently led to the measurement of the ratio. Such measurements show a minimum that obeys the bound given by (4.1) at temperatures $T \approx 2T_c$ [5, 105, 106].

4.2.1 Superfluid Fluctuations in the Unitary Fermi Gas

To better understand η/s within the context of strongly correlated systems, we develop a simple theoretical model to calculate the quasiparticle scattering rates of a strongly correlated quantum liquid above T_c . Such a model differs from past calculations [107–110] in that we include the effects of superfluid fluctuations as $T \to T_c^+$. The model separates the quasiparticle scattering amplitude for the strongly correlated quantum fluid into two components: the superfluid fluctuations term coming from the particle-particle pairing fluctuations in the singlet scattering channel above T_c , and a normal Fermi liquid scattering term calculated from the local version of the induced interaction model [80, 81]. Applying our theory to the unitary Fermi gas, we calculate η/s for the unitary Fermi gas about T_c following the methods used in the transport studies of Landau Fermi-liquid theory [2]. We find a local minimum as $T \to T_c$ of $\eta/s \approx 0.3\hbar/k_{\rm B}$ which agrees with the experimentally measured lower bound [104]. However, an additional intriguing result of η/s dropping to zero at T_c thus violating (4.1). Our work therefore seeks to explain the nature of this violation within the context of Landau Fermi liquid theory. While violations of (4.1) are not uncommon, for example the work done by Alberte et. al. [111] and Jain et. al. [112] both show violation, our work is unique in that our calculation is done for the unitary Fermi gas, a system frequently studied experimentally. Furthermore, we differ from



FIGURE 4.1: Feynman diagram for the temperature vertex function of particle-particle type, \mathscr{T}_s . The bubbles represent the irreducible $(\tilde{\mathscr{T}}_s)$ and fully reducible (\mathscr{T}_s) particle-particle vertex function, the solid lines stand for the fermion Green's functions. The propagators are the quasiparticle propagators with fully renormalized quasiparticle interactions and $K = p_1 + p_2$.

other work on η/s in the unitary Fermi gas, such as that by Samanta et. al. that also showed violation [113], in that the system under consideration was trapped. While our result appears to be in contradiction with the work done by Cao et. al. [104], we find good qualitative agreement with the more recent analysis done by Joseph et. al. [5]. We believe this discrepancy is because the measurements were done over a wide temperature range while the violation of the bound happens in a small window around T_c . Additionally, due to the breakdown of the quasiparticle picture, numerous other methods have been employed such as those performed by Enss et. al. [114] to determine the viscosity. While we don't disagree with these results, we feel our model is valid due to the experimental support of the quasiparticle picture near T_c (as shown in Fig. 4 and will be discussed later). To conclude, we draw on previous work by Alberte, Baggioli, and Pujolàs [94, 111] they present the idea of the viscoelastic nature of holographic solids violating the bound. We expand on their work and provide insight into this high-energy problem from the viewpoint of condensed matter.

The high transition temperature, $T_c \approx 0.167T_F$, of the unitary Fermi gas allows for the experimental measurement of η/s at temperatures close to T_c , where superfluid fluctuations could play a role [6, 104]. For example, previous study of spin transport found that superfluid fluctuations play a significant role in the spin diffusion [3, 103]. As such, our work sets out to understand how the superfluid fluctuations may affect the viscosity and subsequently η/s . The superfluid fluctuations come from the particle-particle pairing fluctuations in the spin singlet quasiparticle scattering channel closely above T_c . This is also known as the Cooper phenomenon [14], where the fluctuations lead to the instability of the ground state of the Fermi system with arbitrarily weak attraction at absolute zero, known as the Cooper instability.

Due to the pairing fluctuations, the quasiparticle scattering amplitudes for small total momentum scattering diverge at T_c . Here we consider only the s-wave (spin singlet) pairing mechanism for the Cooper pairs and incorporate the superfluid fluctuations in the scattering amplitudes by evaluating the temperature vertex function of particle-particle type in the spin singlet channel for small total momentum scattering using standard quantum field theory methods [14]. The spin singlet temperature vertex function $\mathscr{T}_s(K)$ is generated from the diagram shown in Fig. 4.1, leading to the following integral equation:

$$\mathcal{T}_{s}(p_{1}, p_{2}; p_{3}, p_{4}) = \tilde{\mathcal{T}}_{s}(p_{1}, p_{2}; p_{3}, p_{4}) - \frac{T}{2(2\pi)^{3}}$$
$$\times \Sigma_{\omega_{n}} \int \tilde{\mathcal{T}}_{s}(p_{1}, p_{2}; k, K - k) \mathcal{G}(K - k)$$
$$\times \mathcal{G}(k) \mathcal{T}_{s}(k, K - k; p_{3}, p_{4}) \mathrm{d}^{3}k$$
(4.2)

where, $p_i = (\mathbf{p}_i, \omega_i)$ are the four momenta of the scattering particles, and, $K = (\mathbf{K}, \omega_0)$ stands for the total momentum of the incident particles. \mathscr{T}_s depends only on the total momentum K, $\mathscr{T}_s(p_1, p_2; p_3, p_4) \equiv \mathscr{T}_s(K)$, when $|\mathbf{p}_i| = k_F$ for $i = 1, \dots, 4$ and $|\mathbf{K}| \ll k_F$. Solving eqn. (4.2), we can express \mathscr{T}_s in the small K limit as

$$\mathscr{T}_{s}(\mathbf{K},0) = \frac{1}{\frac{mp_{f}}{4\pi^{2}} \left[\ln \frac{T}{T_{c}} - \frac{1}{6} \left(\frac{v_{f}|\mathbf{K}|}{2\omega_{D}} \right)^{2} - \frac{7\zeta(3)}{3\pi^{2}} \left(\frac{v_{f}|\mathbf{K}|}{4T} \right)^{2} \right]}$$
(4.3)

where $T_c = \frac{2\gamma\omega_D}{\pi} e^{-4\pi^2/mp_f|\tilde{\mathscr{F}}_s|}$, γ is the Euler-Mascheroni constant, p_f is the Fermi momentum, and $\omega_D = 0.244\varepsilon_F$ is the cutoff frequency [115]. $\tilde{\mathscr{F}}_s$ is the zero temperature irreducible particleparticle vertex function, which is approximately equal to the spin singlet normal Fermi-liquid scattering amplitude, denoted by a, given diagrammatically in Fig. 4.2 [2]. In order to calculate the viscosity of the unitary Fermi gas, we need the normal Fermi-liquid scattering amplitude. The total quasiparticle scattering probability, $\langle W \rangle \equiv \int \frac{d\Omega}{4\pi} \frac{W(\theta, \phi)}{\cos(\theta/2)}$, is obtained by averaging the quasiparticle scattering amplitudes of different **K**'s over the phase space [2]. For the unitary Fermi gas, $\langle W \rangle$ is separated into a superfluid fluctuations term, $\langle W \rangle_{\rm fluctuations}$, and a normal Fermi-liquid scattering term, $\langle W \rangle_{\rm normal}$:

$$\langle W \rangle = \int_{0}^{\mathbf{K}_{\max}} \frac{\mathrm{d}\Omega}{4\pi} \frac{W_{\mathrm{f}}(\theta,\phi)}{\cos(\theta/2)} + \int_{\mathbf{q}_{\max}}^{2P_{f}} \frac{\mathrm{d}\Omega}{4\pi} \frac{W_{\mathrm{n}}(\theta,\phi)}{\cos(\theta/2)}$$

$$= \langle W \rangle_{\mathrm{fluctuations}} + \langle W \rangle_{\mathrm{normal}}$$

$$(4.4)$$

 \mathbf{K}_{max} stands for the critical value of the total momentum of the incident particles, beyond which Cooper pairs start to break down and the particles scatter off of each other as in the normal Fermi liquid state. It is given by $v_F |\mathbf{K}_{max}| = 6\varpi$, where $\varpi = 2\omega_D e^{-4\pi^2/mp_f |\tilde{\mathscr{I}}_s|}$, from regular quantum field theory analysis [14]. It's important to note that the angular averages in eqn. (4.4) are different due to the different angular dependencies in \mathbf{K}_{max} and \mathbf{q}_{max} [3, 116].

The Landau parameters needed for computing the quasiparticle scattering amplitudes are determined from the local induced interaction model, shown diagrammatically in Fig.4.2. First developed to study the quasiparticle interactions in liquid ³He, it has seen success in applications to other interacting Fermi systems and been further generalized to account for the momentum dependence in the scattering amplitudes [83, 116–119]. According to the model, the quasiparticle interaction parameter, f, is generated from a direct term, d, which is equivalent to a model dependent effective quasiparticle potential, and an induced term coming from the coupling of collective excitations to the quasiparticles. The mechanism is shown diagrammatically in Fig.1 in Li et. al. [3] In this work we use a local, momentum independent, version of the induced


FIGURE 4.2: Diagrammatic representation of the induced interaction model. (a) represents the equation for Landau parameters f decomposed into direct and induced terms; (b) sums all the reducible diagrams. It represents the equation relating f to the scattering amplitudes a = A/N(0). The momentum in the particle-hole channel is represented by $q = p_1 - p_3 = p_4 - p_2$ and the momentum in the exchange particle-hole channel $q' = p_1 - p_4 = p_3 - p_2$.

interaction model where only the l = 0 Landau parameters, $F_0^{s,a}$, are nonzero [3, 80–82] and given as

$$F_0^s = D_0^s + \frac{1}{2}F_0^s A_0^s + \frac{3}{2}F_0^a A_0^a, \tag{4.5}$$

$$F_0^a = D_0^a + \frac{1}{2}F_0^s A_0^s - \frac{1}{2}F_0^a A_0^a, \qquad (4.6)$$

where, $A_0^{s,a} = F_0^{s,a}/(1+F_0^{s,a}) = N(0)a_0^{s,a}$. In the unitary limit, the Landau parameters take on the following values: $F_0^s = -0.5$ and $F_0^a \to +\infty$. These parameters capture the strong interactions and successfully explain various universal thermodynamic properties of the unitary Fermi gas [3, 84].

Following the approach of Landau Fermi-liquid theory [2], with the local induced interaction model, we calculate the quasiparticle scattering amplitudes $W_{\rm f}(\theta, \phi)$ and $W_{\rm n}(\theta, \phi)$:

$$W_{\rm f}(\theta,\phi) = \frac{1}{2} W_{\uparrow\downarrow} = \frac{1}{2} \frac{2\pi}{\hbar} |a_0^{\uparrow\downarrow}|^2 = \frac{1}{2} \frac{2\pi}{\hbar} \left| \frac{\mathscr{T}_s(\mathbf{K},0)}{2} \right|^2 \tag{4.7}$$

$$W_{\rm n}(\theta,\phi) = \frac{1}{2} W_{\uparrow\downarrow} = \frac{1}{2} \frac{2\pi}{\hbar} |a_0^{\uparrow\downarrow}|^2 = \frac{1}{2} \frac{2\pi}{\hbar} \left| \frac{-2A_0^a}{N(0)} \right|^2 \tag{4.8}$$

where $A_0^a = 1$ in the unitary limit. The total scattering probability becomes

$$\begin{split} \langle W \rangle &= \langle W \rangle_{\text{normal}} + \langle W \rangle_{\text{fluctuations}} \\ &= \frac{2\pi}{\hbar} \frac{2}{|N(0)|^2} \cdot 2 \left(1 - \frac{\sqrt{6}\pi}{4\gamma} \frac{T_c}{T_F} \right) |A_0^a|^2 \\ &+ \frac{2\pi}{\hbar} \frac{2}{|N(0)|^2} \\ \times & \left[\frac{\frac{\sqrt{6}\pi T_c}{4\gamma T_F}}{\ln \frac{T}{T_c} \left[\ln \frac{T}{T_c} + \left(\frac{\sqrt{6}\pi T_c}{4\gamma T_F}\right)^2 (11.2 + 0.28 \left(\frac{T_F}{T_c}\right)^2 \right) \right]} \\ &+ \frac{\tan^{-1} \left(\sqrt{\left(\frac{\sqrt{6}\pi T_c}{4\gamma T_F}\right)^2 (11.2 + 0.28 \left(\frac{T_F}{T_c}\right)^2} \right) / \sqrt{\ln \frac{T}{T_c}} \right)}{\left(\ln \frac{T}{T_c} \right)^{3/2} \sqrt{11.2 + 0.28 \left(\frac{T_F}{T_c}\right)^2}} \right]. \end{split}$$
(4.9)

To calculate the viscosity of the unitary Fermi gas within the Landau Fermi-liquid theory, we need the viscous lifetime τ_{η} in addition to the scattering probabilities eqn.(4.9). In the low temperature limit, the viscous lifetime, τ_{η}^{0} , is [2]

$$\tau_{\eta}^{0} = \frac{0.205 \times 8\pi^{4}\hbar^{6}}{m^{3}\langle W \rangle (k_{B}T)^{2}}$$
$$= 0.205\tau \qquad (4.10)$$

where the bare mass and the effective mass are the same since we're operating in a local model, τ without any index is the quasiparticle lifetime, and the factor of 0.205 is from the different angular average of the scattering amplitude in the unitary limit. A finite temperature correction is added to τ_{η}^{0} to give [120]

$$\tau_{\eta} = \frac{\hbar}{k_B T_F} \left(\frac{T_F}{T}\right)^2 \left(\frac{\hbar |N(0)|^2}{0.205 \times 16} \langle W \rangle - 3\pi \zeta(3) \right)$$

×
$$[0.202(A_0^a)^3 + 0.164(A_0^a)^2] \frac{T}{T_F} \right)^{-1}.$$
 (4.11)

The viscosity is then given by:

$$\eta = \begin{cases} \frac{1}{5} n p_f v_f \tau_{\eta}, & T \ll T_F \\ \\ n k_B T \tau_{\eta} = 3.4 n \hbar \left(\frac{T}{T_F}\right)^{3/2}, & T \gg T_F \end{cases}$$

$$(4.12)$$

Eqn. (11) for η is the standard Fermi Liquid result [2]. Eqn. (12) for $T \gg T_F$ can be interpreted as the classical viscosity which is found upon taking a thermal average of eqn. (11). The classical lifetime $\tau \propto \frac{\hbar}{k_B T_F} \left(\frac{T}{T_F}\right)^{1/2}$ [121] is found by fitting to data for the viscosity coefficient [104] and given as $\tau_{\eta} \approx 3.4 \frac{\hbar}{k_B T_F} \left(\frac{T}{T_F}\right)^{1/2}$. A natural concern in our work thus far is our use of the Landau Kinetic equation (LKE) to calculate the viscosity is the short, tending to zero, quasiparticle lifetime. In fact, the validity of Fermi liquid theory close to the transition temperature is still an open question that's still under debate [122]. Typically, the formal derivation of the LKE and subsequent calculations don't allow for arbitrarily short quasiparticle lifetimes and one resorts to other methods, such as the Kubo formalism, to calculate transport quantities when the quasiparticle picture is insufficient. Bruun and Smith performed a calculation [123] and show that corrections to the LKE result are small compared to those using the Kubo formalism. Additionally, the entropy from Ku et. al. [6], shown in Fig.4.4, exhibits Fermi liquid like behavior above T_c . Therefore, in spite of other work that claims Fermi liquid theory isn't valid [124, 125], we justify our approach through the entropy data closely resembling that of a Fermi liquid as well as work done using other methods that yield transport coefficients that minimally differ from LKE results. To calculate the ratio η/s , we also need the entropy density of the unitary Fermi gas. According to Fermi liquid theory [2], the low temperature entropy density is given by

$$s = \frac{\pi^2}{2} n k_B \left(\frac{T}{T_F}\right) \left[1 - \frac{\pi^2}{10} B^s \left(\frac{T}{T_F}\right)^2 \ln\left(\frac{T}{T^*}\right)\right], \qquad T \ll T_F$$
(4.13)

where $T^* \sim v_F q_c/k_B \ll T_F$ is a cutoff temperature [2] $(q_c \text{ is a cutoff momentum defined by} |p - p_F| \ll q_c \ll p_F)$, $B^s = -\frac{1}{2}(4 - \frac{\pi^2}{6})$ for a local Fermi liquid in the unitary limit, and the logarithmic term stands for the finite temperature correction to the low temperature result. In the high temperature limit, the entropy density takes the form of a classical Fermi gas [126]

$$s = nk_B \left\{ \frac{5}{2} - \ln\left(\frac{n\lambda^3}{g}\right) \right\}, \qquad T \gg T_F$$
(4.14)

where $\lambda = h/(2\pi m^* k_B T)^{1/2}$ is the thermal wave length, and g = 2 for a two component Fermi gases.

The ratio η/s is plotted over the entire temperature regime in Fig.4.3. The experimental data of η/s from [5], shown in the inset of Fig.4.3, is measured with respect to reduced temperature $\theta = T/T_F$. Additional data in [127] plot the ratio with respect to E/E_F . A ratio of $E/E_F = 0.6$ corresponds roughly to a temperature ratio of $T/T_F = 0.17$, therefore the low temperature portions of our calculated and the measured ratios of η/s are plotted within the same temperature window. A local minimum, with value $\eta/s \approx 0.3\hbar/k_B$, is found in the calculated ratio η/s at $T \approx 0.36T_F$ (shown by the red curve in Fig.4.3) agrees roughly with the experimental saturation value of η/s for a nearly perfect Fermi gas [93, 104] (in the inset of Fig.4.3) and is not far from the holographic prediction [4] $(\eta/s)_{\text{KSS}} = \hbar/4\pi k_B \approx 0.08\hbar/k_B$. However, as can be seen from Fig. 4.3 and eqns.(4.9) and (4.12), the ratio η/s is not bounded by this local minimum as it appears to drop to zero at T_c due to superfluid fluctuations as

$$\frac{\eta}{s} \sim \left(\frac{T}{T_F}\right)^{-3} \left(\ln\left(\frac{T}{T_c}\right)\right)^{3/2} \tag{4.15}$$

which qualitatively agrees with the behavior in the inset of Fig.4.3. The conjectured universal



FIGURE 4.3: The ratio η/s vs temperature. The ratio η/s is evaluated at $F_0^a = 100$, i.e. close to the unitary limit where $F_0^a \to +\infty$ according to the local model. The black solid curve is the low temperature limit of η/s and the dashed curve is the high temperature limit. The red curve represents the single function that captures the behavior of both curves. The red curve was created through simple interpolation, adding the low temperature and high temperature expressions together with weight factors as was done by Li et. al. [3]. The horizontal blue line indicates the quantum limited lower bound of $\eta/s = \hbar/4\pi k_B$ conjectured [4]. The inset figure shows the data for η/s as a function of $\theta = T/T_F$ obtained by Joseph et. al. [5]. Their data seems to show $\eta/s = 0$ at $\theta = 0.1$, which agrees with our result if one considers that $T_c = 0.1T_F$ in the local model. It should however be stressed that data in this region is inconclusive and cannot be used to justify agreement with our result (For example, one can easily see that the ratio dropping to zero in the inset clearly happens well within the superfluid phase). However, what we can say about the inset is that the general behavior of their data is in good qualitative agreement with our result, albeit with a higher local minimum.

lower bound for η/s is therefore violated in our theory. A concern with our result is if hidden behavior of the entropy density, not captured by eqn.(4.13), is causing $\eta/s \to 0$. While eqn.(4.13) may not be the complete low temperature behavior, the data given by Fig.4.4 suggests that although a kink is present, there is no divergence or singularity. Neither theoretical (eqn.(4.13)) nor experimental result diverges and therefore we believe the entropy density is well behaved and isn't driving the ratio to zero. It is important to note that recent reanalysis of the data in the inset of Fig.4.3 was done by Bluhm et. al. [128]. They observe a minimum slightly above T_c , as many other works do, but unfortunately cannot comment on a minimum at/below T_c . We believe the lack of conclusive results near T_c is due to the volatile behavior of the system in close vicinity to the critical temperature. The two competing phases make it difficult to obtain data and theoretical results, ours included, are model dependent. What we can say however is that there is a finite quasiparticle weight [125] which lends to some validity in our result. Violation of the conjectured bound on η/s within our model begs the following question: why do superfluid fluctuations in the unitary Fermi gas violate the KSS bound?



FIGURE 4.4: Data for the entropy per particle from Ku et. al [6] (red dots). At all temperatures, specifically around $T_c = 0.167T_F$ (blue dotted line), the entropy is a well behaved function without discontinuity. This supports the claim that $\eta/s \to 0$ as $T \to T_c$ is due to lifetime effects and not unusual behavior in the entropy. The remaining three curves are our expressions for the entropy(eqn.(4.13) and eqn.(4.14)). The green solid curve is eqn.(4.13), the black dashed curve is eqn.(4.14) and the purple dashed/dotted curve is eqn.(4.13) with T_F/T^* and B^s being adjustable parameters (fit values given in legend). The purple curve, in spite of agreeing with the low temperature dependence and matching eqn.(4.13), suggests that for more accurate results, we must go beyond the local model for a Fermi liquid. As one can see, the entropy behaves closely to that of a Fermi liquid suggesting a good quasiparticle picture and further validating our use of the LKE regardless of the vanishing quasiparticle lifetime.

4.2.2 Viscoelasticity of the Unitary Fermi Gas

Previous work [111, 129] has led us to study the connection between the viscoelastic behavior of the unitary Fermi gas and η/s . Alberte et. al. have shown that holographic solids, solid massive gravity black branes with nonzero graviton mass, violate the KSS bound [111]. Their work ultimately found that holographic solids with a non-zero bulk modulus, specifically finite shear modulus, violate the KSS bound, with strong evidence for extension to real solids. Our work aims to go a step further by presenting a system where experiment is possible, the unitary Fermi gas, that exhibits viscoelastic behavior and violates the KSS bound.

We must first ask if the viscoelastic model is suitable to describe the unitary Fermi gas, i.e. if the following conditions are met: (i) c_0 , $c_1 \gg v_F$ where c_0 and c_1 are the speeds of zero and first sound respectively and/or (ii) $l \to 0$ as $T \to T_c$ where l is the viscous mean free path. Although (i) is violated for the unitary Fermi gas since $-1 < F_0^s < 0$, (ii) is satisfied since the quasiparticle mean free path goes to zero as $T \to T_c$ and Cooper pairs form. Additionally, provided we are in a regime such that $\omega \tau \gg 1$, according to [130], the fluid behaves as a solid with elastic response. We start with the general form for the stress tensor for a viscoelastic model, different from those found in [2, 129, 130]:

$$-\Pi_{ij} = \sigma_{ij} - \zeta \mathbf{u}_{ll} \delta_{ij} \tag{4.16}$$

where

$$\sigma_{ij} = p\delta_{ij} + 2\mu \left(\mathbf{u}_{ij} - \frac{1}{3}\mathbf{u}_{ll}\delta_{ij} \right)$$

is the stress tensor that shows the two modes (an elastic mode which is $p\delta_{ij}$ and a shear mode which is the remaining terms) and

$$\mathbf{u}_{ij} \simeq \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

is the strain tensor for small displacements and u_i is the flow velocity. ζ is the bulk viscosity and may be ignored since $\zeta/\eta \sim T^4$ at low temperature for a Normal Fermi Liquid [2, 131]. In general, μ is the shear modulus which contains the viscous (viscosity) and elastic (elasticity) behavior (i.e. there are in general two modes μ_{\perp} and μ_{\parallel}). Within the viscoelastic model, due to the short lifetime near T_c in the unitary Fermi gas, we have $\omega \tau_{\eta} \ll 1$, $\eta \sim \tau \mu$ and elasticity is no different from viscosity. Using the LKE we get

$$\omega^2 - c_1^2 q^2 = \frac{2}{15} \left(q v_F \right)^2 \left(1 + \frac{F_2^s}{5} \right) \frac{\nu_2}{\nu_0}$$
(4.17)

$$\frac{\nu_2}{\nu_0} \simeq 2 \left[1 + i \left(1 + \frac{1}{5} F_2^s \right) / (\omega \tau_\eta) \right]^{-1}$$
(4.18)

where the real and imaginary parts of (4.17) are analyzed separately. Letting $\omega = c (q - i\alpha)$, we obtain the following expression for the coefficient of sound attenuation [132] in the unitary Fermi gas

$$\alpha = \frac{2}{15} \left(\frac{v_F}{c}\right)^2 q \left(\frac{\omega \tau_\eta \left(1 + F_2^s/5\right)^2}{\left(\omega \tau_\eta\right)^2 + \left(1 + F_2^s/5\right)^2}\right)$$
(4.19)

The real part of (4.17) gives

$$(c^{2} - c_{1}^{2}) q^{2} - c^{2} \alpha^{2} = \frac{2}{15} (qv_{F})^{2} (1 + F_{2}^{s}/5) \\ \times \left(\frac{2(\omega\tau_{\eta})^{2}}{(\omega\tau_{\eta})^{2} + (1 + F_{2}^{s}/5)^{2}} \right)$$

$$(4.20)$$

Eqns.(4.19) and (4.20) provide experimentally attainable quantities relating to the viscoelasticity of unitary Fermi gases. As the temperature of the unitary Fermi gas approaches T_c , two things happen: (i) $\alpha \to 0$ and (ii) $c \simeq c_1$. From [133] we interpret $\alpha \to 0$ as the penetration depth of c_1 being infinite. Additionally, if we impose the restrictions of the local model, as mentioned earlier when dealing with the unitary Fermi gas near T_c , Fermi liquid parameters F_1^s and higher are zero but the behavior of α and c_1 remain unchanged. As the unitary Fermi gas approaches its transition temperature, the zero sound mode predicted by Landau Fermi Liquid Theory is over damped and not propagating. This leads to the first sound mode propagating through the entire system and is another indicator of viscoelastic behavior. Continuing with the Landau Kinetic equation, we can use conservation laws (momentum and number) to obtain a hydrodynamic equation of motion for the mass density

$$\frac{\partial^2 \rho}{\partial t^2} - c_1^2 \nabla^2 \rho = \frac{4}{3} \frac{\eta}{\rho_0} \frac{\partial}{\partial t} \nabla^2 \rho \tag{4.21}$$

where if $\eta = 0$, as our result suggests, we obtain

$$\frac{\partial^2 \rho}{\partial t^2} - c_1^2 \nabla^2 \rho = 0 \tag{4.22}$$

a standard wave equation for a sound wave propagating at velocity c_1 which is in agreement with our analysis and interpretation of eqn.(4.20).

4.2.3 Summary

Superfluid phase transitions appear to have significant effects on the ratio η/s . Our work, investigating such effects in the unitary Fermi gas, presents a violation of the conjectured KSS bound thus calling into question its proposed universality as well as the role of phase transitions on η/s . In general, strongly coupled systems often exhibit phase transitions leading us to wonder if similar conclusions could be drawn about other strongly correlated quantum fluids. For example, in dense nuclear matter produced in heavy ion collisions, the ratio is found to be obeyed albeit taking on a very small value of $(\eta/s)_{\text{KSS}} \leq \eta/s \leq 2.5(\eta/s)_{\text{KSS}}$ [86]. Based on our model, one could argue that the small value of η/s is related to fluctuations that arise from the strongly interacting quark gluon plasma (QGP) phase [134]. The transition temperature for the QGP phase is predicted from lattice QCD computations [135] to be, $T_{\rm QGP} \sim 170 MeV$, and from the experiments below this temperature, η/s is close to the KSS bound [136]. Therefore, we raise a general question: Is the minimum found in η/s of the nearly perfect quantum fluid due to universal quantum behavior predicted by the AdS/CFT correspondence or is it a local minimum in the ratio η/s caused by the interplay between correlated liquid effects that want the ratio to grow and the fluctuations of a nearby phase that want to drive them to zero at/near the phase transition?

The model developed in this work, which differs from other work by taking into consideration amplitude fluctuations, aims to study the ratio η/s in strongly correlated quantum fluids. While past calculations find a minimum that obeys (4.1), such as those calculated by Wlazłowski et. al. [137, 138], our calculations have shown that fluctuations from the nearby superfluid phase can drive the ratio η/s to very low values, even to zero at the phase boundary, thus violating the conjectured universal lower bound. More precise measurements of η/s near the phase boundaries, in tighter temperature windows around T_c , are needed to establish validity of the KSS bound. Additionally, we expand on the connection between viscoelastic responses and violation of the conjectured bound as was first introduced by Alberte et. al. [111]. In our work and that done by Alberte et. al., two systems that can violate the KSS bound, the unitary Fermi gas and holographic solids, exhibit both viscous and elastic responses implying that complicated viscoelastic behavior, in addition to phase fluctuations, contribute to violation of the KSS bound. In conclusion, our theory provides an alternative and unique way of studying η/s in a strongly correlated quantum fluid by considering the effects of pairing instabilities in the quasiparticle scattering amplitude. We hope this work sheds light on the rich connection between condensed matter and high energy problems through (bottom up) AdS/CFT.

4.3 $\frac{\eta}{s}$ in Dirac Materials

From the previous section, we saw the effects of superfluid fluctuations on the KSS bound. While the interpretation of why $\eta/s \to 0$ as $T \to T_c$ can be debated¹, the result is rigorous and shows at the very least that a phase transition has substantial effects on the bound. Experimentally, J.A. Joseph et. al. [5] saw these effects in the unitary gas, albeit with questionable results near T_c where our result is valid. It would be interesting if future work investigated the effects of phase transitions on η/s in other condensed matter systems to provide more insight into the bound in experimentally viable systems².

The KSS bound is heavily studied in a variety of systems, including DM such as graphene [92, 139]. With the notion that equality in eqn. (4.1) denotes a perfect quantum fluid, the task was to determine how perfect the fluid of electrons in graphene was. Given it's unusual properties, from high mobility and quasi-relativistic behavior (as discussed in **chapter 3**), the problem was of high interest and showed promise and intrigue. While much work on η/s is restricted to areas close to the charge neutrality point, it should be noted that a well established, agreed upon theory for what happens at the charge neutrality point is lacking (to be discussed in **chapter 5**). The best model, based on hydrodynamics [21, 38, 72, 140, 141], is unfortunately sensitive to any disorder/impurity and the theory breaks down. With this in mind, we divert our attention away from the charge neutrality point and ask what η/s looks like in **DM** while they're in the Fermi liquid phase, i.e. when $T \ll T_F$ but still when $\varepsilon_F \neq 0$. To begin, we start with a d = 3 DM in its Fermi liquid state. Right away, we're able to use the more well known three-dimensional Fermi liquid expression entropy density eqn.(4.13). As shown in chapter 2 and in appendix A, the equilibrium properties between normal Fermi liquids and DM maintain the same form (i.e. temperature dependence) albeit with a change of the density of states. The viscosity is a little trickier to deal with. When deriving eqn.(4.12) via the **LKE**, we may mimic the angular integrals part of calculation as our system is also in three dimensions. The appropriate changes come from pre-factors of η . To the best of our knowledge, as of now, those factors haven't been

¹For example, one can make the argument for η/s dropping to zero since the single electrons/**qp** cease to exist as they begin to form Cooper pairs. Another common rebuttal is an attack on the validity of using the Landau kinetic equation to determine viscosity and entropy.

 $^{^{2}}$ Of course obtaining precise data near volatile regions around transition temperatures is a daunting task. Nevertheless, condensed matter systems provide the possibility of experiments more than the examples involving such things as "higher" gravities and black holes

explicitly calculated. However, on first glance, it seems a naive substitution may be made for p_F and $v_F \rightarrow v_g$. The final roadblock comes in prefactors for the **qp** lifetime eqn.(4.10). Again, these have not been explicitly calculated but we believe that as long as we have a good **qp** picture, which we do, then at the very least the temperature dependence will remain the same; our main goal is to determine how η/s behaves as a function of temperature. Barring any sort of corrections, for example those due to superfluid fluctuations as discussed above, the temperature dependence of η , tucked away in the lifetime³ eqn.(4.10), is T^{-2} . The entropy density is linear in temperature as derived in **appendix A**. This leads to $\eta/s \propto T^{-3}$ in the Fermi liquid regime:

$$\frac{\eta}{s} = \frac{\alpha}{T_F^3} \left(\frac{T}{T_F}\right)^{-3} \tag{4.23}$$

where α is a constant that contains all factors from eqn.(4.12), eqn.(4.10), and eqn.(4.13) that don't explicitly depend on temperature. Unfortunately, this expression for η/s is uneventful as it's well-behaved and doesn't violate the bound, as per various experiments. For example, as discussed earlier and in [93, 105, 106], in the unitary Fermi gas, a minimum that satisfies eqn.(4.1) occurs at $T \approx 0.334T_F$, as can be seen in fig.(4.3). A plot showing the qualitative behavior of this curve and its relation to the bound $\eta/s = \hbar/4\pi k_B$ is shown below in fig.(4.5):



FIGURE 4.5: This figure **qualitatively** shows the behavior of the KSS bound as a function of normalized temperature $\overline{T/T_F}$ (blue curve). Additionally, the bound itself (red dashed line), $\hbar/4\pi k_B$ is plotted and easily shows the bound doesn't get violated. Obviously if η/s only behaved as T^{-3} , as shown, there will be no minimum. Adding high temperature corrections, which changes the temperature dependencies of η and s (see eqn.(4.12) & eqn.(4.14) for $T \gg T_F$), causes the blue curve to trend upward showing a well-defined minimum at $T \approx 0.334T_F$ with a value of $\approx 0.3\hbar/k_B$, about 3.7 times larger than the lowest bound.

The three-dimensional case was easy, albeit boring. The question we now face is what

³Expressions for the T^{-2} dependence on the **qp** lifetime are additionally found in other numerous texts such as [2, 7, 12, 56]. Furthermore, the corrections due to superfluid fluctuations are not relevant to this discussion so we're using eqn.(4.10), not eqn.(4.11)

this bound looks like in a two-dimensional Dirac material (for example graphene): i.e. a twodimensional Fermi liquid with linear single particle spectrum. The two challenges that remain are: (1) calculating the temperature dependencies of η and s and (2) calculating the prefactors of η and s. Fortunately, the temperature dependencies in the two-dimensional Fermi liquid have been derived before by Coffey, Bedell, and Qian [23–25, 142]. Their results show that, due to the reduced phase space area in a two-dimensional Fermi liquid, the temperature dependence in the lifetime contains a logarithmic term in addition to the usual T^{-3} . And with the temperature dependence of the entropy density remaining the same, we have :

$$\frac{\eta}{s} = \frac{\beta}{T_F^3} \left(\frac{T}{T_F}\right)^{-3} \left[\ln\left(a \cdot \frac{T_F}{T}\right)\right]^{-1} \tag{4.24}$$

where, similar to the three-dimensional case discussed earlier, β is a constant containing all the factors explicitly independent of temperature. The new parameter $a = T^*/T_F$ is a constant less than 1 and T^* is an adjustable fitting parameter whose value can be determined via experiment or by imposing a microscopic theory. Physically, T^* represents something of a cut-off temperature, above which quantum many-body effects start to die out and high temperature/classical behavior takes over. For example in ³He, $T^*/T_F = .3$ and is set by the spin fluctuation temperature (for more information, consult Hua Li's thesis [17]).

A natural question to ask is if addition of the new logarithmic term makes a difference. It's commonly stated⁴ that logarithmic terms are small and therefore negligible. This was in fact a reason why past calculations for the compressibility using Hartree-Fock methods, which included logarithmic terms, were accepted despite not agreeing with experimental data. To see if this logarithm makes a difference, we first plot the curve below: where each curve is for



FIGURE 4.6: This figure, again, **qualitatively** shows the behavior of the KSS bound as a function of normalized temperature T/T_F . The various colors of the curves corresponds to different values of T^*/T_F . The bound itself (black dashed line), $\hbar/4\pi k_B$ is plotted as well and, although hard to see, is under all the curves seeming to indicate the bound isn't violated. What's interesting however is the existence of minimum **without** including high temperature behavior. Recall that for the d = 3 case in fig.(4.5), a minimum is only seen if expressions for $T \gg T_F$ are included.

 $^{^{4}}$ Whether it's "commonly stated" or "agreed upon", it seems lazy to standby a calculation that doesn't agree with experiment purely based on ease. Furthermore, it looks even worse when you distrust a calculation that agrees with experiment (ours). But I digress

a different value of $a = T^*/T_F$ as dictated by the legend shown. For the plots in fig.(4.5), and subsequent plots related to this discussion, a value of $T_F = 1000$ K was used. This value of the Fermi temperature corresponds to experimentally accepted values of particle density in graphene, roughly $n \sim 10^{12}$ cm⁻² [28, 143]. At first glance, the data also seems uninteresting. The minimum value for η/s is plotted and is clearly below all the curves. However, what one should take away from this is that a minimum exists at all! Recall in fig.(4.5) that a minimum is not seen in the quantum regime and to see a minima, high temperature corrections are needed. The existence of a minimum in d = 2 is due to the logarithmic correction in the **qp** lifetime.

With the clear existence of a minimum, our next step is to find for what temperature that is. Taking the derivative of eqn.(4.24):

$$\frac{d}{dT}\left(\frac{\eta}{s}\right) = \frac{\beta - 3\beta \ln\left(a \cdot \frac{T_F}{T}\right)}{T^4 \left[\ln\left(a \cdot \frac{T_F}{T}\right)\right]^2}$$

and setting this equal to zero gives the temperature for which the minimum exists:

$$T = T_F\left(\frac{a}{e^{1/3}}\right) \tag{4.25}$$

At this stage, we do not have a value of T^* for graphene. Since spin fluctuations also exist in graphene, and T^* was set by the spin fluctuation temperature in ³He, we assume $T^* = .3T_F$ for graphene and continue the discussion. The exact value for T^* doesn't change the main result which is that a minimum does exist. Looking closely at a single curve, when a = .3, from fig.(4.6): where the vertical blue line represents where the minimum is $T = 0.215T_F$. Given the



FIGURE 4.7: Specifically looking at the a = .3 curve from fig.(4.6). Again, the black dashed line shows $(\eta/s) = \hbar/4\pi k_B$. The vertical blue line indicates the temperature where the minimum is located.

behavior of eqn.(4.24), the minimum is difficult to see. If we focus on a small section around the minimum: Now we can see a minimum, along with the minimum value of η/s as predicted. At this point, we cannot make anymore statements about the plots above. The simple addition of the logarithm in the **qp** lifetime was crucial in seeing a minimum in this low temperature regime. The next task is to fully calculate the constant β in eqn.(4.24), as this will determine if the minimum can get close to/become equal to/go below the black dashed line in fig.(4.8).



FIGURE 4.8: A close up version of the minimum from fig.(4.7). Here we can clearly see a well-defined minimum.

Thankfully, due to the work of Qian [24, 25] which calculated the \mathbf{qp} lifetime for a d = 2 electron gas, the only challenge is properly taking into the account the quasi-relativistic nature of **DM**. Nevertheless, the preliminary graphs shown here, together with the already unusual behavior of **DM**, make this an open problem worth exploring, which will hopefully lead to **DM**, graphene specifically, being crowned a perfect fluid.

Chapter 5

Concluding Remarks and Future Work



FIGURE 5.1: Graphic depicting what this thesis was all about.

5.1 A Brief Summary

This thesis sought to provide an overview of the unique properties of **DM**. First, using the Tomanaga-Luttinger liquid theory, we saw the severe consequences a linear single particle dispersion has on a system. In a d = 1 **DM**, i.e. a single walled carbon nanotube, we see static (specific heat)and dynamic (speed of sound) quantities exhibit density independent behavior [1], which isn't physically intuitive at all. Furthermore, we also see this density independent behavior in the so-called anomalous exponents, a signature of Luttinger liquid behavior. Our work also reproduces the well known results in a d = 1 interacting Fermi system: the continuous behavior of the momentum distribution function, rather than a sharp discontinuity at $p = p_F$.

Second, using Fermi liquid theory on **DM** in d = 2, 3, we saw a number of interesting phenomena: an effective velocity relation as opposed to effective mass, nearly identical expressions for equilibrium properties with slight changes to the density of states, atypical behavior in the collective modes (zero sound and first sound) [63, 64]. At the end of the day, we come to similar conclusions that Baym & Chin did [52]: Most of the results from the nonrelativistic case hold for the relativistic case. Similarly for us, most of the results (at least equilibrium ones) from normal Fermi liquid theory are applicable to **DM** with the appropriate change in density of states. The interesting and intriguing differences come from the transport phenomena. A careful analysis and application of **LKE** to a d = 2 Fermi liquid (not a Fermi liquid in **DM**) led to zero sound propagation in the presence of attractive interactions, an absence of Landau damping, and a modified plasmon frequency that captures the instability of a d = 2 Fermi liquid.

Third, we applied the Virial theorem to **DM**, where the linear dispersion (once again) is responsible for unlocking new physics. A closed form for the total ground state energy, including full interaction behavior, was obtained allowing for an uncanny ease in calculating various quantities. As a consequence of the inherent density independence, statements about Fermi liquid parameters could be made and, combined with the work of Baym and Chin [52], it appears that the Fermi liquid in **DM** is a local Fermi liquid (only F_0^s is nonzero) in the unitary limit (Fermi liquid parameter is a constant in density).

Fourth, we investigated the highly studied (and perhaps scrutinized) Kovtun-Son-Starinets (KSS) bound in **DM**. Although the bound has been investigated in **DM** before, we approach η/s in a temperature regime where electron-electron collisions are dominant, as opposed to electron-impurity dominated. While the work is far from finished, our preliminary results in fig.(4.6) show promise in a minimum that can be seen without having to include high temperature corrections.

An additional goal of this thesis was to fill in cracks of knowledge in previous works and make the lives of future students of theoretical physics a little easier. As one can see from the projects explored in this thesis, many of them were born was a simple and careful review of what was done, and applying it in a dimension that it wasn't done in before. Historically, many expressions and results in d = 3 have been incorrectly applied to lower dimensions and we've shown here that great care must be taken when attempting that task. In regards to the d = 2 Fermi liquid, some results of this thesis were meant to provide analysis and discussion. While numerous in-depth texts exist for d = 3 Fermi liquids, the same rigor for d = 2 Fermi liquids is lacking, until now. With both the Fermi liquid analysis and application of the Virial theorem, we hoped to show how interesting and new physics can be discovered from using methods of the past that may have been shelved for newer more complex methods. That isn't to say that current methods are useless, rather we mean to show that we can learn a lot from past methods and models as well¹. This logic followed us into our investigation of the KSS bound where we showed violation in a experimentally testable system by using Fermi liquid theory and temperature Green's functions, methods made popular by [14], while the bound itself was theorized using string theory and conformal field theory techniques. Above all, we hope this work further cemented the idea that Fermi liquid theory is <u>still</u> the standard by which all current and future theoretical models are measured against.

5.2 What lies on the horizon?

The goal of this thesis was to understand the physics of **DM** in the Fermi liquid regime depicted by Fig.(1.10). With the vast amount of work on **DM** that has been mentioned/cited in this thesis, one might naively think that this story is nearing its end. However, as said by the author of everyone's favorite quantum mechanics textbook, R. Shankar: *Every time you think you're done with the laws of physics, somebody does some experiment that doesn't fit what you know, and you have to make up new stuff.*

For starters, an accepted theory in regards to the behavior of electrons/**qp** near the **CNP**, depicted by Fig.(1.9), is still poorly understood. In fact, it's still argued whether or not we can have a Fermi liquid there. The work of Crossno et. al. [38] stated near **CNP**, due to a breakdown of the Wiedemann-Franz law, Fermi liquid theory is invalid (as is commonly accepted). However, later work by Lavasani et. al. [144] show that in presence of strong phonon scattering, the Wiedemann-Franz law can be violated definitively establishing the connection between Fermi liquid behavior (or non-Fermi liquid behavior) and the Wiedemann-Franz law to be incorrect. The unfortunate lack of understanding in what happens at **CNP** should not be confused with a lack of effort as multiple theorists have put forth ideas for how to understand transport in the absence of a Fermi liquid picture [21, 72, 140, 141]. Recent experiment by Sulpizio [145] visualized the hydrodynamic flow of electrons in graphene, however their results are qualitatively Boltzmann indicating their work being done at a temperature/energy away from **CNP**. Unfortunately, most of this work lacks a robustness coveted amongst theories. Nevertheless, the rich interesting physics, for example in conductivity, hall viscosity, and hydrodynamics [145–147], will push the effort forward.

In regards to the KSS bound discussed in the previous chapter, the viscosity in graphene has been discussed many times [92, 139] albeit in a regime where impurity scattering is dominant. Our work investigated what this bound could look like in a regime where electron-electron scattering dominates and although the journey is not complete, fig.(4.6) shows promise and confirms this a problem worth exploring further. As stated in **chapter 4**, the main challenge is in determination of the transport and **qp** lifetimes. Although this task has already been completed by various groups for d = 2 electron liquids [23–25, 148], and well established for d = 3 [2, 12, 13],

¹Ain't no school like the old school

no work has been done for these <u>exact</u> expressions in **DM**; recall that the prefactors for η and s are imperative in that they could potentially lower the minimum to violate the bound. Such work on the **qp** lifetimes would require tackling the problem of scattering in which the induced interaction model could be used [83, 116, 119, 119].

Finally, our work has shown a surprising robustness in Fermi liquid theory. Development of the **qp** picture by Landau and expanded upon by many others, is a bit vague. Technically, the **qp** in Fermi liquid theory should be referred to as "Landau" **qp** to differentiate them from collective modes. The original work by J.R. Schrieffer [149] elegantly describes what a \mathbf{qp} is. More recently, a comprehensive review on \mathbf{qp} in condensed matter was done by P. Wölfle [15] describing different **qp**, and hinting at applicability of Fermi liquid theory in places where it was thought to fail, for example at a quantum critical point. Our work pushed Fermi liquid theory to its furthest extent of applicability and perhaps even too far². However, the expressions we obtained agree with experiment! For example, when calculating the viscosity in the unitary Fermi gas, we use Fermi liquid theory near T_c yet obtain results for the viscosity that bare resemblance to that in experiment and using other methods [105]. All of this hints at a broader range of applicability for Fermi liquid theory. The start of this journey has been taken up by Joshuah Heath who shows a more general Luttinger's theorem and provides a concrete definition for when a Fermi liquid picture is appropriate, going beyond what was previously established by iconic texts [2, 14]. Still in a young phase, the project does have potential, shows how much more work there is to be done, and solidifies Fermi liquid theory as an elite model for understanding interacting Fermi systems.

To conclude, I hope the reader(s) has found this thesis of various values. First, from a research standpoint, the multiple approaches we took to analyze **DM** has shown, even at a surface level, the rich physics that these systems contain. Furthermore, our results have shown that there is more to uncover and pushing the work of chapters 2, 3, and 4 is a task worth taking. Second, from an educational standpoint, I hope the layout, discussions, details, and story that was told were all clear and concise. It was my goal to provide a text that would benefit future students of theoretical physics, and I have tried to do so to the best of my ability.

 $^{^2\}mathrm{As}$ many referees would claim... damn you physical review

Appendix A

A Discussion of Statistical Mechanics

In this appendix we review the necessary statistical mechanics needed before applying **FLT**, much of which can be found in standard books [13, 126](albeit frequently omitting steps and insight, rendering the eager student dazed and confused). Consider an ideal Fermi gas where the Fermions have a parabolic dispersion shown below in Fig. A.1. Now, our goal is to understand



FIGURE A.1: Typical parabolic energy dispersion.

the properties of the system as a whole, typically $10^{19} - 10^{23}$ particles, not just those of one individual fermion. So if we want to study N fermions, how should we go about this?

Recall that the entropy of a system is found using Boltzmann's equation

$$S = k_B \ln \Omega = k_B \sum_{i} \ln \Omega_i \tag{A.1}$$

where Ω_i is the number of microstates in G_i and G_i is the number of states in the *i*-th group. If we omit spin, then we interpret Ω_i as the number of ways to divide N_i particles between G_i states, see Fig. A.2, with only one particle per state [13]. A straightforward combinatorics



FIGURE A.2: Dividing N particles into G states.

argument leads to the following for Ω_i

$$\Omega_i = \frac{G_i!}{N_i \left(G_i - N_i\right)!} \tag{A.2}$$

and plugging this into eqn. (A.1),

$$S = k_B \sum_{i} \ln \Omega_i$$

= $k_B \sum_{i} \left(\frac{G_i!}{N_i (G_i - N_i)!} \right)$
= $k_B \left[\sum_{i} G_i (n_i \ln n_i + (1 - n_i) \ln (1 - n_i)) \right]$ (A.3)

where in the last line we've assumed the thermodynamic limit (i.e. used Stirling's approximation: $\ln x! \approx x \ln x - x$ for $x \gg 1$) and we substitute $n_i = N_i/G_i$. In the continuum limit, we have

$$\sum_{i} G_{i} \to V \sum_{\mathbf{p}\sigma}$$
$$n_{i} \to n_{\mathbf{p}\sigma}$$

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we're thus left with the following expression for the entropy

$$S = -Vk_B \sum_{\mathbf{p}\sigma} \left(n_{\mathbf{p}\sigma} \ln n_{\mathbf{p}\sigma} + (1 - n_{\mathbf{p}\sigma}) \ln (1 - n_{\mathbf{p}\sigma}) \right)$$
$$= Vs \tag{A.4}$$

where s is the entropy density.

What about $n_{\mathbf{p}\sigma}$? To obtain an expression for $n_{\mathbf{p}\sigma}$, we first turn to variational calculus. Recall that the first variation of a function $F = F(x_1, x_2, \dots, x_n)$ is given as [150]

$$\delta F = \frac{\partial F}{\partial x_1} \delta x_1 + \frac{\partial F}{\partial x_2} \delta x_2 + \dots + \frac{\partial F}{\partial x_n} \delta x_n \tag{A.5}$$

Applying eq. (A.5) to eqn. (A.4), we find

$$\delta s = -k_B \sum_{\mathbf{p}\sigma} \delta n_{\mathbf{p}\sigma} \ln \frac{n_{\mathbf{p}\sigma}}{1 - n_{\mathbf{p}\sigma}} \tag{A.6}$$

Applying similar variational techniques to the first law of thermodynamics, we are left with

$$\delta \epsilon = T \delta s + \mu \delta n \tag{A.7}$$

where $\delta \epsilon = \sum_{\mathbf{p}\sigma} \varepsilon_{\mathbf{p}\sigma} \delta n_{\mathbf{p}\sigma}$ and $\delta n = \sum_{\mathbf{p}\sigma} \delta n_{\mathbf{p}\sigma}$. Combining A.6 and A.7 and matching terms, we finally arrive at the well known Fermi-Dirac distribution

$$n_{\mathbf{p}\sigma} = \frac{1}{1 + \exp\left[\beta\left(\varepsilon_{\mathbf{p}\sigma} - \mu\right)\right]}, \qquad \beta = \frac{1}{k_B T}$$
(A.8)

It is useful to note that the while we set a specific form for the dispersion $\varepsilon_{\mathbf{p}\sigma}$, it wasn't necessary in deriving the expressions above. Rather, the most important property in the analysis above was that the particles were fermions; i.e. everything we've done holds for a linear spectrum as long as they're fermions. In the next two sections, we go over some of the properties of an ideal Fermi gas at T = 0 and $T \neq 0$.

A.1 Properties at T = 0

At T = 0, the distribution given in eqn. (A.8) becomes a step function

$$n_{\mathbf{p}\sigma} = \Theta \left(p_F - |\mathbf{p}| \right) \tag{A.9}$$

This allows us to define the Fermi momentum in terms of the number density¹



FIGURE A.3: The Fermi-Dirac distribution function $n_{{\bf p}\sigma}$ at T=0

$$n = \sum_{\mathbf{p}\sigma} n_{\mathbf{p}\sigma}$$

= $\sum_{\sigma} \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} \Theta \left(p_F - |\mathbf{p}| \right)$
= $\frac{k_F^3}{3\pi^2} \Longrightarrow k_F = \left(3\pi^2 n \right)^{\frac{1}{3}}$ (A.10)

where the bounds of the momentum integral are set by the step function. The energy density at T = 0 is

$$\epsilon = \sum_{\mathbf{p}\sigma} \varepsilon_{\mathbf{p}\sigma} n_{\mathbf{p}\sigma}$$
$$= \sum_{\sigma} \int \frac{d\mathbf{p}}{\left(2\pi\hbar\right)^3} \cdot \left(\frac{p^2}{2m}\right) \cdot \Theta\left(p_F - |\mathbf{p}|\right)$$
$$= \frac{3}{5}n\varepsilon_F$$
(A.11)

A.2 Properties at $T \neq 0$

Now we focus on the low temperature properties where $n_{p\sigma}$, Fig. A.4, differs from Fig. A.3 in a small area near p_F . We write the distribution function as

$$n_{\mathbf{p}\sigma}(T) = n_{\mathbf{p}\sigma}\left(T=0\right) + \delta n_{\mathbf{p}\sigma}$$

¹The integrals are all done in d = 3 where $\int d\mathbf{p} = \int p^2 dp \int d\Omega_3$. The expressions for d = 2 are found after a substitution of $\int d\mathbf{p} = \int p dp \int \Omega_2$

where $\delta n_{\mathbf{p}\sigma}$ is some small variation from the ground state distribution function. If we apply the first variation to A.8 and combine it with the first variation in entropy density A.6, we get

$$\begin{split} \delta s &= \frac{1}{T} \sum_{\mathbf{p}\sigma} \left(\varepsilon_{\mathbf{p}\sigma} - \mu \right) \left[\frac{\partial n_{\mathbf{p}\sigma}}{\partial \varepsilon_{\mathbf{p}\sigma}} \left(-\frac{(\varepsilon_{\mathbf{p}\sigma} - \mu)}{T} \delta T + \delta \varepsilon_{\mathbf{p}\sigma} - \delta \mu \right) \right] \\ &= -\sum_{\mathbf{p}\sigma} \frac{\partial n_{\mathbf{p}\sigma}}{\partial \varepsilon_{\mathbf{p}\sigma}} \left(\frac{\varepsilon_{\mathbf{p}\sigma} - \mu}{T} \right)^2 \delta T \end{split}$$

where the last two terms, $\delta \varepsilon_{\mathbf{p}\sigma}$ and $\delta \mu$, are ignored since their lowest order terms vary as T^2 . From the expression for δs above, we can extract an expression for the first derivative of s with



FIGURE A.4: The Fermi-Dirac distribution function $n_{\mathbf{p}\sigma}$ at $T \neq 0$ but low T. A key feature is the smooth transition at the chemical potential μ (which has a value of $n_{\mathbf{p}\sigma} = \frac{1}{2}$). This is clearly diffrent from the discontinuity at μ as shown in Fig. 1.1

respect to T^2 :

$$\begin{aligned} \frac{\partial s}{\partial T} &= -\sum_{\mathbf{p}\sigma} \frac{\partial n_{\mathbf{p}\sigma}}{\partial \varepsilon_{\mathbf{p}\sigma}} \left(\frac{\varepsilon_{\mathbf{p}\sigma} - \mu}{T}\right)^2 \\ &= -k_B^2 \sum_{\sigma} \int p^2 \frac{dp}{d\varepsilon} \frac{4\pi}{(2\pi\hbar)^3} dx \frac{\partial}{\partial x} \left(\frac{1}{1+e^x}\right) x^2 \\ &= -k_B^2 g(0) \int_{-\infty}^{\infty} dx \frac{\partial}{\partial x} \left(\frac{1}{1+e^x}\right) x^2 \end{aligned}$$
(A.12)

²Again, the calculation is done in d = 3. This influences the integral over momentum and g(0). Extending to d = 2 is straightforward.

$$s = \frac{\pi^2}{3}g(0)k_B^2 T$$
 (A.13)

where the density of states at the Fermi surface in d = 3 is ⁴

$$g(0) = \frac{mp_f}{\pi^2 \hbar^3} \tag{A.14}$$

From here, all thermodynamic properties of interest are obtainable. A few remarks worth mentioning:

- 1. Extending our analysis to d = 1, 2 is straightforward (although d = 1 has no meaning in Fermi Liquid theory).
- 2. Everything done was in the absence of interactions. The effects of interactions on a system with linear dispersion are discussed in chapter 2 while the parabolic system is extensively discussed in. [2, 7, 12, 13]
- 3. The results in chapter 2 for a free Dirac gas follow the same analysis shown here.

Although the main purpose of this thesis is with the effects of electron interactions in a system with linear dispersion, it's useful and instructive to go through the non-interacting behavior since the behavior including interactions is suspiciously close to non-interacting.

³The calculation leads to a group of terms that resembles the density of states. We replace this with the density of states at the Fermi surface, g(0), since we're close to T = 0 and the main contribution is from g(0). ⁴Appropriate degeneracy factors have been taken into consideration.

Appendix B

Detailed Calculations of the Fermi Liquid Properties

In this appendix, we go through detailed derivations of a few Fermi liquid properties (both equilibrium and nonequilibrium). The main purpose of this is to provide a reference for future students learning Fermi Liquid theory. Hopefully this will supply aspiring theorists with clarity and a useful resource when the iconic references [2, 12] need supplement.

B.1 Pomeranchuk Instability in two-dimensions

In chapter 1, we discussed the various conditions and scenarios in which **FLT** is applicable. Aside from the examples mentioned, Pomeranchuk derived a constraint on Landau parameters $F_l^{s,a}$ for thermodynamic stability in a three-dimensional Fermi liquid which has been derived multiple times [2, 12, 151, 152]. The relation, now known as the Pomeranchuk Instability, ensures the ground state energy is stationary and a minimum for a given angular momentum channel:

$$F_l^{s,a} > -(2l+1)$$
 (B.1)

If the condition is violated, it simply means that the interaction is "too strong" which leads to an instability of the normal state [12]. An interesting feature of eqn.(B.1) is how the minimum value for each Landau parameter depends on the specific moment. This is in agreement with the general perturbative nature of Fermi liquid theory; higher order interaction terms must be smaller than previous terms for validity of the theory. As we'll see in a two-dimensional Fermi liquid, this is not the case. Here, we derive the Pomeranchuk instability condition in a twodimensional Fermi liquid as this is relevant to **chapter 2** of this thesis. Before continuing, it should be clear that the brunt of this work was done by Joshuah Heath en route to the work we did for the two-dimensional Fermi liquid [63, 64]. Stability requires the free energy density $F = E - \mu n$ to be a minimum for the undistorted distribution function. As such, we look at the change in F, keeping up to second order

$$\delta F = \frac{1}{V} \sum_{\mathbf{p}\sigma} \left(\varepsilon_{\mathbf{p}\sigma} - \mu \right) \delta n_{\mathbf{p}\sigma} + \frac{1}{2V^2} \sum_{\mathbf{p}\sigma, \mathbf{p}'\sigma'} f_{\mathbf{p}\mathbf{p}'}^{\sigma\sigma'} \delta n_{\mathbf{p}\sigma} \delta n_{\mathbf{p}'\sigma'}$$
(B.2)

where V is the area since we're working in two-dimensions. Our goal is to obtain an expression for δF in terms of Landau parameters by investigating the effects of Fermi surface distortions on the ground state and then impose the stability condition. At/close to T = 0, we linearize the $(\varepsilon_{p\sigma} - \mu)$ term and Taylor expand the Heaviside step functions that appear in $\delta n_{p\sigma}$. For a two-dimensional system, the best choice for orthogonal functions are the Chebyshev polynomials $T_l(x)$ which are defined as follows [36]

$$T_l(\cos\theta) = \cos\left(l\theta\right)$$
$$\int_0^{2\pi} d\Omega T_l(\cos\theta) T_{l'}(\cos\theta) = \pi \left(1 + \delta_{l',0}\right) \delta_{l,l'}$$

where $d\Omega$ is the 2d polar angle. The remaining calculation follows from the literature [64] and leads to

$$\delta F = \frac{\pi v_F p_F}{(2\pi\hbar)^2} \sum_l |\nu_l|^2 \left(1 + \delta_{l,0} + \frac{1}{2} F_l^{s,a} \left(1 + \delta_{l,0} \right)^2 \right)$$
(B.3)

Imposing that eqn.(B.3) be greater than zero, we find the condition for stability in a twodimensional Fermi liquid is given by the following constraint:

$$F_l^{s,a} > -\frac{2}{1+\delta_{l,0}} \tag{B.4}$$

The l = 0 Landau parameter obeys the same stability condition as in the three-dimensional Fermi liquid. A stark difference is seen in higher order terms. In a two-dimensional Fermi liquid, the minimum value Landau parameters for $l \ge 1$ is a constant, i.e. $F_l^{s,a} > -2$ for $l \geq 1$. The stability condition is far more strict for a two dimensional Fermi liquid than for its three-dimensional counterpart. This change, a seemingly "constant" constraint on higher order Landau parameters, can be interpreted in a number of ways. Historically, it's been known that Fermi liquid runs into trouble as dimensionality is reduced. The failure of Fermi liquid theory in one-dimensional interacting Fermi systems is well studied [31, 32, 153] where the lowest energy excitation being a collective boson (as opposed to a traditional Landau quasiparticle), and the gap in the particle-hole spectrum at low energies, invalidates Fermi liquid theory and Luttinger liquid theory must be used. In two-dimensional systems, the distinction between a good and bad Fermi liquid requires more care. While a sharp Fermi surface exists and the discontinuity in the distribution function at the Fermi momentum remains, a two-dimensional Fermi liquid theory is not as robust as it is in three-dimensions. For example, the earlier work of Anderson [43, 44] discussed the breakdown of Fermi liquid theory, albeit in two-dimensional Hubbard systems and under specific conditions. More recent work on two-dimensional interacting Fermi systems have alluded to possible marginal Fermi liquid behavior [58, 60, 154] where the low temperature behavior of the self energy is modified. These past results are in line with our derived stability condition, eqn.(B.4), which seems to imply the two-dimensional Fermi liquid is not as robust and more susceptible to instabilities.

This stability condition can be readily observed if we derive specific moments of the LKE. The l = 0 moment leads to the number conservation law

$$s\nu_0 = \frac{\nu_1}{3} \left(1 + \frac{1}{2} F_1^s \right) \tag{B.5}$$

while the l = 1 moment leads to the equation of motion for fluid momentum

$$s\nu_1 - \frac{3}{2}\left(1 + F_0^s\right)\nu_0 - \frac{7}{10}\left(1 + \frac{1}{2}F_2^s\right)\nu_2 = U$$
(B.6)

Eliminating ν_1 from these equations, we get

$$\left(s^{2} - \left(\frac{c_{1}}{v_{f}}\right)^{2}\right)\nu_{0} - \frac{7}{30}\left(1 + \frac{1}{2}F_{1}^{s}\right)\left(1 + \frac{1}{2}F_{2}^{s}\right)\nu_{2} = \left(1 + \frac{1}{2}F_{1}^{s}\right)\frac{U}{3}$$
(B.7)

where the speed of ordinary (first) sound c_1 is

$$c_1^2 = \frac{v_F^2}{2} \left(1 + F_0^s\right) \left(1 + \frac{1}{2}F_1^s\right)$$
(B.8)

Unlike the results derived by Baym and Pethick [2], the coefficients in front of the Landau parameters F_l^s for $l \ge 1$ in eqns.(B.6) and (B.7) are all a constant $\frac{1}{2}$ as opposed to different constants. This, along with the expression for ordinary sound eqn.(B.8), is in agreement with our derived stability condition eqn.(B.4) which further supports the more restrictive nature of the two-dimensional Fermi liquid.

B.2 Compressibility

We start with the formal definition of the compressibility

$$\kappa^{-1} = n^2 \frac{\partial \mu}{\partial n} \tag{B.9}$$

where n is the particle density and μ is the chemical potential. Before evaluating eqn.(B.9), we re-write

$$\kappa^{-1} = n^2 \frac{\partial \mu}{\partial n} = n^2 \frac{\partial \mu}{\partial p_F} \cdot \frac{\partial p_F}{\partial n}$$

where now the derivatives are with respect to the Fermi momentum. We do this for the following reasons:

- Recall in appendix A that the derived expression for the Fermi momentum was in terms
 of the particle density n. Therefore, a change/variation in n is also a change/variation in
 p_F
- We can intuitively understand the compressibility as a "squeezing" or "expanding" of the Fermi surface (as in the figure below)



FIGURE B.1: Popular visual aide for understanding the changes in the Fermi surface when calculating the compressibility; similar figure can be found in Giuliani [7]. In using eqn.(B.9), we evaluate the derivative of the chemical potential μ . This quantity changes in two ways: by a change in the Fermi momentum p_F and/or by the addition of **QP** to the shaded region.

Using the expressions for particle density n found in **appendix A**, we can cast eqn.(B.9) in its final form:

$$\kappa^{-1} = \frac{np_F}{d} \frac{\partial \mu}{\partial p_F} \tag{B.10}$$

Now the task is to evaluate the derivative in eqn.(B.10) within **FLT**. To do so, recall that the chemical potential is the **qp** energy at the Fermi surface and our Fermi surface is changing (e.g. a slight expansion as shown in fig.(B.1) can be written as $p_F \rightarrow p_F + \delta p_F$). The change in μ is therefore

$$\delta\mu = \varepsilon_{\mathbf{p}_F + \delta p_F \sigma} - \varepsilon_{\mathbf{p}_F \sigma} = v_F^* \delta p_F + \sum_{\mathbf{p}', |\mathbf{p}| = p_F \sigma'} f_{\mathbf{p}\mathbf{p}'}^{\sigma\sigma'} \delta n_{\mathbf{p}'\sigma'}$$

These two terms, as per Giuliani [7], can be understood as follows: the first term is the change in the bare **qp** energy when $p_F \rightarrow p_F + \delta p_F$, the second term is the interaction between additional **qp** that arise when the Fermi surface is expanded. Focusing on the second term with the Fermi liquid interaction:

$$\sum_{\mathbf{p}',|\mathbf{p}|=p_F\sigma'} f_{\mathbf{pp}'}^{\sigma\sigma'} \delta n_{\mathbf{p}'\sigma'} = \left(\frac{4\pi}{\left(2\pi\hbar\right)^3} \int_{p_F}^{p_F+\delta p_F} dp'\left(p'\right)^s\right) \left(\int \frac{d\Omega_3}{\Omega_3} \sum_{\sigma'} f_{\mathbf{pp}'}^{\sigma\sigma'}\right)$$

where Ω_3 is the solid angle in d = 3 dimensions. The integral over momentum in the first parentheses is straightforward where one must remember to neglect terms higher than linear order in δp_F . The second integral over the solid angle (angular average) is evaluated by comparing terms and using eqn.(8.14) from [7]. This leaves us with

$$\delta\mu = g(0)v_F\delta p_F + v_F F_0^s\delta p_F \to \frac{\delta\mu}{\delta p_F} = g(0)v_F + v_F F_0^s$$

this result is equivalent to the derivative we sought out to calculate. Finally we arrive at the final expression for compressibility

$$\kappa = \frac{1}{n^2} \frac{g(0)}{1 + F_0^s} \tag{B.11}$$

where the density of states at the Fermi surface g(0) and the Fermi velocity v_F include the interactions, i.e. they contain either m^* if dealing with a normal Fermi liquid of v_g^* if dealing with the Fermi liquid phase of a **DM**. An interesting feature of this derivation, as well as other calculations for equilibrium properties within **FLT** is that the expressions remain the same in both two and three dimensions, albeit with different expressions for the density of states. This is not true for non-equilibrium properties which we derive in the next section and in **chapter 2**.

B.3 Zero Sound & Plasmons using LKE

In order to study the non-equilibrium phenomena of the three-dimensional Fermi liquid, we begin with the LKE¹

$$\frac{\partial n_{\mathbf{p}\sigma}}{\partial t} - \{\varepsilon_{\mathbf{p}\sigma}, n_{\mathbf{p}\sigma}\} = I \tag{B.12}$$

where $\{\cdots, \cdots\}$ are Poisson brackets, $n_{\mathbf{p}\sigma}$ is the quasiparticle distribution function, $\varepsilon_{\mathbf{p}\sigma}$ is the quasiparticle Hamiltonian, and I is the collision integral. Our interest is in small deviations from equilibrim², $\delta \varepsilon_{\mathbf{p}\sigma}$ and $\delta n_{\mathbf{p}\sigma}$, so we write

$$\varepsilon_{\mathbf{p}\sigma}\left(\mathbf{r},t\right) = \varepsilon_{\mathbf{p}\sigma}^{0} + \delta\varepsilon_{\mathbf{p}\sigma}\left(\mathbf{r},t\right)$$
$$n_{\mathbf{p}\sigma}\left(\mathbf{r},t\right) = n_{\mathbf{p}\sigma}^{0} + \delta n_{\mathbf{p}\sigma}\left(\mathbf{r},t\right)$$

¹It's important to note that $\varepsilon_{\mathbf{p}\sigma}$ and $n_{\mathbf{p}\sigma}$ have spatial and time dependence. But in the spirit of conserving space, time, and my sanity, I'm dropping the notation.

 $^{^{2}}$ Note that the spatial dependence is completely in the small deviation. This is achieved by setting a local equilbrium, denoted by the superscript 0, where everything is measured from and greatly simplifies calculations

To further simplify calculations, we linearize the **LKE** keeping terms linear in $\delta n_{\mathbf{p}\sigma}$ and ignoring anything of higher order. This leads to

$$\frac{\partial n_{\mathbf{p}\sigma}}{\partial t} - \mathbf{v}_{\mathbf{p},\sigma} \cdot \nabla_r \left(\delta \varepsilon_{\mathbf{p}\sigma} \frac{\partial n_{\mathbf{p}\sigma}^0}{\partial \varepsilon_{\mathbf{p}\sigma}^0} - \delta n_{\mathbf{p}\sigma} \right) = I$$
(B.13)

at this point it is instructive to point out this formalism differs from the traditional Boltzman theory. As per Baym & Pethick [2] and Bedell [13], the differences from Boltzman theory are (1) in the possible spatial and time dependence of the **qp** velocity $\mathbf{v_p}$ and (2) in the effective field contributions that are seen in the $\delta \varepsilon_{\mathbf{p}\sigma}(\mathbf{r},t)$ term³. While the space and time dependences in the **qp** velocity only arise in noinlinear deviations from equilbrium, the effective field effects are responsible for the unique transport phenomena in Fermi liquids (as we'll see shortly with zero sound).

Proceeding along, we Fourier transform eqn.(B.13) and are left with

$$\left(\omega - \mathbf{q} \cdot \mathbf{v}_{\mathbf{p}\sigma}\right) \delta n_{\mathbf{p}\sigma} \left(\mathbf{q}, \omega\right) + \left(\mathbf{q} \cdot \mathbf{v}_{\mathbf{p}\sigma}\right) \frac{\partial n_{\mathbf{p}\sigma}^{0}}{\partial \varepsilon_{\mathbf{p}\sigma}^{0}} \delta \varepsilon_{\mathbf{p}\sigma} \left(\mathbf{q}, \omega\right) = iI$$
(B.14)

which is the form of the **LKE** that we will deal with going further. We should note that the treatment thus far has been classical. A quantum mechanical version is given in [2] however, as it's stated, the classical and quantum mechanical equations produce identical results in most situations (the one exception being when $\hbar \omega \gtrsim k_B T$). Now using eqn.(B.14) we'll study a unique phenomena predicted by **FLT**, zero sound.

By definition, zero sound is a collision-less sound mode, i.e. in a regime described by $\omega \tau \gg 1$. An alternative viewpoint is considering zero sound to be quantum sound, i.e. zero temperature sound. At zero temperature, no collisions take place. Using either rationale, the collision integral is negligible which allows us to set $I = 0^4$. The idea of collision-less sound almost sounds like an oxymoron as sound, when taught to introductory physics students, is solely dependent on the compression and rarefaction of particles. Collisions are a crucial part of sound. This, however, is what we call first sound and is talked about briefly in the previous section and given in eqn.(B.8). Zero sound is still a density oscillation but it's a direct consequence of the **qp** interaction in the $\delta \varepsilon_{\mathbf{p}\sigma}$ term

$$\delta \varepsilon_{\mathbf{p}\sigma}\left(\mathbf{r},t\right) = U + \sum_{\mathbf{p}',\sigma'} f_{\mathbf{p}\mathbf{p}'}^{\sigma\sigma'} \delta n_{\mathbf{p}'\sigma'}$$

as the interactions cause the density oscillation as opposed to collisions. If the \mathbf{qp} interaction was absent, the excess \mathbf{qp} would diffuse away without driving any elements thus not sustaining a propagating mode.

³This term dictates $\mathbf{q}\mathbf{p}$ drift due to the motion of $\mathbf{q}\mathbf{p}$ in the absence of collisions

⁴This is a great simplification. For other non-equilibrium phenomena, the collision must be separately linearized and taken into consideration. Nevertheless, in spite of setting I = 0, we still see unique effects emerge.

Writing $\delta n_{\mathbf{p}\sigma}$ in terms of Fermi surface distortions $\nu_{\mathbf{p}\sigma}^{5}$, $\delta n_{\mathbf{p}\sigma} \equiv -\frac{\partial n_{\mathbf{p}\sigma}^{0}}{\partial \varepsilon_{\mathbf{p}\sigma}^{0}}\nu_{\mathbf{p}\sigma}$, we can cast eqn.(1.15) in a familiar form as derived by Baym and Pethick [2]

$$\nu_{\mathbf{p}\sigma} + \frac{\mathbf{q} \cdot \mathbf{v}_{\mathbf{p}\sigma}}{\omega - \mathbf{q} \cdot \mathbf{v}_{\mathbf{p}\sigma}} \sum_{\mathbf{p}'\sigma'} f_{\mathbf{p}\mathbf{p}'}^{\sigma\sigma'} \left(\frac{\partial n_{\mathbf{p}'\sigma'}^0}{\partial \varepsilon_{\mathbf{p}'\sigma'}^0} \right) \nu_{\mathbf{p}'\sigma'} = \frac{\mathbf{q} \cdot \mathbf{v}_{\mathbf{p}\sigma}}{\omega - \mathbf{q} \cdot \mathbf{v}_{\mathbf{p}\sigma}} U$$
(B.15)

At low temperature, all momenta are on/close to the Fermi surface so only the direction between momenta matter and we expand $\nu_{\mathbf{p}\sigma}$ in a series of Legendre polynomials similar to what is done for the Fermi liquid parameters⁶

$$\nu_{\mathbf{p}\sigma} = \sum_{l} P_l(\cos\theta) \nu_l$$
$$f^s_{\mathbf{p}\mathbf{p}'} = \sum_{l} P_l(\cos\theta) f^s_l$$

This leaves us with

$$\sum_{l} P_{l}(\cos\theta) \nu_{l} + \frac{qv_{F}\cos\theta}{\omega - qv_{F}\cos\theta} \sum_{\mathbf{p}'\sigma'} \sum_{l''} P_{l''}\left(\hat{p}\cdot\hat{p}'\right) f_{l''}^{s} \left(\frac{\partial n_{\mathbf{p}'\sigma'}^{0}}{\partial\varepsilon_{\mathbf{p}'\sigma'}^{0}}\right) \sum_{l'} P_{l'}\left(\cos\theta'\right) \nu_{l} = \frac{qv_{F}\cos\theta}{\omega - qv_{F}\cos\theta} U$$
(B.16)

To get from eqn.(B.15) to eqn.(B.16) is no trivial and we outline the steps here:

1. Introduce the following variables:

•
$$s = \frac{\omega}{qv_F}$$

•
$$x = \cos \theta = \hat{p} \cdot \hat{q}$$

- $x' = \cos \theta' = \hat{p}' \cdot \hat{q}$
- $x'' = \cos \theta'' = \hat{p} \cdot \hat{p}'$
- 2. Multiply eqn.(B.16) by the Legendre polynomial $P_l(x)$ and integrate each term from -1 to 1 with respect to x
- 3. Exploit the orthogonality relation of the Legendre polynomials
- 4. Use the addition theorem for spherical harmonics (keeping in mind that the m = 0 spherical harmonic is proportional to the Legendre polynomial of the same angular momentum channel)

Then after dealing with the momentum sum in eqn.(B.16) in the standard way (going to the continuum and performing the integration over the applicable phase space), we arrive at

$$\frac{\nu_l}{2l+1} + \sum_{l'} \Omega_{ll'} F_{l'}^s \frac{\nu_{l'}}{2l'+1} = -\Omega_{l0} U \tag{B.17}$$

⁵We may also interpret $\nu_{\mathbf{p}\sigma}$ as teh energy by which $n_{\mathbf{p}\sigma}$ shifts

 $^{^{6}}$ At this point we impose that only the spin symmetric Landau parameter matters as that is the nature of zero sound. If a magnetic field was to be applied, then the antisymmetric term would be considered

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where we defined $\Omega_{ll'}$ as

$$\Omega_{ll'} = \frac{1}{2} \int_{-1}^{1} dx P_l(x) \frac{x}{x-s} P_{l'}(x)$$
(B.18)

To determine the modes of oscillation, we look to poles of the response function ν_l/U , which can now be determined from eqn.(B.17). In general, a pole in the response function at s indicates a mode with dispersion $\omega = sqv_F$. In our case, we would find a sound like dispersion with velocity sv_F .

For zero sound, as a first approximation, we operate in the s-wave channel, i.e. $F_l^s = 0$ for $l \ge 1$. Furthermore, only l = 0 distortions of the Fermi surface contribute and we have

$$\nu_{0} + \sum_{l'} \Omega_{0l'} F_{l'}^{s} \frac{\nu_{l'}}{2l' + 1} = -\Omega_{00} U$$

$$\nu_{0} + \Omega_{00} F_{0}^{s} \nu_{0} = -\Omega_{00} U$$
(B.19)

And the response function can be readily read off

$$\chi(s) = g(0)\frac{\nu_0}{U} = \frac{-\Omega_{00}}{1 + \Omega_{00}F_0^s}g(0)$$
(B.20)

where we're interested in the zeroes of $1 + \Omega_{00} F_0^s$. Our final task is to compute Ω_{00} which is far from trivial. The integral changes depending on the value of s. For s > 1 there is no singularity and the integral is straightforward. For s < 1, a singularity exists which must me moved off the axis so as to not interefere with the counter-clockwise semi-circle contour in the upper half plane. The result is plotted in fig.(B.2) and is given as

$$\Omega_{00}(s) = 1 + \frac{s}{2} \ln \left| \frac{s-1}{s+1} \right| + \frac{i\pi s}{2} \Theta \left(1 - |s| \right)$$

As can be seen from fig.(B.2), Ω_{00} is complex when s < 1 and completely real when s > 1. This tells us that for s < 1, the zero sound mode is damped as it sits in the particle-hole continuum. Furthermore, if $F_0^s < 0$, we again see no purely real solutions and can conclude that in the presence of an attractive interaction, the zero sound mode is damped⁷. The damping found in both these situations is Landau damping: damping due to the interaction between collective modes and the particle-hole excitations. So for the remaining discussion, we set s > 1 and $F_0^s > 0$ and solve the following expression for s:

$$\Omega_{00} = 1 + \frac{s}{2} \ln \left| \frac{s-1}{s+1} \right| = -\frac{1}{F_0^s}$$

To determine the modes of propagation, we study different limits of F_0^{s8} :

• $F_0^s \ll 1$

 $^{^{7}}$ This is vastly different from what happens in a two-dimensional Fermi liquid. For more details see **chapter** 2 and/or [63]

⁸In general, a closed expression for s cannot be found for an arbitrary F_0^s . The equation can be solved numerically



FIGURE B.2: A plot of Ω_{00} as a function of s in a three-dimensional Fermi liquid

For these weaker interactions, $\frac{-1}{F_0^s} \to -\infty$. From fig.(B.2), this corresponds to $s \to 1$ from the right:

$$1 + \frac{s}{2}\ln\left(\frac{s-1}{s+1}\right) = -\frac{1}{F_0^s}$$
$$s\ln\left(\frac{s-1}{s+1}\right) = -2 - \frac{2}{F_0^s}$$

As $s \to 1$, s and $\ln(s+1)$ approach finite values while $\ln(s-1)$ approaches infinity and is the dominant term. Therefore we can approximate the above expression as

$$\ln\left(\frac{s-1}{2}\right) = -2 - \frac{2}{F_0^s}$$
(B.21)

which is now trivial to solve for s:

$$s = 1 + 2\exp\left[-2\left(1 + \frac{1}{F_0^s}\right)\right] \tag{B.22}$$

we should note here that this result is slightly different from that seen in most literature. Frequently the factor of 2 in the square brackets is omitted since $1/F_0^s$ is the dominant term. This does not appreciably change the behavior of the mode.

• $F_0^s \gg 1$

For these stronger interactions, $\frac{-1}{F_0^s} \to 0$. This corresponds to $s \to \infty$:

$$1 + \frac{s}{2} \ln\left(\frac{s-1}{s+1}\right) = -\frac{1}{F_0^s}$$
$$1 + \frac{s}{2} \ln\left(\frac{1-\frac{1}{s}}{1+\frac{1}{s}}\right) = -2 - \frac{2}{F_0^s}$$

In the form given in the line above, the $\frac{1}{s}$ terms are small and we may Taylor expand each natural log keeping up to third order

$$1 + \frac{s}{2}\left(-\frac{2}{s} - \frac{2}{3s^3}\right) = -\frac{1}{F_0^s}$$

which can be easily solved for s

$$s = \sqrt{\frac{F_0^s}{3}} \tag{B.23}$$

In both expressions for s, we see that the speed of zero sound, $c_0 = sv_F$, is larger than the Fermi velocity which is to be expected.

A natural question to ask is: what happens if higher order Landau parameters are kept? One can see that keeping additional terms in eqn.(B.19) simply leads to a polynomial that we need to solve for s. Unfortunately, this high order polynomial is a cumbersome expression which generally is solved numerically. Additionally, the inclusion of Landau parameters $l \ge 2$ leads to the possibility of multiple velocity values satisfying the zero sound dispersion. Regardless, the addition of higher order Landau parameters does not lead to substantial changes in the behavior of zero sound.

The analysis done for zero sound is relevant for a neutral Fermi liquid. While this is possible in liquid helium, most systems are charged in which case a plasmon mode replaces the traditional zero sound mode. A plasmon mode is similar to a zero sound mode in that they are both collective density oscillations, the difference of course being the plasmon takes into account the effects of charge and subsequently the Coulomb interaction. While Baym & Pethick do not discuss the charged Fermi liquid in [2], Pines does in [12] albeit without much quantitative discussion. Here summarize the description of Pines and go a step further and calculate the plasmon frequency.

To see the emergence of plasmons, we apply an external electric field \mathbf{D} to the charged Fermi liquid. This polarizes the system and gives rise to an induced electric field \mathbf{E}_p . The sum of these two fields, \mathbf{D} and \mathbf{E}_p must be included in the **LKE** (specifically, they are included in the driving term U in $\delta \varepsilon_{\mathbf{p}\sigma}$). Lucky for us, the machinery for deriving the plasmon frequency is the same as the machinery used to tackle zero sound developed earlier with a slight change: the Fermi liquid parameter F_0^s is adjusted to include the Coulomb interaction⁹

$$F_0^s \to \tilde{F}_0^s = g(0) \left(\frac{4\pi e^2}{q^2} + f_0^s\right)$$

We can then take this "charged" Fermi liquid parameter, insert it into eqn.(B.17) and proceed as we did earlier by finding the poles of the response function. Ordinarily we're interested in the long wavelength limit for plasmons. This corresponds to a very large $s = \omega/qv_F$ value. Using

⁹Highler order Landau parameters are not shifted by the Coulomb interaction this way

the analysis above eqn.(B.23), we get

$$1 + \frac{s}{2} \left(-\frac{2}{s} - \frac{2}{3s^3} \right) = -\frac{1}{\tilde{F}_0^s}$$

where again we have made the simplification that only the l = 0 Landau parameter contributes. Solving for the frequency explicitly, we arrive at

$$\omega^2 = \omega_p^2 + \frac{1}{3} F_0^s q^2 v_F^2 \tag{B.24}$$

where $\omega_p^2 = 4\pi n e^2/m$ is the RPA plasmon mode frequency. If we were to include the F_1^s Landau parameter,

$$\nu_0 + \left(\Omega_{00} F_0^s \nu_0 + \Omega_{01} F_1^s \frac{\nu_1}{3}\right) = -\Omega_{00} U$$

This equation is appreciably more difficult to deal with as we first must substitute for ν_1 and Ω_{01} before continuing as we previously did when deriving eqn.(B.24). First we replace ν_1 using the l = 0 moment of eqn. (B.17)

$$s\nu_0 = \frac{\nu_1}{3} \left(1 + \frac{1}{3}F_1^s \right)$$

which is a manifestation of the **qp** number conservation law. Then, we re-write Ω_{01} as $s\Omega_{00}$ as per Baym and Pethick [2]. Finally, since we're interested in the long wavelength plasmon, we impose s being very large which leads to a simplification in Ω_{00} :

$$\Omega_{00}\approx -\frac{1}{3s^2}$$

and the mode becomes

$$\omega^{2} = \left(\omega_{p}^{2} + \frac{1}{3}F_{0}^{s}q^{2}v_{F}^{2}\right)\frac{m^{*}}{m}$$
(B.25)

where $m^*/m = \left(1 + \frac{1}{3}F_1^s\right)$. There are a few unique properties one should notice:

- 1. Both eqn.(B.24) and eqn.(B.25) have a second term that is additive to ω_p . This term is the zero sound mode eqn.(B.23) provided $F_0^s \gg 1$.
- 2. Both eqn.(B.24) and eqn.(B.25) have Fermi liquid correction, F_0^s , and eqn.(B.25) has an additional F_1^s correction capturing the effective mass.
- 3. While eqn.(B.25) includes the effect of F_1^s , we should note that the RPA plasmon term remains unchanged. The mass term in the particle density n must have the same effect as the bare mass as per Luttinger's theorem.

Item 1 above is intriguing in that it shows a possible crossover from zero sound to plasmon as a function of wavelength q. This apparent crossover may be of interest in a **DM** where, as a function of doping or gating, the chemical potential μ may be adjusted. As μ is lowered, we go from a charged to neutral system where the collective mode behavior should crossover from a plasmon (charged Fermi liquid) to zero sound (neutral Fermi liquid).

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