

Hyperbolic 3-manifolds of infinite type:

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Hyperbolic 3-manifolds of Infinite type

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A dissertation
submitted to the Faculty of
the department of Mathematics
in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy

Boston College
Morrissey College of Arts and Sciences
Graduate School

March 2019

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Abstract: In this thesis we study the class of 3-manifolds that admit a compact exhaustion by hyperbolizable 3-manifolds with incompressible boundary and such that the genus of the boundary components of the elements in the exhaustion is uniformly bounded. For this class we give necessary and sufficient topological conditions that guarantee the existence of a complete hyperbolic metric.

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Acknowledgments

In the last five years I was lucky enough to meet and interact with many wonderful mathematicians without whom I would not be the researcher I am today. First and foremost I would like to thank my advisors Ian Biringer and Martin Bridgeman for their patience, especially in reading drafts, and their continuous guidance. I would also like to extend my gratitude to Juan Souto for introducing me to this thesis topic and for countless stimulating conversations.

Thanks are also in order to my housemates Matt, Risa and Tee who kept me sane over the last two years.

Chapter 1

Introduction

A question of interest in low-dimensional topology is whether a manifold M admits constant curvature geometric structures and what is the interplay between the geometry and the topology of M . In low-dimensions people have been particularly interested in *hyperbolic* manifolds that is manifolds admitting metrics of constant sectional curvature -1 . Necessary and sufficient topological conditions for the existence of a complete hyperbolic metric in the interior of a compact 3-manifold have been known since Thurston's proof that the interior of every atoroidal Haken 3-manifold is hyperbolizable (1982, [34]). The result was a step in Thurston's program on the study of geometric structures on 3-manifolds, known as the Geometrization conjecture, which was later completed by Perelman (2003, [41, 42, 43]). These results give a topological characterization of compact 3-manifolds admitting complete hyperbolic metrics in their interiors. On the other hand, by the Tameness Theorem (2004, [1, 9]) hyperbolic 3-manifolds with finitely generated fundamental group are *tame*, that is they are homeomorphic to the interiors of compact 3-manifolds. By combining Geometrization and the Tameness Theorem we obtain a complete topological characterisation of hyperbolizable finite type 3-manifolds. We have that an irreducible finite type 3-manifold M is hyperbolizable if and only if M is the interior of a compact atoroidal 3-manifold \overline{M} with infinite fundamental group. In this thesis, we are concerned with the study of *infinite-type* 3-manifolds. Some interesting examples of infinite-type 3-manifolds are Antoine's necklace [2], i.e. an example of a non-tame embedding¹ of a Cantor set in \mathbb{S}^3 , and Whitehead manifolds [61, 62] which were the first examples of non-tame open 3-manifolds. Throughout this work, M is always an oriented, aspherical 3-manifold. A 3-manifold M is said to be *hyperbolizable* if it is homeomorphic to \mathbb{H}^3/Γ for $\Gamma \leq \text{Isom}(\mathbb{H}^3)$ a discrete, torsion free subgroup isomorphic to $\pi_1(M)$.

¹The complement of the embedding in \mathbb{S}^3 is not tame.

Geometric structures on infinite-type 3-manifolds are not widely studied. In particular, not much is known about the topology of hyperbolizable infinite-type 3-manifolds. Nevertheless, some interesting examples of such 3-manifolds are known (see [7, 53, 57]). In [57], they arise as geometric limits of quasi-Fuchsian hyperbolic 3-manifolds. In [7], the authors constructed infinite-type 3-manifolds by gluing together collections of hyperbolic 3-manifolds with bounded combinatorics via complicated pseudo-Anosov maps. An essential element of their proof is the model geometry developed to prove the Ending Lamination Conjecture [8, 39]. The boundedness comes from gluing together manifolds from a finite list of hyperbolizable 3-manifolds with incompressible boundary. Other examples arise in [53] as gluings of acylindrical hyperbolizable 3-manifolds with incompressible boundary and such that their boundary components have uniformly bounded genus.

There are certain obvious obstructions to the existence of a complete hyperbolic metric. Indeed, let $M \cong \mathbb{H}^3/\Gamma$ be an hyperbolizable 3-manifold, then by [23] Γ has no divisible subgroups (see Definition 5.2.1), hence neither does $\pi_1(M)$. Moreover, by definition covering spaces of hyperbolizable manifolds are hyperbolizable as well. We say that a manifold M is *locally hyperbolic* if every covering space $N \twoheadrightarrow M$ with $\pi_1(N)$ finitely generated is hyperbolizable.

Given the known obstructions and inspired by the examples [7, 53], we introduce the class \mathcal{M}^B , where B stands for bounded, of 3-manifolds M so that:

- (i) M admits a nested compact exhaustion $\{M_n\}_{n \in \mathbb{N}}$ by hyperbolizable 3-manifolds;
- (ii) for all $n \in \mathbb{N}$, the submanifold M_n has incompressible boundary in M so that $\pi_1(M_n)$ injects into $\pi_1(M_{n+1})$;
- (iii) each component S of ∂M_n has uniformly bounded genus, that is $\text{genus}(S) \leq g = g(M) \in \mathbb{N}$.

We denote by \mathcal{M} the class of 3-manifolds satisfying (i) and (ii). It is natural to address hyperbolization questions in this class since, by (i) and (ii), every $M \in \mathcal{M}$ is locally hyperbolic. Moreover, one can also show that for every manifold $M \in \mathcal{M}^B$ $\pi_1(M)$ does not contain any divisible subgroup (see Remark 4.3.19). Therefore, it is meaningful to look for a characterisation of hyperbolizable manifolds in \mathcal{M}^B . Since \mathcal{M}^B already contains hyperbolizable 3-manifolds, namely the ones in [7, 53], a first question is whether there exists non-hyperbolizable 3-manifolds in \mathcal{M}^B . In [16] we built an example $M_\infty \in \mathcal{M}^B$ answering the following question of Agol [17, 36]:

Question (Agol). Is there a 3-dimensional manifold M with no divisible subgroups in $\pi_1(M)$ that is locally hyperbolic but not hyperbolic?

However, the 3-manifold M_∞ is homotopy equivalent to a complete hyperbolic 3-manifold. In [15] we improved the above example by building a 3-manifold $N \in \mathcal{M}^B$ such that N is not homotopy equivalent to any complete hyperbolic 3-manifold.

The main result of this paper is a complete topological characterisation of hyperbolizable manifolds in \mathcal{M}^B . Before stating the result we need to introduce some objects and notation.

For all $M \in \mathcal{M}$, we construct a canonical *maximal* bordified manifold $(\overline{M}, \partial\overline{M})$, see Definition 4.1.2, where each component of $\partial\overline{M}$ is a surface, not necessarily of finite type nor closed. To construct \overline{M} , we compactify properly embedded π_1 -injective submanifolds of the form $S \times [0, \infty)$ by adding $\text{int}(S) \times \{\infty\}$ to M . The bordification \overline{M} only depends on the topology of M . Specifically, we have that $\text{int}(\overline{M})$ is homeomorphic to M , and that any two maximal bordifications for M are homeomorphic. We then say that an essential annulus $(\mathbb{A}, \partial\mathbb{A}) \rightarrow (\overline{M}, \partial\overline{M})$ is *doubly peripheral* if both components of $\partial\mathbb{A}$ are peripheral in $\partial\overline{M}$.

Our main result is:

Theorem 1. Let $M \in \mathcal{M}^B$. Then, M is homeomorphic to a complete hyperbolic 3-manifold if and only if the associated maximal bordified manifold \overline{M} does not admit any doubly peripheral annulus.

1.1 Topology of manifolds in \mathcal{M}

For all $M \in \mathcal{M}$, we construct a canonical *maximal* bordified manifold $(\overline{M}, \partial\overline{M})$ where each component of $\partial\overline{M}$ is a, not necessarily of finite type, punctured surface. The bordification \overline{M} only depends on the topology of M , specifically we have that $\text{int}(\overline{M})$ is homeomorphic to M and any two maximal bordifications for M are homeomorphic.

To obtain a bordification \overline{M} of M we compactify properly embedded π_1 -injective products submanifolds $\mathcal{P} : F \times [0, \infty) \hookrightarrow M$, where F is a connected compact surface, by adding the boundary at infinity $\text{int}(F) \times \{\infty\}$ to M . Since we want to have a notion of a “largest”, or in some sense *maximal*, bordification we don’t want to consider products \mathcal{P} in which the base surface is a disk \mathbb{D}^2 . This is because by compactifying properly embedded rays to any bordification \overline{M} we can always join a new \mathbb{D}^2 boundary component. Similarly given a bordification \overline{M} with components open boundary S_1, S_2 we do not want two cusps $C_1 \subseteq S_1, C_2 \subseteq S_2$ to be the side boundaries of a properly embedded submanifold N of the form $(\mathbb{S}^1 \times I) \times [0, \infty)$ since then we can compactify N joining S_1 and S_2 along the cusps to get a new bordified manifold \overline{M}' such that $(\overline{M}, \partial\overline{M}) \hookrightarrow (\overline{M}', \partial\overline{M}')$.

Thus, we can describe a bordification as a pair \overline{M} and a marking homeomorphism $\iota : M \rightarrow \text{int}(\overline{M})$. We also require that bordifications (\overline{M}, ι) have no disk components in $\partial\overline{M}$ and that no two

boundary components S_1, S_2 of $\partial\overline{M}$ contain two cusps neighbourhoods $C_1 \subseteq S_1$ and $C_2 \subseteq S_2$ that co-bound with an annulus C connecting ∂C_1 and ∂C_2 as submanifold of the form $(\mathbb{S}^1 \times I) \times [0, \infty)$.

Definition 1.1.1. Given $M \in \mathcal{M}$ we say that a pair (\overline{M}, ι) , for \overline{M} a 3-manifold with boundary and $\iota : M \rightarrow \text{int}(\overline{M})$ a marking homeomorphism, is a *bordification* for M if the following properties are satisfied:

- (i) $\partial\overline{M}$ has no disk components and every component of $\partial\overline{M}$ is incompressible;
- (ii) there is no properly embedded manifold

$$\psi : (\mathbb{A} \times [0, \infty), \partial\mathbb{A} \times [0, \infty)) \hookrightarrow (\overline{M}, \partial\overline{M})$$

for A an annulus.

Moreover, we say that two bordifications $(\overline{M}, f), (\overline{M}', f')$ are equivalent $(\overline{M}, f) \sim (\overline{M}', f')$ if we have a homeomorphism $\psi : \overline{M} \xrightarrow{\cong} \overline{M}'$ that is compatible with the markings, that is: $\psi|_{\text{int}(\overline{M})} \stackrel{\text{iso}}{\cong} f' \circ f^{-1}$. We denote by $\mathbf{Bor}(M)$ the set of equivalence classes of bordified manifolds

To obtain a *maximal bordification* \overline{M} of $M \in \mathcal{M}$, one takes a maximal collection $\mathcal{P}_{max} \doteq \{\mathcal{P}_n\}_{n \in \mathbb{N}}$ of properly embedded, pairwise disjoint π_1 -injective product submanifolds $\mathcal{P}_n : F_n \times [0, \infty) \hookrightarrow M$ such that no \mathcal{P}_n is isotopic into \mathcal{P}_m for $n \neq m$. Then, we have:

Theorem 4. Let $M \in \mathcal{M}^B$ be an open 3-manifold. Then, there exists a unique maximal bordification $[(\overline{M}, \iota)] \in \mathbf{Bor}(M)$.

This maximal bordification will have the key property that all products submanifolds of the form $F \times [0, \infty)$ that are properly embedded in \overline{M} are properly isotopic into collar neighbourhoods of $\partial\overline{M}$.

Thus, to every $M \in \mathcal{M}$ we can naturally associate a 3-manifold with incompressible boundary \overline{M} . Then, for \overline{M} one wants to recover a notion of characteristic submanifold, extending results of Johansson and Jaco-Shalen [28, 32]. Intuitively a *characteristic submanifold* is the minimal codimension zero submanifold N of M that contains, up to homotopy, all essential annuli and tori of M . In our setting we define:

Definition 1.1.2. Given a 3-manifold $M \in \mathcal{M}$ let $(\overline{M}, \partial\overline{M})$ be the maximal bordification, which could be M itself, then we define the *characteristic submanifold* $(N, R) \hookrightarrow (\overline{M}, \partial\overline{M})$ to be a codimension zero submanifold satisfying the following properties:

(i) every $\Sigma \in \pi_0(N)$ is homeomorphic to either:

- an I -bundle over a compact surface with $\Sigma \cap \partial\overline{M}$ the lids of the I -bundle;
- a solid torus $V \cong \mathbb{S}^1 \times \mathbb{D}^2$ with $V \cap \partial\overline{M}$ a collection of finitely many parallel annuli or a non-compact submanifold V' that compactifies to a solid torus such that $V' \cap \partial\overline{M}$ are infinitely many annuli;
- a thickened boundary torus $T \cong \mathbb{T}^2 \times [0, 1]$ such that $T \cap \partial\overline{M}$ is an essential torus and a, possibly empty, collection of annuli that are parallel in ∂T or a non-compact manifold T' that compactifies to a thickened boundary torus such that $T' \cap \partial\overline{M}$ is an essential boundary torus and infinitely many annuli;

(ii) $\partial N \cap \partial\overline{M} = R$;

(iii) all essential maps of an annulus $(\mathbb{S}^1 \times I, \mathbb{S}^1 \times \partial I)$ or a torus \mathbb{T}^2 into $(\overline{M}, \partial\overline{M})$ are homotopic as maps of pairs into (N, R) ;

(iv) N is minimal i.e. there are no two components of N such that one is homotopic into the other.

and we prove:

Theorem 5. The maximal bordification $(\overline{M}, \partial\overline{M})$ of $M \in \mathcal{M}$ admits a characteristic submanifold (N, R) and any two characteristic submanifolds are properly isotopic.

The characteristic submanifold (N, R) of \overline{M} is obtained by studying how the characteristic submanifold (N_n, R_n) of each compact piece $(\overline{M}_n, \partial\overline{M}_n)$ (see [31, 32]) change as we go through the exhaustion. We construct the characteristic submanifold (N, R) by taking the components Σ of the various (N_n, R_n) that remain “essential” throughout the exhaustion. By this we mean that for all $m \geq n$ Σ is isotopic in M_m to an essential submanifold of a component of N_m .

Given a characteristic submanifold N of \overline{M} we can make sense of the condition in Theorem 1 by looking at the characteristic submanifold of the maximal bordification \overline{M} . By a *doubly peripheral annulus* C we mean a properly embedded annulus C in \overline{M} such that both boundary components of C are peripheral in the components of $\partial\overline{M}$ containing them. Then we only need to check the presence of doubly peripheral annuli in the characteristic submanifold N of \overline{M} .

1.1.1 Hyperbolicity of manifolds in \mathcal{M}^B

Once we have completed the topological constructions we can show how the annulus condition is an obstruction to hyperbolicity.

Definition 1.1.3. We say that an essential annulus $C : (\mathbb{S}^1 \times I, \mathbb{S}^1 \times \partial I) \rightarrow (\overline{M}, \partial \overline{M})$ is *doubly peripheral* if both components of $C(\mathbb{S}^1 \times \partial I)$ are peripheral in $\partial \overline{M}$.

Then our characterisation becomes:

Theorem 6. A 3-manifold $M \in \mathcal{M}^B$ admits a complete hyperbolic metric if and only if the associated maximal bordification \overline{M} does not contain any essential doubly peripheral cylinder.

By using arguments similar to the ones developed in [16] we show:

Proposition. If $M \in \mathcal{M}^B$ is hyperbolizable, then \overline{M} cannot have a properly embedded doubly peripheral essential annulus C .

The proof of Theorem 1 follows from ideas developed in [53]. For simplicity we will now describe the case where \overline{M} is acylindrical so that the characteristic submanifold is empty. This, is obviously the case when the M_i are acylindrical themselves, as in [53], but one can also make examples in which the M_n have non-trivial characteristic submanifold N_i . However, if \overline{M} is acylindrical for all i there exists n_i such that M_i is contained in the *acylindrical part* of M_{n_i} i.e. we have that $M_i \subseteq \overline{M_{n_i}} \setminus N_{n_i}$.

Then, by choosing hyperbolic structures $\rho_i : \pi_1(\overline{M}_i) \rightarrow PSL_2(\mathbb{C})$ on all the \overline{M}_i 's and using the fact that for a pared acylindrical finite type hyperbolic 3-manifold (M, \mathcal{P}) the algebraic topology $AH(M, \mathcal{P})$ is compact, see [56, 7.1], we get that the sequences: $\{\rho_j|_{\pi_1(M_i)}\}_{j \geq n_i}$ have converging subsequences. Then by picking diagonal subsequences we obtain:

Theorem 7. Given a manifold $M \in \mathcal{M}$, if the maximal bordification \overline{M} is acylindrical then there exists a hyperbolic 3-manifold N and a homotopy equivalence $f : M \rightarrow N$.

Then to conclude the proof of the main Theorem 6 we show:

Theorem 8. Given $M \in \mathcal{M}^B$ and $\varphi : \overline{M} \rightarrow N$ a homotopy equivalence with N a complete hyperbolic manifold. If \overline{M} is acylindrical, then we have a homeomorphism $\psi : M \rightarrow N$ homotopic to φ .

1.2 Organisation

The remainder of this thesis is organised into three chapters. The first, Chapter 2, serves mainly to summarise some well-known results about the geometry and topology of 3-manifolds and recall

some facts and properties of hyperbolic 3-manifolds. In Chapter 3 we construct our two examples of locally hyperbolic 3-manifold one of which is not hyperbolizable and the other is not homotopy equivalent to any complete hyperbolic 3-manifold.

In Chapter 4 we study 3-manifolds in \mathcal{M} and prove the hyperbolization result. After constructing a canonical bordification and showing existence and uniqueness of a characteristic submanifold we finally prove Theorem 6 for manifolds in \mathcal{M}^B .

Chapter 2

Background

In this chapter we recall some well-known facts about the topology of 3-manifolds and the geometry of hyperbolic manifolds.

2.1 Notation and Conventions

We use \cong for homeomorphic, \simeq for homotopic and $\stackrel{\text{iso}}{\simeq}$ for properly isotopic. By $S \hookrightarrow M$ we denote an embedding of S into M while $S \looparrowright M$ denotes an immersion. By a proper embedding $(S, \partial S) \hookrightarrow (M, \partial M)$ we mean an embedding of S in M mapping boundary to boundary and such that the pre-image of compact sets is compact. All appearing 3-manifolds are assumed to be aspherical and orientable.

By $\Sigma_{g,n}$ we denote an orientable surface of genus g with n boundary components. We say that a manifold is closed if it is compact and without boundary. Unless otherwise stated we use $I = [0, 1]$ to denote the closed unit interval and generally by A we denote an annulus $A \cong \mathbb{S}^1 \times I$ and with \mathbb{T}^2 a torus. By $\pi_0(M)$ we denote the set of connected components of M .

Given an open manifold M by an *exhaustion* $\{M_i\}_{i \in \mathbb{N}}$ we mean a nested collection of compact submanifolds $M_i \subseteq \text{int}(M_{i+1})$. By *gaps* of an exhaustion $\{M_i\}_{i \in \mathbb{N}}$ we mean the connected components of $\overline{M_i} \setminus \overline{M_{i-1}}$. Given a manifold with non-empty incompressible boundary $(M, \partial M)$ and a π_1 -injective embedding $\iota : N \hookrightarrow M$ we have a decomposition of $\iota(\partial N)$ into two submanifolds meeting along simple closed curves. These two complementary submanifolds are the *outer boundary*: $\partial_{\text{out}} N \doteq \iota(\partial N) \cap \partial M$ and the closure of the complement: $\partial_{\text{int}} N \doteq \overline{\iota(\partial N) \setminus \partial_{\text{out}} N}$ which we call the *interior boundary*.

2.2 Some 3-manifold Topology

We now recall some facts and definitions about 3-manifold topology. For more details on the topology of 3-manifolds some references are [25, 26, 28].

Let M be an orientable 3-manifold, then M is said to be *irreducible* if every embedded sphere \mathbb{S}^2 bounds a 3-ball \mathbb{B}^3 . Given a connected properly immersed surface $S \looparrowright M$ we say it is π_1 -*injective* if the induced map on the fundamental groups is injective. Furthermore, if $S \hookrightarrow M$ is embedded and π_1 -injective we say that the surface S is *incompressible* in M . By the Loop Theorem [26, 28] if $S \hookrightarrow M$ is a two-sided surface that is not incompressible we have that there is an embedded disk $D \hookrightarrow M$ such that $\partial D = D \cap S$ and ∂D is non-trivial in $\pi_1(S)$. Such a disk is called a *compressing* disk.

An irreducible 3-manifold with boundary $(M, \partial M)$ is said to have *incompressible boundary* if every map of a disk: $(D, \partial D) \hookrightarrow (M, \partial M)$ is homotopic via maps of pairs into ∂M . Therefore, a manifold $(M, \partial M)$ has incompressible boundary if and only if each component S of ∂M is incompressible.

Definition 2.2.1. We say that an open 3-manifold M is *tame* if it is homeomorphic to the interior of a compact 3-manifold \overline{M} .

Definition 2.2.2. Given an irreducible, open 3-manifold M we say that a codimension zero submanifold $N \xhookrightarrow{\iota} M$ forms a *Scott core* if the inclusion map is a homotopy equivalence.

If M is an orientable irreducible 3-manifold such that $\pi_1(M)$ is finitely generated we have that a Scott core exists and is unique up to homeomorphism:

Theorem 2.2.3 (Scott's Core, [44, 48, 49]). Let M be an orientable 3-manifold with $\pi_1(M)$ finitely generated, then there exists an embedded compact submanifold $\iota : C \hookrightarrow M$ such that ι is an homotopy equivalence. Moreover, any two cores are homeomorphic.

This Theorem has two important corollaries: one is that finitely generated 3-manifold groups are finitely presented. The second is that if M has $\pi_1(M)$ finitely generated then it has finitely many ends and the components of ∂C give a bijection with the ends of M .

Let M be a tame 3-manifold with manifold compactification \overline{M} . Given a Scott core $C \hookrightarrow M \subseteq \overline{M}$ with incompressible boundary we have that by Waldhausen's cobordism Theorem [60, 5.1] every component of $\overline{M} \setminus C$ is a product submanifold homeomorphic to $S \times I$ for $S \in \pi_0(\partial C)$. We also have a relative version:

Lemma 2.2.4. Let $\iota : (N, A) \rightarrow (M, R)$ be a Scott core for a non-compact irreducible 3-manifold (M, R) that admits a manifold compactification \overline{M} with $R \subseteq \partial\overline{M}$. If ∂N is incompressible in M rel R then $M \cong \text{int}(N)$ and $N \cong \overline{M}$.

Proof. Consider a component U in $\overline{M} \setminus N$. Then U corresponds to an end of M and since $(N, A) \hookrightarrow (M, R)$ is a homotopy equivalence we have that there exists $S \in \pi_0(\partial N \setminus A)$ facing U . Since $\pi_1(S)$ surjects onto $\pi_1(U)$ and S is incompressible in M by Waldhausen's cobordism Theorem [60, 5.1] we get that $U \cong S \times I$ and the result follows. \blacksquare

Definition 2.2.5. Given an open 3-manifold M with Scott core $N \hookrightarrow M$ we say that an end $E \subseteq \overline{M} \setminus N$ is *tame* if it is homeomorphic to $S \times [0, \infty)$ for $S \cong \partial E \in \pi_0(\partial N)$. For the core N we say that a surface $S \in \pi_0(\partial N)$ *faces the end* E if E is the component of $\overline{M} \setminus C$ with boundary S .

It is a known fact that if an end E is exhausted by submanifolds homeomorphic to $S \times I$ then E is a tame end.

Example 2.2.6. Not all manifold with a core are tame, see Example 2, 3 of [59].

Finally we say that a properly embedded annulus $(A, \partial A)$ in a 3-manifold $(M, \partial M)$ is *essential* if A is π_1 -injective and it is not boundary parallel, i.e. not isotopic rel ∂A into the boundary. Moreover, a loop γ in a surface $(S, \partial S)$ is similarly said *essential* if it is not homotopic into the boundary and non zero in $\pi_1(S)$.

2.2.1 Haken Manifolds

An important class of 3-manifolds are Haken manifolds:

Definition 2.2.7. A compact, orientable, irreducible 3-manifold M is said to be *Haken* if every connected component contains a properly embedded 2-sided incompressible surface.

Whenever we have a Haken manifold M with incompressible surface $\Sigma \hookrightarrow M$ we can split M along Σ to get a new Haken manifold $M|\Sigma \doteq \overline{M} \setminus N_\epsilon(\Sigma)$, this is still a compact orientable irreducible 3-manifold and push-offs $\partial N_\epsilon(\Sigma)$ are incompressible two sided surfaces.

The irreducibility follows from the following argument: if $\mathbb{S}^2 \hookrightarrow M|\Sigma$ is a two-sided sphere then it bounds a 3-ball in M , if it does not bound a 3-ball in $M|\Sigma$ it means that $\Sigma \hookrightarrow \mathbb{B}^3$ which gives a contradiction with Σ being incompressible. We can repeat this splitting as long as the resulting manifold is Haken, therefore we introduce:

Definition 2.2.8. If M is Haken a *partial hierarchy* for M is a sequence of surfaces $S_0, \dots, S_{n_1} \hookrightarrow M$ and manifolds:

$$M = M_0 \xrightarrow{S_0} M_1 \xrightarrow{S_1} \dots \xrightarrow{S_{n-1}} M_n$$

where $M_i \doteq M_{i-1} | S_i$ and each S_i is a two-sided incompressible surface that is not boundary parallel. If $M_n \cong \coprod \mathbb{B}^3$ then the sequence is called a *hierarchy* for M .

The reason why Haken manifold have been studied is that they admit finite length hierarchies. This allows one to implement induction techniques to prove results about Haken 3-manifolds.

If M has boundary and is irreducible then it is a Haken manifold with a specific hierarchy. If we have some boundary component $\Sigma \subseteq \partial M$ that is compressible then we can find a compressing disk $D \hookrightarrow M$ which is an embedded 2-sided surface, therefore M is Haken. We can then compress along all possible compressing disks to get a new manifold M' with $\partial M'$ incompressible in M' and this M' is linked to M by a partial hierarchy. If $\partial M' \cong \coprod \mathbb{S}^2$ by irreducibility we get that M' is a disjoint union of 3-balls hence we obtain a full hierarchy for M , otherwise we have some incompressible boundary component Σ .

2.2.2 Finiteness and Decomposition Theorems

We will only state the Finiteness theorem for closed incompressible surface but it also works for properly embedded surfaces as long as we also assume them to be boundary incompressible.

Theorem 2.2.9 (Kneser-Haken Finiteness, [21]). Let M be a compact irreducible 3-manifold, then there exists $h \doteq h(M) \in \mathbb{N}$ such that if $\mathcal{S} \doteq S_1 \coprod \dots \coprod S_n$ is a collection of incompressible closed embedded surfaces in M and $n > h$ then at least two of them are parallel.

In the most general setting we can have the surfaces to have boundary, in which case we need them to be ∂ -incompressible as well, otherwise we might incur in the following phenomena:

Example 2.2.10. Let P be a pair of pants then $P \times I \cong H_2$ and if we pick any simple closed curve $\alpha \hookrightarrow P$ we have that the annulus $\alpha \times I$ is not ∂ -incompressible in H_2 in fact we get that $P \times I | \alpha \times I \cong P \times I \coprod H_1$, hence we can create hierarchies of arbitrary length.

Proposition 2.2.11. There is a constant $c = c(M)$ so that if $\{(M_i, S_i)\}_{i \leq n}$ is a partial hierarchy where every S_i is incompressible and boundary incompressible, then there are at most c surfaces that are not disks.

Note that this proposition does not give an a-priori bound on the length of the hierarchy. But as a consequence we get:

Theorem 2.2.12. Let M be Haken, then any maximal partial hierarchy in which every surface is boundary incompressible must terminate.

2.2.2.1 Decomposition Theorems

A 3-manifold M can be decomposed in various ways to obtain a collection of *smaller* 3-manifolds. Theorem 2.2.9 basically gives an existence result of finite decomposition by cutting along surfaces of fixed genus. We now describe the two most important decompositions.

Prime Decomposition. The prime decomposition is the decomposition in which we cut by genus zero surfaces, i.e. spheres \mathbb{S}^2 , a 3-manifold M . By $M_1 \# M_2$ we denote the connected sum of M_1 and M_2 .

Definition 2.2.13. A 3-manifold M is prime if every decomposition $M = M_1 \# M_2$ implies that either M_1 or M_2 are homeomorphic to \mathbb{S}^3 . A *prime decomposition* of M is a decomposition of $M = M_1 \# M_2 \# \dots \# M_k$ such that each M_i is prime.

Theorem 2.2.14 (Prime Decomposition, [25]). Given a closed, oriented 3-manifold M there exists a prime decomposition, unique up to reordering.

Note that the existence result follows directly from the Kneser-Haken finiteness Theorem 2.2.9.

Torus Decomposition Before stating the the Torus decomposition we need to introduce another important class of compact 3-manifolds.

Definition 2.2.15. A compact, irreducible 3-manifold M is a *Seifert Fibered manifold* (SF) if M admits a fibration by circles in which each fiber C has a closed tubular neighbourhood V such that $V \cong \mathbb{S}^1 \times \mathbb{D}^2$ and the fibration on M induces one on V .

Note that if M is Seifert Fibered then ∂M is a collection of Tori and Klein bottles.

Definition 2.2.16. We say that a surface $S \hookrightarrow M$ with M Seifert Fibered is *horizontal* if S is transverse to all fibers and is *vertical* if it is a union of fibers.

An important property of SF-spaces is:

Proposition 2.2.17 ([25, 1.11]). If M is a connected, compact, irreducible Seifert-fibered manifold, then any essential surface in M is isotopic to a surfaces that is either vertical or horizontal.

Theorem 2.2.18 (Torus Decomposition, [25]). Given M a closed, orientable and irreducible 3-manifold there exists a maximal collection of embedded non-parallel incompressible tori T such that each component of $M|T$ is atoroidal or Seifert fibered. This collection is unique up to isotopy.

JSJ or annulus-torus decomposition The torus decomposition generalises to 3-manifold with boundary by adding another type of SF-piece: I -bundles over compact surfaces: $\Sigma \times I$, $\Sigma \tilde{\times} I$ where by $\Sigma \tilde{\times} I$ we denote any non-trivial I -bundle.

Definition 2.2.19. Given a SF-space N and a submanifold $R \subseteq \partial N$ a map $f : (N, R) \rightarrow (M, \partial M)$ is *essential* if it is not homotopic rel $f(R)$ into ∂M .

By the Kneser-Haken finiteness Theorem 2.2.9 we get that there is a maximal collection of annuli and tori \mathcal{A} such that $M|\mathcal{A}$ is a collection of atoroidal and relatively acylindrical¹ 3-manifolds. The issue, as in the torus decomposition, is to show that there is a minimal collection, unique up to isotopy, such that once we split we get SF-pieces and atoroidal, acylindrical components. Proofs of this can be found in [28, 32]. The collection of the SF pieces is called the characteristic submanifold (N, R) of $(M, \partial M)$.

This characteristic submanifold has the property that any essential map of a Seifert fibered space into $(M, \partial M)$ is properly homotopic into (N, R) .

Definition 2.2.20. Given a compact 3-manifold $(M, \partial M)$ with incompressible boundary a *characteristic submanifold* for M is a codimension zero submanifold $(N, R) \hookrightarrow (M, \partial M)$ satisfying the following properties:

- (i) every $(\Sigma, \partial\Sigma) \in \pi_0(N)$ is an I -bundle or a Seifert fibered manifold;
- (ii) $\partial N \cap \partial M = R$;
- (iii) all essential maps of a Seifert fibered manifold S into $(M, \partial M)$ are homotopic as maps of pairs into (N, R) ;
- (iv) N is minimal, that is no component P of N is homotopic into a component Q of N .

¹They are acylindrical relative the annuli in \mathcal{A} , otherwise we enlarge the collection \mathcal{A} contradicting its maximality. Thus, every essential annulus is isotopic into an annulus in \mathcal{A} .

Definition 2.2.21. A *window* in a compact irreducible 3-manifold $(M, \partial M)$ with incompressible boundary is an essential I -subbundle of the characteristic submanifold.

We now state the theorem:

Theorem 2.2.22 (Existence and Uniqueness, [28, 32]). Let $(M, \partial M)$ be a compact 3-manifold with incompressible boundary. Then there exists a characteristic submanifold $(N, R) \hookrightarrow (M, \partial M)$ and any two characteristic submanifolds are isotopic.

Irreducible 3-manifold with incompressible boundary are Haken and one of the main applications of characteristic submanifold theory is to study homotopy types of Haken 3-manifolds. That is, one wants to classify all 3-manifold N that are homotopy equivalent to a given Haken 3-manifold M .

Remark 2.2.23. We conclude by pointing out that if one allows M to be non-compact, in particular of *infinite type* all these decomposition Theorems are false. It is not too hard to construct examples where the family of tori/spheres that one wants to decompose along is not locally finite, see [35].

2.2.3 Homotopy Equivalence of Haken manifolds

Homotopy equivalent 3-manifolds do not need to be homeomorphic for example if we take $P \times \mathbb{S}^1$, for P a pair of pants, and $\tilde{T} \times \mathbb{S}^1$, for \tilde{T} a punctured torus, we get that they are homotopy equivalent but cannot be homeomorphic since they have a different number of boundary components. Similarly we can have hyperbolic examples, see book of I -bundles [14], and for closed examples we can look at Lens spaces $L(p, q)$ which are homotopy equivalent whenever $|p| = |p'|$ and $q'q = \pm \omega^2 \pmod{p}$ while they are homeomorphic if and only if $|p| = |p'|$ and $q' = \pm q^{\pm 1}$ (this can be done by looking at Reidemeister Torsion).

A Theorem of Waldhausen that uses induction on the length of a hierarchy states that homotopic 3-manifolds are homeomorphic if their boundary structure is preserved, this theorem has been generalised Johansson does a complete classification.

Theorem 2.2.24 (Waldhausen). Let M, N be Haken 3-manifolds and $f : (M, \partial M) \rightarrow (N, \partial N)$ injective on π_1 and such that the map on the boundary is injective on each component (automatically true if ∂M is incompressible). Then f is homotopic through maps of pairs to a covering map $g : (M, \partial M) \rightarrow (N, \partial N)$ such that one of the following holds:

- (i) $g : M \rightarrow N$ is a covering map;
- (ii) M is an I -bundle over a closed surface and $g(M) \subseteq \partial N$;

(iii) N and M are both solid tori and $g : M \rightarrow N$ is a branched covering over a circle.

The hypothesis of both previous results can be reduced to an algebraic statement since any isomorphism of the fundamental group of aspherical manifolds is induced by a continuous map, the issue is that it does not have to be induced by a boundary preserving map (see exotic homotopies). For this reason we introduce a peripheral group system for a 3-manifold. Let $\partial M \doteq B_1 \amalg \dots \amalg B_n$ then the *peripheral group system* is the following collection of data:

$$\{\pi_1(M); \eta_i : \pi_1(B_i) \rightarrow \pi_1(M)\}$$

where each η_i is the homomorphism induced by the inclusion maps and for base-point consideration is only determined up to inner automorphism of $\pi_1(M)$.

Given two 3-manifolds M, N with boundary components B_i, C_i we say that a group homomorphism $\varphi : \pi_1(M) \rightarrow \pi_1(N)$ *preserves the peripheral structure* if $\forall 1 \leq i \leq n$ we can find an integer $j(i)$ and a homomorphism $\psi_i : \pi_1(B_i) \rightarrow \pi_1(C_{j(i)})$ with an inner automorphism $\alpha_i : \pi_1(N) \curvearrowright$ such that:

$$\varphi \circ \eta_i = \alpha_i \circ \theta_i \circ \psi_i$$

for η_i, θ_i the inclusion of the boundary components of M and N respectively. Then the collection $\{\varphi; \psi_1, \dots, \psi_n\}$ is called a *homomorphism of peripheral group systems*. If each boundary component is incompressible i.e. if all the boundary inclusion maps are monic we get that a group homomorphism preserves the peripheral structure if for each i we have that $\varphi(\eta_i(\pi_1(B_i)))$ is conjugated to a subgroup of some $\theta_j(\pi_1(C_j))$. Then, we can rephrase theorem 2.2.24 as:

Theorem 2.2.25. Let M, N be Haken manifolds with a monomorphism of peripheral group systems. Then, there is a boundary preserving map $g : (M, \partial M) \rightarrow (N, \partial N)$ inducing the monomorphism such that g satisfies one of the following:

(i) $g : M \rightarrow N$ is a covering map;

(ii) M is an I -bundle over a closed surface and $g(M) \subseteq \partial N$;

(iii) N and M are both solid tori and $g : M \rightarrow N$ is a branched covering over a circle.

2.2.3.1 Johansson Theory and Exotic Homotopy equivalences

By the use of the JSJ decomposition the previous theorems can be generalized. In general if one drops the assumptions on the boundary a homotopy equivalence needs not to be homotopic to a homeomorphism. These homotopy equivalences are called *exotic* and have been extensively studied by Johansson [32].

Example 2.2.26. Let $(M, \partial M)$ be a 3-manifold with incompressible boundary with a marked simple closed curve $\gamma \hookrightarrow \partial M$. Fix a neighbourhood of γ and glue two copies of M along this annulus to get a manifold M_1 . We can also glue two copies of M via the map $h : \mathbb{S}^1 \times I \rightarrow \mathbb{S}^1 \times I$ that flips the interval: $h(x, t) = (x, 1 - t)$ and let the resulting manifold be M_2 . Then $M_1 \simeq M_2$ but in general they are not homeomorphic because they can have non-homeomorphic boundary. For example say that $\partial M \cong \Sigma_3$ and that $\partial M \setminus \gamma \cong \Sigma_{1,1} \amalg \Sigma_{2,1}$, then $\partial M_1 \cong \Sigma_4 \amalg \Sigma_2$ while $\partial M_2 \cong \Sigma_3 \amalg \Sigma_3$.

Definition 2.2.27. Given a homotopy equivalence $f : M \rightarrow N$ and $M' \subseteq M$ we say that f has *singular support* in M' if and only if there exist $N' \subseteq N$ and $f' : M \rightarrow N$ such that:

- (i) $f \simeq f'$;
- (ii) $f'(M') \subseteq N'$;
- (iii) $f'(M \setminus M') \subseteq N \setminus N'$;
- (iv) $f'|_{M'} : M' \rightarrow N'$ is a homotopy equivalence;
- (v) $f'|_{\overline{M \setminus M'}} : \overline{M \setminus M'} \rightarrow \overline{N \setminus N'}$ is a homeomorphism.

We then have the following amazing theorem:

Theorem 2.2.28. Given a homotopy equivalence $f : M \rightarrow N$ between 3-manifolds with incompressible boundary it has singular support in the characteristic submanifolds meeting the boundary of M and N respectively.

Therefore the study of exotic homotopy equivalence is reduced to the study of exotic homotopy equivalence of the components of the JSJ.

Definition 2.2.29. A 3-manifold $(M, \partial M)$ is *acylindrical* if any map $C : (\mathbb{S}^1 \times I, \mathbb{S}^1 \times \partial I) \rightarrow (M, \partial M)$ is homotopic into ∂M . This, is equivalent to M not having any essential cylinders.

Corollary 2.2.30. If M is acylindrical then any homotopy equivalence $f : M \rightarrow N$ is homotopic to a homeomorphism.

The above Theorem has a more general setting, which involves the use of *boundary patterns*.

Definition 2.2.31. A *boundary pattern* for $(M, \partial M)$ is a finite collection of subsurfaces $S_i \hookrightarrow \partial M$, $1 \leq i \leq n$, such that their pairwise intersection are 1-manifolds. A boundary pattern is said *complete* if $\cup_{i=1}^n S_i = \partial M$. The completed boundary pattern for $\{S_i\}_{i=1}^n$ is $\{S_i\}_{i=1}^n \cup \overline{\partial M \setminus \cup_{i=1}^n S_i}$. We denote a boundary pattern for M by \underline{m} .

We then need to consider pair maps that also preserve boundary patterns.

Definition 2.2.32. An *admissible map* is a continuous map $f : (M, \underline{m}) \rightarrow (N, \underline{n})$ such that $\forall S_i \in \underline{m} : f(S_i) \subseteq G_{j(i)} \in \underline{n}$ and

$$\cup_{i=1}^m S_i = \cup_{j=1}^n \{\text{connected components of } f^{-1}(G_j) \cap \partial M\}$$

Practically this means that \underline{m} is induced by \underline{n} via pull-back along f . We then only consider homotopies that respect the boundary patterns i.e. for all $t : h_t$ is admissible, similarly we define isotopies and homotopy equivalences.

One then needs to translate incompressibility into this new setting. This will in turn yield a definition of essential. The reason why this is useful, other than proving the previous theorem, is that one can talk about *bound* and *free* sides. The elements of a given boundary pattern are the bound sides and the complement are the free sides. Then, bound side will be preserved by the homotopy equivalence while the free sides can be changed. In particular one only needs to consider the submanifold of the characteristic submanifold whose boundary is contained in the free sides.

One then needs to define a *useful boundary pattern* which is the equivalent definition of incompressible and boundary incompressible in the boundary pattern setting. For example the completion of the empty boundary pattern is useful if and only if ∂M is incompressible. In order to be precise one has to define small-faced disks.

A *small faced disk* is a disk with boundary pattern having 3 or fewer sides. Then, a boundary pattern is *useful* if every embedded small faced disk has an admissible isotopy to a point.

The Loop Theorem has an analogous in this setting and it gives an alternate definition of useful:

Proposition 2.2.33. Let \underline{m} be a boundary pattern on an irreducible 3-manifold M . Then \underline{m} is

useful if and only if every admissible map of a small-faced disk into (M, \underline{m}) is admissibly homotopic to a constant map.

Waldhausen's Theorem 2.2.24 can then be reformulated as follows:

Theorem 2.2.34. Let (M, \underline{m}) and (N, \underline{n}) be connected irreducible 3-manifolds with complete and useful boundary patterns. If M has non-empty boundary and is not a 3-ball with one or two bound sides. Then any essential map: $f : (M, \underline{m}) \rightarrow (N, \underline{n})$ is admissibly homotopic to a covering map. Moreover if the restriction of f to ∂M is a covering map then the homotopy can be chosen to be constant on ∂M .

The real strength of boundary patterns is that one can show that any homotopy equivalence $f : (M, \underline{m}) \rightarrow (N, \underline{n})$ has singular support in the characteristic submanifold induced by $\underline{m}, \underline{n}$ and Theorem 2.2.28 becomes an application of this result when we consider the empty boundary pattern \emptyset on M and N .

Moreover, by using boundary patterns, one arrives to a complete characterization of exotic homotopy equivalences, which can be summed in combinations of the following examples:

Example 2.2.35.

- $\Sigma_{1,1} \xrightarrow{f} \Sigma_{0,3}$ hence $\Sigma_{1,1} \times I \xrightarrow{f \times \text{id}} \Sigma_{0,3} \times I$ but there is no homeomorphism in the homotopy class since the two manifolds do not have the same number of boundary components.
- Let M_1, M_2 be two 3-manifolds with essential embeddings $f_i : \mathbb{S}^1 \times I \hookrightarrow \partial M_i$. Let $h : \mathbb{S}^1 \times I \xrightarrow{\cong} \mathbb{S}^1 \times I$ given by $h(x, t) = (x, 1 - t)$. Then if we construct $M = M_1 \cup_{f_1} \mathbb{S}^1 \times I \cup_{f_2} M_2$ and $N = M_1 \cup_{f_1 \circ h} \mathbb{S}^1 \times I \cup_{f_2 \circ h} M_2$ we obtain homotopic manifolds but are in general not homeomorphic. The idea is that you are flipping the annulus so you might get different boundary components. This is called a *Dehn flip*.

In particular if M is acylindrical rel \underline{M} we have that M has no characteristic submanifold therefore any homotopy equivalence $f : (M, \underline{m}) \rightarrow (N, \underline{n})$ is homotopic to a homeomorphism. Moreover, any homotopy equivalence is generated by Dehn flips [32, 29.1].

2.2.3.2 Dehn Flips

From example 2.2.26 we define:

Definition 2.2.36. Given an essential properly embedded annulus $(A, \partial A)$ in $(M, \partial M)$ a *Dehn flip* of M along A is the 3-manifold N obtained by cutting M along A picking a homeomorphism

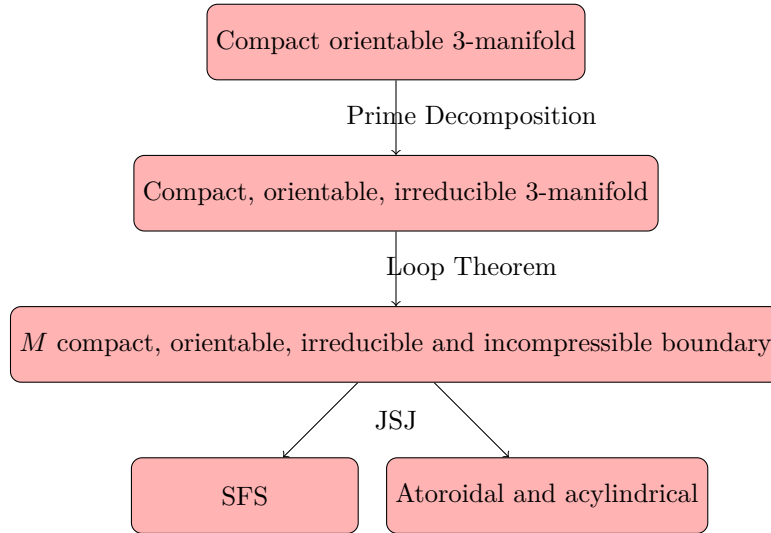
$f : M|A \xrightarrow{\cong} M|A$ that is the identity on A and re-gluing $f(M|A)$ along $f(A)$ either via the identity or via the map $\varphi(x, t) = (x, 1 - t)$ where we parametrised A by $S^1 \times [0, 1] \cong A$.

A Dehn flip of M along A naturally gives a homotopy equivalence $h : M \rightarrow N$ which we will also denote by a *Dehn flip*.

Since homotopy equivalences of M are generated by *Dehn flips* along annuli contained in the boundary of the characteristic submanifold of M , see Theorem [32, 29.1]. As a consequence of Johansson homotopy equivalence theory for Haken 3-manifold we get:

Lemma 2.2.37. If the characteristic submanifold of a Haken 3-manifold M is given by one embedded separating cylinder C then any 3-manifold N homotopy equivalent to M is either homeomorphic to M or to a Dehn flip along C .

To conclude this discussion of finite type manifold if one wants to study compact 3-manifolds one does:



Since SFS are classified one only need to understand M : atoroidal, acylindrical, compact, irreducible 3-manifolds. By a deep Theorem of Thurston these are *hyperbolic*, that is their interior is homeomorphic to \mathbb{H}^3/Γ for $\Gamma \subseteq Isom(\mathbb{H}^3)$ a discrete, torsion free subgroup that is isomorphic to $\pi_1(M)$ (Geometrization Thurston-Pereleman).

Example 2.2.38. Most knot complements are hyperbolic. In fact out of the 1,701,936 prime knots with less than 16 crossings all are hyperbolic except 32.

2.3 Hyperbolic 3-manifolds and Kleinian groups

For proofs see [34, 37].

Definition 2.3.1. A *Kleinian group* is a subgroup $\Gamma \leq \text{Isom}(\mathbb{H}^3)$ that is *discrete*, that is any sequence $\gamma_n \in \Gamma$ such that $\gamma_n \rightarrow \gamma$ is eventually constant.

Definition 2.3.2. A *hyperbolic n -manifold* is a complete Riemannian n -manifold of constant sectional curvature -1 .

For example if $\Gamma \leq \text{Isom}(\mathbb{H}^3)$ is Kleinian and torsion free it acts properly discontinuously on \mathbb{H}^3 , hence if \mathbb{H}^3/Γ is a hyperbolic 3-manifold.

By the study of the developing map and the Holonomy representation [54] we have:

Theorem 2.3.3. Given a hyperbolic 3-manifold M then $M \cong \mathbb{H}^3/\Gamma$ for Γ a torsion free Kleinian group.

2.3.1 Margulis Lemma

We first introduce an important lemma for complete hyperbolic 3-manifolds.

Definition 2.3.4. Let (M, g) a Riemannian manifold, the for $\varepsilon > 0$ define

$$M_{[0, \varepsilon]} \doteq \{x \in M \mid \exists \text{id} \neq \gamma \in \pi_1(M, x) : \ell_g(\gamma) < \varepsilon\}$$

this is the ε -thin part and $M_{[\varepsilon, \infty)} \doteq \{x \in M \mid \forall \text{id} \neq \gamma \in \pi_1(M, x) : \ell_g(\gamma) \geq \varepsilon\}$ is the ε -thick part.

Remark 2.3.5. If M is compact we can always find ε such that $M_{[0, \varepsilon]} = \emptyset$, therefore compact manifolds do not have thin parts.

Heuristically the Margulis Lemma says the following: for any $n \in \mathbb{N}$ there is $\varepsilon_n > 0$ such that for any complete hyperbolic n -manifold and for all $x \in M_{[0, \varepsilon]}$ the subgroups of $\pi_1(M, x)$ consisting of ε_n short loops at x is "simple", i.e. the ε -thin part is generally not complicated.

The Lemma itself will be formulated by means of properly discontinuous subgroups of $\text{Iso}(\mathbb{H}^n)$. See [5, 134-139].

Theorem 2.3.6 (Margulis Lemma). For all $n \in \mathbb{N} : \exists \varepsilon_n \geq 0$ such that for any properly discontinuous subgroup $\Gamma \subseteq \text{Iso}(\mathbb{H}^n)$ and $\forall x \in \mathbb{H}^n$ the group $\Gamma_{\varepsilon_n}(x)$ generated by the following set: $F_{\varepsilon_n}(x) \doteq \{\gamma \in \Gamma \mid d_{\mathbb{H}}(x, \gamma(x)) \leq \varepsilon_n\}$ is almost nilpotent².

²It has a finite index subgroup that is nilpotent.

2.3.1.1 Local Geometry of Hyperbolic Manifolds

We will now look at some geometric consequences of the Margulis Lemma 2.3.6.

Proposition 2.3.7. Let M be a n -dimensional oriented complete hyperbolic manifold and let $x \in M$, $\varepsilon > 0$, let $\Gamma_\varepsilon < \Gamma < Iso(\mathbb{H}^n)$ the subgroup of loops at x of length less or equal to ε . Then the ε -ball at x is isometric to a ball in $\mathbb{H}^n/\Gamma_\varepsilon$.

We now examine what are the possibilities for Γ_ε when $\varepsilon \leq \varepsilon_n$.

Theorem 2.3.8. Let M be a n -dimensional oriented complete hyperbolic manifold and let $x \in M$, $\varepsilon > 0$, let $\Gamma_\varepsilon < \Gamma < Iso(\mathbb{H}^n)$ the subgroup of loops at x of length less or equal to $\varepsilon \leq \varepsilon_n$. Then we have one of the following cases:

1. $\Gamma_\varepsilon = \{\text{id}\}$;
2. $\Gamma_\varepsilon \cong \mathbb{Z}$ generated by a hyperbolic isometry;
3. Γ_ε consists of parabolic elements all having the same fixed point and $\Gamma_\varepsilon < Isom(\mathbb{R}^{n-1})$ (by looking at horospheres).

2.3.2 Simplicial Hyperbolic Surfaces

Given a hyperbolic 3-manifold M , a *useful simplicial hyperbolic surface* is a surface S with a 1-vertex triangulation \mathcal{T} , a preferred edge e and a map $f : S \rightarrow M$, such that:

1. $f(e)$ is a geodesic in M ;
2. every edge of \mathcal{T} is mapped to a geodesic segment in M ;
3. the restriction of f to every face of \mathcal{T} is a totally geodesic immersion.

By [6, 11] every π_1 -injective map $f : S \rightarrow M$ with a 1-vertex triangulation with a preferred edge can be homotoped so that it becomes a useful simplicial surface. Moreover, with the path metric induced by M a useful simplicial surface is negatively curved and the map becomes 1-Lipschitz.

Lemma 2.3.9 (Bounded Diameter Lemma, Thurston). Assume $f : S \rightarrow M$ is simplicial hyperbolic surface and let S_ε be the ε -thick part. Then $\text{diam}(S_\varepsilon) \leq \frac{8}{\varepsilon} |\chi(S)|$.

2.4 Thurston Hyperbolization Theorem

In this section we outline Thurston's Hyperbolization Theorem:

Theorem 2.4.1 (Hyperbolization of Haken manifolds). Given a Haken 3-manifold M then it admits a complete hyperbolic metric if and only if M is atoroidal and $|\pi_1(M)| = \infty$.

The above condition are obviously necessary since if we have a \mathbb{Z}^2 subgroup it has to be parabolic hence homotopic to infinity. The proof of the Hyperbolization theorem follows by induction on the Haken hierarchy for M :

$$M = N_0, N_1, \dots, N_n = \coprod \mathbb{B}^3$$

where $N_{i+1} = N_i|S_i$ for S_i some incompressible, ∂ -incompressible embedded surface that is not boundary parallel in N_i . Then we have two cases:

1. generic case: when N_1 is not an I -bundle, i.e. S_1 is not a vertical fiber of N_1 .
2. exceptional case: we have that $N_1 \cong S_1 \times I$ (or a disjoint union of I -bundles) so that $M \cong M_\psi$ for some homeomorphism $\psi : S_1 \rightarrow S_1$.

These two phenomena split the proof of 2.4.1 into two cases.

2.4.1 Overview of Non-fibered case

Given M satisfying the hypothesis of 2.4.1 we want to prove the Theorem by induction on the length of the Haken hierarchy. Obviously if the hierarchy has length zero i.e. $M \cong \coprod B^3$ then this clearly admits a hyperbolic metric. This will be the base case.

The induction step requires the solution of a gluing problem, i.e. given M, N hyperbolic manifolds with a homeomorphism $\tau : \partial M \rightarrow \partial N$ can we find $\alpha \simeq \tau$ such that α is an isometry?

If we have such a map we can then glue M, N to get a hyperbolic structure on $M \cup_\partial N$, unfortunately this will hardly be the case.

The question we actually want to solve is: are there hyperbolic metrics g, h on M, N such that τ is homotopic to an isometry.

2.4.1.1 Gluing hyperbolic structures: Maskit Combination and Skinning maps

A recipe for gluing two hyperbolic manifolds is given by looking at the corresponding Kleinian groups and using the Maskit combination theorems.

Theorem 2.4.2 (First Maskit combination Theorem). Given $\Gamma_0, \Gamma_1 \leq PSL_2(\mathbb{C})$ such that $\Gamma_0 \cap \Gamma_1 = H$ is a quasi-fuchsian group with limit set Λ_H and domain of discontinuity Ω_H . Write $\Omega_H = B_1 \coprod B_2$. If $\forall i : g \in \Gamma_i \setminus H : gB_i \cap B_i = \emptyset$ then $\Gamma \doteq \langle \Gamma_0, \Gamma_1 \rangle$ is Kleinian and isomorphic to $\Gamma_0 *_H \Gamma_1$. Moreover the Kleinian manifold for Γ is homeomorphic to the gluing of the Kleinian manifolds of Γ_i along the boundary components stabilized by H .

Therefore to solve our gluing problem we need to find hyperbolic structures Γ_0, Γ_1 on M, N such that they satisfy the conditions of Maskit's combination theorem.

Since ∂M is incompressible by the Ahlfors-Bers isomorphism Theorem [37] we have that geometrically finite structures on M are parametrized by $\mathcal{T}(\partial M) \cong \prod_{S \in \pi_0(\partial M)} \mathcal{T}(S)$. Then a hyperbolic metric on M is uniquely determined by the conformal structures on the components of the boundary ∂M . From now on assume that we want to glue M_0, M_1 along a surface $S \subseteq \partial M_0, \partial M_1$ and let $H_i \leq \Gamma_i$ be the stabilizers of S . Let Z_0, Z_1 be the conformal structures corresponding to S in the parametrisation for M_0, M_1 respectively.

Since covers of geometrically finite manifolds are themselves geometrically finite we get that the covers $N_i(S)$ of M_i corresponding to S are in $QF(S) \cong \mathcal{T}(S) \times \mathcal{T}(\bar{S})$ and they have Z_i as one element of their parametrisation. This is because each $N_i(S)$ has limit set Λ_i that splits \mathbb{S}^2 into $\Omega_i \doteq B_i \coprod B'_i$ and $B_i/H_i = Z_i$.

Then if these subgroups were induced by the same group (as in Maskit's theorem) one would hope for:

$$B'_i/H_i = Z_{i+1},^3$$

This is because the gluing map τ has to reverse orientations hence the two surfaces have to correspond to complementary regions in Ω_H .

In the case of $H_0 = H_1$ we have that H_i is the stabilizer of B_i in Γ_i (the domain of discontinuity $\Omega(\Gamma_i)$ is a union of disconnected components that are permuted by Γ_i). Since no element of $\Gamma_i \setminus H_i$ stabilizes B_i we have that for $g \in \Gamma_i \setminus H_i : gB_i \cap B_i = \emptyset$ which is what we needed in Maskit's combination theorem, hence we can glue.

Then we only need to find structures as above. This is what the *skinning map* does. We have a map:

$$\mathcal{T}(\partial M_0 \coprod \partial M_1) \rightarrow CC(M_0 \coprod M_1)$$

³Indices taken mod 2.

any geometric structure on the union gives rise to QF structures H_0, H_1 corresponding to the boundary component $S \in \pi_0(\partial M_i)$.

Each quasi-fuchsian group H_i is itself parametrized by $QF(S) \cong \mathcal{T}(S) \times \mathcal{T}(\bar{S})$ which we denote by: $H_i = (Z_i, Z'_i)$. We then define the skinning map as:

$$\begin{aligned} \sigma : \mathcal{T}(S) \times \mathcal{T}(S) &\rightarrow \mathcal{T}(\bar{S}) \times \mathcal{T}(\bar{S}) \\ (Z_0, Z_1) &\rightarrow (Z'_0, Z'_1) \end{aligned}$$

Then to reformulate the first Maskit combination theorem we want σ to send Z_0 to Z_1 , alternatively we want $\sigma \circ \tau$ to have a fixed point.⁴

The skinning map has always a fixed point if S is incompressible, i.e. $H_i \hookrightarrow \Gamma_i$ (this implies that $[\Gamma_i; H_i] = \infty$). In the acylindrical case this follows from showing that σ has bounded image, since then by iteration we obtain a fixed point. If the manifold is not acylindrical the image of σ is not bounded. Let C be the essential cylinder then by the annulus theorem we have simple loops α, β that are the boundaries of C . For simplicity assume that they are in the same boundary component S . Then by doing Dehn Twists along α, β we get a sequence of homotopy equivalent 3-manifolds such that the conformal structure on S are divergent in $\mathcal{T}(S)$ and therefore $\text{Im}(\sigma)$ has to be unbounded. Nevertheless, McMullen [38] showed the existence of fixed points even in the non-acylindrical case.

2.4.2 Fibered Case

Let S be a surface with $\chi(S) < 0$ and $\psi : S \rightarrow S$ a pseudo-Anosov homeomorphism (pA). We want to find a hyperbolic structure on M_ψ which by Mostow rigidity [5, C.0] will be unique.

In order to get a hyperbolic structure on M_ψ one needs a hyperbolic structure on $S \times \mathbb{R}$ on which $[\psi]$ is represented by an isometry α since then we get: $M_\psi \cong S \times \mathbb{R} / \langle \alpha \rangle$. It is not hard to find hyperbolic structures on $S \times \mathbb{R}$ such that ψ is a quasi-isometry (q.i.), for example given any Fuchsian structure on $S \times \mathbb{R}$ we have that ψ is a quasi-conformal map and hence a quasi-isometry.

In order to get the required structure on $S \times \mathbb{R}$ we will first build a hyperbolic 3-manifold $M_{\psi,Y} \cong S \times \mathbb{R}$ such that one end is geometrically finite and parametrized by Y and the other is geometrically infinite but q.i.-invariant under ψ . Once we have such a $M_{\psi,Y}$ we can look at sequences:

$$M_n \doteq \psi^n(M_{\psi,Y})$$

⁴The map τ is an isometry of $\mathcal{T}(S)$.

Where M_n, M_m only differ by the marking of their fundamental group and as $n \rightarrow \infty$ the generators $\{g_i\}$ of $\pi_1(S)$ are represented by more and more complicated geodesics $\psi^n(g_i)$ (mixing property of pA maps, see [20]) which will be deeper and deeper in the convex core of $M_{\psi,Y}$.

This is where we will apply the Double Limit Theorem [57] which will give us an algebraic limit $M_n \rightarrow M_\psi$, then by taking a sub-sequence we will obtain a geometric limit N which completes the construction.

Chapter 3

Infinite Type hyperbolic 3-manifolds

3.1 Overview

In this chapter we construct two 3-manifolds M_1, M_2 such that $\pi_1(M_i)$, $i = 1, 2$, have no divisible subgroups and they both admit exhaustions by hyperbolic 3-manifolds with incompressible boundary. Moreover, the exhaustions will have the useful property that the boundary components of the exhaustions elements have uniformly bounded genus.

A 3-manifold M is *hyperbolizable* if its interior is homeomorphic to \mathbb{H}^3/Γ for $\Gamma \leq \text{Isom}(\mathbb{H}^3)$ a discrete, torsion free subgroup. An irreducible 3-manifold M is of *finite-type* if $\pi_1(M)$ is finitely generated and we say it is of *infinite-type* otherwise. By Geometrization (2003, [41, 42, 43]) and Tameness (2004, [1, 9]) a finite type 3-manifold M is hyperbolizable if and only if M is the interior of a compact 3-manifold \overline{M} that is atoroidal and with non finite $\pi_1(\overline{M})$. On the other hand, if M is of infinite type not much is known and we are very far from a complete topological characterisation. Nevertheless, some interesting examples of these manifolds have been constructed in [7, 53]. What we do know are necessary condition for a manifold of infinite type to be hyperbolizable. If M is hyperbolizable then $M \cong \mathbb{H}^3/\Gamma$, hence by discreteness of Γ and the classification of isometries of \mathbb{H}^3 we have that no element $\gamma \in \Gamma$ is *divisible* ([23, Lemma 3.2]). Here, $\gamma \in \Gamma$ is divisible if there are infinitely many $\alpha \in \pi_1(M)$ and $n \in \mathbb{N}$ such that: $\gamma = \alpha^n$. We say that a manifold M is *locally hyperbolic* if every cover $N \rightarrow M$ with $\pi_1(N)$ finitely generated is hyperbolizable. Thus, local hyperbolicity and having no divisible subgroups in π_1 are necessary conditions. In [17, 36] Agol asks

whether these conditions could be sufficient for hyperbolization:

Question (Agol). Is there a 3-dimensional manifold M with no divisible elements in $\pi_1(M)$ that is locally hyperbolic but not hyperbolic?

We give a positive answer:

Theorem 3.1.1. There exists a locally hyperbolic 3-manifold with no divisible subgroups in its fundamental group that does not admit any complete hyperbolic metric.

We will prove this theorem by constructing such a manifold which we call M_1 . The second manifold M_2 is a strengthening of this result:

Theorem 3.1.2. There exists a locally hyperbolic 3-manifold with no divisible subgroups in its fundamental group that is not homotopy equivalent to any complete hyperbolic manifold.

3.2 Not homeomorphic

Consider a surface of genus two Σ and denote by α a separating curve that splits it into two punctured tori. To $\Sigma \times I$ we glue a thickened annulus $C \doteq (\mathbb{S}^1 \times I) \times I$ so that $\mathbb{S}^1 \times I \times \{i\}$ is glued to a regular neighbourhood of $\alpha \times i$, for $i = 0, 1$. We call the resulting manifold M :

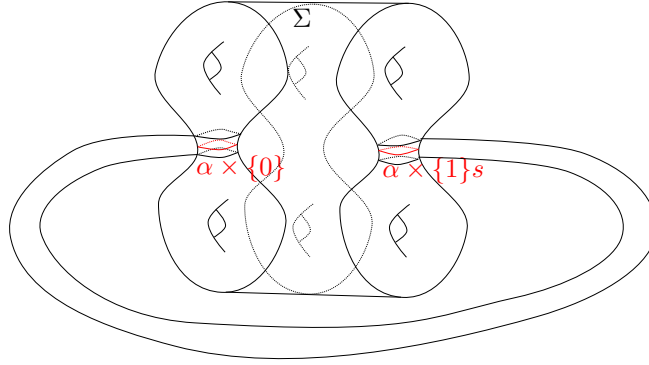


Figure 3.1: The manifold M .

The manifold M is not hyperbolic since it contains an essential torus T coming from the cylinder C . Moreover, M has a surjection p onto S^1 obtained by projecting the surfaces in $\Sigma \times I$ onto I and also mapping the cylinder onto an interval. We denote by H the kernel of the surjection map $p_* : \pi_1(M) \twoheadrightarrow \pi_1(S^1)$.

Consider an infinite cyclic cover M_∞ of M corresponding to the subgroup H . The manifold M_∞ is an infinite collection of $\{\Sigma \times I\}_{i \in \mathbb{Z}}$ glued to each other via annuli along the separating curves $\alpha \times \{0, 1\}$. Therefore, we have the following covering:

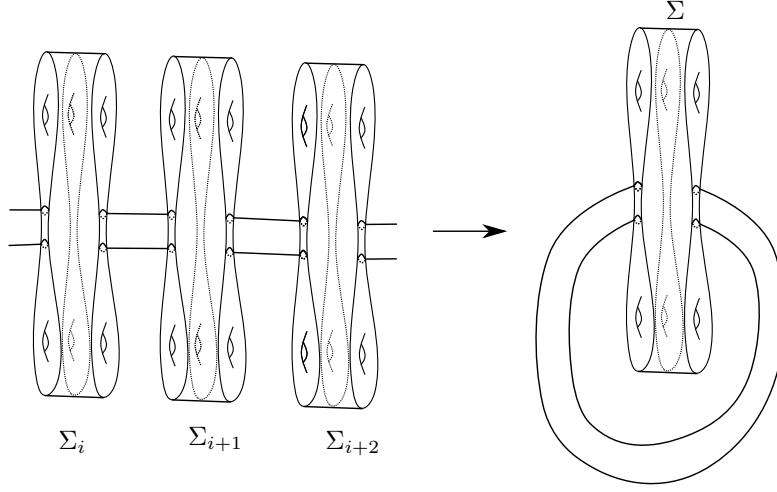


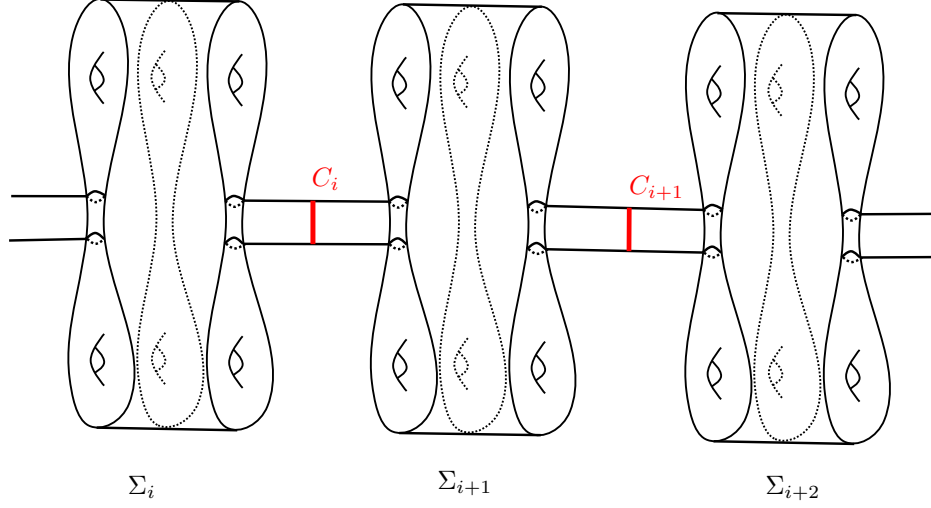
Figure 3.2: The infinite cyclic cover.

where the Σ_i are distinct lifts of Σ and so are incompressible in M_∞ . Since $\pi_1(M_\infty)$ is a subgroup of $\pi_1(M)$ and M is Haken (M contains the incompressible surface Σ) by [50] we have that $\pi_1(M)$ has no divisible elements, thus $\pi_1(M_\infty)$ has no divisible subgroups as well.

Lemma 3.2.1. The manifold M_∞ is locally hyperbolic.

Proof. We claim that M_∞ is atoroidal and exhausted by hyperbolizable manifolds. Let $T^2 \hookrightarrow M_\infty$ be an essential torus with image T . Between the surfaces Σ_i and Σ_{i+1} we have incompressible annuli C_i that separate them. Since T is compact it intersects at most finitely many $\{C_i\}$. Moreover, up to isotopy we can assume that T is transverse to all C_i and it minimizes $|\pi_0(T \cap \cup C_i)|$. If T does not intersect any C_i we have that it is contained in a submanifold homeomorphic to $\Sigma \times I$ which is atoroidal and so T wasn't essential.

Since both C_i and T are incompressible we can isotope T so that the components of the intersection $T \cap C_i$ are essential simple closed curves. Thus, T is divided by $\cup_i T \cap C_i$ into finitely many parallel annuli and $T \cap C_i$ are disjoint core curves for C_i . Consider C_k such that $T \cap C_k \neq \emptyset$ and



$\forall n \geq k : T \cap C_n = \emptyset$. Then T cannot intersect C_k in only one component, so it has to come back through C_k . Thus, we have an annulus $A \subseteq T$ that has both boundaries in C_k and is contained in a submanifold of M_∞ homeomorphic to $\Sigma_{k+1} \times I$. The annulus A gives an isotopy between isotopic curves in $\partial(\Sigma_{k+1} \times I)$ and is therefore boundary parallel. Hence, by an isotopy of T we can reduce $|\pi_0(T \cap \cup C_i)|$ contradicting the fact that it was minimal and non-zero.

We define the submanifold of M_∞ co-bounded by Σ_k and Σ_{-k} by M_k . Since M_∞ is atoroidal so are the M_k . Moreover, since the M_k are compact manifolds with infinite π_1 they are hyperbolizable by Thurston's Hyperbolization Theorem [34].

We now want to prove that M_∞ is locally hyperbolic. To do so it suffices to show that given any finitely generated $H \leq \pi_1(M_\infty)$ the cover $M_\infty(H)$ corresponding to H factors through a cover $N \twoheadrightarrow M_\infty$ that is hyperbolizable. Let $\gamma_1, \dots, \gamma_n \subseteq M_\infty$ be loops generating H . Since the M_k exhaust M_∞ we can find some $k \in \mathbb{N}$ such that $\{\gamma_i\}_{i \leq n} \subseteq M_k$, hence the cover corresponding to H factors through the cover induced by $\pi_1(M_k)$. We now want to show that the cover $M_\infty(k)$ of M_∞ corresponding to $\pi_1(M_k)$ is hyperbolizable.

Since $\pi : M_\infty \twoheadrightarrow M$ is the infinite cyclic cover of M we have that $M_\infty(k)$ is the same as the cover of M corresponding to $\pi_*(\pi_1(M_k))$. The resolution of the Tameness [1, 9] and the Geometrization conjecture [41, 42, 43] imply the Simon's conjecture, that is: covers of compact irreducible 3-manifolds with finitely generated fundamental groups are tame [12, 51]. Therefore, since M is compact by the Simon's Conjecture we have that $M_\infty(k)$ is tame. The submanifold $M_k \hookrightarrow M_\infty$ lifts homeomorphically to $\widetilde{M}_k \hookrightarrow M_\infty(k)$. By Whitehead's Theorem [24] the inclusion is a homotopy equivalence, hence \widetilde{M}_k forms a Scott core for $M_\infty(k)$. Thus, since $\partial \widetilde{M}_k$ is incompressible and $M_\infty(k)$ is tame we have that $M_\infty(k) \cong \text{int}(M_k)$ and so it is hyperbolizable. \blacksquare

In the infinite cyclic cover M_∞ the essential torus T lifts to a π_1 -injective annulus A that is properly embedded: $A = \gamma \times \mathbb{R} \hookrightarrow M_\infty$ for γ the lift of the curve $\alpha \hookrightarrow \Sigma \subseteq M$.

Remark 3.2.2. Consider two distinct lifts Σ_i, Σ_j of the embedded surface $\Sigma \hookrightarrow M$. Then we have that the only essential subsurface of Σ_i homotopic to a subsurface of Σ_j is a neighbourhood of γ . This is because by construction the only curve of Σ_i homotopic into Σ_j is γ .

Proposition 3.2.3. The manifold M_∞ is not hyperbolic.

Proof. The manifold M_∞ has two non tame ends E^\pm and the connected components of the complement of a region co-bounded by distinct lifts of Σ give neighbourhoods of these ends. Let A be the annulus obtained by the lift of the essential torus $T \hookrightarrow M$. The ends E^\pm of M_∞ are in bijection with the ends A^\pm of the annulus A . Let γ be a simple closed curve generating $\pi_1(A)$. Denote by $\{\Sigma_i\}_{i \in \mathbb{Z}} \subseteq M_\infty$ the lifts of $\Sigma \subseteq M$ and let $\{\Sigma_i^\pm\}_{i \in \mathbb{Z}}$ be the lifts of the punctured tori that form the complement of α in $\Sigma \subseteq M$. The proof is by contradiction and it will follow by showing that γ is neither homotopic to a geodesic in M_∞ nor out a cusp.

Step 1 We want to show that the curve γ cannot be represented by a hyperbolic element.

By contradiction assume that γ is represented by a hyperbolic element and let $\bar{\gamma}$ be the unique geodesic representative of γ in M_∞ . Consider the incompressible embeddings $f_i : \Sigma_2 \hookrightarrow M_\infty$ with $f_i(\Sigma_2) = \Sigma_i$ and let $\gamma_i \subseteq \Sigma_i$ be the simple closed curve homotopic to γ . By picking a 1-vertex triangulation of Σ_i where γ_i is represented by a preferred edge we can realise each (f_i, Σ_i) by a useful-simplicial hyperbolic surface $g_i : S_i \rightarrow M_\infty$ with $g_i(S_i) \simeq \Sigma_i$ (see [6, 11]). By an abuse of notation we will also use S_i to denote $g_i(S_i)$. Since all the S_i realise $\bar{\gamma}$ as a geodesic we see the following configuration in M_∞ :

On the simplicial hyperbolic surfaces S_i a maximal one-sided collar neighbourhood of $\bar{\gamma}$ has area bounded by the total area of S_i . Since the simplicial hyperbolic surfaces are all genus two by Gauss-Bonnet we have that $A(S_i) \leq 2\pi |\chi(S_i)| = 4\pi$. Therefore, the radius of a one-sided collar neighbourhood is uniformly bounded by some constant $K = K(\chi(\Sigma_2), \ell(\gamma)) < \infty$. Then for $\xi > 0$ in the simplicial hyperbolic surface S_i the $K + \xi$ two sided neighbourhood of $\bar{\gamma}$ is not embedded and contains a 4-punctured sphere. Since simplicial hyperbolic surfaces are 1-Lipschitz the 4-punctured sphere is contained in a $K + \xi$ neighbourhood C of $\bar{\gamma}$, thus it lies in some fixed set M_h . Therefore

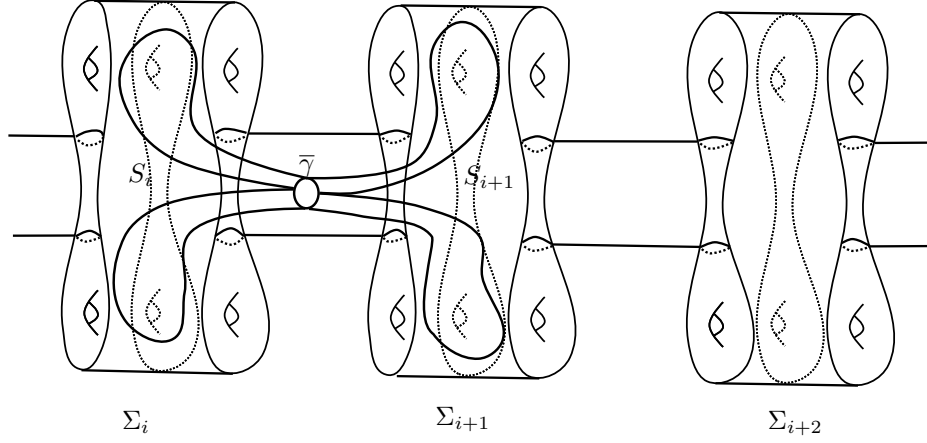


Figure 3.3: The simplicial hyperbolic surfaces S_i exiting the ends.

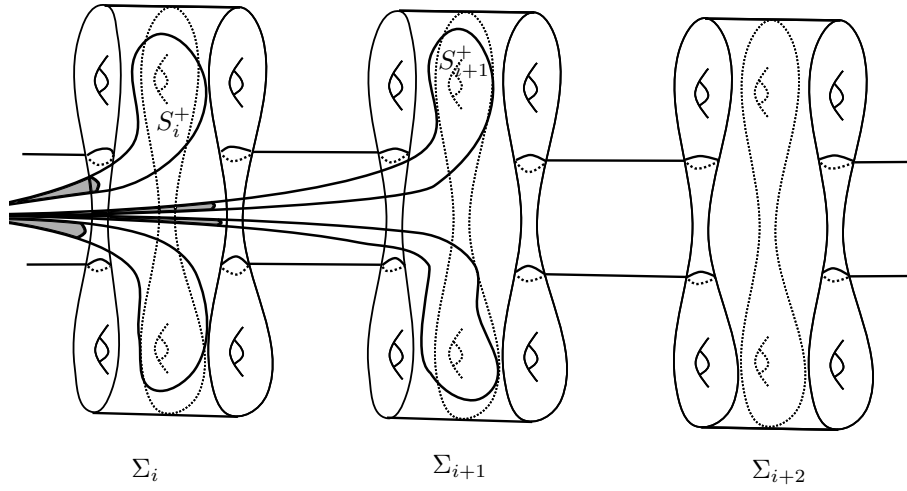
for every $|n| > h$ we have that $\Sigma_{\pm n}$ has an essential subsurface, homeomorphic to a 4-punctured sphere, homotopic into $\Sigma_{\pm h}$ respectively. But this contradicts remark 3.2.2.

Step 2 We now show that γ cannot be represented by a parabolic element.

Let $\varepsilon > 0$ be less than the 3-dimensional Margulis constant μ_3 [5] and let P be a cusp neighbourhood of γ such that the horocycle representing γ in ∂P has length ε . Without loss of generality we can assume that P is contained in the end E^- of M_∞ .

Let $\{\Sigma_i^+\}_{i \geq 0} \subseteq \{\Sigma_i\}_{i \geq 0}$ be the collection of subsurfaces of the Σ_i formed by the punctured tori with boundary γ_i that are exiting E^+ . By picking an ideal triangulation of Σ_i where the cusp γ_i is the only vertex we can realise the embeddings $f_i : \Sigma_i^+ \hookrightarrow M_\infty$ by simplicial hyperbolic surfaces (g_i, S_i^+) in which γ_i is sent to the cusp [6, 11]. The $\{S_i^+\}_{i \geq 0}$ are all punctured tori with cusp represented by γ .

All simplicial hyperbolic surface S_i^+ intersects ∂P in a horocycle $f_i(c_i)$ of length $\ell(f_i(c_i)) = \varepsilon$. Therefore, in each S_i^+ the horocycle c_i has a maximally embedded one sided collar whose radius is bounded by some constant $K = K(\varepsilon, 2\pi)$. Then for $\xi > 0$ we have that a $K + \xi$ neighbourhood of c_i in S_i^+ has to contain a pair of pants $P_i \subseteq S_i^+$. Since simplicial hyperbolic surfaces are 1-Lipschitz the pair of pants of P_i are contained in a $K + \xi$ neighbourhood of $f_i(c_i)$ in M_∞ . Thus, the Σ_i have pair of pants that are homotopic a uniformly bounded distance from ∂P . Let $k \in \mathbb{N}$ be minimal such that Σ_k lies outside a $K + \xi$ neighbourhood of ∂P . Then for any $j > k$ we have that Σ_j has a

Figure 3.4: The ε -thin part is in grey.

pair of pants homotopic into Σ_k contradicting remark 3.2.2. ■

This concludes the proof of Theorem 3.1.1.

3.3 Not Homotopy

Consider the 3-manifold A obtained as a thickening of the 2-complex given by gluing a genus two surface S and a torus T so that a meridian of T is identified with a separating simple closed curve γ of S . Note that ∂A is formed by two genus two surfaces both of which are incompressible in A . Let B, C be two copies of a hyperbolizable, acylindrical 3-manifold with incompressible genus two boundary (for example see [55, 3.3.12]) and glue B, C to the 3-manifold A one to each boundary component. Then we obtain a closed 3-manifold X :

Note that the manifold X is not hyperbolizable since it contains the essential torus T and that the surface S is incompressible and separating in X .

Remark 3.3.1. The 3-manifold $Y \doteq X|S = X \setminus N_\varepsilon(S)$ is hyperbolizable, with incompressible boundary and its characteristic submanifold is given by an annulus connecting the two distinct boundaries. Thus, any annulus with both boundary components on the same surface is boundary parallel.

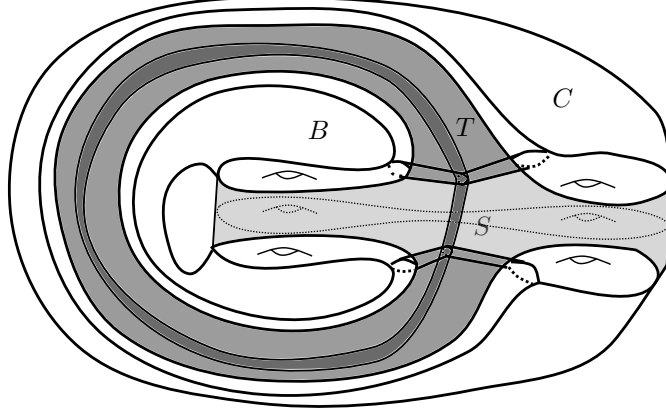


Figure 3.5: Schematic of the manifold X where the starting manifold A is shaded and the essential torus T is dark grey.

The infinite cyclic cover M of X is obtained by gluing infinitely many copies $\{Y_i\}_{i \in \mathbb{N}}$ of Y along their boundaries:

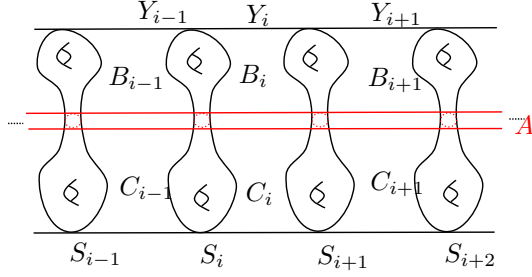


Figure 3.6

Denote by $\{S_i\}_{i \in \mathbb{Z}} \subseteq M$ the lifts of S . The surfaces $\{S_i\}_{i \in \mathbb{Z}}$ are all genus two and incompressible in M . Moreover, we denote by $M_{i,j} \doteq \cup_{i \leq k \leq j} Y_k$ the compact submanifolds of M co-bounded by S_i, S_j for $i < j$ and by A the properly embedded annulus in M that is the lift of the essential torus T and we let $A_{i,j} \doteq M_{i,j} \cap A$. With an abuse of notation we denote by $\gamma \in \pi_1(M_{i,j})$ the elements corresponding to $\pi_1(A)$.

In the remainder of this work we will show that the manifold M satisfies the following three properties:

1. $\pi_1(M)$ has no divisible elements;
2. M is locally hyperbolic;
3. M is not homotopy equivalent to any hyperbolic 3-manifold.

Which will give us Theorem 3.1.2. We will now show that M is locally hyperbolic and that $\pi_1(M)$ has no divisible elements.

Lemma 3.3.2. The manifold M has no divisible elements in $\pi_1(M)$.

Proof. The manifold M is the cover of a compact 3-manifold X thus we have that $\pi_1(M) \subseteq \pi_1(X)$. Since X is irreducible, compact and with infinite π_1 by [50] we have that $\pi_1(X)$ it has no divisible elements in π_1 . ■

Lemma 3.3.3. The manifold M is locally hyperbolic and all covers corresponding to $\pi_1(M_{i,j})$ are homeomorphic to $\text{int}(M_{i,j})$.

Proof. We first claim that M is atoroidal. Let $T \subseteq M$ be an essential torus. Since T is compact it intersects at most finitely many $\{S_i\}_{i \in \mathbb{Z}}$. Moreover, up to an isotopy we can assume that T is transverse to all S_i and that it minimises $|\pi_0(T \cap \cup_{i \in \mathbb{Z}} S_i)|$. If T does not intersect any S_i we have that it is contained in a submanifold homeomorphic to Y , see Remark 3.3.1, which is atoroidal and so T isn't essential.

Since both the S_i 's and T are incompressible by our minimality condition we have that the components of the intersection $T \cap S_i$ are essential pairwise disjoint simple closed curves in T, S_i . Thus, T is decomposed by $\cup_{i \in \mathbb{Z}} T \cap S_i$ into finitely many parallel annuli. Consider S_k such that $T \cap S_k \neq \emptyset$ and $\forall n \geq k : T \cap S_n = \emptyset$. Then T cannot intersect S_k in only one component, so it has to come back through S_k . Thus, we have an annulus $A \subseteq T$ that has both boundaries in S_k and is contained in a submanifold of M homeomorphic to Y . The annulus A gives an isotopy between isotopic curves in ∂Y and is therefore boundary parallel, see Remark 3.3.1. Hence, by an isotopy of T we can reduce $|\pi_0(T \cap \cup_{i \in \mathbb{Z}} S_i)|$ contradicting the fact that it was minimal and non-zero. Therefore, M is atoroidal.

Claim: The $M_{i,j}$ are hyperbolizable.

Proof of Claim: Since M is atoroidal and for $i < j$ the $M_{i,j}$ are π_1 -injective submanifolds they are also atoroidal. Moreover, since the $M_{i,j}$ are compact manifolds with infinite π_1 they are hyperbolizable by Thurston's Hyperbolization Theorem [34]. □

The manifold M is exhausted by the hyperbolizable π_1 -injective submanifolds $M_i \doteq M_{-i,i}$.

Claim: The manifold M is locally hyperbolic.

Proof of Claim: To do so it suffices to show that given any finitely generated $H \leq \pi_1(M)$ the cover $M(H)$ corresponding to H factors through a cover $N \twoheadrightarrow M$ that is hyperbolizable. Let $\gamma_1, \dots, \gamma_n \subseteq M$ be loops generating H . Since the M_i exhaust M we can find some $i \in \mathbb{N}$ such that $\{\gamma_k\}_{1 \leq k \leq n} \subseteq M_i$, hence the cover corresponding to H factors through the cover induced by $\pi_1(M_i)$. We now want to show that the cover $M(i)$ of M corresponding to $\pi_1(M_i)$ is hyperbolizable.

Since $\pi : M \twoheadrightarrow X$ is an infinite cyclic cover of X we have that $M(i)$ is the same as the cover of X corresponding to $\pi_*(\pi_1(M_i))$. The resolution of the Tameness [1, 9] and the Geometrization conjecture [41, 42, 43] imply the Simon's conjecture¹, that is: covers of compact irreducible 3-manifolds with finitely generated fundamental groups are tame [12, 51]. Therefore, since X is compact by the Simon's Conjecture we have that $M(i)$ is tame. The submanifold $M_i \hookrightarrow M$ lifts homeomorphically to $\widetilde{M}_i \hookrightarrow M(i)$. By Whitehead's Theorem [24] the inclusion is a homotopy equivalence, hence \widetilde{M}_i forms a Scott core for $M(i)$. Thus, since $\partial \widetilde{M}_i$ is incompressible and $M(i)$ is tame we have that $M(i) \cong \text{int}(M_i)$ and so it is hyperbolizable. \square

Which concludes the proof. \blacksquare

Remark 3.3.4. Note that in the manifold M the surfaces S_i, S_j have no homotopic simple closed curve except for the loops $\gamma_i \doteq S_i \cap A$. If not we would have an embedded cylinder C not homotopic into $A_{i,j}$ which contradicts the fact that the characteristic submanifold of $M_{i,j}$ is given by a thickening of $A_{i,j}$. In particular this gives us the important fact that for any homotopy equivalence $f : M \rightarrow N$ and any essential subsurface $F \subseteq S_i$ not isotopic to a neighbourhood of γ we cannot homotope $f(F)$ through any $f(S_j)$ for $i \neq j$.

Lemma 3.3.5 (Homotopy Equivalences). Given a tame 3-manifold N let \overline{N} be its compactification and let $g : M_{i,j} \rightarrow \overline{N}$ be a homotopy equivalence. Then, there exists a homotopy equivalence $f : M_{i,j} \rightarrow \overline{N}$ such that $f \simeq g$ and:

1. $f(S_k)$ is embedded for all $i \leq k \leq j$;
2. there are essential subsurfaces T_m, T_n of S_m, S_n , respectively, whose components are homeomorphic to punctured tori, and where for all $i \leq m < k < n \leq j$, the images $f(T_n), f(T_m)$ are separated in N by $f(S_k)$. Moreover, the same holds for any surface $\Sigma_k \xrightarrow{\text{iso}} f(S_k)$ intersecting $f(S_n), f(T_m)$ minimally.

¹Final steps completed by Long and Reid, see [12].

Proof. By Lemma 2.2.37 we get that $g : M_{i,j} \rightarrow \bar{N}$ is either homotopic to a homeomorphism f or is given by a Dehn flip along the annulus $A_{i,j}$ of $M_{i,j}$. If N is homeomorphic to $M_{i,j}$ we have nothing to do since the required map f is the homeomorphism and (1) and (2) are true for $M_{i,j}$.

Therefore, we only need to deal with the case in which $f : M_{i,j} \rightarrow \bar{N}$ is a Dehn flip of M along the annulus $A_{i,j}$. We will now explicitly write the Dehn flip f . Let $V \cong S^1 \times I_s \times I_t$ be a regular neighbourhood of the annulus $A_{i,j}$ in $M_{i,j}$ such that $V \cap S_k$, $i \leq k \leq j$, are regular neighbourhoods $S^1 \times \{s_k\} \times I_t$ of γ in S_k . Similarly let $W \cong S^1 \times I \times I$ be a regular neighbourhood of $A_{i,j}$ in \bar{N} . Let $F : V \rightarrow W$ be given by:

$$F(x, s, t) \doteq \begin{cases} (x, 2t(1-s) + (1-2t)s, t), & 0 \leq t \leq \frac{1}{2} \\ (x, (2-2t)(1-s) + (2t-1)s, t), & \frac{1}{2} \leq t \leq 1 \end{cases}$$

and $f : M_{i,j} \rightarrow \bar{N}$ be the homotopy equivalence obtained by extending F via the homeomorphism of $\overline{M_{i,j} \setminus V} \rightarrow \overline{\bar{N} \setminus W}$ coming from Lemma 2.2.37. Moreover, for $M'_{i,j} \doteq \overline{M_{i,j} \setminus V}$ the homeomorphism F is the identity on $V \cap M'_{i,j}$. Then f realises the Dehn flip from $M_{i,j}$ to \bar{N} . The homeomorphism of $M'_{i,j}$ preserves the order of the surfaces, it is the identity on $\partial V \cap M'_{i,j}$, hence for all $i \leq k \leq j$ the surfaces $f(S_k)$ are embedded. This concludes the proof of (1).

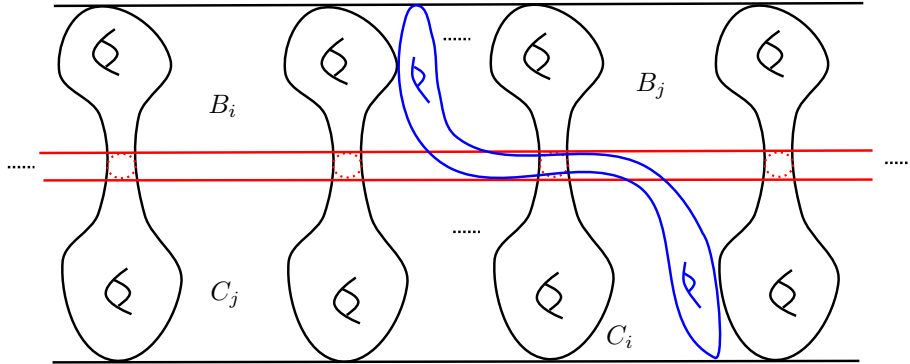
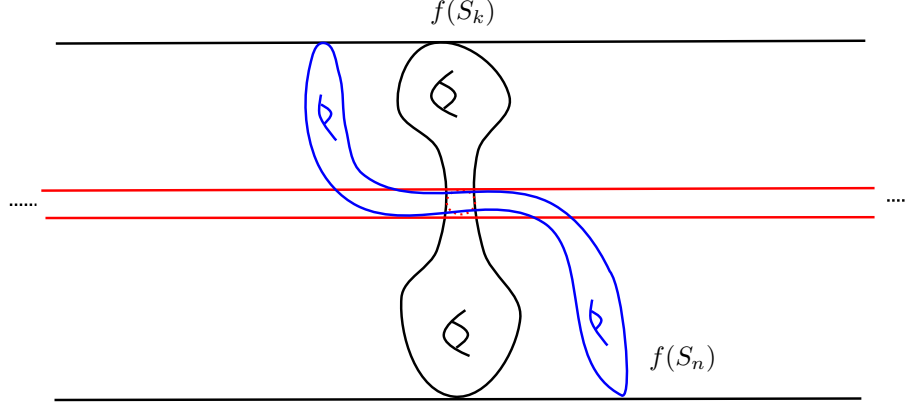


Figure 3.7: The surface in blue is an embedded push-off of $f(S_i)$ in $N \simeq \text{int}(M_{i,j})$ when \bar{N} is a Dehn flip of $M_{i,j}$.

For (2) note that for all $i \leq k \leq j$ we have that $S_k \setminus V$ is given by two essential punctured tori T_k^\pm . Moreover, for all $i \leq n \neq k \leq j$ we have that the essential tori T_n^\pm are separated by $f(S_k)$, this

again follows from the fact that f is the identity on $\partial V \cap M'_{i,j}$ and so it preserves ordering. Thus, we always see, up to isotopy, the following configuration:



Finally if $\Sigma_k \stackrel{\text{iso}}{\cong} f(S_k)$ and intersects $f(S_n), f(S_m)$ minimally we have that all components of intersections of $f(S_n), f(S_m)$ and Σ_k are isotopic to the intersection of $f(S_n), f(S_m)$ with the annulus $A_{i,j}$. If the subsurfaces $f(T_n), f(T_m)$ are not separated by Σ_k it means that they all lie in the same component N_1 of $N \setminus \Sigma_k$.

By the isotopy extension Theorem [27, 8.1.3] we have Σ_n, Σ_m isotopic to $f(S_n)$ and $f(S_m)$ respectively and subsurfaces T'_n, T'_m isotopic to $f(T_n), f(T_m)$ that are separated in N by Σ_k . Therefore, we can find a simple closed loop α in one of the essential subsurfaces T'_n, T'_m that is not contained in N_1 . Since we assumed that $f(T_n), f(T_m)$ are contained in N_1 the loop α is homotopic into Σ_k contradicting Remark 3.3.4. ■

Definition 3.3.6. Given a hyperbolic 3-manifold M , a *useful simplicial hyperbolic surface* is a surface S with a 1-vertex triangulation \mathcal{T} , a preferred edge e and a map $f : S \rightarrow M$, such that:

1. $f(e)$ is a geodesic in M ;
2. every edge of \mathcal{T} is mapped to a geodesic segment in M ;
3. the restriction of f to every face of \mathcal{T} is a totally geodesic immersion.

By [6, 11] every π_1 -injective map $f : S \rightarrow M$ with a 1-vertex triangulation with a preferred edge can be homotoped so that it becomes a useful simplicial surface. Moreover, with the path metric induced by M a useful simplicial surface is negatively curved and the map becomes 1-Lipschitz.

Lemma 3.3.7. Let $N \cong \mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold homotopy equivalent to M then γ is represented by a parabolic element in Γ .

Proof. Assume that $f_*(\gamma)$ is represented by a hyperbolic element and let $A \subseteq M$ be the essential bi-infinite annulus obtained as the limit of the $A_{-i,i}$. Since all S_i are incompressible in M and f is a homotopy equivalence the maps: $f : S_i \rightarrow N$ are π_1 -injective. Let τ_i be 1-vertex triangulations of S_i realising $\gamma_i \doteq S_i \cap A$ as an edge in the 1-skeleton. Then, by [6, 11] we can realise the maps $f : S_i \rightarrow N$ by useful simplicial hyperbolic surfaces $\Sigma_i \subseteq N$ such that $\Sigma_i \simeq f(S_i)$ and the image of γ_i is the unique geodesic representative $\bar{\gamma}$ of $f_*(\gamma)$.

In the simplicial hyperbolic surfaces Σ_i a maximal one-sided collar neighbourhood of $\bar{\gamma}$ has area bounded by the total area of Σ_i . Since the simplicial hyperbolic surfaces are all genus two by Gauss-Bonnet we have that $A(\Sigma_i) \leq 2\pi |\chi(\Sigma_i)| = 4\pi$. Therefore, the radius of a one-sided collar neighbourhood is uniformly bounded by some constant $K = K(\chi(\Sigma_i), \ell_N(\bar{\gamma})) < \infty$ for $\ell_N(\bar{\gamma})$ the hyperbolic length of γ in N . Then for $\xi > 0$ in the simplicial hyperbolic surface Σ_i the $K + \xi$ two sided neighbourhood of $\bar{\gamma}$ is not embedded and contains an essential 4-punctured sphere $X_i \subseteq \Sigma_i$. Since simplicial hyperbolic surfaces are 1-Lipschitz the 4-punctured sphere is contained in a $K + \xi$ neighbourhood C of $\bar{\gamma}$ in N . The curves α_i obtained by joining the seams of the pants decomposition of X_i induced by γ_i have length bounded by $L \doteq 2K + 2\xi + \ell_N(\bar{\gamma})$. Since there are infinitely many α_i and they are all homotopically distinct we have that Γ is not discrete since the α_i move a lift of $\bar{\gamma}$ a uniformly bounded amount. \blacksquare

Proposition 3.3.8. Let $f : M \rightarrow N$ be a homotopy equivalence, then for all i the maps: $f : S_i \rightarrow N$ have embedded representatives $\Sigma_i \simeq f(S_i)$ in N .

Proof. Fix a triangulation τ of N . Since the $f(S_i)$ are π_1 -injective by taking a refinement τ_i of τ outside a pre-compact neighbourhood U_i of $f(S_i)$ we can homotope $f(S_i)$ to be a PL-least area surface Σ_i with respect to a weight system induce by τ_i , see [22, 30, 34].

We now want to show that they are embedded. To do so it suffices to show that they have embedded representatives in some cover, see [22, 30, 34]. Consider the cover $\pi_i : N_{i,i+1} \rightarrow N$ with the triangulation $\tilde{\tau}_i$ induced by τ_i . The PL-least area surface Σ_i lifts homeomorphically to $\tilde{\Sigma}_i \subseteq N_{i,i+1}$ and it is still minimal with respect to $\tilde{\tau}_i$. Since $f(S_i) \simeq \Sigma_i$ has embedded representatives in $N_{i,i+1}$, see Lemma 3.3.5, by [22, 30, 34] we have that $\tilde{\Sigma}_i$ is embedded as well, hence $\Sigma_i = \pi_i(\tilde{\Sigma}_i)$ is. \blacksquare

We will now prove Theorem 3.1.2 which we now restate.

Theorem 3.3.9. The manifold M is locally hyperbolic and without divisible elements in $\pi_1(M)$ but is not homotopy equivalent to any hyperbolic 3-manifold.

Proof. By Lemma 3.3.2 and Lemma 3.3.3 we only need to show that M is not homotopy equivalent to any hyperbolic 3-manifold N .

The proof will be by contradiction. Assume that we have a homotopy equivalence $f : M \rightarrow N$ for $N \cong \mathbb{H}^3/\Gamma$ a hyperbolic 3-manifold. By Lemma 3.3.7 we have that for γ the element of $\pi_1(M)$ generating the fundamental group of the bi-infinite essential annulus $A \subseteq M$ $f_*(\gamma)$ is represented by a parabolic element in Γ (with an abuse of notation we will refer to this element by γ as well). Thus, in N we have a cusp $E \doteq E_\gamma$ corresponding to γ . Moreover, Proposition 3.3.8 gives us a collection $\{\Sigma_i\}_{i \in \mathbb{Z}}$ of embedded genus two surfaces contained in neighbourhoods U_i of $f(S_i)$ in N such that $\Sigma_i \simeq f(S_i)$. Moreover the Σ_i 's are incompressible and separating in N . The fact that they are separating follows from f_* being an isomorphism in homology and the fact that the S_i 's are not dual to 1-cycles in M , similarly they are incompressible since the S_i are and f is a homotopy equivalence. Therefore, if we take the surface Σ_0 we have that $N|\Sigma_0 \doteq N \setminus \text{int}(N_r(\Sigma_0))$ is given by two manifolds N_1, N_2 with boundary a surface isotopic in N to Σ_0 .

The element $\gamma \in \Gamma$ is parabolic with cusp E and each Σ_i has a simple closed loop γ_i homotopic in N to γ such that $\overline{\Sigma_i \setminus N_r(\gamma_i)}$ is given by two punctured tori T_i^\pm with boundary isotopic to γ_i . Without loss of generality we can assume that $E \subseteq N_2$. Moreover, up to an isotopy of each Σ_i we can also assume that for all $i \in \mathbb{Z}$ the surfaces Σ_i are transverse to Σ_0 and that $|\pi_0(\Sigma_i \cap \Sigma_0)|$ is minimal.

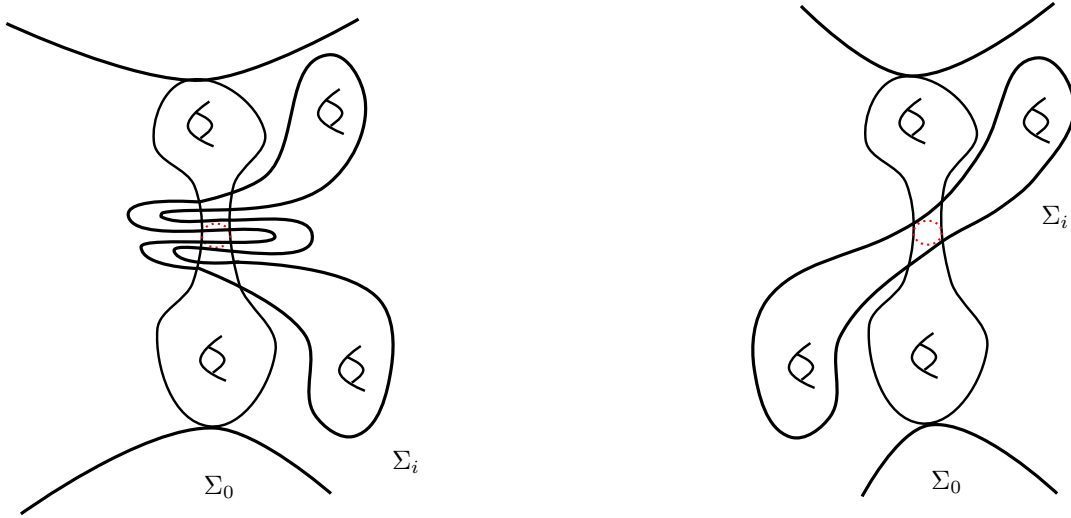
Claim: Every component $\alpha \in \pi_0(\Sigma_i \cap \Sigma_0)$ is isotopic to γ_i and γ_0 .

Proof of Claim: Since Σ_i, Σ_0 are incompressible and we minimised $\Sigma_i \cap \Sigma_0$ we have that every $\alpha \in \pi_0(\Sigma_i \cap \Sigma_0)$ has to be essential in both surfaces. By Remark 3.3.4 we have that the only simple closed curve in S_i homotopic into S_0 is γ_i which is homotopic to γ_0 . \square

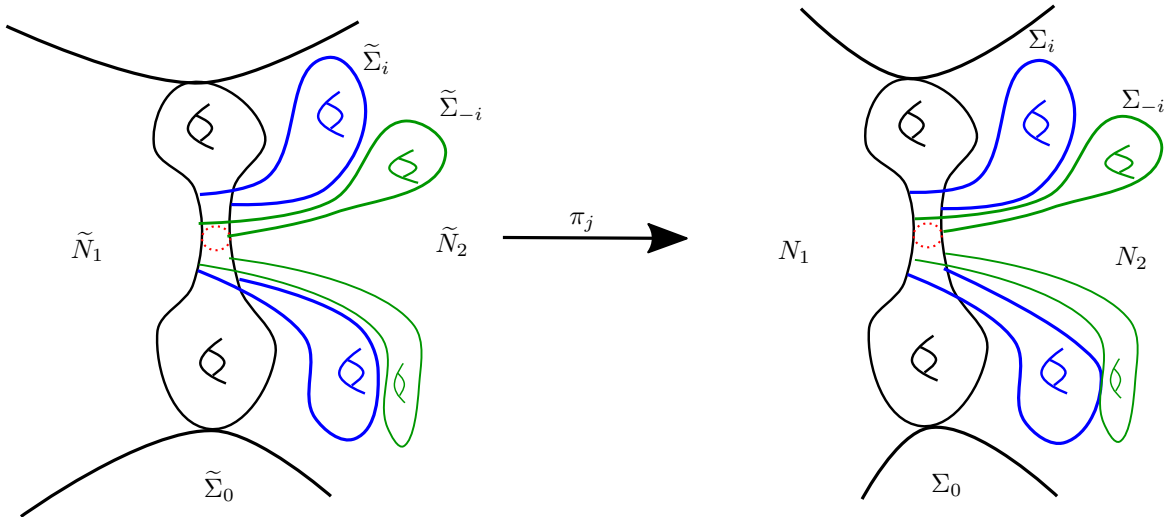
Thus in N we have that the punctured tori T_i^\pm to Σ_i are either on the same side of Σ_0 or on opposite sides:

Moreover, all the components of $\Sigma_i \cap \Sigma_0$ are contained in neighbourhoods of γ_0 and γ_i .

Claim: There are infinitely many punctured tori $\{T_n\}_{n \in \mathbb{N}} \subseteq N_1$ such that T_n is a component of $T_{i_n}^\pm$.

Figure 3.8: Possible configurations of the embedded surfaces Σ_0, Σ_i in N .

Proof of Claim: Consider $\Sigma_{-i}, \Sigma_0, \Sigma_i$ and a cover $\pi_j : N_{-i,i} \rightarrow N$ corresponding to the subgroup $f_*(\pi_1(M_{-j,j})) \subseteq \pi_1(N)$ where $\Sigma_0 \cup \Sigma_i \cup \Sigma_{-i}$ lifts homeomorphically. Assume that there only finitely many T_i^\pm that are contained in N_1 . Then, for infinitely many T_i^\pm in the covers N_j we see the following configuration:

Figure 3.9: The tori T_i^\pm are marked by the surfaces Σ_i that they are subsurfaces of.

Let g be as in Lemma 3.3.5 and homotopic to the homotopy equivalence $\tilde{f} : M_{-j,j} \rightarrow \overline{N}_{-j,j}$. Since $\tilde{\Sigma}_k$ and $g(S_k)$ are incompressible closed surfaces by [60] we have that $\tilde{\Sigma}_k \simeq g(S_k)$ and by (2) of Lemma 3.3.5 we have that $\tilde{\Sigma}_k$ separates the punctured tori $g(Q_i^\pm), g(Q_{-i}^\pm)$ in $g(S_i)$ and $g(S_{-i})$. Thus, we have a punctured torus, say $f(Q_i^+)$, that is contained in \tilde{N}_1 and such that the corresponding punctured torus $\tilde{T}_i^+ \subseteq \tilde{\Sigma}_i$ is contained in \tilde{N}_2 . Let $\alpha \subseteq T_i^+$ be any essential non peripheral curve, then since α is homotopic into $f(Q_i^+)$ and $\tilde{\Sigma}_0$ separates we have that α is homotopic into $\tilde{\Sigma}_0$ contradicting Remark 3.3.4. Therefore, we have infinitely many punctured tori $\{T_n\}_{n \in \mathbb{N}}$ with boundary $\partial T_n \doteq \gamma_n$ isotopic to γ such that $\forall n : T_n \subseteq N_1$. \square

We can now reach a contradiction with the fact that Γ is a discrete group. Let $\mu \doteq \text{inj}_N(\Sigma_0)$ and let $\varepsilon \doteq \min\{\mu, \mu_3\}$ where μ_3 is the 3-dimensional Margulis constant (see [5]). Since the T_n 's are π_1 -injective by picking a 1-vertex triangulation τ_n with preferred edge corresponding to γ_n we can realise the T_n 's by useful simplicial hyperbolic surfaces P_n .

The surfaces P_n are mapping γ_n to the cusps E and cannot be homotoped through Σ_0 since again we would contradict Remark 3.3.4. Hence, we get that for all n $P_n \cap \Sigma_0 \neq \emptyset$. Let $x_n \in P_n \cap \Sigma_n$, then $\text{inj}_{x_n}(P_n) \geq \varepsilon$. Since the surfaces P_n are negatively curved by the Bounded Diameter Lemma [6, 54] we get that we can find loops $\alpha_n, \beta_n \in \pi_1(P_n, x_n)$ whose length is bounded by $\frac{8}{\varepsilon}$ and such that they generate a rank two free group. Since $\langle \alpha_n, \beta_n \rangle \cong \mathbb{F}_2$ we have that at least one of α_n, β_n is not homotopic to γ_n . Without loss of generality we can assume that this element is always α_n . By Remark 3.3.4 the collection $\{\alpha_n\}_{n \in \mathbb{N}}$ are all distinct elements in Γ . Moreover, we have that $\ell_N(\alpha_n) \leq \frac{8}{\varepsilon}$. Let $D \doteq \text{diam}(\Sigma_0)$, pick $x \in \Sigma_0$ and fix \tilde{x} to be a lift in $\tilde{\Sigma}_0 \subseteq \mathbb{H}^3$. Then for lifts \tilde{x}_n of x_n we have that:

$$\begin{aligned} d_{\mathbb{H}^3}(\alpha_n(\tilde{x}), \tilde{x}) &\leq d_{\mathbb{H}^3}(\alpha_n(\tilde{x}), \alpha_n(\tilde{x}_n)) + d_{\mathbb{H}^3}(\alpha_n(\tilde{x}_n), \tilde{x}_n) + d_{\mathbb{H}^3}(\tilde{x}_n, \tilde{x}) \\ &\leq D + \frac{8}{\varepsilon} + D \\ &= 2D + \frac{8}{\varepsilon} \end{aligned}$$

Thus the family $\{\alpha_n\}_{n \in \mathbb{N}}$ has an accumulation point in $\text{PSL}_2(\mathbb{C})$ contradicting the discreteness of Γ . \blacksquare

Chapter 4

Hyperbolization results for \mathcal{M}^B

4.1 Topological Constructions

We now study the topology of manifolds in the class \mathcal{M}^B . Specifically we will show that any such manifold admits a canonical *manifold bordification* and a characteristic submanifold.

4.1.1 Existence of maximal bordifications for manifolds in \mathcal{M}

The aim of this section is to show that an open 3-manifold M with a compact exhaustion by hyperbolizable 3-manifolds with incompressible boundary admits a “maximal” manifold bordification \overline{M} . The boundary components of \overline{M} are in general open surfaces and come from compactifying properly embedded *product submanifolds* of the form $F \times [0, \infty)$ where F is an incompressible surface.

We will work in the following class of 3-manifolds:

Definition 4.1.1. An open 3-manifold M lies in the class \mathcal{M} if it is irreducible, orientable and satisfies the following properties:

- (i) $M = \cup_{i \in \mathbb{N}} M_i$ where each M_i is a compact, orientable and hyperbolizable 3-manifold;
- (ii) for all i : ∂M_i is incompressible in M .

Moreover, we say that $M \in \mathcal{M}_g$ if $M \in \mathcal{M}$ and for all $i \in \mathbb{N}$ all components of ∂M_i have genus at most g . We write \mathcal{M}^B for the class $\cup_{g \geq 2} \mathcal{M}_g$.

Definition 4.1.2. Given $M \in \mathcal{M}$ we say that a pair (\overline{M}, ι) , for \overline{M} a 3-manifold with boundary and $\iota : M \rightarrow \text{int}(\overline{M})$ a marking homeomorphism, is a *bordification* for M if the following properties are satisfied:

- (i) $\partial\overline{M}$ has no disk components and every component of $\partial\overline{M}$ is incompressible;
- (ii) there is no properly embedded manifold

$$(\mathbb{A} \times [0, \infty), \partial\mathbb{A} \times [0, \infty)) \hookrightarrow (\overline{M}, \partial\overline{M})$$

Moreover, we say that two bordifications $(\overline{M}, f), (\overline{M}', f')$ are equivalent, $(\overline{M}, f) \sim (\overline{M}', f')$, if we have a homeomorphism $\psi : \overline{M} \xrightarrow{\cong} \overline{M}'$ that is compatible with the markings, that is: $\psi|_{\text{int}(\overline{M})} \xrightarrow{\text{iso}} f' \circ f^{-1}$. We denote by **Bor**(M) the set of equivalence classes of bordified manifolds.

Condition (ii) is so that $(\overline{M}, \partial\overline{M})$ does not embed into any $(\overline{M}', \partial\overline{M}')$ so that two cusps in $\partial\overline{M}$ are joined by an annulus in $\partial\overline{M}'$. Condition (i) is so that we can have *maximal* bordification since it is always possible to add disk components to $\partial\overline{M}$ by compactifying properly embedded rays.

We will build a *maximal* bordification $\overline{M}_m \in \mathbf{Bord}(M)$. The bordified manifold \overline{M}_m has the key property that every properly embedded product submanifold of M is compactified in \overline{M} and M is homeomorphic to $\text{int}(\overline{M})$. The main result of the section is:

Theorem 4.1.3. Let M be an orientable, irreducible 3-manifold such that $M = \cup_{i \in \mathbb{N}} M_i$ and $M \in \mathcal{M}$ then, there exists a unique bordification $[\overline{M}] \in \mathbf{Bor}(M)$ such that every properly embedded submanifold $F \times [0, \infty)$ in \overline{M} is properly isotopic into a collar neighbourhood of a subsurface of $\partial\overline{M}$.

We will first deal with the case in which the manifolds $M = \cup_{i \in \mathbb{N}} M_i$ have the property that the genus of $S \in \pi_0(M_i)$ is uniformly bounded, i.e. $M \in \mathcal{M}^B$. Then we will show how to generalize the main technical results to deal with manifolds that have exhaustions with arbitrarily large boundary components.

4.1.1.1 Existence of maximal bordification for manifolds in \mathcal{M}^B

For open manifolds we define:

Definition 4.1.4. Given an open 3-manifold M a *product* \mathcal{P} is a proper π_1 -injective embedding $\mathcal{P} : F \times [0, \infty) \hookrightarrow M$ for F a, possibly disconnected and of infinite type, surface with no disk components. Given products \mathcal{P}, \mathcal{Q} then \mathcal{P} is a *subproduct* of \mathcal{Q} if \mathcal{P} is properly isotopic to a restriction of \mathcal{Q} to a subbundle. Whenever \mathcal{P} is a subproduct of \mathcal{Q} we write $\mathcal{P} \subseteq \mathcal{Q}$. We say

that a product $\mathcal{P} : F \times [0, \infty) \hookrightarrow M$ is *simple* if for $\{F_i\}_{i \in \mathbb{N}}$ the connected components of F no $\mathcal{P}_i \doteq \mathcal{P}|_{F_i \times [0, \infty)}$ is a subproduct of a \mathcal{P}_j .

Note that the image of every level surface of a product \mathcal{P} in M is incompressible in M . With an abuse of notation we will often use \mathcal{P} for the image of the embedding.

Given a compact, irreducible and atoroidal 3-manifold $(M, \partial M)$ with incompressible boundary and two essential properly embedded I -bundles P, Q we can find a characteristic submanifold [31, 32] $N(P)$ extending P , i.e. P is contained in a component of $N(P)$. Then by JSJ Theory [31, 32] we can isotope Q into the characteristic submanifold $N(P)$. Then, up to another isotopy of Q supported in $N(P)$, either Q and P are disjoint or they intersect in one of the following ways:

- (i) their union forms a larger connected I -bundle in $N(P)$;
- (ii) both P and Q are products over annuli and they intersect ‘transversally’, that is $V \in \pi_0(P \cap Q)$ is a solid torus containing a fiber of both P and Q .

In case (ii) we have that P and Q are contained in a essential solid torus component of $N(P)$. We will refer to the second type of intersection as a *cross shape*.

Lemma 4.1.5. Let P_1, P_2 be essential properly embedded thickened annuli in a compact irreducible 3-manifold M with incompressible boundary. If, up to isotopy, P_1, P_2 intersect in cross shapes more than twice then M is not atoroidal.

Proof. By JSJ theory we can isotope them, relative to the boundary, into the characteristic submanifold. Then they are either disjoint or they intersect in a cross shape. In the latter case, by JSJ theory their union gives a solid torus piece in the JSJ decomposition. However, if the ambient manifold is atoroidal we have that the P_i cannot intersect essentially more than twice. If they intersect at least twice their union contains two essential tori that are joined by an annulus. This configuration of essential tori and annuli (see Figure 4.1) contradicts the fact that M is atoroidal.

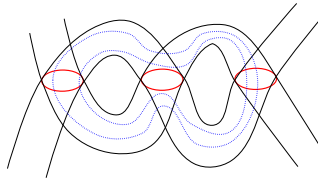


Figure 4.1: The essential torus is dotted in blue.

■

Remark 4.1.6. Let P_1, P_2 be infinite products over annuli in $M \in \mathcal{M}$ whose component of intersection are cross shapes. If P_1, P_2 have at least two components of intersections we can find k such that M_k contains an embedded copy of Figure 4.1 contradicting the atoroidality of M_k . Thus, since $P_1 \cap P_2$ intersect in fibers at most once, after a proper isotopy, we can find a compact set K such that outside K the products P_1, P_2 are either parallel or have disjoint representatives.

Definition 4.1.7. Let $M \in \mathcal{M}$ and let $X_k \doteq \overline{M_k} \setminus \overline{M_{k-1}}$ be the gaps of the exhaustion $\{M_k\}_{k \in \mathbb{N}}$. Given characteristic submanifolds $(N_k, R_k) \subseteq (X_k, \partial X_k)$, for $R_k \doteq \partial N_k \cap \partial X_k$, we say that they form a *normal family* $\mathcal{N} \doteq \{N_k\}_{k \in \mathbb{N}}$ if: whenever we have essential subsurfaces S_1, S_2 of R_k, R_{k+1} respectively that are isotopic in ∂M_k we have an essential subsurface $S \subseteq R_k \cap R_{k+1}$ such that $S \stackrel{\text{iso}}{\simeq} S_1 \stackrel{\text{iso}}{\simeq} S_2$ in ∂M_k . Thus, if $\{N_k\}_{k \in \mathbb{N}}$ forms a normal family we can assume that if a component of R_{k+1} is isotopic into R_k then it is contained in R_k , i.e. the N_k 's match up along the ∂M_k 's.

Lemma 4.1.8. Given $M = \cup_{k \in \mathbb{N}} M_k \in \mathcal{M}$ there exists a normal family \mathcal{N} of characteristic submanifolds.

Proof. By [31, 32] each $X_k \doteq \overline{M_k} \setminus \overline{M_{k-1}}$ has a characteristic submanifolds N_k and define $R_k \doteq \partial N_k \cap \partial X_k$. Consider (N_2, R_2) and let $S \subseteq R_2$ be the maximal, up to isotopy, essential subsurface of R_2 that is isotopic in ∂M_1 into R_1 . Let Σ be a component of S . If Σ is the boundary of a wing of a solid torus of N_2 we can isotope N_2 so that $\Sigma \subseteq R_1$. If Σ is contained into an I -bundle P we can isotope P so that $P \cap R_1$ contains a subsurface isotopic to Σ . By doing this for all components of S we obtain that N_1, N_2 form a normal pair. We then iterate this construction for all X_k and N_k to obtain the required collection of characteristic submanifolds. \blacksquare

Definition 4.1.9. Given $M \in \mathcal{M}$ and a normal family $\mathcal{N} = \{N_k\}_{k \in \mathbb{N}}$ of characteristic submanifolds for $X_k \doteq \overline{M_k} \setminus \overline{M_{k-1}}$ a product $\mathcal{P} : F \times [0, \infty) \hookrightarrow M$ is in *standard form* if for all $k \in \mathbb{N}$ every component of $\text{Im}(\mathcal{P}) \cap X_k$ is an essential I -bundle contained in N_k or an essential sub-surface of ∂M_k .

Definition 4.1.10. We say that a product $\mathcal{P} : F \times [0, \infty) \hookrightarrow M$ is of finite type if the base surface F is compact and of infinite type if the base surface F is of infinite type.

In the case that $\mathcal{P} : F \times [0, \infty) \hookrightarrow M$ is of finite type, i.e. F is a compact surface, we have that \mathcal{P} is in standard form if and only if, up to reparametrization, we have $\text{Im}(\mathcal{P}) \cap \cup_{k \in \mathbb{N}} \partial M_k = \cup_{i \in \mathbb{N}} \mathcal{P}(F \times \{i\})$ and each submanifold $\mathcal{P}(F \times [i, i+1])$ is an essential I -bundle contained in N_{k_i} for some $k_i \in \mathbb{N}$.

From now on we will focus on the case of manifolds M that are in \mathcal{M}^B , i.e. M is in \mathcal{M}_g for some $g \in \mathbb{N}$. This is so that every product $\mathcal{P} : F \times [0, \infty) \hookrightarrow M$ has the property that every component is a finite type product, see Corollary 4.1.20. The next pages will be dedicated to the proof of the following Theorem:

Theorem 3.18. Consider a product $\mathcal{P} : \Sigma \times [0, \infty) \hookrightarrow M$ with $M = \cup_{k \in \mathbb{N}} M_k \in \mathcal{M}^B$ and $P_i \doteq \mathcal{P}|_{\Sigma_i \times [0, \infty)}$ for Σ_i a connected component of Σ . Given a normal family $\mathcal{N} = \{N_k\}_{k \in \mathbb{N}}$ of characteristic submanifold for $X_k \doteq \overline{M_k} \setminus \overline{M_{k-1}}$ then, there is a proper isotopy Ψ^t of \mathcal{P} such that Ψ^1 is in standard form.

The proof is fairly technical and involves ideas and techniques coming from standard minimal position argument. We start by showing that we can properly isotope π_1 -injective submanifolds so that the intersections with the boundaries of the exhaustion are π_1 -injective surfaces.

Lemma 4.1.11. Let $\Psi : N \hookrightarrow M \in \mathcal{M}$ be a π_1 -injective proper embedding of an irreducible 3-manifold N . Then, there exists a proper isotopy Ψ^t of the embedding $\Psi^0 = \Psi$ such that all components of $\mathcal{S} \doteq \Psi^1(N) \cap \cup_{k \in \mathbb{N}} M_k$ are π_1 -injective surfaces and no component S of \mathcal{S} is a disk.

Proof. By a proper isotopy of Ψ , supported in ε -neighbourhoods of the ∂M_k 's we can assume that $\forall k : \partial \text{Im}(\Psi) \pitchfork \partial M_k$ so that $\partial M_k \cap \text{Im}(\Psi)$ are properly embedded surfaces in $\text{Im}(\Psi)$. Thus, we only need to show:

Claim: Up to a proper isotopy of Ψ we have that every component of $\mathcal{S} \doteq \cup_{k \in \mathbb{N}} \partial M_k \cap \text{Im}(\Psi)$ is a properly embedded incompressible surface in $\text{Im}(\Psi)$ and \mathcal{S} has no disk components.

Since every component S of \mathcal{S} is a subsurface of some ∂M_k and ∂M_k is incompressible in M it suffices to show that up to a proper isotopy of Ψ we have that every component of \mathcal{S} is an essential subsurface of some ∂M_k . Therefore, we have to show that for every component S of \mathcal{S} we have that ∂S is essential in M .

Define $B_k \doteq \pi_0(\partial M_k \cap \Psi(\partial N))$, since Ψ is proper embedding we have that: for all $k \in \mathbb{N} |B_k| < \infty$. The first step is to show that up to isotopies we can remove all inessential components of B_k . To do the isotopies we will need *good balls* for $\partial M_k \cap \partial \text{Im}(\Psi)$. These are embedded closed 3-balls $B \subseteq \overline{M} \setminus \overline{M_{k-1}}$ with $\partial B = D_1 \cup_\partial D_2$ where D_1, D_2 are disks such that $D_1 \subseteq \partial M_k$, $D_2 \subseteq \partial \text{Im}(\Psi)$ and $\partial B \cap \partial \text{Im}(\Psi) = D_2$. Given a good ball B we can push Ψ through B to reduce B_k . Pushing

through a 3-ball effectively adds/deletes a 3-ball from $\text{Im}(\Psi)$. We now define:

$$\mathcal{D}_k \doteq \{(D_1, D_2) | D_1 \subseteq \partial M_k, D_2 \subseteq \partial \text{Im}(\Psi) \text{ disks with: } \partial D_1 = \partial D_2\}$$

Notice that by the Loop Theorem [26, 28] and incompressibility of $\cup_{k \in \mathbb{N}} \partial M_k$ and of $\partial \text{Im}(\Psi)$ if $\mathcal{D}_k = \emptyset$ it means that every component of B_k is essential.

By an iterative argument the key thing to show is that if for all $n < k$ we have that $\mathcal{D}_n = \emptyset$, then if $\mathcal{D}_k \neq \emptyset$ it contains a good ball for ∂M_k .

Since ∂M_k and $\partial \text{Im}(\Psi)$ are properly embedded in every compact subset we see finitely many components of intersection. Therefore, we can take an innermost component in ∂M_k . Thus, we have a disk $D_1 \subseteq \partial M_k$ such that $D_1 \cap \partial \text{Im}(\Psi) = \partial D_1$ and the loop $\gamma \doteq \partial D_1$ is contained in $\partial \text{Im}(\Psi)$. Since $\partial \text{Im}(\Psi)$ is incompressible we see that γ bounds a disk $D_2 \subseteq \partial \text{Im}(\Psi)$ and since D_1 was picked to be innermost we have that $D_2 \cap D_1 = \gamma$. By irreducibility of M we have that the embedded 2-sphere $\mathbb{S}^2 \doteq D_1 \cup_\gamma D_2$ bounds a 3-ball B . The only thing left to check is that $B \subseteq \overline{M \setminus M_{k-1}}$.

The disk $D_2 \subseteq \partial \Psi$ does not intersect any component of ∂M_n with $n < k$, otherwise by incompressibility of ∂M_n and by taking an innermost disk of intersection we would have $\mathcal{D}_n \neq \emptyset$. Hence, B is a good ball for ∂M_k .

Thus, we can push $\text{Im}(\Psi)$ through the good ball B to reduce B_k without changing any B_n for $n < k$. This process either adds or deletes a 3-ball to $\text{Im}(\Psi)$ therefore the homeomorphism type does not change. Moreover, since B_k is finite and every time we remove a good ball it goes down by at least one we have that by pushing through finitely many good 3-balls, i.e. after a proper isotopy Ψ_k^t of Ψ supported in $\overline{M \setminus M_{k-1}}$, we have that every component of B_k is essential. The composition in $k \in \mathbb{N}$ of all the isotopies Ψ_k^t is still proper since the support of Ψ_k^t is contained in $\overline{M \setminus M_{k-1}}$. Thus we obtain a proper isotopy $\Psi^t \doteq \lim_{\substack{\longrightarrow \\ k \in \mathbb{N}}} \Psi_k^t$ of Ψ such that for all $k \in \mathbb{N}$ $\mathcal{D}_k = \emptyset$. \blacksquare

Before we can prove Theorem 4.1.19 we will need some technical Lemmas about isotopies of annuli and I -bundles in 3-manifolds.

Lemma 4.1.12. Let $(M, \partial M)$ be an irreducible 3-manifold with a collection of properly embedded pairwise disjoint boundary parallel annuli A_1, \dots, A_n with $\partial A_i \subseteq \partial M$, for $1 \leq i \leq n$. Then there exists pairwise disjoint solid tori V_1, \dots, V_ℓ in M such that $\cup_{i=1}^n A_i \subseteq \cup_{k=1}^\ell V_k$ and for all $1 \leq k \leq \ell$ we have that $\partial V_k = C_k \cup_\partial A_{i_k}$ for C_k 's pairwise disjoint annuli in ∂M .

Proof. Every annulus A_i is properly isotopic rel ∂A_i to an annulus $C_i \subseteq \partial M$ and for all $1 \leq i \leq n$ each pair A_i, C_i co-bounds a solid torus $V_i \subseteq M$.

Claim: For $i \neq j$ the annuli $C_i, C_j \subseteq \partial M$ are either disjoint or $C_i \subsetneq C_j$.

Proof of Claim: Since $\partial C_i \cap \partial C_j = \emptyset$ if $C_i \cap C_j \neq \emptyset$ we have that at least one component of $\partial C_i \doteq \alpha_1 \cup \alpha_2$ is contained in $\text{int}(C_j)$. Then, if we look at the solid torus V_j we see that $V_j \cap A_i \neq \emptyset$. Thus, either $A_i \subseteq V_j$, which gives us that $C_i \subsetneq C_j$, or it escapes. In the latter case we have that $\partial V_j \cap \text{int}(A_i) \neq \emptyset$ but $\partial V_j = C_j \cup_{\partial} A_j$ and $\text{int}(A_i) \cap C_j = \emptyset$. Hence, we must have that $A_i \cap A_j \neq \emptyset$ contradicting the fact that the annuli A_1, \dots, A_n were pairwise disjoint in M . \square

By taking maximal pairs (C_{i_k}, V_k) , $1 \leq k \leq \ell$, with respect to inclusions, we get a collection $\mathcal{V} = \{V_1, \dots, V_\ell\}$ of finitely many solid tori that contain all the annuli A_1, \dots, A_n .

Moreover, for $1 \leq k \leq \ell$ the V_k 's are pairwise disjoint. If not we would have two solid tori V_i, V_h with intersecting boundary and again we contradict the fact that the A_i 's are pairwise disjoint or the fact that the V_k 's were maximal with respect to inclusion. Thus the required collection of solid tori is given by \mathcal{V} . \blacksquare

Say we have $N \subseteq \text{int}(M)$ and $F \times I \subseteq \text{int}(M)$ where N, M are irreducible manifolds with incompressible boundary. If $\partial N \pitchfork F \times I$ by applying Lemma 4.1.12 to $F \times I$ we can remove ∂ -parallel annuli of $\partial N \cap F \times I$ by pushing $F \times I$ through the solid tori. Therefore, we have:

Corollary 4.1.13. Given $F \times I \subseteq \text{int}(M)$, with $N \subseteq \text{int}(M)$ and ∂ -parallel annuli $A_1, \dots, A_n \subseteq \partial N \cap F \times I$ then there is an isotopy Ψ_t of $F \times I$ that is the identity outside neighbourhoods of the V_k 's such that for all $t \in [0, 1]$ $\Psi_t(F \times I) \subseteq F \times I$ and $A_i \cap \Psi_1(F \times I) = \emptyset$.

Lemma 4.1.14. Let V be a solid torus with $\partial V \doteq C_1 \cup_{\partial} C_2$, for C_1, C_2 annuli and let A_1, \dots, A_n be properly embedded π_1 -injective annuli in V such that for $1 \leq i \leq n$ $\partial A_i \subseteq C_1$. Given a properly embedded annulus $S \subseteq V$ with $\partial S = \partial C_1 = \partial C_2$ there exists an isotopy Ψ_t of V that is constant on ∂V such that $\Psi_1(\cup_{i=1}^n A_i) \cap S = \emptyset$.

Proof. The annulus S splits the solid torus V into two solid tori V_1 and V_2 such that $\partial V_k = S \cup_{\partial} C_k$ for $k = 1, 2$. Since there are finitely many A_i 's and they all have boundary in $C_1 \subseteq \partial V$ we can find an annulus $S' \subseteq V_2$ with $\partial S' = \partial S$ such that all A_i are contained in the component of $V \setminus S'$ containing C_1 . By pushing S' to S we obtain the required isotopy. \blacksquare

Lemma 4.1.15. Let $\mathcal{P} : \Sigma \times [0, \infty) \hookrightarrow M$ be a product, for $M = \cup_{k \in \mathbb{N}} M_k \in \mathcal{M}$ and let $\mathcal{P}_i : \Sigma_i \times [0, \infty) \rightarrow M$ be the restriction of \mathcal{P} to the connected components of $\Sigma \times [0, \infty)$ in which we assume that the Σ_i 's are compact and let $A \subseteq \Sigma$ be the essential subsurface containing all annular components. Assume that every component of $\mathcal{S} \doteq \cup_{k \in \mathbb{N}} \partial M_k \cap \text{Im}(\mathcal{P})$ is properly embedded and

incompressible and no component S of \mathcal{S} is a boundary parallel annulus in $\text{Im}(\mathcal{P}) \setminus \mathcal{P}(A \times \{0\})$. Then, there exists a proper isotopy Ψ^t of the embedding \mathcal{P} such that for all $t \in [0, 1]$ $\Psi^t(\Sigma \times [0, \infty)) \subseteq \text{Im}(\mathcal{P})$ and for every component S of $\cup_{k \in \mathbb{N}} \partial M_k \cap \text{Im}(\Psi^1)$ the surface $(\Psi^1)^{-1}(S)$ is a horizontal fiber in some component of $\Sigma \times [0, \infty)$. Moreover, we have that $\Psi^1 : \Sigma \times [0, \infty) \hookrightarrow M$ maps $\Sigma \times \{0\}$ into \mathcal{S} and if $\mathcal{P}(\Sigma \times \{0\}) \subseteq \mathcal{S}$ then the isotopy can be assumed to be constant on $\mathcal{P}(\Sigma \times \{0\})$.

Proof. Since we will do proper isotopies supported in \mathcal{P}_i with image in \mathcal{P}_i we can work connected component by connected component. Therefore, it suffices to prove the proposition in the case that $\Sigma \times [0, \infty)$ is connected. We define $\widehat{\mathcal{S}}$ be the collection of surfaces $S \subseteq \mathcal{S}$ such that $\mathcal{P}^{-1}(S) \cap \Sigma \times \{0\} = \emptyset$. Since \mathcal{P} is a proper embedding and $\Sigma \times \{0\}$ is compact the set $\widehat{\mathcal{S}}$ is not empty.

Claim 1: Given a component S of $\widehat{\mathcal{S}}$, then $\mathcal{P}^{-1}(S)$ is isotopic to a horizontal surface in $\Sigma \times [0, \infty)$.

Proof of Claim: Since S is compact and \mathcal{P} is a proper embedding we have $0 < t_1 < \infty$ such that $S \subseteq \mathcal{P}(\Sigma \times [0, t_1])$ and since $\mathcal{P}^{-1}(S) \cap \Sigma \times \{0\} = \emptyset$ we have $\partial S \subseteq \mathcal{P}(\partial \Sigma \times (0, t_1])$. By [60, 3.1, 3.2]¹ we have an isotopy φ_t of $\mathcal{P}^{-1}(S)$ supported in $\Sigma \times [0, t_1]$ that is the identity on $\partial(\Sigma \times [0, t_1])$ such that the natural projection map: $p : \Sigma \times [0, t_1] \rightarrow \Sigma$ is a homeomorphism on $\varphi_1(\mathcal{P}^{-1}(S))$. Moreover, the surface $\mathcal{P}^{-1}(S)$ is also isotopic to a subsurface S' of $\Sigma \times \{t_1\}$. Since p is a homeomorphism on $\mathcal{P}^{-1}(S)$ and S is not a ∂ -annulus we have that the boundary components of S are in bijection with a subset of boundary components of $\partial \Sigma$. Thus, we have that S' is a clopen subset of Σ , hence S must be homeomorphic to Σ . \square

All surfaces $\widehat{\mathcal{S}}$ are pairwise disjoint and isotopic to a fiber. Thus, we can label them by $\{S_n\}_{n \in \mathbb{N}}$ such that for $n < m$ we have that S_n is contained in the bounded component of $\text{Im}(\mathcal{P}) \setminus N_\varepsilon(S_m)$.

Consider S_1 then we have positive real numbers a_1, b_1 with: $0 < a_1 < b_1 < \infty$ such that $\mathcal{P}^{-1}(S_1) \subseteq \Sigma \times [a_1, b_1]$. By Waldhausen Cobordism Theorem [60, 5.1] we can change the fibration so that $\mathcal{P}^{-1}(S_1)$ is a horizontal fiber in $\Sigma_i \times [a_1, b_1]$ hence in \mathcal{P} . Then, since $\mathcal{P}^{-1}(S_1)$ is a horizontal fiber in \mathcal{P} by a proper isotopy of \mathcal{P} supported in $\text{Im}(\mathcal{P})$ and with image in $\text{Im}(\mathcal{P})$ we can ‘raise’ $\mathcal{P}(\Sigma \times \{0\})$ to S_1 so that $\cup_{k \in \mathbb{N}} \partial M_k \cap \text{Im}(\mathcal{P}) = \widehat{\mathcal{S}}$. Note that, these isotopy preserves all properties of \mathcal{S} and $\widehat{\mathcal{S}}$. Moreover, since the last isotopy removed all components of intersection of $\cup_{k \in \mathbb{N}} \partial M_k \cap \mathcal{P}_i$ that were not in $\widehat{\mathcal{S}}$ we obtain that $\mathcal{S} = \widehat{\mathcal{S}}$.

Also note that if $\mathcal{P}(\Sigma \times \{0\}) \subseteq S$ we don’t have to do ‘raise’ isotopy and we automatically have that $\mathcal{S} = \widehat{\mathcal{S}}$.

¹The exact statement we are using here is an easy consequence of the ones cited. In particular we are applying Waldhausen’s result to a isotopic fiber structure on the I -bundle.

Claim 2: Assume that for $1 \leq k \leq n$ the surfaces $\mathcal{P}^{-1}(S_k)$ are the horizontal fibers $\Sigma \times \{k-1\}$ in $\Sigma \times [0, \infty)$. Then, by a proper isotopy that is the identity on $\Sigma \times [0, n-1]$ we can make $\mathcal{P}^{-1}(S_{n+1})$ equal to $\Sigma \times \{n\}$.

Proof of Claim: Since S_{n+1} is compact and \mathcal{P} is properly embedded we have $0 < n-1 < t_n < \infty$ such that the surface S_{n+1} is contained in $\Sigma \times [n-1, t_n]$. Then, by Waldhausen Cobordism Theorem [60, 5.1] the submanifolds bounded by S_n, S_{n+1} and $\partial\Sigma \times [n-1, t_n]$ is homeomorphic to $\Sigma \times [0, 1]$ with $\Sigma \times \{0\} = S_n$ and $\Sigma \times \{1\} = S_{n+1}$ thus by changing the fiber structure we get that S_{n+1} is also horizontal and equal to $\Sigma \times \{n\}$. \square

By iterating Claim 2 we get that all components of \mathcal{S} are horizontal in \mathcal{P} . Moreover, since $\mathcal{P}(\Sigma \times \{0\}) \subseteq \mathcal{S}$ we complete the proof. The last statement of the Lemma holds by the observation before Claim 2 and the fact that the isotopies in Claim 2 are constant on $\Sigma \times \{0\}$. \blacksquare

From now on we will use *annular product* to indicate a product $\mathcal{A} : \mathbb{A} \times [0, \infty) \hookrightarrow M$ where $\mathbb{A} = \mathbb{S}^1 \times I$ is an annulus.

Definition 4.1.16. Given a connected product $\mathcal{P} : \Sigma \times [0, \infty) \hookrightarrow M$ such that for every component S of $\cup_{k \in \mathbb{N}} \partial M_k \cap \text{Im}(\mathcal{P})$ the surface $\mathcal{P}^{-1}(S)$ is a horizontal fiber of a component of $\Sigma \times [0, \infty)$ we say that $Q \doteq \mathcal{P}(\Sigma \times [a, b])$ is a *compact region of \mathcal{P} at ∂M_k* if $Q \cap \partial M_k = \mathcal{P}(\Sigma \times \{a, b\})$. Whenever the product and the level is clear we will just write compact region.

Proposition 4.1.17. Let $M = \cup_{k \in \mathbb{N}} M_k \in \mathcal{M}$ and $\mathcal{P} : \Sigma \times [0, \infty) \hookrightarrow M$ be a product such that for each component S of $\mathcal{S} \doteq \cup_{k \in \mathbb{N}} \partial M_k \cap \text{Im}(\mathcal{P})$ then $\mathcal{P}^{-1}(S)$ is a horizontal surface in some component of $\Sigma \times [0, \infty)$. Consider the subproduct $\mathcal{A} \subseteq \mathcal{P}$ consisting of all annular products of \mathcal{P} . If for $k < m$ all compact regions of \mathcal{A} at ∂M_k are essential in either M_k or $\overline{M \setminus M_k}$ then, there is a proper isotopy Ψ_m^t of \mathcal{P} supported in $\overline{M \setminus M_{m-1}}$ such that all compact regions of \mathcal{A} at ∂M_m are essential.

Proof. Since $\text{Im}(\mathcal{P})$ and $\cup_{k \in \mathbb{N}} \partial M_k$ are properly embedded there are finitely many annular products of \mathcal{A} that intersect ∂M_m . Consider a compact region $Q \doteq \mathcal{P}(\mathbb{A} \times [a, b])$ of \mathcal{A} at ∂M_m , since $Q \cap \partial M_m = \mathcal{P}(\mathbb{A} \times \{a, b\})$ we have that Q is either contained in M_m or in $\overline{M \setminus M_m}$. Let \mathcal{A}_m be the collection of all compact regions of \mathcal{A} at ∂M_m that are boundary parallel in either M_m or $\overline{M \setminus M_m}$. We have that $|\mathcal{A}_m| < \infty$ and is bounded by $b_m \doteq |\pi_0(\partial M_m \cap \partial \text{Im}(\mathcal{P}))|$ which is finite by properness of the embedding.

Claim: Let $\mathcal{P}(\mathbb{A} \times [a, b])$ be a compact region in \mathcal{A}_n , for $n \in \mathbb{N}$ such that:

$$\mathcal{P}(\mathbb{A} \times I) \cap \bigcup_{k=1}^n \partial M_k = \mathcal{P}(\mathbb{A} \times \{a, b\}) \subseteq \partial M_n$$

and $\mathcal{P}(\mathbb{A} \times [a, b])$ is inessential. Then, there is a solid torus $V \subseteq \overline{M \setminus M_{n-1}}$ containing $\mathcal{P}(\mathbb{A} \times [a, b])$ such that all components of $\text{Im}(\mathcal{P}) \cap V$ are ∂ -parallel I -bundles contained in $\text{Im}(\mathcal{A})$.

Proof of Claim: Consider $\mathcal{P}(\partial \mathbb{A} \times [a, b])$ then these are embedded annuli C_1, C_2 in either X_n or $\overline{M \setminus M_n}$. If $\mathcal{P}(\mathbb{A} \times [a, b])$ is ∂ -parallel so are C_1, C_2 , hence we have that one of them co-bounds with an annulus $C \subseteq \partial M_n$ a solid torus V containing $\mathcal{P}(\mathbb{A} \times [a, b])$. Without loss of generality we can assume that $\partial V = C_1 \cup_{\partial} C$. Since every component of $\bigcup_{k \in \mathbb{N}} \partial M_k \cap \text{Im}(\mathcal{P})$ is a horizontal fiber in some component of $\text{Im}(\mathcal{P})$ we have that no component of $\partial M_n \cap \text{Im}(\mathcal{P})$ is a boundary parallel annulus or a disk. Since every properly embedded π_1 -injective surface in a solid torus V is either a disk or an annulus we see that $\text{Im}(\mathcal{P}) \cap V$ have to be subbundles Q_1, \dots, Q_n of annular products in $\text{Im}(\mathcal{P})$. Moreover, all Q_i , for $1 \leq i \leq n$, are inessential I -bundles. \square

Let $Q = \mathcal{P}(\mathbb{A} \times [h, \ell])$ in \mathcal{A}_k be a compact region. Assume that $Q \subseteq M_m$ and that $Q \cap \partial M_{m-1} \neq \emptyset$. Since components of intersection of $Q \cap \bigcup_{k \in \mathbb{N}} \partial M_k$ are horizontal fibers of Q let $\mathcal{P}(\mathbb{A} \times \{a\}), \mathcal{P}(\mathbb{A} \times \{b\})$ with $h < a < b < \ell$ be the first and last component of intersections in Q of $Q \cap \partial M_{m-1}$ and let $Q' \doteq \mathcal{P}(\mathbb{A} \times [a, b]) \subseteq Q$. Since $\mathcal{P}(\mathbb{A} \times [h, a]) \subseteq Q$ and $\mathcal{P}(\mathbb{A} \times [b, \ell]) \subseteq Q$ have boundaries on distinct components of $X_m \doteq \overline{M_m \setminus M_{m-1}}$ we get that they are essential I -bundles. Therefore, since Q is inessential in M_m we have some $k < m$ such that $Q' \cap (X_k \amalg X_{k+1})$ has a component $T \doteq \mathcal{P}(\mathbb{A} \times [t_1, t_2])$ that is inessential and T is a thickened annulus intersecting ∂M_k in $\mathcal{P}(\mathbb{A} \times \{t_1, t_2\})$. Since $T \subseteq Q'$ we have that Q' has a compact region that is ∂ -parallel in either M_{k-1} or $\overline{M \setminus M_{k-1}}$ contradicting the hypothesis that for all $k < m$ all compact regions of $\mathcal{A} \cap X_k$ were essential.

Therefore, we have that Q is boundary parallel in either X_m or in $\overline{M \setminus M_m}$. By the Claim we have a solid torus V such that $V \cap \text{Im}(\mathcal{P}) = \text{Im}(\mathcal{A}) \cap V$. By Lemma 4.1.14 we have a proper isotopy of \mathcal{P} supported in a solid torus $N_\varepsilon(V)$ contained in $\overline{M \setminus M_{m-1}}$ that removes Q from \mathcal{A}_m and reduces b_m by at least two.

Thus, we obtain a proper isotopy of \mathcal{P} supported in $\overline{M \setminus M_{m-1}}$ that removes Q from \mathcal{A}_m . Finally, since \mathcal{A}_m has finitely many elements the composition of these isotopies gives us a proper isotopy Ψ_m^t of \mathcal{P} that makes all sub-bundles of $\mathcal{A} \cap X_k$ for $k \leq m$ minimal. Moreover, since all the isotopies are supported in $\overline{M \setminus M_{m-1}}$ we get that Ψ_m^t is also supported outside M_{m-1} . \blacksquare

The last thing we need to prove Theorem 4.1.19 is:

Proposition 4.1.18. Let $\mathcal{P} : \Sigma \times [0, \infty) \hookrightarrow M$ be a product with $M = \cup_{k \in \mathbb{N}} M_k \in \mathcal{M}$ and let $A \subseteq \Sigma$ be the collection of components of Σ that are homeomorphic to annuli. Then, there is a proper isotopy of $\mathcal{P} \xrightarrow{\text{iso}} \mathcal{Q}$ such that all components of $\mathcal{S} \doteq \text{Im}(\mathcal{Q}) \cap \cup_{k \in \mathbb{N}} M_k$ are properly embedded π_1 -injective surfaces in $\text{Im}(\mathcal{P})$ such that no $S \in \mathcal{S}$ is a ∂ -parallel annulus in $\text{Im}(\mathcal{Q}) \setminus \mathcal{Q}(A \times \{0\})$ or a disk.

Proof. Since products are π_1 -injective by Lemma 4.1.11 we have that:

Step 1: Up to a proper isotopy of \mathcal{P} we have that every component of $\mathcal{S} \doteq \cup_{k \in \mathbb{N}} \partial M_k \cap \text{Im}(\mathcal{P})$ is a properly embedded incompressible surface in $\text{Im}(\mathcal{P})$ and \mathcal{S} has no disk components.

Let $\mathcal{A} : A \times [0, \infty) \hookrightarrow M$ be the restriction of \mathcal{P} to $A \subseteq \Sigma$. We first isotope $\mathcal{A}(A \times \{0\})$ so that every component of $\mathcal{A}(A \times \{0\})$ is an essential annulus in some ∂M_k . Let $\mathcal{A}_1 \doteq \mathcal{A}(A_1 \times [0, \infty))$, $A_1 \in \pi_0(A)$, be a component of $\text{Im}(\mathcal{A})$ then not all π_1 -injective annuli $\mathcal{S} \cap \mathcal{A}_1$ can have boundary on a component of $\partial \mathcal{A}_1 \setminus \mathcal{A}(A \times \{0\})$ since otherwise by a proper isotopy of \mathcal{A}_1 supported in \mathcal{A}_1 we would have that $\mathcal{A}_1 \cap \cup_{k \in \mathbb{N}} \partial M_k$ would be compact which contradicts the fact that \mathcal{A}_1 is a proper embedding. Therefore, we must have an essential annulus $S \subseteq \partial M_k$ of $\mathcal{S} \cap \mathcal{A}_1$ whose boundaries are on distinct components of $\partial \mathcal{A}_1 \setminus \mathcal{A}(A \times \{0\})$. Therefore, since $S \cap \mathcal{A}_1$ and $\mathcal{A}(A_1 \times \{0\})$ are isotopic in \mathcal{A}_1 we can isotope \mathcal{A}_1 so that $\mathcal{A}(A_1 \times \{0\})$ is mapped to $S \subseteq \partial M_k$. By doing this for all components of \mathcal{A} we can assume that $\mathcal{A}(A \times \{0\}) \subseteq \cup_{k \in \mathbb{N}} \partial M_k$.

Step 2: Up to a proper isotopy of \mathcal{P} supported in $\text{Im}(\mathcal{P})$ we have that no component S of \mathcal{S} is a boundary parallel annulus in $\text{Im}(\mathcal{P}) \setminus \mathcal{A}(A \times \{0\})$.

Let \mathcal{A}_k be the collection of annuli of $\mathcal{S}_k \doteq \mathcal{S}^0 \cap \partial M_k$ that are ∂ -parallel in $\text{Im}(\mathcal{P}) \setminus \mathcal{A}(A \times \{0\})$. Since \mathcal{P} is a proper embedding we have that for all $k \in \mathbb{N}$ $|\pi_0(\mathcal{A}_k)| < \infty$. By an iterative argument it suffices to show the following:

Claim 1: If for $1 \leq n < k$ $\mathcal{A}_n = \emptyset$ then via an isotopy φ_k^t of \mathcal{P} supported in $\overline{M \setminus M_{k-1}} \cap \text{Im}(\mathcal{P})$ we can make $\mathcal{A}_k = \emptyset$.

Proof of Claim: For all $k \in \mathbb{N}$ we have $0 < a_k < b_k < \infty$ such that $\mathcal{A}_k \subseteq \mathcal{P}(F_k \times [a_k, b_k])$ for $F_k \subseteq \Sigma$ a finite collection of connected components of Σ .

Denote by A_1, \dots, A_n the ∂ -parallel annuli in \mathcal{A}_k . By applying Corollary 4.1.13 to each component of $\mathcal{P}(F_k \times [a_k, b_k])$ we have a local isotopy φ_k^t of \mathcal{P} that removes all these intersections. The

isotopy φ_k^t is supported in a collection of solid tori $\mathcal{V}_k \subseteq F_k \times [a_k, b_k]$ thus it can be extended to the whole of \mathcal{P} . Moreover, if we consider for $n < k$ a component of intersection of $\partial M_n \cap \mathcal{P}(\mathcal{V}_k)$ then it is either a boundary parallel annulus or a disk. However, we assumed that for $n < k$ $\mathcal{A}_n = \emptyset$ and by **Step 1** no component of $\cup_{k \in \mathbb{N}} \partial M_k \cap \text{Im}(\mathcal{P})$ is a disk thus, the solid tori \mathcal{V}_k that we push along are contained in $\text{Im}(\mathcal{P}) \cap \overline{M \setminus M_{k-1}}$. Therefore, we get a collection of solid tori $\mathcal{V}_k \subseteq \text{Im}(\mathcal{P}) \cap \overline{M \setminus M_{k-1}}$ such that pushing through them gives us an isotopy φ_k^t of \mathcal{P} that makes $\mathcal{A}_k = \emptyset$. \square

Since for all $k \in \mathbb{N}$ $\text{supp}(\varphi_k^t) = \mathcal{V}_k$ is contained in $\overline{M \setminus M_{k-1}}$ the limit φ^t of the φ_k^t gives us a proper isotopy of \mathcal{P} such that for all $k \in \mathbb{N}$ $\mathcal{A}_k = \emptyset$.

This concludes the proof of **Step 2** and the Lemma follows. \blacksquare

We can now show that products whose components are of finite type can be put in standard form.

Theorem 4.1.19. Consider a product $\mathcal{P} : \Sigma \times [0, \infty) \hookrightarrow M$ with $M = \cup_{k \in \mathbb{N}} M_k \in \mathcal{M}$ where the Σ_i are the connected component of Σ and are compact. Given a normal family $\mathcal{N} = \{N_k\}_{k \in \mathbb{N}}$ of characteristic submanifold for $X_k \doteq \overline{M_k \setminus M_{k-1}}$ there is a proper isotopy Ψ^t of \mathcal{P} such that $\Psi^1 : \Sigma \times [0, \infty) \hookrightarrow M$ is in standard form.

Proof. From now on we denote the gaps of the exhaustion $\{M_k\}_{k \in \mathbb{N}}$ of M by $X_k \doteq \overline{M_k \setminus M_{k-1}}$, by definition we have that $N_k \subseteq X_k$. With an abuse of notation we will often confuse a product \mathcal{P} with its image $\text{Im}(\mathcal{P})$ and we define $\mathcal{P}_i \doteq \mathcal{P}|_{\Sigma_i \times [0, \infty)}$. By Lemma 4.1.18 up to a proper isotopy of \mathcal{P} all components of \mathcal{S} are properly embedded π_1 -injective surfaces in $\text{Im}(\mathcal{P})$ such that no component S of \mathcal{S} is a ∂ -parallel annulus or a disk. Then, we are in the setting of Lemma 4.1.15, thus by a proper isotopy of \mathcal{P} and a reparametrization we have that $\mathcal{S} = \{S_i^n\}_{i, n \in \mathbb{N}}$ where $S_i^n \doteq \mathcal{P}(\Sigma_i \times \{n\})$ and we let $I_i^n \doteq \mathcal{P}(\Sigma_i \times [n, n+1])$.

Step 1: Up to a proper isotopy of \mathcal{P} we can make for all $k \in \mathbb{N}$ all I -bundle components I_i^n , $i, n \in \mathbb{N}$, of $\text{Im}(\mathcal{P}) \cap X_k$ essential.

Since every I -bundle I_i^n over a surface Σ_i with $\chi(\Sigma_i) < 0$ is automatically essential we only need to deal with annular components of \mathcal{P} , i.e. products $\mathcal{P}_i : \Sigma_i \times [0, \infty) \hookrightarrow M$ where $\Sigma_i \cong \mathbb{A}$. We denote by $\mathcal{A} \subseteq \mathcal{P}$ the collection of all annular products. By Proposition 4.1.17 and an iterative argument we will show that by isotopies supported in $\overline{M \setminus M_k}$ we can make $\mathcal{A} \cap X_k$ essential.

Let \mathcal{A}_k be the ∂ -parallel compact regions of \mathcal{A} at ∂M_k . Since $\mathcal{A} \subseteq \mathcal{P}$ is properly embedded we have that for all k each \mathcal{A}_k has finitely many components each of which is a compact region over an

annulus. By applying Proposition 4.1.17 to $\mathcal{A}_1 \subseteq \mathcal{P}$ we obtain a proper isotopy Ψ_1^t that makes all compact regions Q in ∂M_1 essential in either M_1 or $\overline{M \setminus M_1}$. In particular this gives us that every I -bundle in $\text{Im}(\mathcal{P}) \cap X_1 = \text{Im}(\mathcal{P}) \cap M_1$ is essential.

We now proceed iteratively. Assume that we made for $1 \leq n < k$ all compact regions $Q \in \mathcal{A}_n$ essential at ∂M_n . Then, by applying Proposition 4.1.17 to $\mathcal{A}_k \subseteq \mathcal{P}$ we get a proper isotopy Ψ_k^t supported in $\overline{M \setminus M_{k-1}}$ that makes all compact regions $Q \in \mathcal{A}_k$ essential at ∂M_k . In particular we get that for all $1 \leq n \leq k$ all components of $\text{Im}(\mathcal{P}) \cap X_n$ are essential I -bundles or essential subsurfaces of ∂X_n .

Since the isotopies Ψ_k^t are supported in $\overline{M \setminus M_{k-1}}$ their composition yields a proper isotopy $\Psi^t \doteq \varinjlim_{k \in \mathbb{N}} \Psi_k^t$ of \mathcal{P} such that for all $k \in \mathbb{N}$ every component of $\text{Im}(\mathcal{P}) \cap X_k$ is an essential I -bundle or an essential subsurfaces of ∂X_n given by $\mathcal{P}(\Sigma \times \{0\})$.

Step 2: By a proper isotopy of \mathcal{P} we have that $\text{Im}(\mathcal{P}) \subseteq \cup_{k \in \mathbb{N}} N_k$.

By Step 1 we have that for all $k \in \mathbb{N}$ the I -bundle components of $\text{Im}(\mathcal{P}) \cap X_k$ are essential and pairwise disjoint. Consider $X_1 = M_1$ then by JSJ theory we can isotope $\text{Im}(\mathcal{P}) \cap X_1$ so that $\text{Im}(\mathcal{P}) \cap X_1 \subseteq N_1$. Moreover, since $\mathcal{P}(\Sigma \times (0, \infty)) \cap \partial M_1$ is, up to isotopy, contained in both R_1 and R_2 ² by definition of normal family we can assume that $\mathcal{P}(\Sigma \times (0, \infty)) \cap \partial M_1$ is contained in $R_{1,2} \doteq R_1 \cap R_2$. This isotopy is supported in a neighbourhood of X_1 , hence it can be extended to a proper isotopy Ψ_1^t of \mathcal{P} . Noting that each component of $\mathcal{P}(\Sigma \times \{0\})$ is isotoped at most once to obtain the required proper isotopy it suffices to work iteratively by doing isotopies relative $R_{k,k+1} \doteq R_k \cap R_{k+1}$.

Assume that we isotoped \mathcal{P} such that for all $1 \leq n \leq k$ we have that $\mathcal{P} \cap X_n \subseteq N_n$ and such that $\mathcal{P}(\Sigma \times (0, \infty)) \cap \partial M_n$ is contained $R_{n,n+1}$. Since the components of $\text{Im}(\mathcal{P}) \cap X_{k+1}$ are essential I -bundles of X_{k+1} with some boundary components contained in $R_{k,k+1}$ we can isotope them rel $R_{k,k+1}$ inside N_{k+1} so that their boundaries are contained in $R_{k,k+1} \amalg R_{k+1,k+2}$. This can be extended to an isotopy Ψ_{k+1}^t of \mathcal{P} whose support is contained in $\overline{M \setminus M_k}$, hence the composition of these isotopies gives a proper isotopy of \mathcal{P} such that $\forall k \in \mathbb{N} : \text{Im}(\mathcal{P}) \cap X_k \subseteq N_k$, thus completing the proof. ■

As a consequence of the Theorem we have:

Corollary 4.1.20. If $\mathcal{P} : \coprod_{i=1}^{\infty} F_i \times [0, \infty) \hookrightarrow M$ is a product in $M \in \mathcal{M}_g$ then, every F_i is of finite type and $|\chi(F_i)|$ is uniformly bounded by $2g - 2$.

²We remind the reader that $R_i \doteq \partial N_i \cap \partial X_i$.

Proof. It suffices to show that the statement holds for connected products. Assume that we have a connected product $\mathcal{P} : F \times [0, \infty) \hookrightarrow M$ with $|\chi(F)| \not\leq 2g - 2$. Without loss of generality we can assume that:

$$|\chi(F)| = n > 2g - 2$$

since even if \mathcal{P} is not a product of finite type we can find a subproduct $\mathcal{P}_n \doteq \mathcal{P}|_{F_n \times [0, \infty)}$ where F_n is an essential connected finite type subsurface of Σ with $|\chi(F_n)| = n$.

By Theorem 4.1.19 up to a proper isotopy of \mathcal{P} we can assume \mathcal{P} to be in standard form. Then, the surface F is an essential subsurface of $\Sigma_h \in \pi_0(\partial M_k)$ for some $k \in \mathbb{N}$. Since $h \leq g$ we have that $n = |\chi(F)| \leq 2g - 2$, which gives us a contradiction. \blacksquare

Thus, we have that:

Corollary 4.1.21. Given a product $\mathcal{P} : \Sigma \times [0, \infty) \hookrightarrow M$ if M is in $\mathcal{M}^B = \cup_{g \geq 2} \mathcal{M}_g$, then there is a proper isotopy Ψ^t of \mathcal{P} such that Ψ^1 is in standard form.

By isotopying surfaces in general position we have:

Lemma 4.1.22. Let $\varphi_i : (F_i \times I, F_i \times \partial I) \hookrightarrow (F \times I, F \times \partial I)$, $i = 1, 2$ be essential embeddings in which $\varphi_i(F_i \times \{0\})$. Then by a proper isotopy of φ_1, φ_2 we have that $\text{Im}(\varphi_1) \cup \text{Im}(\varphi_2) = \text{Im}(\varphi_3)$ where $\varphi_3 : (F_3 \times I, F_3 \times \partial I) \hookrightarrow (F \times I, F \times \partial I)$ is an essential embedding. Moreover, if $\varphi_2(F_2 \times \{0\}) \subseteq \varphi_1(F_1 \times \{0\})$ we can do the isotopy $\text{rel } \varphi_1(F_1 \times \{0\})$.

Definition 4.1.23. Given a normal family of characteristic submanifolds $\mathcal{N} = \{N_k\}_{k \in \mathbb{N}}$ for $M = \cup_{k \in \mathbb{N}} M_k \in \mathcal{M}$ and a π_1 -injective subbundle $w \doteq F \times I \hookrightarrow N_k$ with $N_k \subseteq X_k \doteq \overline{M_k} \setminus \overline{M_{k-1}}$ we say that w goes to infinity if it can be extended via I -bundles $w_i \hookrightarrow N_{k_i}$, $\{k_i\}_{i \in \mathbb{N}} \subseteq \mathbb{N}$ and $w_0 = w$, to a product $F \times [0, \infty) \hookrightarrow M$.

Note that each $F \times I \cong w \subseteq \pi_0(N_k)$ with $\chi(F) < 0$ has at most two extensions to infinity since these I -bundles do not branch in any N_k . On the other hand annular products can branch off in solid tori in N_k and thus may have infinitely many extensions to infinity.

Remark 4.1.24. Let w_1, w_2 be subbundles of $w \subseteq N_k$ going to infinity. Say that $w \stackrel{\varphi}{\cong} F \times I$ and $w_i \stackrel{\varphi_i}{\cong} F_i \times I$, then if $\varphi_1(F_1 \times \{0\}) \cap \varphi_2(F_2 \times \{0\})$ is an essential subsurface $F_{1,2}$, π_1 -injective and not ∂ -parallel, then $w_3 \doteq w_1 \cup w_2$ gives a product going to infinity containing the ones given by w_1, w_2 as subproducts.

Definition 4.1.25. For $M = \cup_{k \in \mathbb{N}} M_k \in \mathcal{M}$ we say that a product $\mathcal{P} : F \times [0, \infty) \hookrightarrow M$ starts at $X_k \doteq \overline{M_k} \setminus \overline{M_{k-1}}$ if $\text{Im}(\mathcal{P}) \cap X_k$ contains a component homeomorphic to $F \times I$ and k is minimal with respect to this property.

Recall that a simple product \mathcal{P} is a product such that no component of $\text{Im}(\mathcal{P})$ is properly isotopic into any other one, see Definition 4.1.4.

Lemma 4.1.26. Let $M = \cup_{k=1}^{\infty} M_k \in \mathcal{M}$ and $\mathcal{N} = \{N_k\}_{k \in \mathbb{N}}$ be a normal family of characteristic submanifolds for $X_k \doteq \overline{M_k} \setminus \overline{M_{k-1}}$. Then, for all $k \in \mathbb{N}$ there exists a simple product \mathcal{P}_k , in standard form, starting at X_k that contains, up to proper isotopy, all products at X_k generated by sub-bundles over hyperbolic surfaces of windows of N_k .

Proof. Let $W \subseteq N_k$ be the collection of I -bundles over hyperbolic surfaces of N_k . Then, W is homeomorphic, via a map φ , to $F \times I$. If no sub-bundle $F' \times I$ of $F \times I$ goes to infinity there is nothing to do and \mathcal{P}_k is just the empty product.

Otherwise let $S \times I \subseteq F \times I$ be a sub-bundle in which S has maximal Euler characteristic and fewest number of boundary components going to infinity through $F \times \{1\}$ such that the product $\mathcal{Q} : S \times [0, \infty) \hookrightarrow M$ it generates is simple and $\mathcal{Q}(S \times \{0\}) \subseteq F \times \{1\}$. By definition we see that \mathcal{Q} is also in standard form. We now need to show that \mathcal{Q} contains all subbundles going to infinity. Let $w' \doteq \varphi(S' \times I)$ be a product going to infinity not properly isotopic into a product given by some components of \mathcal{Q} . Via an isotopy of S' we can assume that S' is in general position with respect to S . If $S' \subseteq S$ we are done. Otherwise since S' and S are in general position no component of $S' \setminus S$ is a disk D such that $\partial D = \alpha \cup_{\partial} \beta$ with $\alpha \subseteq \partial S$ and $\beta \subseteq \partial S'$. Say we have a disk D component in $S' \setminus S$ then ∂D is decomposed into arcs $\alpha_1, \dots, \alpha_{2n}$ such that the odd ones are in $\partial S \cap S'$ and the even ones are in ∂S and $n \geq 2$. Thus, by adding D to S we get that $\chi(S \cup D) = \chi(S) - n + 1 < \chi(S)$ contradicting the maximality of $|\chi(S)|$. Thus, all components Σ of $S' \setminus S$ have $\chi(\Sigma) \leq 0$.

If we have one $\Sigma \in \pi_0(S' \setminus S)$ that is not an annulus with a boundary component in ∂S and one in $\partial S'$ we would also get that by adding it to S we would get $|\chi(\Sigma \cup_{\partial} S)| > |\chi(S)|$. Thus, we must have that all components of $S' \setminus S$ are annuli A with one boundary component in ∂S and the other in $\partial S'$ or with both boundary components in ∂S . The latter case cannot happen since then by adding $A \times I$ to $S \times I$ we would have gotten a new sub-bundle $\Sigma \times I$ going to infinity through $F \times \{1\}$ such that $\chi(\Sigma) = \chi(S)$ but $|\pi_0(\partial \Sigma)| < |\pi_0(\partial S)|$. Hence, S' is isotopic to a subsurface of S .

Therefore, we obtain a product \mathcal{P}_1 containing all windows going to infinity going through $F \times \{1\}$. By doing the same proof for $F \times \{0\}$ we obtain another product \mathcal{P}_0 , hence the required product \mathcal{P}_k is $\mathcal{P}_0 \amalg \mathcal{P}_1$. ■

Example 4.1.27. For example for the manifold constructed in Section 3.3.3 the two annular products start at $X_1 = M_1$.

We will now define maximal products which are the products that we will compactify to construct the maximal bordification.

Definition 4.1.28. A simple product $\mathcal{P} : \coprod_{i=1}^{\infty} \Sigma_i \times [0, \infty) \hookrightarrow M \in \mathcal{M}$ is *maximal* if given any other product \mathcal{Q} in M then \mathcal{Q} is properly isotopic to a subproduct of \mathcal{P} .

Theorem 4.1.29. Given $M = \cup_{k \in \mathbb{N}} M_k \in \mathcal{M}^B$ with \mathcal{N} a normal family of characteristic submanifolds there exists a product in standard form $\mathcal{P}_{max} : F \times [0, \infty) \hookrightarrow M$ such that any other product \mathcal{Q} is properly isotopic to a sub-product of \mathcal{P}_{max} .

Proof. Let $\mathcal{N} \doteq \{N_k\}_{k \in \mathbb{N}}$ be the normal family of characteristic submanifolds for $X_k \doteq \overline{M_k} \setminus \overline{M_{k-1}}$. With an abuse of notation in the proof we will often confuse a product with its image. By taking a maximal collection of product starting at X_i we will build collections of pairwise disjoint, disconnected products P_i such that:

- (i) for all $i \in \mathbb{N} : P_i \subseteq \cup_{k \in \mathbb{N}} N_k$;
- (ii) for all $n \in \mathbb{N} : \cup_{i=1}^n P_i$ is a simple product;
- (iii) for all $k \in \mathbb{N} : \cup_{i=1}^{\infty} P_i \cap X_k$ is closed.

Then by defining $\mathcal{P}_{max} \doteq \cup_{i=1}^{\infty} P_i$ we obtain a product that we will show to be maximal by our choice of P_i . The fact that \mathcal{P}_{max} is a simple product follows by (ii) and (iii) while (i) gives us that \mathcal{P}_{max} is in standard form.

With an abuse of notation we will also use P_i to denote the image of the product.

Existence: Consider $N_1 \subseteq X_1 = M_1$. We add to P_1 the products coming from Lemma 4.1.26 applied to the windows of $X_1 = M_1$. Note that P_1 contains finitely many products since all base surfaces are isotopically distinct subsurfaces of ∂M_1 and each such submanifold generates at most two non-properly isotopic products. Also note that all such products are necessarily pairwise disjoint and in standard form since they are in every N_k , $k \geq 1$.

Next, consider submanifolds of the form $\mathbb{A} \times I \subseteq N_1$ that go to infinity. Potentially every such manifold has countably many extensions. If this is the case we choose one representative A_h^1 , $h \in \mathbb{N}$, for each extension not isotopic into a subproduct of P_1 and we add it to P_1 . However, there is no reason why two such products are not intersecting. So far P_1 satisfies (i) and the only obstruction

to (ii) is that annular products may intersect in cross shapes inside solid tori components of N_k . Moreover, since in each X_k there are finitely many distinct isotopy classes of pairwise disjoint annular products we can choose the representatives $\{A_h^i\}_{h \in \mathbb{N}}$ so that $\forall k : |\{h \in \mathbb{N} | A_h^1 \cap M_k \neq \emptyset\}| < \infty$. Therefore, we can also assume that P_1 satisfies (iii).

Let $\{Q_n\}_{n \in \mathbb{N}} \subseteq P_1$ be all annular subproducts that are not pairwise disjoint. By Remark 4.1.6 we have that all the intersections of Q_n with Q_j are contained in some compact set (each annular product can intersect another one at most twice). Therefore, by flowing each Q_j in the “time” direction so that Q_j is disjoint from Q_n with $1 \leq n \leq j$ we get a proper isotopy of the Q_j ’s so that in the image they are pairwise disjoint. Moreover, we can also assume that the new $\{A_h^1\}_{h \in \mathbb{N}}$ still satisfy $\forall k : |\{h \in \mathbb{N} | A_h^1 \cap M_k \neq \emptyset\}| < \infty$. By construction P_1 satisfies condition (i) and since the annular products do not accumulate it is a product. Moreover, P_1 is simple since by construction no annular product is isotopic into a product over a hyperbolic surface and by Lemma 4.1.26 the subproduct of P_1 given by products over hyperbolic surfaces is simple. Thus, P_1 satisfies conditions (i), (ii) and (iii).

We now proceed inductively. Assume we defined P_j , $1 \leq j \leq n$, satisfying (i)-(iii) and so that we have representatives of products that start at X_j , $k \leq n$, and the annular products $\{A_h^j\}_{h \in \mathbb{N}}$ of P_j intersecting any given M_k are finite.

Consider X_{n+1} and add to P_{n+1} , as for P_1 , the collection of products going to infinity coming from Lemma 4.1.26 applied to X_{n+1} that are not properly isotopic into subproducts of P_j , for $1 \leq j \leq n$. Every product in P_{n+1} is, by construction, a sub-bundle of N_k for all $k \in \mathbb{N}$. Thus, P_{n+1} satisfies condition (i). We now need to make sure that P_{n+1} satisfies condition (ii), i.e. that $\cup_{i=1}^{n+1} P_i$ is a simple product. Condition (iii) follows from the fact that no product of P_{n+1} intersects M_n , otherwise it would have been included in P_n . The problem is that the union $\cup_{i=1}^{n+1} P_i$ might not be an embedding, however up to an isotopy of P_{n+1} we can make it an embedding so that P_1, \dots, P_{n+1} satisfy (i), (ii) and (iii).

Let $\mathcal{Q} \subseteq P_{n+1}$ be a product over a connected compact surface F with $\chi(F) < 0$. For all $k \geq n+1$ $\mathcal{Q} \cap N_k$ is a sub-bundle of a, not necessarily connected, window $w_k \in \pi_0(N_k)$. Therefore, \mathcal{Q} can only intersect products $T \subseteq \cup_{i=1}^n P_i$ that are also sub-bundles of the same window w_k . Let $\{w_{k_n}\}_{n \in \mathbb{N}} \subseteq \{w_k\}_{k \in \mathbb{N}}$ be the windows containing the intersections of T and \mathcal{Q} . Then, by Lemma 4.1.22 we have an isotopy of \mathcal{Q} supported in w_{n_1} such that $(\mathcal{Q} \cup T) \cap w_{n_1}$ is a sub-bundle of w_{n_1} . By an iterative argument using Lemma 4.1.22 on $w_{n_{k+1}}$ and doing isotopies rel $w_{n_k} \cap w_{n_{k+1}}$ we get a proper isotopy of \mathcal{Q} such that now $\mathcal{Q} \cup \cup_{i=1}^n P_i$ is a product.

By repeating this for the finitely many such \mathcal{Q} ’s in P_{n+1} we obtain a collection of products P_{n+1}

such that all products over surfaces F with $\chi(F) < 0$ can be added to $\cup_{i=1}^n P_i$ to define a, possibly disconnected, product $\cup_{i=1}^{n+1} P_i$. Moreover, this product is still simple by Lemma 4.1.26.

Finally, we add, as for P_1 , one representative A_h^{n+1} , $h \in \mathbb{N}$, for each extension to infinity of annular products starting in X_{n+1} and not properly isotopic into any subproduct of P_i , $1 \leq i \leq n+1$. Note that this condition necessarily implies that for $k \leq n$: $A_h^{n+1} \cap X_k = \emptyset$, otherwise it would have been added in some P_k with $k \leq n$. Therefore, we can assume that:

- $\forall k > n : |\{h \in \mathbb{N} | A_h^{n+1} \cap M_k \neq \emptyset\}| < \infty$;
- $\forall k \leq n : \{h \in \mathbb{N} | A_h^{n+1} \cap M_k \neq \emptyset\} = \emptyset$

For the same reasons as before condition (ii) might still fail, however by doing the same isotopies as for $\{A_h^{n+1}\}_{h \in \mathbb{N}}$ as for P_1 so that they become pairwise disjoint and are also disjoint from all annular products $\{A_h^i\}_{i \in \mathbb{N}}$ in P_i , $1 \leq i \leq n$. Moreover, since no connected subproduct of P_{n+1} intersects M_n , otherwise it would have defined a product starting at X_n and so it would have been added to P_n , we have that:

$$\forall k \leq n : \cup_{i=1}^{n+1} P_i \cap X_k = \cup_{i=1}^n P_i \cap X_k$$

and that $\cup_{i=1}^{n+1} P_i \cap X_{n+1}$ is compact, hence it also satisfies (iii).

We then define $\mathcal{P}_{max} \doteq \cup_{i=1}^{\infty} P_i$, and \mathcal{P}_{max} satisfies (i) and (ii). Thus, $\mathcal{P}_{max} \doteq \cup_{i=1}^{\infty} P_i$ is homeomorphic to $F \times [0, \infty)$ for F , in general, some disconnected surface $F = \coprod_{n \in \mathbb{N}} F_n$ where the F_n are all essential subsurface of a fixed genus $g = g(M)$ surface. Property (iii) follows from the previous remark since:

$$\forall k \in \mathbb{N} : \mathcal{P}_{max} \cap X_k = \cup_{i=1}^{\infty} P_i \cap X_k = \cup_{i=1}^k P_i \cap X_k$$

which is compact by (ii). Therefore, by construction \mathcal{P} is a simple a product.

Maximality: Let \mathcal{Q} be a product in $M \in \mathcal{M}^B$. Since we are only interested in \mathcal{Q} up to proper isotopy by Corollary 4.1.21 we can assume that it is in standard form with respect to \mathcal{N} . Let $Q_i \cong F_i \times [0, \infty)$ be a connected subproduct of \mathcal{Q} . This means that there is a minimal k_i such that $Q_i \cap X_{k_i}$ is a collection of essential I -bundles each one homeomorphic to $F \times I$. Hence it is, up to proper isotopy, contained in a component of \mathcal{P}_{max} . Therefore, we get that each connected finite type product Q_i is properly isotopic into \mathcal{P}_{max} .

Let $\mathcal{P} \subseteq \mathcal{P}_{max}$ be a connected subproduct and let $\mathcal{Q}_{\mathcal{P}}$ be all the connected subproduct of \mathcal{Q}

isotopic into subproducts of \mathcal{P} . Since \mathcal{P} and $\mathcal{Q}_{\mathcal{P}}$ are in standard form, up to a proper isotopy of $\mathcal{Q}_{\mathcal{P}}$ flowing in the ‘time’ direction, they are contained in the same collection of components $\{w_n\}_{n \in \mathbb{N}}$ for $w_n \subseteq N_{k_n}$. Then, by doing isotopies in each w_n rel w_{n-1} we can properly isotope $\mathcal{Q}_{\mathcal{P}}$ into \mathcal{P} . By doing this for all $\mathcal{P} \subseteq \mathcal{P}_{max}$ we complete the proof. \blacksquare

By the maximality condition we get:

Corollary 4.1.30. If \mathcal{P} and \mathcal{Q} are both maximal products in $M \in \mathcal{M}^B$ then they are properly isotopic.

Definition 4.1.31. Given an irreducible 3-manifold $(M, \partial M)$ and a product $\mathcal{P} : F \times [0, \infty) \hookrightarrow M \in \mathcal{M}$ we say that it is ∂ -parallel if $\text{Im}(\mathcal{P})$ is properly isotopic into a collar neighbourhood of a subsurface of ∂M . If \mathcal{P} is not ∂ -parallel we say that it is *essential*.

Example 4.1.32. Given $(M, \partial M)$ with $S \in \pi_0(\partial M)$ a punctured surface we can build a ∂ -parallel product \mathcal{P} by taking a collar neighbourhood of a puncture of S and pushing it via a proper isotopy inside $\text{int}(M)$.

For convenience we recall the definition of a bordification in the following way:

Definition 4.1.33. Given $M \in \mathcal{M}$ we say that a pair (\overline{M}, ι) , for \overline{M} a 3-manifold with boundary and $\iota : M \rightarrow \text{int}(\overline{M})$ a marking homeomorphism, is a *bordification* for M if the following properties are satisfied:

- (i) $\partial \overline{M}$ has no disk components and every component of $\partial \overline{M}$ is incompressible;
- (ii) there is no properly embedded manifold

$$(\mathbb{A} \times [0, \infty), \partial \mathbb{A} \times [0, \infty)) \hookrightarrow (\overline{M}, \partial \overline{M})$$

Moreover, we say that two bordifications $(\overline{M}, f), (\overline{M}', f')$ are equivalent $(\overline{M}, f) \sim (\overline{M}', f')$ if we have a homeomorphism $\psi : \overline{M} \xrightarrow{\cong} \overline{M}'$ that is compatible with the markings, that is: $\psi|_{\text{int}(\overline{M})} \stackrel{\text{iso}}{\cong} f' \circ f^{-1}$.

We denote by $\mathbf{Bor}(M)$ the set of equivalence classes of bordified manifolds

Condition (ii) is so that $(\overline{M}, \partial \overline{M})$ does not embed into any $(\overline{M}', \partial \overline{M}')$ in a way that two cusps in $\partial \overline{M}$ are joined by an annulus in $\partial \overline{M}'$. Condition (i) is so that we can have ‘maximal’ bordification since it is always possible to add disk components to $\partial \overline{M}$ by compactifying properly embedded rays and so that collar neighbourhoods of $\partial \overline{M}$ correspond to products in M .

Definition 4.1.34. We say that a bordification $[(\overline{M}, f)] \in \mathbf{Bor}(M)$ is *maximal* if \overline{M} has no essential products.

Lemma 4.1.35. A bordification $[(\overline{M}, f)] \in \mathbf{Bord}(M)$ is maximal if and only if the preimage of a collar of $\partial\overline{M}$ in M via f is a maximal product.

Proof. Let (\overline{M}, f) be a maximal bordification and let $\partial\overline{M} = \coprod_{i=1}^{\infty} S_i$ and $\mathcal{P}_i \doteq f^{-1}(N_{\varepsilon}(S_i))$. Then, we get a product $\mathcal{P} \doteq \coprod \mathcal{P}_i$ in M . Moreover, \mathcal{P} is simple since otherwise we would have two component S_1, S_2 of $\partial\overline{M}$ that can be joined by a submanifold homeomorphic to $\mathbb{A} \times [0, \infty)$, contradicting property (ii) of the definition of a bordification. Finally, by property (i) of a bordification we see that \mathcal{P} is π_1 -injective and by maximality of the bordification every product \mathcal{Q} in M is isotopic into \mathcal{P} and hence \mathcal{P} is a maximal product.

Similarly if for (\overline{M}, f) a bordification we have that $\mathcal{P} \doteq f^{-1}(N_{\varepsilon}(\partial\overline{M}))$ is a maximal product then \overline{M} is maximal since if not we would have another bordification: (\overline{M}', f') and an embedding:

$$\psi : (\overline{M}, \partial\overline{M}) \hookrightarrow (\overline{M}', \partial\overline{M}')$$

such that $\partial\overline{M}' \setminus \psi(\partial\overline{M})$ contains a non-annular component contradicting the maximality of \mathcal{P} . ■

We can now prove the main result of the section:

Theorem 4.1.36. Let $M \in \mathcal{M}^B$ then there exists a unique maximal bordification $[(\overline{M}, \iota)] \in \mathbf{Bor}(M)$.

Proof. Since $M \in \mathcal{M}^B$ by Theorem 4.1.29 we have a maximal product $\mathcal{P}_{max} : F \times [0, \infty) \hookrightarrow M$. We now want to compactify \mathcal{P}_{max} by adding $\text{int}(F) \times \{\infty\}$ to M . Topologically the subproduct $\mathcal{P} \doteq \mathcal{P}_{max}|_{\text{int}(F) \times [0, \infty)}$ can be naturally compactified to $\overline{\mathcal{P}} : \text{int}(F) \times [0, \infty] \hookrightarrow M \cup \text{int}(F) \times \{\infty\}$ by adding the boundary at infinity $\text{int}(F) \times \{\infty\}$ to M .

Define $\overline{M} \doteq M \coprod \text{int}(F) \times \{\infty\}$ with the topology that makes $\overline{\mathcal{P}} : \text{int}(F) \times [0, \infty] \hookrightarrow \overline{M}$ into a homeomorphism onto its image. To see that \overline{M} is a 3-manifold it suffices to show that $F \times [0, \infty) \cup \text{int}(F) \times \{\infty\}$ is a 3-manifold. This follows from the fact that $F \times [0, \infty) \cup \text{int}(F) \times \{\infty\}$ is naturally an open submanifold of $F \times [0, 1]$ and so the smooth structures agree. Moreover, we have that the inclusion: $\text{id} : M \hookrightarrow \overline{M}$ is an embedding. Since products have no disk components we have that $[(\overline{M}, \text{id})] \in \mathbf{Bord}(M)$ and by Lemma 4.1.35 we get that $[(\overline{M}, \text{id})]$ is a maximal bordification.

Uniqueness: Say we have another maximal bordification $[(\overline{M}', \iota')]$, then by Lemma 4.1.35 we obtain a maximal product \mathcal{P}' . Since the products \mathcal{P} and \mathcal{P}' are maximal by Corollary 4.1.30 we have a proper isotopy H_t from \mathcal{P} to \mathcal{P}' .

We can then extend this proper isotopy to a proper isotopy $\widehat{H}_t : M \rightarrow M$. The diffeomorphism $H_1 : M \rightarrow M$ mapping $\text{Im}(\mathcal{P}_m)$ to $\text{Im}(\mathcal{P}')$ extends to a diffeomorphism $\psi : \overline{M} \rightarrow \overline{M}'$ mapping $\partial\overline{M}$ to $\partial\overline{M}'$. By construction we have that this gives an equivalence of bordifications concluding the proof. \blacksquare

4.1.1.2 Extension to Manifolds in \mathcal{M}

Note that in Theorem 4.1.36 we used the fact that the manifold was in \mathcal{M}^B just to say that we had maximal products via Theorem 4.1.29. Thus, the aim of this subsection is to show how one can extend Theorem 4.1.29 to deal with infinite type products. To do so it suffices to show that infinite type product can be put in standard form, i.e. extending Theorem 4.1.19.

We will use the term *window* to denote an essential I -subbundle of a component of the characteristic submanifold.

Definition 4.1.37. We say that an I -bundle $F \times I$ embedded in an irreducible 3-manifold $(M, \partial M)$ with incompressible boundary, not necessarily boundary to boundary, is *mixed* if it is π_1 -injective and it contains a window of M .

Example 4.1.38. Let M be a compact, irreducible 3-manifold with incompressible boundary and let $w \stackrel{\varphi}{\cong} F \times I$ be a window in M with $\varphi(F \times \{i\}) \subseteq S_i \in \pi_0(\partial M)$, for $i = 0, 1$, and $S_0 \neq S_1$. If we denote by N_1 a collar neighbourhood of S_1 we have that $w \cup N_1$ is a mixed I -bundle with fiber structure $S_1 \times I$.

We now extend Theorem 4.1.19:

Lemma 4.1.39. Given $\mathcal{P} : \Sigma \times [0, \infty) \hookrightarrow M$ a product in $M \in \mathcal{M}$ and a normal family of characteristic submanifolds $\mathcal{N} = \{N_k\}_{k \in \mathbb{N}}$ with $N_k \subseteq \overline{M_k} \setminus \overline{M_{k-1}}$, then there is a proper isotopy Ψ^t of \mathcal{P} such that Ψ^1 is in standard form.

Proof. By Lemma 4.1.18 we can assume that after a proper isotopy of \mathcal{P} all components of $\mathcal{S} \doteq \bigcup_{k \in \mathbb{N}} \partial M_k \cap \text{Im}(\mathcal{P})$ are properly embedded π_1 -injective surfaces in $\text{Im}(\mathcal{P})$ and the components S of \mathcal{S} are neither disk nor ∂ -parallel annuli in $\text{Im}(\mathcal{P}) \setminus \mathcal{P}(A \times \{0\})$ for A the collection of annular components of Σ .

Step 1 Up to a proper isotopy of \mathcal{P} for every component S of \mathcal{S} we have that $\mathcal{P}^{-1}(S)$ is isotopic to an essential subsurface of $\Sigma \times \{0\}$.

By applying **Claim 1** of Lemma 4.1.15 to a finite type sub-product $\mathcal{P}(\Sigma_S \times [0, \infty))$ containing $\mathcal{P}^{-1}(S)$ we get that every $\mathcal{P}^{-1}(S)$ is isotopic rel ∂ to an essential subsurface F of Σ .

Since $\Sigma = \cup_{i=1}^{\infty} \Sigma_i \cup \cup_{i=1}^{\infty} \Delta_i$ where the Δ_i 's are components of infinite type we have an essential subsurface $T \subseteq \Sigma$ such that $T \doteq \cup_{i=1}^{\infty} \Sigma_i \cup \cup_{i=1}^{\infty} T_i$ where the $T_i \neq \emptyset$ are finite type hyperbolic essential subsurfaces of the Δ_i 's. We denote by $\mathcal{P}_T \doteq \mathcal{P}|_{T \times [0, \infty)} \subseteq \mathcal{P}$ the subproduct it generates. We denote by $\mathcal{S}_T \subseteq \mathcal{S}$ the subcollection of components of \mathcal{S} that do not intersect $\mathcal{P}(T \times \{0\})$. By properness of the embedding \mathcal{P} we see that for all $i \in \mathbb{N}$:

$$\mathcal{P}(\Sigma_i \times [0, \infty)) \cap \mathcal{S} \setminus \mathcal{S}_T \coprod \mathcal{P}(\Delta_i \times [0, \infty)) \cap \mathcal{S} \setminus \mathcal{S}_T$$

has finitely many components.

Note that not all surfaces of $\mathcal{P}(\Delta_i \times [0, \infty)) \cap \mathcal{S}_T$ can be ∂ -parallel annuli or disks in \mathcal{P}_T since then by Lemma 4.1.12 and an iterative argument we have a proper isotopy of $\mathcal{P}|_{T_i \times [0, \infty)}$ such that $\text{Im}(\mathcal{P}|_{T_i \times [0, \infty)})$ does not intersect $\cup_{k \in \mathbb{N}} \partial M_k$. Therefore, in each $\mathcal{P}(\Delta_i \times [0, \infty))$ we have a sequence $\{S_n^i\}_{n \in \mathbb{N}} \subseteq \mathcal{S}_T$ such that for all $n \in \mathbb{N}$: $S_n^i \cap \text{Im}(\mathcal{P}_T)$ is an essential properly embedded surface in $\text{Im}(\mathcal{P}_T)$. Thus, in $\Sigma \times [0, \infty)$ we have the configuration depicted in Figure 4.2.

We denote by F_n^i the essential subsurface of Σ that $\mathcal{P}^{-1}(S_n^i)$ is isotopic to rel ∂ . Since Lemma 4.1.15 does isotopies supported in the image of \mathcal{P} we can apply it connected component by connected component and we can assume that for all n $\mathcal{P}^{-1}(S_n^i) \cap T_i \times [0, \infty) = T_i \times \{a_n^i\}$. Thus, we have that for all $n \in \mathbb{N}$ the surfaces $\mathcal{P}^{-1}(S_n^i)$ co-bound with Σ I -bundles H_n^i . Moreover, since for all $n \in \mathbb{N}$: $T_i \times [0, \infty) \cap H_n^i = T_i \times [0, a_n]$ and $S_n^i \cap S_{n+1}^i = \emptyset$ we get that $H_n^i \subsetneq H_{n+1}^i$. Moreover, by properness of \mathcal{P} we have that $\cup_{n \in \mathbb{N}} H_n^i$ is an open and closed submanifold of $\Sigma \times [0, \infty)$ thus, we have that $\cup_{n=1}^{\infty} H_n^i = \Sigma \times [0, \infty)$.

Moreover, since each component of $\overline{\text{Im}(\mathcal{P}) \setminus \cup_{n,i \in \mathbb{N}} S_n^i}$ is essential it is either an I -bundle, if $S_n^i \xrightarrow{\text{iso}} S_{n+1}^i$ or a mixed I -bundle homeomorphic to $S_{n+1}^i \times I$.

We now want to remove all components of intersection of $\text{Im}(\mathcal{P}) \cap \cup_{k \in \mathbb{N}} \partial M_k$ that are not S_n^i for some i, n . By Lemma 4.1.15 we have an isotopy supported in the image of the products of finite type so that they are in standard form. Therefore, we only need to worry about the components of

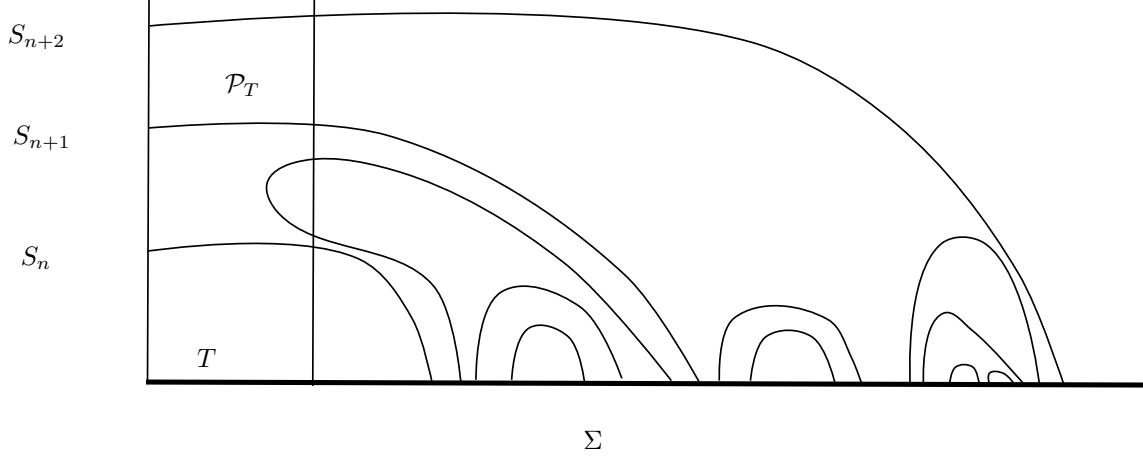


Figure 4.2: Schematic of the intersection of $\cup_{k \in \mathbb{N}} \mathcal{P}^{-1}(\partial M_k)$ in $\Sigma \times [0, \infty)$ where F, G are surfaces in $\mathcal{P}^{-1}(\mathcal{S} \setminus \mathcal{S}_T)$ and we assume that Σ is a connected infinite type surface.

\mathcal{P} that are of infinite type.

Claim: Let Δ_i be a component of infinite type, then up to a proper isotopy of \mathcal{P} supported in $\text{Im}(\mathcal{P}|_{\Delta_i \times [0, \infty)})$ we have that $\cup_{k \in \mathbb{N}} \partial M_k \cap \text{Im}(\mathcal{P}|_{\Delta_i \times [0, \infty)}) = \cup_{n \in \mathbb{N}} S_n^i$.

Proof of Claim: Via a proper isotopy of \mathcal{P} supported in $\text{Im}(\mathcal{P}|_{\Delta_i \times [0, \infty)})$ we can assume that

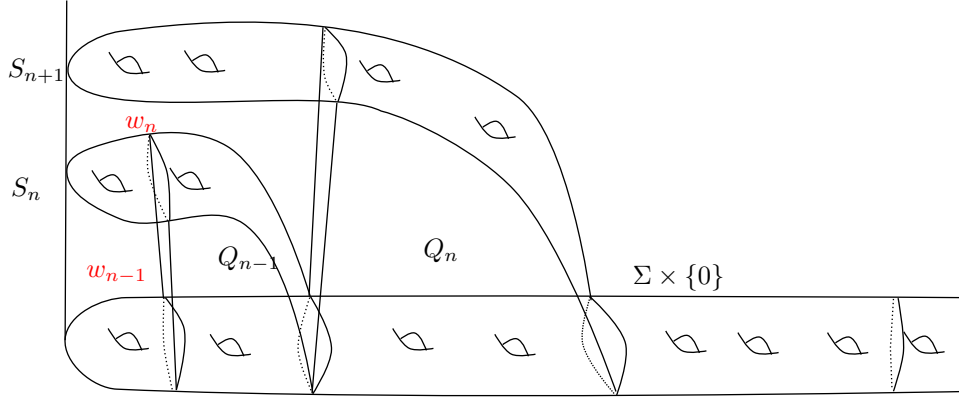
$$\mathcal{P}^{-1}(S_1) \subseteq \Sigma \times \{0\}$$

so that $\mathcal{P}(\Delta_i \times [0, \infty)) \cap \mathcal{S} = \mathcal{P}(\Delta_i \times [0, \infty)) \cap \mathcal{S}_T$. We will now do isotopies of the H_n^i relative to S_n^i, S_{n+1}^i . All components of $\mathcal{S}^i \doteq \mathcal{P}(\Delta_i \times [0, \infty)) \cap \mathcal{S}_T$ that are not $\{S_n^i\}_{n \in \mathbb{N}}$ are contained in a $H_n^i \cong S_{n+1} \times I$ and are essential. We denote this collection of components \mathcal{S}_n^i . By properness of \mathcal{P} and the fact that H_n^i is compact we get that \mathcal{S}_n^i has finitely many components L_1, \dots, L_k and $\mathcal{P}^{-1}(L_j)$ is isotopic to a subsurface $F_j \subseteq \Sigma \times \{0\}$. Moreover, since the $\mathcal{P}^{-1}(L_j)$ are pairwise disjoint and separating in H_n^i we can find an innermost one. That is if $F_j \subseteq \Sigma$ is the surface that $\mathcal{P}^{-1}(L_j)$ is properly isotopic to then there are no other components L_h of \mathcal{S}_n^i such that $\mathcal{P}^{-1}(L_h)$ is contained in the submanifold J bounded by $\mathcal{P}^{-1}(L_j) \cup F_j$. Then, by a proper isotopy of $\mathcal{P}|_{H_n^i}$ supported in $\text{Im}(\mathcal{P}|_{\Delta_i \times [0, \infty)})$ that is the identity on S_n^i, S_{n+1}^i we can push F_j to $\mathcal{P}^{-1}(L_j)$ to reduce $|\pi_0(\mathcal{S}_n^i)|$. Thus, by concatenating these finitely many isotopies we obtain a proper isotopy ψ_n^i of \mathcal{P} supported in $\text{Im}(\mathcal{P}|_{H_n^i})$ that is constant on S_n^i, S_{n+1}^i such that $\mathcal{P}^{-1}(\mathcal{S}_n^i) = S_n^i \amalg S_{n+1}^i$. Since the ψ_n^i are constant on S_n^i, S_{n+1}^i they can be glued together to obtain a proper isotopy of \mathcal{P} supported in

$\text{Im}(\mathcal{P}|_{\Delta_i \times [0, \infty)})$ so that $\cup_{k \in \mathbb{N}} \partial M_k \cap \text{Im}(\mathcal{P}|_{\Delta_i \times [0, \infty)}) = \cup_{n \in \mathbb{N}} S_n^i$. \square

So far we have $\Sigma \times [0, \infty) = \cup_{n \in \mathbb{N}} J_n$ and for all n $\mathcal{P}(J_n)$ are mixed I -bundles such that there exists k_n with $\mathcal{P}(J_n) \subseteq X_{k_n}$. Since $\mathcal{P}(J_n)$ is a mixed I -bundle we can decompose it in the window w_n and Q_n the non-window part, with an abuse of notation we denote their preimages in J_n by the same name.

Thus for $\Sigma_i \doteq \mathcal{P}^{-1}(S_i)$ in $\Sigma \times [0, \infty)$ we have the following configuration:



Step 2: Up to a proper isotopy of \mathcal{P} we have that for all k all components of $\text{Im}(\mathcal{P}) \cap X_k$ are essential I -bundles.

We will again do isotopies supported in $\text{Im}(\mathcal{P})$ so we can assume that all products are connected. We see that S_n is the boundary of the window w_n and the non-window part Q_{n-1} of the mixed I -bundle I_{n-1} is isotopic into w_n . Via an isotopy of \mathcal{P} supported in $X_{k_n} \cup X_{k_{n-1}}$ that is the identity on Σ_{n-1} and S_{n+1} we can isotope Q_{n-1} into $w_n \subseteq \mathcal{P}(J_n)$. The n -th isotopy is supported in neighbourhoods of the non-window part Q_n and the image is contained in w_{n+1} . Therefore, the support of the $n+1$ isotopy does not intersect w_{n-1} thus the composition of these isotopies yields a proper isotopy of \mathcal{P} .

Thus one gets a "staircase picture" in which every step is isotopic to a window in N_k , hence it is an essential I -bundle, and $\Sigma \times \{0\}$ is the boundary of the "stairs".

Step 3: Up to proper isotopy $\text{Im}(\mathcal{P}) \cap X_k \subseteq N_k$.

This follows from the corresponding **Step** in Theorem 4.1.19. \blacksquare

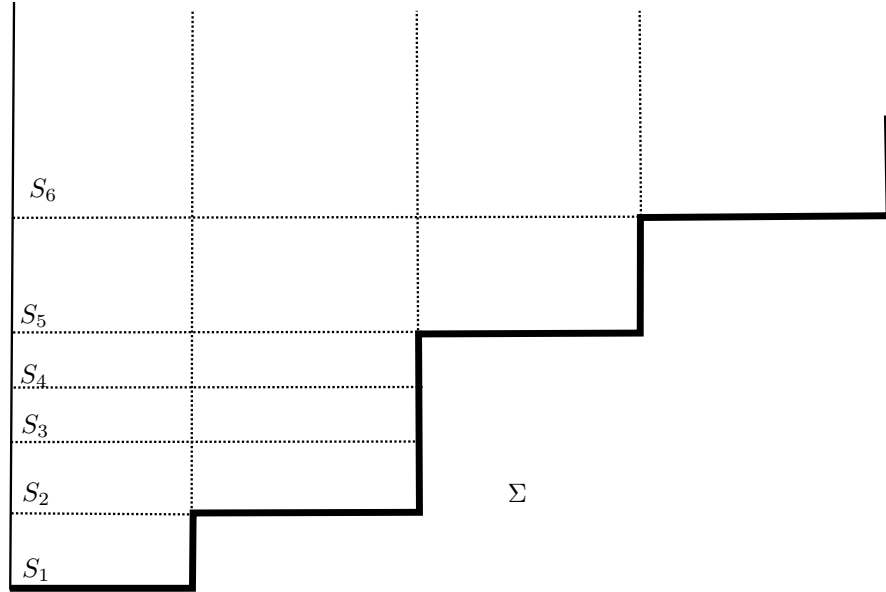


Figure 4.3: Where the surfaces S_i coming from the exhaustion induce products P_i whose union is P .

Thus we obtain:

Theorem 4.1.40. Let $M \in \mathcal{M}$ be an open 3-manifold. Then, there exists a unique maximal bordification $[(\overline{M}, \iota)] \in \mathbf{Bor}(M)$.

4.1.1.3 Minimal Exhaustions

If $M \in \mathcal{M}$ we have two types of I -bundles between the boundaries of the gaps in the compact exhaustion: *type I* products have as bases compact surfaces while *type II* products have as bases closed surfaces.

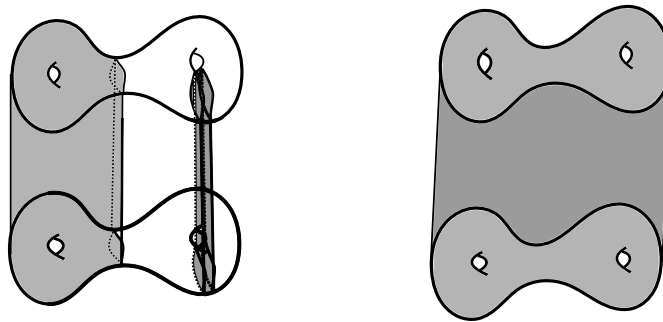


Figure 4.4: Type I and type II products.

We now want to show that type II products either correspond to tame ends of M or can be thrown away by modifying the exhaustion.

Definition 4.1.41. We say that a compact exhaustion $\{M_i\}_{i \in \mathbb{N}}$ of M is *minimal* if the two following conditions hold:

- (i) for all $i < j$: there are no pairs of closed orientable surfaces $F \in \pi_0(\partial M_i)$, $F' \in \pi_0(\partial M_j)$ such that $F \simeq F'$, unless they bound neighbourhoods U, V of the same tame end E of M_i ;
- (ii) for all i no component of $\overline{M \setminus M_i}$ is compact.

These conditions are so that the exhaustion has minimal redundancy.

Lemma 4.1.42. Let M be an irreducible 3-manifold with a compact exhaustion $\{M_i\}_{i \in \mathbb{N}}$ where each M_i has incompressible boundary then, M has a minimal exhaustion.

Proof. For the second condition of a minimal exhaustion we just look at the various $\overline{M \setminus M_i}$ and whenever we see a compact component we add it to all M_j with $j \geq i$. By repeating this process for all components of every ∂M_k we obtain an exhaustion that satisfies the first condition of a minimal exhaustion. With an abuse of notation we still denote this new exhaustion by $\{M_i\}_{i \in \mathbb{N}}$.

We now deal with the second condition. Since $M \setminus \text{int}(M_i)$ has no compact component no component of ∂M_i is homotopic to a component of ∂M_i in $M \setminus M_i$ since then by Lemma [60, 5.1] we would get that it is homeomorphic to an I -bundle. Assume that for $i < j$ we have two distinct closed incompressible surfaces F_i, F_j in $\partial M_i, \partial M_j$, respectively, that are homotopic in M_j . By replacing F_i and i , if needed, we can assume that i is minimal. By Lemma [60, 5.1] the surfaces F_i, F_j bound an I -bundle J in M_j . Up to an isotopy of J rel ∂J we can assume that $J \cap \partial M_k$, for $i \leq k \leq j$, are level surfaces in J . Then, either $J \subseteq \overline{M_j \setminus M_i}$ or by Lemma [60, 5.1] $M_i \cong F_i \times I$ and we have $J' \subseteq \overline{M_j \setminus M_i}$ given an isotopy from a component of ∂M_i to F_j .

Consider the connected component U of the gap $\overline{M_j \setminus M_i}$ containing the two surfaces. By Lemma [60, 5.1] we have that: $U \cong F_i \times I$. Then there are two cases:

- (i) either there is $k > j$ such that F_j is not homotopic to any other $F_k \in \pi_0(\partial M_k)$;
- (ii) $\forall k > j$ there is $F_k \in \partial M_k$ with $F_k \simeq F_i$.

In the first case we have a minimal $k \in \mathbb{N}$ with $k > j > i$ such that F_i is not homotopic to any $F_k \in \pi_0(\partial M_k)$. Then, by Lemma [60, 5.1] the connected component U of $\overline{M_{k-1} \setminus M_i}$ containing

F_i and the surface F_{k-1} that it is homotopic to is an I -bundle over F_i . Hence, we can modify our exhaustion by adding U to all M_s with $i \leq s < k-1$ and leave the other elements of the exhaustion unchanged.

In the latter case for all $k > i$ there is a boundary component $F_k \in \pi_0(\partial M_k)$ homotopic to F_i . Therefore, $\forall k > i$: the connected component $U_{k,i} \in \pi_0(\overline{M_k \setminus M_i})$ containing F_i, F_k is an I -bundle over F_i . Hence, we obtain an exhaustion by submanifolds homeomorphic to $F_i \times I$ of a connected component E of $\overline{M \setminus M_i}$. Thus $E \cong F_i \times [0, \infty)$ and E is a tame end of M . ■

4.1.2 Characteristic submanifold for bordifications of manifolds in \mathcal{M}

In this section we construct the *characteristic submanifold* (N, R) of \overline{M} . Specifically we will prove the following Theorem:

Theorem 4.1.43. The maximal bordification $(\overline{M}, \partial \overline{M})$ of $M \in \mathcal{M}$ admits a characteristic submanifold (N, R) and any two characteristic submanifolds are properly isotopic.

We will first define characteristic submanifolds for bordifications of manifolds in \mathcal{M} and postponing the proof of existence we prove some general facts about characteristic submanifolds and construct families of characteristic submanifolds for the exhaustion. The proof of Theorem 4.1.43 is divided into two sections, in which we first prove existence of characteristic submanifolds and then uniqueness.

4.1.2.1 Characteristic submanifolds

In this subsection we define characteristic submanifolds for the bordifications of manifolds in \mathcal{M} and describe their components.

Definition 4.1.44. Given 3-manifolds M, N and a π_1 -injective submanifold $R \subseteq \partial N$ a continuous map $f : (N, R) \rightarrow (M, \partial M)$ is *essential* if f is not homotopic via map of pairs to a map g such that $g(N) \subseteq \partial M$. Similarly we say that a submanifold N is *essential* in M if by taking $R \doteq N \cap \partial M$ then the embedding is essential.

In Definition ?? we defined a characteristic submanifold N for a compact irreducible 3-manifold with incompressible boundary M . In this setting characteristic submanifolds exists and are unique up to isotopy by work of Johannson [32] and Jaco-Shalen [31]. In the case that M is atoroidal, see [14, 2.10.2], we get that all components of N fall into the following types:

- (1) I -bundles over compact surfaces;
- (2) solid tori $V \cong \mathbb{S}^1 \times \mathbb{D}^2$ such that $V \cap \partial M$ is a collection of finitely many annuli;
- (3) thickened tori $T \cong \mathbb{T}^2 \times I$ such that $T \cap \partial M$ is a collection of annuli contained in $\mathbb{T}^2 \times \{0\}$ and the torus $\mathbb{T}^2 \times \{1\}$.

We now show that manifolds in \mathcal{M} are also atoroidal.

Lemma 4.1.45. Let $\overline{M} \in \mathbf{Bord}(M)$, for $M \in \mathcal{M}^B$, be the maximal bordification for $M \in \mathcal{M}$ then \overline{M} is atoroidal.

Proof. Let $\mathcal{T} : \mathbb{T}^2 \rightarrow \overline{M}$ be an essential torus and $\{M_i\}_{i \in \mathbb{N}}$ the exhaustion of M . By compactness of $\mathcal{T}(\mathbb{T}^2)$ we have that, up to a homotopy pushing $\mathcal{T}(\mathbb{T}^2)$ off of $\partial \overline{M}$, $\mathcal{T} : \mathbb{T}^2 \rightarrow \overline{M}$ factors through some M_i . Since M_i is atoroidal and $\mathcal{T}(\mathbb{T}^2) \subseteq M_i$ we have that $\mathcal{T}(\mathbb{T}^2)$ is homotopic into a torus component T of ∂M_i . For all $j > i$: by Waldhausen Cobordism's Theorem [60] T is isotopic in M_j to a torus component T_j of ∂M_j and so T, T_j cobound an I -bundle I_j . By the arguments of Theorem 4.1.19 up to an isotopy of I_j we can assume that $I_j \cap \partial M_k$ are level surfaces of I_j for $i \leq k \leq j$. Thus, either $I_j \cap M_i = T$ or $M_i \cong \mathbb{T}^2 \times I$ and then for all $j > i$ we have that $\overline{M_j} \setminus \overline{M_i} \cong \mathbb{T}^2 \times I \amalg \mathbb{T}^2 \times I$. In either case, we get that the component of $\overline{M_j} \setminus \overline{M_i}$ containing T, T_j is homeomorphic to $\mathbb{T}^2 \times I$ and since j was arbitrary $\mathcal{T}(\mathbb{T}^2)$ is homotopic into $\partial \overline{M}$. Therefore, every π_1 -injective torus in \overline{M} is homotopic into $\partial \overline{M}$ and so inessential. ■

In our setting we have $\overline{M} \in \mathbf{Bord}(M)$, for $M \in \mathcal{M}$, with $\text{int}(\overline{M})$ exhausted by compact hyperbolizable 3-manifolds M_i with incompressible boundary. Therefore, we have a collection $(N_i, R_i) \hookrightarrow (M_i, \partial M_i)$ of characteristic submanifolds whose components are of the form (1)-(3). Thus, since \overline{M} is atoroidal for a characteristic submanifold N of \overline{M} we expect the components of N to be of the following types:

- (i) I -bundles over compact incompressible surfaces;
- (ii) solid tori V with finitely many wings³;
- (iii) thickened essential tori $\mathbb{T}^2 \times I$ corresponding to a torus component of $\partial \overline{M}$ possibly with finitely many wings;
- (iv) limit of nested solid tori or of nested thickened essential tori.

³Recall that a *wing* is a thickening of an essential annulus A with one boundary component on the solid torus V and one on $\partial \overline{M}$.

Except for (iv) these are the same components that one finds in the usual JSJ decomposition of compact atoroidal 3-manifolds with incompressible boundary.

The difference for manifolds in \mathcal{M} is that we can have a countable family of nested solid tori or thickened essential tori having no parallel wings. One can think of these as *infinitely winged solid tori* (IWSD) or *infinitely winged essential tori* (IWET). These are solid tori V , or thickened tori T , with infinitely many wings, specifically in each M_i we have that $V \cap M_i$, or $T \cap M_i$, has a component that is isotopic to a solid torus $V_i \subseteq N_i$ or an essential thickened torus $T_i \subseteq N_i$ with a_i -wings and $a_i \nearrow \infty$.

We will now build such an example.

Example 4.1.46 (A 3-manifold with an infinitely winged solid torus.). Let $(N, \partial N)$ be an acylindrical and atoroidal compact 3-manifold with boundary an incompressible genus two surface (for example see [55, 3.3.12] or Appendix 5.1).

Let T be a solid torus with three wings winding once around the soul of the solid torus. The boundary of T is decomposed into 6 annuli, one for each wing and one between each pair of wings.

Consider the manifold obtained by gluing the annular end of the wings of T to three copies of $\Sigma_2 \times I^4$ along a neighbourhood of a curve $\gamma \subseteq \Sigma_2 \times \{0\}$ separating $\Sigma_2 \times \{0\}$ into two punctured tori. The resulting 3-manifold has for boundary six copies of Σ_2 . Three boundary components are coming from the three copies of $\Sigma_2 \times \{1\}$ and the other three are coming from gluing two punctured tori in the $\Sigma_2 \times \{0\}$'s along an annulus in the boundary of the solid torus.

By gluing 3 copies of N along the second type of Σ_2 we obtain a 3-manifold X as in the picture:

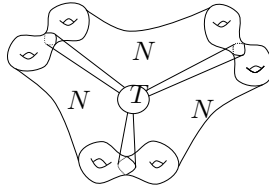


Figure 4.5: An X -piece.

The 3-manifold X is hyperbolizable with incompressible boundary and has the property that its characteristic submanifold is given by the solid torus with three wings T . We now construct a 3-manifold M by gluing together countably many copies $\{X_i\}_{i=1}^{\infty}$ of X and product manifolds

⁴By Σ_2 we mean a genus two surface.

$P \doteq \Sigma_2 \times [0, \infty)$. We denote by T_i the three winged solid torus in X_i . The gluing is given by the following tree pattern in which the gluing maps are just the identity:

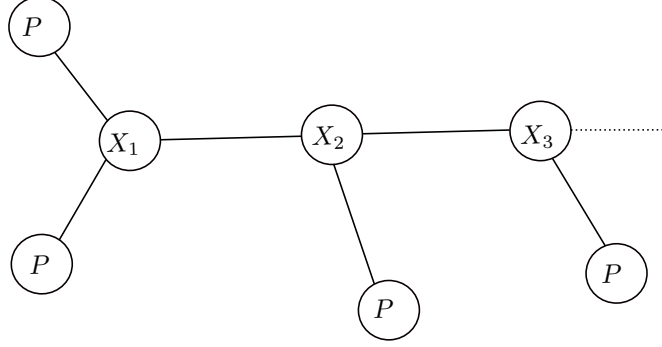


Figure 4.6: The gluing pattern for M .

The manifold M has a compact exhaustion given by taking M_i to be the manifold up to the i -th copy of X and the compact submanifolds of the product ends given by $\Sigma_2 \times [0, i]$. Hence, ∂M_i is formed by $2+i$ copies of Σ_2 all of which are incompressible. At each M_i the characteristic submanifold is a solid torus τ_i with $2+i$ wings. Moreover, since the M_i are atoroidal Haken 3-manifold by the Hyperbolization theorem [34] they are hyperbolizable. Therefore, since all boundaries of the M_i are incompressible and of genus two M is a manifold in \mathcal{M} .

Moreover, the JSJ submanifold of M_1 is given by the solid torus with 3-wings T_1 and the JSJ submanifold of the component of $\overline{M_j} \setminus \overline{M_{j-1}}$ that is not an I -bundle is also given by the solid torus T_j . Let T_∞ be the submanifold of M obtained by taking all the $\{T_j\}_{j \geq 1}$ and adding to it cylinders going to infinity in all the tame ends. Then T_∞ is an example of an *infinitely winged solid torus* since it is an open 3-manifold that compactifies to a solid torus V and is homeomorphic to $V \setminus L$ where $L \subseteq \partial V$ is a collection of pairwise disjoint isotopic simple closed curves forming a closed subset of ∂V . Namely $L = \left\{ L_{\frac{1}{n}} \right\}_{n \in \mathbb{N}} \cup L_0$ is in bijection with the ends of M where L_0 is the non-tame end and the $\left\{ L_{\frac{1}{n}} \right\}_{n \in \mathbb{N}}$ correspond to the tame ends. Moreover, it is topologised so that $L_{\frac{1}{n}} \rightarrow L_0$ as $n \rightarrow \infty$.

The maximal bordification \overline{M} of M has for boundary components an open annulus A and countably many genus two surfaces $\{\Sigma_i\}_{i \in \mathbb{N}}$. The annulus comes from the compactification of a product $P : \mathbb{A} \times [0, \infty) \hookrightarrow T_\infty$ going out the non-tame end that is contained in the interior of T_∞ . The genus

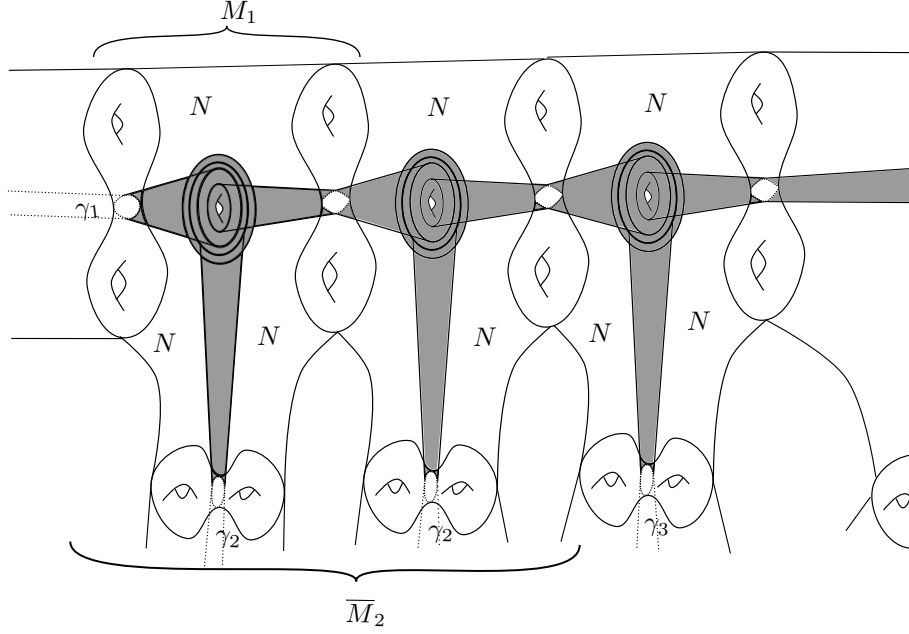


Figure 4.7: The tori T_j 's are in grey and the extra cylinders in the tame ends are dotted.

two surfaces Σ_i come from compactifying all the tame ends P_i . In the maximal bordification \overline{M} we have that the characteristic submanifold N is given by $N \doteq T_\infty \cup_{i \in \mathbb{N}} A_i \cup A'$ where the $A_i \subseteq \Sigma_i$ are the annuli that T_∞ limits to and $A' \subseteq A$ is a core annulus for A .

It is easy to modify the above example to obtain a 3-manifold containing an IWET by adding to any tame end P_i , along the boundary S_i , a compact hyperbolizable 3-manifold Y with incompressible boundary $\partial Y \cong T \cup \Sigma_2$ and such that a simple closed loop β in the boundary torus T is isotopic, in Y , to the separating curve of the genus two boundary Σ_2 glued to the separating loop of P_i .

Thus, we define:

Definition 4.1.47. Given a 3-manifold $M \in \mathcal{M}$ let $(\overline{M}, \partial \overline{M})$ be the maximal bordification, which could be M itself, then we define the *characteristic submanifold* $(N, R) \hookrightarrow (\overline{M}, \partial \overline{M})$ to be a codimension-zero submanifold satisfying the following properties:

- (i) every $\Sigma \in \pi_0(N)$ is homeomorphic to either:
 - an essential I -bundle over a compact surface;
 - an essential solid torus $V \cong \mathbb{S}^1 \times \mathbb{D}^2$ with $V \cap \partial \overline{M}$ a collection of finitely many parallel annuli or a non-compact submanifold V' that compactifies to a solid torus such that $V' \cap \partial \overline{M}$ are infinitely many annuli;

- an essential thickened torus $T \cong \mathbb{T}^2 \times [0, 1]$ such that $T \cap \partial \overline{M}$ is an essential torus and a, possibly empty, collection of parallel annuli in ∂T or a non-compact manifold T' that compactifies to a thickened torus such that $T' \cap \partial \overline{M}$ is an essential torus and infinitely many annuli;

(ii) $\partial N \cap \partial \overline{M} = R$;

(iii) all essential maps of an annulus $(\mathbb{S}^1 \times I, \mathbb{S}^1 \times \partial I)$ or a torus \mathbb{T}^2 into $(\overline{M}, \partial \overline{M})$ are homotopic as maps of pairs into (N, R) ;

(iv) N is minimal i.e. there are no two components of N such that one is homotopic into the other.

Lemma 4.1.48. Let $M = \cup_{i \in \mathbb{N}} M_i \in \mathcal{M}$ and $\{T_i\}_{i \in \mathbb{N}}$ be a collection of essential solid tori $T_i \subseteq M_i$ such that for $j > i$ $T_j \cap M_i$ is compact, contains T_i and $\overline{T_{i+1}} \setminus T_i$ are essential solid tori. Moreover, assume that the inclusion maps $\iota_i : T_i \hookrightarrow T_{i+1}$ induce isomorphisms on π_1 . Then the direct limit $T \doteq \varinjlim_i T_i$ is a properly embedded submanifold of M such that $T \cong \mathbb{S}^1 \times \mathbb{D}^2 \setminus L$ for L a closed subset of $\partial(\mathbb{S}^1 \times \mathbb{D}^2)$ consisting of parallel simple closed curves.

Proof. Since all inclusions induce isomorphism on the fundamental groups and $\pi_1(T) = \varinjlim_i \pi_1(T_i)$ we have that $\pi_1(T) \cong \mathbb{Z}$. A non-compact manifold N , possibly with boundary, is a *missing boundary manifold* if $N \cong \overline{N} \setminus L$ where \overline{N} is a manifold compactification of N and L is a closed subset of $\partial \overline{N}$. By Tucker's Theorem [59] the manifold T is a missing boundary manifold if the complement of every compact submanifold has finitely generated fundamental group. Since $T = \cup_{i=1}^\infty T_i$ it suffices to check the above condition for the T_i .

Let $Q \in \pi_0(\overline{T} \setminus T_i)$, then since $\overline{T_j} \setminus T_i$ are solid tori Q is either a solid torus or another direct limit of nested solid tori in which the inclusions induce isomorphism in π_1 . In either case $\pi_1(Q) \cong \mathbb{Z}$. Therefore T compactifies to \widehat{T} and T is homeomorphic to $\widehat{T} \setminus L$ where $L \subseteq \partial \widehat{T}$ is a closed set. Since \widehat{T} is compact, irreducible and $\pi_1(\widehat{T}) \cong \mathbb{Z}$ we have that $\widehat{T} \xrightarrow{\psi} V \doteq \mathbb{S}^1 \times \mathbb{D}^2$, see [26, Theorem 5.2].

Claim: Up to a homeomorphism of V the set L is a union of parallel curves.

Proof of Claim: Given the homeomorphism $\psi : T \rightarrow V \setminus L$ for $V = \mathbb{S}^1 \times \mathbb{D}^2$ we see that every $T_i \subseteq T$ is mapped to a solid torus $V_i \subseteq V \setminus L$ such that $\partial V_i = S_i \cup A_1^i \cup \dots \cup A_{n_i}^i$ where $S_i \subseteq \partial V \setminus L$ and the A_j^i 's are compact properly embedded annuli in V . Moreover, since the A_j^i 's are π_1 -injective embedded annuli in ∂V_i they are isotopic annuli in ∂V_i and are the images of the annuli of T_i contained

in ∂M_i to which new solid tori get glued in $\overline{M_{i+1} \setminus M_i}$ to obtain T_{i+1} . Moreover, since $V \setminus V_i$ is a collection of solid tori we have that each $A_{j_k}^i$ is ∂ -parallel in V . We define $\mathcal{A}_i \doteq \widehat{A}_1^i \cup \dots \cup \widehat{A}_{m_i}^i \subseteq \partial V$ for $\widehat{A}_{j_k}^i$ the annulus in $\overline{\partial V \setminus V_i}$ co-bounded by $\partial A_{j_k}^i$. Every component γ of L is given as a countable intersection of a sequence of the annuli \widehat{A}_j^i . Moreover, we fix a fiber structure on V such that it is fibered by circles. Then, for each V_i we want to construct a homeomorphism φ_i of V such that:

- $\varphi_i|_{V_{i-1}} = \varphi_{i-1}$;
- $\varphi_i(\mathcal{A}_i)$ are fibered annuli contained in $\varphi_{i-1}(\mathcal{A}_{i-1})$.

Assume we defined such a φ_j for all $j \leq i$. To define φ_{i+1} we only need to change φ_i in the solid tori co-bounded by $\varphi_i(\mathcal{A}_i)$ and $\varphi_i(\cup_{j=1}^{n_i} A_j^i)$ which lie in the complement of $\varphi_i(V_i)$. Each such solid torus W_k has boundary given by $\varphi_i(\widehat{A}_k^i \cup A_{j_k}^i)$ and in $\varphi_i(\widehat{A}_{j_k}^i)$ contains some of the annuli $\{\varphi_i(\widehat{A}_k^{i+1})\}_{1 \leq k \leq m_{i+1}}$. Then by an isotopy $\psi_{i,k}^t$ supported in $\varphi_i(\widehat{A}_k^i)$ that is the identity on $\partial \varphi_i(\widehat{A}_k^i)$ we can make the $\varphi_i(\widehat{A}_k^{i+1})$ fibered in $\varphi_i(\widehat{A}_j^i)$. By extending the isotopy $\psi_{i,k}^t$ to the solid torus and taking the time one map we obtain the required homeomorphism $\varphi_{i+1} \doteq \psi_{i,k}^1$.

Finally, the map $\varphi \doteq \varinjlim \varphi_i$ is a homeomorphism from $V \setminus L$ to $V \setminus L$ such that now $L = \cap_i \varphi_i(\cup_j \widehat{A}_j^i)$ where the $\varphi_i(\widehat{A}_j^i)$ are now compatibly fibered annuli. Thus every component of L is also fibered and if we assume that the φ_i are strictly contracting on the annuli we get that the components of L do not contain any annuli and are indeed parallel loops. \square

By taking the homeomorphism $\varphi \circ \psi : T \rightarrow V \setminus L$ we obtain the required conclusion. \blacksquare

Similarly we obtain:

Corollary 4.1.49. Let $M = \cup_{i \in \mathbb{N}} M_i \in \mathcal{M}$ and $\{T_i\}_{i \in \mathbb{N}}$ be a collection of essential tori such that for $j > i$ $T_j \cap M_i$ is compact, contains T_i and $\overline{T_{i+1} \setminus T_i}$ are essential solid tori or thickened essential tori. Moreover, assume that the inclusion maps $\iota_i : T_i \hookrightarrow T_{i+1}$ induce isomorphisms on π_1 . Then the direct limit $T \doteq \varinjlim T_i$ is a properly embedded submanifold of M such that $T \cong \mathbb{T}^2 \times (0, 1] \setminus L$ for L a closed subset of $\mathbb{T}^2 \times \{1\}$ consisting of parallel simple closed curves.

The two above Lemma deal with two of the types of components we expect to have. For I -bundles we have:

Lemma 4.1.50. Let $\overline{M} \in \mathbf{Bord}(M)$ be the maximal bordification of $M \in \mathcal{M}$ and let $\iota : (F \times I, F \times \partial I) \hookrightarrow (\overline{M}, \partial \overline{M})$ be an essential I -bundle over a connected surface F . Then, the surface F is compact.

Proof. Let \mathcal{P} denote the proper embedding $\iota|_{F \times (0,1)}$ and let $\{M_i\}_{i \in \mathbb{N}}$ be the exhaustion of $M \in \mathcal{M}$. Since \mathcal{P} is a proper π_1 -injective embedding by Lemma 4.1.11 we have that:

Step 1: Up to a proper isotopy of \mathcal{P} we can make all components of $\cup_{i \in \mathbb{N}} \partial M_i \cap \mathcal{P}$ be π_1 -injective subsurfaces of \mathcal{P} and no component is a disk.

We now claim that:

Step 2: Up to a proper isotopy of \mathcal{P} , supported in $\text{Im}(\mathcal{P})$, no component S of $\mathcal{S} \doteq \cup_{i \in \mathbb{N}} \partial M_i \cap \text{Im}(\mathcal{P})$ is a ∂ -parallel annulus.

Let \mathcal{A}_i be the collection of annuli of $\mathcal{S}_i \doteq \partial M_i \cap \text{Im}(\mathcal{P})$ that are ∂ -parallel in $\text{Im}(\mathcal{P})$. Since \mathcal{P} is a proper embedding we have that for all $i \in \mathbb{N}$ $|\pi_0(\mathcal{A}_i)| < \infty$. By an iterative argument it suffices to show the following:

Claim If for $1 \leq n < i$ $\mathcal{A}_n = \emptyset$ then via an isotopy φ_i^t of \mathcal{P} supported in $\overline{M \setminus M_{i-1}} \cap \text{Im}(\mathcal{P})$ we can make $\mathcal{A}_i = \emptyset$.

Proof of Claim: For all $i \in \mathbb{N}$ we have $0 < a_i < b_i < \infty$ such that $\mathcal{A}_i \subseteq \mathcal{P}(F_i \times [a_i, b_i])$ for $F_i \subseteq F$ a compact essential subsurface of F .

Denote by A_1, \dots, A_n the ∂ -parallel annuli in \mathcal{A}_i . By applying Corollary 4.1.13 to $\mathcal{P}(F_i \times [a_i, b_i])$ we have a local isotopy φ_i^t of \mathcal{P} that removes all these intersections. The isotopy φ_i^t is supported in a collection of solid tori $\mathcal{V}_i \subseteq F_i \times [a_i, b_i]$ such that $\partial \mathcal{V}_i \cap \partial F_i \times [a_i, b_i] \subseteq \partial F \times [a_i, b_i]$ and φ_i^t is the identity outside a neighbourhood of $\partial \mathcal{V}_i \setminus \partial F \times \mathbb{R}$ thus it can be extended to the whole of \mathcal{P} . Moreover, if we consider for $n < i$ a component of intersection of $\partial M_n \cap \mathcal{P}(\mathcal{V}_i)$ then it is either a boundary parallel annulus or a disk. However, we assumed that for $n < i$ $\mathcal{A}_n = \emptyset$ and by **Step 1** no component of $\cup_{k \in \mathbb{N}} \partial M_k \cap \text{Im}(\mathcal{P})$ is a disk thus, the solid tori \mathcal{V}_i that we push along are contained in $\text{Im}(\mathcal{P}) \cap \overline{M \setminus M_{i-1}}$. Therefore, we get a collection of solid tori $\mathcal{V}_i \subseteq \text{Im}(\mathcal{P}) \cap \overline{M \setminus M_{i-1}}$ such that pushing through them gives us an isotopy φ_i^t of \mathcal{P} that makes $\mathcal{A}_i = \emptyset$. \square

Since for all $i \in \mathbb{N}$ $\text{supp}(\varphi_i^t) = N_\varepsilon(\mathcal{V}_i)$ is contained in $\overline{M \setminus M_{i-1}}$ the limit φ^t of the φ_i^t gives us a proper isotopy of \mathcal{P} such that for all $i \in \mathbb{N}$ $\mathcal{A}_i = \emptyset$.

By **Step 2** every component S of \mathcal{S} the surface $\iota^{-1}(S)$ is an essential surface in $F \times \mathbb{R}$. In particular, since \mathcal{P} is properly embedded we have a component Σ of some ∂M_i such that $S \doteq \Sigma \cap \text{Im}(\mathcal{P}) \neq \emptyset$. If $S = \Sigma$ we get a contradiction since then we have an incompressible closed surface

$\iota^{-1}(S)$ in the I -bundle $F \times I$. Therefore, we must have that S is an essential proper subsurface of Σ with $\partial S \subseteq \mathcal{P}(\partial F \times \mathbb{R})$.

However, since the surface S is compact and properly embedded in \mathcal{P} we can find $0 < t_1 < t_2 < 1$ such that $\iota^{-1}(S) \subseteq F' \times (t_1, t_2)$ for $F' \subseteq F$ a compact surface such that not all boundary components of F' are boundary components of F . Since $\iota^{-1}(S)$ is an essential properly embedded subsurface of $F' \times [t_1, t_2]$ by [60, 3.1, 3.2] we have that $\iota^{-1}(S)$ is isotopic to an essential sub-surface S' of $F' \times \{t_1\}$. However, $\partial \iota^{-1}(S) \subseteq \partial F \times I$ which are a subset of $\partial F' \times I$ and so we get a contradiction since then $\partial S' \subsetneq \partial F'$ are zero in $H_1(F')$. \blacksquare

4.1.2.2 Existence of characteristic submanifolds

In this section we prove that if $M \in \mathcal{M}$ then the maximal bordification \overline{M} admits a characteristic submanifold. Before proving this existence statement we need to show that given $M \in \mathcal{M}$ and a normal family of characteristic submanifolds for the gaps we can find a family of characteristic submanifolds N_i of the M_i that are compatible with each others:

Proposition 4.1.51. Given $M = \cup_{i \in \mathbb{N}} M_i \in \mathcal{M}$ and a normal family of characteristic submanifolds $\{N_{i-1,i}\}_{i \in \mathbb{N}}$ for $X_i \doteq \overline{M_i} \setminus \overline{M_{i-1}}$, then we have characteristic submanifolds $C_i \subseteq M_i$ such that for all $i \in \mathbb{N}$ we have that for $j \geq i$: $C_j \cap M_i \subseteq C_i$ and $C_n \subseteq C_{n-1} \cup N_{n-1,n}$.

In the next series of Lemmas we will construct a family N_i 's of characteristic submanifolds for the M_i 's such that $N_i \cap M_j \subseteq N_j$ whenever $i > j$.

Lemma 4.1.52. Let $M_1 \subseteq \text{int}(M_2)$ be hyperbolizable 3-manifolds with incompressible boundary and let N_1, N_2 be their characteristic submanifolds. Given distinct components $P, Q \in \pi_0(N_2)$ if every component $P \cap M_1$ is an essential submanifold of M_1 and $P \cap M_1$ has a component isotopic into $Q \cap M_1$ then one of P or Q is an I -bundle over a surface F with $\chi(F) < 0$ and the other is either a solid torus or a thickened essential torus.

Proof. Let $S \in \pi_0(P \cap M_i)$ be a component isotopic into $S' \in \pi_0(Q \cap M_1)$. If $S \cong F \times I$, with $\chi(F) < 0$, then S and S' are isotopic into an I -bundle component of N_1 , thus P (or Q) is a sub-bundle of Q (or P) and we reach a contradiction since then they are not distinct components of N_2 .

If $S \cong \mathbb{S}^1 \times \mathbb{D}^2$ is a solid torus we have that either S' is an I -bundle $F' \times I$, with $\chi(F') < 0$, and S is homotopic into $\partial F' \times I$ or S' is either a solid torus or an essential thickened torus. In the first case we have that P is either a solid torus component or a thickened essential torus component of N_2 while Q is an I -bundle over a surface of negative Euler characteristic and we are done.

In the second case we have that S' is homeomorphic to a solid torus. Since S' is homotopic into S we can find an embedded annulus A in $M_1 \setminus S \cup S'$ connecting ∂S to $\partial S'$ and denote by A' a regular neighbourhood A intersecting P, Q only in neighbourhoods of $A \cap \partial P \cup \partial Q$. Since both P, Q are solid tori we get that $P \cup A' \cup Q$ is homeomorphic to a solid torus V . Thus we get an essential map: $f : V \rightarrow M_2$ whose image is $P \cup A' \cup Q$. By properties of characteristic submanifolds we have that V is homotopic into a component T of $N_2 \setminus P \cup Q$. However, this contradicts the minimality properties of N_2 since then $N_2 \setminus P \cup Q$ would also be characteristic.

Finally if $S \cong \mathbb{T}^2 \times I$ is isotopic into S' we have that S' is also homeomorphic to $\mathbb{T}^2 \times I$ and since they contain the same \mathbb{Z}^2 subgroup of $\pi_1(M_2)$ they are the same component of N_2 . \blacksquare

We now prove the iterative step of constructing a compatible family of characteristic submanifold.

Lemma 4.1.53. Let $M_1 \subseteq \text{int}(M_2)$ be hyperbolizable 3-manifolds with incompressible boundary and let (N_1, R_1) , (N_2, R_2) and (N_{12}, R_{12}) be characteristic submanifold of M_1, M_2 and $\overline{M_2 \setminus M_1}$ respectively. Moreover, assume that N_1, N_{12} form a normal family, then we can isotope N_2 in M_2 such that $N_2 \subseteq N_1 \amalg N_{12}$.

Proof. If, up to isotopy, $N_2 \cap M_1 = \emptyset$ then we can isotope N_2 so that $N_2 \subseteq N_{12}$ and there is nothing else to do. So can we assume that the intersection, up to isotopy, is not empty thus, some component of $N_2 \cap M_1$ is essential in M_1 .

Step 1: Up to an isotopy of N_2 we have that every component of N_2 intersects M_1 and $\overline{M_2 \setminus M_1}$ in essential I -bundles, essential solid tori or thickened essential tori.

By an isotopy of N_2 and a general position argument we can minimise $|\pi_0(\partial M_1 \cap N_2)|$ and have that $\partial M_1 \cap N_2$ are π_1 -injective surfaces, see Lemma 4.1.11.

Let $P \in \pi_0(N_2 \cap M_1)$ be a component of intersection coming from an I -bundle component $P' \cong F \times I$, with $\chi(F) < 0$, of N_2 . Since the components S of $P' \cap \partial M_1$ are essential and with boundary in the side boundary of the I -bundle P' by [60, 3.1, 3.2] they are isotopic to subsurfaces of the lids of the I -bundle region. Therefore, we have that $P \cong F \times I$ is an I -bundle and is essential since it is π_1 -injective.

If $P' \cong \mathbb{S}^1 \times \mathbb{D}^2$ is a solid torus component of N_2 then $A \doteq P' \cap \partial M_1$ is a collection of ∂ -parallel annuli in P' . The annuli A decompose P' into a collection of solid tori each of which is contained in either M_1 or $\overline{M_2 \setminus M_1}$. If a solid torus component T of $P' \cap M_1$ is inessential, i.e. it either is ∂ -parallel or it has, at least, two wings w_1, w_2 in M_1 that are parallel, then by an isotopy of N_2 that

either pushes P' outside of M_1 or pushes w_2 along w_1 outside of M_1 we can decrease $|\pi_0(\partial M_1 \cap N_2)|$ contradicting the assumption that it was minimal.

Similarly if $P \cong \mathbb{T}^2 \times I$ we have that ∂M_1 decomposes P into one essential thickened essential torus in M_1 and essential solid tori contained in M_1 and $\overline{M_2 \setminus M_1}$.

Moreover, by properties of a normal family we can assume that up to another isotopy supported in a neighbourhood U_1 of ∂M_1 we have that $N_2 \cap \partial M_1 \subseteq R_1 \cap N_{12}$.

Step 2: Up to an isotopy of N_2 we have that $N_2 \cap M_1 \subseteq N_1$.

Since $N_2 \cap M_1$ is a collection of essential Seifert-fibered 3-manifolds and I -bundles such that $N_2 \cap \partial M_1 \subseteq R_1$ by JSJ theory we can isotope them rel R_1 into N_1 .

Let $P \cong F \times I$ be an I -bundle component of N_2 with $\chi(F) < 0$. By Step 1 we can assume that $P \pitchfork \partial M_1$ and $K_P \doteq |\pi_0(P \cap \partial M_1)|$ is minimal. Moreover, we can assume that every component of $P \cap \partial M_1$ is in $R_1 = \partial N_1 \cap \partial M_1$. Then $P \cap M_1 = P_1 \amalg P_2 \amalg \dots \amalg P_n$ are essential I -bundles in M_1 . Thus, we can isotope the $\amalg_{i=1}^n P_i$ rel $\partial P_i \cap R_1$ into N_1 . We repeat this for all I -bundles of N_2 and by Lemma 4.1.52 we do not need to worry of them being parallel in M_1 . We denote by N'_2 the resulting submanifold. The submanifold N'_2 is isotopic to N_2 hence characteristic for M_2 .

Let $P \in \pi_0(N_2)$ be a solid torus component. By Step 1 and the fact that N_1, N_{12} form a normal family we have that each component of $P \cap \partial M_1$ is in R_1 . Then, P is decomposed by ∂M_1 into solid tori and annuli that are contained in M_1 and $\overline{M_2 \setminus M_1}$. Moreover, each such component is essential, thus every component of $P \cap M_1$ is either an essential solid torus with $k \geq 3$ wings in N_1 or a thickened cylinder. Each solid torus component is then isotopic into a solid torus component of N_1 and each annular component is isotopic into a solid torus or an I -bundle. Say that an annular component A of $P \cap M_1$ is isotopic into the side boundary of an I -bundle component $Q \cong F \times I$ of $N_2 \cap M_1$. Then, up to a further isotopy of Q we can assume that both A and Q are contained in N_1 .

The same process applies when $P \in \pi_0(N_2)$ is a thickened essential torus, the only difference is that if the boundary torus T is in M_1 we also have an essential thickened torus component in $N_2 \cap M_1$.

Step 3: Up to an isotopy supported in $\overline{M_2 \setminus M_1}$ of N_2 we have that $N_2 \cap \overline{M_2 \setminus M_1} \subseteq N_{12}$.

By Step 1 every component of $N_2 \cap \overline{M_2 \setminus M_1}$ is an essential I -bundle, a solid torus or a thickened

essential torus. Then, by JSJ theory we can isotope $N_2 \cap \overline{M_2 \setminus M_1}$ into N_{12} . Moreover, since the components of $N_2 \cap \partial M_1$ isotopic into $N_{12} \cap \partial M_1$ are already contained in $N_{12} \cap \partial M_1$, by properties of a normal family, we can assume that the isotopy is the identity on ∂M_1 .

The composition of the isotopies yields the required characteristic submanifold $N_2 \subseteq N_1 \cup N_{12}$. ■

Thus, we have:

Proposition 4.1.54. Given $M = \cup_{i \in \mathbb{N}} M_i \in \mathcal{M}$ and a normal family of characteristic submanifolds $\{N_{i-1,i}\}_{i \in \mathbb{N}}$ for $X_i \doteq \overline{M_i \setminus M_{i-1}}$, then we have characteristic submanifolds $C_i \subseteq M_i$ such that for all $i \in \mathbb{N}$ we have that for $j \geq i$: $C_j \cap M_i \subseteq C_i$ and $C_n \subseteq C_{n-1} \cup N_{n-1,n}$.

Proof. We start by defining $C_1 = N_1$ which obviously satisfies all the properties. Now suppose that we constructed the required collection up to level $n - 1$. Let \widehat{C}_n be any characteristic submanifold for M_n and apply Lemma 4.1.53 to C_{n-1}, \widehat{C}_n and $N_{n-1,n}$ to obtain a new characteristic submanifold C_n of M_n such that $C_n \subseteq C_{n-1} \cup N_{n-1,n}$. Then, for all $j < n$:

$$C_n \cap M_j \subseteq (C_{n-1} \cup N_{n-1,n}) \cap M_j = C_{n-1} \cap M_j \subseteq C_j$$

By iterating this step the result follows. ■

We construct the characteristic submanifold (N, R) of $(\overline{M}, \partial \overline{M})$ by picking specific components of the various (N_i, R_i) . Precisely, we want to pick the components that remain essential throughout the exhaustion, we call these components *admissible*. These will be components P of N_i with enough components of $P \cap R_i$ that generate a product in $M \setminus \text{int}(P)$. That is, if $S \cong \Sigma_{g,n}$ is a component of $P \cap R_i$ then we have a product $\mathcal{P} : \Sigma_{g,n} \times [0, \infty) \hookrightarrow M \setminus \text{int}(P)$ such that $\mathcal{P}(\Sigma_{g,n} \times \{0\}) = S \subseteq \partial M \setminus \text{int}(P)$.

Definition 4.1.55. Let $M = \cup_{i \in \mathbb{N}} M_i \in \mathcal{M}$ and let (N_i, R_i) be characteristic submanifolds of the $(M_i, \partial M_i)$'s. We say that an essential submanifold (P, Q) of (N_i, R_i) homeomorphic to a sub-bundle, solid torus or a thickened torus is *admissible* if one of the following holds:

- (i) Q has two components A_1, A_2 that generate in $M \setminus \text{int}(P)$ a product \mathcal{A} ;
- (ii) P is homeomorphic to an essential solid torus and Q has one component A that generates a product in $M \setminus \text{int}(P)$ and another component B such that B is the boundary of solid torus $V \subseteq M \setminus \text{int}(P)$ whose wings wrap $n > 1$ times around the soul of V ;

- (iii) P is a solid torus whose wings wrap $n > 1$ times around the soul of P and a component of Q generates in $M \setminus \text{int}(P)$ a product \mathcal{P} ;
- (iv) P is homeomorphic to an essential thickened torus.

Lemma 4.1.56. Let $M = \cup_{i \in \mathbb{N}} M_i \in \mathcal{M}$ and let (N_i, R_i) be characteristic submanifolds of the $(M_i, \partial M_i)$'s. Then, for each i there exists an admissible submanifold (P_i, Q_i) of (N_i, R_i) such that any admissible submanifold of (N_i, R_i) is isotopic into P_i .

Proof. Since N_i has finitely many component and any admissible submanifold of M_i is isotopic into N_i it suffices to prove the Lemma for a component of N_i . By Lemma 4.1.26 for every window $W \xrightarrow{\psi} F \times I$ over a hyperbolic surface F we get a maximal submanifold $Q \subseteq W$ such that, up to isotopy, Q contains all sub-bundles of W going to infinity. The manifold Q , up to isotopy, is homeomorphic via ψ to $F_1 \times [0, \frac{1}{3}] \amalg F_2 \times [\frac{2}{3}, 1]$ where F_1, F_2 are essential subsurfaces of F , which we can assume to be in general position. Then, $\Sigma \doteq F_1 \cap F_2$ is an essential sub-surface of F such that we have a proper embedding $\iota : \Sigma \times \mathbb{R} \hookrightarrow M$ in which $\iota(\Sigma \times [0, 1]) \subseteq N_i$, thus it is an admissible submanifold and we denote it by Q . We then add to Q a maximal collection of pairwise disjoint admissible solid tori contained N_i and all thickened tori components of N_i . Note that an essential torus $V \in \pi_0(Q)$ can be isotopic into a side bundle of a window $w \in \pi_0(Q)$ over a hyperbolic surface. We now show that any admissible submanifolds is isotopic into Q .

Let $P \xrightarrow{\psi} S \times I \subseteq F \times I$, $|\chi(S)| < 0$, be any admissible submanifold of W then by Lemma 4.1.26, up to isotopy, we have that $S \times \{1\} \subseteq F_2 \times \{1\}$ and $S \times \{0\} \subseteq F_1 \times \{0\}$, thus S is isotopic into $F_1 \cap F_2$ and hence P is isotopic into Q .

If P is an admissible solid torus or thickened torus then by JSJ theory is isotopic in a component of N_i that is either a component of Q or $P \cong \mathbb{S}^1 \times \mathbb{D}^2$ is isotopic into an I -bundle component of N_i . Since all other cases are contained in Q by construction we only need to show it for the latter case. Thus, we can assume that we have P isotopic to a vertical thickened annulus P' in a window $w \cong F \times I$ of N_i . We need to show that P is isotopic into Q . If $P' \cap Q = \emptyset$ and is not isotopic into an essential torus component of Q we contradict the maximality of Q . Therefore, we have that $P' \cap Q \neq \emptyset$. Thus, in the window $w \xrightarrow{\varphi} F \times I$ of N_i we have that $P' \xrightarrow{\varphi} A \times I$ for A an annulus and we have a component $W \xrightarrow{\varphi} S \times I$ of Q in which up to isotopy $A \cap S \neq \emptyset$ and are in minimal position with respect to each other so that $S \cup A$ is a subsurface of F . We now need to deal with various cases.

Say that W is admissible and of type (i) so that $\varphi(S \times \partial I)$ generate a product $\mathcal{P} : (S \times \partial I) \times [0, \infty) \hookrightarrow M \setminus \text{int}(W)$. If P is also of type (i) we get that it also generates a product $\mathcal{Q} : (A \times$

$\partial I) \times [0, \infty) \hookrightarrow M \setminus \text{int}(P)$. Thus, by adjoining Q to \mathcal{P} we can enlarge \mathcal{P} to a new product $\mathcal{P}' : (S' \times \partial I) \times [0, \infty) \hookrightarrow M \setminus \text{int}(W')$ for S' the essential sub-surface of F filled by $S \cup A$ and $W' \doteq \varphi(S' \times I)$ contradicting the maximality of Q .

Now assume that P is of type (ii) so that we have only one component A_1 of $\varphi(A \times \partial I)$ generating a product while A_2 has a root in $M \setminus \text{int}(P)$ which is contained in some solid torus V . Since V is compact let $k > i$ be such that $V \subseteq M_k$ and let $L \subseteq \partial M_i$ be the component containing A_2 . Let $M'_k \doteq \overline{M_k \setminus N_r(L)}$, then M'_k is irreducible, with incompressible boundary and atoroidal thus it has a characteristic submanifold N . Moreover, we have that $V \subseteq N$ and we also have an I -bundle induced $J \cong S \times I$ by \mathcal{P} such that they intersect essentially in a component of $\partial M'_k$. Then, we have a component of N containing both an I -bundle and a root of its boundary which is impossible. Similarly this takes care of the case in which W is of type (ii) and P of type (i). Thus, we are only left with the case in which both W and P are of type (ii).

Let $k > i$ be such that M_k contains both roots of the elements of $W \cap \partial M_i$ and $P \cap \partial M_i$ and consider as before M'_k and its characteristic submanifold N . Moreover, let S_1, S_2 be the surfaces induced by the regular neighbourhood of the component L of ∂M_i . Then, we either have two simple closed loops α, β both having a root in M'_k such that $\iota(\alpha, \beta) > 0$ which cannot happen or we have a component of N containing an I -bundle and a root of its boundary which also cannot happen.

Therefore, we get that every admissible submanifold of N_i is indeed isotopic into Q completing the proof. ■

The following two Lemmas says that essential annuli in \overline{M} are eventually essential in some M_i .

Lemma 4.1.57. Let $M \in \mathcal{M}$ and $C : (\mathbb{A}, \partial \mathbb{A}) \hookrightarrow (\overline{M}, \partial \overline{M})$ for $\overline{M} \in \mathbf{Bord}(M)$ the maximal bordification. If C is essential in \overline{M} there exists a minimal n and a proper isotopy of C such that all compact annuli of $\text{Im}(C) \cap M_n$ and $\text{Im}(C) \cap M \setminus \text{int}(M_n)$ are essential.

Proof. With an abuse of notation we will use C to denote $\text{Im}(C)$. Up to a proper isotopy of C that is the identity on $\partial \overline{M}$ we can assume that $C \pitchfork \partial M_i$ for all $i \in \mathbb{N}$. Now consider the minimal i such that $M_i \cap C \neq \emptyset$ and look at the components of $C \cap M_i$. If we have a component H of $C \cap M_n$ that is essential in M_i up to another proper isotopy of C we can push outside M_i all inessential components. Then, by looking at $C \cap M \setminus \text{int}(M_i)$ by a proper isotopy we can push inside M_i all inessential components. Note that by pushing components of $C \cap M \setminus \text{int}(M_i)$ into M_i we might change H to a component H' which is however isotopic to it hence still essential. Since all these isotopies decrease the number of components of $C \cap \partial M_i$ eventually we terminate and all annular components of $C \cap M_i$ and $C \cap M \setminus \text{int}(M_i)$ are essential. Therefore, by picking $n = i$ we are done.

If not it means that all components of $C \cap M_i$ are inessential and via an isotopy H_i of C we can push C outside of M_i so that for all $k \leq i$ we have that $C \cap M_k = \emptyset$. This process either stops at some $k \geq i$ and by picking $n = k$ we are done by the above case or we obtain a collection of isotopies $\{H_t^k\}_{k \geq i}$ that push C outside every compact subset of M and are the identity on $\partial \overline{M}$. We will denote by \widehat{C} the properly embedded annulus $C(\mathbb{S}^1 \times (0, 1)) \subseteq M = \text{int}(\overline{M})$ and without loss of generality we assume that M_1 is disjoint from \widehat{C} .

Claim: The annulus \widehat{C} is separating in M .

Proof of Claim: If $M \setminus \widehat{C}$ is connected there exists a loop $\alpha \subseteq \overline{M}$ such that $\alpha \cap \widehat{C} \neq \emptyset$. Moreover, for any isotopy H_t of \widehat{C} we still have that for all t : $H_t(\widehat{C}) \cap \alpha \neq \emptyset$. By compactness of α there exists i such that $\alpha \subseteq M_i$. Then since \widehat{C} can be isotoped outside every M_i we reach a contradiction and so \widehat{C} is separating in M . \square

Let E, \widehat{M} be the components of $\overline{M \setminus \widehat{C}}$ and assume that $M_1 \subseteq \widehat{M}$. For all $i \in \mathbb{N}$ there is a proper isotopy H_t^i of \widehat{C} , namely the one that pushes $\widehat{C} \cap M_i$ outside M_i , such that $H_1^i(\widehat{C}) \cap M_i = \emptyset$. Moreover, we have that $E \xrightarrow{\text{iso}} E_i$ for E_i the component of $\overline{M \setminus \text{Im}(H_1^i)}$ not containing M_1 and for all $i \in \mathbb{N}$ we have:

- (i) $E_i \cap M_i = \emptyset$;
- (ii) $E_{i+1} \subseteq E_i$;
- (iii) $E_{i+1} \setminus \text{int}(E_i)$ is compact and homeomorphic to a finite collection of solid tori.

Claim: The inclusion $\iota : \widehat{C} \hookrightarrow E$ induces a homotopy equivalence.

Proof of Claim: Since \widehat{C} and E are aspherical by Whitehead Theorem [24] it suffices to show that the map ι induces an isomorphism in π_1 . Since $\pi_1(\widehat{C})$ injects in M we only need to show that ι_* is a surjection. If ι_* is not surjective let $\alpha \subseteq E$ be a non-trivial loop that is not in the image $\iota_*(\pi_1(\widehat{C}))$ and let i be minimal such that $\alpha \subseteq M_i$. Then, we have a homotopy φ_t from α into $E_i \simeq E$ and since $H_1^i(\widehat{C})$ is separating we have that α is homotopic in E_i into $\partial E_i \xrightarrow{\text{iso}} \widehat{C}$ and so the inclusion map is a homotopy equivalence. \square

Claim: The submanifold E is tame, hence $E \cong V \setminus L$ where V is a solid torus and L is a simple closed curve in ∂V .

Proof of Claim: If we show that E is tame the fact that $E \cong V \setminus L$ follows by the fact that $\pi_1(E) \cong \mathbb{Z}$ and $\partial E = \widehat{C} \cong \mathbb{S}^1 \times (0, 1)$. To show that E is tame we will use the fact that $E \stackrel{\text{iso}}{\cong} E_i$, $E_{i+1} \subseteq E_i$ and Tucker's Theorem [59]. To show that E is tame we need to show that for any compact submanifold $K \subseteq E$ the fundamental group $\pi_1(E \setminus K)$ is finitely generated. Let i be such that $E_i \cap K = \emptyset$ and so that $E \setminus K = E_i \cup K'$ where by (iii) K' is a compact submanifold of E . Then by Van-Kampen's Theorem [24] we have:

$$\pi_1(E_i) * \pi_1(K') \twoheadrightarrow \pi_1(E \setminus K)$$

and so $\pi_1(E \setminus K)$ is finitely generated. \square

Since $E \cong \widehat{C} \times [0, \infty)$ by Theorem 4.1.36 we have a maximal bordification in which \widehat{C} compactifies to C' and is ∂ -parallel. Moreover, by uniqueness of the maximal bordification we have that $M' \stackrel{\psi}{\cong} \overline{M}$ and ψ induces an isotopy from C to C' . Contradicting the fact that C was essential in \overline{M} . \blacksquare

By Lemma 4.1.26 we define:

Definition 4.1.58. Given $M = \cup_{i \in \mathbb{N}} M_i \in \mathcal{M}$ we define the *boundary at infinity* of M_i to be the submanifold $\partial_\infty M_i \subseteq M_i$ to be the maximal, up to isotopy, submanifold of ∂M_i such that we have a simple product $\mathcal{P} : \partial_\infty M_i \times [0, \infty) \hookrightarrow M \setminus \text{int}(M_i)$ with the property that every other product $(F \times [0, \infty), F \times \{0\}) \hookrightarrow (M \setminus \text{int}(M_i), \partial M_i)$ is isotopic into $\mathcal{P}(\partial_\infty M_i \times [0, \infty))$. We also define the *bounded boundary* to be $\partial_b M_i \doteq \overline{\partial M_i} \setminus \overline{\partial_\infty M_i}$.

Example 4.1.59. For the manifold M of Example 4.1.46 for the elements of the exhaustion M_i we have that $\partial_\infty M_i$ is given by the collection of genus two surfaces corresponding to tame ends and an annulus in the genus two surface facing the non-tame end. The sub-surface $\partial_b M_i$ is given by two punctured tori contained in the genus two surface bounding the non-tame end.

We now extend the previous Lemma to non-embedded annuli.

Proposition 4.1.60. Let $M \in \mathcal{M}$ and $C : (\mathbb{A}, \partial \mathbb{A}) \rightarrow (\overline{M}, \partial \overline{M})$ for $\overline{M} \in \mathbf{Bord}(M)$ the maximal bordification. If C is essential in \overline{M} there exists a minimal i and a proper homotopy of C such that all compact components of $\text{Im}(C) \cap M_i$ and $\text{Im}(C) \cap M \setminus \text{int}(M_i)$ are essential and any $\mathbb{Z}^2 \subseteq \pi_1(\text{Im}(C))$ is induced by an annulus in $\text{Im}(C) \cap M_i$.

Proof. By compactness of the annulus we have a proper homotopy of C in \overline{M} so that we can assume that $C : (\mathbb{A}, \partial \mathbb{A}) \rightarrow (\overline{M}, \partial \overline{M})$ is an immersion that is in general position with $\cup_{k \in \mathbb{N}} \partial M_k$.

Case 1: Assume that $\pi_1(\text{Im}(C))$ does not contain any \mathbb{Z}^2 , so that, up to homotopy, the singular locus of C does not contain any essential double curve.

Up to homotopy we can find $i \in \mathbb{N}$ such that $\text{Im}(C) \setminus K_i \subseteq \partial_\infty M_i \times [0, \infty]$ for K_i a compact subset of $\text{Im}(C)$ and $\text{Im}(C) \cap \partial_\infty M_i \times [0, \infty] = \gamma_1 \times [0, \infty] \amalg \gamma_2 \times [0, \infty]$ for γ_1, γ_2 two, not necessarily simple, closed curves in $\partial_\infty M_i$.

If the γ_i are simple then by the fact that essential annuli in M_j and $\overline{M_j} \setminus \overline{M_{j-1}}$ with an embedded boundary component are homotopic to embedded essential annuli we obtain a compactly supported homotopy that makes C an embedding. Thus, we are done by Lemma 4.1.57. With an abuse of notation we will use C for $\text{Im}(C)$ and we will now deal with the case in which the γ_i are not simple.

Let $F_i \doteq \text{Fill}(\gamma_i)$ be the essential sub-surface of $\partial_\infty M_i$ filled by γ_i , for $i = 1, 2$. If F_i is not an annulus we have that $C \cap M_i$ is essential. Moreover, every compact component of $C \cap M \setminus \text{int}(M_i)$ is also essential, it is induced by a map of an I -bundle over the surface F_i , and so we are done. Thus, we can assume that F_i is homeomorphic to an annulus.

If F_i are annuli, we have that $\gamma_i \simeq \alpha_i^{n_i}$ with α_i simple. Then, by the previous argument we have an embedded annulus $C' : (\mathbb{A}, \partial\mathbb{A}) \hookrightarrow (\overline{M}, \partial\overline{M})$ such that C is properly homotopic into C' . By Lemma 4.1.57 there is a proper isotopy of C' and i such that all compact components of $C' \cap M_i$ and $C' \cap \overline{M} \setminus \overline{M_i}$ are essential. Thus, since up to a proper homotopy of C it is contained in a thickening of C' the result follows.

Case 2: Assume that $\pi_1(\text{Im}(C))$ contains a \mathbb{Z}^2 subgroup G .

Since, $\pi_1(M) = \cup_{i \in \mathbb{N}} \pi_1(M_i)$ there exists a minimal i such that $\pi_1(M_i)$ contains G . By hyperbolicity of the M_i we have that G is conjugated into a subgroup of $\pi_1(T)$ for T a torus in ∂M_i . Since, the torus T is compactified in \overline{M} we have a torus T_∞ such that T_∞, T cobound an I -bundle Q in \overline{M} . Moreover, up to an isotopy of Q each M_j intersects Q into a level surface.

If, up to homotopy, $C \subseteq Q$ then there is some M_j such that up to homotopy $M_j \cap C$ and the compact components of $\overline{M} \setminus \overline{M_j} \cap C$ are essential.

If C cannot be homotoped into Q we have a minimal $j \geq i$ such that $C \cap M_j \neq \emptyset$ and we claim it contains an essential component. If all components of $C \cap M_j$ are inessential we can homotope C such that $C \cap M_j = \emptyset$ and since C cannot be homotoped into Q we have that $C \subseteq \overline{M} \setminus \overline{M_j} \cup Q$. But $\pi_1(\text{Im}(C))$ contains G and G is conjugated into $\pi_1(T)$ with T a torus in ∂M_i . By tracing the homotopy from G into $\pi_1(T)$, we have a component S of $\partial M_i \setminus T$ that contains a \mathbb{Z}^2 subgroup that is homotopic in M_i into $\pi_1(T)$. Thus, we get that $M_i \cong \mathbb{T}^2 \times [-i, i]$ and so $M \cong M \times \mathbb{R}$ in

which $M_j \cong \mathbb{T}^2 \times [-j, j]$ and the result follows. Thus, we can assume that $C \cap M_i$ has essential components. Then, as in Lemma 4.1.57, up to a homotopy we can assume that $M_i \cap C$ and all compact components of $\overline{M \setminus M_i} \cap C$ are essential. \blacksquare

In the proof of existence of characteristic submanifolds for manifolds in M we will need the following fact about characteristic submanifold for compact 3-manifolds with incompressible boundary.

Corollary 4.1.61. If $\psi_n : (F \times I, \partial I) \hookrightarrow (M, \partial M)$, $n = 1, 2$, are essential I -bundles and M is compact, irreducible with incompressible boundary. If $\chi(F) < 0$ and $\psi_1(F \times \{0\}) = \psi_2(F \times \{0\})$ then up to isotopy we have that $\psi_1 = \psi_2$. If F is an annulus and we have a collection $\{\psi_n\}_{n \in \mathbb{N}}$ then the result is true up to sub-sequence.

We will use the following Lemma to show that the submanifold that we build in Proposition 4.1.63 contains, up to homotopy, all essential cylinders.

Lemma 4.1.62. Let $C : (\mathbb{A}, \partial \mathbb{A}) \rightarrow (\overline{M}, \partial \overline{M})$ be an essential cylinder such that every compact sub-annulus of $\text{Im}(C) \cap \overline{M \setminus M_i}$ and $\text{Im}(C) \cap M_i$ is essential and if $\mathbb{Z}^2 \subseteq \pi_1(\text{Im}(C))$ then it is induced by a sub-annulus contained in M_i . Then, every $A \subseteq \text{Im}(C) \cap M_i$ is homotopic into an admissible submanifold P of the characteristic submanifold N_i of M_i .

Proof. We $0 < a < b < 1$ such that $A = C(\mathbb{S}^1 \times [a, b])$ is an essential annulus in M_i , hence by JSJ theory is homotopic into a component Q of N_i . Moreover, we let C_i be the collection of essential annuli induced by C contained in M_i .

Case I Assume that $Q \stackrel{\varphi}{\cong} F \times I$ for F a hyperbolic surface.

Then, up to homotopy, we have that $A \stackrel{\varphi}{\cong} \gamma \times I$ for $\gamma \subseteq F$ a π_1 -injective closed curve. Thus, up to an ulterior homotopy we have that all compact components C_c of $\text{Im}(C) \cap \overline{M \setminus M_i}$ are also of the form $\gamma \times I$ and so are the other components of C_i . Thus, in $\overline{M \setminus \varphi(N_\varepsilon(\varphi(\gamma \times I)))}$ we get two I -bundles P_1, P_2 such that $C_i \cup C_c \subseteq P_1 \cup P_2 \cup \varphi(N_\varepsilon(\gamma \times I))$. After this homotopy we have that $\text{Im}(C) \setminus \text{int}(M_i)$ has two unbounded components C_1, C_2 that have for boundary on ∂M_i loops α_1, α_2 homeomorphic to γ . Then, by applying Corollary 4.1.61 to $\overline{M_n \setminus M_i}$ and a diagonal argument we obtain an embedded product $\widehat{C}_1, \widehat{C}_2 \subseteq \overline{M \setminus M_i}$ such that $\widehat{C}_i \cong \gamma \times [0, \infty)$ and so we get that $N_\varepsilon(A)$ is admissible.

Case II Assume that $Q \stackrel{\varphi}{\cong} \mathbb{S}^1 \times \mathbb{D}^2$ a solid torus of type T_k^n of N_i where n is the number of wings and k is the number of times that they wrap around the soul.

If $k > 1$ we have that T_k^n is admissible if a component B of $Q \cap \partial M_i$ is isotopic in $M \setminus \text{int}(Q)$ to infinity. Let A_Q be the collection of annuli of C_i homotopic into Q . Since C is a proper map we have finitely many such components and thus we can assume that there is $t > 0$ such that $C(\mathbb{S}^1 \times [t, 1)) \cap M_i$ has no components homotopic into Q and $C(\mathbb{S}^1 \times \{t\}) \subseteq Q \cap \partial M_i$. Thus we can assume that up to homotopy it is disjoint from Q . The loop $C(\mathbb{S}^1 \times \{t\})$ is homotopic to α^m for α the core curve of a component B of $V \cap \partial M_i$. Then, by Corollary 4.1.61 and a diagonal argument we get that $N_\varepsilon(\alpha) \subseteq \partial Q \subseteq \partial M_i$ generates a product in $M \setminus \text{int}(V)$ and so Q is admissible.

Similarly if $k = 1$ for each $A \subseteq A_V$ we have that $A \cap \partial Q$ is, up to homotopy, homeomorphic to α_1^m, α_2^m for α_1, α_2 core curves of components of $\partial V \cap \partial M_i$. Since $A_V \subseteq \text{Im}(C)$ is compact we have two components $A_1, A_2 \in \pi_0(A_V)$ such that $\text{Im}(C) \setminus A_1 \cup A_2$ has two unbounded component C_1, C_2 such that $\partial C_1, \partial C_2$ are α_1^m, α_2^k for α_i core curves of components w_1, w_2 of $\partial V \cap \partial M_i$. If the components w_1, w_2 are distinct by using Corollary 4.1.61 and a diagonal argument we get that $N_\varepsilon(\alpha_1 \cup \alpha_2) \subseteq \partial Q \subseteq \partial M_i$ generates a product in $M \setminus \text{int}(V)$ and so Q is admissible.

If $w_1 = w_2$ it means that either the annulus $C \setminus C_1 \cup C_2$ contains a \mathbb{Z}^2 subgroup and so it is contained in M_i and thus Q was not a solid torus or a compact component of $C \cap \overline{M} \setminus \overline{M}_i$ is contained in a T_k^n torus. Thus, by using Corollary 4.1.61 and a diagonal argument we get that $w_1 \subseteq \partial Q \subseteq \partial M_i$ generates a product in $M \setminus \text{int}(V)$ and so Q is admissible.

Case III Assume that $Q \cong \mathbb{T}^2 \times I$.

By definition these components are admissible and there is nothing to do. ■

We can now prove the existence of the characteristic submanifold for bordifications of manifolds in \mathcal{M} .

Theorem 4.1.63 (Existence of JSJ). Given $M \in \mathcal{M}$ there exists a maximal bordification \overline{M} with a characteristic submanifold (N, R) .

Proof. Let $\{N_i\}_{i \in \mathbb{N}}$ be a collection of characteristic submanifold of the M_i coming from Corollary 4.1.54 applied to a normal family $\{N_{i-1,i}\}_{i \in \mathbb{N}}$ of characteristic submanifolds for the $X_i \doteq \overline{M}_i \setminus \overline{M}_{i-1}$. Thus, we can assume that the $(N_i, R_i) \subseteq (M_i, \partial M_i)$ satisfy for all $i > j$: $N_i \cap M_j \subseteq N_j$ and for all i $N_i \subseteq N_{i-1} \cup N_{i-1,i}$.

We will construct N as a bordification of a nested union of codimension-zero submanifolds \widehat{N}_i . The submanifolds \widehat{N}_i will be obtained by taking admissible submanifolds of the characteristic submanifold $N_i \subseteq M_i$ and the \widehat{N}_i will satisfy the following properties:

- (i) $\forall j \geq k : \widehat{N}_j \cap M_k \subseteq N_k$ is compact;

(ii) if $P \subseteq N_j$ is an admissible submanifold then, up to isotopy, $P \subseteq \widehat{N}_j$;

(iii) $\forall k \leq j : \widehat{N}_j \cap M_k = \widehat{N}_k$.

Let \widehat{N}_1 be the maximal submanifold of N_1 containing all admissible submanifolds, see Lemma 4.1.56. Then, \widehat{N}_1 clearly satisfies (i)-(iii). We then proceed iteratively. Assume we have constructed \widehat{N}_i and start by defining $\widehat{N}_{i+1} \doteq \widehat{N}_i$.

Let $P \stackrel{\psi}{\cong} F \times I$ be an I -bundle component, with $\chi(F) < 0$, of \widehat{N}_i , then $\psi(F \times \partial I) \subseteq \partial M_i$. Since P is admissible we have that the surface $\psi(F \times \partial I)$ generates a product:

$$\mathcal{P} : (F \times \partial I) \times [0, \infty) \hookrightarrow M \setminus \text{int}(\psi(F \times I))$$

such that $\mathcal{P}(F \times \{0, 1\}) = \psi(F \times \partial I)$. Since $\mathcal{P}(F \times \{0, 1\})$ are already essential sub-surfaces of ∂M_i by Theorem 4.1.19 we have a proper isotopy of \mathcal{P} rel $\mathcal{P}(F \times \{0, 1\})$ such that \mathcal{P} is in standard form. Thus, in X_{i+1} there are finitely many essential I -bundles P_1, \dots, P_n in $N_{i,i+1}$ that connect a component of $\psi(F \times \partial I)$ to either ∂M_{i+1} or to another I -bundle P' in N_i . Since P is admissible we have that P' is also admissible and so it is contained in a component Q of \widehat{N}_i . Moreover, Q is also homeomorphic to $F \times I$. If not, we would have that P is an essential submanifold of a submanifold of M homeomorphic to $F' \times \mathbb{R}$ where up to isotopy $P \stackrel{\varphi}{\cong} F \times [0, 1] \subseteq F' \times [0, 1]$ is a sub-bundle. Thus, we would have that P is contained in a larger admissible submanifold of \widehat{N}_i contradicting the construction.

By adding all such P_ℓ 's to \widehat{N}_{i+1} and repeating it for all such I -bundles we have that \widehat{N}_{i+1} satisfies (i) and (iii) by construction.

Let $P \stackrel{\psi}{\cong} \mathbb{S}^1 \times \mathbb{D}^2$ a solid torus component of \widehat{N}_i . As before, we add to \widehat{N}_{i+1} all solid tori components of $N_{i,i+1}$ and thickened annuli contained in I -bundles of $N_{i,i+1}$ that match up with component of $P \cap \partial M_i$. Properties (i) and (iii) are still satisfied by construction. Similarly we do the case where $P \stackrel{\psi}{\cong} \mathbb{T}^2 \times I$. We now claim that the only admissible submanifolds of N_{i+1} that we are missing in \widehat{N}_{i+1} are contained in $N_{i,i+1}$

Claim: If $Q \subseteq N_{i+1}$ is admissible and, up to isotopy, $Q \cap M_i \neq \emptyset$ then we have that Q is isotopic into \widehat{N}_{i+1} .

Proof of Claim: Let Q be such a component then by Corollary 4.1.54 we have that, up to isotopy, $Q \subseteq N_i \cup N_{i,i+1}$ and $Q \cap N_i$ is an essential submanifold. Since Q is admissible in N_{i+1} and $Q_i \doteq Q \cap N_i$

is an essential submanifold we have that Q_i is also admissible in N_i . Therefore, Q_i is, up to isotopy, contained in \widehat{N}_i . Hence, by the above construction we get that $Q \subseteq \widehat{N}_{i+1}$. \square

Finally, we add to \widehat{N}_{i+1} all admissible solid tori, thickened essential tori and I -subbundles contained in N_{i+1} .

By construction we have that $\widehat{N}_{i+1} \cap M_i = \widehat{N}_i$ and \widehat{N}_i is compact thus (i) and (iii) are satisfied. Moreover, for all $i \in \mathbb{N}$ all components of \widehat{N}_i are I -bundles over hyperbolic surfaces, solid tori or thickened essential tori. Let $P \subseteq N_{i+1}$ be admissible then, up to isotopy, we have that $P \subseteq \widehat{N}_{i+1}$ and so \widehat{N}_{i+1} satisfies (i)-(iii).

Since by construction \widehat{N}_i does not change as we go through the construction we obtain a collection $\{\widehat{N}_i\}_{i \in \mathbb{N}}$ of nested codimension-zero submanifold satisfying (i)-(iii).

Let, $\widehat{N} \doteq \cup_{i=1}^{\infty} \widehat{N}_i \subseteq M$. Since every component of N has a natural 3-manifold structure and for all $k \in \mathbb{N}$:

$$\widehat{N} \cap M_k = \cup_{i=1}^{\infty} \widehat{N}_i \cap M_k \stackrel{(ii)}{=} \widehat{N}_k$$

is compact we get that N is a properly embedded codimension-zero submanifold. Moreover, \widehat{N} contains, up to isotopy, all admissible submanifolds since they appear in some \widehat{N}_j .

Let $P \in \pi_0(\widehat{N})$ then, by construction, P is either:

- homeomorphic to $F \times \mathbb{R}$ for F a compact surface;
- a nested union of solid tori $T_i \subseteq N_i$ such that $\overline{T_{i+1} \setminus T_i}$ are essential solid tori;
- a nested union of manifolds $Q_i \subseteq N_i$ each homeomorphic to $\mathbb{T}^2 \times [0, 1]$ and such that $\overline{Q_{i+1} \setminus Q_i}$ are essential solid tori.

If P is the limit of solid tori T_i then by Lemma 4.1.48 we have that $P \cong V \setminus L$ for V a solid torus and L a closed collection of parallel loops in ∂V . Similarly, if P is the limit of thickened essential tori: $\mathbb{T}^2 \times [0, 1]$ we get by Corollary 4.1.49 that P is homeomorphic to $\mathbb{T}^2 \times [0, 1] \setminus L$ for $L \subseteq \mathbb{T}^2 \times \{0\}$ a collection of parallel loops.

We will now add boundary to \widehat{N} . Let $P \stackrel{\psi}{\cong} F \times \mathbb{R}$ be an I -bundle component of \widehat{N} since P is properly embedded by adding $\text{int}(F) \times \{\pm\infty\}$ to M we can compactify P to \overline{P} in $\widehat{M} \in \mathbf{Bord}(M)$ so that $\overline{P} \stackrel{\bar{\psi}}{\cong} F \times I$ is an essential I -bundle in \widehat{M} . By repeating this for all components of \widehat{N} homeomorphic to $F \times \mathbb{R}$ we obtain a new manifold, which we still denote by \widehat{N} , properly embedded in \widehat{M} such that all I -bundles components are essential and compact.

Let $P \cong V \setminus L$ or $P \cong \mathbb{T}^2 \times [0, 1] \setminus L$ and consider the subset of loops $L^{iso} \subseteq L$ that are not

accumulated by any family of loops $\gamma_i \rightarrow \gamma$. Since each $\gamma \in L^{iso}$ is isolated it means that if we take a closed end neighbourhood U of γ it is homeomorphic to a properly embedded annular product $\mathbb{A} \times [0, \infty)$ which we can compactify in \widehat{M} by adding an open annulus to $\partial\widehat{M}$. Moreover, for all components homeomorphic to $\mathbb{T}^2 \times [0, 1) \setminus L$, $L \subseteq \mathbb{T}^2 \times \{0\}$, we add the corresponding boundary torus $\mathbb{T}^2 \times \{1\}$ to \widehat{M} . We still denote by \widehat{N} the resulting bordified manifold contained in $\widehat{M} \in \mathbf{Bord}(M)$.

Finally, for each γ in the set $L' \doteq L \setminus L^{iso}$ of an IWSL or an IWET by tracing through the gluing of the solid tori we obtain an embedded product $\mathcal{P}_\gamma : \mathbb{A} \times [0, \infty) \hookrightarrow \widehat{M}$. We also compactify \mathcal{P}_γ by adding an open annulus A_γ to $\partial\widehat{M}$ so that \mathcal{P}_γ is partially compactified i.e. :

$$\overline{\mathcal{P}}_\gamma : (\mathbb{A} \times [0, \infty) \cup A' \times \infty, A' \times \infty) \hookrightarrow (\widehat{M} \cup A_\gamma, A_\gamma)$$

where $A' \subsetneq \mathbb{A}$ is an annulus sharing only one boundary component with \mathbb{A} . Moreover, we have that \mathcal{P}_γ is properly isotopic into a collar neighbourhood of A_γ .

Then, we extend the bordification $\widehat{M} \in \mathbf{Bord}(M)$ to obtain a maximal bordification \overline{M} , see Theorem 4.1.36. Finally from N we remove any essential torus T that is properly homotopic into the side boundary of an I -bundle component.

To show that N is a characteristic submanifold we need to show that any essential annulus $A : (\mathbb{A}, \partial\mathbb{A}) \rightarrow (\overline{M}, \partial\overline{M})$ and essential torus $\mathcal{T} : \mathbb{T}^2 \rightarrow \overline{M}$ is homotopic into N . We first show that annuli can be homotoped into N .

By Proposition 4.1.60 for any essential annulus $A \doteq \mathcal{A}(\mathbb{A})$ in \overline{M} we have a proper homotopy of A and $i \in \mathbb{N}$ such that all compact sub-annuli of $A_i \doteq M_i \cap A$ and $A \cap \overline{M} \setminus \overline{M}_i$ are essential in M_i , $\overline{M} \setminus \overline{M}_i$ respectively. Thus, by Lemma 4.1.62 we have that A_i is homotopic not just into N_i but into \widehat{N}_i . Thus, we can assume that $A_i \subseteq \widehat{N}_i$.

Since all compact components A_c of $A \cap \overline{M} \setminus \overline{M}_i$ are essential there is a proper homotopy of A_c rel ∂M_i such that for all $k > i$ $A_c \cap X_k \subseteq N_{k,k-1}$. Moreover, since all components Q of $N_{k,k-1}$ containing sub-annuli of A_c math up with admissible components of N_i we get that $Q \subseteq \widehat{N}_k$. Thus, we have a proper homotopy such that $A_i \cup A_c \subseteq N$.

We will now do an iterative argument to construct homotopies rel ∂M_{j-1} , $j > i$, supported in $M \setminus \text{int}(M_{j-1})$ such that $A \cap X_j \subseteq \widehat{N}_j \cap X_j$.

Claim: If for $n < j$ we have that $A \cap X_n \subseteq \widehat{N}_n \cap X_n$ and all annuli of $A \cap X_j$ with boundary on ∂M_{j-1} are essential. Then, there is a proper homotopy rel ∂M_{j-1} supported in $M \setminus \text{int}(M_{j-1})$ such that $A \cap \overline{M} \setminus \overline{M}_{j-1}$ are essential and contained in $\widehat{N}_j \cap X_j$ and all annuli of $A \cap \overline{M} \setminus \overline{M}_j$ with

boundary on ∂M_j are essential.

Proof of Claim: All annuli of $A \cap X_j$ that are ∂ -parallel are induced by the unbounded components A_1, A_2 and have boundary on ∂M_j . Thus, by a proper homotopy φ_1 supported in $M \setminus \text{int}(M_{j-1})$ we can remove the ∂ -parallel annuli and guarantee that all compact components of $(A_1 \cup A_2) \cap \overline{M} \setminus \overline{M_j}$ with boundary on M_j are essential so that all annuli of $A \cap \overline{M} \setminus \overline{M_j}$ with boundary on ∂M_j are essential.

Since every annulus of $A \cap X_j$ is essential we have a homotopy φ_2 , supported in X_j , that by properties of normal family is the identity on ∂M_{j-1} such that $A \cap X_j \subseteq N_{j,j-1}$. If a sub-annulus C of $A \cap X_j$ satisfies $\partial C \subseteq \partial M_j$ then by Lemma 4.1.62 we have that A is contained in an admissible component and so $A \subseteq \widehat{N}_j$. If ∂A has a component contained in ∂M_{j-1} by properties of normal families and the fact that for all $n < j$ $A \cap X_n \subseteq \widehat{N}_n \cap X_n$ we get that A matches up with an admissible component and is so admissible. \square

The composition $\varphi \doteq \varinjlim_{j \geq i} \varphi_j$ gives a proper homotopy of A such that $A \subseteq \widehat{N}$.

Finally, let $\mathcal{T} : \mathbb{T}^2 \rightarrow \overline{M}$ be an essential torus, then by a homotopy we can assume that $\text{Im}(\mathcal{T}) \cap \partial \overline{M} = \emptyset$ and by compactness of $\text{Im}(\mathcal{T})$ we have M_i such that $\text{Im}(\mathcal{T}) \subseteq M_i$. Since the M_i are atoroidal we have that $\text{Im}(\mathcal{T})$ is homotopic into a torus component of $\partial_\infty M_i$ which is isotopic into a torus component T_∞ of $\partial \overline{M}$. Hence, we have a homotopy from $\text{Im}(\mathcal{T})$ into the component of N corresponding to T_∞ .

Claim: The manifold N is minimal, that is no component is homotopic into another.

Assume that $P, Q \in \pi_0(N)$ are such that P is properly homotopic into Q . By fundamental group reasons we get that $P \cong F \times I$ and $Q \cong \mathbb{S}^1 \times \mathbb{D}^2$ and that Q is properly homotopic into a side boundary of P and we removed all these redundancies. \blacksquare

4.1.2.3 Uniqueness of characteristic submanifolds

We will now show that any characteristic submanifold of \overline{M} can be put in a *normal form* so that they are contained in a pre-scribed family of normal characteristic submanifolds for the gaps of the exhaustion $\{M_i\}_{i \in \mathbb{N}}$. We will use this fact to show that any characteristic submanifold for \overline{M} is properly isotopic to the one constructed in Proposition 4.1.63.

Definition 4.1.64. Let $M = \cup_{k \in \mathbb{N}} M_k \in \mathcal{M}$ and $\iota : (N, R) \hookrightarrow (\overline{M}, \partial \overline{M})$ be a characteristic submanifold for the maximal bordification \overline{M} . Let $N' \doteq \iota(N \setminus R)$ then N is in *pre-normal form* if every

component of $\overline{N' \setminus \bigcup_{k \in \mathbb{N}} \partial M_k}$ is an I -bundle, a solid torus or a thickened torus that π_1 -injects in N' and every component of $N' \cap \bigcup_{k \in \mathbb{N}} \partial M_k$ is a π_1 -injective surface and no component is a ∂ -parallel annulus or a disk.

Given a normal family of characteristic submanifolds $\{N_k\}_{k \in \mathbb{N}}$ we say that N is in *normal form* with respect to the N_k 's if for all $k \in \mathbb{N}$ we have that each component of $N' \cap \overline{M_k \setminus M_{k-1}} \subseteq N_k$ is an essential submanifold of N_k .

Example 4.1.65. Note that by construction the characteristic submanifold constructed in Proposition 4.1.63 is in normal form.

Remark 4.1.66. The difference between pre-normal form and normal form is that if N is in pre-normal form but not in normal form then there exists a $k \in \mathbb{N}$ and a component Q of $N' \cap M_k$ such that Q is homeomorphic to either a solid torus or a thickened torus and it has at least two parallel wings. Equivalently it means that an annular component of $\partial Q \setminus \partial M_k$ is ∂ -parallel.

Moreover, if two wings of a solid torus $Q \subseteq M_k$ or $Q \subseteq \overline{M \setminus M_k}$ are parallel by an isotopy supported in $X_k \cup X_{k+1}$ we can slide one over the other and push it in $\overline{M \setminus M_k}$ or M_k respectively.

We now prove a Lemma needed to show that characteristic submanifolds can be put in pre-normal form.

Lemma 4.1.67. Let $\iota : (N, R) \hookrightarrow (\overline{M}, \partial \overline{M})$ be a characteristic submanifold and let $N' \doteq N \setminus R$. Let $\mathcal{S} \doteq \text{Im}(\iota) \cap \bigcup_{k \in \mathbb{N}} \partial M_k$ and assume that every component of \mathcal{S} is π_1 -injective and no component of \mathcal{S} is a disk. Then, there is a proper isotopy ψ_t of ι supported in $\iota(N')$ such that no component S of \mathcal{S} is a boundary parallel annulus in $\iota(N')$.

Proof. Let \mathcal{A}_k be the collection of annuli of $\mathcal{S}_k \doteq \mathcal{S} \cap \partial M_k$ that are ∂ -parallel in $\iota(N')$. Since ι is a proper embedding we have that for all $k \in \mathbb{N}$ $|\pi_0(\mathcal{A}_k)| < \infty$. Moreover, since N is a characteristic submanifold every ∂ -parallel annulus $A \subseteq \mathcal{A}_k$ is contained in a component of N' homeomorphic to either an \mathbb{R} -bundle, a missing boundary solid torus V or thickened essential torus T . By an iterative argument it suffices to show the following:

Claim: If for $1 \leq n < k$ $\mathcal{A}_n = \emptyset$ then via an isotopy φ_k^t of ι supported in $\overline{M \setminus M_{k-1}} \cap \text{Im}(\iota)$ we can make $\mathcal{A}_k = \emptyset$.

Proof of Claim: Denote by A_1, \dots, A_n the ∂ -parallel annuli in \mathcal{A}_k and assume that A_1, \dots, A_{n_1} are contained in \mathbb{R} -bundle components of N' and A_{n_1+1}, \dots, A_n are contained in missing boundary solid tori or thickened tori.

Since the annuli contained in \mathbb{R} -bundles are finitely many we have a disconnected compact horizontal surface F_k in N' such that A_1, \dots, A_{n_1} are contained in $\iota(F_k \times [a_k, b_k])$. By applying Corollary 4.1.13 to each component of $F_k \times [a_k, b_k]$ we have a local isotopy φ_k^t of ι that removes all these intersections. The isotopy φ_k^t is supported in a collection of solid tori $\mathcal{V}'_k \subseteq F_k \times [a_k, b_k]$ thus it can be extended to the whole of ι . Moreover, if we consider for $n < k$ a component of intersection of $\partial M_n \cap \mathcal{P}(\mathcal{V}_k)$ then it is either a boundary parallel annulus or a disk. However, we assumed that for $n < k$ $\mathcal{A}_n = \emptyset$ and by hypothesis no component of $\cup_{k \in \mathbb{N}} \partial M_k \cap \text{Im}(N')$ is a disk thus, the solid tori $\iota(\mathcal{V}'_k)$ that we push along are contained in $\iota(N') \cap \overline{M \setminus M_{k-1}}$.

Similarly, consider the annuli A_{n_1+1}, \dots, A_n then by Lemma 4.1.12 we have a collection of solid tori $\mathcal{V}''_k \subseteq N'$ such that by pushing along $\iota(V''_k)$ we obtain an isotopy φ_k^t of ι so that $\mathcal{A}_k = \emptyset$ and as before the solid tori $\iota(\mathcal{V}''_k)$ are contained in $\iota(N') \cap \overline{M \setminus M_{k-1}}$ thus φ_k^t is supported in $\overline{M \setminus M_{k-1}} \cap \text{Im}(\iota)$. Moreover, note that $\mathcal{V}'_k \cap \mathcal{V}''_k = \emptyset$ since they are contained in disjoint components of N .

Therefore, we get a collection of solid tori $\mathcal{V}_k \doteq \mathcal{V}'_k \amalg \mathcal{V}''_k$ contained in $\text{Im}(\iota) \cap \overline{M \setminus M_{k-1}}$ such that pushing through them gives us an isotopy φ_k^t of \mathcal{P} that makes $\mathcal{A}_k = \emptyset$. \square

Since for all $k \in \mathbb{N}$ $\text{supp}(\varphi_k^t) = \mathcal{V}_k$ is contained in $\overline{M \setminus M_{k-1}}$ the limit φ^t of the φ_k^t gives us a proper isotopy of \mathcal{P} such that for all $k \in \mathbb{N}$ $\mathcal{A}_k = \emptyset$. \blacksquare

By the Lemma we have:

Proposition 4.1.68. Given a characteristic submanifold N of the maximal bordification \overline{M} of $M \in \mathcal{M}$ there is a proper isotopy such that N is in pre-normal form.

Proof. Since $N \cap \text{int}(\overline{M})$ is a π_1 -injective submanifold of $M \cong \text{int}(\overline{M})$ by Lemma 4.1.11 we have a proper isotopy of N such that for all $k \in \mathbb{N}$ $\partial M_k \cap N$ are π_1 -injective surfaces in N and no component of $\mathcal{S} \doteq \text{Im}(\iota) \cap \cup_{k \in \mathbb{N}} \partial M_k$ is a disk.

Then, by Lemma 4.1.67 we have a proper isotopy of N' such that no component of \mathcal{S} is a ∂ -parallel annulus. Therefore, for all components S of \mathcal{S} the surface $\iota^{-1}(S)$ is an essential surface in N' . By the proof of Lemma 4.1.15 we get that up to a proper isotopy of N' supported in the \mathbb{R} -bundle components every essential surface $\iota^{-1}(S)$ in an \mathbb{R} -bundle component is horizontal. Thus, \mathbb{R} -bundles components of N' are decomposed by $\cup_{k \in \mathbb{N}} \partial M_k$ into I -bundles contained in $X_k \doteq \overline{M_k \setminus M_{k-1}}$.

Let $\mathcal{S}' \subseteq \mathcal{S}$ be the collection of components S of \mathcal{S} such that $\iota^{-1}(S)$ is not contained in an \mathbb{R} -bundle. Each component S of \mathcal{S}' is either an essential annulus or an essential torus. Since all essential tori are contained in products $\mathbb{T}^2 \times [0, \infty)$ by the proof of Lemma 4.1.15 we have a proper

isotopy supported inside a, possibly infinite, collection of products over essential tori such that the pre-images under ι are essential tori of \mathcal{S}' which co-bound I -bundles.

Let $\mathcal{A}_k \subseteq \mathcal{S}'$ be the collection of essential annuli of $\mathcal{S}' \cap X_k$. Then $\iota^{-1}(\mathcal{A}_k)$ are essential annuli contained in a component Q of N' that is homeomorphic to either a missing boundary solid torus V or a missing boundary thickened essential torus T . In either case they co-bound either a solid torus or a thickened torus and the Lemma follows. \blacksquare

We now show that characteristic submanifolds in pre-normal form can be isotoped to be in normal-form. To prove the iterative step we need:

Lemma 4.1.69. Let $M = \cup_{k \in \mathbb{N}} M_k \in \mathcal{M}$ and $\iota : (N, R) \hookrightarrow (\overline{M}, \partial \overline{M})$ be a characteristic submanifold in pre-normal form for the maximal bordification \overline{M} and let $N' \doteq \iota(N \setminus R)$. Let \mathcal{A}_n be the collection of annuli of $\overline{\partial N' \setminus \partial M_n}$ that are ∂ -parallel in either M_n or $M \setminus \text{int}(M_n)$. If for all $1 \leq n < k$ we have that $\mathcal{A}_n = \emptyset$ then there is a proper isotopy Ψ_k^t of ι supported in $\overline{M \setminus M_{k-1}}$ such that $\mathcal{A}_k = \emptyset$.

Proof. We have that $|\mathcal{A}_m|$ is bounded by $b_m \doteq |\pi_0(\partial M_m \cap \partial N')|$ which is finite by properness of the embedding.

Claim: Let A be an annulus in \mathcal{A}_n , for $n \in \mathbb{N}$ such that:

$$A \cap \cup_{m=1}^n \partial M_m = \partial A \subseteq \partial M_n$$

and A is inessential. Then, there exists a solid torus $V \subseteq \overline{M \setminus M_{n-1}}$ containing $\iota(A)$ such that all components of $\iota(N) \cap V$ are inessential solid tori in either M_n or $\overline{M \setminus M_n}$.

Proof of Claim: The annulus $\iota(A)$ is ∂ -parallel so it co-bounds with an annulus $C \subseteq \partial M_n$ a solid torus $V \subseteq \overline{M \setminus M_{n-1}}$. Consider a component Q of $\iota(N) \setminus A \cap V$, then $Q \cap \partial M_n \neq \emptyset$ and since $Q \cap \partial M_n \subseteq C$ it is a collection of annuli B . Since $\iota(N) \cap \partial M_n$ has no ∂ -parallel annuli we have that all the annuli B are essential in $\iota(N)$ hence since $Q \subseteq V$ it must be a solid torus contained in a component of N' homeomorphic to either a solid torus or a thickened essential torus, both potentially missing boundary. Moreover, since $Q \subseteq V$ and $\partial V \cap \partial M_n$ is an annulus we have that Q is inessential in either M_n or $\overline{M \setminus M_n}$. \square

Let A be an element of \mathcal{A}_k and assume that $A \subseteq M_k$ and that $A \cap \partial M_{k-1} \neq \emptyset$. Denote by Q the component of $N \cap M_k$ containing A . Since A is ∂ -parallel we have that any essential sub-annulus $A' \subseteq A$ with boundaries on ∂M_{k-1} , such that $A' \subseteq M_{k-1}$ is also ∂ -parallel. Since $A \cap \partial M_{k-1} \neq \emptyset$

and $A \pitchfork \partial M_{k-1}$ the annulus A is decomposed by ∂M_{k-1} into annuli $A_1^{k-1}, \dots, A_h^{k-1}$ such that $A_j^{k-1} \cap \partial M_{k-1} \subseteq A_j^{k-1} \cap \partial M_{k-1}$ and for $1 < j < h$ A_j^{k-1} has both boundary components on ∂M_{k-1} . Moreover, since $h \geq 3$ and A is ∂ -parallel in M_k we get that $\mathcal{A}_{k-1} \neq \emptyset$ reaching a contradiction. Thus, we can assume that any annulus A satisfies:

$$A \cap \bigcup_{m=1}^k \partial M_m = \partial A \subseteq \partial M_k$$

and is inessential. By the Claim we have a solid torus V such that $V \cap \iota(N)$ are inessential solid tori Q_1, \dots, Q_{m_k} . By a proper isotopy of $\iota(N)$ supported in $V \subseteq \overline{M \setminus M_{k-1}}$ we can push all inessential tori Q_1, \dots, Q_{m_k} contained in V in either M_k or $\overline{M \setminus M_k}$ and in either case we reduce b_k by at least $2m_k > 0$. Thus, we can assume that $V \cap \iota(N) = A$ and A is ∂ -parallel. Let Q be the solid torus or thickened torus of $\iota(N) \cap M_k$ or $\iota(N) \cap \overline{M \setminus M_k}$ containing A . Then by Remark 4.1.66 we have a proper isotopy supported in $\overline{M \setminus M_{k-1}}$ that reduces b_k by at least one and removes A from \mathcal{A}_k . Finally, since \mathcal{A}_k has finitely many elements the composition of these isotopies gives us a proper isotopy Ψ_k^t of ι such that $\mathcal{A}_m = \emptyset$ for $m \leq k$. Moreover, since all the isotopies are supported in $\overline{M \setminus M_{k-1}}$ we get that Ψ_k^t is also supported outside M_{k-1} . ■

Proposition 4.1.70. Let $\iota : (N, R) \hookrightarrow (\overline{M}, \partial \overline{M})$ be a characteristic submanifold for the maximal bordification \overline{M} of $M = \bigcup_{i \in \mathbb{N}} M_i \in \mathcal{M}$. Given a normal family of characteristic submanifolds $\{N_i\}_{i \in \mathbb{N}}$ for $X_i \doteq \overline{M_i \setminus M_{i-1}}$, there is a proper isotopy of ι such that N is in normal form.

Proof. By Lemma 4.1.68 we can assume that N is in pre-normal form and let $N' \doteq \iota(N \setminus R)$.

Step 1: Up to a proper isotopy we have that for all $i \in \mathbb{N}$ $\iota(N') \cap X_i$ is a collection of essential, pairwise disjoint I -bundles, solid tori and thickened tori.

Since $\iota(N') \cap X_i$ is in pre-normal form we have that $\iota(N') \cap X_1$ is a collection of I -bundles, solid tori and thickened tori.

Let \mathcal{A}_1 be the collection of annuli of $\iota(\partial N') \cap X_1$ and $\iota(\partial N') \cap \overline{M \setminus M_1}$ that are ∂ -parallel. By properness of the embedding we have that \mathcal{A}_1 has finitely many components. Then, by Lemma 4.1.69 we get a proper isotopy Ψ_1^t such that all annuli of $\iota(\partial N') \cap M_1$ and $\iota(\partial N') \cap \overline{M \setminus M_1}$ are essential. Therefore, by Remark 4.1.66 we have that all components of $\iota(N') \cap X_1$ are essential.

We now proceed iteratively. Assume that we made for $1 \leq n < k$ all annuli $Q \in \mathcal{A}_n$ essential. Then, by applying Lemma 4.1.69 to \mathcal{A}_k we obtain a proper isotopy Ψ_k^t supported in $\overline{M \setminus M_{k-1}}$ that

makes all annuli $Q \subseteq \iota(\partial N') \cap M_k \cup \iota(\partial N') \cap \overline{M \setminus M_k}$. In particular we get that for all $1 \leq n \leq k$ all components of $\iota(N') \cap X_n$ are essential.

Since the isotopies Ψ_k^t are supported in $\overline{M \setminus M_{k-1}}$ the limit composition is a proper isotopy $\Psi^t \doteq \varinjlim_{k \in \mathbb{N}} \Psi_k^t$ of ι such that for all $k \in \mathbb{N}$ every component of $N' \cap X_k$ is either an essential I -bundle, essential solid torus or an essential thickened torus.

Step 2: Up to a proper isotopy we have that N is in normal form.

By **Step 1** we have that for all $i \in \mathbb{N}$ the components of $N' \cap X_i$ are a collection of essential, pairwise disjoint I -bundles, solid tori and thickened tori. Consider $X_1 = M_1$ then by JSJ theory we can isotope $N' \cap X_1$ so that $N' \cap X_1 \subseteq N_1$. Moreover, since $N' \cap \partial M_1$ is, up to isotopy, contained in both R_1 and R_2 ⁵ by definition of normal family we can assume that $N' \cap \partial M_1$ is contained in $R_{1,2} \doteq R_1 \cap R_2$. This isotopy is supported in a neighbourhood of X_1 , hence it can be extended to a proper isotopy Ψ_1^t of N' . We will now work iteratively by doing isotopies relative $R_{k,k+1} \doteq R_k \cap R_{k+1}$.

Assume that we isotoped N' such that for all $1 \leq n \leq k$ we have that $N' \cap X_n \subseteq N_n$ and such that $N' \cap \partial M_n$ is contained $R_{n,n+1}$. Since the components of $N' \cap X_{k+1}$ are essential, pairwise disjoint I -bundles, solid tori and thickened tori of X_{k+1} with some boundary components contained in $R_{k,k+1}$ we can isotope them rel $R_{k,k+1}$ inside N_{k+1} so that their boundaries are contained in $R_{k,k+1} \coprod R_{k+1,k+2}$. This can be extended to an isotopy Ψ_{k+1}^t of \mathcal{P} whose support is contained in $\overline{M \setminus M_k}$, hence the composition of these isotopies gives a proper isotopy of N' such that $\forall i \in \mathbb{N} : N' \cap X_i \subseteq N_i$, thus completing the proof. \blacksquare

We now show that characteristic submanifolds are unique up to isotopy.

Proposition 4.1.71. If N and N' are two characteristic submanifolds for $\overline{M} \in \mathbf{Bord}(M)$, $M \in \mathcal{M}$, then they are properly isotopic.

Proof. It suffices to show that any characteristic submanifolds N' is properly isotopic to the one constructed in Theorem 4.1.63, which is in normal form. Say we have another submanifold $(N', R') \subseteq (\overline{M}, \partial \overline{M})$ satisfying the same properties of N but not properly isotopic to it. By applying Proposition 4.1.70 to N' we get that up to proper isotopy we can assume that for all i $N' \cap X_i \subseteq N_{i-1,i}$.

By definition each component of $N' \cap X_i$ is admissible hence it is isotopic into a component of $N \cap X_i$. Then, by an iterative argument and properties of a normal family we can isotope $N' \cap X_i$

⁵We remind the reader that $R_i \doteq \partial N_i \cap \partial X_i$.

into $N \cap X_i$ rel ∂M_{i-1} to get a proper isotopy of N' into N . Moreover, $N' \subseteq N$ has to be equal to N , up to another proper isotopy, since otherwise N' does not satisfy the engulfing property. ■

With this we conclude the proof of Theorem:

Theorem 4.1.72. Given $M \in \mathcal{M}$ there exists a maximal bordification \overline{M} with a characteristic submanifold (N, R) . Moreover, any two characteristic submanifolds are properly isotopic.

4.1.3 Acylindricity Conditions

Given the existence of characteristic submanifolds we can now study acylindricity properties of manifolds in \mathcal{M}^B . In particular we want to construct a system of simple closed curves P in $\partial \overline{M}$, for $M \in \mathcal{M}^B$, such that \overline{M} is acylindrical with respect to P .

Definition 4.1.73. We say that an irreducible 3-manifold $(M, \partial M)$ with incompressible boundary is acylindrical rel $P \subseteq \partial M$ if M has no essential cylinders C with boundary in $\partial M \setminus P$.

The example of Section 3.3.3 shows that not all manifolds in \mathcal{M}^B admit such a system, but the existence of a doubly peripheral annulus is the only obstruction.

Definition 4.1.74. A manifold $M \in \mathcal{M}^B$ has *property (\star)* if the characteristic submanifold (N, R) of \overline{M} does not contain any essential annulus $\mathcal{A} : (\mathbb{A}, \partial \mathbb{A}) \rightarrow (\overline{M}, \partial \overline{M})$ such that $\mathcal{A}(\partial \mathbb{A})$ are both peripheral in $\partial \overline{M}$. We call such an annulus *doubly peripheral*.

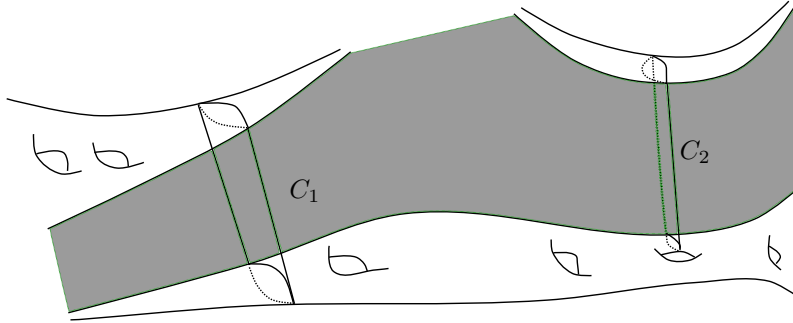


Figure 4.8: The annulus C_1 is an example of a doubly peripheral cylinder while C_2 is not since ∂C_2 has only one peripheral component in $\partial \overline{M}$. The interior of \overline{M} is shaded.

We will show that manifolds $M \in \mathcal{M}^B$ having property (\star) have a system of simple closed curves P that make \overline{M} acylindrical relative to P . Once we add all the torus and annular components of

$\partial\overline{M}$ to P we get that (\overline{M}, P) becomes what is known as a pared manifold (see [56] and definition 7).

In order to find a collection of curves P in $\partial\overline{M}$ such that \overline{M} is acylindrical relative to P it suffices to show that the curves in P ‘pierce’ all the cylinders of the characteristic submanifold $(N, R) \subseteq (\overline{M}, \partial\overline{M})$. Before we go on with the construction, we make the following remarks:

- (i) since we assumed that M satisfies property (\star) every essential cylinder $C \subseteq \overline{M}$ must have at least a non-peripheral boundary component;
- (ii) in order to construct P it suffices to find simple closed curves $\{\gamma_i\}_{i \in \mathcal{I}} \subseteq \partial M$ such that for any closed curve $\alpha \subseteq R$ there is $i \in \mathcal{I}$ such that $i(\gamma_i, \alpha) > 0$;
- (iii) given a solid torus component V of the characteristic submanifold N we cannot have more than one wing being peripheral in $\partial\overline{M}$ otherwise we can find a doubly peripheral cylinder, see figure 4.9, and we violate condition (\star) . Moreover, if V has n -wings it suffices to kill all wings but one. Therefore we can always assume that every curve coming from a solid torus to be non-peripheral in ∂M .

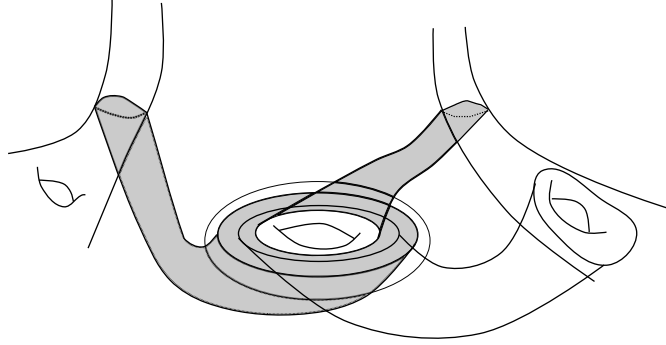


Figure 4.9: The doubly peripheral cylinder is shaded.

The construction of the system of curves P will be highly non-canonical since it involves, among other choices, the choice of filling simple closed curves on essential subsurfaces of $\partial\overline{M}$.

Proposition 4.1.75. Let $(\overline{M}, \partial\overline{M}) \in \mathbf{Bord}(M)$ be maximal bordification for $M \in \mathcal{M}^B$. If \overline{M} satisfies property (\star) then we can find a collection $P \subseteq \partial\overline{M}$ of pairwise disjoint simple closed curves such that \overline{M} is acylindrical relative P .

Proof. By Theorem 4.1.72 let $(N, R) \subseteq (\overline{M}, \partial\overline{M})$ be a characteristic submanifold. Consider all components of N that are homeomorphic to $F \times I$ for F a hyperbolic surface, i.e. $\chi(F) < 0$, and

denote this collection by N_1 . Denote by N_2 the collection of solid and thickened tori in N so that $N = N_1 \coprod N_2$. We define $R_1 \doteq R \cap N_1$ and we let R_2 be the intersection of N_2 with the non-tori components of R so that all components of R_2 are annuli in $\partial\overline{M}$.

For every component $F \times I \xrightarrow{\varphi} Q \subseteq N_1$ that is not a pair of pants we have two filling essential simple closed curves $\alpha_0, \alpha_1 \hookrightarrow F$, see [20, 3.5]. We can embed α_i in $\varphi(F \times \{i\}) \subseteq \partial\overline{M}$, $i = 0, 1$ and we denote the collection of these simple closed curves in $\partial\overline{M}$ by P_1 . Let \mathcal{S} be collection of non-annular components of $\partial\overline{M}$ containing components of $R_1 \cup R_2$. Thus, no $S \in \pi_0(\mathcal{S})$ is a torus or an annulus. Since $\partial\overline{M}$ has at most countably many boundary components we can label \mathcal{S} by the natural numbers so that $\mathcal{S} = \{S^n\}_{n \in \mathbb{N}}$. Consider $S^n \subseteq \mathcal{S}$ and let $\Sigma^n \doteq S^n \setminus N_\varepsilon(P_1)$.

Consider the collection of simple closed curves $\Gamma^n \subseteq \Sigma^n$ induced by the boundaries of the subsurfaces $R_1 \cap S^n$ and the core curves of the annuli in $R_2 \cap S^n$. From Γ^n throw away all simple closed curves that are peripheral in Σ^n and remove redundancies, i.e. if $\beta, \gamma \subseteq \Gamma^n$ are isotopic we only keep one of them.

Claim: Any component S of Σ^n containing a component of Γ^n has $\chi(S) < 0$ and is not a pair of pants.

Proof of Claim: Since no two components of P_1 are isotopic and no component of P_1 is peripheral in S^n we have that any component S of Σ_n has $\chi(S) < 0$. Say we have a pair of pants $Q \in \pi_0(\Sigma^n)$ then ∂Q are either peripheral in $\partial\overline{M}$ or isotopic to elements of P_1 . Since any component γ of Γ^n is a simple closed curve we have that γ is peripheral in Q hence it is either peripheral in $\partial\overline{M}$ or isotopic to an element of P_1 and neither case can happen. \square

The curves $\Gamma^n \subseteq \Sigma^n$ are pairwise disjoint, not isotopic and not peripheral. Since $M \in \mathcal{M}^B$ all components of $\partial\overline{M}$ are of finite type, thus so are the Σ^n 's. By the previous claim every component of Σ^n containing elements of Γ^n is not a pair of pants and we denote this components by X_k^n , $1 \leq k \leq j_n$.

Since for all k the curve complex $\mathcal{C}(X_k^n)$ has infinite diameter [45, 2.25] we can pick an essential simple closed curve $\gamma_k^n \subseteq X_k^n$ such that for all simple closed curves $\gamma \in \pi_0(\Gamma^n \cap X_k^n)$ we have that the geometric intersection number $i(\gamma, \gamma_k^n) > 0$. We then add, for all $n, k \in \mathbb{N}$, the curves γ_k^n to P_1 and denote this new collection by P .

By construction we have that every essential cylinder $(C, \partial C) \rightarrow (\overline{M}, \partial\overline{M})$ intersects some component of P . Therefore, we have that \overline{M} is acylindrical with respect to P . \blacksquare

Definition 4.1.76. We say that a manifold (\overline{M}, P) , with $\overline{M} \in \mathbf{Bord}(M)$ the maximal bordification for $M \in \mathcal{M}^B$, is an *infinite-type acylindrical pared 3-manifold* if:

- (i) the components of P are tori, annuli (closed, open, half open) and all annular and tori components of $\partial\overline{M}$ are in P ;
- (ii) cusp neighbourhoods of $\partial\overline{M}$ are contained in P ;
- (iii) the manifold \overline{M} is acylindrical rel P .

Example 4.1.77. Consider the manifold M of Example 4.1.46 and let \overline{M} be its maximal bordification. Then, $\partial\overline{M} = A \amalg \coprod_{n=1}^{\infty} S_n$ where the S_n are genus two surfaces, A is an open annulus and the characteristic submanifold N is given by an infinitely winged solid T_{∞} such that $T_{\infty} \cap \partial\overline{M}$ are neighbourhoods of the core curve β of the annulus A and the separating loops $\alpha_n \subseteq S_n$ that split the genus two surface into two punctured tori.

Let Γ be a collection of simple closed curves $\gamma_n \subseteq S_n$ such that $\iota(\gamma_n, \alpha_n) = 2$ and $N_{\varepsilon}(\gamma_n, \alpha_n)$ is an essential 4-punctured sphere in S_n . Then, by defining $\mathcal{P} \doteq \{N_{\varepsilon}(\gamma_n)\}_{n \in \mathbb{N}} \amalg A$ we obtain that (\overline{M}, P) is an infinite type acylindrical pared 3-manifold. In particular, every M_i is acylindrical rel P_i where P_i are the components of P isotopic in ∂M_i .

4.1.3.1 Eventual acylindricity of the M_i

In Definition 4.1.58 we gave a decomposition of ∂M_i into two essential sub-surfaces $\partial_{\infty} M_i$ and $\partial_b M_i$ such that $\partial_{\infty} M_i$ is the essential subsurface isotopic to infinity in $M \setminus \text{int}(M_i)$. We will now show that $\partial_b M_i$ has "bounded homotopy class". That is, $\partial_b M_i$ has no essential loops⁶ homotopic arbitrarily far into M .

Definition 4.1.78. Let (\overline{M}, P) be an infinite pared acylindrical 3-manifold for $M = \cup_{i \in \mathbb{N}} M_i \in \mathcal{M}$. Define, $P_i \subseteq \partial M_i$ to be the collection of annuli $\widehat{A} \subseteq \partial M_i$ such that in \overline{M} we have I -bundles $\psi : \mathbb{A} \times I \hookrightarrow \overline{M}$ such that $\psi(\mathbb{A} \times 0) \in \pi_0(\widehat{A})$, $\psi(\mathbb{A} \times \{1\})$ is a compact annular component of P and for some $\varepsilon > 0$ $\psi(\mathbb{A} \times [0, \varepsilon)) \subseteq \overline{M} \setminus \overline{M_i}$. Let Q_{n_i} be the characteristic submanifold of M_{n_i} rel P_i and define $M_{n_i}^{\text{acyl rel } P}$ to be $\overline{M_{n_i}} \setminus \overline{Q_{n_i}}$.

The definition is so that if we consider the cover K_i of (\overline{M}, P) corresponding to $\pi_1(M_i)$ we have that in the compactification $\overline{K_i} \cong M_i$ of K_i the lifts of P are isotopic to the P_i 's in $\overline{K_i} \setminus M_i$.

⁶Recall that by an essential loop we mean a π_1 -injective loop in a surface S not homotopic into ∂S .

Proposition 4.1.79. Given an exhaustion $\{M_i\}_{i \in \mathbb{N}}$ of M by hyperbolizable 3-manifolds with incompressible boundary for all i there exists n_i such that for $n \geq n_i$ no essential loop of $\partial_b M_i$ is homotopic into ∂M_n in $M \setminus \text{int}(M_i)$. Moreover, if (\overline{M}, P) is an infinite-type pared acylindrical 3-manifold there exists n_i such that $M_i \subseteq M_{n_i}^{\text{acyl rel } P}$.

Proof. For $k \in \mathbb{N}$ we define $X_k \doteq \overline{M_{k+1}} \setminus \overline{M_k}$.

Step 1: For all i there exists n_i such that no essential curve in $\partial_b M_i$ is homotopic in $M \setminus \text{int}(M_i)$ into ∂M_m for $m \geq n_i$.

Assume that we have a collection $\{\gamma_k\}_{k \in \mathbb{N}}$ of homotopically distinct essential curves in $\partial_b M_i$ such that for each $n > i$ we can find $k_i \in \mathbb{N}$ such that for $k \geq k_i$ the curve γ_k is homotopic into ∂M_n . We want to show that a tail of $\{\gamma_k\}_{k \in \mathbb{N}}$ always fills a fixed essential subsurface F of S . That is, for all $k \in \mathbb{N}$ we have that $\text{Fill}(\cup_{n \geq k} \gamma_n) = F$. Let n_k be such that γ_k is homotopic into ∂M_{n_k} via a homotopy H_k . Without loss of generality we can assume that $\text{Im}(H_k)$ is contained in $\overline{M_{n_k}} \setminus \overline{M_i}$.

Moreover, up to reordering, we can assume that for $i < j$ we have $n_i \leq n_j$ and $n_i \rightarrow \infty$. Let F_0 be the essential compact subsurface of $\partial_b M_i$ filled by the $\{\gamma_k\}_{k \in \mathbb{N}}$. Since the $\{\gamma_k\}_{k \in \mathbb{N}}$ fill F_0 and are homotopic in $\overline{M_{n_1}} \setminus \overline{M_i}$ into ∂M_{n_1} we have that F_0 is homotopic into ∂M_{n_1} in $\overline{M_{n_1}} \setminus \overline{M_i}$. Thus, by JSJ theory we have an essential I -bundle $F_0 \times I \rightarrow \overline{M_{n_1}} \setminus \overline{M_i}$ and let F'_0 be the induced surface in ∂M_{n_1} .

Let $k_1 \in \mathbb{N}$ be such that γ_k is homotopic to ∂M_{n_1+1} for $k \geq k_1$. Denote by $\{\gamma_k^{(2)}\}_{k \geq k_1}$ the curves in ∂M_{n_1} homotopic to the γ_k via the I -bundle. Then the curves $\{\gamma_k^{(2)}\}_{k \geq k_1}$ fill an essential subsurface $F'_1 \subseteq F'_0$ in ∂M_{n_1} which is isotopic in $\overline{M} \setminus \overline{M_i}$ to an essential subsurface $F_1 \subseteq F_0$ in ∂M_i . Thus we have $|\chi(F_1)| \leq |\chi(F_0)|$. By iterating this process we obtain a nested sequence of connected essential subsurfaces $\{F_j\}_{j \in \mathbb{N}}$ of $\partial_b M_i$ such that $\forall j \in \mathbb{N}$: $0 \leq |\chi(F_{j+1})| \leq |\chi(F_j)|$ and F_j is isotopic in $\overline{M} \setminus \overline{M_i}$ into M_{m_j} for $m_j \rightarrow \infty$. Since every-time the surface shrinks the absolute value of the Euler characteristic goes down the sequence $\{F_j\}_{j \in \mathbb{N}}$ must stabilise to an essential subsurface F . Moreover, since no γ_k is peripheral F is not a peripheral annulus in $\partial_b M_i$. Therefore, the tail of the $\{\gamma_k\}_{k \in \mathbb{N}}$ always fills a fixed subsurface F of $\partial_b M_i$.

By making the homotopies H_k immersions and transverse to ∂M_n 's we can homotope them to be essential in each $\overline{M_{n+1}} \setminus \overline{M_n}$ for $i \leq n \leq n_k - 1$. Then, by picking a collection of the γ_k 's that fill F we get that F is homotopic in $M \setminus \text{int}(M_i)$ into ∂M_{i+n} for all $n \in \mathbb{N}$.

Then by JSJ theory we have a collection of essential I -bundles:

$$\varphi_n : F \times [0, a_n] \hookrightarrow M_{i+n} \setminus \text{int}(M_i)$$

$a_n \in \mathbb{N}$ and $a_n \geq n$, such that $\varphi_n(F \times \{0\}) = \text{Fill}(\gamma) \subseteq \partial_b M_i$ and $\varphi_n(F \times \{a_n\}) \subseteq M_{i+n}$. Moreover, up to an isotopy of the φ_n 's we can assume that for all $n \in \mathbb{N}$:

$$\varphi_n^{-1}(\text{Im}(\varphi_n) \cap \cup_{m=i}^{i+n} \partial M_m) = \cup_{0 \leq a \leq a_n} F \times \{a\}$$

and such that for all $0 \leq a \leq a_n$ $\varphi_n(F \times [a, a+1])$ is an essential I -bundle in some X_k , $i+1 \leq k \leq i+n$, see the proof of Theorem 4.1.19.

By applying Lemma 4.1.61 to $\{\varphi_n|_{F \times [0,1]}\}_{n \in \mathbb{N}}$, i.e. to the restriction of the φ_n so that $\varphi_n(F \times [0,1]) \subseteq X_{i+1}$, we obtain, up to isotopy, a sub-sequence such that for all $n \in \mathbb{N}$: $\varphi_n(F \times [0,1]) = P_1 \subseteq X_{i+1}$. We then repeat the argument to $\{\varphi_n|_{F \times [1,2]}\}$ in X_{i+2} to obtain a subsequence such that $\varphi_n(F \times [1,2]) = P_2$. Moreover, we have that $P_1 \cup P_2$ is naturally homeomorphic to an I -bundle over F . Then, one works in either X_{i+1} or X_{i+2} depending on whether the lid $\varphi_n(F \times \{2\})$ of P_2 is contained in ∂M_{i+2} or ∂M_{i+1} . Note that the new I -bundle P_3 obtained in X_{i+1} is disjoint from P_1 since otherwise the φ_n were not isotopic to embeddings.

By iterating this argument we obtain a collection of pairwise disjoint I -bundles P_n such that P_n and P_{n+1} have a matching lid. Then $P \doteq \cup_{n \in \mathbb{N}} P_n \subseteq M \setminus \text{int}(M_i)$ gives a product $\mathcal{P} : F \times [0, \infty) \hookrightarrow M$, contradicting the fact that $F \subseteq \partial_b M_i$ was not peripheral.

We only need to prove the last claim. By **Step 1** we get that the only cylinders from ∂M_i to ∂M_n with $n \geq n_i$ have boundaries that are peripheral in $\partial_b M_i$.

If M_i is not in $M_n^{\text{acyl rel } P}$, $n \geq n_i$ by Lemma 4.1.53 it means that we can find an annulus of the following type:

- an embedded essential cylinder $C \subseteq M_n$ such that $C_i^n \doteq C \cap M_i$ is essential in M_i with boundary homotopic to $\partial_b M_i$;
- an immersed annulus C formed by an embedded annulus $C_1 \subseteq M_n \setminus \text{int}(M_i)$ with one boundary γ homotopic to $\partial_b M_i$ and the non-trivial homotopy contained in a solid torus $C_n^i(r)$ obtained by collapsing $\gamma \simeq r^n$ to the root r .

Step 2: If (\overline{M}, P) is an infinite-type pared acylindrical 3-manifold there exists n_i such that $M_i \subseteq M_{n_i}^{\text{acyl rel } P}$.

Assume that the statement is not true and assume that no peripheral element of $\partial_b M_i$ has a root in M_i . Then, by Lemma 4.1.53 we have essential cylinders:

$$C_n : (\mathbb{S}^1 \times I, \mathbb{S}^1 \times \partial I) \hookrightarrow (M_n, \partial M_n \setminus P_n), \quad n \geq n_i$$

such that $\text{Im}(C_n) \cap M_i$ is a collection of pairwise disjoint essential cylinders whose boundaries are peripheral in $\partial_b M_i$. By doing isotopies, supported in M_i , of the $\{C_n\}_{n \in \mathbb{N}}$ and picking a subsequence $\{C_n^{(1)}\}_{n \in \mathbb{N}}$ we can assume that the $\{C_n^{(1)}\}_{n \in \mathbb{N}}$ satisfy $\text{Im}(C_n^{(1)}) \cap M_i = \text{Im}(C_m^{(1)}) \cap M_i$ for all $n, m \geq n_i$.

By repeating this argument in M_k , $k \geq i$, and picking a diagonal subsequence $\{C_n\}_{n \in \mathbb{N}}$ we obtain a bi-infinite cylinder $\widehat{C} = \varinjlim_n C_n$ with $\widehat{C} : \mathbb{S}^1 \times \mathbb{R} \hookrightarrow M$ such that for all $k \in \mathbb{N}$ $\widehat{C}_k \doteq \widehat{C} \cap M_k$ are cylinders and \widehat{C}_i contains an essential cylinder. Let \widehat{M} be a bordification where \widehat{C} compactifies and denote by α_1, α_2 the boundaries of \widehat{C} . Since \widehat{C}_i is essential we have that \widehat{C} is also essential in the bordification \widehat{M} . By uniqueness of the maximal bordification we can assume that $\overline{M} = \widehat{M}$. Since (\overline{M}, P) is an infinite-type pared acylindrical 3-manifold and $\widehat{M} \cong \overline{M}$ we have at least one component γ of P such that $\iota(\gamma, \partial \widehat{C}) > 0$, say that $\iota(\gamma, \alpha_1) > 0$. Moreover, since γ is not peripheral in $\partial \overline{M}$ so is α_1 and let $S \in \pi_0(\partial \overline{M})$ be the component containing them.

Pick $k > n_i$ such that we have $\gamma' \stackrel{\text{iso}}{\cong} \gamma$ contained in $P_k \subseteq \partial M_k$ ⁷ and such that the unbounded component \widehat{C}^1 of $\widehat{C} \cap \overline{M \setminus M_k}$ compactifying to α_1 is contained in a product in standard form:

$$\mathcal{Q} : (F \times [0, \infty), F \times \{0\}) \hookrightarrow (\overline{M \setminus M_k}, \partial_\infty M_k)$$

where F is a surface isotopic into $S \subseteq \partial \overline{M}$ in $\overline{M \setminus M_k}$ containing γ' . Moreover, up to a proper isotopy of \widehat{C} , we can assume that $\widehat{C}^1 = \mathcal{Q}(\alpha_1 \times [0, \infty))$.

Since $\varinjlim_n C_n = \widehat{C}$ we can pick a cylinder C_n such that $\text{Im}(\widehat{C}) \cap M_k \subseteq \text{Im}(C_n) \cap M_k$ and let C_n^1 be the component of $\text{Im}(C_n) \cap \overline{M \setminus M_k}$ containing $\mathcal{Q}(\alpha_1 \times \{0\})$. Because $\text{Im}(\mathcal{Q}) \subseteq \overline{M \setminus M_k}$ and \mathcal{Q} is in standard form we have a minimal $t = t_n > 0$ such that $\mathcal{Q}(F \times \{t\}) \subseteq \partial M_n$, $n \geq k$, hence:

$$\mathcal{Q} : (F \times [0, t], F \times \{0, t\}) \hookrightarrow (\overline{M_n \setminus M_k}, \partial(\overline{M_n \setminus M_k}))$$

⁷See Definition .

is an essential I -bundle.

Since α_1 is not peripheral in F and $C_n^1 \cap \partial M_k = \mathcal{Q}(\alpha_1 \times \{0\})$ we have that the annulus C_n^1 , is up to isotopy, vertical in $\mathcal{Q}(F \times [0, t])$. The simple closed curve $\mathcal{Q}(\gamma' \times \{t\})$ is isotopic through \mathcal{Q} to $\gamma \subseteq P$ and we have some $\varepsilon > 0$ such that $\mathcal{Q}(\gamma' \times [t, t + \varepsilon])$ is contained in $M \setminus \text{int}(M_n)$. Hence, we have that $\mathcal{Q}(\gamma' \times \{t\}) \subseteq \mathcal{P}_n$. Therefore, ∂C_n is not in $\partial M_n \setminus P_n$ reaching a contradiction.

Say that peripheral elements of $\partial_b M_i$ have roots in M_i and that we do not have any collection of embedded cylinders so that we have:

$$C_n : (\mathbb{S}^1 \times I, \mathbb{S}^1 \times \partial I) \rightarrow (M_n, \partial M_n \setminus P_n), \quad n \geq n_i$$

whose images in M_i contain a root of $\partial_b M_i$. By Lemma 4.1.53, up to a homotopy, we have that all C_n contain a sub-cylinder $C_n^i(r) \subseteq M_i$ which is the non-trivial homotopy obtained by collapsing a wing of a solid torus to a power of the core and re-expanding it and are embedded otherwise. Since $\partial_b M_i$ has finitely many peripheral elements and they have unique roots up to a subsequence we can assume that all $C_n^i(r)$ are the same and then the previous argument applies to the embedded part of the C_n 's. ■

By combining Proposition 4.1.79 and Theorem 4.1.19 we obtain the following corollary:

Corollary 4.1.80. Let $M = \cup_{i=1}^{\infty} M_i \in \mathcal{M}^B$ then for all i M_i is contained in an open 3-manifold $\widehat{M}_i \cong M_i \cup \partial_{\infty} M_i \times [0, \infty)$ such that $\partial \widehat{M}_i$ is isotopic to $\partial_b M_i$ and $\partial_{\infty} M_i \times [0, \infty)$ is in standard form. Moreover, for all $n > i$ we have that $\partial M_n \cap M \setminus \text{int}(\widehat{M}_i)$ are properly embedded surfaces and there is $j = j(i) \in \mathbb{N}$ such that for all $n \geq j$ no essential loop of $\partial \widehat{M}_i$ is homotopic into ∂M_n .

Proof. By Lemma 4.1.79 we have a maximal essential subsurface $S \cong \partial_{\infty} M_i \subseteq \partial M_i$ that generates a properly embedded product

$$\mathcal{P} : (S \times [0, \infty), S \times \{0\}) \hookrightarrow (M, \partial_b M_i)$$

in $M \setminus \text{int}(M_i)$ and we have $j = j(i) \in \mathbb{N}$ such that for all $n \geq j$ no essential loop of $\partial_b M_i \xrightarrow{\text{iso}} \partial \widehat{M}_i$ is homotopic into ∂M_n . By Theorem 4.1.19 we can assume \mathcal{P} to be in standard form. Thus, $\widehat{M}_i \doteq M_i \cup_{\partial_{\infty} M_i} \mathcal{P} \cong M_i \cup \partial_{\infty} M_i \times [0, \infty)$ is the required submanifold. ■

4.1.3.2 Homotopy equivalences of 3-manifolds

We now proceed with two topological Lemmata that we need in the final proof. The first is a generalisation of Lemma 2.2 in [53] and the latter is a relative version of Johansson homeomorphism Theorem [32].

Lemma 4.1.81. Let M be a complete, open hyperbolic 3-manifold and K a compact, atoroidal, aspherical, 3-manifold with incompressible boundary such that $\iota : K \rightarrow M$ is a homotopy equivalence and $\iota|_{\partial K} : \partial K \hookrightarrow M$ is an embedding. Then ι is homotopic relative to ∂K to an embedding $\iota' : K \rightarrow M$.

Proof. Since M is homotopy equivalent to K we have that $\pi_1(M)$ is finitely generated. By the Tameness Theorem [1, 9] it follows that $M \cong \text{int}(\overline{M})$ for some compact manifold \overline{M} . Therefore, we have $\iota : K \rightarrow \overline{M}$ with the same properties as in the statement.

We need to show that $\iota(\partial K)$ is peripheral in \overline{M} since then by Waldhausen's Theorem [60] follows that ι is homotopic to an embedding. Since ι is a homotopy equivalence the map on homology ι_* induces an isomorphism: $\iota_* : H_*(K) \rightarrow H_*(\overline{M})$. Let $\partial K = \coprod_{i=1}^n S_i$ then we can write: $H_3(K, \partial K) \cong \mathbb{Z}\langle [K, \partial K] \rangle$ with $\partial[K, \partial K] = \sum_{i=1}^n [S_i]$ and $H_2(\partial K) \cong \mathbb{Z}\langle [S_i] \rangle$. By the long exact sequence of the pair $(K, \partial K)$:

$$0 \rightarrow H_3(K, \partial K) \rightarrow H_2(\partial K) \rightarrow H_2(K)$$

we obtain the following injection:

$$\mathbb{Z}\langle [S_i] \rangle / \sum_{i=1}^n [S_i] \hookrightarrow H_2(K) \xrightarrow{\iota_*} H_2(\overline{M}) \cong H_2(M)$$

This means that no linear combination of the $[S_i]$ except $\sum_{i=1}^n [S_i]$ is null-homologous in \overline{M} . Moreover, the $\{S_i\}_{i=1}^n$ are separating in K since they are not dual to any 1-cycle in $H_1(K)$. Since ι preserves homological conditions the same holds for the $\{\iota(S_i)\}_{i=1}^n$. Therefore, all the $\iota(S_i)_{i=1}^n$ are separating in M and no linear combination except $\sum_{i=1}^n [\iota(S_i)]$ is null-homologous in M . Hence, if we start splitting along the $\{[\iota(S_i)]\}_{i=1}^n$ we get a connected submanifold $N \subseteq M$ whose boundary is $\sum_{i=1}^n [\iota(S_i)]$. If we show that $\overline{M} \setminus N$ is a product manifold over $\partial N \cong \partial K$ we are done.

Consider a homotopy inverse $f : \overline{M} \rightarrow K$ then $f|_N : N \rightarrow K$ is a π_1 -injective map and up to homotopy it sends $\partial N \rightarrow \partial K$ homeomorphically. The map $f|_N$ is also degree one since $f|_N = f \circ \iota_N$ and f has degree one. Thus $f|_N : N \rightarrow K$ is a π_1 -injective degree one map. We now claim that $f|_N$ is a surjection on π_1 and so a homotopy equivalence. If f_* is not surjective let $H \doteq f_*(\pi_1(N)) \leq \pi_1(K)$ and consider the cover $\pi : K_H \rightarrow K$ corresponding to H . Since by construction the map $f|_N$ lifts

to $\tilde{f}|_N$ we have: $f|_N = \pi \circ \tilde{f}|_N$, but $f|_N$ has degree one hence $\deg(\pi) = \pm 1$. Therefore, $K_H \cong K$ which implies that $(f|_N)_*$ is a surjection on π_1 , hence an isomorphism.

By Whitehead's Theorem we get that $f|_N$ is a homotopy equivalence which implies that $\iota : N \hookrightarrow M$ is also a homotopy equivalence and so N is a Scott core. Moreover, since $\iota(\partial K) = \partial N$ is incompressible by Corollary [60, 5.5] we have that each component of $\overline{M \setminus N}$ is a product over $\iota(S_i)$. Hence the homotopy equivalence $\iota : K \rightarrow \overline{M}$ is homotopic rel boundary to a homotopy equivalence $\iota' : K \rightarrow N \subseteq M$ that is a homeomorphism on $\partial K \rightarrow \partial N$. Thus by Waldhausen's Theorem [60] we get that it is homotopic rel boundary to a homeomorphism from K to N . Hence, we get an embedding $\iota' : K \hookrightarrow M$ with $\iota'|_{\partial K} = \iota|_{\partial K}$. \blacksquare

The following is an application of the classification theorem of Johansson [14, 2.11.1] for homotopy equivalences between 3-manifolds.

Lemma 4.1.82. Let $\varphi : M \rightarrow N$ be a homotopy equivalence between compact, irreducible, orientable 3-manifolds and let $X \subseteq M$ be a codimension-zero submanifold. If S is a collection of essential subsurfaces of ∂M such that X is contained in the acylindrical part of M relative to S and $\varphi|_S : S \rightarrow \partial N$ is an embedding, then we can homotope φ to ψ so that $\psi|_X : X \rightarrow N$ is an embedding and the homotopy is constant on S .

Proof. Complete S and $\varphi(S)$ to useful boundary patterns for $\partial M, \partial N$, which we denote by \overline{S} and $\overline{\varphi(S)}$ respectively. Let V, Z be the characteristic submanifolds corresponding to \overline{S} and $\overline{\varphi(S)}$ then we have that $X \subseteq \overline{M \setminus V}$. By the Johansson Classification theorem [32] we have that φ is admissibly homotopic to a homeomorphism $\psi : (\overline{M \setminus V}, S) \rightarrow (\overline{N \setminus Z}, \varphi(S))$. An admissible homotopy is a homotopy by pair maps hence since $\varphi|_S$ is already a homeomorphism we can choose it to be constant on S . Since we assumed that $X \subseteq \overline{M \setminus V}$ we get that X is embedded by ψ and $\psi|_S = \varphi|_S$. \blacksquare

4.2 Proof of the Main Theorem

In this section we prove our main Theorem:

Theorem 1. Let $M \in \mathcal{M}^B$. Then, M is homeomorphic to a complete hyperbolic 3-manifold if and only if the associated maximal bordified manifold \overline{M} does not admit any doubly peripheral annulus.

In the next subsection we show that not having doubly peripheral annuli is a necessary condition. Specifically, we prove that if $M \in \mathcal{M}^B$ does not have a doubly peripheral cylinders then it is homotopy equivalent to a hyperbolic 3-manifolds N . Finally we show that particula homotopy equivalences between M and N are homotopic to homeomorphisms.

4.2.1 Necessary condition in the main Theorem

Using the same techniques of [16] we prove the necessary condition on the annulus in Theorem 1. We start with a remark on characteristic submanifolds of manifolds in \mathcal{M}^B .

Remark 4.2.1. By Theorem 4.1.72 for a manifold $M \in \mathcal{M}^B$ we have a characteristic submanifold N for the maximal bordification \overline{M} . Then, any doubly peripheral cylinder C is homotopic into one of the following components of N :

- (i) a solid torus with at least one peripheral wing in $\partial\overline{M}$ that wraps around the soul $n > 2$ times;
- (ii) a solid torus with at least two peripheral wings in $\partial\overline{M}$ each of which wraps around the soul once;
- (iii) a thickened essential torus with at least one wing that is peripheral in $\partial\overline{M}$;
- (iv) an I -bundle $P \cong F \times I$ such that at least one component of $\partial F \times I$ is doubly peripheral.

For (i),(ii) and (iii) the cases with infinitely many wings are also allowed. However, for all cases except (i) if C is a doubly peripheral annulus then there exists a properly embedded annulus C' that is also doubly peripheral and such that C is homotopic into C' , i.e. C is a *power* of C' .

We first show that if M is hyperbolizable, i.e. $M \cong \mathbb{H}^3/\Gamma$, the elements of $\pi_1(M)$ that are peripheral in $\partial\overline{M}$ are represented by parabolic elements in the Kleinian group Γ .

Lemma 4.2.2. For $M \in \mathcal{M}^B$ let $\overline{M} \in \mathbf{Bor}(M)$ be the maximal bordification. If $M \cong \mathbb{H}^3/\Gamma$ admits a complete hyperbolic metric and $\gamma \in \pi_1(M)$ is homotopic to $\overline{\gamma} \subseteq \partial\overline{M}$ such that $\overline{\gamma}$ is peripheral in $\partial\overline{M}$ then γ is represented by a parabolic element in Γ .

Proof. Let $\{M_i\}_{i \in \mathbb{N}}$ be the exhaustion of M and let G be the bound on the Euler characteristic of the boundary components of the M_i . Without loss of generality it suffices to consider the case where $\overline{\gamma}$ is a simple closed curve. If γ is peripheral in $\partial\overline{M}$ the components of $\cup_{i \in \mathbb{N}} \partial M_i$ that have a simple closed curve γ_n isotopic to γ in $\overline{M} \setminus \text{int}(M_i)$ form a properly embedded sequence of hyperbolic incompressible surfaces $\{\Sigma_n\}_{n \in \mathbb{N}}$ with $\Sigma_n \in \pi_0(\partial M_{i_n})$. Moreover, up to picking a subsequence we can assume that $i_n < i_{n+1}$.

If γ is not represented by a parabolic element it has a geodesic representative $\hat{\gamma}$ in M which is contained in some M_i . Let τ_n be a 1-vertex triangulation of Σ_n realising γ_n . Since the Σ_n are incompressible closed surfaces we can realise them, in their homotopy class, via simplicial hyperbolic

surfaces (S_n, f_n) in which γ_n is mapped to $\hat{\gamma}$ (see [6, 11]). Since the Σ_n are hyperbolic surfaces and γ_n is simple each S_n contains at least one pair of pants with one boundary component γ_n .

By Gauss-Bonnet for all n we have:

$$A(S_n) \leq 2\pi |\chi(S_n)| \leq 2\pi G$$

Since the S_n have uniformly bounded area, we have that any maximally embedded *one-sided* collar neighbourhood of γ_n in S_n has radius uniformly bounded by $K \doteq K(\ell_M(\hat{\gamma}), G)$. Therefore, for any $\varepsilon > 0$ an essential pair of pants $P_n \subseteq S_n$ with γ_n in its boundary is contained into a $K + \varepsilon$ neighbourhood of $\gamma_n \subseteq S_n$. Since the maps f_n are 1-Lipschitz for all n the $f_n(P_n)$ are contained in a $K + \varepsilon$ neighbourhood of $\hat{\gamma} = f_n(\gamma_n)$ in M .

Since for any pair of pants $P_n \subseteq S_n$ with $\gamma_n \subseteq \partial P_n$ the $f_n(P_n)$'s are at a uniformly bounded distance from $\hat{\gamma}$ we can assume that all such pair of pants are contained in \widehat{M}_k for some k , where \widehat{M}_k is as in Corollary 4.1.80 and we also let $j \doteq j(k)$ be as in the Corollary. Moreover, we can assume that there is a cusp component Q of $\partial \widehat{M}_k$ that compactifies to a simple closed loop $\alpha \subseteq \partial \overline{M}$ isotopic to $\bar{\gamma}$ in $\partial \overline{M}$. We now want to show that we can find $n \in \mathbb{N}$ such that Σ_n has a pair of pants P_n with boundary γ_n contained outside \widehat{M}_k .

Claim: There is a component of $\cup_{n \geq j} \partial M_n \setminus \text{int}(\widehat{M}_k)$ that has $\gamma_n \subseteq Q$ as a boundary component and is not an annulus.

Proof of Claim: Since $\partial_\infty M_k \times [0, \infty)$ is in standard form we have that all components S of $\cup_{i \in \mathbb{N}} M_i$ intersect $\partial_\infty M_k \times [0, \infty)$ in level surfaces or are disjoint from it. If the claim is not true we have $j \in \mathbb{N}$ such that every component of $\cup_{i \geq j} M_i \setminus \text{int}(\widehat{M}_k)$ having γ_n as a boundary component is an annulus. Each such component A has boundary on $\Lambda \doteq \partial(\partial_\infty M_k) \times [0, \infty)$.

Since Λ has finitely many components we can assume that we have a collection of annuli $\{A_\ell\}_{\ell \in \mathbb{N}}$ with $\partial A_\ell = \alpha \times \{t_\ell\} \cup \beta \times \{t_\ell\} \subseteq \Lambda$. Then, the $\{A_\ell\}_{\ell \in \mathbb{N}}$ have to be in at most two homotopy classes. If not we have two essential tori that have homotopic boundaries and this cannot happen in hyperbolic 3-manifolds, see Remark 4.1.6. Thus, we get that eventually we can enlarge the product $\partial_\infty M_i \times [0, \infty)$ so that Q is not peripheral anymore. This gives a new bordification M' such that $\overline{M} \subsetneq M'$ contradicting the maximality of \overline{M} and the fact that $\bar{\gamma}$ was peripheral. \square

Thus, we have a surface $F_n \subseteq \pi_0(\partial M_n)$ with $n \geq j$ such that $F_n \cap \widehat{M}_k^C$ contains a pair of pants P_n with a boundary component homotopic in M to $\hat{\gamma}$. Therefore, the corresponding simplicial

hyperbolic surface $f_n : S_n \rightarrow M$ has a pair of pants P_n with $\gamma_n \subseteq P_n$ such that $f_n(P_n) \subseteq \widehat{M}_k$. However, this means that P_n is homotopic into $\partial \widehat{M}_k \stackrel{\text{iso}}{\cong} \partial_b M_k$ and since $\pi_1(P_n)$ cannot inject into \mathbb{Z} we reach a contradiction with Corollary 4.1.80. \blacksquare

We will now show that manifolds in \mathcal{M}^B with a doubly peripheral annulus are not hyperbolic. First we need the following topological Lemma saying that if $\alpha, \beta \subseteq \partial \overline{M}$ are peripheral simple closed curves and isotopic in \overline{M} then we can separate their homotopy class by a compact subset.

Lemma 4.2.3. Let $\overline{M} \in \mathbf{Bord}(M)$, $M \in \mathcal{M}^B$ be the maximal bordification and $\mathcal{A} : (\mathbb{A}, \partial \mathbb{A}) \hookrightarrow (\overline{M}, \partial \overline{M})$ be an essential doubly peripheral annulus. Then, there exists M_i such that in $\overline{M} \setminus \text{int}(M_i)$ the peripheral loops $\mathcal{A}(\partial \mathbb{A}) \doteq \gamma_0 \amalg \gamma_1$ have no essential homotopy and are isotopic in $\overline{M} \setminus \text{int}(M_i)$ to peripheral loops in $\partial_\infty M_i$.

Proof. Since A is embedded it is isotopic in a component P of the characteristic submanifold N of \overline{M} . Thus, by Remark 4.2.1 we have three possibilities for P :

- (i) a solid torus with at least two peripheral wings in $\partial \overline{M}$;
- (ii) an I -bundle $P \cong F \times I$ such that at least one component of $\partial F \times I$ is doubly peripheral;
- (iii) an essential torus with at least one wing that is peripheral in $\partial \overline{M}$.

Case (iii). Let P be the component of the characteristic submanifold N corresponding to the essential torus $T \subseteq \partial \overline{M}$ that $A \doteq \mathcal{A}(\mathbb{A})$ is homotopic into. Then, by Lemma 4.1.57 there exists a minimal i such that the essential torus T is isotopic in $\overline{M} \setminus \text{int}(M_i)$ into an essential torus \overline{T} of $\partial_\infty M_i$ and such that the compact components of $A \cap M_i$ and $A \cap M \setminus \text{int}(M_i)$ are essential and γ_0, γ_1 are isotopic in $\overline{M} \setminus \text{int}(M_i)$ into peripheral loops γ_0^i, γ_1^i of $\partial_\infty M_i$. For now assume that γ_0 is not isotopic to γ_1 in $\partial \overline{M}$.

For $j \geq i$ let $X_j^\infty \doteq \overline{M} \setminus \text{int}(M_j)$ and assume we have an essential homotopy $C_j : (\mathbb{A}, \partial \mathbb{A}) \hookrightarrow (X_j^\infty, \partial X_j^\infty)$ from γ_0 to γ_1 . Then, $C_j \cup_{\partial} A$ forms a torus \widehat{T} which is essential since otherwise $A \cap M_i$ was inessential. Moreover, up to a homotopy of \widehat{T} pushing it off $\partial \overline{M}$ we can assume that \widehat{T} and \overline{T} are contained in M_k for some $k > i$. Thus, since \widehat{T} and \overline{T} have homotopic simple closed curves by hyperbolicity of M_k we must have that \widehat{T} is homotopic into \overline{T} in M_k . Hence, we have that $A \cap X_i^\infty$ does not contain any compact annuli since they would be homotopic into M_i contradicting Lemma 4.1.57.

Let $C'_j \subseteq X_i^\infty$ be the subannulus of \widehat{T} obtained by taking C_j and going to $\gamma_0^i, \gamma_1^i \subseteq \partial M_i$ along A . Since \widehat{T} is homotopic into M_i we get that C' is inessential in X_i^∞ , thus $\partial C' = \gamma_0^i \amalg \gamma_1^i$ are parallel in

∂M_i and co-bound an annulus $A' \subseteq \partial M_i$. Moreover, we have that C_j is parallel to A' in X_i^∞ . Thus we proved:

Claim: If $C_j : (\mathbb{A}, \partial\mathbb{A}) \hookrightarrow (X_j^\infty, \partial X_j^\infty)$ is an essential annulus connecting γ_0, γ_1 then it is isotopic to the annulus $A' \subseteq \partial M_i$ connecting γ_0^i, γ_1^i . Moreover, for $j \neq \ell$ we also have that $C_j \stackrel{\text{iso}}{\simeq} C_\ell$.

If we cannot separate the homotopy class of γ_0, γ_1 we have a collection of annuli $C_j : (\mathbb{A}, \partial\mathbb{A}) \hookrightarrow (X_j^\infty, \partial X_j^\infty)$ such that for all $k \geq i$ there exists j such that $C_j \cap M_k = \emptyset$. Then, we get that in $\partial \overline{M}$ there are cusps neighbourhoods P_1, P_2 of the components S_1, S_2 of $\partial \overline{M}$ co-bounding with A a submanifold of the form $\mathbb{A} \times [0, \infty)$ contradicting the properties of a maximal bordification.

Now assume that γ_0, γ_1 are isotopic in $\partial \overline{M}$ so that they co-bound an annulus $C \subseteq \partial \overline{M}$ and assume that we have an essential annulus $C' : (\mathbb{A}, \partial\mathbb{A}) \hookrightarrow (X_i^\infty, \partial X_i^\infty)$ with boundary ∂C . Since C' is essential we have that $\widehat{T} \doteq C \cup_{\partial} C'(\mathbb{A})$ is an essential torus which is then homotopic to the torus $T \subseteq P$ in \overline{M} . Moreover, since T, \widehat{T} are embedded, incompressible and homotopic we can assume by [60, 5.1] that they are isotopic in M so they co-bound an I -bundle J . Since $\widehat{T} \subseteq X_i^\infty$ up to an isotopy of J we can assume that $J \cap \partial M_i$ are level surfaces hence all components of $J \cap \partial M_i$ are essential tori.

Then, either \widehat{T} is contained in the I -bundle $Q \cong \mathbb{T}^2 \times I$ generated by the boundary torus T of ∂M_i or it is contained in some other component of X_i^∞ . If it is contained in Q we get a contradiction since then γ_0, γ_1 are contained in a torus component of $\partial \overline{M}$. Thus, since $\widehat{T} \subseteq X_i^\infty \setminus Q$ and $J \cap \partial M_i \setminus Q$ are essential tori we get that $M_i \cong \mathbb{T}^2 \times I$. In turn, this gives us that $M \cong \mathbb{T}^2 \times \mathbb{R}$ and $\overline{M} \cong \mathbb{T}^2 \times I$ which does not contain any doubly peripheral annulus.

We will now deal with annuli of type (i) and (ii) and we can assume that we have no doubly peripheral annulus of type (iii).

Case (i) and (ii). Let A be as before an essential annulus connecting γ_0 to γ_1 in \overline{M} . By Lemma 4.1.57 we have an isotopy of A and a minimal M_i such that compact components of $A \cap M_i$ and $A \cap X_i^\infty$ are essential annuli.

Assume we have an essential annulus C connecting γ_0 to γ_1 in X_i^∞ . The annuli C and A cannot be parallel since otherwise $A \cap M_i$ would have no essential components. Therefore, by taking a push-off C' of C and connecting it to A along γ_1 we obtain an essential annulus A' that has both boundaries isotopic to γ_0 in $\partial \overline{M}$. Therefore, we contradict the fact that we had no type (iii) annuli

and so M_i disconnects the homotopy class of γ_0 and γ_1 in M . ■

Theorem 4.2.4. If $M \in \mathcal{M}^B$ is hyperbolic, $M \cong \mathbb{H}^3/\Gamma$, then \overline{M} cannot have an essential doubly peripheral annulus.

Proof. Since $M \in \mathcal{M}^B$ we have $G \in \mathbb{N}$ such that for all $i \in \mathbb{N}$ all components Σ of ∂M_n have $|\chi(\Sigma)| \leq G$. Let $\mathcal{A} : (\mathbb{A}, \partial\mathbb{A}) \rightarrow (\overline{M}, \partial\overline{M})$ be an essential annulus such that $\mathcal{A}(\partial\mathbb{A}) \doteq \gamma_1 \cup \gamma_2$ are peripheral in $\partial\overline{M}$. Let $\gamma \in \pi_1(\overline{M})$ be the element that generates $\pi_1(\mathbb{A}) \hookrightarrow \pi_1(\overline{M})$. By Lemma 4.2.2 γ has to be represented by a parabolic element and by Remark 4.2.1 we only have to consider the following four cases:

- (i) a solid torus with at least one peripheral wing in $\partial\overline{M}$ that wraps around the soul $n > 2$ times;
- (ii) a solid torus with at least two peripheral wings in $\partial\overline{M}$ each of which wraps around the soul once;
- (iii) an I -bundle $P \cong F \times I$ such that at least one component of $\partial F \times I$ is doubly peripheral;
- (iv) an essential torus with at least one wing that is peripheral in $\partial\overline{M}$.

Except for (i) we can assume that \mathcal{A} is an embedding.

Step 1 M cannot have a doubly peripheral cylinder $\mathcal{A} : (\mathbb{A}, \partial\mathbb{A}) \rightarrow (\overline{M}, \partial\overline{M})$ with $\mathcal{A}(\partial\mathbb{A}) \doteq \gamma_1 \cup \gamma_2$ of type (i).

In this case we have a doubly peripheral cylinder $C \doteq \mathcal{A}(\mathbb{A})$ whose boundaries are isotopic in $\partial\overline{M}$. Let $S \subseteq \partial\overline{M}$ be the component containing ∂C . By construction of the characteristic submanifold and of the maximal bordification we have M_i such that $\partial_\infty M_i$ contains a component isotopic to S in $\overline{M} \setminus \text{int}(M_i)$ and an essential solid torus $V \subseteq M_i \cup \partial_\infty M_i \times [0, \infty)$ with a wing w whose boundary ∂C is isotopic to a collar neighbourhood of γ_i and such that w wraps around the soul γ of V $n > 1$ times. Also note that in this case γ is primitive in $\pi_1(M_i)$.

Since the cover \widetilde{M} of M corresponding to $\pi_1(M_i)$ is homeomorphic to $\text{int}(M_i)$, see Lemma 3.3.3, we have that in the pared hyperbolic 3-manifold (N, P) , $N \cong M_i$, such that $\text{int}(N) = \widetilde{M}$ there are disjoint embedded annuli $A, B \in \pi_0(P)$ such that γ is homotopic to the soul b of B and the soul a of A is isotopic to γ_i .

Thus, the n -th power of the soul b of B is homotopic to the soul a of A . Therefore, we have a component of the characteristic submanifold of N that realises this homotopy. However, since N is hyperbolizable this cannot happen because no component Q of the characteristic submanifold has elements in the boundary such that one is a root of the other in Q .

Since we dealt with the non-embedded case from now on we can assume that the doubly peripheral cylinder is embedded. The idea is to use a collection of simplicial hyperbolic surfaces S_n 's, as in Lemma 4.2.2, all intersecting the doubly peripheral annulus in simple loops γ_n isotopic to a peripheral loop γ in the boundary $\partial\overline{M}$. By Lemma 4.2.3 the simplicial hyperbolic surfaces S_n 's will be forced to go through some M_i . By using the hyperbolicity of M this will force loops α_n transverse to the γ_n to have uniformly bound length and this will allow us to construct a product \mathcal{P} whose compactification makes γ not peripheral.

Step 2 A hyperbolic M cannot have a doubly peripheral cylinder $(A, \partial A) \rightarrow (\overline{M}, \partial\overline{M})$ with $\partial A \doteq \gamma_1 \cup \gamma_2$ of type (ii)-(iv).

Let M_i be as in Lemma 4.2.3 so that $\partial_\infty M_i$ contains peripheral loops γ_1^i, γ_2^i isotopic to γ_1, γ_2 respectively in $\overline{M} \setminus \text{int}(M_i)$. Moreover, if $\gamma_1 \stackrel{\text{iso}}{\simeq} \gamma_2$ in $\partial\overline{M}$ we can assume, by picking a larger i , that the essential torus T induced by the doubly peripheral annulus C is contained in M_i . Let Q be the cusp neighbourhood, in M , of the parabolic element corresponding to the homotopy class of γ .

In the case that we have an essential torus T the cusp Q is contained in a component of $\overline{M} \setminus \overline{M_i}$ homeomorphic to $\mathbb{T}^2 \times [0, \infty)$. Otherwise, it corresponds to a neighbourhood of γ_1, γ_2 in $\partial\overline{M}$ in which case we will assume it is γ_2 .

Let \widehat{M}_i be the manifold from Lemma 4.1.80 and let $\{\Sigma_n\}_{n>i \in \mathbb{N}}$ be the sequence of surfaces in $\partial M_n \cap \widehat{M}_i^C$ coming from **Claim 1** of Lemma 4.2.2 and let $F_n \subseteq \partial M_n$ the component containing Σ_n .

Let τ_n be ideal triangulations of the $\widehat{\Sigma}_n \doteq \Sigma_n \setminus \gamma_1^n$ where the cusps corresponding to γ_1^n have exactly one vertex each and homotope them so to obtain proper maps of the punctured surfaces. Then by [6, 11] we can realise the embeddings $\widehat{\Sigma}_n \hookrightarrow M$ by simplicial hyperbolic surfaces (S_n, f_n) in which γ_1^n is sent to the cusp Q . Therefore, since M_i separates in M the homotopy class of γ_1 and γ_2 we have that all the S_n 's must intersect ∂M_i . Moreover, we still have that $\left| \chi(\widehat{\Sigma}_n) \right| \leq G$.

Let $\mu \doteq \min \{ \mu_3, \text{inj}_M(\partial M_i) \}$ then for all n the μ -thick part of S_n has a component intersecting ∂M_i . Moreover, each such component contains the image $f_n(\widehat{P}_n)$ of a pair of pants $\widehat{P}_n \subseteq P_n$ that has γ_1^n in its boundary. Then, by the Bounded diameter Lemma [54] we have that \widehat{P}_n has diameter bounded by $D_1 \doteq D_1(G, \mu)$ and let D_2 be maximal diameter of a component of ∂M_i .

Consider the loops $\{\alpha_n\}_{n \in \mathbb{N}}$ that are contained in the surfaces F_n with $i(\alpha_n, \gamma_1^n) = 2$ and such that $N_r(\alpha_n \cup \gamma_1^n) \subseteq F_n$, $r > 0$, is an essential four punctured sphere. Then, the $\{\alpha_n\}$'s have representatives in M whose length is bounded by:

$$\ell_M(\alpha_n) \leq D \doteq D_1 + D_2$$

this is because we can push α_n to be obtained as two arcs $\alpha \subseteq \partial_b M_i$ and $\beta_n \subseteq S_n$ that meet along $\partial_b M_i$.

Therefore, by discreteness of Γ we have that they are in finitely many homotopy classes. Thus, we have a subsequence $\{\alpha_{n_k}\}_{k \in \mathbb{N}}$ such that for all $k, h \in \mathbb{N}$ $\alpha_{n_k} \simeq \alpha_{n_h}$ in $M \setminus \text{int}(M_i)$. Thus, by taking a sub-sequence of embedded annuli connecting α_{n_1} to α_{n_k} we obtain an embedded product with base a neighbourhood of α_{n_1} in Σ_{n_1} . This means that in the compactification \overline{M} we have that γ_1 was not peripheral since it is a separating curve of a four punctured sphere embedded in $\partial \overline{M}$. ■

In order to prove the characterisation Theorem 1 we only need to show that if $\overline{M} \in \mathbf{Bord}(M)$, for $M \in \mathcal{M}^B$, does not have any double peripheral cylinder C then M admits a complete hyperbolic metric.

4.3 Hyperbolization results

4.3.1 Relatively Acylindrical are Hyperbolic

Now that we have completed the necessary topological construction we can show that manifolds $M \in \mathcal{M}^B$ with (\overline{M}, P) an infinite-type pared acylindrical 3-manifold admit a complete hyperbolic metric. We first show that such manifolds are homotopic equivalent to a complete hyperbolic manifold N with $\pi_1(P)$ represented by parabolic elements. We achieve this by using the relative compactness of algebraic sequences developed by Thurston. Afterwards, with techniques similar to [53], we show that the homotopy equivalence is homotopy to a homeomorphism $\psi : M \cong N$.

4.3.1.1 Relatively acylindrical are homotopy equivalent to hyperbolic

To prove the homotopy equivalence we need a couple of technical result about sequences of non-elementary representations.

Lemma 4.3.1. Let $\rho_n : G \rightarrow \text{PSL}_2(\mathbb{C})$ be non-elementary discrete and faithful representations such that $\rho_n \rightarrow \rho$ and let $\{g_n\}_{n \in \mathbb{N}} \subseteq \text{PSL}_2(\mathbb{C})$. Then, if we have a converging sub-sequence $g_{n_k} \rho_{n_k} g_{n_k}^{-1} \rightarrow$

ρ' we have that up to an ulterior sub-sequence: $g_{n'_k} \rightarrow g$ and $\rho' = g\rho g^{-1}$. The converse also holds.

Proof. If the $\{g_n\}_{n \in \mathbb{N}}$ have a converging subsequence we are done. So assume that $g_n \rho_n g_n$ has a converging subsequence, which we denote by $g_n \rho_n g_n \rightarrow \rho'$. Since the $g_n \rho_n g_n$ are non-elementary their algebraic limit ρ' is non-elementary as well (see [33]). Therefore, we can find $\alpha, \beta \in G$ loxodromic elements that generate a discrete free subgroup $\langle \rho'(\alpha), \rho'(\beta) \rangle$. By algebraic convergence we have $g_n \rho_n(\alpha) g_n^{-1} \rightarrow \rho'(\alpha)$, $g_n \rho_n(\beta) g_n^{-1} \rightarrow \rho'(\beta)$ with $\rho_n(\alpha) \rightarrow \rho(\alpha)$ and $\rho_n(\beta) \rightarrow \rho(\beta)$.

Since traces are preserved under conjugation and we assumed that $\rho'(\alpha), \rho'(\beta)$ were loxodromic so are $\rho(\alpha), \rho(\beta)$. Denote by $x_\infty^\pm, y_\infty^\pm$ the attracting/repelling fixed points of $\rho(\alpha), \rho(\beta)$ respectively and similarly define a^\pm, b^\pm for $\rho'(\alpha), \rho'(\beta)$. Moreover, we have that eventually $g_n \rho_n(\alpha) g_n^{-1}$ are all loxodromic and similarly for $g_n \rho_n(\beta) g_n^{-1}$. Therefore, the attracting (repelling) fixed points of $g_n \rho_n(\alpha) g_n^{-1}$ converge to the attracting (repelling) fixed point of $\rho'(\alpha)$. The fixed points of $g_n \rho_n(\alpha) g_n^{-1}$ are $g_n(x_n^\pm)$ for x_n^\pm the fixed point of $\rho_n(\alpha)$, hence we have that $x_n^\pm \rightarrow x_\infty^\pm$ and $g_n(x_n^\pm) \rightarrow a^\pm$. By triangle inequality:

$$\begin{aligned} d_{\mathbb{H}^3}(g_n x_\infty^\pm, a^\pm) &\leq d_{\mathbb{H}^3}(g_n x_\infty^\pm, g_n x_n^\pm) + d_{\mathbb{H}^3}(g_n x_n^\pm, a^\pm) \\ &= d_{\mathbb{H}^3}(x_\infty^\pm, x_n^\pm) + d_{\mathbb{H}^3}(g_n x_n^\pm, a^\pm) \end{aligned}$$

Thus it follows that $g_n(x_\infty^\pm) \rightarrow a^\pm$ and this also holds for the y_∞^\pm and b^\pm . Since $\langle \rho(\alpha), \rho(\beta) \rangle$ is discrete at least three of the x_∞^\pm and y_∞^\pm are distinct. Then, by Theorem [4, 3.6.5] we have that the $\{g_n\}_{n \in \mathbb{N}}$ form a normal family. ■

Proposition 4.3.2. Let $\rho_i : G = \cup_{j=1}^\infty G_j \rightarrow \text{PSL}_2(\mathbb{C})$ be non-elementary discrete and faithful representations and let ρ_i^j be the restriction of ρ_i to G_j . Then, given $g_i^j \in \text{PSL}_2(\mathbb{C})$ such that $\forall i, j : g_i^j \rho_i^j (g_i^j)^{-1} : G_j \rightarrow \text{PSL}_2(\mathbb{C})$ converge up to subsequence we have that $\forall j : g_i^1 \rho_i^j (g_i^1)^{-1}$ converge up to subsequence.

Proof. We first show that $\forall j : \{g_i^1 (g_i^j)^{-1}\}_{i \geq j}$ converge up to subsequence. For all j consider:

$$g_i^1 \rho_i^1 (g_i^1)^{-1} = \left(g_i^1 (g_i^j)^{-1} \right) g_i^j \rho_i^j (g_i^j)^{-1} \left(g_i^j (g_i^1)^{-1} \right) \quad i \geq j$$

By assumption we have the the left-hand side has a converging subsequence which we call i_n . Since the $\{g_i^j \rho_i^j (g_i^j)^{-1}\}_{i \geq j}$ have a converging subsequence so do their restrictions on G_1 : $\{g_i^j \rho_i^1 (g_i^j)^{-1}\}_{i \geq j}$. Therefore, we can extract another subsequence $\{i'_n\}_{n \in \mathbb{N}}$ such that both $g_{i'_n}^1 \rho_{i'_n}^1 (g_{i'_n}^1)^{-1}$ and $g_{i'_n}^j \rho_{i'_n}^j (g_{i'_n}^j)^{-1}$ are converging. Then we are in the setting of Lemma 4.3.1 thus, we have that $g_{i'_n}^1 (g_{i'_n}^j)^{-1}$ are con-

verging, up to an ulterior subsequence, as well and we call the limit g_1^j . Since we are only concerned about subsequences we assume that $g_i^1(g_i^j)^{-1}$ is the converging subsequence for which also $g_i^j \rho_i^j(g_i^j)^{-1}$ is converging. Then we have:

$$\forall j : g_i^1 \rho_i^j(g_i^1)^{-1} = g_i^1(g_i^j)^{-1} \left(g_i^j \rho_i^j(g_i^j)^{-1} \right) g_i^j(g_i^1)^{-1}$$

but everything on the right hand side is converging, hence the left hand side does. Since the j was arbitrary this concludes the proof. \blacksquare

We recall the following Theorem by Thurston [58, 0.1] for hyperbolizable compact pared 3-manifolds:

Theorem. Let (M, P) be a pared compact hyperbolizable 3-manifold. Then the set of representations induced by $AH(M, P)$ on the fundamental group of any component of $M^{\text{acyl}} \text{ rel } P$ $AH(M, P)$ is compact up to conjugation.

Where he shows that given a sequence of discrete and faithful representations $\{\rho_n\}_{n \in \mathbb{N}}$ of an acylindrical 3-manifold we can find elements $\{g_n\}_{n \in \mathbb{N}}$ of $\text{PSL}_2(\mathbb{C})$ so that the sequence $\{g_n \rho_n g_n^{-1}\}_{n \in \mathbb{N}}$ has a converging subsequence.

Then for (\overline{M}, P) we have:

Theorem 4.3.3. Let (\overline{M}, P) be an infinite-type pared acylindrical hyperbolic 3-manifold for \overline{M} the bordification of a manifold in \mathcal{M}^B . Then M is homotopic to a complete hyperbolic 3-manifold N such that P is represented by parabolic elements.

Proof. Let $\{M_i\}_{i \in \mathbb{N}}$ be the exhaustion of M , then by Proposition 4.1.79 for each i we can find n_i such that $M_i \subseteq M_{n_i}^{\text{acyl}} \text{ rel } P$. Since each M_i is hyperbolizable we have a discrete and faithful representation $\rho_i \in AH(M_i, P)$.

Let $X_i \doteq M_{n_i}^{\text{acyl}} \text{ rel } P$, then by Theorem [58, 0.1] applied to the sequence $\{\rho_k|_{\pi_1(X_i)}\}_{k \geq n_i}$ we can find $\{g_k^j\} \subseteq \text{PSL}_2(\mathbb{C})$ such that for all j the restriction of $\{g_k^j \rho_k (g_k^j)^{-1}\}$ to $\pi_1(M_j) \leq \pi_1(X_j)$ have a converging subsequence. By Proposition 4.3.2 we can assume that the g_k^j do not depend on j so that we have representations $\{g_k \rho_k g_k^{-1}\}_{k \in \mathbb{N}}$ that subconverge on each $\pi_1(M_j)$. By picking a diagonal subsequence we can define:

$$\forall \gamma \in \pi_1(M), \gamma \in \pi_1(M_i) : \rho_\infty(\gamma) \doteq \lim_{n \geq n_i} g_n \rho_n(\gamma) g_n^{-1}$$

Since $\pi_1(M) = \cup_{i \in \mathbb{N}} \pi_1(M_i)$ we get a representation $\rho_\infty : \pi_1(M) \rightarrow PSL_2(\mathbb{C})$ which is discrete and faithful by [33].

Thus if we define $N \doteq \mathbb{H}^3 / \rho_\infty(\pi_1(M))$ we have $\pi \doteq \pi_1(N) \cong \pi_1(M)$ and since they are both $K(\pi, 1)$ there is a homotopy equivalence between them. By construction all elements of P are parabolic in ρ_∞ . \blacksquare

We define:

Definition 4.3.4. Let (\overline{M}, P) with $\overline{M} \in \mathbf{Bord}(M)$ an infinite type pared acylindrical 3-manifold and $N \cong \mathbb{H}^3 / \Gamma$ be a hyperbolic 3-manifold. Then, a homotopy equivalence $\varphi : M \rightarrow N$ is said to *preserve parabolics* if $\forall \gamma \in \pi_1(M)$ homotopic in a component of P we have that $\varphi_*(\gamma)$ is represented by a parabolic element in Γ .

Lemma 4.3.5. Let (\overline{M}, P) be a pared infinite type acylindrical 3-manifold for $M \in \mathcal{M}^B$ and let $\varphi : M \rightarrow N$ be a homotopy equivalence preserving P . Then, for all $n \in \mathbb{N}$ we have that the cover $N_n \rightarrow N$ corresponding to $\varphi_*(\pi_1(M_n))$ the lift: $\varphi : M_n \rightarrow N_n$ is homotopic to an embedding and $N_n \cong \text{int}(M_n)$.

Proof. By Proposition 4.1.79 for all n we have k_n such that $M_n \subseteq M_{k_n}^{\text{acyl rel } P}$ and let P_{k_n} be the annuli in ∂M_{k_n} induced by P . Consider the cover $N_{k_n} \rightarrow N$ corresponding to $\pi_1(M_{k_n})$ and let N'_{k_n} be its manifold compactification, which exists by Tameness [1, 9], and let $Q \subseteq \partial N'_{k_n}$ the parabolic locus. Since φ preserves parabolics we can homotope $\tilde{\varphi}|_{P_{k_n}} : P_{k_n} \rightarrow Q$ to be a homeomorphism onto its image. Then, by Lemma 4.1.82 the homotopy equivalence:

$$\tilde{\varphi} : M_{k_n} \rightarrow N'_{k_n}$$

is homotopic to a map ψ that is an embedding on M_n . Then, $\psi|_{M_n} \simeq \varphi|_{M_n}$ lifts to the cover $N_n \rightarrow N_{k_n} \rightarrow N$ and its image forms a Scott core for N_n . Since the homotopy equivalence $\tilde{\psi}|_{M_n} : M_n \rightarrow N_n$ is an embedding and $\tilde{\psi}(\partial M_n)$ is incompressible by Lemma 2.2.4 we get that $N_n \cong \text{int}(M_n)$. \blacksquare

4.3.1.2 Relatively hyperbolic are homeomorphic to hyperbolic

A key step in the proof of Theorem 1 is that the homotopy equivalence $\varphi : M \rightarrow N$ mapping the elements corresponding to $P \subseteq \partial \overline{M}$ to parabolics in N is homotopic to a proper homotopy equivalence and it embeds the boundary components of a subsequence of a minimal exhaustion⁸ $\{M_n\}_{n \in \mathbb{N}}$.

⁸See Definition 4.1.41.

Our first objective is to show the following Theorem:

Theorem 4.3.6. Given the maximal bordification $\overline{M} \in \mathbf{Bord}(M)$ of $M \in \mathcal{M}^B$ and a minimal exhaustion $\{M_i\}_{i \in \mathbb{N}}$, if \overline{M} satisfies condition (\star) let $P \hookrightarrow \partial \overline{M}$ be such that (\overline{M}, P) is an infinite-type pared acylindrical manifold. Then, for any hyperbolic N and $\varphi : M \rightarrow N$ a homotopy equivalence preserving P we have a proper homotopy equivalence $\widehat{\varphi} : M \rightarrow N$ preserving P such that $\widehat{\varphi}$ is a proper embedding on tame ends of M and on $\mathcal{S} \doteq \cup_{i \in \mathbb{N}} \partial M_{a_i} \setminus \mathcal{P}$ for $\{a_i\}_{i \in \mathbb{N}}$ an increasing subsequence.

Before doing a full proof we deal with a couple of preliminary Lemmata. The first Lemma says that if \overline{M} is a maximal bordification induced by a maximal product $\mathcal{P}_{max} : S \times [0, \infty) \hookrightarrow M$ in standard form then we cannot have products $\mathcal{P} : \mathbb{A} \times [0, \infty) \hookrightarrow M$, also in standard form, such that for k sufficiently large the components of $\text{Im}(\mathcal{P}) \cap \cup_{k \in \mathbb{N}} \partial M_k$ are not peripheral in $\partial M_k \setminus \text{Im}(\mathcal{P}_{max})$.

Lemma 4.3.7. Let \mathcal{P}_{max} be a maximal product in $M \in \mathcal{M}$ and let $\mathcal{P} : \mathbb{A} \times [0, \infty) \hookrightarrow M$ be a product. If they are both in standard form then there exists some $i \in \mathbb{N}$ such that for $k > i$ the component of intersections of $\partial M_k \cap \text{Im}(\mathcal{P})$ are peripheral in $\partial M_k \setminus \text{Im}(\mathcal{P}_{max})$.

Proof. By maximality of \mathcal{P}_{max} we have a proper isotopy of \mathcal{P} and a connected sub-product \mathcal{Q} of \mathcal{P}_{max} such that \mathcal{P} is contained in an r -neighbourhood \mathcal{Q}' of \mathcal{Q} . Moreover, without loss of generality we can also assume that this new \mathcal{P} is in standard form. Let i be the minimal i such that $\text{Im}(\mathcal{P}) \cap \partial M_i \neq \emptyset$. Then, for $k \geq i$ all components of $\partial M_k \cap \text{Im}(\mathcal{P})$ are isotopic in \mathcal{Q}' into $\mathcal{Q} \cap S$ and so they are peripheral in $\partial M_k \setminus \text{Im}(\mathcal{P}_{max})$ reaching a contradiction. \blacksquare

The next Lemma says that tame ends of M relative to P embed, up to homotopy, into N for any homotopy equivalence $\varphi : M \rightarrow N$ preserving P .

Lemma 4.3.8. Let (\overline{M}, P) with $\overline{M} \in \mathbf{Bord}(M)$ an infinite-type pared acylindrical 3-manifold, N be a hyperbolic 3-manifold, $\varphi : M \rightarrow N$ be a homotopy equivalence preserving P and let $\mathcal{P}_{max} : S \times [0, \infty) \hookrightarrow M$ be a maximal product in M inducing $\partial \overline{M}$. Given $\varepsilon < \mu_3$ ⁹For μ_3 the 3-dimensional Margulis constant. we have a homotopy equivalence $\psi : M \rightarrow N$, homotopic to φ , that is a proper embedding on $\text{Im}(\mathcal{P}_{max})$ and with the property that $\psi \circ \mathcal{P}_{max}(\partial S \times [0, \infty)) \subseteq \partial Q_\varepsilon$ for Q_ε the ε -boundary of the parabolic locus of N .

Proof. Let $\mathcal{P}' \subseteq \text{Im}(\mathcal{P}_{max})$ be a regular neighbourhood of $P \subseteq \partial \overline{M}$ in $\text{Im}(\mathcal{P}_{max})$ and let $N^0 \doteq N \setminus \text{int}(Q_\varepsilon)$. We will now show that φ can be homotoped to be an embedding on \mathcal{P}' .

Step 1: We can homotope φ to be a homeomorphism on \mathcal{P}' mapping $\partial\mathcal{P}'$ onto ∂Q_ε .

Each component of \mathcal{P}' is homeomorphic to either $\mathbb{T}^2 \times [0, \infty)$ or $\mathbb{A} \times [0, \infty)$. Since (\overline{M}, P) is relatively acylindrical no two components of \mathcal{P}' are mapped by φ to the same component of Q_ε . If a component of \mathcal{P}' is homeomorphic to $\mathbb{T}^2 \times [0, \infty)$ the fact that φ can be homotoped to be an embedding follows directly by the hyperbolicity of N . So from now on we only consider components P_i of \mathcal{P}' homeomorphic to $\mathbb{A} \times [0, \infty)$. Since φ preserves parabolics, the image of the fundamental group of any component P_i of \mathcal{P}' is contained in a parabolic subgroup $\langle \gamma \rangle$ in $\pi_1(N)$ for γ a primitive element. Moreover, since (\overline{M}, P) is an infinite-type pared acylindrical 3-manifold any generator of $\pi_1(P_i)$ is primitive in $\pi_1(M)$ and hence so is its image in $\pi_1(N)$ thus $\varphi_* : \pi_1(P) \rightarrow \langle \gamma \rangle$ is an isomorphism.

Therefore, we see that the core of P_i is homotopic through φ to a simple closed curve γ in ∂Q_ε . We claim that γ is contained in an annular component of ∂Q_ε . If γ is in a torus component then we would have that $\pi_1(P)$ is contained in a \mathbb{Z}^2 factor in $\pi_1(M)$ which again is not possible by the fact that (\overline{M}, P) is an infinite-type pared acylindrical 3-manifold. Since γ is in an annular component the claim follows and we can assume that, up to a homotopy, φ is a homeomorphism on \mathcal{P}' . Moreover, we can assume that $\varphi(\partial\mathcal{P}') \subseteq \partial Q_\varepsilon$.

Let $S \subseteq M$ be a push-in of a component of $\partial\overline{M} \setminus P$. Since S is incompressible and φ is a homotopy equivalence we have that up to a homotopy of φ that is constant on \mathcal{P}' we have that $\varphi|_S : (S, \partial S) \rightarrow (N^0, \partial Q_\varepsilon)$ and let $U \doteq N_r(\varphi(S)) \subseteq N^0$ be a regular neighbourhood of $\varphi(S)$. By the existence results for PL-least area surfaces in [29] and [34, 1.26] for a triangulation τ_S of N^0 there are PL-least area representatives S' of $\varphi(S)$ such that $S' \subseteq U$. Consider the cover $\pi_j : N_j \rightarrow sN^0$ corresponding to $\varphi_*(\pi_1(M_j))$ such that the set U lifts homeomorphically to \tilde{U} in the cover and denote by \tilde{S}' the lift of S' . By Lemma 4.3.5, up to homotopy we have that the homotopy equivalence $\tilde{\varphi} : M_j \rightarrow N_j$ is an embedding, thus we see that all the \tilde{S}' are homotopic to embedded surfaces. Hence, by the results of [29] we have that the S' 's are embedded as well. Since the covering projection π_j is a homeomorphism on \tilde{U} with image U we get that $S' = \pi_j(\tilde{S}')$ is embedded as well.

Therefore, up to a homotopy of φ we can assume that φ embeds in N^0 a collection of surfaces $S \doteq \{S_n\}_{n \in \mathbb{N}}$ that are pushed-in of components of $\partial\overline{M} \setminus P$. Since $[S] \in H_2(M, \partial\mathcal{P}')$ is a separating π_1 -injective surface so is $[\varphi(S)] \in H_2(N, \partial Q_\varepsilon)$.

Let $\{E_n\}_{n \in \mathbb{N}}$ be the collection of tame ends of $M \setminus \mathcal{P}'$ facing the surfaces S_n and let $\Sigma_n \doteq \varphi(S_n)$.

Step 2: Up to a homotopy of φ we have that for all n $\varphi : E_n \rightarrow N$ is an embedding, $\varphi(E_n)$ are pairwise disjoint and $\varphi(\partial\mathcal{P}') \subseteq \partial Q_\varepsilon$.

We first show that each E_n embeds. Let X_1, X_2 be the connected components of $N^0|\Sigma_n$ and assume that $\pi_1(\Sigma_n) \hookrightarrow \pi_1(X_h)$ is not a surjection for $h = 1, 2$. Thus, neither one of X_1, X_2 is homeomorphic to $\Sigma_n \times [0, \infty)$. Then, we can find some compact submanifold $K_n \subseteq N$ containing Σ_n such that K_n is not homeomorphic to a product and it contains topology on both sides of Σ_n . Let $m > n$ such that $\pi_1(K_n) \subseteq \varphi_*(\pi_1(M_m))$ and consider the cover $N_m \rightarrow N$ corresponding to $\varphi_*(\pi_1(M_m))$. By Lemma 4.3.5 we have that $\tilde{\varphi} : M_m \rightarrow N_m$ is homotopic to a homeomorphism $\Phi : \text{int}(M_m) \rightarrow N_m$. In M_m the incompressible surfaces $S_n, \Phi^{-1}(\Sigma_n)$ are homotopic and since they are incompressible they are isotopic by [60]. Thus, at least one of X_1, X_2 is homeomorphic to a product and so we can homotope $\varphi \text{ rel } \mathcal{P}'$ so that $\varphi : E_n \rightarrow N$ is an embedding and we denote by \mathcal{E}_n its image. Therefore, up to a homotopy of φ we get that all ends E_n embed in N .

We now need to show that up to an ulterior homotopy we have that the $\varphi(E_n)$'s are pairwise disjoint. Up to relabelling the E_n we can assume that for all n there exists k_n such that the ends E_1, \dots, E_n are ends of $\widehat{M}_{k_n} \setminus \mathcal{P}'$ for $\widehat{M}_{k_n} \cong M_{k_n} \cup \partial_\infty M_{k_n} \times [0, \infty)$. Consider the covers N_{k_n} of N corresponding to $\varphi_*(\pi_1(M_{k_n}))$ and denote the homeomorphic lifts of \mathcal{E}_n in N_{k_n} by $\tilde{\mathcal{E}}_n$. By Lemma 4.3.5 $\varphi : \text{int}(M_{k_n}) \rightarrow N_{k_n}$ is homotopic to a homeomorphism. Hence, all ends E_1, \dots, E_n are mapped to distinct ends in N_{k_n} and so the $\tilde{\mathcal{E}}_n$ correspond to distinct ends of $N_{k_n}^0$. Therefore, up to pushing the $\varphi(E_n)$ inside the $\tilde{\mathcal{E}}_n$ we can assume that $\varphi : M_{k_n} \rightarrow N_{k_n}$ is an embedding on $E_1 \cup \dots \cup E_n$. Since the projection is a homeomorphism on $\tilde{\mathcal{E}}_n$ and maps $\varphi(\partial\mathcal{P}')$ into ∂Q_ε by iterating this construction we conclude the proof. \blacksquare

Before showing that if we have a homotopy equivalence $\varphi : M \rightarrow N$ respecting parabolics we can homotope it so that it is a proper homotopy equivalence we need to understand how loops in components S of $\cup_{k \in \mathbb{N}} \partial M_k$ are homotopic into P .

Definition 4.3.9. Let $\mathcal{P}_m : S \hookrightarrow M$, $M \in \mathcal{M}^B$, a maximal product in standard form with respect to a normal family $\{(N_k, R_k)\}_{k \in \mathbb{N}}$ and such that $\mathcal{P}_m(S \times \{0\}) \subseteq \cup_{k \in \mathbb{N}} \partial M_k$ and let $\mathcal{P}_s \doteq \mathcal{P}_m(\partial S \times [0, \infty))$. We say that an essential torus $V \subseteq N_k \setminus \text{int}(\text{Im}(\mathcal{P}_m))$ is a *parabolic solid torus* (PST) if:

- (i) no annulus of $\partial V \setminus R_k$ is parallel to \mathcal{P}_s ;
- (ii) $\partial V \cap R_k$ has a component that is isotopic into $N_\varepsilon(\mathcal{P}_s) \cap \partial X_k$ in ∂X_k ;
- (iii) V is maximal with respect to (ii), i.e. if $V, Q \subseteq N \in \pi_0(N_k)$ are both PST then $V = Q$.

Remark 4.3.10. If M has property (\star) , i.e. the maximal bordification does not have any doubly peripheral cylinder, then if V is a PST we have that by (ii) the component of $V \cap R_k$ isotopic into $N_\varepsilon(\mathcal{P}_s) \cap \partial X_k$ is unique.

Definition 4.3.11. Let $\{N_k\}_{k \in \mathbb{N}}$ be a normal family of JSJ for $M = \cup_{k \in \mathbb{N}} M_k \in \mathcal{M}^B$ and let $V \subseteq N_k$ be a parabolic solid torus. We define a *maximal parabolic solid torus* (MPST) \widehat{V} as the direct limit $\varinjlim V_i$ where:

- (i) $V_1 = V$;
- (ii) $V_i \setminus V_{i-1}$ are essential solid tori contained in N_j , $j \in \mathbb{N}$, whose wings wrap once around the soul;
- (iii) V_i is obtained from V_{i-1} by adding all essential solid tori $Q \subseteq N_j$, $j \in \mathbb{N}$, that have a wing matching up with one of V_{i-1} and such that $\partial Q \setminus R_j$ has no annuli parallel to \mathcal{P}_s in X_j .

Remark 4.3.12. Let \widehat{V} be a maximal parabolic solid torus, since for all k we have that by (ii) $V \cap X_k$ are essential tori by Lemma 4.1.48 we get that $\widehat{V} \cong \mathbb{S}^1 \times \mathbb{D}^2 \setminus L$ for $L \subseteq \partial(\mathbb{S}^1 \times \mathbb{D}^2)$ a collection of parallel simple closed curves.

We now show that if we do not have doubly peripheral cylinders then maximal parabolic solid tori are compact.

Lemma 4.3.13. Let $M \in \mathcal{M}^B$ satisfying property (\star) and let V be a parabolic solid torus. Then \widehat{V} is compact.

Proof. If \widehat{V} is not compact by Lemma 4.1.48 we have that $\widehat{V} \cong \mathbb{S}^1 \times \mathbb{D}^2 \setminus L$ for $L \neq \partial(\mathbb{S}^1 \times \mathbb{D}^2)$ and $L \neq \emptyset$. Thus, we obtain a product in standard form $\mathcal{P} : \mathbb{A} \times [0, \infty) \hookrightarrow M$ whose image is contained in \widehat{V} . By construction, in particular property (iii), we have that no annular component of $\partial \widehat{V} \cap X_k$ is parallel in X_k into \mathcal{P}_s , thus no component of $\text{Im}(\mathcal{P}) \cap X_k$ is isotopic in X_k into \mathcal{P}_m . Therefore, by Lemma 4.3.7 we reach a contradiction with the fact that \mathcal{P}_m was maximal. \blacksquare

We now show that if $M \in \mathcal{M}^B$ has no double peripheral annuli maximal parabolic solid tori corresponding to distinct parabolic solid tori are disjoint.

Lemma 4.3.14. Let $V \neq Q$ be parabolic solid tori contained in X_k, X_j respectively and assume that M has no doubly peripheral annuli. Then, $\widehat{V} \cap \widehat{Q} = \emptyset$.

Proof. If $\widehat{V} \cap \widehat{Q} \neq \emptyset$ then by construction we get that $\widehat{V} = \widehat{Q}$. Let $A_1 \subseteq \partial V \cap R_k$ and $A_2 \subseteq \partial Q \cap R_j$ be the annuli isotopic into $N_\varepsilon(\mathcal{P}_s) \cap X_k$ and $N_\varepsilon(\mathcal{P}_s) \cap X_j$ respectively. Since $V \neq Q$ by (iii) of the definition of PST we get that if $j = k$ then A_1 and A_2 are non-isotopic annuli in R_k .

Since $\widehat{V} = \widehat{Q}$ by (ii) of the definition we have an annulus $C \subseteq \widehat{V}$ connecting A_1 to A_2 . By extending the annulus C to an annulus \widehat{C} by going to infinity along the components of \mathcal{P}_s that A_1, A_2 are homotopic to we get a properly embedded annulus $\widehat{C} \subseteq M$ which compactifies to an annulus \overline{C} in \overline{M} .

Claim: The annulus \overline{C} is essential.

Proof of Claim: If A_1, A_2 are isotopic into distinct components of \mathcal{P}_s then \overline{C} is essential in \overline{M} and we are done. Thus, we can assume that $\partial \overline{C}$ are isotopic in $\partial \overline{M}$. Then, if \overline{C} is ∂ -parallel we have $k \in \mathbb{N}$ such that $\overline{C} \cap X_k$ is isotopic into \mathcal{P}_s contradicting the construction of \widehat{V} and \widehat{Q} or the fact that A_1 was not isotopic to A_2 in X_k and so that V and Q were distinct parabolic solid tori (we contradict property (iii)). \square

Thus, since \overline{C} is essential and has both boundaries peripheral in $\partial \overline{M}$ we get that \overline{M} has a doubly peripheral annulus reaching a contradiction. \blacksquare

Our final preparatory Lemma is:

Lemma 4.3.15. Let $\mathcal{P}_m \subseteq M \in \mathcal{M}^B$ be the image of a maximal product in standard form and let \mathcal{V} be a maximal collection of MPST. Then, for $S \in \pi_0(\cup_{k \in \mathbb{N}} \partial M_k \setminus (\mathcal{P}_m \cup \mathcal{V}))$ we have that any essential non-peripheral simple loop $\gamma \subseteq S$ is not homotopic into \mathcal{P}_s , for \mathcal{P}_s the side boundary of \mathcal{P}_m .

Proof. Since $\gamma \subseteq S \in \pi_0(\cup_{k \in \mathbb{N}} \partial M_k \setminus (\mathcal{P}_m \cup \mathcal{V}))$ is non-peripheral it is not isotopic into any torus of \mathcal{V} . Let H be the cylinder connecting γ to \mathcal{P}_s , up to a homotopy of H rel ∂H we can assume that for all $k \in \mathbb{N}$ $H \cap X_k$ is essential. By an iterative argument and the Annulus Theorem we have that for all k $H \cap X_k$ is homotopic to an embedded annulus. Then a thickening P of H is a PST and since $\gamma \cap \mathcal{V} = \emptyset$ by Lemma 4.3.14 we have that $\widehat{P} \cap \mathcal{V} = \emptyset$ contradicting the maximality of \mathcal{V} . \blacksquare

We can now show that given a homotopy equivalence φ between the interior of an infinite-type acylindrical pared 3-manifold and a hyperbolic 3-manifold respecting parabolics we have that φ is homotopic to a proper homotopy equivalence.

Theorem 4.3.16. Let (\overline{M}, P) with $\overline{M} \in \mathbf{Bord}(M)$ and $M \in \mathcal{M}^B$ an infinite-type pared acylindrical 3-manifold. Then, there exists a complete hyperbolic 3-manifold N and a proper homotopy

equivalence $\varphi : M \rightarrow N$ respecting P such that φ is an embedding on $\mathcal{S} \doteq \cup_{i \in \mathbb{N}} \partial M_{a_i}$ for $\{a_i\}_{i \in \mathbb{N}}$ an increasing subsequence. Moreover, we can also assume φ to be a proper embedding on any tame end of M .

Proof. By Theorem 4.3.3 we have a homotopy equivalence $\varphi : M \rightarrow N$ respecting P . Let \mathcal{P}_{max} be a maximal product inducing \overline{M} , $\mathcal{P}' \subseteq \text{Im}(\mathcal{P}_{max})$ be a neighbourhood of P and for $\varepsilon < \mu_3$ let Q_ε be the ε -thin part of N . By Lemma 4.3.8 we have:

Step 1: Up to a homotopy of φ we can assume that $\varphi|_{\text{Im}(\mathcal{P}_{max})}$ is an embedding and that it maps $\partial \mathcal{P}'$ into ∂Q_ε .

Let $\mathcal{S} \doteq \cup_{k \in \mathbb{N}} \partial M_k \setminus \text{Im}(\mathcal{P}_{max})$. Then, for any component S of \mathcal{S} we have that $\varphi|_S : S \rightarrow N$ maps ∂S homeomorphically into ∂Q_ε and without loss of generality we can assume each component of $\varphi(\partial S)$ to be a horocycle in ∂Q_ε .

Let $\widehat{\mathcal{V}} \subseteq M$ be a maximal collection of MPST and define:

$$\widehat{\mathcal{P}} \doteq N_r(\text{Im}(\mathcal{P}_{max})) \cup \widehat{\mathcal{V}}$$

then $\widehat{\mathcal{P}} \cong \text{Im}(\mathcal{P}_{max})$ and for all $k \in \mathbb{N}$: $\partial M_k \setminus \widehat{\mathcal{P}}$ is a collection of essential subsurfaces of ∂M_k , thus since $M \in \mathcal{M}^B$ they have a uniform bound on their complexity:

$$\forall k, \forall S \in \pi_0(\partial M_k \setminus \widehat{\mathcal{P}}) : |\chi(S)| \leq G$$

and we define $\widehat{\mathcal{S}} \doteq \cup_{k \in \mathbb{N}} \partial M_k \setminus \widehat{\mathcal{P}}$.

Since (\overline{M}, P) is an infinite-type acylindrical 3-manifold we have that no component S of $\widehat{\mathcal{S}}$ is an annulus.

Step 2: There exists a proper homotopy equivalence $\psi : M \rightarrow N$ with $\psi \simeq \varphi$.

The aim will be to show that up to homotopy we have that φ is proper when restricted on $\cup_{k \in \mathbb{N}} \partial M_k$. We first show that we can make $\varphi|_{\widehat{\mathcal{S}}}$ proper and then by doing homotopies of the annuli of $\mathcal{S} \setminus \widehat{\mathcal{S}}$ we will get φ is proper when restricted on \mathcal{S} and then by doing homotopies in the tame ends of N we will obtain the required result.

Claim: Up to homotopy $\psi|_{\widehat{\mathcal{S}}}$ is proper map.

For any essential subsurface $S \subseteq \partial M_k$ in $\pi_0(\widehat{\mathcal{S}})$ we can pick a triangulation τ such that each component of ∂S is realised as a single edge in τ and all vertices are contained in ∂S . Each component

of $\varphi(\partial S)$ is homotopic into a unique component of ∂Q_ε . Let S' be an open regular neighbourhood of S in ∂M_k , then we can homotope $\varphi|_{S'}$ so that φ maps the cusps of S' into cusps region contained in $\varphi(\mathcal{P}_m) \cap Q_\varepsilon$, i.e. into cusps of N . Since $\varphi|_{S'}$ is type preserving proper map we can realise the proper homotopy class of $\varphi(S')$ by a simplicial hyperbolic surface sending cusps to cusps, see [6, 11]. Moreover, we can do this for all S in $\widehat{\mathcal{S}}$ via a homotopy of φ . With an abuse of notation we still denote by φ the resulting map. We now claim that φ is a proper map when restricted to $\widehat{\mathcal{S}} = \{\Sigma_k\}_{k \in \mathbb{N}}$ whose image is contained in the simplicial hyperbolic surfaces $\{S_k\}_{k \in \mathbb{N}}$ we constructed.

If φ is not proper we can find a sequence $\{p_k \in \Sigma_k\}_{k \in \mathbb{N}} \subseteq \widehat{\mathcal{S}}$ of points and surfaces such that for $i \neq k$ we have $\Sigma_i \neq \Sigma_k$ and $\varphi(p_k)$ has a limit point $p \in N$. Each Σ_k is contained in a simplicial hyperbolic surface S_k of the same topological type. Since the S_k have uniformly bounded complexity by Gauss-Bonnet we get that their area is uniformly bounded by some $A \doteq A(G)$.

Case 1: There is a sub-sequence of the Σ_k such that Σ_k is not a pair of pants.

Since $\text{Area}(S_k) \leq A$ and S_k is not a pair of pants we can find a constant D such that for all $k \in \mathbb{N}$ there is a non-peripheral essential simple closed loop $\gamma_k \subseteq S_k$ based at p_k such that $\ell_N(\varphi(\gamma_k)) \leq D$. Since we assumed that the points $\varphi(p_k) \rightarrow p$ in N^0 we have that the $\{\varphi(\gamma_k)\}_{k \in \mathbb{N}}$ have to be in finitely many distinct homotopy classes. Since φ is a homotopy equivalence the same must happen to the $\{\gamma_k\}_{k \in \mathbb{N}}$.

Then by picking a subsequence of the Σ_k 's we can assume that they all have a homotopic curve γ . This curve was essential and non-peripheral in each Σ_k and so is not homotopic in $\partial \Sigma_k$, thus we get a product \mathcal{P} that either is not contained in \mathcal{P}_{max} and is not peripheral $\partial M_m \setminus \text{Im}(\mathcal{P}_{max})$ for $m \geq n$, for n the smallest m such that a component of $\{\Sigma_k\}_{k \in \mathbb{N}}$ is in ∂M_m , thus contradicting Lemma 4.3.7 or γ is isotopic in \mathcal{P}_s the side boundary of \mathcal{P}_m contradicting Lemma 4.3.15.

Case 2: All but finitely many Σ_k are pair of pants.

Let $\varphi_k \doteq \varphi|_{S_k}$ be the simplicial hyperbolic surface corresponding to the thrice punctured sphere Σ , so that we have simplicial hyperbolic surfaces:

$$\psi_k : (\Sigma, p_k) \rightarrow N$$

such that $\psi_k(p_k) \rightarrow p \in N$. Since $\text{inj}_N(p) > 0$ and the φ_k are 1-Lipschitz maps we have that since $\liminf \text{inj}_\Sigma(p_k) > 0$ the $\{p_k\}_{k \in \mathbb{N}}$ are contained in a compact core $K \subseteq \Sigma$, homeomorphic to a pair

of pants. This, means that we can find a compact set $K' \subseteq N$ containing p and with the property that for all $k \in \mathbb{N}$ $\psi_k(K) \subseteq K'$. Pick i such that $\pi_1(K') \subseteq \pi_1(\varphi_*(\pi_1(M_i)))$, then all the Σ_k lift to the cover M_i and we get that they are eventually parallel by the Kneser-Haken finiteness theroem, giving us a product over a pair of pants which cannot be properly isotopic into \mathcal{P}_m .

Thus φ is a proper map when restricted on $\widehat{\mathcal{S}}$ and $\text{Im}(\mathcal{P}_m) \cap \cup_{k \in \mathbb{N}} \partial M_k$ and every component A of $\cup_{k \in \mathbb{N}} \partial M_k \setminus (\widehat{\mathcal{S}} \cup \text{Im}(\mathcal{P}_m))$ is an annulus that is mapped into a cusp region of N . Then, by mapping the annuli further and further in the cusp end we obtain that φ is a proper map when restricted on $\cup_{k \in \mathbb{N}} \partial M_k$.

Since the restriction of ψ to $\mathcal{S}' \doteq \cup_{k \in \mathbb{N}} M_k$ is a proper map for every compact set $K \subseteq N$ the preimage $\varphi|_{\mathcal{S}'}^{-1}(K) = \varphi^{-1}(K) \cap \cup_{k \in \mathbb{N}} M_k$ is compact and so is contained in $\cup_{i \leq k} \partial M_i$ for some $k \in \mathbb{N}$. Thus we have that $\varphi^{-1}(K) \subseteq M_k$ and so $\varphi^{-1}(K)$ is a compact since it is closed.

Step 3: Up to picking a sub-sequence of the $\{M_i\}_{i \in \mathbb{N}}$ and a proper homotopy of ψ we can assume that all surfaces $\mathcal{S} \doteq \cup_{k \in \mathbb{N}} M_k$ are properly embedded in N .

Since ψ is a proper map when restricted to $\cup_{i \in \mathbb{N}} \partial M_i$ we have that for all i we can find neighbourhoods $U_i \subseteq N$ of $\psi(\partial M_i)$ with compact closure such that the open sets $\{U_i\}_{i \in \mathbb{N}} \subseteq N$ are properly embedded. This means that up to picking a subsequence, which we still denote by $i \in \mathbb{N}$, we can assume that the $\{U_i\}_{i \in \mathbb{N}} \subseteq N$ are pairwise disjoint.

Then we have a π_1 -injective map: $\psi : \partial M_i \rightarrow U_i$. By the existence results for PL-least area surfaces in [29] and [34, 1.26] for a triangulation τ_i of N there are PL-least area representatives S' of the $\psi(S)$, $S \in \pi_0(\partial M_i)$, such that $S' \subseteq U_i$. Now consider the cover N_j of N corresponding to $\pi_1(M_j)$ such that the set U_i lifts homeomorphically to \tilde{U}_i in the cover and denote by \tilde{S}' the lift of S' . By Lemma 4.3.5, up to homotopy we have that $\tilde{\psi} : M_j \rightarrow N_j$ is an embedding, thus we see that all the \tilde{S}' are homotopic to pairwise disjoint embedded surfaces Σ . Moreover, since by properties of a minimal exhaustion we have that $[\Sigma] \neq 0$ in $H_2(M)$ by the results of [29, Thm 6] the S' 's are embedded as well. By properties of a minimal exhaustion we have that no two \tilde{S}' are covering of an embedded surface thus by [29, Thm 7] the PL-least area surfaces are disjoint. Since the covering projection is a homeomorphism on $\pi_j : \tilde{U}_i \rightarrow U_i$ we get that $S' = \pi_i(\tilde{S}')$ are embedded as well. By repeating this for all $i \in \mathbb{N}$ we obtain a proper homotopy of ψ such that for $\{M_i\}_{i \in \mathbb{N}}$ the restriction of ψ to ∂M_i is an embedding. Moreover, since the U_i 's were pairwise disjoint we see that $\cup_{i \in \mathbb{N}} \partial M_i$ actually embeds in N .

The last claim in the statement, that ψ is a proper embedding on tame ends of M follows by the

same argument of Lemma 4.3.8. ■

We now want to promote φ to a homeomorphism from M to N . This will complete the proof of the main theorem:

Theorem 4.3.17. Let $M \in \mathcal{M}^B$, then M is homeomorphic to a complete hyperbolic 3-manifold if and only if the maximal bordification \overline{M} does not admit any doubly peripheral cylinder.

We now prove the final part of Theorem 1 which is that if the maximal bordification \overline{M} of M does not admit any doubly peripheral cylinders then M is hyperbolizable.

Theorem 4.3.18. Let $M \in \mathcal{M}^B$ and $\varphi : M \rightarrow N$ be a homotopy equivalence with N a complete hyperbolic manifold. If \overline{M} does not have any doubly peripheral annulus, then we have a homeomorphism $\psi : M \rightarrow N$.

Proof. By Lemma 4.1.80 let $\{M_i\}_{i \in \mathbb{N}}$ be a minimal exhaustion of M . By Theorem 4.3.16 we have a proper homotopy equivalence preserving P :

$$\varphi : M \rightarrow N$$

that is an embedding on $\cup_{k \in \mathbb{N}} \partial M_{i_k}$ and tame ends of M . The submanifold $P \subseteq \partial \overline{M}$ is a collection of annuli and tori that make (\overline{M}, P) an infinite-type pared acylindrical 3-manifold. Thus, without loss of generality we can assume that the exhaustion of M is the one given by $\{\partial M_{i_k}\}_{k \in \mathbb{N}}$. Therefore, we have a proper homotopy equivalence $\varphi : M \rightarrow N$ respecting P that is an embedding on the boundary components of a minimal exhaustion $\{M_i\}_{i \in \mathbb{N}}$ and any tame end of M .

Since φ is a homotopy equivalence we get that $\forall i : \varphi(\partial M_i)$ bounds a 3-dimensional compact submanifold K_i of N . We now want to show that the K_i are nested. Since $\varphi(\partial M_i) \cap \varphi(\partial M_j) = \emptyset$ we only need to show that $\varphi(\partial M_{i+1}) \not\subseteq K_i$.

Claim: For all $i : \varphi(\partial M_{i+1}) \not\subseteq K_i$ and up to a proper homotopy M_i embeds in N with image K_i .

Proof of Claim: Assume we have some i such that the above does not happen so that there is $S \in \pi_0(\partial M_{i+1})$ such that $\varphi(S) \subseteq K_i$. Pick $L > i$ so that K_i lifts homeomorphically $\tilde{\iota}(K_i) \hookrightarrow N_L$ for N_L the cover corresponding to $\varphi_*(\pi_1(M_L))$. Then in the cover we see $\tilde{\iota}(K_i)$ and $\tilde{\varphi}(S)$ inside it. By Lemma 4.3.5 we have that the map $\tilde{\varphi} : M_L \rightarrow \overline{N}_L$, for \overline{N}_L the manifold compactification [1, 9] of N_L , is homotopic to a homeomorphism ψ . Particularly, we have $\psi(M_i) \subseteq \overline{N}_L$ and, up to isotopy, we can assume $\psi(\partial M_i) = \tilde{\iota}(\partial K_i)$. Since \overline{N}_L is not a closed 3-manifold we must have

$\tilde{\iota}(K_i) = \psi(M_i)$. Moreover, since $\tilde{\iota}(K_i)$ projects down homeomorphically we get that up to a proper homotopy φ embeds M_i in N .

In particular from the homeomorphism $\psi : M_L \rightarrow \bar{N}_L$ we see that $\tilde{\varphi}(S)$ is homotopic outside $\tilde{\iota}(K_i) = \psi(M_i)$ and therefore it must be homotopic into $\tilde{\iota}(\partial K_i) = \psi(\partial M_i)$. Since we had a minimal exhaustion this can only happen if S co-bounds with $S' \subseteq \partial M_i$ an I -bundle contained in a tame end of M . Therefore, since φ was a proper embedding on tame ends of M we reach a contradiction. \square

Thus we can assume that we have an exhaustion $\{K_i\}_{i \in \mathbb{N}}$ of N with $\varphi(\partial M_i) = \partial K_i$ and we define $K_{j,i} \doteq \overline{K_j \setminus K_i}$. Moreover, by the claim we also have that $\pi_1(K_{j,i}) \xrightarrow{\varphi_*} \pi_1(U_{j,i})$ for $U_{j,i} \doteq \overline{M_j \setminus M_i}$ and $U_{1,0} = M_1$.

By the claim we can also assume that up to a proper homotopy of φ the restriction $\varphi|_{M_1}$ is an embedding with image K_1 .

To conclude the proof we need to show that the map φ is properly homotopic to an embedding.

We will now show with an inductive argument that up to proper homotopy φ is an embedding. Our base case is that M_1 embeds. By an iterative argument we need to show that we can embed $U_{i+1,i}$ relative to the previous embedding, hence rel ∂M_i .

Consider the following diagram:

$$\begin{array}{ccc}
 & \mathbb{H}^3 / \rho_\infty(\pi_1(U_{i+1,i})) & \\
 \varphi \nearrow & \downarrow \pi & \nwarrow \tilde{\iota} \\
 U_{i+1,i} & & K_{i+1,i} \\
 \varphi \searrow & & \nwarrow \iota \\
 & N &
 \end{array}$$

By Lemma 4.1.81 we have that $\tilde{\varphi}$ is homotopic to an embedding ψ rel boundary and we have that $\psi(\partial M_i) = \psi(\partial K_i)$. Then we can isotope ψ so that $\psi(\partial M_{i+1}) = \tilde{\iota}(\partial K_{i+1})$. Hence we have that $\psi(U_{i+1,i}) = \tilde{\iota}(K_{i+1,i})$, they are compact submanifolds with the same boundary in an open manifold. Therefore we get that $\pi \circ \psi$ is properly homotopic to φ , the homotopy is constant outside a compact set, and embeds $U_{i+1,i}$ rel the previous embedding. We can then glue all this proper homotopies together to get a proper embedding $\psi : M \hookrightarrow N$. Since the embedding is proper and N is connected we get that ψ is a homeomorphism from M to N completing the proof. \blacksquare

By combining Theorem 4.2.4 and Theorem 4.3.18 we complete the proof of Theorem 4.3.17.

Remark 4.3.19. Using our main result we can show that manifolds in \mathcal{M} have **CAT**(0) metric in which $\pi_1(M)$ acts by semisimple isometries. Then by [3, p.86] for $\gamma \in \pi_1(M)$ we have that the

centraliser $C(\gamma)$ is isomorphic to \mathbb{Z} . Since all roots of γ are in $C(\gamma)$ we would get that \mathbb{Z} has a divisible element which is impossible.

To construct the **CAT**(0) structure let \mathcal{A} be the collection of doubly peripheral annuli in \overline{M} . Let $X \doteq \coprod_{i \in \mathbb{N}} X_i$ be the manifold obtained by splitting \overline{M} along the annuli \mathcal{A} . Each manifold X_i has a collection \mathcal{A}_i of annuli in ∂X_i corresponding to annuli in \mathcal{A} . By Theorem 1 we can construct a complete hyperbolic metric on X_i . Moreover, we can rig the hyperbolic metric so that all \mathcal{A}_i correspond to rank one cusps¹⁰. Then by flattening all the cusps we obtain complete **CAT**(0) metrics in every X_i . Then by gluing back by euclidean isometries along the \mathcal{A}_i one can obtain a singular **CAT**(0) metric on M in which every element is represented by an hyperbolic isometry, since they all have an axis.

¹⁰This is achieved by choosing the curves \mathcal{P}_i that make X_i acylindrical in $\partial X_i \setminus \mathcal{A}_i$ and then adding \mathcal{A}_i to the collection \mathcal{P}_i .

Chapter 5

Appendix

5.1 The manifold N and X

A *knot* is an embedding $K : \mathbb{S}^1 \hookrightarrow M$. Given a knot we denote the complement of a regular neighbourhood $N_r(K)$ of $\text{Im}(K)$ in M by $M_K \doteq \overline{M \setminus N_r(K)}$. We recall the following theorem by Myers [40, 6.1]:

Theorem. Let M be a compact, orientable, 3-manifold whose boundary contains no 2-spheres. Then M has a knot $K : \mathbb{S}^1 \hookrightarrow M$ such that M_K is irreducible, with incompressible boundary and without any non-boundary parallel annuli or tori.

We call such a knot a *simple knot*.

Consider the hyperbolizable 3-manifold $M \doteq \Sigma_{1,1} \times I$. By Myer's Theorem we can pick a simple knot K in M such that M_K is acylindrical, atoroidal, irreducible and with incompressible boundary. We now want to fill in the torus boundary while keeping the resulting manifold acylindrical and hyperbolizable.

Consider the manifold M' obtained by gluing two copies of M_K along the genus two boundaries. The manifold M' has two torus boundaries corresponding to the copies of the knot K and since each M_K is acylindrical and atoroidal we have that M' is atoroidal as well. Thus, by the Hyperbolization Theorem we have that M' is hyperbolizable. By Hyperbolic Dehn Filling Theory [5, 54] we can find a high enough Dehn Filling of type $\frac{p}{q}$ so that the manifold M' filled by $\frac{p}{q}$ surgery on the tori is hyperbolic. We denote by $M' \left(\frac{p}{q}, \frac{p}{q} \right)$ the resulting manifold.

The manifold $M' \left(\frac{p}{q}, \frac{p}{q} \right)$ is homeomorphic to the double of M_K along the genus two boundary with $\frac{p}{q}$ filling in the tori boundaries. The double being atoroidal implies that $M_K \left(\frac{p}{q} \right)$ is acylindrical.

Therefore, by doing $\frac{p}{q}$ Dehn Filling on $M(K)$ we obtain an acylindrical and hyperbolizable 3-manifold N that has for boundary an incompressible genus two surface.

By gluing two copies of N along a separating annulus A in their boundary we get a manifold X such that ∂X are two genus two surfaces. We also denote by A the essential annulus in X obtained by the gluing. We now claim that X is hyperbolizable, since $\pi_1(X)$ is infinite by the Hyperbolization Theorem [34] it suffices to show that X is atoroidal. Let $T \subseteq X$ be an essential torus. Then, up to isotopy, $T|A \doteq \overline{T \setminus A}$ is a collection of embedded cylinders in N that are π_1 -injective. Therefore, since N is acylindrical we have that all the components of $T|A$ are boundary parallel. Moreover, since their boundaries are contained in A all components of $T|A$ are isotopic into A . Therefore, the torus T is not essential since $\iota(\pi_1(T)) \cong \mathbb{Z}$.

Hence, X is an hyperbolizable 3-manifold with a unique essential cylinder A and two genus two incompressible boundaries.

5.2 Example with Divisible Element

Recall:

Definition 5.2.1. An element $\gamma \in G$ is said to be *divisible* if for all $n \in \mathbb{N}$ there is $a \in G$ such that $\gamma = a^n$.

Let N be the acylindrical hyperbolizable 3-manifold with genus two incompressible boundary constructed in Appendix 5.1. Consider an infinite one sided *thick* cylinder $C \cong (\mathbb{S}^1 \times [0, 1]) \times [0, \infty)$ and let γ be the generator of $\pi_1(C)$.

Let $\{T_n\}_{n=2}^\infty$ be a collection of solid tori such that T_n has one wing winding around the soul n times. Glue the boundary of the wing of T_n to C along a small neighbourhood of $\mathbb{S}^1 \times \{1\} \times \{n - \frac{3}{2}\}$. The resulting 3-manifold \widehat{C} has a divisible element given by γ and is not atoroidal since if we consider the portion containing T_n, T_{n+1} it contains an incompressible non-boundary parallel torus.

By construction we still have that: $\partial \widehat{C} \cong \mathbb{S}^1 \times (-\infty, \infty)$ and we can think of the boundary of the solid torus as being a neighbourhood of $\mathbb{S}^1 \times \{1\} \times \{n - \frac{3}{2}\}$ so that the $\mathbb{S}^1 \times \{1\} \times [n, n+1]$ pieces now contain part of the boundary of the solid tori.

On each N we mark a closed neighbourhood A of the simple closed curve in ∂N splitting the genus two surface into two punctured tori Σ^\pm . We then glue countably many copies $\{N_n\}_{n \in \mathbb{N}}$ of N such that the marked annulus A_n in ∂N_n is glued to $\mathbb{S}^1 \times \{1\} \times [n, n+1]$ and then glue Σ_n^+ to Σ_{n+1}^- via the identity. On $\mathbb{S}^1 \times \{0\} \times [0, \infty)$ we glue countably many copies $\{N_k\}_{k \in \mathbb{N}}$ of N by gluing A_k

in ∂N_k to $\mathbb{S}^1 \times \{1\} \times [k, k+1]$ and Σ_k^+ to Σ_{k+1}^- via the identity.

Finally we glue another N to the remaining genus 2 boundary component. The result is a 3-manifold X that has an exhaustion X_i , $i \in \mathbb{N}$, given by taking the manifolds $\{N_k\}_{k \leq i}$, $\{N_n\}_{n \leq i}$, $\mathbb{S}^1 \times I \times [0, n]$ and the bottom copy of N . A schematic of the manifold is given in Figure 5.1.

The gaps $\overline{X_n \setminus X_{n-1}}$ are hyperbolizable since they are homeomorphic to two copies of N glued along the solid torus T_{n+1} (the proof of the manifold being atoroidal is similar to the proof of the atoroidality of X). The element of the exhaustion are not hyperbolizable, for example if we look at X_2 we see that it has a Torus subgroup $\langle \alpha, \beta | \alpha^2 = \beta^3 \rangle$.

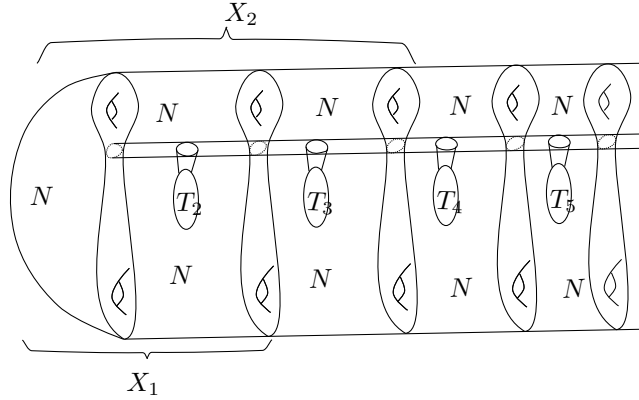


Figure 5.1: The manifold X with the first two elements of the exhaustion.

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