Endogenous Alliances in Survival Contests

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Abstract

Esteban and Sákovics (2003) showed in their three-person game that an alliance never appears in a possibly multi-stage contest game for an indivisible prize when allies' efforts are perfectly substitutable. In this paper, we introduce allies' effort complementarity by using a CES effort aggregator function. We consider an open-membership alliance formation game followed by two contests: the one played by alliances, and the one within the winning alliance. We show that if allies' efforts are too substitutable or too complementary, no meaningful alliance appears in equilibrium. However, if allies' efforts are moderately complementary to each other, then competition between two alliances is a subgame perfect equilibrium, which Pareto-dominates the equilibrium in a no-alliance single-stage contest. We also show that if forming more than two alliances is supported in equilibrium, then it Pareto-dominates two-alliance equilibrium. Nevertheless, the parameter space for such an allocation to be supported as an equilibrium shrinks when the number of alliances increases.

1 Introduction

In their influential paper, Esteban and Sákovics (2003) consider a three-person strategic alliance formation in a Tullock contest model in which players compete for an indivisible prize, and demonstrate that an alliance involves strategic disadvantages (see also Konrad 2009). There are two main disadvantageous forces against forming an alliance: First, if an alliance is formed, there will be a further contest that dissipates the members' rents even if the alliance wins the

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first race. Because of this *rent-dissipation* effect, the members of the alliance have lower valuations for winning in the first race, reducing their efforts and the winning probability. Second, even without the rent-dissipation problem, e.g., if the winning prize is shared equally, there are still *free-riding* incentives for the alliance members to reduce efforts, and consequently, the winning probability. As a result, they conclude that it is hard to materialize strategic alliances in a Tullock contest model.[1] Although Wärneryd (1998) shows that forming alliances and competing in a multi-stage competition reduce wasteful competition and increase total welfare, this resource saving effect is difficult to realize due to the disadvantageous effect on alliances when members' individual efforts are perfectly substitutable. Konrad (2009) points out that these disincentive effects are not specific to Tullock contest models—these effects also appear in first price all-pay auctions.

In this paper, we provide a simple solution for this alliance paradox by using complementarity in efforts in a general but symmetric n-person game.[2] To analyze complementarity, we introduce a simple, tractable CES effort aggregator function to translate alliance members' individual efforts into the alliance's joint effort. We assume that each individual member's marginal effort cost is constant in order to limit the benefits of forming an alliance to effort complementarity only.[3] With complementarity in efforts, a larger alliance can achieve a larger amount of joint effort among other benefits. Although there are aforementioned disincentives, it makes sense to form an alliance as long as the benefits from complementarity exceed the costs. The complementarity parameter in the CES aggregator provides a simple measure of the strength of incentive to form alliances as its value increases from 0 to 1.[4]

[1]Konrad (2004) considers an asymmetric all-pay auction game with exogenously determined hierarchical tournament structure, and shows that the highest valuation player may have no chance to become the final winner depending on the hierarchical structure.

[2]There are at least a few ways to resolve this alliance paradox (Konrad and Leininger 2007 and Konrad and Kovenock 2008: see the literature review).

[3]In general, forming an alliance can reach higher total efforts with less individual cost when cost functions are convex. For example, Esteban and Sákovics (2003) assume quadratic individual effort functions but still got their alliance paradox.

[4]Complementarity in efforts within a group in Esteban and Ray (2011) is more subtle. They analyze the conflict between two ethnic groups by assuming that players have heterogeneous financial and human opportunity costs, and they can contribute financially to a conflict or they can directly participate as activists. This generates some sort of complementarity: more activism requires more of the two inputs, time and money. They find that within-group inequality leads to more activism. This is because alliance members are specializing: poor with available cheap activist time and rich with money. They can contribute at a lower marginal utility cost. Their result can be interpreted that an increase in complementarity within groups intensifies group competition.

We are not the first to present the idea of using CES aggregator function to show that there are incentives to form an alliance. Following Cornes (1993) and Cornes and Hartley (2007) in the literature of private provision of public goods, Kolmer and Rommeswinkel (2013) and Choi, Chowdhury, and Kim (2016) have already demonstrated the presence of such incentives in alliance formation (see next section). This paper goes one step further. Since players' payoffs are related to the whole alliance structure, it is important to know how other players react to the alliance structure and whether or not the alliance structure could be stable. Therefore, we need to see players' and alliances' strategic interactions, and what happens in equilibrium: in particular, we ask whether or not there exists an equilibrium alliance structure.

We set up a simple alliance (coalition) formation game with multiple stages. In stage 1, players form alliances. In stage 2, alliances compete with each other, and in stage 3, the winning alliance members compete with each other for the indivisible prize. The solution concept is the standard subgame perfect Nash equilibrium. Two things should be noted. First, we model the alliance formation stage as an "open membership" game (Yi 1997 and Bogomolnaia and Jackson 2002) in which players can freely choose their alliance without being excluded.[5] Second, given the way we set up the multi-stage game, a singleton-only alliance structure and a grand alliance structure are practically identical, since the former does not have the third stage competition, and the latter does not have the second stage competition. The outcome of these two alliance structures coincides with the one of a grand standard Tullock contest. Thus, our focus will be finding subgame perfect equilibria with non-trivial alliance structures.

We first analyze the third stage game, which is just a Tullock contest within the winning alliance from stage 2. The rate of rent dissipation increases with the size of alliance increases (Proposition 1). Substituting this as the winning payoff of stage 2, we analyze equilibrium payoffs and strategies in stage 2 (Theorem 1). Using these building blocks we analyze for which values of CES parameter σ when *nontrivial alliance structures* emerge in equilibrium. We show that when the complementarity parameter in CES function is small, there are spin-off incentives for alliance members, while when complementarity parameter is large, players want to join a bigger alliance, and end up with a trivial grand alliance. Therefore, in those ranges, there in no nontrivial equilibrium structure. In order to show the existence of a nontrivial equilibrium

[5] In a companion paper, Konishi and Pan (2019), we consider a sequential alliance formation game à la Bloch (1996), and compare the resulting alliance structures (see Conclusion section).

alliance structure, we provide sufficient conditions for the existence of equilibrium with two alliances, and its uniqueness (Theorem 2). The condition is namely that σ is in the middle range. The intuition is that, if σ is small, the disadvantages of forming an alliance surpass the benefits, so that players would rather stay on as a singleton. But, if σ is large, a larger alliance becomes too attractive, ending up with a trivial grand coalition. However, in the middle range, players cannot benefit from being a singleton, but the rent dissipates more in a larger alliance. These two forces make a structure with two similar-sized alliances stable. Moreover, we show that such a two-alliance equilibrium allocation always Pareto-dominates the Tullock contest allocation (Theorem 3). That is, nontrivial alliances are not only an equilibrium phenomenon but also provide benefits to their members. Equilibria with more than two alliances are also analyzed. Although it is harder to satisfy equilibrium conditions as the number of alliances increases, it is shown that allocations with more alliances achieve higher payoffs.

The rest of the paper is organized as follows. In the next subsection, we review the relevant literature. Section 2 introduces the model, and Sections 3 and 4 investigate subgames in stages 3 and 2, respectively. Section 5 presents results on equilibrium alliance structures. Section 6 concludes, commenting on other alliance formation games.

1.1 Literature Review

There have been attempts to resolve the alliance paradox in Esteban and Sákovics (2003). Konrad and Leininger (2011) consider a dynamic all-pay auction game with possible side payments and endogenous timing of effort in which a group of players (alliance members) fight against a threat from an external enemy player. They show that there is a subgame perfect equilibrium, in which the alliance members exert efficient efforts against the enemy, followed by peaceful side payments from a leader of the alliance to the members in the equilibrium path. In this setup, the free-riding problem and redistributive conflicts are avoided by potential wasteful internal fighting. Konrad and Kovenock (2009) introduce budget constraints for efforts (resources) for each contest in a three-person all-pay auction game, and show that there can be an alliance that is beneficial for two players with tighter budgets. Konrad (2012) considers an all-pay-auction game in which each player's budget constraint is private information, considering forming an alliance as a tool of information sharing. Assuming a common winningness-to-pay, Konrad (2012) finds that merging alliances is weakly Pareto-improving, and the grand alliance emerges as equilibrium. An asymmetric three-player alliance formation game by Skaperdas (1998) may appear to be the closest to our model in the sense that he considers complementarity in members' efforts. He shows that alliance formation is beneficial if and only if the effort aggregator function exhibits increasing returns to scale in the members' efforts, but he assumes that effort levels of players are exogenously fixed.[6] In a general symmetric n-player game, Garfinkel (2004) adopts a farsighted solution concept (in the spirit of farsighted stability in Chwe 1994), i.e., a player spins off from an alliance structure only when the eventual outcome after such a move is more preferable than the original alliance structure. With her solution concept, she shows that with a large number of players, there are stable alliance structures with similar alliance sizes. In contrast, in our paper, we use the standard subgame perfect Nash equilibrium as the solution concept of our alliance formation game, and derive a stable alliance structure with similar sizes.

There are papers that use a CES aggregator function to capture effort complementarity. In the public good context, Cornes (1993) and Cornes and Hartley (2007) introduce complementarity in the famous voluntary public good contribution game in Bergstrom, Blume, and Varian (1986). Cornes and Hartley (2007) examine this problem extensively. In contest games, Kolmer and Rommeswinkel (2013) consider a group contest played by exogenously formed groups using a CES effort-aggregator function when group-members have heterogeneous abilities. Assuming that the winning prize is enjoyed by all members of a winning team as a public good, they analyze how complementarity of efforts affect members' efforts. They find that complementarity parameter has no effect on equilibrium efforts if groups are homogeneous. If groups are heterogeneous, then the divergence of efforts among group-members decreases as complementarity of efforts goes up, contradicting with common intuitions that complementarity of efforts solves free-riding problem. In contrast, Choi, Chowdhury, and Kim (2016) consider an indivisible private good award à la Esteban and Sákovics (2003) in an exogenous two-group model with two members each, who are heterogeneous in within-group powers. They find that the weaker player may get higher payoff under effort complementarity. Crutzen and Sahuguet (forthcoming) and Crutzen et al. (2019) compare political party competition with multiple party candidates under different voting systems using CES aggregator functions.

There is literature on contests among exogenously formed groups, concern-

[6] Tan and Wang (2010) also analyze an asymmetric model with exogenously fixed efforts. In their framework, they show that equilibrium alliance structure has only two-alliance with balanced power in a three- or four-player game. Herbst et al. (2015) experimentally study a three-player alliance formation game when the winning alliance members share the prize equally.

ing how group size and group sharing rules affect incentives to exert efforts (the prize is divisible). In his pioneering work, Olson (1965) argues that due to a free-riding problem sharing private benefits from the prize with the members of a group, larger groups are less effective in collective effort making than smaller groups. This is the so-called "group-size paradox." Assuming individual efforts are contractable, Nitzan (1991) considers a two-part reward system that combines an egalitarian and a relative-effort-sharing system, and analyzes how the combination affects members' incentives for players in large and small groups. Lee (1995) and Ueda (2002) endogeneize group sharing rules in this class. Esteban and Ray (2001) allow for allocating the prize among the members into public and private benefits (a mixed prize), and show that the group-size paradox disappears even if private benefits are allocated in an egalitarian manner,[7] as long as each member's marginal cost of effort is increasing in a sufficient speed (sufficient condition for this is their cost functions are quadratic). Nitzan and Ueda (2011) show that if private benefits can be allocated by an endogenously chosen relative-effort-sharing rule, then the group–size paradox disappears entirely in their class of effort functions, and larger groups tend to have more egalitarian rules.

Based on the line of research above, Baik and Lee (1997, 2001) endogenize the alliance formation in Nitzan's (1991) game with endogenous group sharing rules, and analyze two- and multiple-alliance cases, respectively. They use open-membership games to describe alliance formation. Bloch et al. (2006) generalize the model substantially to analyze the stability of the grand alliance in different alliance formation games, including a sequential coalition formation game in Bloch (1996), Okada (1996), and Ray and Vohra (2001). Sánchez-Páges (2007a) explore different types of stability concepts including sequential coalition formation games in alliance formation in contests where efforts are perfect substitutes. Sánchez-Páges (2007b) considers various stability concepts in a model where players allocate endowment into productive and exploitive activities. These papers assume the award is divisible, and alliance members can write a binding contract of sharing rule in the case of the alliance's winning. In our paper, we do not allow for any side payment, and players cannot credibly commit to any intra-alliance distribution rule as in Esteban and Sákovics (2003). We only focus on the benefits of forming a larger group through complementarity of effort and analyze endogenous formation of alliances in Tullock contests.

[7] Kolmer and Rommeswinkel (2013) is also a follow-up analysis of this line of research.

2 The Model

There are N players who seek to get an indivisible prize (say, to be the head of an organization). There is no side-payment allowed. The set of players is also denoted by $N = \{1, ..., N\}$, and they can form alliances exclusively for the purpose of being elected. Each player $i \in N$ can make an effort to enhance the popularity of her alliance and that of herself. We assume that each player has an identical linear cost function $C(e_i) = e_i$ for all $e_i \geq 0$.

We introduce potential benefits for players who belong to an alliance—complementarity in aggregating efforts by all alliance members. That is, if player i belongs to alliance j with N_j as the set of members, and these members make efforts $(e_{hj})_{h\in N_j}$, then the aggregated effort of alliance j, E_j, is described by a CES aggregator function

$$E_j = \left(\sum_{h \in N_j} e_{hj}^{1-\sigma} \right)^{\frac{1}{1-\sigma}}, \tag{1}$$

where $\sigma \in (0, 1]$ is a parameter that describes the degree of complementarity: if $\sigma = 0$ it is a linear aggregator function as in Esteban and Sákovics (2003), and if $\sigma = 1$ it is a Cobb-Douglas function. Thus, as σ goes up, the complementarity of members' efforts increases.

Candidate i in alliance j decides how much effort e_{ij} to contribute to her alliance j. The winning probabilities of an alliance is a Tullock-style contest. That is, an alliance j's "winning probability" given its members' efforts is

$$p_j = \frac{E_j}{\sum_{k \in J} E_k}. \tag{2}$$

An indivisible prize is valued as $V > 0$, which is common to all players. Since the prize is indivisible, one player in the winning alliance in the second stage must be selected as the final winner in the third-stage contest.

In the third-stage competition, we assume that a Tullock contest takes place within the winning alliance N_j. Denoting the second-stage effort as $\hat{e}_i$, the winning probability of player $i \in N_j$ is

$$p_i = \frac{\hat{e}_i}{\sum_{h \in N_j} \hat{e}_h} \tag{3}$$

Formally, a partition of the set of players N, $\pi = \{N_1, ..., N_J\}$ is an alliance structure, where each alliance j consists of a set of players N_j, where $\cup_{j \in J} N_j =$

N and $N_{j'} \cap N_j = \emptyset$ for any $j, j' \in J$ with $j \neq j'$. Since we assume that players are ex-ante homogenous, we also call $\{n_1, ..., n_J\}$ an alliance structure with $n_j = |N_j|$ for all $j = 1, ..., J$. We consider a dynamic contest game with endogenous alliances: it starts with players' forming alliances, the alliances compete for an indivisible prize in the first contest, and then the players in the winning alliance compete with each other to determine the final winner in the second contest. Our dynamic contest game with endogenous alliances has three stages:

Stage 1. All players $i \in N$ choose one of locations $z_i \in Z$ simultaneously, where the number of locations is at least as many as the number of players $|Z| \geq N$. Players choosing the same integer form an alliance: $N(z) \equiv \{i \in N : z_i = z\}$ for all $z \in \mathbb{Z}$, and a collection of nonempty alliances is an alliance structure $\pi = \{N_j\}_{j=1}^{J}$.[8]

Stage 2. All players $i \in N$ choose effort $e_i \in \mathbb{R}_+$ simultaneously, knowing the aggregated effort of her alliance is (1). The inter-alliance contest is a Tullock contest with winning probabilities equal to (2).

Stage 3. All members of the winning alliance N_j choose effort $\hat{e}_i \in \mathbb{R}_+$ simultaneously. The ultimate winner is selected in a simple Tullock contest with winning probabilities equal to (3)

We use standard subgame perfect Nash equilibrium as the solution of this dynamic game. We consider equilibria in pure strategies only. We will analyze this game by backward induction.

3 Stage 3: Final Contest within the Winning Alliance

In the third stage, all members in the winning alliance N_j in the first stage engage in a Tullock contest by exerting effort $\hat{e}_i \geq 0$. Thus, player i's winning probability is

$$p_i = \frac{\hat{e}_i}{\sum_{h \in N_j} \hat{e}_h}.$$

[8]This is called an open-membership game, in which (i) players can freely move from alliance to alliance, and (ii) players are free to spin off unilaterally. See Yi (1997) and Bogomolnaia and Jackson (2002).

For any player i in the winning group j, the expect payoff in stage 3 is

$$\tilde{V}_i = \frac{\hat{e}_i}{\hat{e}_i + \sum_{h \neq i} \hat{e}_h} V - \hat{e}_i$$

The first-order condition implies that

$$\frac{1 - p_i}{\hat{e}_i + \sum_{h \neq i} \hat{e}_h} - 1 = 0 \Rightarrow \frac{1}{\hat{e}_i} p_i (1 - p_i) V - 1 = 0$$

Since players are homogeneous, $p_i(1 - p_i) = \frac{n_j - 1}{n_j^2}$ is the same for all i in the winning group j. Then, we have the following proposition.

Proposition 1. *Suppose that the winning coalition of the first stage has size n_j. Then, the second-stage equilibrium strategy and payoff are*

$$\hat{e}^j = \frac{n_j - 1}{n_j^2} V \text{ and } \tilde{V}^j = \frac{V}{n_j} \left(1 - \frac{n_j - 1}{n_j} \right) = \frac{V}{n_j^2}$$

4 Stage 2: Contest between Alliances

Consider an inter-alliance contest problem. We assume that all players are ex ante identical, and all players in each alliance exert the same amount of effort in equilibrium. Therefore, the related information for an alliance N_j is the number of members in it, n_j. We simplify our analysis using this property. From Proposition 1, we know that for a given size of alliance n_j the payoff of intra-alliance contest is determined by $\tilde{V}_j = \frac{V}{n_j^2}$ Thus, the second stage maximization problem of a player ij in alliance j is to maximize the payoff

$$\begin{aligned} V_{ij} &= \frac{\left(e_{ij}^{1-\sigma} + \sum_{h \neq i} e_{hj}^{1-\sigma} \right)^{\frac{1}{1-\sigma}}}{\left(e_{ij}^{1-\sigma} + \sum_{h \neq i} e_{hj}^{1-\sigma} \right)^{\frac{1}{1-\sigma}} + \sum_{j' \neq j} E_{j'}} \tilde{V}_j - e_{ij} \\ &= \frac{\left(e_{ij}^{1-\sigma} + \sum_{h \neq i} e_{hj}^{1-\sigma} \right)^{\frac{1}{1-\sigma}}}{\left(e_{ij}^{1-\sigma} + \sum_{h \neq i} e_{hj}^{1-\sigma} \right)^{\frac{1}{1-\sigma}} + \sum_{j' \neq j} E_{j'}} \frac{V}{n_j^2} - e_{ij} \end{aligned}$$

The first-order condition with respect to e_{ij} (if an interior solution) is

$$\frac{\left(\sum_{j'} E_{j'} - E_j \right)}{\left(\sum_{j'} E_{j'} \right)^2} e_{ij}^{-\sigma} E_j^{\sigma} \frac{V}{n_j^2} - 1 = 0$$

Since members of alliance j are symmetric, they exert the same amount of effort due to a CES aggregator. Let $e_j = e_{ij}$ for all $ij \in N_j$. Then, the aggregated effort can be written as $E_j = \left(n_j e_j^{1-\sigma}\right)^{\frac{1}{1-\sigma}} = n_j^{\frac{1}{1-\sigma}} e_j$. Substituting this back into the above condition, we have

$$\frac{\left(\sum_{j' \neq j} n_{j'}^{\frac{1}{1-\sigma}} e_{j'}\right)}{\left(\sum_{j'} n_{j'}^{\frac{1}{1-\sigma}} e_{j'}\right)^2} n_j^{\frac{\sigma}{1-\sigma}} \frac{V}{n_j^2} - 1 = 0,$$

or

$$\frac{\left(\sum_{j' \neq j} n_{j'}^{\frac{1}{1-\sigma}} e_{j'}\right)}{\left(\sum_{j'} n_{j'}^{\frac{1}{1-\sigma}} e_{j'}\right)^2} V - n_j^{\frac{2-3\sigma}{1-\sigma}} = 0,$$

for all $j = 1, ..., J$. This is a set of conditions that characterize the first stage equilibrium if all coalitions exert positive efforts. Using the share function approach by Cornes and Hartley (2005), we convert our first-stage team competition model by J alliances to an artificial J-person Tullock contest model. We can prove the existence and uniqueness of equilibrium in the second stage under any π. Note that in the equilibrium, there may be alliances that do not exert effort. Using this property, we will also identify a sufficient condition for the nonexistence of single-player alliances in equilibrium.

4.1 Artificial Tullock Contest Game and Share Function

To apply a method called the "share function" approach that is systematically analyzed in Cornes and Hartley (2005), we rewrite the second-stage competition as a Tolluck contest with heterogeneous marginal costs.[9] Formally, let $w_j = n_j^{\frac{2-3\sigma}{1-\sigma}}$ (marginal cost) and $x_j = n_j^{\frac{1}{1-\sigma}} e_j$ (effort) for each $j = 1, ..., J$. *An artificial Tullock contest game* $(J, V, (w_j)_{j=1}^J)$ *corresponding to our second-stage game* is a J person game in which each player j exerts effort x_j with constant marginal cost $w_j > 0$. Her winning probability is specified by $\pi_j = \frac{x_j}{\sum_{j'=1}^J x_{j'}}$, and her payoff is

$$u_j = \frac{x_j}{\sum_{j'=1}^J x_{j'}} V - w_j x_j.$$

[9]Esteban and Ray (2001) and Ueda (2002) used the same method in their papers.

The payoff function is strictly concave in x_j, and the first-order condition is

$$\frac{\left(\sum_{j'\neq j} x_{j'}\right)}{\left(\sum_{j'} x_{j'}\right)^2} V - w_j = 0,$$

for $j = 1, ..., J$. This set of equations are the (*interior*) first-order conditions for the artificial game which is identical to the set of first-order conditions for the original game. Thus, in order to analyze the properties of the equilibrium in the original game, it suffices to analyze the properties of the corresponding artificial game. To do that, we follow the share function approach in Cornes and Hartley (2005).

Let $X_{-j} = \sum_{j'\neq j} x_{j'}$. Then, $x_j > 0$ is a unique best response to X_{-j} if and only if

$$x_j^2 + 2X_{-j}x_j + X_{-j}^2 - \frac{VX_{-j}}{w_j} = 0$$

Noting that some players may have too high marginal cost for an interior solution, player j's best response to X_{-j} is

$$\beta_j(X_{-j}) = \max\left\{-X_{-j} + \sqrt{\frac{VX_{-j}}{w_j}}, 0\right\}$$

We define player j's *replacement function* following Cornes and Hartley (2005): a replacement function $r_j(X)$ is a function of total effort $X = \sum_{j'} x_{j'}$ such that $r_j(X)$ is the best response to $X - r_j(X)$: i.e., $r_j(X) = \beta_j(X - r_j(X))$. Thus we obtain

$$r_j(X) = \max\left\{X - w_j\frac{X^2}{V}, 0\right\}$$

Let group j's *share function* be $s_j(X) = \frac{1}{X}r_j(X)$:

$$s_j(X) = \max\left\{1 - w_j\frac{X}{V}, 0\right\}.$$

Clearly, $X = X^*$ is a unique equilibrium at $\sum_{j'} s_{j'}(X^*) = 1$. Let $s(X) = \sum_{j'} s_{j'}(X)$. This is a decreasing function. Order players by $w_1 \leq w_2 \leq .... \leq w_J$. The share function $s(X)$ is a piecewise linear function with kinks at $\hat{X}^{n_j} = \frac{V}{w_j}$ for each $j = 1, ..., J$. Moreover, at the equilibrium X^*, $s_j(X^*)$ is also the winning probability of player j. Figure 1 depicts share functions for $j = 1, ..., J$ and $s(X)$. As is easily seen from Figure 1, if $\hat{X}^{n_j} = \frac{V}{w_j} < X^*$, then $s_j(X^*) = 0$ must hold. The following lemma summarizes the result of this artificial Tullock game.

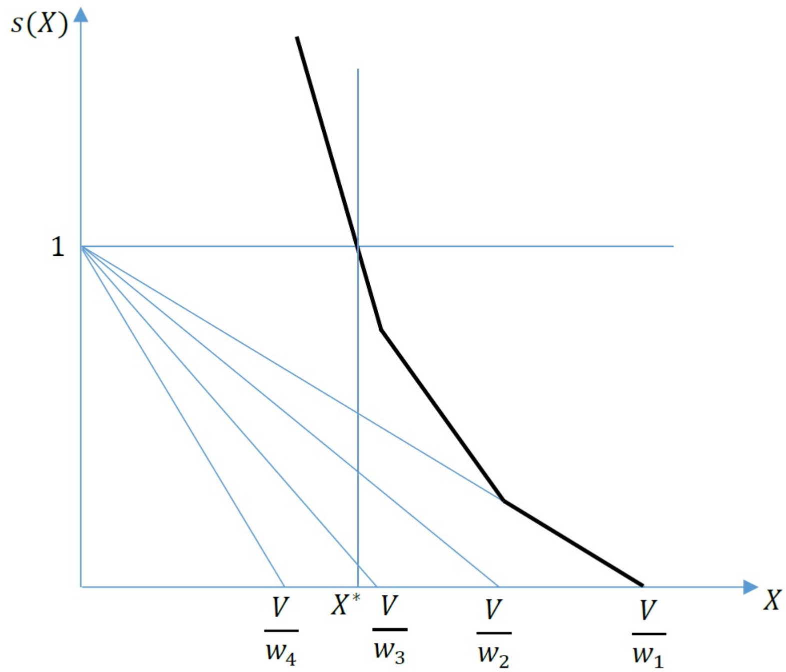

Figure 1: $s_j(X)$ *and* $s(X)$ *when* $J = 4$. *For the alliance with large* $w_j = n_j^{\frac{2-3\sigma}{1-\sigma}}$, *the equilibrium effort is* 0, *i.e., it is inactive.*

Lemma 1. [Cornes and Hartley, 2005] *An artificial Tullock game has a unique equilibrium* $X = X^*$ *at* $\sum_j s_j(X^*) = 1$. *Moreover, there exist* j^* *such that, for each* $j = 1, ..., j^*$, $x_j = X^* - w_j \frac{(X^*)^2}{V}$, *and for each* $j = j^* + 1, ..., J$, $\hat{X}^{n_j} \leq X^*$ *(or* $\sum_{j'} s_{j'}(\hat{X}^{n_j}) \geq 1$*) and* $x_j = 0$ *hold.*

4.2 Equilibrium in Stage 2

Lemma 1 shows the existence and uniqueness of the equilibrium for the artificial contest and, therefore, for the original game's stage 2 contest. Given an alliance structure, the following theorem further derives players' equilibrium efforts and utilities in stage 2.

Theorem 1. *There exists a unique equilibrium in the second-stage game for any partition of players* $\pi = \{n_1, ..., n_j\}$ *characterized by the share function* $s(X^*) = 1$. *There is* $j^* \in \{1, ..., J\}$ *such that* $p_j^* = s_j(X^*) > 0$ *(active alliance)* *for all* $j \leq j^*$ *(*$\hat{X}_j > X^*$*), while* $p_j^* = s_j(X^*) = 0$ *(inactive alliance) for all* $j > j^*$ *(*$\hat{X}_j \leq X^*$*). Then, the members of alliance* $j = 1, ..., J$ *obtain payoff*

$$u_j = \begin{cases} \frac{1}{n_j^2}\left[1-(j^*-1)\frac{n_j^{\frac{2-3\sigma}{1-\sigma}}}{\sum_{j'=1}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right]\left[1-(j^*-1)\frac{n_j^{\frac{1-2\sigma}{1-\sigma}}}{\sum_{j'=1}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right] & \text{if } j \leq j^* \\ 0 & \text{if } j > j^* \end{cases}$$

by exerting effort

$$e_j = \begin{cases} \frac{1}{n_j^{\frac{1}{1-\sigma}}}\left[1-(j^*-1)\frac{n_j^{\frac{2-3\sigma}{1-\sigma}}}{\sum_{j'=1}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right]\frac{(j^*-1)V}{\sum_{j'=1}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}} & \text{if } j \leq j^* \\ 0 & \text{if } j > j^* \end{cases}$$

Moreover, the equilibrium total efforts are

$$X^* = \frac{(j^*-1)V}{\sum_{j'}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}},$$

and

$$(j^*-1)n_j^{\frac{2-3\sigma}{1-\sigma}} < \sum_{j'}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}$$

holds for all $j = 1, ..., j^*$.

5 Stage 1: Alliance Formation

Before proceeding to the equilibrium analysis of the first-stage game of this dynamic contest game, we need to clarify the implications of no alliance (alliances are all singletons) and the grand alliance. If each player forms a singleton alliance, $\pi^0 = \{1, ..., 1\}$, and from Theorem 1, the resulting payoff of π^0 is $u^0 = \frac{V}{N^2}$. If players form the grand alliance, then the game will directly proceed to Stage 3 which is just a regular Tullock contest. Thus, having the grand alliance and having no alliance are practically the same. We denote a single grand alliance structure and its resulting payoff by $\pi^N = \{N\}$ and $u^N = \frac{V}{N^2}$, respectively. To answer the alliance paradox, our analysis is focused on the incentives for players forming stable alliance structures other than π^0 and π^N due to the complementarity we introduced.

5.1 Examples of Alliance Formation

In this section, we consider a case of four players and $V = 1$. If they do not form an alliance, everybody gets $u^0 = \frac{1}{16}$. Since there are only four identical players, we only need to consider the following coalition structures: (i) $\pi^0 = \{1, 1, 1, 1\}$, (ii) $\pi^1 = \{2, 1, 1\}$, (iii) $\pi^2 = \{2, 2\}$, (iv) $\pi^3 = \{3, 1\}$, and (v) $\pi^N = \{4\}$. Let us denote the payoff of a player in size n alliance in partition π by $u(n, \pi)$. Since the key parameter in a CES aggregator function is $\sigma \in [0, 1)$, and the complementarity of team efforts increases as σ increases, we consider three values of σ in order: $\sigma = \frac{1}{2}$ (weak complementarity), $\frac{3}{4}$ (moderate complementarity), and $\frac{4}{5}$ (strong complementarity). We check how alliance structure is affected by the complementarity of team efforts.

5.1.1 Weak Complementarity $\sigma = \frac{1}{2}$

In this case, we have $\frac{2-3\sigma}{1-\sigma} = 1$ and $\frac{1-2\sigma}{1-\sigma} = 0$. Using Theorem 1, we know the following:

	$u(1, \pi^0)$	$u(2, \pi^1)$	$u(1, \pi^1)$	$u(2, \pi^2)$	$u(3, \pi^3)$	$u(1, \pi^3)$	$u(4, \pi^N)$
payoff	$\frac{1}{16}$	0	$\frac{1}{4}$	$\frac{3}{32}$	$\frac{1}{48}$	$\frac{9}{16}$	$\frac{1}{16}$

Note that under π^1 and π^3, smaller alliances perform better than larger ones.

We analyze which partition can be a Nash equilibrium of stage 1:

1. $\pi^0 = \{1, 1, 1, 1\}$: This is a Nash equilibrium.

2. $\pi^1 = \{2, 1, 1\}$: There is a unilateral spin-off from the size 2 alliance, resulting in π^0.

3. $\pi^2 = \{2, 2\}$: There is a unilateral spin-off from one of the size 2 alliances, resulting in π^1.

4. $\pi^3 = \{3, 1\}$: There is a unilateral spin-off from the size 3 alliance, resulting in π^1.

5. $\pi^N = \{4\}$: There is a unilateral spin-off from the grand alliance, resulting in π^3.

Thus, when $\sigma = \frac{1}{2}$, the complementarity of team efforts is too weak to form a nontrivial alliance.

5.1.2 Medium Complementarity $\sigma = \frac{3}{4}$

In this case, we have $\frac{2-3\sigma}{1-\sigma} = -1$ and $\frac{1-2\sigma}{1-\sigma} = -2$. Using Theorem 1, we know the following:

	$u(1,\pi^0)$	$u(2,\pi^1)$	$u(1,\pi^1)$	$u(2,\pi^2)$	$u(3,\pi^3)$	$u(1,\pi^3)$	$u(4,\pi^N)$
payoff	$\frac{1}{16}$	$\frac{3}{25}$	$\frac{1}{50}$	$\frac{3}{32}$	$\frac{11}{144}$	$\frac{1}{16}$	$\frac{1}{16}$

Note that under π^1 and π^3, larger alliances perform better than smaller ones.

We analyze which partition can be a Nash equilibrium of stage 1:

1. $\pi^0 = \{1,1,1,1\}$: Two singletons merge to form an alliance, resulting in π^1.
2. $\pi^1 = \{2,1,1\}$: Two singletons merge to form an alliance, resulting in π^2.
3. $\pi^2 = \{2,2\}$: This is a Nash equilibrium.
4. $\pi^3 = \{3,1\}$: One of the size 3 alliance members moves to merge with a singleton, resulting in π^2.
5. $\pi^N = \{4\}$: This is a Nash equilibrium.

This case allows for two Nash equilibria: a trivial grand alliance equilibrium, and an equally sized two-alliance equilibrium. One important observation is that π^2 Pareto-dominates π^N.

5.1.3 Strong Complementarity $\sigma = \frac{4}{5}$

In this case, we have $\frac{2-3\sigma}{1-\sigma} = 1$ and $\frac{1-2\sigma}{1-\sigma} = 0$. Using Theorem 1, we know the following:

	$u(1,\pi^0)$	$u(2,\pi^1)$	$u(1,\pi^1)$	$u(2,\pi^2)$	$u(3,\pi^3)$	$u(1,\pi^3)$	$u(4,\pi^N)$
payoff	$\frac{1}{16}$	$\frac{14}{81}$	$\frac{1}{162}$	$\frac{3}{32}$	$\frac{29}{300}$	$\frac{1}{100}$	$\frac{1}{16}$

We analyze which partition can be a Nash equilibrium of stage 1:

1. $\pi^0 = \{1,1,1,1\}$: Two singletons merge to form an alliance, resulting in π^1.
2. $\pi^1 = \{2,1,1\}$: Two singletons merge to form an alliance, resulting in π^2.

3. $\pi^2 = \{2, 2\}$: One of the size 2 alliance members moves to the other alliance, resulting in π^3.

4. $\pi^3 = \{3, 1\}$: A singleton merges into the size 3 alliance, resulting in π^N.

5. $\pi^N = \{4\}$: This is a Nash equilibrium.

Thus, when $\sigma = \frac{4}{5}$, the trivial grand alliance is the unique Nash equilibrium.

5.1.4 Observations

The above examples show that when σ is small, there is no gravity to sustain an alliance since the complementarity of efforts is not sufficient enough to compensate Olson's inefficiency of alliances.[10] In this case, players prefer standing alone and competing with other single players and/or alliances. In contrast, if σ is large, a larger alliance is always relatively more attractive than a smaller alliance, resulting in the grand alliance. When σ is in the middle range, nontrivial alliances can appear and Pareto-dominate trivial allocation. For nontrivial equilibria, the complementarity is strong enough to make a singleton player not profitable. At the same time, it is not strong enough that players prefer a smaller group to avoid severe competition in the final stage. These two forces jointly ensure stability. We will show that this is not a coincidence.

5.2 No Spin-off Condition

Before we move on to the alliance formation stage (Stage 1), we provide preliminary analysis. From now on, we consider only those structures where $J = j^*$, since no one is attracted by an inactive alliance in equilibrium.[11] Then, first note that if $\sigma < \frac{2}{3}$, then $n_h < n_k$ implies $w_h < w_k$. Since $s_j(X^*) = 1 - w_j \frac{X^*}{V}$ is the winning probability, the larger alliance has a lower winning probability, and needs to share the prize with many members, even if it wins. Thus, although team production is more efficient due to complementarity, we may conjecture that a non-singleton alliance cannot survive if σ is small. The following proposition confirms the intuition for σ small enough.

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[10] Skaperdas (1998) shows that forming an alliance is beneficial if and only if the effort aggregator function exhibits increasing returns to scale. However, in his model, players' efforts are exogenously fixed. As a result, our model requires strong complementarity to offset the free-rider problem.

[11] Note that a player's payoff is positive as long as the alliance she belongs to has a positive probability of winning, which generates a positive payoff for the members. If an alliance is inactive, it means a zero payoff for the members.

Proposition 2. *Suppose that $\sigma \leq \frac{1}{2}$. Then, from any alliance structure π with a non-singleton alliance, there is a player with an incentive to spin-off to form a singleton alliance.*

Recall that as long as $\sigma < \frac{2}{3}$, then $n_h < n_k$ still implies $w_h < w_k$. Thus, it is natural to assume that even if $\sigma > \frac{1}{2}$, the same result from Proposition 2 could hold. However, even if it may be beneficial to have a smaller alliance, it does not means that a player has an incentive to spin off as a singleton, especially when there are many alliances or some alliances with a relatively large membership. We demonstrate this in the following example, showing that a spin-off may not be profitable when σ is close to $\frac{2}{3}$ and the number of alliances is large.

Example 1. Suppose $\sigma = \frac{2}{3}$, π being a structure with J n-member alliances, and π' being the structure that one player spins off to form a singleton alliance from π. We can greatly simplify $u(1, \pi')$ and $u(n, \pi)$ in this case:

$$u(1,\pi') = \frac{1}{1}\left[1 - J\frac{1}{J-1+1+1}\right]\left[1 - J\frac{1}{J-1+1+1}\right] = \frac{1}{(J+1)^2}$$

$$\begin{aligned} u(n,\pi) &= \frac{1}{n^2}\left[1-(J-1)\frac{1}{J}\right]\left[1-(J-1)\frac{\frac{1}{n}}{J}\right] \\ &= \frac{1}{n^2}\frac{1}{(J)^2}\left(J - \frac{J-1}{n}\right) \end{aligned}$$

Note that $u(1, \pi') > u(n, \pi)$ holds for all $n \geq 2$ and all $J \leq 4$; i.e., there exist spin-off incentives, and π cannot be a subgame perfect equilibrium outcome. However, when $J = 5$ and $n = 2$, $u(1, \pi') = \frac{1}{36}$ and $u(2, \pi) = \frac{1}{4}\frac{1}{25}(5 - \frac{4}{2}) = \frac{3}{100} > \frac{1}{36}$, no player has incentives to spin off and form a singleton alliance. Moreover, since the size of an alliance has no effect when $\sigma = \frac{2}{3}$, the payoff of deviating from a two-player alliance and forming a three-player alliance is $\frac{1}{9}\frac{1}{5^2}\left(5 - \frac{4}{3}\right) < \frac{3}{100}$. Therefore, $\{2, 2, 2, 2, 2\}$ is in fact a stable structure. Since payoffs are continuous in σ, this example can be extended to those σs that are close to but smaller than $\frac{2}{3}$. ■

Now, we consider the case of $\sigma > \frac{2}{3}$. In this case, $n_h < n_k$ implies $w_k < w_h$, and a larger coalition is more efficient (the marginal cost is lower in the artificial game). In this case, if existing alliances are large enough, then a singleton alliance's marginal cost is too high to exert any effort, resulting in zero winning probability and zero utility. The following Lemma identifies a *sufficient condition* by using Lemma 1.

Lemma 2. **[No Spin-off Condition]** *Suppose that $J = 2$ and that an alliance structure is $\{n_j, n_k\}$. If the following is satisfied:*

$$V \leq X^*(n_j - 1, n_k) = \frac{(n_j - 1)^{\frac{3\sigma-2}{1-\sigma}} n_k^{\frac{3\sigma-2}{1-\sigma}}}{(n_j - 1)^{\frac{3\sigma-2}{1-\sigma}} + n_k^{\frac{3\sigma-2}{1-\sigma}}} V,$$

there is no unilateral spin-off incentive for a member of alliance j, where $X^*(n_j - 1, n_k)$ is the equilibrium total effort when the alliances of sizes $n_j - 1$ and n_k are the only active ones. For any $\sigma > \frac{2}{3}$, there is an integer $\bar{n}$ such that $V \leq X^*(n_j - 1, n_k)$ is satisfied for any $\min\{n_j - 1, n_k\} \geq \bar{n}$.

When $J \geq 3$, let an alliance structure be $\pi = \{n_1, ..., n_J\}$. When $\sigma > \frac{2}{3}$, there is no unilateral spin-off incentive for a member of alliance j if the following sufficient condition holds:

$$V\left(1 + n_j^{\frac{3\sigma-2}{1-\sigma}} - (n_j - 1)^{\frac{3\sigma-2}{1-\sigma}}\right) \leq X^* = \frac{(j^* - 1)V}{\sum_{j'=1}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}$$

When $J = 2$, the No Spin-off Condition is not satisfied for any n_j as long as $\sigma = \frac{2}{3}$, and it is not satisfied unless n_1 and n_2 are very large for $\sigma > \frac{2}{3}$ but close to $\frac{2}{3}$. As σ goes up, it is more easily satisfied for smaller population sizes (say, if $\sigma = \frac{5}{7}$, then No Spin-off Condition is satisfied for all $n_1 > n_2 \geq 4$), and at $\sigma = \frac{3}{4}$, it is satisfied for all $n_j \geq 2$.[12]

5.3 Two Competing Alliances

We start with the case where the number of (active) alliances is two. We argue that when the value of the complementarity parameter is moderate, there is a unique two-alliance equilibrium in which the maximal difference in sizes is one.

[12]Since it is a sufficient condition, there exist some cases where the No Spin-off Condition in Lemma 2 is violated, but players still do not have incentives to be singleton. For example, when $\sigma = \frac{5}{7}$, all $n_1 = n_2 = n < 4$ violates this condition. However, players still do not have incentives to spin-off from any symmetric alliance size n. Say, for $n = 3$, we have

$$u(\{3, 3\}) = 0.0273 \text{ and } u(\{1, 2, 3\}) = 0.0014.$$

Therefore, players do not spin off. Please see Figures 3 and 4 and the Appendix to see how tight the No Spin-Off Condition is.

Theorem 2. *When $J = 2$, $\frac{2}{3} \leq \sigma \leq \frac{3}{4}$ and the No Spin-off Condition are satisfied, there exists a unique two-alliance equilibrium with its alliance structure π^* that satisfies $|n_1 - n_2| \leq 1$.*

To show this, we need the following two lemmas.

Lemma 3. *When $J = 2$, $\frac{2}{3} \leq \sigma \leq \frac{3}{4}$, players in the smaller alliance do not have an incentive to move to a larger alliance. When alliance sizes are equal, players do not move to create a larger alliance.*

However, players in a larger alliance have incentives to move. Formally:

Lemma 4. *Suppose $J = 2$ and $\frac{2}{3} \leq \sigma \leq \frac{3}{4}$, players in a larger alliance have an incentive to move to the smaller one.*

Proof of Theorem 2. From Lemmas 3 and 4, for any coalition structure with $|n_1 - n_2| > 1$, a player in a larger alliance moves to the smaller one. This cannot be an equilibrium alliance structure. As a result, the only alliance structure that is immune to moving incentives is the one with $|n_1 - n_2| \leq 1$. Since the No Spin-off condition is assumed to be satisfied, there is no profitable unilateral deviation from π^*. $\square$

This theorem confirms the intuition in the examples in Section 5.1. The two-alliance equilibrium exists and is unique when complementarity is moderate. Although it is more difficult to extend Theorem 2 to $\sigma \geq \frac{3}{4}$, we can still make a partial generalization without uniqueness due to the monotonicity of $\frac{P_0(t)}{P_1(t)}$.

Proposition 3. *If the two-alliance structure $\{n, n\}$ or $\{n, n+1\}$ is immune to forming a larger alliance for some $\bar{\sigma} > \frac{3}{4}$, then it is immune for all $\sigma \in [\frac{2}{3}, \bar{\sigma}]$.*

To apply this result: e.g., when $\sigma = 7/9$ and $n \geq 3$ numerical calculations show that both $\{n, n\}$ and $\{n, n+1\}$ are immune to forming a larger alliance and satisfy the No Spin-off Condition in Lemma 2.[13] Proposition 3 implies that they are stable for all smaller $\sigma \geq \frac{2}{3}$ given that the Non-Spin-off Condition is satisfied.

However, the stability of two-alliance structure has an upper bound.[14] Notice that a higher σ means the complementarity in a coalition is relatively

[13]In fact, in this case, even when $n = 2$, both $\{2, 2\}$ and $\{2, 3\}$ are stable.

[14]According to our numerical analysis, the two-alliance structure is stable when $\sigma < \frac{19}{24}$. But, it is not stable when, say, $\sigma = \frac{19.1}{24}$ for $\pi = \{2, 2\}$. In fact, when $\sigma \leq 0.79$ our numerical results show that the two-alliance structure with a maximal population difference 1 is an equilibrium for all n. See Figure 2 and Figure 5.

strong, which gives players more benefits in a larger group. This effect will eventually dominate. The following example illustrates this point.

Example 2. Suppose that $\frac{2-3\sigma}{1-\sigma} = t = -2$ or $\sigma = \frac{4}{5} > \frac{3}{4}$. In this case, we cannot support equal division for every alliance size n when $J^* = 2$. That is,

$$\begin{aligned} u_0 - u_1 &= \frac{n+1}{2\left(n^2+2n+2\right)(n+2)} + \frac{n+2}{4\left(n^2+2n+2\right)^2} - \frac{2n+1}{4\left(n+1\right)^3} \\ &= \frac{1}{4}\frac{n^2}{\left(n+1\right)^3(n+2)\left(n^2+2n+2\right)^2}\left(n^2+3n+3\right) \\ &> 0. \end{aligned}$$

Therefore, as σ increases, the preference for a larger coalition is strengthened, so that players prefers to form a larger alliance.

This example shows that somewhere between $\sigma = \frac{3}{4}$ and $\frac{4}{5}$, the preference for a larger coalition starts to dominate, and the grand alliance will be the only stable alliance structure. The following Figure 2 depicts the parameter space in which $\{n, n\}$ is stable.

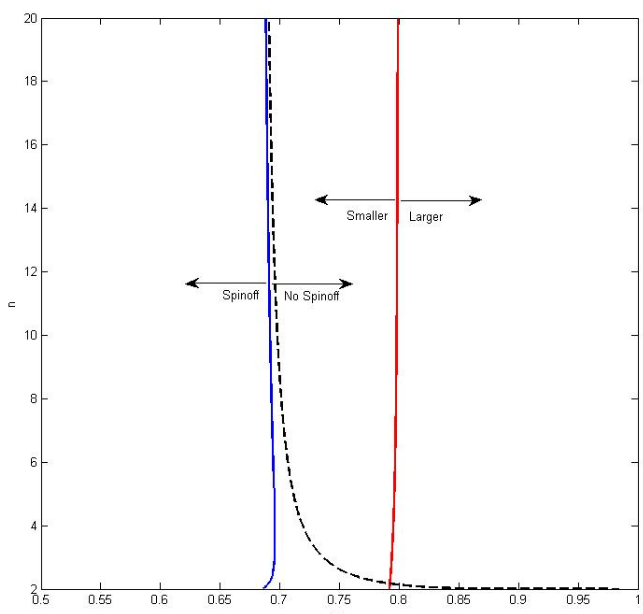

Figure 2: *The parameter space for 2 n-sized alliances to be an equilibrium structure. The red line stands for the maximal σ for preferring a smaller*

alliance. The black-dashed line is the No Spin-Off Condition in Lemma 2. The blue line is the exact No Spin-Off Condition from our numerical analysis.

The following theorem shows an important welfare implication of having a chance to form alliances. The emergence of alliances in subgame perfect equilibrium is not only an equilibrium phenomenon (like prisoners' dilemma games), but also a Pareto-improvement for players' welfare, because it has dynamic contests instead of a single round contest.

Theorem 3. *Every two-alliance equilibrium $\{n_1, n_2\}$ with $|n_1 - n_2| \leq 1$ Pareto-dominates a no-alliance contest outcome.*

5.4 Multi-Alliance Case

Is a symmetric alliance structure stable when $J > 2$? First of all, forming multiple alliances may be welfare-improving. In fact, if the alliances are symmetric, players' welfare improves as the number of alliances increases. Formally,

Proposition 4. *Let symmetric alliance structure π_J be a symmetric alliance structure that has $\frac{N}{J} \geq 2$ players in each alliance. If $\pi_{J'}$* and *$\pi_{J''}$ with $J'' > J'$ are both equilibrium alliance structures, then $\pi_{J''}$ Pareto dominates $\pi_{J'}$.*

However, the remaining question is whether a multi-alliance structure is stable or not. The benefit from a larger alliance is that the new alliance has a higher winning probability in the first stage. However, this effect is offset by a stronger intra group competition in the second stage. But, the first effect seems to be stronger when the number of alliances is large and each alliance only has relatively few members. Thus, we expect that when the number of alliances is more than two, it requires a larger membership in each alliance to be a symmetric equilibrium allocation. This intuition leads us to the following example.

Example 3. Consider the case when $J = 3$, $n = 7$ or 8, and $\sigma = \frac{3}{4}$

$$u(7, \{7, 7, 7\}) = 0.0061548 < u(8, \{8, 6, 7\}) = 0.0061581$$

$$u(8, \{8, 8, 8\}) = 0.0047743 > u(9, \{9, 7, 8\}) = 0.0047736$$

The above example shows that even when the complementarity between players is moderate, a symmetric three-alliance structure is not immune to a unilateral move if $n = 7$. But, a larger membership ($n = 8$) again guarantees the stability. In fact, our numerical results show that when $\sigma = \frac{3}{4}$, a symmetric four-alliance structure is not immune to a unilateral move for all size n. Moreover, suppose we consider a lower $\sigma = \frac{5}{7}$, a symmetric four-alliance structure is stable as long as $n \geq 5$. In the following figure, we demonstrate the parameter space for stable symmetric three-alliance structures. Compared with Figure 2, we can see that both borderlines for no spin-off and forming a larger alliance are shifted to the left, and the parameter space for the stable symmetric three-alliance structure shrinks.[15]

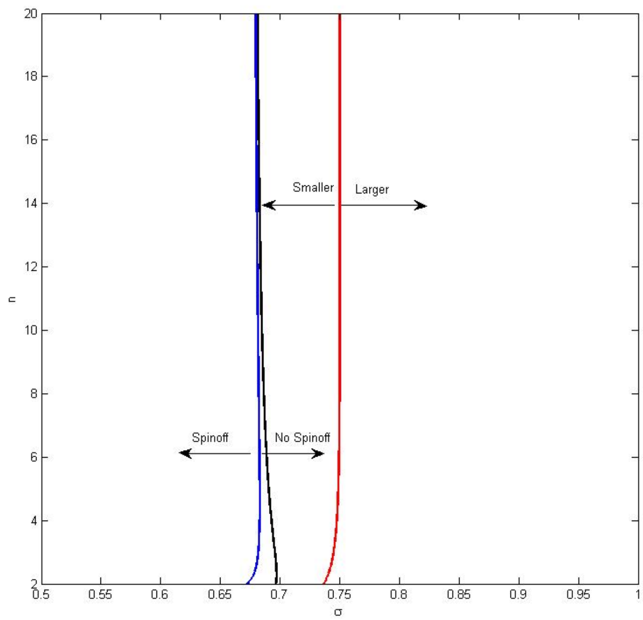

Figure 3: *The parameter space for three size-n alliances to be an equilibrium structure. Note that the viable parameter space shrinks relative to the one in Figure 2.*

In summary, the symmetric alliance structure gives players stronger incentives to join a larger group when there are more alliances. Therefore, a symmetric structure is stable only in a smaller range of σ, which implies less complementarity. As a result, although more alliances potentially improve welfare,[16] it becomes more difficult for players to form a stable multi-alliance

[15] Our numerical analysis shows that this ia a general phenomenon. In the Appendix, the parameter space for a structure of four equal-sized alliances is an equilibrium allocation. It shrinks further compared to the one in Figure 4.

[16] See also Wärneryd (1998).

structure.

6 Concluding Remarks

In this paper, we used a CES effort aggregator function to show that if the effort complementarity within an alliance is strong, players can have incentives to form an alliance. Moreover, we show that there exist stable alliances in an open-membership two-stage alliance formation game when the complementarity of efforts is strong but not too strong. The reason why a nontrivial alliance cannot be formed in these cases is that a large alliance becomes too attractive, and all players end up forming a grand alliance, which simply defers the the noncooperative contest by one period.

There are alternative alliance formation games in the literature (see Hart and Kurz 1983). Using a noncooperative game approach, Bloch (1996), Okada (1996), and Ray and Vohra (1999) consider an interesting sequential coalition formation game. In a companion paper, Konishi and Pan (2019), we study equilibrium alliance structure adopting their game: In this game, the alliance formation stage has multiple steps, and a player proposes an alliance at each step, and if all called upon members agree to form a group, then an alliance is formed, and multiple alliances are formed sequentially. That is, these alliances can exclude outsiders in this alternative setup. Allowing for side payments, Bloch et al. (2006) consider a sequential alliance formation game in contests, which allows alliances to limit their memberships (exclusion), and show that the grand alliance would be formed by sharing the prize peacefully. However, in our game without side payments (an indivisible prize), the grand alliance would not be formed, since this is identical to not forming an alliance. We show that there is always a subgame perfect equilibrium and that there can be at most two alliances in equilibrium, one large and one small without any fringe players (all players belong to one of the two alliances) if the complementarity parameter σ is large enough. In this case, the large alliance is formed first, and the leftover players form the second smaller alliance, and the former alliance achieves higher payoffs than the latter.[17]

[17] Note that this cannot be an equilibrium in the open-membership game used in the present paper, since players in the latter alliance want to move to the former since the memberships of alliances are not exclusive. In open-membership games, the sizes of alliances need to be more or less the same.

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Appendix A (Proofs)

Proof of Theorem 1. The artificial game we constructed has the same first-order conditions as the original first-stage game. This implies that j^* is uniquely defined, as in the statement of Lemma 1, only $j = 1, ..., j^*$ exert efforts in equilibrium. Since $p_j^* = 1 - \frac{\sum_{j' \neq j} x_{j'}}{\sum_{j'=1}^{j^*} x_{j'}}$, the first-order conditions can be written as

$$\frac{\left(1 - p_j^*\right)}{\left(\sum_{j'=1}^{j^*} x_{j'}\right)} V - n_j^{\frac{2-3\sigma}{1-\sigma}} = 0$$

or

$$1 - p_j^* = \frac{\sum_{j'=1}^{j^*} x_{j'}}{V} n_j^{\frac{2-3\sigma}{1-\sigma}}.$$

Summing up the above from $j = 1$ to j^*, we have

$$j^* - 1 = \frac{\sum_{j'=1}^{j^*} x_{j'}}{V} \sum_{j'=1}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}$$

Eliminating $\frac{\sum_{j'=1}^{j^*} x_{j'}}{V}$ from the first-order condition, we obtain:

$$p_j^* = 1 - (j^* - 1) \frac{n_j^{\frac{2-3\sigma}{1-\sigma}}}{\sum_{j'=1}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}$$

Since $p_j \left(\sum_{j'} x_{j'}\right) = x_j$, we have

$$x_j = \left[1 - (j^* - 1) \frac{n_j^{\frac{2-3\sigma}{1-\sigma}}}{\sum_{j'=1}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right] \frac{(j^* - 1) V}{\sum_{j'=1}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}$$

Notice that $x_j = n_j^{\frac{1}{1-\sigma}} e_j$, which means the equilibrium e_j in the original problem is

$$e_j = \frac{1}{n_j^{\frac{1}{1-\sigma}}} \left[1 - (j^* - 1) \frac{n_j^{\frac{2-3\sigma}{1-\sigma}}}{\sum_{j'=1}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right] \frac{(j^* - 1) V}{\sum_{j'=1}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}$$

Therefore, the equilibrium payoff of the original problem is

$$\begin{aligned}
u_j &= p_j^* V_j - e_j \\
&= \left[1-(j^*-1)\frac{n_j^{\frac{2-3\sigma}{1-\sigma}}}{\sum_{j'=1}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right]\frac{V}{n_j^2} - \left[\frac{1}{n_j^{\frac{1}{1-\sigma}}}\left[1-(j^*-1)\frac{n_j^{\frac{2-3\sigma}{1-\sigma}}}{\sum_{j'=1}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right]\frac{(j^*-1)\,V}{\sum_{j'=1}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right] \\
&= \left[1-(j^*-1)\frac{n_j^{\frac{2-3\sigma}{1-\sigma}}}{\sum_{j'=1}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right]\left[\frac{1}{n_j^2} - \frac{1}{n_j^{\frac{1}{1-\sigma}}}\frac{(j^*-1)}{\sum_{j'=1}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right]V \\
&= \frac{1}{n_j^2}\left[1-(j^*-1)\frac{n_j^{\frac{2-3\sigma}{1-\sigma}}}{\sum_{j'=1}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right]\left[1-(j^*-1)\frac{n_j^{\frac{1-2\sigma}{1-\sigma}}}{\sum_{j'=1}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right]V
\end{aligned}$$

We completed the proof.■

Proof of Proposition 2. From Theorem 1, we know that the payoff of a player who is one of n_j is

$$u(n_j,\pi) = \frac{1}{n_j^2}\left[1-(J-1)\frac{n_j^{\frac{2-3\sigma}{1-\sigma}}}{\sum_{j'=1}^{J} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right]\left[1-(J-1)\frac{n_j^{\frac{1-2\sigma}{1-\sigma}}}{\sum_{j'=1}^{J} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right]$$

Let π'_{n_j} stand for the structure after one player in alliance j spins off to form a singleton alliance. This player has a payoff equal to

$$\begin{aligned}
u(1,\pi'_{n_j}) = \frac{1}{1}&\left[1-J\frac{1}{\sum_{j'=1,j'\neq j}^{J} n_{j'}^{\frac{2-3\sigma}{1-\sigma}} + (n_j-1)^{\frac{2-3\sigma}{1-\sigma}}+1}\right] \\
\times&\left[1-J\frac{1}{\sum_{j'=1,j'\neq j}^{J} n_{j'}^{\frac{2-3\sigma}{1-\sigma}} + (n_j-1)^{\frac{2-3\sigma}{1-\sigma}}+1}\right]
\end{aligned}$$

Since $\frac{1-2\sigma}{1-\sigma}\geq 0$, $n_j^{\frac{2-3\sigma}{1-\sigma}} \geq (n_j-1)^{\frac{2-3\sigma}{1-\sigma}} \geq 1^{\frac{2-3\sigma}{1-\sigma}} = 1$ and $n_j^{\frac{1-2\sigma}{1-\sigma}} \geq 1^{\frac{1-2\sigma}{1-\sigma}} = 1$ hold for all $n_j \geq 2$. Since $n_j^{\frac{2-3\sigma}{1-\sigma}}$ is convex function for $\sigma \in [0,\frac{1}{2}]$ ($\frac{2-3\sigma}{1-\sigma}\in[1,2]$), we have

$$\sum_{j'=1,j'\neq j}^{J} n_{j'}^{\frac{2-3\sigma}{1-\sigma}} + (n_j-1)^{\frac{2-3\sigma}{1-\sigma}} + 1 \leq \sum_{j'=1}^{J} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}$$

This implies

$$\frac{\sum_{j'=1,j'\neq j}^{J} n_{j'}^{\frac{2-3\sigma}{1-\sigma}} + (n_j-1)^{\frac{2-3\sigma}{1-\sigma}} + 1}{J} < \frac{\sum_{j'=1}^{J} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}{J-1}.$$

Thus, we have

$$u(1,\pi'_{n_j}) > \frac{1}{1}\left[1-(J-1)\frac{1}{\sum_{j'=1}^{J} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right]\left[1-(J-1)\frac{1}{\sum_{j'=1}^{J} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right]$$

We want to show the RHS that the above inequality is not exceeded by u_j for any $\sigma \in [0,\frac{1}{2}]$ Note that $\frac{2-3\sigma}{1-\sigma} \geq 1$ and $\frac{1-2\sigma}{1-\sigma} \geq 0$ for any $\sigma \in [0,\frac{1}{2}]$. Thus, we have

$$\begin{aligned} u(n_j,\pi) &= \frac{1}{n_j^2}\left[1-(J-1)\frac{n_j^{\frac{2-3\sigma}{1-\sigma}}}{\sum_{j'=1}^{J} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right]\left[1-(J-1)\frac{n_j^{\frac{1-2\sigma}{1-\sigma}}}{\sum_{j'=1}^{J} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right] \\ &< \frac{1}{n_j^2}\left[1-(J-1)\frac{1}{\sum_{j'=1}^{J} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right]\left[1-(J-1)\frac{1}{\sum_{j'=1}^{J} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right] \end{aligned}$$

Therefore, we conclude that for any $\sigma \in [0,\frac{1}{2}]$, a player has an incentive to spin off from any alliance with $n_j \geq 2$.■

Proof of Lemma 2. For the first part, the proof follows by Figure 4.

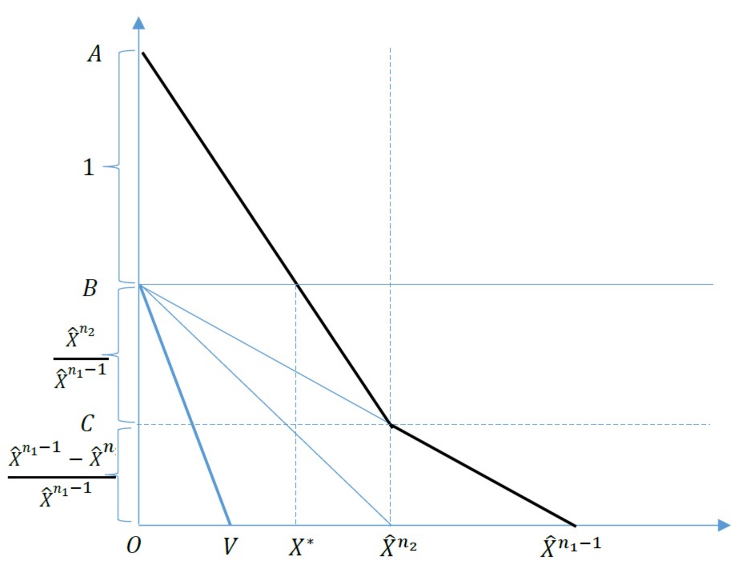

Figure 4: *No Spin-Off Condition in Lemma 2.*

First, recall that $\hat{X}^{n_j} = \frac{V}{w_j} = n_j^{\frac{3\sigma-2}{1-\sigma}} V$. Without loss of generality, we will consider spin-off incentives from alliance 1. Suppose $n_1 - 1 \geq n_2$. Then, we have $\hat{X}^{n_1-1} \geq \hat{X}^{n_2}$. Alliance 1's share function satisfies $s_1(\hat{X}^{n_2}) = \frac{\hat{X}^{n_1-1}-\hat{X}^{n_2}}{\hat{X}^{n_1-1}} < 1$. Since $\overline{AB} : \overline{BC} = 1 : \frac{\hat{X}^{n_2}}{\hat{X}^{n_1-1}}$, we have $\overline{OX^*} : \overline{X^*\hat{X}^{n_2}} = 1 : \frac{\hat{X}^{n_2}}{\hat{X}^{n_1-1}}$ as well. Since $\overline{O\hat{X}^{n_2}} = \hat{X}^{n_2}$, we have

$$X^* = \frac{1}{1+\frac{\hat{X}^{n_2}}{\hat{X}^{n_1-1}}}\hat{X}^{n_2} = \frac{\hat{X}^{n_1-1}\hat{X}^{n_2}}{\hat{X}^{n_1-1}+\hat{X}^{n_2}} = \frac{(n_1-1)^{\frac{3\sigma-2}{1-\sigma}} V n_2^{\frac{3\sigma-2}{1-\sigma}} V}{(n_1-1)^{\frac{3\sigma-2}{1-\sigma}} V + n_2^{\frac{3\sigma-2}{1-\sigma}} V} = \frac{(n_1-1)^{\frac{3\sigma-2}{1-\sigma}} n_2^{\frac{3\sigma-2}{1-\sigma}}}{(n_1-1)^{\frac{3\sigma-2}{1-\sigma}} + n_2^{\frac{3\sigma-2}{1-\sigma}}} V.$$

From Figure 4, if $\hat{X}^1 = 1^{\frac{3\sigma-2}{1-\sigma}} V = V \leq X^*$, then the effort is zero in the inter-alliance contest, and the alliance's winning probability and payoff are both zero. As we have shown in Theorem 1, the equilibrium payoff for belonging to an alliance with a positive probability of winning is positive. Thus, under this condition, there is no unilateral spin-off.

We now move on to the latter part. If alliance j is divided into two alliances with sizes $n_j - 1$ and 1 when the latter spins off, then the share functions' intercepts on the X-axis are $\hat{X}^{n_j-1} = (n_j-1)^{\frac{3\sigma-2}{1-\sigma}} V$ and $\hat{X}^1 = V$, respectively. Note that if the equilibrium total effort is X^* under π, the new equilibrium total effort after the spin-off, X^{**}, satisfies $X^{**} \geq X^* - \left(\hat{X}^{n_j} - \hat{X}^{n_j-1}\right)$. Therefore, if $\hat{X}^1 = V \leq X^* - \left(\hat{X}^{n_j} - \hat{X}^{n_j-1}\right) = X^* - \left(n_j^{\frac{3\sigma-2}{1-\sigma}} V - (n_j-1)^{\frac{3\sigma-2}{1-\sigma}} V\right)$, then $\hat{X}^1 \leq X^{**}$ and the singleton alliances do not exert effort and the winning probability is zero. ■

Proof of Lemma 3. Consider two alliance structures: $\pi_0 = \{n+d+1, n+1\}$ and $\pi_1 = \{n+d+2, n\}$ with $d \geq 0$. Let u_0 and u_1 stand for the utilities in the $(n+1)$-sized alliance at π_0 and the $(n+d+2)$-sized alliance at π_1, respectively. We will show that $u_0 > u_1$ when $\frac{2}{3} \leq \sigma \leq \frac{3}{4}$ for all d. First, we introduce a notation $t = \frac{2-3\sigma}{1-\sigma}$ to simplify formulae. Note that when $\frac{2}{3} \leq \sigma \leq \frac{3}{4}$,

$-1 \leq t \leq 0$ holds. Then,

$$\begin{aligned}
u_0 &= \frac{1}{(n+1)^2}\left[1-\frac{(n+1)^t}{(n+d+1)^t+(n+1)^t}\right]\left[1-\frac{(n+1)^{t-1}}{(n+d+1)^t+(n+1)^t}\right] \\
&= \frac{1}{(n+1)^3}\left[\frac{(n+d+1)^t}{(n+d+1)^t+(n+1)^t}\right]\left[(n+1)-\frac{(n+1)^t}{(n+d+1)^t+(n+1)^t}\right] \\
&= \frac{1}{(n+1)^3}\left[\frac{(n+d+1)^t}{(n+d+1)^t+(n+1)^t}\right]\left[n+\frac{(n+d+1)^t}{(n+d+1)^t+(n+1)^t}\right] \\
&> \frac{1}{(n+1)^3}\left[\frac{(n+d+1)^t}{(n+d+1)^t+(n+1)^t}\right]\left[n+\frac{(n+d+1)^t}{2(n+1)^t}\right] \\
&= \frac{1}{(n+1)^3}P_0(t)\left[(n+d)+\frac{(n+d+1)^t}{2(n+1)^t}\right] \qquad (4)
\end{aligned}$$

and similarly

$$\begin{aligned}
u_1 &= \frac{1}{(n+d+2)^2}\left[1-\frac{(n+d+2)^t}{(n+d+2)^t+n^b}\right]\left[1-\frac{(n+d+2)^{t-1}}{(n+d+2)^t+n^t}\right] \\
&= \frac{1}{(n+d+2)^3}\left[\frac{n^t}{(n+d+2)^t+n^t}\right]\left[(n+d+2)-\frac{(n+d+2)^t}{(n+d+2)^t+n^t}\right] \\
&< \frac{1}{(n+d+2)^3}\left[\frac{n^t}{(n+d+2)^t+n^t}\right](n+d+2) \\
&= \frac{1}{(n+d+2)^2}P_1(t) \qquad (5)
\end{aligned}$$

Note that

$$\frac{P_0(t)}{P_1(t)} = \frac{\frac{(n+d+1)^t}{(n+d+1)^t+(n+1)^t}}{\frac{n^t}{(n+d+2)^t+n^t}}$$

is monotonically decreasing in $|t|$ in $0 > t > -1$. Also, $\left[(n+d)+\frac{(n+d+1)^t}{2(n+1)^t}\right]$ is monotonically decreasing in $|t|$ since $t < 0$ and $d > 0$. Thus, when $t = -1$, if $\frac{1}{(n+1)^3}P_0(-1)\left[(n+d)+\frac{(n+d+1)^{-1}}{2(n+1)^{-1}}\right] \geq \frac{1}{(n+d+2)^2}P_1(-1)$ holds, then $\frac{1}{(n+1)^3}P_0(t)[(n+d)+\frac{(n+d+1)^t}{2(n+1)^t}] \geq \frac{1}{(n+d+2)^2}P_1(t)$ for all $0 > t > -1$. Thus, $u_0(t) > u_1(t)$ for all $0 > t > -1$. Since $P_1(-1) = \frac{n+d+2}{2n+d+2}$ and $P_0(-1) = \frac{n+1}{2n+d+2}$, we have

$$
\begin{aligned}
\frac{u_0(-1)}{u_1(-1)} &\geq \frac{\frac{1}{(n+1)^3}P_0(t)}{\frac{1}{(n+d+2)^2}P_1(t)}\left[\frac{2n(n+d+1)+(n+1)}{2(n+d+1)}\right] \\
&= \frac{(n+d+2)^2}{(n+1)^3}\left[\frac{n+1}{n+d+2}\right]\left[\frac{2n(n+d+1)+(n+1)}{2(n+d+1)}\right] \\
&= \frac{n+(d+2)}{(n+1)^2}\left[\frac{2n^2+(2d+3)n+1}{2n+2(d+1)}\right] \\
&= \frac{2n^3+(4d+7)n^2+(2d^2+7d+7)n+d+2}{2n^3+(2d+6)n^2+(4d+6)n+(2d+2)}
\end{aligned}
$$

It is clear that $\frac{u_1(-1)}{u_0(-1)} \geq 1$ for all $d \geq 0$ and $n \geq 1$. Therefore, there are no incentives for players to unilaterally move from a smaller alliance to the larger one. ■

Proof of Lemma 4. All we need to show is that when a player i is in an alliance with size $n+\tilde{d}+1$ while the other coalition has size equal to $n+1$, he has incentives to move, unless $\tilde{d}=1$. From Lemma 1 we know that for any player i in the alliance with size $n+d+1$, $u_1 = u(n+1, \{n+d+1, n+1\}) > u(n+d+2, \{n+d+2, n\})$ which immediately implies that a player in a larger alliance has an incentive to move to a smaller one. □

Proof of Theorem 3. There are two cases: Case 1 with two equally sized alliances $\{n, n\}$, and Case 2 with two alliances whose sizes differ by one $\{n, n+1\}$. We start with Case 1. The payoff from $\{n, n\}$ is $\frac{V}{n^2}\frac{1}{2}\left(1-\frac{1}{2n}\right) = \frac{2n-1}{4n}V$, and the one from $\{2n\}$ is $\frac{V}{(2n)^2} = \frac{V}{4n^2}$. Thus, the two-alliance equilibrium dominates no alliance case.

Case 2: Consider allocation $\pi = \{n+1, n\}$. First, the payoff from belonging to size n alliance is

$$
\begin{aligned}
u(n;\pi) &= \frac{V}{n^2}\left[1-\frac{n^{\frac{2-3\sigma}{1-\sigma}}}{n^{\frac{2-3\sigma}{1-\sigma}}+(n+1)^{\frac{2-3\sigma}{1-\sigma}}}\right]\left[1-\frac{n^{\frac{1-2\sigma}{1-\sigma}}}{n^{\frac{2-3\sigma}{1-\sigma}}+(n+1)^{\frac{2-3\sigma}{1-\sigma}}}\right] \\
&= \frac{V}{n^2}\left[1-\frac{n^{\frac{2-3\sigma}{1-\sigma}}}{n^{\frac{2-3\sigma}{1-\sigma}}+(n+1)^{\frac{2-3\sigma}{1-\sigma}}}\right]\left[\frac{n-1}{n}+\frac{1}{n}-\frac{n^{\frac{1-2\sigma}{1-\sigma}}}{n^{\frac{2-3\sigma}{1-\sigma}}+(n+1)^{\frac{2-3\sigma}{1-\sigma}}}\right] \\
&= \frac{V}{n^2}\left[1-\frac{n^{\frac{2-3\sigma}{1-\sigma}}}{n^{\frac{2-3\sigma}{1-\sigma}}+(n+1)^{\frac{2-3\sigma}{1-\sigma}}}\right]\left[\frac{n-1}{n}+\frac{1}{n}\left(1-\frac{n^{\frac{2-3\sigma}{1-\sigma}}}{n^{\frac{2-3\sigma}{1-\sigma}}+(n+1)^{\frac{2-3\sigma}{1-\sigma}}}\right)\right] \\
&= \frac{V}{n^2}\left[\frac{(n+1)^{\frac{2-3\sigma}{1-\sigma}}}{n^{\frac{2-3\sigma}{1-\sigma}}+(n+1)^{\frac{2-3\sigma}{1-\sigma}}}\right]\left[\frac{n-1}{n}+\frac{1}{n}\left(\frac{(n+1)^{\frac{2-3\sigma}{1-\sigma}}}{n^{\frac{2-3\sigma}{1-\sigma}}+(n+1)^{\frac{2-3\sigma}{1-\sigma}}}\right)\right]
\end{aligned}
$$

Since $u(n;\pi)$ is decreasing in σ for $\sigma \geq \frac{2}{3}$, and since ther is no two-alliance equilibrium for $\sigma > \frac{4}{5}$ (see Example 2), it suffices to show that $u(n;\pi)$ exceeds $\frac{V}{(2n+1)^2}$ when $\sigma = \frac{4}{5}$. Substituting $\sigma = \frac{4}{5}$ into $u(n;\pi)$, we obtain

$$
\begin{aligned}
u(n;\pi) &= \frac{V}{n^2}\left[\frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}+\frac{1}{(n+1)^2}}\right]\left[\frac{n-1}{n}+\frac{1}{n}\left(\frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}+\frac{1}{(n+1)^2}}\right)\right] \\
&= \frac{V}{n^2}\frac{n^2}{2n^2+2n+1}\left[\frac{n-1}{n}+\frac{1}{n}\left(\frac{n^2}{2n^2+2n+1}\right)\right] \\
&= \frac{V}{2n^2+2n+1}\times\frac{(2n^3+n^2-n-1)}{n\left(2n^2+2n+1\right)} \\
&= V\frac{(2n^3+n^2-n-1)}{n\left(2n^2+2n+1\right)^2}.
\end{aligned}
$$

Subtracting $\frac{V}{(2n+1)^2}$ from the above, we obtain,

$$
V\frac{(2n^3+n^2-n-1)}{n\left(2n^2+2n+1\right)^2}-\frac{V}{(2n+1)^2}=V\frac{4n^5+4n^4-6n^3-11n^2-6n-1}{n\left(4n^3+6n^2+4n+1\right)^2}.
$$

Let $f_{(n;\pi)}(n) \equiv 4n^5+4n^4-6n^3-11n^2-6n-1$. Since $f_{(n;\pi)}(2) > 0$ and $f'_{(n;\pi)}(n) > 0$ for $n \geq 2$, we conclude $u(n;\pi) > \frac{V}{(2n+1)^2}$.

Second, we check $u(n+1;\pi)$. We have

$$
\begin{aligned}
u(n+1;\pi) &= \frac{V}{(n+1)^2}\left[1-\frac{(n+1)^{\frac{2-3\sigma}{1-\sigma}}}{n^{\frac{2-3\sigma}{1-\sigma}}+(n+1)^{\frac{2-3\sigma}{1-\sigma}}}\right]\left[1-\frac{(n+1)^{\frac{1-2\sigma}{1-\sigma}}}{n^{\frac{2-3\sigma}{1-\sigma}}+(n+1)^{\frac{2-3\sigma}{1-\sigma}}}\right] \\
&\quad \frac{V}{(n+1)^2}\left[\frac{n^{\frac{2-3\sigma}{1-\sigma}}}{n^{\frac{2-3\sigma}{1-\sigma}}+(n+1)^{\frac{2-3\sigma}{1-\sigma}}}\right]\left[\frac{n-1}{n}+\frac{1}{n}\left(\frac{n^{\frac{2-3\sigma}{1-\sigma}}}{n^{\frac{2-3\sigma}{1-\sigma}}+(n+1)^{\frac{2-3\sigma}{1-\sigma}}}\right)\right]
\end{aligned}
$$

Since $u(n+1;\pi)$ is increasing in σ for $\sigma \geq \frac{2}{3}$, we check whether or not $u(n+1;\pi) > \frac{V}{(2n+1)^2}$ for the smallest relevant sigma, $\sigma = \frac{2}{3}$.

Substituting $\sigma = \frac{2}{3}$ into $u(n+1;\pi)$, we obtain:

$$
\begin{aligned}
u(n+1;\pi) &= \frac{V}{(n+1)^2}\frac{1}{2}\left[\frac{n-1}{n}+\frac{1}{2n}\right] \\
&= \frac{V}{2\,(n+1)^2}\left[\frac{2n-1}{2n}\right]
\end{aligned}
$$

Subtracting $\frac{V}{(2n+1)^2}$ from the above, we obtain

$$
\begin{aligned}
& \frac{V}{2\,(n+1)^2}\left[\frac{2n-1}{2n}\right]-\frac{V}{(2n+1)^2} \\
= \; & V \times \frac{(2n-1)\,(2n+1)^2-4\,(n+1)^2\,n}{4\,(n+1)^2\,n\,(2n+1)^2}
\end{aligned}
$$

Denoting the numerator by $f_{(n+1;\pi)}(n)$, we have

$$
\begin{aligned}
f(n) &= \left(4n^2-1\right)(2n+1)-\left(4n^3+8n^2+4n\right) \\
&= 8n^3+4n^2-2n-1-4n^3-8n^2-4n \\
&= 4n^3-4n^2-6n-1
\end{aligned}
$$

Since $f_{(n+1;\pi)}(2) = 3 > 0$ and $f'_{(n+1;\pi)}(n) >$ for $n \geq 2$, we conclude that $u(n+1;\pi) > \frac{V}{(2n+1)^2}$ for all $n \geq 2$. We completed the proof.$\blacksquare$

Proof of Proposition 4. This can be shown by the utility in symmetric alliance structure

$$
u(\pi_J) = \frac{V}{\left(\frac{N}{J}\right)^2}\frac{1}{J}\left(1-\frac{J-1}{N}\right) = \frac{V}{N^3}J\,(N-J+1)
$$

$$
\frac{\partial u(\pi_J)}{\partial J} = \frac{V}{N^3}\,(N-2J+1)
$$

Therefore, $\frac{\partial u(\pi_J)}{\partial J} > 0$ holds for all $J \leq \frac{N+1}{2}$. Also, notice that a group of N players can only sustain $\frac{N}{2}$ alliance. Therefore, a symmetric structure with more alliances Pareto-dominates one with less. $\square$

Appendix B

The Exact No Spin-off Condition in a Two-Alliance Equilibrium

The No Spin-off Condition in the main text is a sufficient condition where no member of an alliance spins off by forming a singleton alliance. For the sake of easier calculations, the condition assures that if a player spins off, then she will not exert effort, and her winning probability is exactly zero. Clearly, this is sufficient to assure that no spin-off occurs, but it is not a necessary condition. Even if a player still exerts effort after spinning off, as long as her winning probability is very small, she will not deviate. In this Appendix, we consider the exact version of No Spin-off Condition, and check how it differs from our main text No Spin-off Condition by using numerical methods in a two-alliance equilibrium structure with $|n_1 - n_2| \leq 1$. First, we need to identify the exact conditions for different cases:

Case 1: Consider $\pi = \{n, n\}$ and $\pi' = \{1, n, n-1\}$,

$$u(n, \pi, t) = \frac{V}{n^2}\frac{1}{2}\left[1 - \frac{1}{2n}\right]$$

and

$$u(1, \pi', t) = V\left[1 - \frac{2}{1 + n^t + (n-1)^t}\right]^2.$$

The exact No Spin-Off consider is defined by $\frac{u(n,\pi,t)}{u(1,\pi',t)} > 1$. This is the condition shown in the Figure 2. The No Spin-Off Condition in Lemma 2 performs relatively well when n is large and σ is not too close to $\sigma = \frac{2}{3}$.

Case 2: Consider $\pi = \{n, n+1\}$ and $\pi' = \{1, n-1, n+1\}$. This is the case of an asymmetric equilibrium. Then,

$$\begin{aligned}
u(n, \pi, t) &= \frac{V}{n^2}\left[\frac{(n+1)^t}{n^t + (n+1)^t}\right]\left[1 - \frac{n^{t-1}}{n^t + (n+1)^t}\right] \\
&= \frac{V}{n^3}\left[\frac{(n+1)^t}{n^t + (n+1)^t}\right]\left[n - 1 + 1 - \frac{n^t}{n^t + (n+1)^t}\right] \\
&= \frac{V}{n^3}\left[\frac{(n+1)^t}{n^t + (n+1)^t}\right]\left[n - 1 + \frac{(n+1)^t}{n^t + (n+1)^t}\right]
\end{aligned}$$

$$u(1, \pi', t) = V\left[1 - \frac{2}{(n-1)^t + 1 + (n+1)^t}\right]^2$$

If $\frac{u(n,\pi,t)}{u(1,\pi',t)} \geq 1$, players in $n-$sized alliance has no incentives to spin off.

Next, let $\pi'' = \{1, n, n\}$

$$u(n+1, \pi, t) = \frac{V}{(n+1)^3}\left[\frac{n^t}{n^t + (n+1)^t}\right]\left[n + \frac{n^t}{n^t + (n+1)^t}\right]$$

$$u(1, \pi'', t) = V\left[1 - \frac{2}{2n^t + 1}\right]^2$$

If If $\frac{u(n+1,\pi,t)}{u(1,\pi'',t)} \geq 1$, players in $(n+1)-$sized alliance has no incentives to spin off.

Stability for Asymmetric Case and $J = 4$

We now consider the stability for $\{n, n+1\}$. The parameter space is depicted in Figure 5. The green line stands for the indifference between $\{n, n+1\}$ and $\{1, n, n\}$. This incentive is weaker in the sense that players in the size n alliance has a spin-off incentive in a wider parameter space. Comparing to the one in Figure 2, the difference is that, e.g., $\{2, 2\}$ is stable when $0.793 > \sigma > 0.686$, but $\{2, 3\}$ is stable when $0.787 > \sigma > 0.6825$. The reason for this difference is that the no spin-off incentive is very sensitive to asymmetry when n is small (see Case 1 and 2 above). Since $\{2, 2\}$ is very different from $\{2, 3\}$. The latter has 50% more population, after all. Except for this difference, the parameter space for stability is similar to Figure 2.

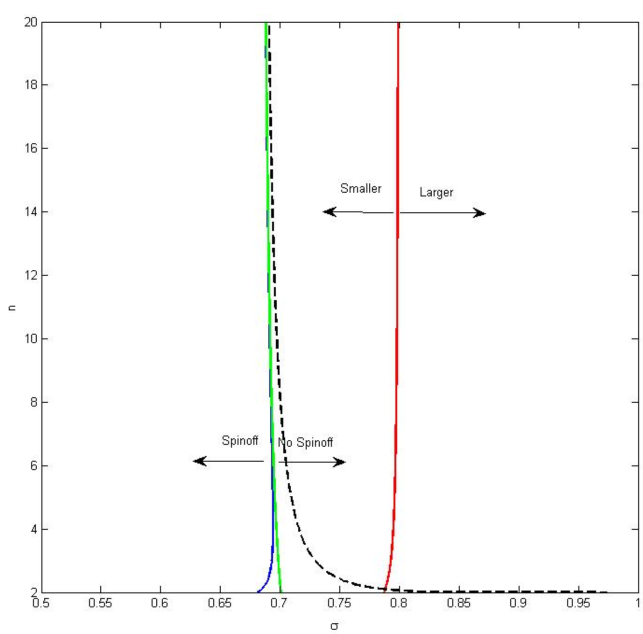

Figure 5: *The parameter space for $\{n, n+1\}$ being an equilibrium structure.*

Next, we present the parameter space for stability when $J = 4$. It is clear that the parameter space shrinks even more compared with Figure 3.

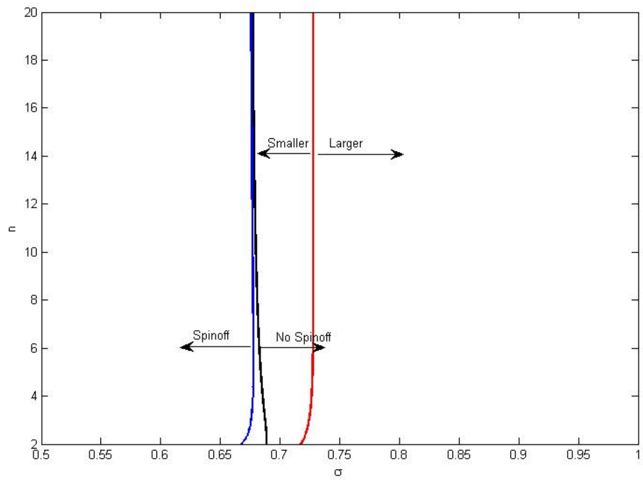

Figure 6: *The parameter space for* 4 *n-sized alliances to be an equilibrium structure.*

Cooperative Alliances

In this section, we briefly analyze the situation where alliance members choose their effort levels cooperatively. In alliance j, the first-stage maximization problem of alliance j is to maximize the total payoff

$$\begin{aligned}
\sum_{ij\in N_j} V_{ij} &= \sum_{ij\in N_j} \frac{\left(\sum_{ij\in N_j} e_{ij}^{1-\sigma}\right)^{\frac{1}{1-\sigma}}}{\left(\sum_{ij\in N_j} e_{ij}^{1-\sigma}\right)^{\frac{1}{1-\sigma}} + \sum_{j'\neq j} E_{j'}} \tilde{V}_j - \sum_{ij\in N_j} e_{ij} \\
&= n_j \frac{\left(n_j e_j^{1-\sigma}\right)^{\frac{1}{1-\sigma}}}{\left(n_j e_j^{1-\sigma}\right)^{\frac{1}{1-\sigma}} + \sum_{j'\neq j} E_{j'}} \frac{V}{n_j^2} - n_j e_j \\
&= n_j \left[\frac{n_j^{\frac{1}{1-\sigma}} e_j}{n_j^{\frac{1}{1-\sigma}} e_j + \sum_{j'\neq j} E_{j'}} \frac{V}{n_j^2} - e_j \right]
\end{aligned}$$

The first-order condition with respect to e_j (if it is an interior solution) is

$$\frac{\left(\sum_{j'} E_{j'} - E_j\right)}{\left(\sum_{j'} E_{j'}\right)^2} n_j^{\frac{1}{1-\sigma}} \frac{V}{n_j^2} - 1 = 0$$

or

$$\frac{\left(\sum_{j' \neq j} n_{j'}^{\frac{1}{1-\sigma}} e_{j'}\right)}{\left(\sum_{j'} n_{j'}^{\frac{1}{1-\sigma}} e_{j'}\right)^2} V - n_j^{\frac{1-2\sigma}{1-\sigma}} = 0,$$

for all $j = 1, ..., J$. Letting $w_j^c = n_j^{\frac{1-2\sigma}{1-\sigma}}$ and $x_j = n_j^{\frac{1}{1-\sigma}} e_j$ (effort) for each $j = 1, ..., J$, *an artificial Tullock contest game corresponding to this cooperative alliance game* is a J-person game in which each player j exerts effort x_j with constant marginal cost $w_j^c = n_j^{\frac{1-2\sigma}{1-\sigma}} > 0$. By solving this, we can show that there exists a unique equilibrium in the first-stage game for any partition of players $\pi = \{n_1, ..., n_j\}$ characterized by the share function $s(X^*) = 1$. There is $j^* \in \{1, ..., J\}$ such that $s_j(X^*) > 0$ for all $j \leq j^*$, while $s_j(X^*) = 0$ for all $j > j^*$ ($\hat{X}_j \leq X^*$). Candidates in alliance $j \leq j^*$ obtain payoff

$$u_j = \begin{cases} \frac{1}{n_j^2}\left[1 - (j^* - 1)\frac{n_j^{\frac{1-2\sigma}{1-\sigma}}}{\sum_{j'=1}^{j^*} n_{j'}^{\frac{1-2\sigma}{1-\sigma}}}\right]^2 & \text{if } j \leq j^* \\ 0 & \text{if } j > j^* \end{cases}$$

As in the main text example, we consider a four-person game $N = \{1, 2, 3, 4\}$, and three values of σ in order: $\sigma = \frac{1}{2}$ (weak complementarity), $\frac{3}{4}$ (moderate complementarity), and $\frac{4}{5}$ (strong complementarity). We check how alliance structure is affected by the complementarity of team effort.

Weak Complementarity $\sigma = \frac{1}{2}$

In this case, we have $\frac{1-2\sigma}{1-\sigma} = 0$. Thus, we know the following:

	$u(1, \pi^0)$	$u(2, \pi^1)$	$u(1, \pi^1)$	$u(2, \pi^2)$	$u(3, \pi^3)$	$u(1, \pi^3)$	$u(4, \pi^N)$
payoff	$\frac{1}{16}$	$\frac{1}{36}$	$\frac{1}{9}$	$\frac{1}{16}$	$\frac{1}{36}$	$\frac{1}{4}$	$\frac{1}{16}$

Note that under π^1 and π^3, smaller alliances perform better than larger ones.

We analyze which partition can be a Nash equilibrium of stage 1:

1. $\pi^0 = \{1,1,1,1\}$: This is a Nash equilibrium.

2. $\pi^1 = \{2,1,1\}$: There is a unilateral spin-off from the size 2 alliance, resulting in π^0.

3. $\pi^2 = \{2,2\}$: This is a Nash equilibrium.

4. $\pi^3 = \{3,1\}$: There is a unilateral move from the size 3 alliance to a singleton, resulting in π^1, or there is a unilateral move from the size 3 coalition to join in a singleton, resulting in π^2.

5. $\pi^N = \{4\}$: There is a unilateral spin-off from the grand alliance, resulting in π^3.

Thus, when $\sigma = \frac{1}{2}$, π^2 becomes a Nash equilibrium alliance structure in addition to π^0 in the noncooperative case.

Medium Complementarity $\sigma = \frac{3}{4}$

In this case, we have $\frac{1-2\sigma}{1-\sigma} = -2$. Thus, we know the following:

	$u(1,\pi^0)$	$u(2,\pi^1)$	$u(1,\pi^1)$	$u(2,\pi^2)$	$u(3,\pi^3)$	$u(1,\pi^3)$	$u(4,\pi^N)$
payoff	$\frac{1}{16}$	$\frac{49}{324}$	$\frac{1}{81}$	$\frac{1}{16}$	$\frac{9}{100}$	$\frac{1}{100}$	$\frac{1}{16}$

Note that under π^1 and π^3, larger alliances perform better than smaller ones.

We analyze which partition can be a Nash equilibrium of stage 1:

1. $\pi^0 = \{1,1,1,1\}$: Two singletons merge to form an alliance, resulting in π^1.

2. $\pi^1 = \{2,1,1\}$: Two singletons merge to form an alliance, resulting in π^2, or one singleton merges into a size 2 alliance, resulting in π^3.

3. $\pi^2 = \{2,2\}$: One of the size 2 alliance members moves to the other size 2 alliance, resulting in π^3.

4. $\pi^3 = \{3,1\}$: A singleton player moves to a size 3 alliance, resulting in π^N.

5. $\pi^N = \{4\}$: This is a Nash equilibrium.

Unlike in the noncooperative case, nontrivial Nash equilibrium disappears. The benefits of belonging to a larger coalition already overwhelm other factors.

Strong Complementarity $\sigma = \frac{4}{5}$

In this case, we have $\frac{1-2\sigma}{1-\sigma} = -3$. Using Theorem 1, we know the following:

	$u(1,\pi^0)$	$u(2,\pi^1)$	$u(1,\pi^1)$	$u(2,\pi^2)$	$u(3,\pi^3)$	$u(1,\pi^3)$	$u(4,\pi^N)$
payoff	$\frac{1}{16}$	$\frac{15}{68}$	$\frac{1}{17}$	$\frac{1}{16}$	$\frac{3}{28}$	$\frac{1}{28}$	$\frac{1}{16}$

We analyze which partition can be a Nash equilibrium of stage 1:

1. $\pi^0 = \{1,1,1,1\}$: Two singletons merge to form an alliance, resulting in π^1.
2. $\pi^1 = \{2,1,1\}$: Two singletons merge to form an alliance, resulting in π^2, or one singleton moves to a size 2 alliance, resulting in π^3.
3. $\pi^2 = \{2,2\}$: One of the size 2 alliance members moves to the other alliance, resulting in π^3.
4. $\pi^3 = \{3,1\}$: A singleton merges into size 3 alliance, resulting in π^N.
5. $\pi^N = \{4\}$: This is a Nash equilibrium.

Thus, when $\sigma = \frac{4}{5}$, the trivial grand alliance is the unique Nash equilibrium.

Observations

Unlike in a noncooperative case, a nontrivial alliance structure appears only when $\sigma = \frac{1}{2}$. When $\sigma = \frac{3}{4}$ or $\frac{4}{5}$, then agglomeration forces are too strong and a smaller alliance tries to join a larger alliance, ending up with a grand alliance, which is a trivial alliance structure.