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# Regret in the Small and in the Large

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## Abstract

We propose a model of local-regret behavior that allows the separation of regret behavior between random variables that are close to each other and between random variables that are far apart. This enables a reinterpretation of evidence related to intransitive behavior in the laboratory. When viewed through this paper's analysis of regret, the laboratory evidence need not imply intransitive behavior for large risky decisions such as investment choices and insurance.

Keywords: Regret, intransitivity, preference reversal

## 1 Introduction

Transitivity is a fundamental assumption of decision theory, both at the individual and at the social level. The requirement that if  $A$  precedes  $B$  and  $B$  precedes  $C$  then  $A$  precedes  $C$  seems almost obvious. Yet we know that not all decision rules satisfy transitivity. The voting paradox of Condorcet (see, e.g., Muller [15]) shows that the majority rule may lead to a cyclic violation of transitivity. Nor do decision makers always behave according to this

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rule. Many experiments show that individual preferences are often intransitive, especially preferences over random variables. The preference reversal phenomenon (Lichtenstein and Slovic [10] and Grether and Plott [8]) shows that decision makers prefer a random variable like  $P = (4, \frac{35}{36}; -1, \frac{1}{36})$  to  $S = (16, \frac{11}{36}; -1.5, \frac{25}{36})$ , yet they set a higher selling price  $r_S$  on  $S$  than the selling price  $r_P$  they set on  $P$ . This behaviour violates transitivity, because presumably  $r_P \sim P \succ S \sim r_S \succ r_P$ . For other documented violations of transitivity, see e.g. Loomes, Starmer, and Sugden [12], Starmer [18], Birnbaum and Schmidt [5], and Regenwetter, Dana, and Davis-Stober [17].

Bell [2] and Loomes and Sugden [11] independently offered a simple idea to explain violations of transitivity. Unlike standard models of economics, where the value of an outcome depends only on the outcome itself, regret theory postulates that decision makers evaluate each possible outcome they may receive by comparing it to the alternative outcome they could have received by choosing differently. When comparing the random variable  $X = (x_1, s_1; \dots; x_n, s_n)$  with  $Y = (y_1, s_1; \dots; y_n, s_n)$ , the decision maker computes the expected value of a (subjective) elation/regret function  $\psi(x, y)$  and will choose  $X$  over  $Y$  if and only if this value is positive. Formally, he prefers  $X$  to  $Y$  if and only if  $\sum_i \Pr(s_i)\psi(x_i, y_i) \geq 0$ .

Regret theory often assumes that decision makers are *regret averse*. That is, if  $x > y > z$  then  $\psi(x, z) > \psi(x, y) + \psi(y, z)$ . The justification for this assumption is that large differences between what one obtained and what one would have obtained from an alternative choice give rise to disproportionately larger regret and elation. Several violations of expected utility can be explained by regret aversion. In particular, it implies that preferences are not transitive, see e.g. Loomes, Starmer, and Sugden [12] or Starmer [18]. For an axiomatization of regret aversion, see Diecidue and Somasundaram [6].

But do these aforementioned experiments prove that decision makers violate transitivity on a large range of random variables? The justification for the cycles presented by Loomes et al. [12] is regret aversion. For instance, the regret/elation difference between 8 and 0 is significant, while the differences between 8 and 4 and between 4 and 0 are not. But suppose that the payoffs are multiplied by 1,000. Can one really argue that any of these differences are insignificant? The argument that  $\psi(8, 0)$  is significantly larger than  $\psi(8, 4) + \psi(4, 0)$  seems less convincing when the outcomes are 8,000, 4,000, and 0.

The purpose of the present paper is two fold. First, we formalize this

last argument. We show that violations of transitivity on triples of random variables that are close to each other do not imply violations of transitivity everywhere. Moreover, we show that one can have regret behaviour with respect to random variables that are close to each other together with transitive behaviour for random variables that are not too close. In this we do not claim that regret theory is not a valid theory. What we claim is that applying this theory to decisions involving “large” random variables (like insurance or investment decisions) cannot be justified based on these experiments.

Our second objective is to establish that even if regret theory, as originally formulated, applies only to random variables that are close to each other, it is still a very powerful theory and preferences in different neighborhoods are not independent of each other. Formally, suppose that for each random variable  $W$  there is a neighborhood around  $W$  on which preferences are induced by regret theory, but the regret function may change from one random variable  $W$  to another. We show that to a certain extent, all these “local” regret functions are tightly knitted to each other. In other words, even if experiments showing regret “in the small” do not prove regret “in the large,” they should still indicate strong connection between local behaviors around different random variables.

The original papers on regret, [2] and [11], assumed linearity in probabilities, that is, they evaluated regret by taking expected values of a regret function. Much of the subsequent literature also assumed regret that is linear in probabilities. We show that with linear regret, intransitivity is pervasive in the sense that the existence of one intransitive cycle implies the existence of intransitive cycles everywhere. Moreover, observing one cycle the theory predicts many other specific cycles. Although this is not an experimental paper, we offer behavioral predictions that can support or refute the original form of regret theory. In fact, it turns out that even for more general forms of regret (as in [3]), some predictions can be made regarding the connection between cycles of preferences in different neighborhoods.

The paper is organized as follows. The linear-regret model is presented in Section 2, where it is shown that intransitivity in one part of the domain implies intransitivity everywhere. A more general regret model, which permits a decoupling of regret between random variables that are close to each other and regret between random variables that are far apart, is considered in Section 3. All proofs are in an appendix.

## 2 Linear Regret

Consider a set  $\mathcal{L}$  of finite-valued random variables  $X$  of the form  $X = (x_1, s_1; \dots; x_n, s_n)$  where the outcomes are monetary payoffs (which may be positive or negative).<sup>1</sup> The events  $s_1, \dots, s_n$  are pairwise disjoint and their union is the sure event. The probability of  $s_i$  is  $p_i$ . The set  $\mathcal{L}$  is endowed with the  $L^2$  norm. Thus, for  $X = (x_1, s_1; \dots; x_n, s_n)$  and  $W = (w_1, s_1; \dots; w_n, s_n)$ ,<sup>2</sup> we have  $\|X - W\| = \sum_{i=1}^n p_i(x_i - w_i)^2$ . An  $\varepsilon$ -neighborhood of  $W$  is the set  $\mathcal{B}(W, \varepsilon) = \{X : \|X - W\| < \varepsilon\}$ .

A decision maker's preferences on  $\mathcal{L}$  are represented by a complete binary relation  $\succeq$  which may or may not be transitive. The standard definition of transitivity, if  $X \succeq Y$  and  $Y \succeq Z$ , then  $X \succeq Z$ , implies that preferences are *not* transitive if there exists even one triplet  $X, Y, Z \in \mathcal{L}$  such that  $X \succeq Y$ ,  $Y \succeq Z$ , yet  $Z \succ X$ . We are however interested in more complex situations, where preferences may be transitive on some domains but intransitive on others. More importantly, we want to investigate how pervasive are such intransitive cycles, and if they exist, whether they are sporadic or must they appear everywhere.

Regret is a convenient way to model intransitive preferences.<sup>3</sup> Bell [2] and Loomes and Sugden [11] suggested a model of *linear regret*: for two random variables  $X = (x_1, s_1; \dots; x_n, s_n)$  and  $Y = (y_1, s_1; \dots; y_n, s_n)$  over the same set of events,

$$X \succeq Y \text{ if and only if } \sum_i p_i \psi(x_i, y_i) \geq 0 \quad (1)$$

where  $p_i$  is the probability of  $s_i$  and  $\psi$  is a *regret function* which is continuous and for all  $x$  and  $y$ ,

- (i)  $\psi(x, y) = -\psi(y, x)$ ,
- (ii)  $\psi$  is increasing in its first argument,
- (iii)  $\psi$  is decreasing in the second argument.

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<sup>1</sup>We assume an unbounded domain of payoffs, but our analysis holds for the case of bounded domain as well.

<sup>2</sup>Two random variables can be written on the same list of events without loss of generality. See Appendix A.

<sup>3</sup>Tserenjigmid [19] provides a related model of intransitivity using intra-dimensional comparisons. See also Nishimura [16].

The function  $\psi$  represents the feelings of the decision maker when he wins  $x$ , knowing that had he chosen differently his outcome would have been  $y$ . If  $x > y$  he will be elated (and  $\psi(x, y) > 0$ ), but if  $x < y$  he will be disappointed and regretful (hence  $\psi(x, y) < 0$ ). Condition (i) simply says that the elation from winning  $x$  greater than  $y$  equals the regret of winning  $y$  less than  $x$ . The two other conditions assert that elation is increasing with the winning outcome and decreasing with the foregone one. A consequence of the first condition is that  $\psi(x, x) = 0$ .

The basic linear model of regret, as suggested by Bell [2] and by Loomes and Sugden [11], was extended by Bikhchandani and Segal [3] to more general evaluations. Let  $\Psi(X, Y) = (\psi(x_1, y_1), p_1; \dots; \psi(x_n, y_n), p_n)$  be a regret lottery. Define

$$X \succeq Y \text{ if and only if } V(\Psi(X, Y)) \geq 0 \quad (2)$$

where the function  $V$  is any general function evaluating regret lotteries  $\Psi$ . If  $\succeq$ , represented as in eq. (2), is transitive, then [3] showed that it must be expected utility. Thus regret-based behavior is not consistent with any transitive non-expected utility choice.<sup>4</sup>

Assume linear regret. If for some  $x_1, x_2, x_3$  and  $s_1, s_2, s_3$  such that  $\Pr(s_1) = \Pr(s_2) = \Pr(s_3) = \frac{1}{3}$ ,  $(x_1, s_1; x_2, s_2; x_3, s_3) \sim (x_3, s_1; x_1, s_2; x_2, s_3) \sim (x_2, s_1; x_3, s_2; x_1, s_3) \sim (x_1, s_1; x_2, s_2; x_3, s_3)$ , then eq. (1) implies that  $\psi(x_1, x_3) = \psi(x_1, x_2) + \psi(x_2, x_3)$ . If the above indifferences hold for all  $x_1, x_2, x_3$ , it follows from the proof of [3, Lemma 7] that there exists  $u : \mathfrak{R} \rightarrow \mathfrak{R}$  such that  $\psi(x, y) = u(x) - u(y)$  which yields expected utility. Therefore, if linear regret preferences violate transitivity, then there exist  $x_1, x_2, x_3$  such that

$$\begin{aligned} (x_1, s_1; x_2, s_2; x_3, s_3) &\succ (x_3, s_1; x_1, s_2; x_2, s_3) \\ &\succ (x_2, s_1; x_3, s_2; x_1, s_3) \succ (x_1, s_1; x_2, s_2; x_3, s_3) \end{aligned} \quad (3)$$

As the next result shows, if non-expected utility preferences can be represented by linear regret, then intransitivities are pervasive. Not only does the existence of one cycle imply the existence of many other cycles, but it also implies that in the neighborhood of each random variable there are such cycles.

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<sup>4</sup>However, regret over pairs of independent lotteries (rather than random variables) is compatible with betweenness and other transitive non-expected utility models. See Bikhchandani and Segal [4].

**Proposition 1** *Suppose that preferences can be represented by linear regret and are non-expected utility. Then for every  $W$  and  $\varepsilon > 0$  there are  $X, Y, Z \in \mathcal{B}(W, \varepsilon)$  such that  $X \succ Y \succ Z \succ X$ .*

The proof of this proposition makes specific predictions that can be checked experimentally. A violation of transitivity implies a cycle as in eq. (3). Then for every  $y$  there is a sufficiently small  $\varepsilon > 0$  such that for  $s_0, \dots, s_3$  where  $\Pr(s_0) = 1 - \varepsilon$  and  $\Pr(s_1) = \Pr(s_2) = \Pr(s_3) = \frac{\varepsilon}{3}$ ,

$$\begin{aligned} (y, s_0; x_1, s_1; x_2, s_2; x_3, s_3) &\succ (y, s_0; x_3, s_1; x_1, s_2; x_2, s_3) \succ \\ &(y, s_0; x_2, s_1; x_3, s_2; x_1, s_3) \succ (y, s_0; x_1, s_1; x_2, s_2; x_3, s_3) \end{aligned}$$

Proposition 1 strongly depends on the assumption that regret is linear in probabilities, but it does not hold for the general model of regret. Example 1 in Appendix B presents a model of regret which is expected utility in all small neighborhoods, yet has a lot of intransitive cycles when random variables are sufficiently far apart from each other. In fact, once regret is not linear in probabilities, the opposite is also possible. Example 2 provides a regret relation that is transitive for random variables that are far away from each other, yet violates transitivity in all sufficiently small neighborhoods.

Although non-linear regret permits a separation between the attitudes towards transitivity in the small and in the large, it nevertheless imposes some strict restrictions over preferences in small neighborhoods. We analyze such preferences in the next section.

### 3 Local preferences and regret

To facilitate a distinction between intransitive cycles where random variables are far away from each other and cycles where random variables are all in a small neighborhood, define preferences to be *locally regret-based* (or to satisfy *regret in the small*) if they can be represented as in eq. (2) above in a neighborhood around each random variable  $W$ , albeit with different functions  $\psi$  and  $V$ . Formally, a binary relation is locally regret-based if for every  $W$  there is  $\varepsilon > 0$  such that for all  $X, Y \in \mathcal{B}(W, \varepsilon)$ ,

$$X \succeq Y \quad \text{if and only if} \quad V_W(\Psi_W(X, Y)) \geq 0$$

For clarity, hereafter we refer to preferences as defined in eqs. (1) or (2) as satisfying *regret in the large*.

As we show below, regret in the small does not imply regret in the large. Yet local regret does impose some restrictions on  $\psi_W$  and  $V_W$  across different values of  $W$ . In particular, local regret satisfies two consistency properties. First, all the local-regret functions  $\psi_W$  can be taken to be the same (Proposition 2). Secondly, the signs of any two local-regret functionals  $V_W, V_{W'}$  agree on regret lotteries that are generated in neighborhoods of both  $W$  and  $W'$  (Proposition 4). These consistency properties are implied by the fact that (i) any pair of random variables that are close to each other belong to many neighborhoods of nearby random variables and (ii) all local-regret preferences over the pair must be in agreement.

**Proposition 2** *If preferences are locally regret-based, then all the  $\psi_W$  functions are ordinally equivalent and can be taken to be the same.*

The reason that all the ordinally equivalent  $\psi_W$  functions can be taken to be the same is that the functions  $V_W$  can be adjusted so that a transformation of the regret function  $\psi_W$  can be achieved by letting the function  $V_W$  transform the values of  $\psi_W$ . It is therefore clear that the local-regret functionals,  $V_W$ , may be different. However, if each local regret is linear in probabilities, then local-regret lotteries are identically evaluated.

**Proposition 3** *If preferences are locally regret-based and each local-regret functional  $V_W$  is linear in probabilities, then there is a common (up to positive multiplication) local-regret function  $\psi$  for all  $W$ .*

Proposition 2 makes some simple behavioral predictions. Let  $W^j = (w_1^j, s_1; \dots; w_n^j, s_n)$ ,  $j = 1, 2$  where  $\Pr(s_1) = \Pr(s_2) = \delta$ . Let  $\varepsilon > 0$  and let  $X^j = (x_1, s_1; x_2, s_2; w_3^j, s_3; \dots; w_n^j, s_n)$  and  $Y^j = (y_1, s_1; y_2, s_2; w_3^j, s_3; \dots; w_n^j, s_n)$  be in  $\mathcal{B}(W^j, \varepsilon)$ ,  $j = 1, 2$ .

**Fact 1** *If  $X^1 \sim Y^1$ , then  $\psi_{W^1}(x_1, y_1) = -\psi_{W^1}(x_2, y_2)$ .*

By Proposition 2,  $\psi_{W^2}$  is an increasing ordinal transformation of  $\psi_{W^1}$ . Therefore, by Fact 1,  $X^1 \sim Y^1$  implies that  $\psi_{W^2}(x_1, y_1) = -\psi_{W^2}(x_2, y_2)$  and thus  $X^2 \sim Y^2$ . Using monotonicity, this provides another test for preferences satisfying local regret:  $X^1 \succeq Y^1$  if and only if  $X^2 \succeq Y^2$ .

If some or all of the  $V_W$  functions are not linear, then there may be local differences. Nevertheless, local regret is “concordant” in the following sense. Consider two regret functionals  $V_1$  and  $V_2$  defined on sets of regret lotteries  $\mathcal{R}_1$  and  $\mathcal{R}_2$ . Then  $V_1$  and  $V_2$  are *concordant* if they have the same sign on regret lotteries common to both domains. That is, for all  $R \in \mathcal{R}_1 \cap \mathcal{R}_2$ ,  $V_1(R) \gtrless 0$  if and only if  $V_2(R) \gtrless 0$ . Note that  $\mathcal{R}_1 \cap \mathcal{R}_2$  is non-empty as the regret lottery that yields 0 with probability 1 is in this open set.

**Proposition 4** *If preferences are locally regret-based, then any pair of local-regret functionals  $V_W$  and  $V_{W'}$  are concordant.*

Regret in the large, being universal, implies (the same) regret in the small, but the converse statement does not hold. As already noted, Examples 1 and 2 in Appendix B demonstrate the compatibility of local transitivity with intransitivity in the large, and vice versa. These examples also show that regret in the small does not pin down regret in the large. That is, there may be multiple regret-based extensions of local-regret preferences to the entire domain. In fact, preferences may satisfy regret in the small but may not be regret-based in the large (see Example 3 in Appendix B).

Our distinction between preferences in the small and in the large should not be confused with Machina’s [14] model of Fréchet differentiable representations, where preferences violate the independence axiom while converging at each point to expected utility. Intransitive regret models of the type discussed in this paper do not permit a representation function (which necessarily implies transitivity), hence are orthogonal to Machina’s analysis.

## 4 Discussion

Example 2 proves that violations of transitivity when random variables are close to each other do not imply the existence of intransitive cycles when random variables are far apart from each other. And as experiments are done with “small” random variables, then it is questionable to what extent one may deduce from these experiments that individuals violate transitivity in “big” decisions like financial investments, real-estate purchases, or retirement planning.

But isn’t this true for all experimental results? For example, when real payments are involved, experiments regarding the Allais paradox (Allais [1];

see also MacCrimmon and Larsson [13], Kahneman and Tversky [9], and Starmer [18]) are conducted, for obvious reasons, with small amounts of money. A similar argument to the one made in the paper will lead to the conclusion that we cannot learn from these experiments that the Allais paradox really exists.

We believe that there is an important difference between the analysis of transitivity and regret and that of phenomena like the Allais paradox. A standard presentation of this decision problem asks the decision maker to choose between  $A = (5M, 0.1; 0, 0.9)$  and  $B = (1M, 0.11; 0, 0.89)$ , and then between  $C = (5M, 0.1; 1M, 0.89; 0, 0.01)$  and  $D = (1M, 1)$ . The common preferences  $A \succ B$  together with  $D \succ C$  violate expected utility maximization. The psychological rationale behind these preferences is simple. Random variables  $A$  and  $B$  offer similar probabilities of success, but  $A$  offers a much higher payoff. This argument applies to random variables  $C$  and  $D$  as well, but there another factor that tilts the scales in favour of  $D$ , and this is the possibility of winning zero in random variable  $C$ . Before making the choice the decision maker knows that he will feel devastated if after choosing  $C$  he were to win zero, when he could have avoided all risk by choosing  $D$  (and receiving  $1M$ ).

This argument becomes less powerful if all outcomes are scaled down, yet such preferences persist even after such modifications (see for example problems 1 and 2 in [9]). The actual experimental data regarding the Allais paradox therefore supports that hypothesis that this phenomenon exists not only in the small but also in the large.

Regret theory, on the other hand, is based on the intuition that for  $x > y > z$ , the elation from obtaining  $x$  when the alternative is  $z$  is greater than the sum of the two smaller elations, from receiving  $x$  when the alternative is  $y$  and from receiving  $y$  when the alternative is  $z$ . This intuition is convincing when there is a certain threshold above which the decision maker's feelings of elation or regret become relevant. But it is much less obvious that this property also holds for large numbers. So even if it is true that  $\psi(8000, 0) > \psi(8000, 4000) + \psi(4000, 0)$ , it may well happen (and indeed, is quite reasonable to expect) that

$$\frac{\psi(8000, 0)}{\psi(8000, 4000) + \psi(4000, 0)} < \frac{\psi(8, 0)}{\psi(8, 4) + \psi(4, 0)}$$

To summarize, our argument that a certain behavior in the small may not

necessarily indicate a similar behavior in the large can be formally extended to other violations of expected utility theory. However, with respect to other phenomena, violations in the small are less likely to happen than violations in the large, and therefore, experiments showing violations in the small correctly predict violations in the large. In contrast, with respect to regret theory the opposite may be true. Violations in the small are more likely to happen than violations in the large, and therefore experiments showing violations in the small do not necessarily indicate similar violations in the large.

## Appendix A: Proofs

First, we show that a finite number of finite-valued random variables may be written on the same list of events. For two random variables

$$X^j = (x_1^j, s_1^j; \dots; x_{i_j}^j, s_{i_j}^j), \quad j = 1, 2$$

define  $s_{1,1}, \dots, s_{1,i_2}, \dots, s_{i_1,1}, \dots, s_{i_1,i_2}$  by  $s_{i,j} = s_i^1 \cap s_j^2$ . Note that

$$\begin{aligned} X^1 &= (x_1^1, \cup_k s_{1,k}; \dots; x_{i_1}^1, \cup_k s_{i_1,k}) \\ X^2 &= (x_1^2, \cup_k s_{k,1}; \dots; x_{i_2}^2, \cup_k s_{k,i_2}) \end{aligned}$$

Therefore, we can assume without loss of generality that any finite number of random variables can be defined on the same list of events.

Second, any event may be partitioned into two sub-events with any probability ratio. For an event  $s_i$  and  $\alpha \in [0, 1]$ , define  $\beta(s_i, \alpha)$  such that  $\Pr(s_{i,\alpha}) := \Pr(s_i \cap [0, \beta(s_i, \alpha)]) = \alpha \Pr(s_i)$ , and let  $s'_{i,1-\alpha} = s_i \setminus s_{i,\alpha}$ . (Note that  $\beta(s_i, \alpha)$  exists because the probability measure is atomless).

**Proof of Proposition 1:** As preferences are non-expected utility and represented by a linear functional, there exist  $x_1, x_2, x_3$  which admit the intransitive cycle of eq. (3). Hence

$$\begin{aligned} V(\psi(x_1, x_3), \frac{1}{3}; \psi(x_2, x_1), \frac{1}{3}; \psi(x_3, x_2), \frac{1}{3}) &= \\ \frac{\psi(x_1, x_3) + \psi(x_2, x_1) + \psi(x_3, x_2)}{3} &> 0 \end{aligned} \tag{4}$$

Let  $W = (w_1, t_1; \dots; w_\ell, t_\ell) \in \mathcal{L}$ . For any  $m > \frac{1}{\varepsilon}$ , let  $s_1, \dots, s_{3m}$  be pairwise disjoint with the probabilities  $\frac{1}{3m}$  each. The random variables  $X, Y, Z$  defined below are in an  $\varepsilon$ -neighborhood of  $W$ :

$$\begin{aligned} X &= (x_1, s_1; x_2, s_2; x_3, s_3; w_1, t_1 \cap (\cup_{j=4}^{3m} s_j); \dots; w_\ell, t_\ell \cap (\cup_{j=4}^{3m} s_j)) \\ Y &= (x_3, s_1; x_1, s_2; x_2, s_3; w_1, t_1 \cap (\cup_{j=4}^{3m} s_j); \dots; w_\ell, t_\ell \cap (\cup_{j=4}^{3m} s_j)) \\ Z &= (x_2, s_1; x_3, s_2; x_1, s_3; w_1, t_1 \cap (\cup_{j=4}^{3m} s_j); \dots; w_\ell, t_\ell \cap (\cup_{j=4}^{3m} s_j)) \end{aligned}$$

That  $X \succ Y \succ Z \succ X$  follows from

$$\begin{aligned} &V(\psi(x_1, x_3), \frac{1}{3m}; \psi(x_2, x_1), \frac{1}{3m}; \psi(x_3, x_2), \frac{1}{3m}; 0, \frac{m-1}{m}) \\ &= \frac{\psi(x_1, x_3) + \psi(x_2, x_1) + \psi(x_3, x_2)}{3m} > 0 \end{aligned}$$

where the inequality follows from (4). ■

**Proof of Proposition 2:** First, we show that for any regret functional  $V$ ,

$$V\left(r, \frac{1}{\ell}; -r, \frac{1}{\ell}; 0, \frac{\ell-2}{\ell}\right) = V\left(-r, \frac{1}{\ell}; r, \frac{1}{\ell}; 0, \frac{\ell-2}{\ell}\right) = 0 \quad (5)$$

for any regret level  $r$  and integer  $\ell \geq 2$ . The first equality is true as  $(r, \frac{1}{\ell}; -r, \frac{1}{\ell}; 0, \frac{\ell-2}{\ell})$  and  $(-r, \frac{1}{\ell}; r, \frac{1}{\ell}; 0, \frac{\ell-2}{\ell})$  are the same regret lottery. Suppose that  $V\left(r, \frac{1}{\ell}; -r, \frac{1}{\ell}; 0, \frac{\ell-2}{\ell}\right) > 0$ . Let  $x_1, x_2$  be such that  $\psi(x_1, x_2) = r$ . By skew symmetry,  $\psi(x_2, x_1) = -r$ . With equiprobable events  $s_1, \dots, s_\ell$ , we have

$$\begin{aligned} (x_1, s_1; x_2, s_2; x_3, s_3; \dots; x_\ell, s_\ell) &\succ (x_2, s_1; x_1, s_2; x_3, s_3; \dots; x_\ell, s_\ell) \\ &\succ (x_1, s_1; x_2, s_2; x_3, s_3; \dots; x_\ell, s_\ell) \end{aligned}$$

which is a violation of irreflexivity. Hence (5).

Define the  $\oplus$  operation as follows. Let  $X = (x_1, s_1; \dots; x_n, s_n)$  and  $Y = (y_1, s_1; \dots; y_n, s_n)$ . Then

$$\alpha X \oplus (1 - \alpha)Y = (x_1, s_{1,\alpha}; \dots; x_n, s_{n,\alpha}, y_1, s'_{1,1-\alpha}, \dots; y_n, s'_{n,1-\alpha})$$

Let  $[W, W'] = \{\alpha W \oplus (1 - \alpha)W' : \alpha \in [0, 1]\}$ . The set  $\{\alpha \in [0, 1] : \alpha W \oplus (1 - \alpha)W' \in \mathcal{B}(\beta W \oplus (1 - \beta)W', \varepsilon)\}$  is open. As  $[0, 1]$  is compact, there is a finite sequence of overlapping neighborhoods  $\mathcal{B}(W, \varepsilon) = \mathcal{B}(W_1, \varepsilon), \dots, \mathcal{B}(W_n, \varepsilon) =$

$\mathcal{B}(W', \varepsilon)$  covering  $[W, W']$ . We show that for  $i = 1, \dots, n - 1$ ,  $\psi_i$  and  $\psi_{i+1}$  are ordinally equivalent. Suppose not. Then there are  $(x, y), (x', y')$  such that  $\psi_i(x, y) > \psi_i(x', y')$  but  $\psi_{i+1}(x, y) \leq \psi_{i+1}(x', y')$ . Let  $Z \in \mathcal{B}(W_i, \varepsilon) \cap \mathcal{B}(W_{i+1}, \varepsilon)$ . As this intersection is open, there is a sufficiently small  $\beta$  such that

$$X = \beta(x, H; y', T) \oplus (1 - \beta)Z, \quad Y = \beta(y, H; x', T) \oplus (1 - \beta)Z$$

and  $X, Y \in \mathcal{B}(W_i, \varepsilon_i) \cap \mathcal{B}(W_{i+1}, \varepsilon_{i+1})$ . With  $\ell \geq \frac{2}{\beta}$  we obtain that

$$\begin{aligned} V_i(\Psi_i(X, Y)) &= V_i\left(\psi_i(x, y), \frac{1}{\ell}; \psi_i(y', x'), \frac{1}{\ell}; 0, \frac{\ell - 2}{\ell}\right) \\ &= V_i\left(\psi_i(x, y), \frac{1}{\ell}; -\psi_i(x', y'), \frac{1}{\ell}; 0, \frac{\ell - 2}{\ell}\right) \\ &> V_i\left(\psi_i(x, y), \frac{1}{\ell}; -\psi_i(x, y), \frac{1}{\ell}; 0, \frac{\ell - 2}{\ell}\right) \\ &= 0 \end{aligned}$$

where we use the fact that  $\psi$  is skew symmetric,  $-\psi_i(x, y) < -\psi_i(x', y')$ , the monotonicity of  $V(\cdot)$ , and equation (5). Therefore,  $X \succ Y$ . Similarly,

$$\begin{aligned} V_{i+1}(\Psi_{i+1}(X, Y)) &= V_{i+1}\left(\psi_{i+1}(x, y), \frac{1}{\ell}; \psi_{i+1}(y', x'), \frac{1}{\ell}; 0, \frac{\ell - 2}{\ell}\right) \\ &= V_{i+1}\left(\psi_{i+1}(x, y), \frac{1}{\ell}; -\psi_{i+1}(x', y'), \frac{1}{\ell}; 0, \frac{\ell - 2}{\ell}\right) \\ &\leq V_{i+1}\left(\psi_{i+1}(x, y), \frac{1}{\ell}; -\psi_{i+1}(x, y), \frac{1}{\ell}; 0, \frac{\ell - 2}{\ell}\right) \\ &= 0 \end{aligned}$$

and therefore  $X \preceq Y$ , a contradiction.

Therefore, for all  $(x, y), (x', y')$  such that  $\psi_i(x, y) > \psi_i(x', y')$  we have  $\psi_{i+1}(x, y) > \psi_{i+1}(x', y')$ . Hence there exists a well-defined increasing function  $h_{i+1, i}$  such that  $h_{i+1, i}(\psi_i(x, y)) = \psi_{i+1}(x, y)$ , for all  $x, y$ . Since  $\psi_1, \dots, \psi_n$  are ordinally equivalent, so are  $\psi_W$  and  $\psi_{W'}$ .

Fix a random variable  $W_0$  and let  $\psi_0$  be the local-regret function at  $W_0$ . Ordinal equivalence of local-regret functions implies that for each  $W \in \mathcal{L}$  there is an increasing function  $h_W : \mathfrak{R} \rightarrow \mathfrak{R}$  such that  $\psi_W = h_W \circ \psi_0$ , where

$\psi_W$  is the local-regret function at  $W$ . Define a new local-regret functional at  $W$

$$\begin{aligned}\hat{V}_W(\psi_0(x_1, y_1), p_1; \dots; \psi_0(x_n, y_n), p_n) &\equiv \\ V_W(h_W \circ \psi_0(x_1, y_1), p_1; \dots; h_w \circ \psi_0(x_n, y_n), p_n)\end{aligned}$$

Thus, for each  $W$ , local-regret preferences may be represented by  $\psi_0, \hat{V}_W$ . ■

**Proof of Proposition 3:** Let  $W = (w_1, s_1; w_2, s_2; \dots; w_n, s_n)$ . Let  $\psi_W$  be a local-regret function at  $W$ . First, we show that the linearity of  $V_W$  implies that each  $\psi_W$  is unique up to positive multiples.

For two regret levels  $r_1, r_2 > 0$ , let  $x, y > 0$  be monetary outcomes such that  $r_1 = \psi_W(x, -x)$  and  $r_2 = \psi_W(y, -y)$ . Define

$$\begin{aligned}X &= (x, s_{1,\varepsilon_1}; -y, s_{1,\varepsilon_2}; w_1, s_{1,1-\varepsilon_1-\varepsilon_2}; w_2, s_2; \dots; w_n, s_n) \\ Y &= (-x, s_{1,\varepsilon_1}; y, s_{1,\varepsilon_2}; w_1, s_{1,1-\varepsilon_1-\varepsilon_2}; w_2, s_2; \dots; w_n, s_n)\end{aligned}$$

where  $\varepsilon_1, \varepsilon_2 > 0$ ,  $\varepsilon_1 + \varepsilon_2 < 1$ ,  $\varepsilon_2/\varepsilon_1 = r_1/r_2$ , and  $\Pr[s_{1,\varepsilon_\ell}] = \varepsilon_\ell$ ,  $\ell = 1, 2$ . Choose  $\varepsilon_1, \varepsilon_2$  small enough so that  $X, Y \in \mathcal{B}(W, \varepsilon)$ . Thus,

$$\begin{aligned}V_W(\Psi_W(X, Y)) &= \psi_W(x, -x) \Pr[s_{1,\varepsilon_1}] + \psi_W(-y, y) \Pr[s_{1,\varepsilon_2}] \\ &= r_1 \varepsilon_1 - r_2 \varepsilon_2 \\ &= 0\end{aligned}$$

Hence,  $X \sim Y$ .

Let  $\hat{\psi}_W$  be another local-regret function at  $W$  for which the regret functional  $\hat{V}_W$  is linear in probability. From  $X \sim Y$  we conclude

$$\begin{aligned}\hat{V}_W(\hat{\Psi}_W(X, Y)) &= \hat{\psi}_W(x, -x) \Pr[s_{1,\varepsilon_1}] + \hat{\psi}_W(-y, y) \Pr[s_{1,\varepsilon_2}] \\ &= \hat{r}_1 \varepsilon_1 - \hat{r}_2 \varepsilon_2 \\ &= 0\end{aligned}$$

Thus, there exists  $a$  such that  $\hat{\psi}_W(x, -x) = a\psi_W(x, -x) = ar_1$  and  $\hat{\psi}_W(y, -y) = a\psi_W(y, -y) = ar_2$ . That,  $a > 0$  follows from the fact that  $\hat{\psi}_W(x, -x) > 0$ . By varying  $r_2$  (and  $y, \varepsilon_2$  in the above construction), we conclude that  $\hat{\psi}_W \equiv a\psi_W$  for some  $a > 0$ .

Let  $W, W' \in \mathcal{L}$ . We show that  $\psi_W$  and  $\psi_{W'}$  are positive multiples of each other. As in the proof of Proposition 2, there is a finite sequence of random

variables  $W = W_1, W_2, \dots, W_n = W'$  such that the open neighborhoods  $\mathcal{B}(W_j, \varepsilon)$ ,  $j = 1, \dots, m$  cover the line segment  $[W, W']$ . Thus, for each  $j$ , there exists

$$\hat{W}_j = (\hat{w}_{1j}, s_1; \hat{w}_{2j}, s_2; \dots; \hat{w}_{nj}, s_n)$$

on the line segment joining  $W_j$  and  $W_{j+1}$  such that  $\hat{W}_j \in \mathcal{B}(W_j, \varepsilon) \cap \mathcal{B}(W_{j+1}, \varepsilon)$ .

Let  $\psi_j$  be the local-regret functional at  $W_j$ . Define

$$\begin{aligned} X_j &= (x, s_{1,\varepsilon_1}; -y, s_{1,\varepsilon_2}; \hat{w}_{1j}, s_{1,1-\varepsilon_1-\varepsilon_2}; \hat{w}_{2j}, s_2; \dots; \hat{w}_{nj}, s_n) \\ Y_j &= (-x, s_{1,\varepsilon_1}; y, s_{1,\varepsilon_2}; \hat{w}_{1j}, s_{1,1-\varepsilon_1-\varepsilon_2}; \hat{w}_{2j}, s_2; \dots; \hat{w}_{nj}, s_n) \end{aligned}$$

where  $\varepsilon_1, \varepsilon_2 > 0$ ,  $\varepsilon_2/\varepsilon_1 = \psi_j(x, -x)/\psi_j(y, -y)$ , and  $\Pr[s_{1,\varepsilon_\ell}] = \varepsilon_\ell$ ,  $\ell = 1, 2$ . Choose  $\varepsilon_1, \varepsilon_2$  small enough so that  $X, Y \in \mathcal{B}(W_j, \varepsilon) \cap \mathcal{B}(W_{j+1}, \varepsilon)$ . Thus, local regret evaluated at  $W_j$  implies that  $X_j \sim Y_j$ . Hence, the local regret evaluated at  $W_{j+1}$  also implies  $X_j \sim Y_j$ . As  $x$  and  $y$  are arbitrary, we conclude that  $\psi_{j+1}$  is a positive multiple of  $\psi_j$ . Consequently,  $\psi_{W'}$  is a positive multiple of  $\psi_W$ .  $\blacksquare$

**Proof of Fact 1** By the skew-symmetry of the functions  $\psi$ ,

$$\begin{aligned} X^1 \sim Y^1 &\iff V_{W^1}(\psi_{W^1}(x_1, y_1), \delta; \psi_{W^1}(x_2, y_2), \delta; 0, 1 - 2\delta) = 0 \\ &\iff V_{W^1}(-\psi_{W^1}(y_1, x_1), \delta; -\psi_{W^1}(y_2, x_2), \delta; 0, 1 - 2\delta) = 0 \end{aligned}$$

Therefore,  $\psi_{W^1}(x_1, y_1) \geq -\psi_{W^1}(y_2, x_2)$  iff  $\psi_{W^1}(x_2, y_2) \leq -\psi_{W^1}(y_1, x_1)$ . However,  $\psi_{W^1}(x_1, y_1) > -\psi_{W^1}(x_2, y_2)$  implies

$$\begin{aligned} -\psi_{W^1}(x_1, y_1) = \psi_{W^1}(y_1, x_1) &< -\psi_{W^1}(y_2, x_2) = \psi_{W^1}(x_2, y_2) \implies \\ V_{W^1}(\psi_{W^1}(y_1, x_1), \delta; \psi_{W^1}(y_2, x_2), \delta; 0, 1 - 2\delta) &< \\ V_{W^1}(\psi_{W^1}(x_1, y_1), \delta; \psi_{W^1}(x_2, y_2), \delta; 0, 1 - 2\delta) &= 0 \end{aligned}$$

A contradiction to  $Y^1 \sim X^1$ .  $\blacksquare$

**Proof of Proposition 4:** Let  $W, W' \in \mathcal{L}$ . We know from Proposition 2 that there exists a regret function  $\psi$  such that preferences are locally regret with  $\psi$  and  $V_W$  on  $\mathcal{B}(W, \varepsilon)$  and locally regret with  $\psi$  and  $V_{W'}$  on  $\mathcal{B}(W', \varepsilon)$ .

Let  $\mathcal{R}_W$  be the set of regret lotteries generated by  $X, Y \in \mathcal{B}(W, \varepsilon)$  and  $\mathcal{R}_{W'}$  be the set of regret lotteries generated by  $X', Y' \in \mathcal{B}(W', \varepsilon)$ . Take  $R \in \mathcal{R}_W \cap \mathcal{R}_{W'}$ . If  $R = (0, 1)$  then  $V_W(R) = V_{W'}(R) = 0$ .

Suppose that  $(0, 1) \neq R \in \mathcal{R}_W \cap \mathcal{R}_{W'}$ . Thus, there exist  $X, Y \in \mathcal{B}(W, \varepsilon)$ ,  $X \neq Y$  and  $X', Y' \in \mathcal{B}(W', \varepsilon)$ ,  $X' \neq Y'$  such that  $R = \Psi(X, Y) = \Psi(X', Y')$ . Without loss of generality we may write  $X, X', W$ , and  $W'$  on the same list of events  $s_1, \dots, s_n$ . We can partition each  $s_i$  into two sub-events,  $s_{i,\alpha}$  and  $s_{i,1-\alpha}$ , such that  $\Pr(s_{i,\alpha}) = \alpha \Pr(s_i)$  and  $\Pr(s_{i,1-\alpha}) = (1 - \alpha) \Pr(s_i)$ . Thus,

$$\begin{aligned} \alpha X \oplus (1 - \alpha) X' &= (x_1, s_{1,\alpha}; \dots; x_n, s_{n,\alpha}; x'_1, s_{1,1-\alpha}; \dots; x'_n, s_{n,1-\alpha}) \\ \alpha Y \oplus (1 - \alpha) Y' &= (y_1, s_{1,\alpha}; \dots; y_n, s_{n,\alpha}; y'_1, s_{1,1-\alpha}; \dots; y'_n, s_{n,1-\alpha}) \\ \alpha W \oplus (1 - \alpha) W' &= (w_1, s_{1,\alpha}; \dots; w_n, s_{n,\alpha}; w'_1, s_{1,1-\alpha}; \dots; w'_n, s_{n,1-\alpha}) \end{aligned} \quad (6)$$

and

$$\begin{aligned} & \| \alpha X \oplus (1 - \alpha) X' - [\alpha W \oplus (1 - \alpha) W'] \| \\ &= \sum_{i=1}^n (x_i - w_i)^2 \Pr(s_{i,\alpha}) + \sum_{i=1}^n (x'_i - w'_i)^2 \Pr(s_{i,1-\alpha}) \\ &= \alpha \sum_{i=1}^n (x_i - w_i)^2 \Pr(s_i) + (1 - \alpha) \sum_{i=1}^n (x'_i - w'_i)^2 \Pr(s_i) \\ &= \alpha \| X - W \| + (1 - \alpha) \| X' - W' \| \end{aligned}$$

Consequently, if  $X \in \mathcal{B}(W, \varepsilon)$  and  $X' \in \mathcal{B}(W', \varepsilon)$  then

$$\alpha X \oplus (1 - \alpha) X' \in \mathcal{B}(\alpha W \oplus (1 - \alpha) W', \varepsilon)$$

Similarly,  $\alpha Y \oplus (1 - \alpha) Y' \in \mathcal{B}(\alpha W \oplus (1 - \alpha) W', \varepsilon)$  if  $Y \in \mathcal{B}(W, \varepsilon)$  and  $Y' \in \mathcal{B}(W', \varepsilon)$ . From eq. (6), it follows that

$$\Psi(\alpha X \oplus (1 - \alpha) X', \alpha Y \oplus (1 - \alpha) Y') = R$$

Hence, if  $R \in \mathcal{R}_W \cap \mathcal{R}_{W'}$  then  $R \in \mathcal{R}_{\alpha W \oplus (1-\alpha) W'}$  for any  $\alpha \in (0, 1)$ . That is, if the regret lottery  $R$  is locally generated in the neighborhoods of  $W$  and  $W'$ , then  $R$  is locally generated in the neighborhood of each random variable on the line segment joining  $W$  and  $W'$ .

Suppose that  $V_W$  and  $V_{W'}$  are not concordant. In particular,  $V_W(R) > 0$  and  $V_{W'}(R) \leq 0$ . Let

$$\underline{\alpha} = \sup \{ \alpha \in [0, 1] : V_{\alpha W \oplus (1-\alpha) W'}(R) \leq 0 \}$$

As the regret lottery  $R$  is locally generated at each  $\alpha W \oplus (1 - \alpha) W'$ ,  $\underline{\alpha}$  is well-defined. Further,  $V_{W'}(R) \leq 0$  implies that the sup is taken over a non-empty set.

From the continuity of  $\succeq$  it follows that  $\underline{\alpha} < 1$ ,  $V_{\underline{\alpha}W \oplus (1-\underline{\alpha})W'}(R) = 0$ , and that

$$V_{\alpha W \oplus (1-\alpha)W'}(R) > 0, \quad \forall \alpha \in (\underline{\alpha}, 1]$$

We know that

$$\underline{\alpha}X \oplus (1 - \underline{\alpha})X', \underline{\alpha}Y \oplus (1 - \underline{\alpha})Y' \in \mathcal{B}(\underline{\alpha}W \oplus (1 - \underline{\alpha})W', \varepsilon) \quad (7)$$

As

$$R = \Psi(\underline{\alpha}X \oplus (1 - \underline{\alpha})X', \underline{\alpha}Y \oplus (1 - \underline{\alpha})Y')$$

eq. (7) together with  $V_{\underline{\alpha}W \oplus (1-\underline{\alpha})W'}(R) = 0$  implies that

$$\underline{\alpha}X \oplus (1 - \underline{\alpha})X' \sim \underline{\alpha}Y \oplus (1 - \underline{\alpha})Y'$$

For  $\alpha_1$  close to  $\underline{\alpha}$ ,

$$\underline{\alpha}X \oplus (1 - \underline{\alpha})X', \underline{\alpha}Y \oplus (1 - \underline{\alpha})Y' \in \mathcal{B}(\alpha_1 W \oplus (1 - \alpha_1)W', \varepsilon)$$

Take such a  $\alpha_1 > \underline{\alpha}$ . Then  $V_{\alpha_1 W \oplus (1-\alpha_1)W'}(R) > 0$  implies that

$$\underline{\alpha}X \oplus (1 - \underline{\alpha})X' \succ \alpha_1 Y \oplus (1 - \alpha_1)Y'$$

Contradiction. ■

## Appendix B: Examples

**Example 1** TRANSITIVE IN THE SMALL, INTRANSITIVE IN THE LARGE  
Let  $\psi(x, y) = x - y$  be a regret function. For  $X = (x_1, s_1; \dots; x_n, s_n)$ ,  $Y = (y_1, s_1; \dots; y_n, s_n)$ ,  $\Pr(s_i) = p_i$  define

$$V(\Psi(X, Y)) = \begin{cases} \sum_{i=1}^n p_i \psi(x_i, y_i) & \|\Psi(X, Y)\| < \varepsilon \\ 1 - \alpha(X, Y) \sum_{i=1}^n p_i \psi(x_i, y_i) + \\ \quad \alpha(X, Y) \sum_{i=1}^n p_i [\psi(x_i, y_i)]^3 & \varepsilon \leq \|\Psi(X, Y)\| < 1 \\ \sum_{i=1}^n p_i [\psi(x_i, y_i)]^3 & 1 \leq \|\Psi(X, Y)\| \end{cases}$$

where  $\alpha(X, Y) \equiv \frac{\|\Psi(X, Y)\| - \varepsilon}{1 - \varepsilon}$ . If  $\|\Psi(X, Y)\| > \varepsilon$ , then  $0 < \alpha(X, Y) < 1$  and if  $\|\Psi(X, Y)\| = \varepsilon$ , then  $\alpha(X, Y) = 1$ . Hence,  $V$  is a continuous functional.

If  $\|\Psi(X, Y)\| < \varepsilon$  then  $V(\Psi(X, Y)) = \mathbb{E}[X] - \mathbb{E}[Y]$ . Therefore, for each  $W$  local regret in  $\mathcal{B}(W, 0.5\varepsilon)$  is transitive.

Take  $a > 0$ . Define  $X = (-a, s_1; 0, s_2; a, s_3)$ ,  $Y = (0, s_1; a, s_2; -a, s_3)$ , and  $Z = (a, s_1; -a, s_2; 0, s_3)$ , where  $s_1, s_2, s_3$  are equiprobable, disjoint events with  $s_1 \cup s_2 \cup s_3 = S$ . For large enough  $a$ ,  $\|\Psi(X, Y)\| > 1$ ,  $\|\Psi(Y, Z)\| > 1$ , and  $\|\Psi(X, Z)\| > 1$ . Therefore,

$$V(\Psi(X, Y)) = V(\Psi(Y, Z)) = V(\Psi(Z, X)) = 2a^3 > 0$$

which implies  $X \succ Y \succ Z \succ X$ . Regret in the large is intransitive.

Note that the regret functional  $V$  is not linear in probabilities. That is, there exist regret lotteries  $R_1, R_2$ , and a constant  $c \in (0, 1)$ , such that

$$V\left(cR_1 + (1 - c)R_2\right) \neq cV(R_1) + (1 - c)V(R_2)$$

To see this, take two regret lotteries,  $R_1, R_2$ , with  $\|R_1\| < \varepsilon < \|R_2\|$ . Let  $c$  be sufficiently close to 1 so that  $\|cR_1 + (1 - c)R_2\| < \varepsilon$ . Then

$$\begin{aligned} & V(cR_1 + (1 - c)R_2) \\ &= c\mathbb{E}[R_1] + (1 - c)\mathbb{E}[R_2] \\ &\neq c\mathbb{E}[R_1] + (1 - c)\left(\left(\frac{1 - \|R_2\|}{1 - \varepsilon}\right)\mathbb{E}[R_2] + \left(\frac{\|R_2\| - \varepsilon}{1 - \varepsilon}\right)(\mathbb{E}[R_2])^3\right) \\ &= cV(R_1) + (1 - c)V(R_2) \end{aligned} \quad \blacksquare$$

Local-regret preferences do not uniquely determine regret preferences in the large. For instance,  $V(\Psi(X, Y)) = \mathbb{E}[X] - \mathbb{E}[Y]$  for all  $\Psi(X, Y)$  has the same local-regret preferences as Example 1.

**Example 2** INTRANSITIVE IN THE SMALL, TRANSITIVE IN THE LARGE

Intransitive cycles between random variables close to each other, but not between random variables that are far apart, are also possible. Let  $X, Y$ , and  $\psi(x, y)$  be as in Example 1. Define

$$V(\Psi(X, Y)) = \begin{cases} \sum_{i=1}^n p_i \psi(x_i, y_i) & \|\Psi(X, Y)\| > \varepsilon \\ \beta(X, Y) \sum_{i=1}^n p_i \psi(x_i, y_i) + (1 - \beta(X, Y)) \sum_{i=1}^n p_i [\psi(x_i, y_i)]^3 & \text{otherwise} \end{cases}$$

where  $\beta(X, Y) \equiv \min\{1, \|\Psi(X, Y)\|/\varepsilon\}$ . \blacksquare

**Example 3** REGRET IN THE SMALL BUT NOT REGRET-BASED

Let  $\psi_\ell$  and  $\psi_g$  be two regret functions that are **not** ordinally equivalent. We define  $\succeq$  over pairs of random variables based on whether the random variables are close to each other, neither close nor far, or far away from each other.

Let

$$\begin{aligned} X &= (x_1, s_1; \dots; x_n, s_n) \\ Y &= (y_1, s_1; \dots; y_n, s_n) \end{aligned}$$

Define  $\gamma(X, Y) \equiv \frac{\|X-Y\|-\varepsilon}{2\varepsilon-\varepsilon}$ , where  $\varepsilon > 0$ .

If  $\|X - Y\| < \varepsilon$  then

$$X \succeq Y \quad \text{iff} \quad \sum_i p_i \psi_\ell(x_i, y_i) \geq 0 \quad (8)$$

If  $\varepsilon \leq \|X - Y\| \leq 2\varepsilon$  then

$$X \succeq Y \quad \text{iff} \quad (1 - \gamma(X, Y)) \sum_i p_i \psi_\ell(x_i, y_i) + \gamma(X, Y) \sum_i p_i \psi_g(x_i, y_i) \geq 0$$

If  $\|X - Y\| > 2\varepsilon$  then

$$X \succeq Y \quad \text{iff} \quad \sum_i p_i \psi_g(x_i, y_i) \geq 0 \quad (9)$$

The binary relation  $\succeq$  is complete. It is also continuous as  $\gamma(X, Y)$  increases from 0 to 1 as  $\|X - Y\|$  increases from  $\varepsilon$  to  $2\varepsilon$ .

Suppose that  $\succeq$  is regret based. That is, there exists a regret function  $\psi$  and regret functional  $V$  such that  $X \succeq Y$  if and only if  $V(\Psi(X, Y)) \geq 0$ . Suppose that  $\psi$  is ordinally equivalent to  $\psi_\ell$ . Then it is not ordinally equivalent to  $\psi_g$ . That is, there are  $(x, y), (x', y')$  such that  $\psi(x, y) > \psi(x', y')$  but  $\psi_g(x, y) \leq \psi_g(x', y')$ . Let  $s_1, \dots, s_4$  be equiprobable, disjoint events and let

$$\begin{aligned} X &= (x, s_1; x', s_2; z, s_3; -z, s_4) \\ Y &= (y, s_1; y', s_2; -z, s_3; z, s_4) \end{aligned}$$

Choose  $z$  large enough that  $\|X - Y\| > 2\varepsilon$ . Then

$$\begin{aligned} V(\Psi(X, Y)) &= V(\psi(x, y), 0.25; \psi(x', y'), 0.25; \psi(z, -z), 0.25; \psi(-z, z), 0.25) \\ &> 0 \end{aligned}$$

as  $\psi(x, y) > \psi(x', y')$ . Thus  $X \succ Y$ . However, (9) implies that  $X \preceq Y$  as

$$\frac{\psi_g(x, y) + \psi_g(x', y') + \psi_g(z, -z) + \psi_g(-z, z)}{4} \leq 0$$

which follows from  $\psi_g(x, y) \leq \psi_g(x', y')$ . Contradiction.

Suppose, instead, that  $\psi$  is ordinally equivalent to  $\psi_g$ . Then it is not ordinally equivalent to  $\psi_\ell$ . That is, there there are  $(x, y), (x', y')$  such that  $\psi(x, y) > \psi(x', y')$  but  $\psi_\ell(x, y) \leq \psi_\ell(x', y')$ . Let  $s_1, s_2, s_3$  be disjoint events with probability  $\Pr[s_1] = \Pr[s_2] = \delta$  and  $\Pr[s_3] = 1 - 2\delta$ . Let

$$\begin{aligned} \hat{X} &= (x, s_1; x', s_2; z, s_3) \\ \hat{Y} &= (y, s_1; y', s_2; z, s_3) \end{aligned}$$

Choose  $\delta$  small enough that  $\|\hat{X} - \hat{Y}\| < \varepsilon$ . Then (8) implies that  $\hat{X} \preceq \hat{Y}$  while an evaluation of regret with  $V(\Psi(X, Y))$  implies that  $\hat{X} \succ \hat{Y}$ . Contradiction.

Thus,  $\succeq$  is not regret based. ■

## References

- [1] Allais, M., 1953. “Le comportement de l’homme rationnel devant le risque: Critique des postulats et axiomes de l’école Americaine,” *Econometrica* 21:503–546.
- [2] Bell, D., 1982. “Regret in decision making,” *Operations Research* 30:961–981.
- [3] Bikhchandani, S. and U. Segal, 2010. “Transitive regret,” *Theoretical Economics* 6:95–108.
- [4] Bikhchandani, S. and U. Segal, 2014. “Transitive regret over statistically independent lotteries,” *Journal of Economic Theory* 152, 237–248.

- [5] Birnbaum, M.H. and U. Schmidt, 2008. “An experimental investigation of violations of transitivity in choice under uncertainty,” *Journal of Risk and Uncertainty* 37:77–91.
- [6] Diecidue, E. and J. Somasundaram, 2017. “Regret theory: A new foundation,” *Journal of Economic Theory* 172:88–119.
- [7] Fishburn, P.C., 1989. “Non-transitive measurable utility for decision under uncertainty,” *Journal of Mathematical Economics* 18:187–207.
- [8] Grether, D. and Plott, C., 1979. “Economic theory of choice and the preference reversal phenomenon,” *American Economic Review* 69:623–638.
- [9] Kahneman, D. and A. Tversky, 1979. “Prospect theory: An analysis of decision under risk,” *Econometrica* 47:263–291.
- [10] Lichtenstein, S. and Slovic, P., 1971. “Reversal of Preference Between Bids and Choices in Gambling Decisions,” *Journal of Experimental Psychology* 89:46–55.
- [11] Loomes, G. and R. Sugden, 1992. “Regret theory: An alternative theory of rational choice under uncertainty,” *Economic Journal* 92:805–824.
- [12] Loomes, G., Starmer, C., and Sugden, R., 1991. “Observing Violations of Transitivity by Experimental Methods,” *Econometrica* 59:425–439.
- [13] MacCrimmon, K.R., and S. Larsson, 1979. “Utility Theory: Axioms Versus ‘Paradoxes’,” in M. Allais and O. Hagen, Eds.: *Expected Utility Hypotheses and the Allais Paradox*. Dordrecht: D. Reidel.
- [14] Machina, M.J., 1982. “‘Expected utility’ analysis without the independence axiom,” *Econometrica* 50:277–323.
- [15] Mueller, D.C., 2003. *Public Choice III*. Cambridge University Press.
- [16] Nishimura, H., 2018. “The transitive core: inference of welfare from nontransitive preference relations,” *Theoretical Economics* 13:579–606.
- [17] Regenwetter, M., J. Dana, and C.P. Davis-Stober, 2011. “Transitivity of preferences,” *Psychological Review* 118:42–56.

- [18] Starmer, C., 2000. “Developments in non-expected utility theory: The hunt for a descriptive theory of choice under risk,” *Journal of Economic Literature* 38:332–382.
- [19] Tserenjigmid, G., 2015. “Theory of decisions by intra-dimensional comparisons,” *Journal of Economic Theory* 159:326–338.