Baskets, Staircases and Sutured Khovanov Homology

Author: Ian Matthew Banfield

Boston College Electronic Thesis or Dissertation, 2017

BASKETS, STAIRCASES AND SUTURED KHOVANOV HOMOLOGY

Ian Matthew Banfield

A dissertation

submitted to the Faculty of

the department of Mathematics

in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

Boston College
Morrissey College of Arts and Sciences
Graduate School

July 2017

BASKETS, STAIRCASES AND SUTURED KHOVANOV HOMOLOGY

Ian Matthew Banfield

Advisor: Julia Elisenda Grigsby, PhD

We use the Birman-Ko-Lee presentation of the braid group to show that all closures of strongly quasipositive braids whose normal form contains a positive power of the dual Garside element δ are fibered. We classify links which admit such a braid representative in geometric terms as boundaries of plumbings of positive Hopf bands to a disk. Rudolph constructed fibered strongly quasipositive links as closures of positive words on certain generating sets of B_n and we prove that Rudolph's condition is equivalent to ours. We compute the sutured Khovanov homology groups of positive braid closures in homological degrees $i = 0,1$ as $\mathfrak{sl}_2(\mathbb{C})$-modules. Given a condition on the sutured Khovanov homology of strongly quasipositive braids, we show that the sutured Khovanov homology of the closure of strongly quasipositive braids whose normal form contains a positive power of the dual Garside element agrees with that of positive braid closures in homological degrees $i \leq 1$ and show this holds for the class of such braids on three strands.

Contents

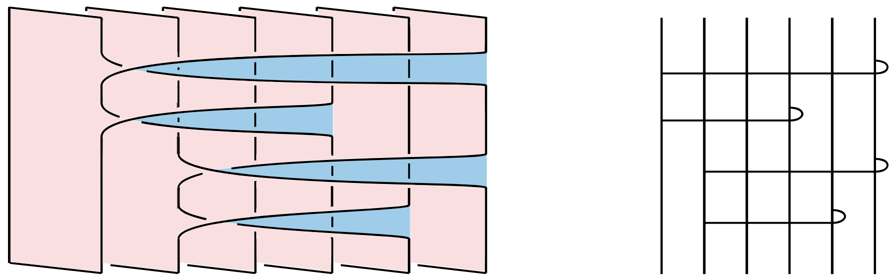
List of Figures

Acknowledgements

The author is deeply grateful to his advisor Eli Grigsby for her inspiration and support, and her thoughtful suggestions. The author thanks Sebastian Baader, Peter Feller, Tao Li and Filep Misev for helpful conversations and their encouragement. Section 2.4 was inspired by a conversation and subsequent email exchange with Lee Rudolph after meeting him at the Boston Topology Graduate Student conference at MIT in October 2016. The author would like to thank Peter Feller for pointing out a mistake in the original proof of Corollary 2.9. Lastly, the author appreciates the many helpful comments and detailed reading of the referee.

Dedication

To my wife, Sarah.



(A) Bennequin surface Σ_w (B) Charged fence diagram

FIGURE 1. The closure of the strongly quasipositive braid $\beta = a_{6,1}a_{6,3}a_{5,1}a_{5,2}$ as the boundary of a quasipositive surface.

1. Introduction

Strongly quasipositive links, i.e. links that are representable as closures of strongly quasipositive braids, are an interesting class of links first studied by Rudolph, cf. [Rud98]. Geometrically, they can be described as boundaries of quasipositive surfaces in S^3, see Figure 1a.

It is known that the slice Bennequin inequality [Rud93] is sharp for quasipositive links in general; indeed for strongly quasipositive links the Bennequin surface is actually embedded in S^3, which implies that the slice genus equals the genus of the link [KM93, Rud93]. This can be used to combinatorially compute the genus of a strongly quasipositive link from any strongly quasipositive braid representation. It also implies that the only slice strongly quasipositive knot is the unknot. There are connections to invariants coming from knot homology theories; e.g. the slice genus bounds from Rasmussen's s-invariant in Khovanov homology and the τ-invariant in knot Floer homology are sharp [Shu07].

Particularly interesting are fibered strongly quasipositive links: By work of Hedden and Rudolph, it is known that these are exactly the fibered links that induce the unique tight contact structure on S^3 [Hed10]. Knot Floer homology detects if a link is fibered [Ni07], and further if a fibered link can be represented as a strongly quasipositive braid closure [Hed10].

We will show that "almost all" strongly quasipositive braid closures are fibered. Our proof uses a presentation of the braid group due to Birman-Ko-Lee, associated to the so-called dual Garside structure on the braid group.

Khovanov homology is a homological invariant for a link in $K \in S^3$, defined combinatorially using resolutions of a diagram of K. In [APS04], Asaeda-Przytycki-Sikora extended Khovanov's theory to links in I-bundles over surfaces. For links in the thickened annulus, Roberts [Rob13] defined an "axis filtration" on the Khovanov chain complex, whose associated graded complex recovers the

Asaeda-Przytycki-Sikora theory. This theory, known as "sutured annular Khovanov homology" [1] is particularly suited to studying braids and braid closures.

These invariants possess a richness of algebraic structures acting on the complex and its homology. The filtration naturally gives rise to a spectral sequence converging to Khovanov homology. Grigsby [ELW15] discovered a $\mathfrak{sl}_2(\mathbb{C})$ action, and Grigsby-Licata-Wehrli in [ELW15] explain how the homology can be given a Lie super-algebra structure. This allows us to make use of the beautiful structure of semi-simple Lie algebra representations over the complex numbers.

Lemma 2.7. *Let $\beta = \delta P$, where $\delta = \sigma_{n-1}\sigma_{n-2}\ldots\sigma_1$ is the dual Garside element and P is a BKL-positive word. Then the braid closure $\hat{\beta}$ is fibered.*

See Section 2.1 for the definition of a BKL-positive word and the dual Garside normal form. We prove the following theorem.

Theorem 2.8. *A link L which can be represented as the closure of a braid whose normal form contains a positive power of the dual Garside element is fibered.*

In Section 2.4 we classify which fibered strongly quasipositive braids arise from our construction and relate the condition that a link L is represented as a braid closure whose normal form contains the dual Garside element to a construction of fibered links due to Rudolph [Rud01].

Theorem 2.11. *A link L is the boundary of a plumbing of positive Hopf bands to a disk D along arcs $\alpha_i \subset D$ if and only if L admits a strongly quasipositive representative $\beta \in B_n$ which contains the dual Garside element δ.*

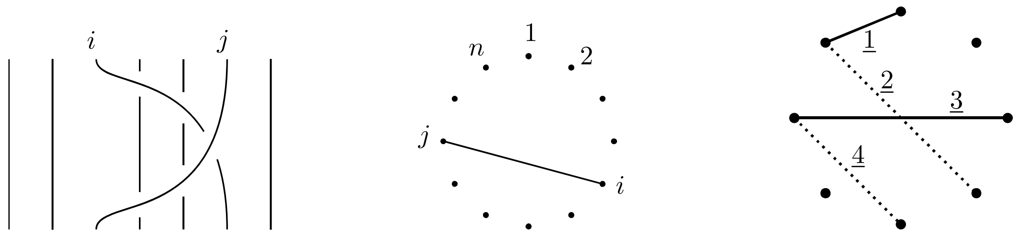
In particular, by appealing to a result of Rudolph, we show that positive braid closures are of this form.

Theorem 2.21. *A non-split positive braid link L is the closure of a strongly quasipositive braid $\beta \in B_n$ whose normal form contains a positive power of the dual Garside element.*

Further, for positive braids and certain 3-braids we show the following structure theorems for the sutured Khovanov homology.

---

[1]In [EW09], Grigsby-Wehrli discovered a connection with sutured Floer homology, giving rise to the title "sutured Khovanov homology".



(A) Braid presentation (B) Circle presentation (C) $\beta = a_{8,1}a_{8,4}^{-1}a_{7,3}a_{7,5}^{-1}$

Figure 2. The generators $a_{i,j}$. For convenience, define $a_{i,j} = a_{j,i}$. The underlined number next to a line indicates the position in the braid word.

Theorem 3.12. *Let $\beta \in B_n$ be a positive non-split braid. The sutured Khovanov homology of the braid closure $\widehat{\beta}$ in homological degree $i = 0$ and $i = 1$ is*

$$\mathrm{Skh}_0(\widehat{\beta}) \simeq \mathrm{Sym}^n(V)$$

$$\mathrm{Skh}_1(\widehat{\beta}) \simeq \mathrm{Sym}^{n-2}(V).$$

Corollary 3.27. *Let $\beta \in B_3$ be a strongly quasipositive 3-braid whose normal form contains a positive power of the dual Garside element $\delta = \sigma_{n-1}\sigma_{n-2}\dots\sigma_1$. Then the sutured Khovanov homlogy groups of the closure $\widehat{\beta}$ in homological degrees $i < 0$, $i = 0$ and $i = 1$ are*

$$\mathrm{Skh}_i(\widehat{\beta}) = 0 \textit{ for } i < 0,$$

$$\mathrm{Skh}_0(\widehat{\beta}) \simeq \mathrm{Sym}^n(V),$$

$$\mathrm{Skh}_1(\widehat{\beta}) \simeq \mathrm{Sym}^{n-2}(V).$$

2. Strongy quasipositive braids, baskets and staircases

2.1. **The dual Garside structure of the braid group.** In [BKL98], Birman, Ko and Lee gave a solution to the word-problem in the braid group using a new presentation of B_n. The generators $a_{i,j}$ in this presentation correspond to pairs of strands (see Figure 2), called band generators or Birman-Ko-Lee generators, and the relations correspond to Reidemeister moves of type II and III.

There is a nice pictorial way to describe the generators and relators, using dots labeled $1, 2, \dots, n$ arranged in a circle [IW15]. The generator $a_{i,j}$ is represented by a straight line between the dots labeled i, j, as pictured in Figure 2b, and its inverse by a dashed line. An element of B_n is represented by a sequence of such lines, see Figure 2c. The Birman-Ko-Lee relations can be stated as follows.

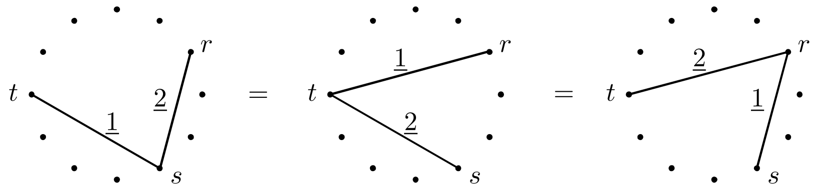


FIGURE 3. Going counter-clockwise, any two consecutive boundaries of the triangle with vertices at r, s, t are equivalent in B_n, i.e. $a_{t,s}a_{s,r} = a_{t,r}a_{t,s} = a_{s,r}a_{t,r}$.

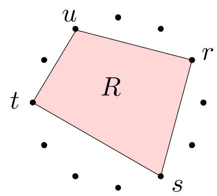


FIGURE 4. The representations of the braid β_R associated to the 4-gon R arising from cyclic counterclockwise orders of its vertices.
$\beta_R = a_{u,r}a_{t,u}a_{t,s} = a_{u,t}a_{t,s}a_{s,r} = a_{t,s}a_{s,r}a_{u,r} = a_{s,r}a_{u,r}a_{t,u}$.

- Consecutive lines in the sequence representing the braid word which do not intersect can be re-ordered arbitrarily.
- ("Cup Product") Consider a triangle with vertices at the dots labeled r, s, t. Moving counter-clockwise, any two consecutive boundaries of the triangle are equivalent. See Figure 3.

The second relation implies that one can associate to a polygon R a well-defined element $\beta_R \in B_n$ as follows: Let the vertices of R be at $(q_1, \ldots, q_r)$ listed in any cyclic counterclockwise order and let $\beta_R = a_{q_r,q_{r-1}} \ldots a_{q_2,q_1}$. See Figure 4 for an example when R is a 4-gon.

A braid β is **BKL-positive** if it can be written as a word in only positive powers of the generators $a_{i,j}$. BKL-positive braids form a monoid, denoted by B_n^+, and Birman-Ko-Lee proved that two BKL-positive elements are equivalent in B_n if and only if they are equivalent in B_n^+.

Definition 2.1 ([BKL98])**.** *The dual Garside element is* $\delta = a_{n,n-1}a_{n-1,n-2} \ldots a_{2,1}$.

The dual Garside element is called the *fundamental braid* in [BKL98]. As the name implies, δ is in fact a Garside element for a Garside structure on B_n. In the circle presentation, δ corresponds to the n-gon spanned by all vertices.

Definition 2.2. *The starting set $S(\beta)$ of a BKL-positive braid word β is the set of generators $a_{i,j}$ that can appear at the start of a BKL-positive word representing β. Similarly, the finishing set $F(\beta)$ is the set of generators $a_{i,j}$ that can appear at the end of β.*

We will need the following facts.

Fact 2.3 ([BKL98], Corollary 3.7). *The finishing and starting sets of the dual Garside Element* δ *are the set of all generators, i.e.* $F(\delta) = \{a_{j,i} \mid i, j \in \{1, \dots n\}, i \neq j\}$.

Fact 2.4 (Dual Garside Normal Form, [BKL98], Theorem 3.10). *Every element* $\beta \in B_n$ *can be uniquely written as* $\beta = \delta^k A_1 \dots A_m$*, where* δ *is the dual Garside element, all* A_i *are BKL-positive and* $A_i s$ *is not a canonical factor for any* $s \in S(A_{i+1})$.

The A_i in the previous theorem are called canonical factors and correspond to disjoint union of polygons in the circle presentation. The condition that $A_i s$ is not a canonical factor for any $s \in S(A_{i+1})$ ensures uniqueness of the normal form and is denoted by $A_i \lceil A_{i+1}$.

A braid is strongly quasipositive if and only if the power of the dual Garside element δ in the normal form is non-negative. A link is strongly quasipositive if it admits a braid representative with a non-negative power of δ.

2.2. **Detecting fibered braids.** A link L is fibered with fiber F if $\partial F = L$ and its exterior $S^3 \setminus \nu(L)$ fibers over the circle such that F is a fiber. This is a strong condition; it implies that all fiber surfaces are isotopic and minimal genus. In fact, for Seifert surfaces F of a fibered link the following are equivalent: F is a fiber surface, F is genus-minimizing and F is incompressible [Kaw96].

In [Gab86], Gabai established an algorithm to detect if a link $L \subset S^3$ is fibered. The idea is as follows: Let F be a minimal genus Seifert surface for L. The link L is fibered if and only if the complementary sutured manifold $(Y, \gamma) = (S^3 \setminus (F \times I), \partial F)$ associated to F is a product sutured manifold, that is, $(Y, \gamma) \simeq (\Sigma \times I, \partial \Sigma)$ for a surface Σ. Gabai proves the following theorem [Gab86].

Fact 2.5 ([Gab86], Theorem 1.9). *The link* $L \subset S^3$ *is fibered with fiber surface* F *if and only if there exists a sequence of product disk decompositions of* $(Y, \gamma) = (S^3 \setminus (F \times I), \partial F)$ *that terminates in the trivial sutured manifold* (B^3, α)*, where* α *is a single curve on* ∂B^3.

We will apply this theorem to a braid closure $\hat{\beta}$ of a braid word expressed in the Birman-Ko-Lee generators. Recall from Section 2.1 that we can express a braid $\beta = w$ as a word w in the Birman-Ko-Lee generators. For such a word w, we can construct a canonical Seifert surface, Σ_w, called the Bennequin surface, as follows.

- Draw n parallel disks, one for every strand of the braid β.

- For each generator $a_{j,i}$ (inverse of $a_{j,i}$, resp.) attach a positively (negatively, resp.) twisted band between the disks corresponding to strands i and j, going over all other disks.

See Figure 1a for an example of the Bennequin surface for $\beta = a_{6,1}a_{6,3}a_{5,1}a_{5,2}$. For strongly quasipositive braid closures, the Bennequin surface is minimal genus among all smoothly embedded surfaces in B^4 bounded by the oriented link; this follows from Rudolph's proof of the slice Bennequin inequality [Rud93], which in turn relies on Kronheimer-Mrowka's proof of the local Thom conjecture. [KM93]

Lemma 2.6 (Cancellation)**.** *Suppose a braid word w contains a term $a_{i,j}^2$, and let w' be the word obtained from w by replacing the $a_{i,j}^2$ with $a_{i,j}$ (i.e. a square of a generator is replaced with just the generator). Then the complementary sutured manifold $(Y_w, \gamma_w) = (S^3 \setminus (\Sigma_w \times I), \partial\Sigma_w)$ is a product if and only if $(Y_{w'}, \gamma_{w'}) = (S^3 \setminus (\Sigma_{w'} \times I), \partial\Sigma_{w'})$ is a product.*

Proof. According to Gabai ([Gab86], Lemma 2.2), if $(Y, \gamma) \rightsquigarrow (Y', \gamma')$ is a product disk decomposition, then (Y, γ) is a product if and only if (Y', γ') is a product. The lemma will follow immediately once we show that $(Y_{w'}, \gamma_{w'})$ is obtained from (Y_w, γ_w) by a product disk decomposition. See figures 5a, 5b, 5c for the proof. □

Lemma 2.6 and Theorem 2.8 were inspired by Ni's work on fibered 3-braids [Ni09]. The operation of canceling repeated generators of B_3 was called "Untwisting" in Ni's work.

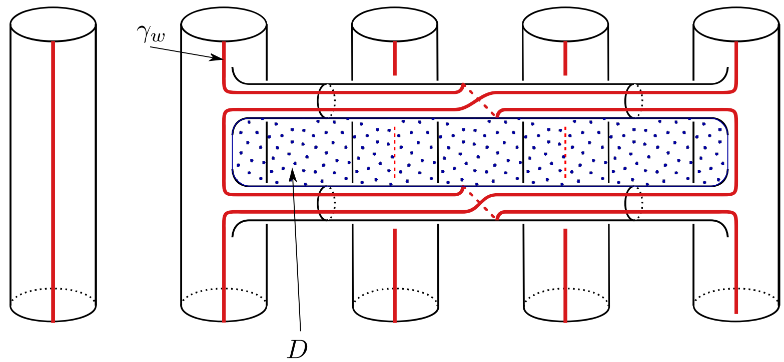
For strongly quasipositive braid closures the Bennequin surface is minimal genus and hence a fiber surface if the braid closure is fibered. This lemma then immediately implies that the closure of $\beta = w$ is fibered if and only if the closure of β' is fibered, meaning we may cancel powers of generators in deciding if strongly quasipositive braid closures are fibered.

2.3. **Strongly quasipositive fibered braid closures.**

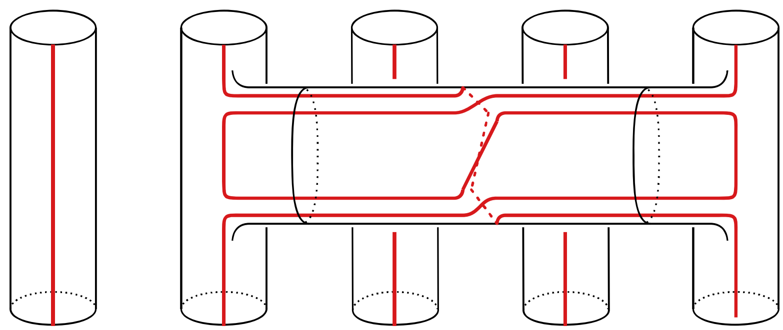
Lemma 2.7. *Let $\beta = \delta P$, where $\delta = \sigma_{n-1}\sigma_{n-2}\ldots\sigma_1$ is the dual Garside element and P is a BKL positive word. Then the braid closure $\hat{\beta}$ is fibered.*

Proof. We use induction on the word length $L = ln(P)$ of the BKL-positive word P in the BKL generators. If $L = 0$, then $\beta = \delta^k$ is a non-split Artin-positive braid, whose closure is fibered by a result of Stallings [Sta78].

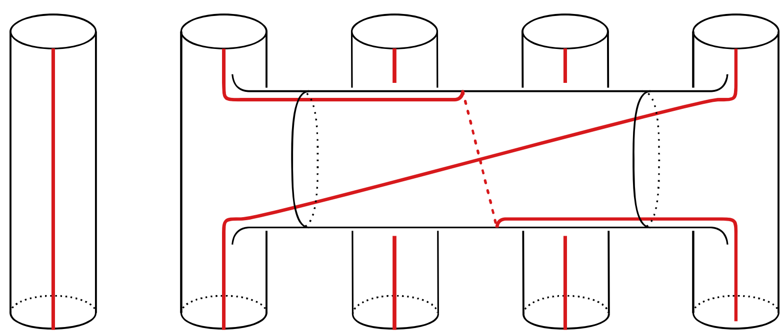
Otherwise, if $L > 0$, write $P = a_{s,t}P'$ for some generator $a_{s,t} \in S(P)$ and a BKL-positive word P'. Since $a_{s,t}$ is in the finishing set $F(\delta)$ of the dual Garside element δ by Fact 2.3, we can write



(A) The piece of (Y_w, γ_w) corresponding to $a_{i,j}^2$. The dotted blue disk D is the product disk and the solid red arcs are part of the sutures γ_w.



(B) Applying the product disk decomposition to (Y_w, γ_w).



(C) An isotopy of the suture from Figure 5b shows that the decomposed sutured manifold is $(Y_{w'}, \gamma_{w'})$.

Figure 5. Proof of Lemma 2.6

$\delta = P''a_{s,t}$, where P'' is a BKL-positive word. Then

$$\begin{aligned}\beta &= \delta^k P \\ &= \delta^{k-1}(P''a_{r,s})(a_{r,s}P') \\ &= \delta^{k-1}(P''(a_{r,s}a_{r,s})P').\end{aligned}$$

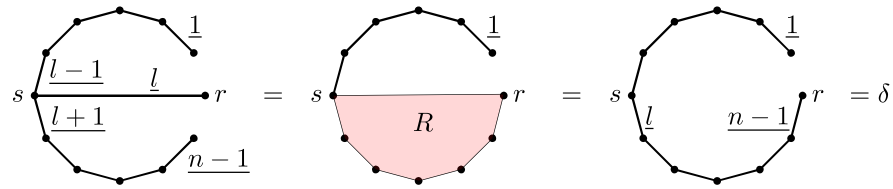


FIGURE 6. We rewrite the braid β_R corresponding to the polygon R as $\beta_R = a_{s,s-1} \dots a_{r+1,r}$, where $l = s - r \pmod n$.

Canceling the repeated generator $a_{r,s}$, we obtain the braid $\beta' = (\delta^{k-1} P'' a_{r,s} P') = \delta^k P'$. Note that $\ln(\beta') = L - 1 < L$, and by induction we conclude that the braid closure $\hat{\beta}$ is fibered. By Fact 2.5 and Lemma 2.6, if $\hat{\beta}'$ is fibered, then $\hat{\beta}$ is also fibered. □

Theorem 2.8. *Let $\beta \in B_n$ be strongly quasipositive and let $\beta = \delta^k A_1 \dots A_m$ be its normal form. If $k \geq 1$, then the braid closure $\hat{\beta}$ is fibered.*

Proof. Apply Lemma 2.7 to $\beta = \delta P$ with $P = A_1 \dots A_m$. □

The converse of Theorem 2.8 is not true. In Section 2.4 we explain a geometric classification of braid closures whose normal form contains a positive power of the dual Garside element, and use this in Corollary 2.12 to show that the $(2, 1)$ cable of the trefoil is a fibered strongly quasipositive braid closure whose normal form does not contain a positive power of the dual Garside element.

The following corollary is known for braids on 3 strands. [Ni09, Sto06]

Corollary 2.9. *After adding at most $n-2$ crossings to $\beta \in B_n$, every non-split strongly quasipositive braid closure $\hat{\beta}$ on n strands becomes fibered.*

Proof. By the previous theorem, we may assume that the normal form is
$\beta = A_1 \dots A_m$. Let $A_m = P a_{s,r}$ for a BKL-positive word P and a generator $a_{s,r} \in F(A_m)$. Consider the strongly quasipositive braid β' obtained from β by adding $n-2$ crossings (all subscripts are mod n):

$$\begin{aligned} \beta' &= A_1 \dots A_{m-1} P (a_{r-2,r-1} a_{r-3,r-2} \dots a_{s+1,s}) a_{s,r} (a_{s,s-1} a_{s-1,s-2} \dots a_{r+2,r+1}) \\ &= A_1 \dots A_{m-1} P \delta \text{ (see Figure 6)} \end{aligned}$$

The braid β' is conjugate to $\delta A_1 \dots A_{m-1} P$ and hence its closure $\hat{\beta}'$ is fibered by Theorem 2.8. □

Theorem 2.8 has the following probabilistic interpretation: Fixing the number of strands, the probability that a randomly generated strongly quasipositive braid word will contain δ, or in fact

(A) $G(\mathcal{T}) = \{a_{2,1}, a_{4,2}, a_{5,2}, a_{4,3}, a_{6,5}\}$ (B) $G(\mathcal{Y}_6) = \{a_{2,1}, a_{3,1}, a_{4,1}, a_{5,1}, a_{6,1}\}$

FIGURE 7. Espaliers and associated generating sets.

any subword of fixed length, approaches 1 as the word length increases to ∞. This justifies our claim that almost all strongly quasipositive braid closures are fibered.

The above observations may lead the reader to conclude that *all* strongly quasipositive braids are fibered, but this is not true. In fact, on the class of strongly quasipositive links, all Seifert forms are realized, so the leading coefficient of the Alexander polynomial Δ_L may be arbitrary! [Rud05]

In particular, if Δ_L is not monic, then the link L is not fibered. An example of a non-fibered strongly quasipositive braid closure is given by $\beta_3 = a_{3,1}a_{4,2}a_{3,1}a_{4,2}$. The Bennequin surface Σ_{β_3} consists of two once-linked Hopf annuli. The complement of a thickening of Σ_{β_3} is the Hopf-link exterior, which implies that the closure $\hat{\beta}_3$ is not fibered as the Hopf-link exterior is not a handlebody.

2.4. **Hopf-plumbed baskets.** In [Rud01], Rudolph constructs fibered links that arise as closures of certain homogeneous braids $\beta \in B_n$, generalizing earlier work of Stallings on closures of homogeneous braids in the Artin generators [Sta78]. Rudolph constructs generating set $G(\mathcal{T})$ of B_n as follows:

- Let $\mathcal{T}$ be a tree with n vertices and $n-1$ edges. Embed the tree into $\mathbb{C}$ with vertices at $1, 2, \dots n \in \mathbb{R} \subset \mathbb{C}$ and edges in the lower-half plane. Note that the assumptions imply that every vertex $1, \dots, n$ is the endpoint of at least one edge. These trees are called espaliers.
- To an edge $e \in E(\mathcal{T})$ of $\mathcal{T}$ with endpoints at the vertices r, s associate the BKL-generator $a(e) = a_{s,r} \in B_n$.
- Let $G(\mathcal{T}) = \{a(e) \mid e \in E(\mathcal{T})\}$ be the set of $\mathcal{T}$ generators.

See Figure 7 for examples of espaliers and their generating sets. The espalier in Figure 7b is maximal [2] among espaliers on n vertices.

Let $H = \langle S|R \rangle$ be a group presentation with generating set S and relations R. A word w is **homogeneous** (**positive**) with respect to the generating set S if every generator $s \in S$ occurs in w with either only positive or only negative (only positive) powers. A word β in the generating set $G(\mathcal{T})$ associated to a tree $\mathcal{T}$ is called a **$\mathcal{T}$-bandword**. A **strict** $\mathcal{T}$-bandword is a homogeneous $\mathcal{T}$-bandword β such that every generator $g \in G(\mathcal{T})$ occurs in β.

[2] $\mathcal{Y}_n$ was denoted $\mathcal{Y}_X$ (for $X = \{1, 2, \dots, n\}$) in [Rud01]. The definition of the partial order is given in Chapter 6 of Rudolph's paper [Rud01].

Definition 2.10 ([Rud01]). *A surface S is a Hopf-plumbed basket if S is a plumbing of Hopf bands along arcs $\alpha_i \subset D$, all of which lie in D. A (+) Hopf-plumbed basket is a Hopf-plumbed basket such that all plumbands are positive Hopf bands.*

Theorem 2.11. *A link L is the boundary of (+) Hopf plumbed basket if and only if L admits a strongly quasipositive representative $\beta \in B_n$ which contains the dual Garside element δ.*

In particular, this implies that the converse to Theorem 2.8 is not true. The author would like to thank Sebastian Baader for providing the following counterexample.

Corollary 2.12. *There exist fibered strongly quasipositive braid closures which do not contain the dual Garside element δ.*

Proof. Let L be the $(2,1)$ cable of the trefoil, and let F be the fiber surface for L. By work of Hedden [Hed08] and earlier work of Melvin-Morton [MM86], F is quasipositive but does not deplum a Hopf band. By definition, Hopf-plumbed baskets always deplum Hopf bands, so F can not be a (+) Hopf-plumbed basket. Theorem 2.11 then implies the boundary $L = \partial F$ does not admit a strongly quasipositive representative which contains the dual Garside element δ. □

Lemma 2.13. *Let*

$$\beta = a_{r_1,1} a_{r_2,1} \dots a_{r_M,1} \in B_n$$

be a positive $\mathcal{Y}_n$-bandword of word length M. Consider the sequence $(r_i) = (r_1, r_2, \dots, r_M)$. If there exists $1 \leq k < n$ and $1 \leq L \leq P \leq U \leq M$ such that

- *$r_L = 1$, $r_P = k$, $r_U = n$,*
- *for all i such that $1 \leq i \leq k$, there exists $Q(i)$ satisfying $L \leq Q(i) \leq P$ such that $r_{Q(i)} = i$ and further, $Q(i) < T \leq P$ implies that $r_T > r_{Q(i)}$,*
- *for all i such that $k \leq i \leq n$, there exists $Q(i)$ satisfying $P \leq Q(i) \leq U$ such that $r_{Q(i)} = i$ and further, $P \leq T < Q(i)$ implies that $r_T < r_{Q(i)}$,*

then β contains the dual Garside element δ.

Proof. Informally, the idea of the proof is to slide $a_{i,1}$ "down" for $i < k$, and slide $a_{j,1}$ "up" for $j > k$ so they form δ. We introduce the following notation: Let $w(i,j)$ be the subword of w from position i to position j.[3] Now let w be a positive $\mathcal{T}$-bandword for β satisfying the statement of the lemma.

[3]For example, take $z = a_{2,1} a_{6,3} a_{4,1} a_{4,2}$. Then $z(2,3) = a_{6,3} a_{4,1}$.

We will prove that the subword $w(Q(1), Q(n)$ of β contains δ. For convenience, define $Q(k) = P$. For $s \in S = \{r_{Q(i)}, \ldots, r_{Q(i+1)-1}\}$, the relations in the braid group imply that $a_{i,1}a_{s,1} = a_{i,s}a_{i,1}$ as $s > i$ by our assumptions. Consider

$$\begin{aligned} a_{i,1}w(Q(i), Q(i+1)-1) &= a_{i,1}a_{r_{Q(i)},1}a_{r_{Q(i)+1},1}\ldots a_{r_{Q(i+1)-1},1} \\ &= \underbrace{a_{r_{Q(i)},i}a_{r_{Q(i)+1},i}\ldots a_{r_{Q(i+1)}-1,i}}_{w_i'}a_{i,1}, \end{aligned}$$

and note that w_i' commutes with $a_{j,1}$ for $j < i$. We apply this equation in the next step:

$$\begin{aligned} w(Q(1), Q(k)-1) &= \prod_{i}^{k-1} a_{i,1}w(Q(i), w(Q(i+1)-1) \\ &= \prod_{i}^{k-1} w_i'a_{i,1} \\ &= w_1'w_2'\ldots w_{k-1}'a_{1,1}a_{2,1}\ldots a_{k-1,1} \\ &= T_La_{1,1}a_{2,1}\ldots a_{k-1,1}, \end{aligned}$$

for a BKL-positive word T_L. Moreover, a similar reasoning shows that for some BKL-positive word T_U,

$$w(Q(k)+1, Q(n)) = a_{k+1,1}a_{k+1,1}\ldots a_{n,1}T_U.$$

To finish the proof, we note that the dual Garside element δ can be written as $\delta = a_{1,1}a_{2,1}a_{3,1}\ldots a_{n,1}$. We can now show that the subword $w(Q(1), Q(n)$ of β contains δ:

$$\begin{aligned} w(Q(1), Q(n)) &= w(Q(1), Q(k)-1)Q(k)w(Q(k)+1, Q(n)) \\ &= T_La_{1,1}a_{2,1}\ldots a_{k-1,1}a_{k,1}a_{k+1,1}\ldots a_{n,1}T_U \\ &= T_L\delta T_U. \end{aligned}$$

Remark The converse is true: If β contains the dual Garside element δ then there is a word w for β which satisfies the assumptions of the lemma. □

For the proof of Theorem 2.11, it will be convenient to use so-called charged fence diagrams, which represent quasipositive surfaces S and define a braid word for the strongly quasipositive link

∂S. See figures 1b and 11d for examples of charged fence diagrams, and [Rud98] for a thorough discussion of quasipositive surfaces and charged fence diagrams.

Lemma 2.14. *Let $\beta = a_{s_1,r_1} \dots a_{s_k,r_k} \delta \in B_n$ be a braid which contains the dual Garside element δ. Then the fiber surface for $\hat{\beta}$ is a (+) Hopf-plumbed basket.*

Proof. We construct a (+) Hopf-plumbed basket whose boundary is $\hat{\beta}$. Let

$$D = (D_1 \cup \dots \cup D_n) \cup (T_{\sigma_1} \cup \dots \cup T_{\sigma_{n-1}}),$$

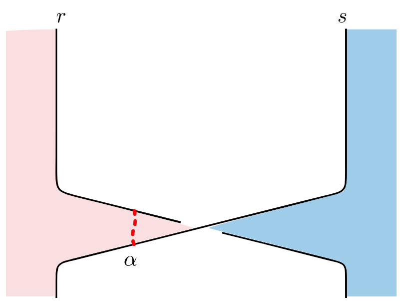
be the Bennequin surface for the subword $\delta = \sigma_{n-1} \dots \sigma_1$ of β, where D_i are the disks corresponding to the strands and T_i are the positively twisted bands corresponding to the generators $\sigma_i = a_{i-1,i}$. Using fact 2.3, write $\delta = a_{s_1,r_1} P$ for a BKL-positive word P and isotope D accordingly.

Consider the surface $D \star_{\alpha_1} A_+$ given by plumbing a positive Hopf band A_+ to this disk D along the arc $\alpha_1 = \alpha$ in Figure 8a. The surface $D \star_{\alpha_1} A_+$ is pictured in Figure 8b, and the sequence of isotopies in figures 8c and 8d show that $D \star_{\alpha_1} A_+$ is the fiber surface for the closure of the braid $a_{s_1,r_1}\delta$. Informally, this plumbing "adds the generator $a_{s,r}$ to the braid word."

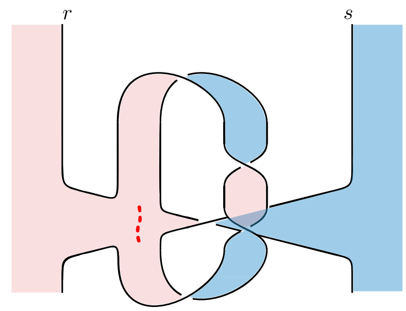
Now repeat this process by plumbing positive Hopf bands for the generators $(a_{s_2,r_2}), \dots, (a_{s_k,r_k})$. By construction, the fiber surface $((D \star_{\alpha_1} A_+) \cdots \star_{\alpha_k} A_+)$ is a (+) Hopf-plumbed basket bounding $\hat{\beta}$. □

Given a braid $\beta = a_{r_1,s_1} \dots a_{r_k,s_k} \delta \in B_n$ representing $L = \hat{\beta}$, the lemma provides an explicit construction of a (+) Hopf-plumbed basket $(D \star_{\alpha_1} A_+) \cdots \star_{\alpha_k} A_+$ whose boundary is the link L. The normal form of the representative $\beta \in B_n$ induces additional structure on D, which is pictured in Figure 9.

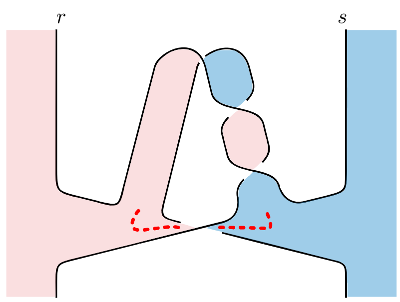
- The disk D is partitioned into disjoint disks D_i for $1 \le i \le n$ and T_j for $1 \le j < n$ as in Figure 9c. The disks D_1 and D_n are distinguished by noting that $\partial D \cap D_i$ has a single component for $i = 1$ and $i = n$ but two components for $1 < i < n$. There are two components in $\partial D \cap T_j$ for every $i \le j < n$.
- There exist disks $AT_i \subset D_i$ which satisfy (1) $\cup \partial \alpha_j \subset \cup \partial AT_i$ and (2) $AT_i \cap \partial D_i$ is connected, where $\{\alpha_j\}$ is an ordered set of compatible arcs (defined below). We refer to AT_i as an "attaching region." In figures 9b and 9c, the attaching regions AT_i are the dotted regions contained in D_i.



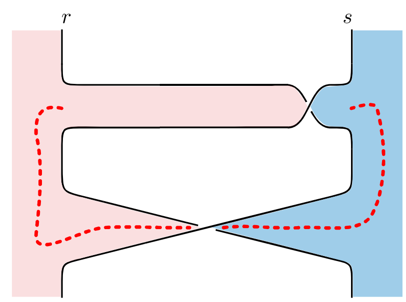
(A) A piece of the disk D and $\alpha \subset D$.



(B) Plumbing a positive Hopf along α.

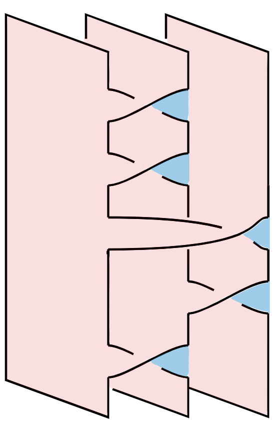


(C) Isotopy.

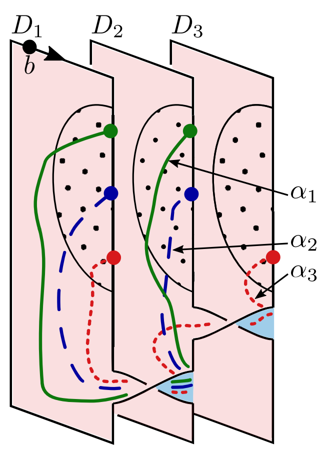


(D) Further isotopy.

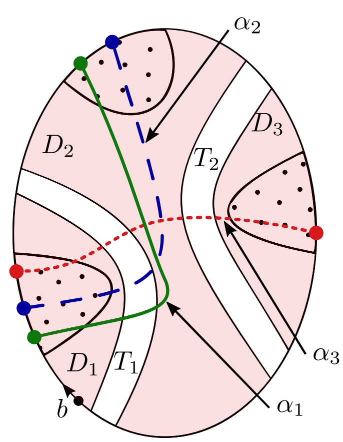
FIGURE 8. Plumbing a positive Hopf band along $\alpha \in D$ "adds the generator $a_{s,r}$."



(A) Fiber surface F bounding $\beta = a_{2,1}a_{2,1}a_{3,1}\delta$.



(B) $D = D_1 \cup D_2 \cup D_3 \cup T_1 \cup T_2$ and the plumbing arcs $\alpha_i \subset D$. $F = ((D \star_{\alpha_1} A_+) \star_{\alpha_2} A_+) \star_{\alpha_3} A_+$.



(C) Folding structure for D.

FIGURE 9

- Orient the arcs $\gamma_i = AT_i \cap \partial D_i$ as submanifolds of ∂D endowed with the boundary orientation. We say that an ordered set $\{\tau_j \subset D\}$ of properly embedded arcs whose endpoints are contained in $\cup_i \gamma_i$ is **compatible** if (1) $|\gamma_i \cap \partial \tau_j| < 2$ for all i, j; and (2) for $k > l$, if $a \in \partial \tau_k \cap \gamma_i$ and $b \in \partial \tau_l \cap \gamma_i$, then $a < b$ in the order on $\gamma_i \simeq (0, 1)$ induced by the orientation.

The construction in the proof of the lemma implies that the plumbing arcs $\{\alpha_i\}$ are compatible and that the disks D_i (T_j, respectively) correspond to the disks D_i (twisted bands for $\sigma_j = a_{j+1,j}$, respectively) in the Bennequin surface Σ_δ.

We call such a structure a **folding structure** and denote it by (D, D_i, α_j), where D_i are disks in a partition of a disk D as described above and $\{\alpha_j\}$ is an ordered set of compatible arcs. To justify the omission of T_j, AT_i and γ_i from the notation, we note that a partition satisfying the conditions is completely determined by the disks D_i, essentially by the smooth Jordan curve theorem. The attaching regions AT_i and the arcs $\gamma_i \subset \partial AT_i$ are defined in terms of the disks D_i and the ordered set of compatible arcs. A priori, (D, D_i, α_j) may not be arising from a braid $\beta \in B_n$ whose normal form contains δ through the construction described in the proof of Lemma 2.14, however, the following lemma asserts this is the case and justifies the term "folding structure."

Lemma 2.15. *Let (D, D_i, α_j) be a folding structure. Then there exists a braid $\beta \in B_n$ which contains the dual Garside δ such that the folding structure is induced by β.*

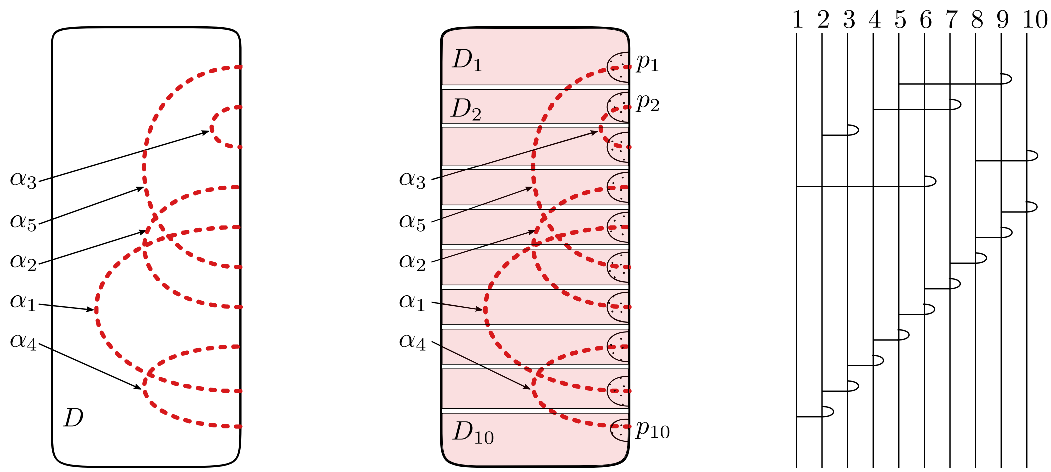
Proof. Let AT_i be the attaching regions for the folding structure and let $k = |\{\alpha_j\}|$ be the number of arcs and $n = |\{D_i\}|$ be the number of disks. Define $r(j)$ and $s(j)$ by $\partial\alpha_j \in AT_{r(j)} \cup AT_{s(j)}$. Let $\beta = a_{r(1),s(1)} \dots a_{r(k),s(k)}\delta \in B_n$. It is straightforward to verify that the folding structure induced by β is (D, D_i, α_j). □

Lemma 2.16. *The boundary ∂F of a (+) Hopf plumbed basket F admits a representative which contains the dual Garside element δ.*

Proof. The idea of the proof is to define a folding structure (D, D_i, α_j) and appeal to the previous lemma. We begin by noting that according to Section 3 of [Rud01], a Hopf-plumbed basket is completely described by the plumbing arcs $S = \{\alpha_j \subset D\}$ and the order in which the Hopf bands are plumbed. By re-indexing S, we may assume the Hopf bands are plumbed in increasing order of the plumbing arcs' indices.

Let $k = |S|$ be the number of arcs and let $p_1, p_2, \dots, p_{2k} \in \partial D$ be the endpoints of the arcs $\{\alpha_j\}$ in some cyclic order. Let $\tau \subset \partial D$ be the arc with endpoints at p_1 and p_{2k} which is disjoint from all other endpoints p_i. Choose points $q_2, \dots, q_{2k-1} \in \tau$ in increasing order on $\tau \simeq (0, 1)$.

We now define the disks D_i of the folding structure. For $i = 1$ and $i = 2k$, let D_i be a small disk containing p_i; and for $1 < i < 2k$, let D_i be a tubular neighborhood of an arc connecting p_i to q_i. This is illustrated in Figure 10b. The arcs $\{\alpha_j\}$ are compatible, as each attaching region contains



(A) Plumbing arcs $\{\alpha_j \subset D\}$ for F.

(B) Folding structure.

(C) The braid $\hat{\beta} = \partial F$ given by Lemma 2.15.

FIGURE 10. Proof of Lemma 2.16. A Hopf-plumbed basket F is described by an ordered collection of arcs $\{\alpha_j \subset D\}$. The attaching regions (dotted) for the folding structure in 10b contain a single endpoint each to make $\{\alpha_j\}$ compatible.

exactly one endpoint, which implies that the compatibility condition is vacuous. By Lemma 2.15, the folding structure (D, D_i, α_j) is induced by a braid $\beta \in B_{2k}$ which contains the dual Garside element δ. □

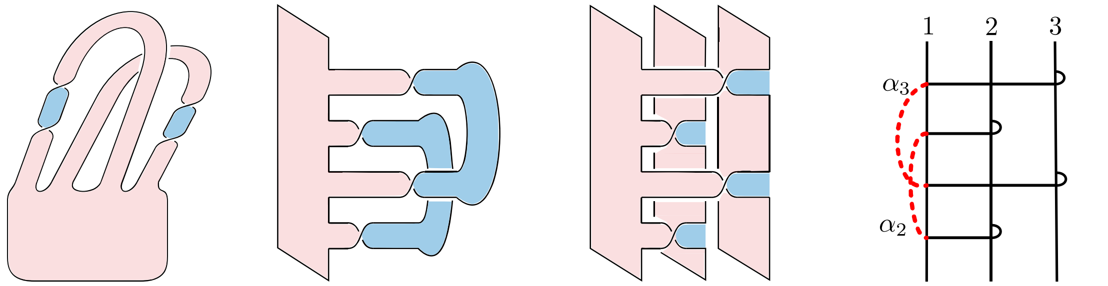
Corollary 2.17. *The boundary ∂F of a $(+)$ Hopf-plumbed basket F with plumbing arcs $\{\alpha_i\}$ is the closure of a strongly quasipositive braid $\beta = \gamma\delta$, where γ is an unlink on $k = |\{\alpha_i\}|$ components.*

Proof of Theorem 2.11. Immediate from Lemma 2.14 and Lemma 2.16. □

Theorem 2.18 ([Rud01], Theorems 6.1.6, 6.2.4)**.** *If S is a Hopf-plumbed basket, then there is an espalier $\mathcal{T}$ and a strict homogeneous $\mathcal{T}$-bandword β such that $\hat{\beta} = \partial S$. Conversely, the fiber surface of a strict homogeneous $\mathcal{T}$-bandword is a Hopf-plumbed basket.*

See Figure 11 for the idea of the proof in case of $\mathcal{Y}_n$-bandwords. It is straightforward to check that Rudolph's proof holds when one restricts to strict positive $\mathcal{T}$-bandwords, which are strongly quasipositive braids:

Theorem 2.19 ([Rud01])**.** *If S is a $(+)$ Hopf-plumbed basket, then there is an espalier $\mathcal{T}$ and a positive strict $\mathcal{T}$-bandword β such that $\hat{\beta} = \partial S$. Conversely, the fiber surface of a positive strict $\mathcal{T}$-bandword is a $(+)$ Hopf-plumbed basket.*



(A) F (B) Isotopy of F (C) Isotopy to a quasipositive surface (D) Charged fence diagram

FIGURE 11. The fiber surface F for the trefoil and a charged fence diagram for F. The positive Hopf band A_+ plumbed to $\alpha_i \in D$ is represented by the strand $i > 1$ together with the two horizontal lines connecting strand 1 to i and the dotted arc α_i.

Corollary 2.20. *A braid closure $\hat{\beta}$ can be represented by a positive strict $\mathcal{T}$-bandword if and only if it can be represented by a strongly quasipositive braid which contains the dual Garside element.*

Proof. Immediate from Theorem 2.11 and Theorem 2.19. □

Theorem 2.21. *A non-split positive braid closure L can be represented by a strongly quasipositive braid which contains a positive power of the dual Garside element δ.*

Proof. Let $\mathcal{T}_p$ [4] be the espalier with n vertices and with edges connecting i to $i+1$ for $1 \le i < n$. The set of generators $G(\mathcal{T}_p) = \{a_{2,1} = \sigma_1, \ldots, a_{n,n-1} = \sigma_{n-1}\}$ is the set of Artin generators of B_n, which shows that a non-split positive braid is a positive strict $\mathcal{T}_p$-bandword. Now apply Corollary 2.20. □

2.4.1. *Some examples and applications.*

Definition 2.22. *A staircase braid is a strongly quasipositive braid whose dual Garside normal form contains a positive power of the dual Garside element $\delta = \sigma_{n-1} \ldots \sigma_2 \sigma_1$.*

Lemma 2.23. *The link 10_{145} is a staircase braid closure, but not a positive braid link.*

Proof. A strongly quasipositive braid representative for 10_{145} is given by $\beta = a_{2,4}a_{2,3}a_{1,2}a_{2,4}^2a_{2,3}a_{1,2}$ [CL]. A sequence of braid relations and conjugacy moves is exhibited in Figure 12, showing that β is conjugate to a staircase braid. However, the knot 10_{145} is not a positive braid knot [CL]. □

[4]We follow Rudolph's notation in [Rud01] for the espalier corresponding to positive braids, hence the subscript "P"

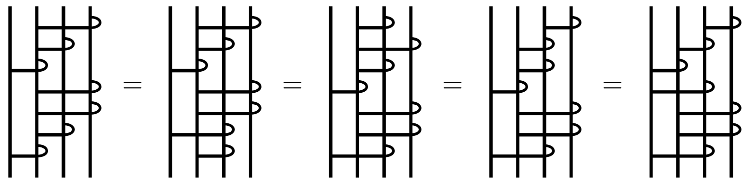


FIGURE 12. The braid $a_{2,4}a_{2,3}a_{1,2}a_{2,4}^2a_{2,3}a_{1,2}$ is conjugate to a staircase braid.

The previous lemma and Corollary 2.12 imply the following strict inclusions.

$$\{\text{Positive braids links}\} \subsetneq \{\text{Staircase braid closures}\} \subsetneq \{\text{Tight links}\}$$

The following application of our work was suggested by Lee Rudolph.

Corollary 2.24. *Staircase braid closures are stably positive braid links, that is, if F is the fiber surface of a staircase braid closure, then there exists an iterated positive Hopf plumbing* $((F \star_{\alpha_1} A_+) \cdots \star_{\alpha_m} A_+$ *which is isotopic to the fiber surface of a positive braid.*

Proof. Let $\beta \in B_n$ be a staircase braid, let F be the fiber surface for the closure $\widehat{\beta}$ and let $\Delta \in B_n$ be the full twist. Fact 2.3 implies that $\Delta^m\beta$ is a positive braid for sufficiently large m. In Lemma 2.14 it was shown that adding a band generator to the braid word for β corresponds to plumbing a positive Hopf band to the fiber surface F. The corollary now follows from the observation that the full twist Δ is a product of band generators. □

The following corollary of Theorem 2.11 was independently obtained by Filip Misev.

Corollary 2.25. *There are only finitely many staircase braid closures in each concordance class.*

Proof. The 4-ball genus $g_4(K)$ is a concordance invariant, which coincides with the ordinary genus for strongly quasipositive links [Rud93]. A staircase braid closure is entirely determined by the collection of plumbing arcs $\{\alpha_i \subset D\}$. By Theorem 2.11, the number of plumbing arcs $|\{\alpha_i\}|$ depends only on the genus of the knot. The count of possible arrangements of a finite number of indexed arcs in the disk is clearly finite and the corollary follows. □

3. Sutured Khovanov homology

3.1. **Sutured Khovanov homology and the $\mathfrak{sl}_2(\mathbb{C})$-module structure.** The sutured Khovanov complex is a combinatorially defined, triply graded chain complex, whose chain homotopy type is an annular link invariant [APS04, Rob13].

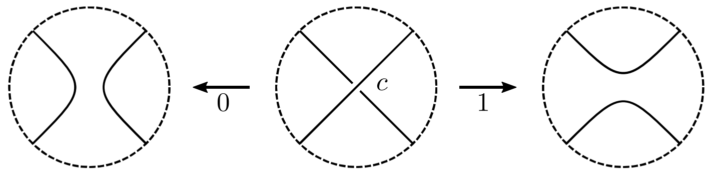


FIGURE 13. The 0-smoothings and 1−-smoothing of a crossing.

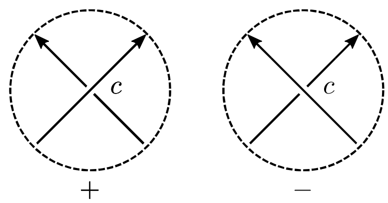


FIGURE 14. Sign of a crossing.

Let A be an oriented annulus. Let $L \subset A \times I$ be an annular link and choose a diagram $D = p(L) \subset A$ with projection $p : A \times I \to A$. A crossing c is a double point of the projection p together with over/under strand information. Endowing the annular link L with an orientation, define the sign of a crossing as in Figure 14. Let n_+ (n_-, respectively) be the number of positive (negative) crossings in D and let $n = n_+ + n_-$ be the total number of crossings.

The two ways to smooth a crossing c are illustrated in Figure 13 and called the 0 and 1 smoothing, respectively.

Definition 3.1 ([Kau83] [Kho00])**.** *A **Kauffman state** is a choice of smoothing for each crossing of a diagram D; that is, a complete resolution of D. An **oriented Kauffman state** v is a Kauffman state with a choice of orientation for all components. The **weight** of the state v is the number of* 1*-smoothings in the complete resolution and denoted by $|v|$.*

We refer to the components of a Kauffman state as "circles". Note that a Kauffman state is determined by the choice of smoothings and is therefore in 1-1 correspondence with elements of the hypercube $\mathcal{H} = \{0, 1\}^n$, where n is the number of crossings of the diagram.

Definition 3.2. *Let v be an oriented Kauffman state. A circle γ of v is trivial if $[\gamma] = 0 \in H_1(A)$ and non-trivial otherwise.*

3.1.1. *Generators.* The sutured Khovanov chain complex is freely generated by oriented Kauffman states. Formally, let $\mathbb{F}$ be the ground ring and let $V = \mathbb{F}\langle v_-, v_+ \rangle$. The sutured Khovanov chain complex groups C_i are

$$C_i = \bigoplus_v V^{\otimes n(v)},$$

where the summation is over all Kauffman states v satisfying $i = |v| - n_-$ and $n(v)$ is the number of circles in the Kauffman state.

Pictorial representation of the generators Specific circles in the complete resolution are associated to specific factors in the tensor product $V^{\otimes n(v)}$. In a complete resolution of a diagram, labeling a circle γ with "+" ("−", respectively) indicates that the component in the tensor for the factor corresponding to γ is v_+ (v_-, respectively).

3.1.2. *Gradings.* The generators of the sutured Khovanov complex inherit the quantum and homological grading from the ordinary Khovanov complex. Specifically, let v be an oriented Kauffman state and define the i-grading (the homological grading) and j-grading (the quantum grading) as follows. [Kho00]

$$i(v) = |v| - n_-,$$

$$j(v) = \#(\text{counterclockwise circles}) - \#(\text{clockwise circles}) + i(v) + n_+ - n_-.$$

The third grading is given by the singular homology class $[v] \in H_1(A)$ of the oriented Kauffman state $v \subset A$. An integer-valued grading is obtained by fixing an isomorphism $\phi : H_1(A) \simeq Z$ and setting $k(v) = \Psi([v])$. This grading is called the k-grading, or $\mathfrak{sl}_2(\mathbb{C})$ weight space grading, for reasons explained in Section 3.1.5.

3.1.3. *Differentials.* The differential measures how oriented Kauffman states behave under a change of the resolution of a crossing from a 0-smoothing to a 1-smoothing. This operation either merges two circles into a single circle, giving rise to the "merge" map $m : V \otimes V \to V$, or splits one circle into two circles, the "split map" $\Delta : V \to V \otimes V$. These maps are defined as follows. [Kho00]

$$m(v_+ \otimes v_+) = v_+ \qquad \Delta(v_+) = v_+ \otimes v_- + v_- \otimes v_+$$

$$m(v_+ \otimes v_-) = v_- = m(v_- \otimes v_+) \qquad \Delta(v_-) = v_- \otimes v_-$$

$$m(v_- \otimes v_-) = 0$$

Consider a Kauffman state and identify it by the corresponding vertex in the hypercube $\mathcal{H} = \{0,1\}^n$. Let E be the set of edges of the cube and associate a sign $\zeta_e \in \{+,-\}$ to each edge $e \in E$ such that each square has an odd number of minus signs, see [BN02]. The Khovanov differential is

$$d_{\mathrm{Kh}}(v) = \sum_e \zeta_e d_e(v),$$

where the summation is over all edges e connecting the Kauffman state v to a state obtained by changing a single 0-resolution to a 1-resolution, and $d_e(v)$ is the corresponding merge or split map. The differential can be written as a sum

$$d_{\mathrm{Kh}} = d_0 + d_{-2}, \tag{3.1}$$

where d_0 (d_{-2}, respectively) is $\mathfrak{sl}_2(\mathbb{C})$ weight space grading-preserving (lowers the $\mathfrak{sl}_2(\mathbb{C})$ weight space grading by 2, respectively). The differential of the sutured Khovanov complex is then defined as $d = d_0$, the grading-preserving part.

Definition 3.3. *The sutured Khovanov homology of an annular link $L \subset A \times I$ is* $\mathrm{Skh}_i(L) = H_i(\mathrm{Ckh}(L), d)$.

Notation We denote the sutured Khovanov homology in homological degree i, quantum degree j and $\mathfrak{sl}_2(\mathbb{C})$ weight space grading k by $\mathrm{Skh}_{i,j,k}(L)$.

3.1.4. *Filtration and spectral sequence to Khovanov homology.* The $\mathfrak{sl}_2(\mathbb{C})$ weight space grading can be exploited to construct a finite-length filtration on the complex. Let the $\mathbb{F}$-module, generated by all generators of k-grading at most K, be $F_K = \langle v \mid k(v) \leq K \rangle$ and consider the filtration

$$\mathfrak{F} : \mathrm{Ckh}(L) = \mathfrak{F}_n \supset \mathfrak{F}_{n-2} \supset \cdots \supset \mathfrak{F}_{-n} \supset \mathfrak{F}_{-n-2} = 0.$$

Equation (3.1) implies that the Khovanov differential is a filtered map and that the sutured Khovanov differential is the grading-preserving part. This implies that the sutured Khovanov complex is the associated graded complex of the filtration $\mathfrak{F}$. Via the standard procedure [Hut, Bal11], the filtration gives rise to a spectral sequence from the annular Khovanov homology converging to ordinary Khovanov homology.

3.1.5. $\mathfrak{sl}_2(\mathbb{C})$*-module structure.*

Definition 3.4. *The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is the complex vector space generated by $\langle X, Y, H\rangle$ and endowed with a bracket $[-,-] : \mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{sl}_2(\mathbb{C})$ defined by the following equations.*

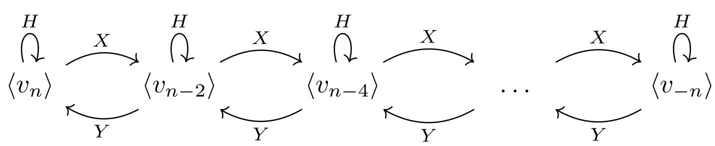
$$[H, X] = 2X,$$

$$[H, Y] = -2Y,$$

$$[X, Y] = H.$$

Fact 3.5. [FH91] *The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is semi-simple and thus every finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-representation is semi-simple.*

Fact 3.6. [FH91] *The unique $(n+1)$-dimensional irreducible $\mathfrak{sl}_2(\mathbb{C})$ representation is the complex vector space $\mathrm{Sym}^n(V) = \langle v_n\rangle \oplus \langle v_{n-2}\rangle \oplus \cdots \oplus \langle v_{-n+2}\rangle \oplus \langle v_{-n}\rangle$. The operator H acts diagonalizable, with eigenvectors v_k and corresponding eigenvalue k. The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ action is thus completely described by the following diagram.*



Note that the vector space $V = \langle v_{+1}\rangle \oplus \langle v_{-1}\rangle$ used in the definition of the chain groups of the chain complex is the unique 2-dimensional irreducible $\mathfrak{sl}_2(\mathbb{C})$ representation $\mathrm{Sym}^1(V)$. The dual representation $\mathrm{Sym}^1(V)^*$ is isomorphic to $\mathrm{Sym}^1(V)$, with the action of the operators X and Y modified by a multiplication with a factor of (-1). The trivial representation of any Lie algebra $\mathfrak{g}$ on a vector space W is defined by $g \otimes w \mapsto 0 \in W$ for all $g \in \mathfrak{g}$ and all $w \in W$. We denote this representation by 0_W.

From now on, we take as our ground ring the field of complex numbers, that is, $\mathbb{F} = \mathbb{C}$.

Definition 3.7 ([ELW15])**.** *Let v be a Kauffman state and let $NT_1, \ldots, NT_m$ be the nested non-trivial circles of v, from innermost to outermost, and let $T_1, \ldots, T_l$ be the trivial circles. The $\mathfrak{sl}_2(\mathbb{C})$ representation associated to v is the tensor representation*

$$\underbrace{Sym^1(V) \otimes \mathrm{Sym}^1(V)^* \otimes \cdots \otimes Sym^1(V)}_{m\ alternating\ Sym^1(V)\ and\ Sym^1(V)^*\ factors} \otimes 0_W^{\otimes l}$$

Lemma 3.8 ([ELW15]). *The sutured Khovanov differential d is an $\mathfrak{sl}_2(\mathbb{C})$-module map and thus the $\mathfrak{sl}_2(\mathbb{C})$-module structure is well defined on the level of homology.*

3.1.6. *Khovanov homology under change of a crossing marking.* Khovanov homology and sutured Khovanov homology are invariants of oriented links and annular links, respectively. However, the orientation of the link is merely necessary to mark crossings as either positive or negative. While these markings determine an overall shift of the complex in the homological and quantum grading, they play no role in the definition of either the generators or differential. It is often useful to define Khovanov homology (sutured Khovanov homology, respectively) as an invariant of an unoriented link (unoriented annular link, respectively) together with the counts of positive and negative markings of the crossings.

Lemma 3.9. *Let (Ckh, d) be a sutured Khovanov complex with at least one crossing, say c, marked positive and let (Ckh', d) be the complex with the crossing c marked negative. Then $(\mathrm{Ckh}, d) = (\mathrm{Ckh}', d)[1, 3]$, where the homological grading is shifted by 1 and the quantum grading shifted by 3.*

Proof. Let n_+, n_-, n'_+, n'_- be the count of positive and negative markings in (Ckh, d) and (Ckh', d), respectively. Then $n'_+ = n_+ - 1$ and $n'_- = n_- + 1$. The claim is then immediate from the definition of the homological and quantum grading in Section 3.1.2. □

3.1.7. *Plamenevskaya's element.* In [Pla06], Plamenevskaya describes an invariant of transverse links $L \subset (S^3, \xi_{\mathrm{std}})$, defined using Khovanov homology. This invariant, called Plamenevskaya's invariant $\Psi(L)$, is defined as a special class in Khovanov homology. Concretely, choose a braid representative $\beta \in B_n$ of the transverse link L, and consider the element $w = v_- \otimes v_- \otimes \cdots \otimes v_- \in \mathrm{Ckh}^{0,\mathrm{sl}(L)}$, living in the braid-like resolution of the braid closure $\hat{\beta}$. Plamenevskaya's invariant is defined as the homology class $[w] \in \mathrm{Kh}^{0,\mathrm{sl}(L)}$.

From the sutured Khovanov point of view, the element w is the unique generator in minimal k-grading. Of course, this implies that $0 \neq [w] \in \mathrm{Skh}_0(\widehat{\beta})$ and the $\mathfrak{sl}_2(\mathbb{C})$-module structure on sutured Khovanov homology then implies that Plamenevskaya's element generates a factor $\langle w \rangle = \mathrm{Sym}^n(V) \subset \mathrm{Skh}_0$. We have proved the following lemma.

Lemma 3.10. *Let $\beta \in B_n$ be a braid. Then $\mathrm{Sym}^n(V) \leq \mathrm{Skh}_0(\beta)$.*

Abusing notation, in the context of the sutured theory, we will use $\Psi(\widehat{\beta})$ to denote Plamenevskaya's element in the *complex* $\mathrm{Ckh}(\widehat{\beta})$.

3.2. **Computer program to calculate sutured Khovanov homology.** The combinatorial nature of Khovanov-type homology theories lend themselves to implementation on a computer. A computer program to calculate the sutured Khovanov homology of braid closures as $\mathfrak{sl}_2(\mathbb{C})$ representations, written by the author of this thesis, is available at `https://www2.bc.edu/ian-banfield/`. See [HKLM15] for another application to calculate sutured Khovanov homology, using Mathematica and the KnotTheory package.

For example, this is the sutured Khovanov homology of a strongly quasipositive braid representative of the fibered link 10_{161}, namely of the closure of $\beta = \sigma_1\sigma_1\sigma_1\sigma_2\sigma_1^{-1}\sigma_2\sigma_1\sigma_1\sigma_2\sigma_2$.

$$i = 0: \ \mathrm{Sym}^3(V)_{11}$$

$$i = 1: \ \mathrm{Sym}^1(V)_{11}$$

$$i = 2: \ \mathrm{Sym}^1(V)_{11}$$

$$i = 3: \ \mathrm{Sym}^1(V)_{13} \oplus \mathrm{Sym}^1(V)_{11}$$

$$i = 4: \ 2\,\mathrm{Sym}^1(V)_{13}$$

$$i = 5: \ \mathrm{Sym}^1(V)_{17} \oplus \mathrm{Sym}^1(V)_{15}$$

$$i = 6: \ 2\,\mathrm{Sym}^1(V)_{17}$$

$$i = 7: \ \mathrm{Sym}^1(V)_{19}$$

$$i = 8: \ \mathrm{Sym}^1(V)_{21}$$

$$i = 9: \ \mathrm{Sym}^1(V)_{23}$$

The subscripts denote the quantum grading of the highest weight vector in the representation.

3.3. **Positive braids.**

Definition 3.11. *A braid $\beta \in B_n$ is split if for a representative w, there exists $1 \leq j < n$ such that the braid word w contains neither the generator σ_j nor its inverse σ_j^{-1}, and is non-split otherwise.*

Note that some authors reserve the term "split braid" to denote braids whose closure is a connected sum in a prescribed way.

Theorem 3.12. *Let $\beta \in B_n$ be a non-split positive braid. The sutured Khovanov homology groups of the braid closure $\widehat{\beta}$ in homological degree $i < 0$, $i = 0$ and $i = 1$ are*

$$\mathrm{Skh}_i(\widehat{\beta}) \simeq 0 \text{ for } i < 0, \tag{3.2}$$

$$\mathrm{Skh}_0(\widehat{\beta}) \simeq \mathrm{Sym}^n(V) = \langle [\Psi(\widehat{\beta})] \rangle, \tag{3.3}$$

$$\mathrm{Skh}_1(\widehat{\beta}) \simeq \mathrm{Sym}^{n-2}(V) = \langle [\delta^+_{\mathrm{Lee}}(\Psi(\widehat{\beta}))] \rangle = \langle v_2([\Psi(\widehat{\beta})]) \rangle, \tag{3.4}$$

where $\Psi(\widehat{\beta})$ is Plamenevskaya's element [Pla06]*, δ^+_{Lee} is the k-grading increasing part of the Lee differential* [Lee05, ELW15] *and v_2 is the element of the Lie superalgebra described in* [ELW15].

This motivates the following definition.

Definition 3.13. *If a braid $\beta \in B_n$ satisfies Equations* (3.2) - (3.4)*, then we say that β is Skh-positive.*

Throughout this section, we work over the field of complex numbers. With suitable adaptations, the results of this section also hold over $\mathbb{Z}_2$. For technical reasons, it will be easier to work with negative braids rather than positive braids. Appealing to a result of Khovanov relating the Khovanov homology of the mirror of a link with the dual complex [Kho00] enables us to prove Lemmas 3.3 and 3.4.

Lemma 3.14. *Let $\beta \in B_n$ be a negative, non-split braid. The sutured Khovanov homology of the braid closure in homological degree $i = 0$ is $\mathrm{Skh}_0(\widehat{\beta}) = \mathrm{Sym}^n(V) = \langle \Psi(\beta) \rangle$, where $\Psi(\beta)$ is Plamenevskaya's element.*

Proof. Let $d_{-1} : \mathrm{Ckh}_{-1} \to \mathrm{Ckh}_0$ and $d_0 : \mathrm{Ckh}_0 \to \mathrm{Ckh}_1$ be sutured Khovanov differentials. We explicitly describe the kernel $\ker(d_0)$ and the image $\mathrm{im}(d_{-1})$ and conclude that the dimension of $\mathrm{Skh}_0(\widehat{\beta}) = \ker(d_0)/\mathrm{im}(d_{-1})$ is bounded above by $\dim(\mathrm{Skh}_0(\widehat{\beta})) \leq (n+1)$, where n is the number of strands of the braid. This part of the sutured Khovanov complex for negative braids is pictured in Figure 15. It is convenient to work in the basis given by the generators of the sutured Khovanov chain complex as described in Section 3.1.1. The map d_0 is trivial and thus the kernel of d_0 is

$$\ker(d_0) = \mathrm{Ckh}_0 = V^{\otimes n} = \langle v_{\epsilon_1} \otimes v_{\epsilon_2} \otimes \cdots \otimes v_{\epsilon_n} \mid \epsilon_i \in \{+, -\} \rangle.$$

From Figure 15, we note that the differential d_{-1} is a sum of split maps; specifically of the maps associated to splitting a trivial circle into non-trivial circles:

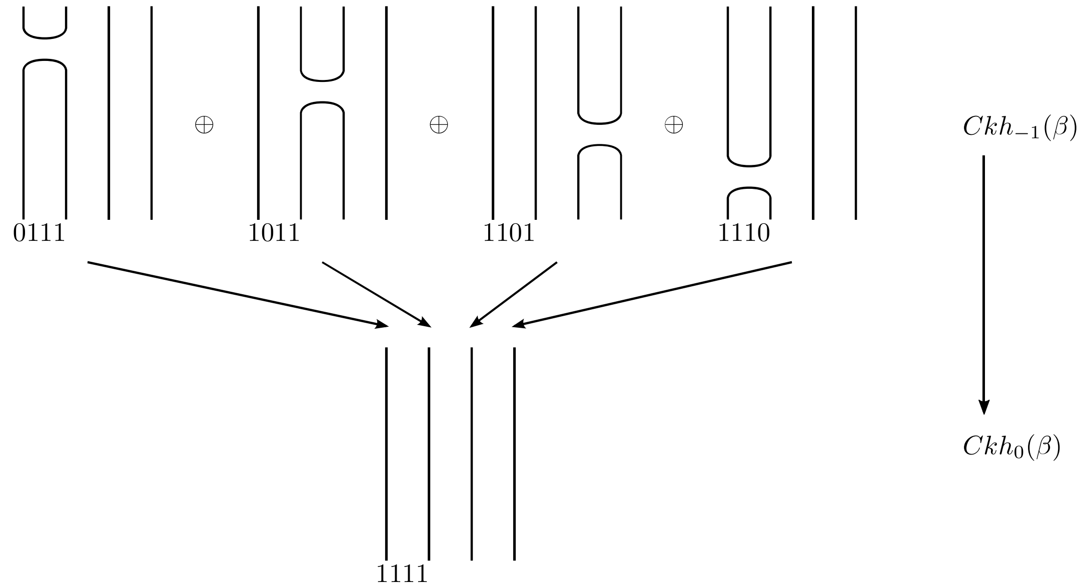


FIGURE 15. Example for $\beta = \sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_1^{-1}$.

- $d_{-1}(v_-) = 0$ as the k-grading is not preserved, and
- $d_{-1}(v_+) = v_+ \otimes v_- + v_- \otimes v_+$.

This shows that the image of d_{-1} is generated by the following elements.

$$\operatorname{im}(d_{-1}) = \langle \cdots \otimes v_- \otimes v_+ \cdots - \cdots \otimes v_+ \otimes v_- \ldots \rangle.$$

On the level of homology, the assumption that the braid is non-split yields the "sign swapping" relation.

$$[\cdots \otimes v_- \otimes v_+ \otimes \ldots] = -[\cdots \otimes v_+ \otimes v_- \otimes \ldots] \in \ker(d_0)/\operatorname{im}(d_{-1}) \tag{3.5}$$

Consider the basis elements

$$v_k = \underbrace{v_+ \otimes \cdots \otimes v_+}_{k} \otimes \underbrace{v_- \otimes \cdots \otimes v_-}_{n-k}.$$

The dimension of $\mathrm{Skh}_0 = \ker(d_0)/\operatorname{im}(d_{-1})$ is bounded above by $\dim(\mathrm{Skh}_0) \leq n+1$, because repeated application of Equation (3.5) to a basis element v of $\ker(d_{-1})$ shows that v is homologous to $\pm v_k$ for some $1 \leq k \leq n$. Appealing to Lemma 3.10 now implies the desired result. □

Lemma 3.15. *Let $\beta \in B_n$ be a negative, non-split braid. The sutured Khovanov homology of the braid closure in homological degree $i = -1$ is $\mathrm{Skh}_{-1}(\widehat{\beta}) = \mathrm{Sym}^{n-2}(V)$.*

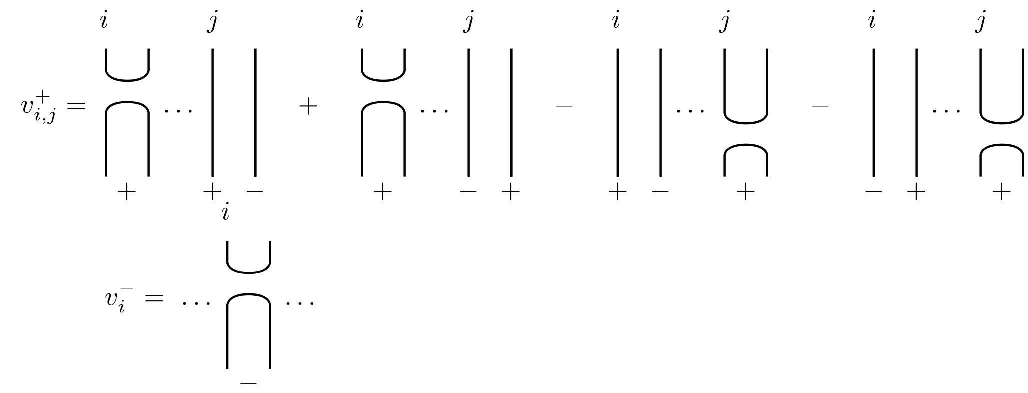


FIGURE 16. Generators of V_+ and V_-.

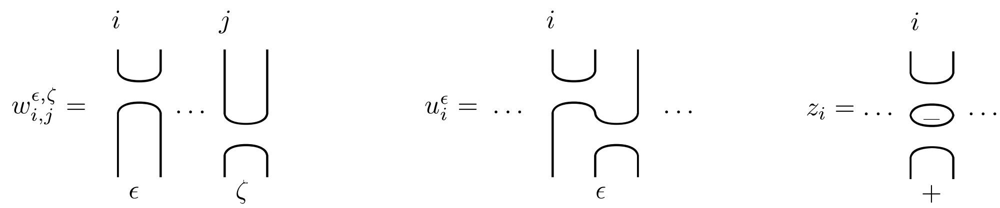


FIGURE 17

Proof. We proceed in two steps. First, we prove the lemma for $\beta = \sigma_1^{-1}\sigma_2^{-1}\dots\sigma_{n-1}^{-1}$, which are negatively stabilized unknots. This is part of Proposition 14 in [ELW15], though we use a slightly different approach. The second step is to show that the operation of injecting σ_j^{-1} into a negative braid word preserves the homology group Skh_{-1}. Let $d_{-2} : \mathrm{Ckh}_{-2} \to \mathrm{Ckh}_{-1}$ and $d_{-1} : \mathrm{Ckh}_{-1} \to \mathrm{Ckh}_0$ be sutured Khovanov differentials.

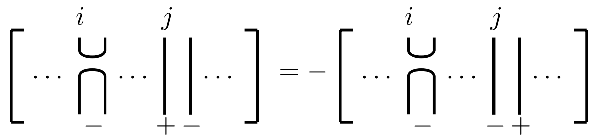
We begin by explicitly describing $\ker(d_{-1})$ in the case of $\beta = \sigma_1^{-1}\sigma_2^{-1}\dots\sigma_{n-1}^{-1} \in B_n$. Consider the following subspaces $V_-, V_+ \leq \mathrm{Ckh}_{-1}$, where the elements $v_{i,j}^+$ and v_i^- are pictured in Figure 16.

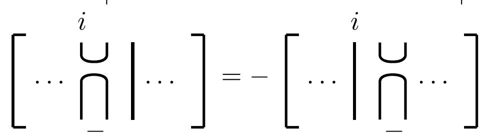
$$V_+ = \langle v_{i,j}^+ \mid i,j \in \{1,\dots,n-1\}, |i-j| > 1\rangle,$$
$$V_- = \langle v_i^- \mid i \in \{1,\dots,n-1\}\rangle.$$

We claim that $\ker(d_{-1}) = V_- \oplus V_+$. A straightforward calculation shows that $d_{-1}(v) = 0$ for $v \in V_- \oplus V_+$. The reverse containment follows from Proposition 14 [ELW15]; or more explicitly from an elementary, if lengthy, linear algebra argument inducting on n. Now consider the elements $w_{i,j}^{\epsilon,\zeta}, u_i^\epsilon \in \mathrm{Ckh}_{-2}$ pictured in Figure 17, where $\epsilon, \zeta \in \{+,-\}$. Calculating the differential shows that

$$d_{-1}(w_{i,j}^{+,+}) = \pm v_{i,j}^+,$$

and this implies that $V_+ \leq \operatorname{im}(d_{-2})$. differentials $d_{-2}(w^{+,-})$ and $d_{-2}(u^+\pm)$ and the assumption that the braid is non-split imply the relations pictured in Equations (3.6) and (3.7) on the level of homology.

(3.6)


(3.7)


Now consider the following elements w_k in the direct summand of Ckh_{-1} corresponding to the complete resolution with the crossing corresponding to σ_1^{-1} resolved with a 0-smoothing and all other crossings resolved with a 1-smoothing.

$$w_k = \underbrace{v_-}_{\text{for } \gamma} \otimes \underbrace{v_+ \otimes \cdots \otimes v_+}_{k} \otimes \underbrace{v_- \otimes \cdots \otimes v_-}_{n-k-2},$$

where γ is the unique trivial circle in the resolution. Note that $w_k \in V_-$ and further, that iterated applications of Equations (3.6) and (3.7) imply that every generator of V_- is homologous to $\pm w_k$ for some $1 \leq k \leq n-2$. There are $n-2+1 = n-1$ such elements, and hence the dimension is bounded above by $\dim(\mathrm{Skh}_{-1}) \leq n-1$. To obtain the lower-bound we note that all non-trivial circles w_{n-2} are marked with a "$-$". However, elements in the image of the differential $\operatorname{im}(d_{-2})$ have at least one non-trivial circle marked "$+$" and thus $w_{n-2} \notin \operatorname{im}(d_{-1})$. The element w_{n-2} generates an irreducible summand $\langle w_{n-2} \rangle = \mathrm{Sym}^{n-2}(V) \leq \mathrm{Skh}_{-1}$ as the k-grading is $k(w_{n-2}) = -n+2$. This completes the first step.

In the second step we generalize the argument to non-split negative braids. Let $\alpha \in B_n$ be such a braid and let c be the number of crossings of the closure $\widehat{\alpha}$. The rank-nullity theorem and Lemma 3.14 show that

$$\dim(\ker(d_{-1})) = n + 1 + (c-2)2^{n-1}. \tag{3.8}$$

Let us now inductively inject the inverse of generators into the braid word for β to obtain α, in any order. Equation (3.8) shows that the operation of injecting the inverse of a single generator into the braid word increases $\dim(\ker d_{-1})$ by 2^{n-1}.

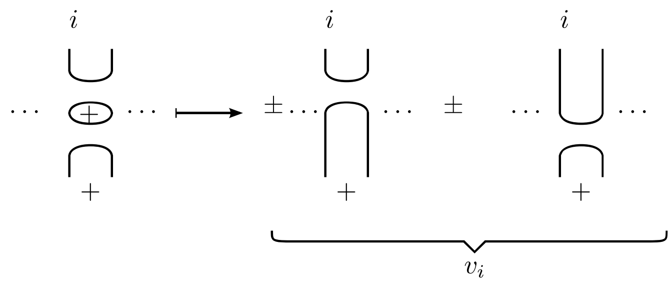
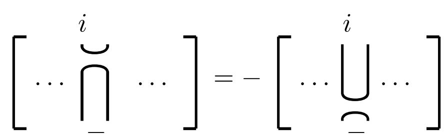


FIGURE 18. The elements $v_i \in V_+ \leq \ker(d_{-1})$ lie in the image of the differential.

We enlarge V_- to contain all additional generators with the trivial circle in the resolution marked with a "$-$". These generators are in the kernel $\ker d_{-1}$ and contribute 2^{n-2} to its dimension. The differential of the elements z_i, pictured in Figure 17, implies Equation (3.9) on the level of homology.

(3.9)



Combining the relation in Equation 3.9 with the previous relations shows that the added generators are homologous to w_k for some $1 \leq k \leq n-2$. Further, from an explicit computation of the sign along the edges, it follows that $w_0 \notin \mathrm{im}(d_{-1})$ is equivalent to

$$x_1 \notin \langle x_i + (-1)^{i-j} x_j \mid 1 \leq i, j \leq M \rangle,$$

where M is a positive integer and the x_i freely generate a complex vectorspace. This is a straightforward exercise which we omit. As noted in step one, $w_0 \notin \mathrm{im}(d_{-1})$ implies that $\dim(\mathrm{Skh}_{-i}) \geq n-1$. Now enlarge V_+ by adding to it the elements v_i shown in Figure 18, where the signs depend on the signs along the edge maps in Section 3.1.3. Counting linearly independent such elements, note that the dimension of V_+ increases by 2^{k-2}. Figure 18 shows that $v_i \in \mathrm{im}(d_{-2})$. The generators added to V_- and V_+ increase the dimension of the kernel by $2 * 2^{n-2} = 2^{n-1}$, accounting for the entire change in the dimension of the kernel without generating additional elements in homology. Thus $\dim(\mathrm{Skh}_{-1}) \leq n-1$, which concludes the second step.

□

To prove Theorem 3.12 for positive braids, we appeal to the following fact, relating the sutured Khovanov homology of a braid with that of its mirror.

Fact 3.16. *Let K be an annular link and $K^!$ its mirror. Then* $\mathrm{Skh}_i(K) \simeq \mathrm{Skh}_{-i}(K^!)$.

Proof. Corollary 1 in [ELW15] implies that $\mathrm{Skh}_i(\beta, k) \simeq \mathrm{Skh}_i(\beta, -k)$ [ELW15]. The claim now follows immediately from [Kho00], Proposition 31. □

Proof of Theorem 3.12. Let $\beta \in B_n$ be a non-split positive braid and let $\beta^!$ be its mirror. The braid $\beta^!$ is a non-split negative braid. Fact 3.16 and Lemmas 3.14 and 3.15 imply

$$\mathrm{Skh}_0(\beta) \simeq \mathrm{Skh}_0(\beta^!) \simeq \mathrm{Sym}^n(V),$$
$$\mathrm{Skh}_1(\beta) \simeq \mathrm{Skh}_{-1}(\beta^!) \simeq \mathrm{Sym}^{n-2}(V).$$

Lastly, it remains to show these homology groups are generated as claimed in the statement of the Theorem 3.12.

$\mathrm{Skh}_0(\widehat{\beta}) = \langle \Psi(\widehat{\beta}) \rangle$**:** Immediate from Section 3.1.7, as the class of Plamenevskaya's element generates a factor of $\mathrm{Sym}^n(V)$ for any braid closure.

$\mathrm{Skh}_1(\widehat{\beta}) = \langle \delta^+_{\mathrm{Lee}}(\Psi(\widehat{\beta}))] \rangle$**:** Lemma 3 of [ELW15] implies that

$$d(\delta^+_{\mathrm{Lee}}(\Psi(\widehat{\beta}))) = -\delta^+_{\mathrm{Lee}}(d(\Psi(\widehat{\beta}))) = -\delta^+_{\mathrm{Lee}}(0) = 0,$$

and therefore $\delta^+_{\mathrm{Lee}}(\Psi(\widehat{\beta}))$ is a cycle. The term $\delta^+_{\mathrm{Lee}}(\Psi(\widehat{\beta}))$ is a signed sum of oriented Kauffman states with a single trivial circle γ marked "+" and $n-2$ non-trivial circles marked "−". In the case of positive braids, all maps associated to the edges $\mathrm{Ckh}_0 \to \mathrm{Ckh}_1$ are maps merging two non-trivial circles into the trivial circle γ. The condition that the sutured Khovanov differential preserves the k-grading implies that $\delta^+_{\mathrm{Lee}}(\Psi(\widehat{\beta}))$ can not be in the image of the differential. Hence $[\delta^+_{\mathrm{Lee}}(\Psi(\widehat{\beta}))] \neq 0$. □

3.4. **Applications.** In [Cro93], Cromwell gave a proof using spanning surfaces of the following fact.

Theorem 3.17 (Cromwell)**.** *Suppose that Seifert's algorithm constructs a minimal genus spanning surface for L when applied to a diagram $\pi(L) = D$ of L. Then L is a split link if and only if D is disconnected.*

When applied to a positive braid link, the braided surface constructed by Seifert's algorithm is indeed of minimal genus, hence positive braid links satisfy the assumptions of Cromwell's theorem. However, for strongly quasipositive links, the spanning surface constructed may not be minimal genus.

The key assumption in Theorem 3.12 is that the braid β is non-split. It follows from the definition of the sutured Khovanov homology that the homology of a split link is given by tensoring the sutured Khovanov homologies of the components. We are now ready to prove a version of Cromwell's result for positive braids links.

Corollary 3.18. *A positive braid β is conjugate to a non-split braid if and only if β is non-split.*

Proof. If the positive braid β is non-split, then apply Theorem 3.12 to deduce that $\mathrm{Skh}_0(\widehat{(\beta)})$ is an irreducible $\mathfrak{sl}_2(\mathbb{C})$-module. Sutured Khovanov homology is a conjugacy class. If β was conjugate to a split braid, then $\mathrm{Skh}_0(\widehat{(\beta)})$ would be a tensor representation, contradicting that tensor representations are not irreducible [FH91]. □

Theorem 3.19 ([Sta78]). *Let β be a positive braid. The closure $\widehat{\beta}$ is fibered if and only if $\widehat{\beta}$ is non-split.*

Corollary 3.20. *Sutured Khovanov homology detects if a positive braid closure is fibered.*

Proof. Let $\beta \in B_n$ be positive. The sutured Khovanov homology in degree $i = 0$ is $\mathrm{Skh}_0(\widehat{\beta}) \simeq \mathrm{Sym}^n(V)$ if and only if $\widehat{\beta}$ is non-split if and only if $\widehat{\beta}$ is fibered. □

3.5. **Strongly quasipositive braids and staircase braids.**

3.5.1. *The mapping cone description of sutured Khovanov homology and the long exact sequence for a crossing.* Let D be a diagram for an annular link L and choose a crossing c. Let D_0 (D_1) be the diagrams obtained from D by resolving the crossing c with a 0-smoothing (1-smoothing, respectively). The sutured Khovanov complex of D can be described in terms of the sutured Khovanov complex of the diagrams D_0 and D_1 via a mapping cone construction. Let $[\times]$, $[)(]$, $[\asymp]$ be the sutured Khovanov complexes associated to D, D_0 and D_1. Then

$$[\times] = C([)(] \xrightarrow{d} [\asymp]),$$

where d is the differential. The associated short exact sequence of chain complexes

$$0 \to [\asymp][1] \xrightarrow{i} [\times] \xrightarrow{\pi} [)(] \to 0$$

gives rise to the long exact sequence of homology groups,

$$\cdots \to H_{-1}[\asymp][1] \to H_{-1}[\times] \to H_{-1}[)(] \to H_0[\asymp][1] \to \dots \quad . \tag{3.10}$$

Lemma 3.21. *The maps in the long exact sequence of homology groups are $\mathfrak{sl}_2(\mathbb{C})$-module homomorphisms.*

Proof. The maps in the exact sequence are either inclusion maps, projection maps or the connecting homomorphism, which is the differential d. All these maps commute with the $\mathfrak{sl}_2(\mathbb{C})$ action. □

3.5.2. *The sutured Khovanov homology of staircase braids.*

Lemma 3.22. *Let $\beta \in B_n$ be Skh-positive and let D be a diagram for the braid closure $\widehat{\beta}$. Let c be a positive crossing, and let D_0 (D_1) be the diagram obtained by resolving the crossing c with a 0-smoothing (1-smoothing). Denote the sutured Khovanov complex for D, D_0, D_1 by $[\times]$, $[)(]$, $[\asymp]$, respectively. The homology groups for $[)(]$ and $[\asymp]$ satisfy $H_i([)(]) \simeq H_i([\asymp])$ for $i \leq -1$.*

Proof. Consider the long exact sequence for homology groups (3.10). For $i \leq -1$, the relevant part of the long exact sequence reads

$$\begin{array}{ccccccc} H_{i-1}[\times] & \longrightarrow & H_{i-1}[)(] & \longrightarrow & H_i[\asymp][1] & \longrightarrow & H_i[\times] \\ \| & & & & & & \| \\ 0 & & & & & & 0 \end{array}$$

and hence $H_{i-1}[)(] \simeq H_i[\asymp][1] = H_{i-1}[\asymp]$. Now consider the exact sequence

$$\begin{array}{ccccccc} H_{-1}[\times] & \longrightarrow & H_{-1}[)(] & \xrightarrow{\alpha} & H_0[\asymp][1] & \xrightarrow{\phi} & H_0[\times] \\ \| & & & & & & \| \\ 0 & & & & & & \mathrm{Sym}^n(V) \end{array}$$

It is immediate that the map α is injective. We claim that α is also surjective. Note that $H_0[\asymp][1]$ is a semi-simple $\mathfrak{sl}_2(\mathbb{C})$-module whose irreducible factors are of dimension at most $n-2$. If the map ϕ was non-zero, then pre-composing with an injection into one of the irreducible factors of $H_0[\asymp][1]$ yields a non-trivial map between irreducible $\mathfrak{sl}_2(\mathbb{C})$-modules of different dimension. This contradicts the fact that a map between irreducible Lie algebra modules must be either an isomorphism or trivial. As the sequence is exact, this implies that image of α is $\mathrm{im}(\alpha) = \ker(\phi) = H_0[\asymp][1]$. Thus $\alpha : H_{-1}[)(] \to H_0[\asymp][1] = H_{-1}[\asymp]$ provides the desired isomorphism. □

Recall that we defined staircase braids as the strongly quasipositive braids whose normal form contains a positive power of the dual Garside element.

Theorem 3.23. *Assume that $\mathrm{Skh}_i(\widehat{\beta}) = 0$ for $i < 0$ if $\beta \in B_n$ is a strongly quasipositive braid. Then the class of staircase braid closures in B_n is Skh-positive, that is, if $\gamma \in B_n$ is a staircase braid then $\widehat{\gamma}$ is Skh-positive.*

Proof. Recall from Lemma 2.7 that if γ is a staircase braid, then there is a sequence of braids

$$\gamma = \gamma_m \rightsquigarrow \gamma_{m-1} \rightsquigarrow \cdots \rightsquigarrow \gamma_0,$$

where all γ_i are strongly quasipositive, γ_0 is a positive braid and $\gamma_{i+1} \rightsquigarrow \gamma_i$ indicates that γ_i is obtained from γ_{i+1} by canceling a repeated band generator. The braid γ_0 is Skh-positive by Theorem 3.12. We now proceed by induction and show that if γ_i being Skh-positive implies that γ_{i+1} is also Skh-positive.

The braid word for γ_{i+1} contains a square of a band generator, say $a^2_{r,s}$, as $\gamma_{i+1} \rightsquigarrow \gamma_i$. Expressed in the Artin generators, a square of a band generator is

$$\begin{aligned} a^2_{r,s} &= \big((\sigma_r)^{-1}(\sigma_{r+1})^{-1} \ldots \sigma_s\sigma_{s-1} \ldots \sigma_r\big)^2 \\ &= (\sigma_r)^{-1}(\sigma_{r+1})^{-1} \ldots \sigma_s^2\sigma_{s-1} \ldots \sigma_r. \end{aligned}$$

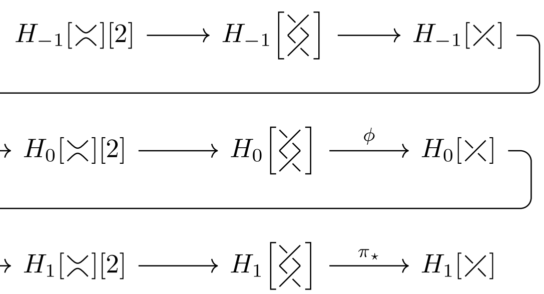
This implies that to obtain a braid word for γ_i we replace the square of the Artin generator o_s^2 in the braid word for γ_{i+1} with a single power of the Artin generator σ_s. Let D (D') be the diagram for the braid closure of γ_i (γ_{i+1}, respectively). By these remarks, these diagrams are identical except in a small neighborhood, which contains σ_s in D and σ_s^2 in D'.

Now consider the short exact sequence of chain complexes

$$\begin{array}{ccccccccc} 0 \longrightarrow & \left[\substack{\asymp \\ \times}\right][1] & \longrightarrow & \left[\substack{\times \\ \times}\right] & \longrightarrow & \left[\substack{)(\\ \times}\right] & \longrightarrow 0\,. \\ & \| & & \| & & \| & \\ & [\asymp][2] & & \mathrm{Ckh}(D') & & \mathrm{Ckh}(D) & \end{array} \tag{3.11}$$

The complex $[\asymp]$ ($[)(]$, respectively) is the sutured Khovanov complex associated to resolving the crossing c corresponding to σ_s in D with a 1-smoothing (0-smoothing, respectively) and $[\times] = \mathrm{Ckh}(D)$. For $i < 0$, the homology groups satisfy $H_i[\asymp] = H_i[)(] = 0$. The first equality follows from Lemma 3.22 and the second is immediate from our assumption that the sutured Khovanov homology of strongly quasipositive braids in B_n is supported only in non-negative homological grading.

Consider the part of the long exact sequence of homology groups associated to the short exact sequence (3.11)



It is immediate that ϕ is an isomorphism as $H_0[\asymp][2] = H_{-2}[\asymp] = 0$ and $H_1[\asymp][2] = H_{-1}[\asymp] = 0$. This also immediately implies that $\pi_\star$ is injective. The map $\pi_\star$ is the map induced by the projection onto a quotient complex and we claim it is surjective. Write $x_j = \Psi(\widehat{\gamma_j})$ for the Plamenevskaya element of γ_j. By Equation (3.4) and the inductive hypothesis, we may assume that $H_1[\times] = \mathrm{Sym}^{n-2}(V) = \langle([\delta^+_{\text{Lee}}(x_i)])\rangle$, where δ^+_{Lee} is the k-grading increasing part of Lee's differential [Lee05, ELW15]. We claim that $\pi_\star([\delta^+_{\text{Lee}}(x_{i+1})]) = [\delta^+_{\text{Lee}}(x_i)]$. To start, recall from the proof of Theorem 3.12 that $d(\delta^+_{\text{Lee}}(x_i)) = 0$, using Lemma 3 of [ELW15], so the elements $\delta^+_{\text{Lee}}(x_i)$ represent homology classes. Identify oriented Kauffman states for the resolutions of γ_{i+1} where c is resolved with a 0-smoothing with the oriented Kauffman states of γ_i. For convenience, index the crossings in such a way that c is the last crossing. Note that $\delta^+_{\text{Lee}}(x_{i+1})$ is a signed sum of oriented Kauffman states, each containing a single trivial circle marked "+" (corresponding to a 1-smoothing of a positive crossing), and $n-2$ non-trivial circles marked "-". Our identification of Kauffman states and choice of indexing shows that $\delta^+_{\text{Lee}}(x_{i+1}) = \delta^+_{\text{Lee}}(x_i) \pm v$. Here, v is the state that corresponds to resolving the crossing c with a 1-smoothing and marking the trivial circle with a "+" and the non-trivial circles with a "−". Note that $\pi(v) = 0$ and therefore,

$$\pi_\star([\delta^+_{\text{Lee}}(x_{i+1})]) = [\pi(\delta^+_{\text{Lee}}(x_i) \pm v)] = [\pi(\delta^+_{\text{Lee}}(x_i))] = [\delta^+_{\text{Lee}}(x_i)].$$

and the lemma now follows. □

The next technical lemma provides a convenient criterion to verify if a strongly quasipositive braid closure has sutured Khovanov homology supported in non-negative homological grading.

Lemma 3.24. *Let $D_{1\ldots1}$ be the annular link diagram obtained by resolving all band generators of the closure of a strongly quasipositive braid $\beta \in B_n$ with a 1-smoothing and let $m(\beta)$[5] be the number of band generators in β. If $H_{i-m(\beta)}[D_{1\ldots1}] = 0$ for $i < 0$ for all strongly quasipositive braids $\beta \in B_n$,*

[5]Note that for strongly quasipositive braids the writhe and the number of band generators coincide.

then strongly quasipositive braids closures in B_n are supported in non-negative homological grading.

Proof. We proceed by induction. Consider a strongly quasipositive braid with one band generator,

$$\beta = a_{r,s} = (\sigma_r)^{-1}(\sigma_{r+1})^{-1} \dots \sigma_s \sigma_{s-1} \dots \sigma_r.$$

Let D be the diagram of the braid closure and let c_1 be the positive crossing associated to the Artin generator σ_s. Let D_0 and D_1 be the annular link diagrams that are obtained by resolving c_1 with a 0-smoothing and 1-smoothing, respectively, and let $[\times], [)(], [\asymp]$ be the sutured Khovanov complexes for D, D_0, D_1. The diagram D_1 represents the trivial braid closure on n strands whose sutured Khovanov homology is supported only in homological grading $i = 0$. Applying the long exact sequence from Equation (3.10) we obtain, for $i < 0$,

$$\begin{array}{ccccccc} H_{i-1}[)(] & \longrightarrow & H_{i-1}[\asymp] & \longrightarrow & H_i[\times] & \longrightarrow & H_i[)(] \\ \| & & & & & & \| \\ 0 & & & & & & 0 \end{array} \quad .$$

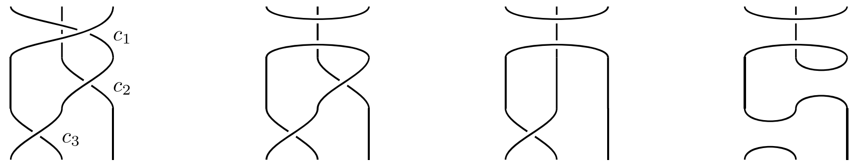
We now assume that the result is true for strongly quasipositive braids in B_n with less than M band generators and consider a strongly quasipositive braid β with M band generator. Label the positive crossings of β corresponding to the band generators by $c_1, \dots, c_M$, and let $D_{j_1 j_2 \dots j_L}$ be the diagram obtained by resolving the crossing c_i with a j_i-smoothing for $1 \leq i \leq L$, see Figure 19. We claim that for $i < 0$, the homology of the sutured Khovanov complex for these diagrams satisfies the following equations.

$$H_{i-k}[D_{\underbrace{1 \dots 10}_{k+1}}] = 0, \tag{3.12}$$

$$H_{i-k}[D_{\underbrace{1 \dots 1}_{k}}] = H_{i-(k+1)}[D_{\underbrace{1 \dots 1}_{k+1}}]. \tag{3.13}$$

Let β' be the strongly quasipositive braid that is obtained by removing the band generator corresponding to the positive crossing c_{k+1} from the braid word for β and let D' be an diagram for the braid closure of β'. By induction and noting that the base case is immediate, we may assume that equations (3.12) and (3.13) hold for the diagram D'.

Note that the diagram $D_{1\dots 10}$ (count of 1s in the subscript is k) is isotopic to the diagram $D'_{1\dots 1}$ (count of 1s in the subscript is k). Hence Equation (3.12) is equivalent to $H_{i-k}(D'_1) = 0$ for $i < 0$



(A) D (B) D_1 (C) D_{10} (D) D_{111}

FIGURE 19. Diagrams obtained by partially resolving the braid $\beta = a_{1,3}a_{2,3}a_{1,2}$, closures omitted.

(count of 1s in the subscript is k). For $k = M - 1$ this claim follows from the assumption in the statement of the lemma, and for $k < M - 1$ the claim follows from this observation and repeated applications of Equation (3.13).

It remains to prove Equation (3.13) for the diagram D. For this, we apply the long exact sequence (3.10) corresponding to the short exact sequence of chain complexes

$$0 \to [D_{\underbrace{1\dots1}_{k+1}}][1] \to [D_{\underbrace{1\dots1}_{k}}] \to [D_{\underbrace{1\dots10}_{k+1}}] \to 0 \quad .$$

We can now prove Equation (3.13) by considering the following part of the long exact sequence (for $i < 0$).

$$\begin{array}{ccccccc} H_{i-k-1}[D_{\underbrace{1\dots10}_{k+1}}] & \longrightarrow & H_{i-k}[D_{\underbrace{1\dots1}_{k+1}}][1] & \longrightarrow & H_{i-k}[D_{\underbrace{1\dots1}_{k}}] & \longrightarrow & H_{i-k}[D_{\underbrace{1\dots10}_{k+1}}] \\ \|(3.12) & & & & & & \|(3.12) \\ 0 & & & & & & 0 \end{array}$$

Using Equation (3.13), we see that for $i < 0$ the sutured Khovanov homology of $\widehat{\beta}$ satisfies $H_i[D] = 0$.

$$H_i[D] = H_{i-1}[D_1] = H_{i-2}[D_{11}] = \cdots = H_{i-m(\beta)}[D_{1\dots1}] = 0$$

□

Lemma 3.25. *The sutured Khovanov homology of strongly quasipositive* 3*-braids is supported in non-negative homological grading.*

Proof. Let $\beta \in B_3$ be a strongly quasipositive 3-braid, let $m(\beta)$ be the number of band generators of β and let $D_{1\dots1}$ be the annular link diagram obtained by resolving all band generators of the closure $\widehat{\beta}$. Then $D_{1\dots1}$ has at most $m(\beta)$ negatively marked crossings and appealing to Lemma 3.24 implies that $\mathrm{Skh}_i(\widehat{\beta}) = 0$. □

We finish with two immediate corollaries of this lemma.

Corollary 3.26. *The Khovanov homology of strongly quasipositive* 3*-braids is supported in non-negative homological grading.*

Proof. The spectral sequence from sutured Khovanov homology to Khovanov homology respects homological grading. □

Corollary 3.27. *Staircase* 3*-braids are Skh-positive.*

Proof. Immediate from Lemma 3.25 and Theorem 3.23. □

Corollary 3.28. *Staircase* 3*-braids are conjugate to non-split braids if and only if they are non-split.*

Proof. Identical to Corollary 3.18. □

References

[APS04] Marta M. Asaeda, Józef H. Przytycki, and Adam S. Sikora. Categorification of the Kauffman bracket skein module of I-bundles over surfaces. *Algebr. Geom. Topol.*, 4:1177–1210 (electronic), 2004.

[Bal11] John A. Baldwin. On the spectral sequence from Khovanov homology to Heegaard Floer homology. *Int. Math. Res. Not. IMRN*, (15):3426–3470, 2011.

[BKL98] Joan Birman, Ki Hyoung Ko, and Sang Jin Lee. A new approach to the word and conjugacy problems in the braid groups. *Adv. Math.*, 139(2):322–353, 1998.

[BN02] Dror Bar-Natan. On Khovanov's categorification of the Jones polynomial. *Algebr. Geom. Topol.*, 2:337–370, 2002.

[CL] J. C. Cha and C. Livingston. Knotinfo: Table of knot invariants. `http://www.indiana.edu/~knotinfo`. Accessed: 2017-07-18.

[Cro93] Peter R. Cromwell. Positive braids are visually prime. *Proc. London Math. Soc. (3)*, 67(2):384–424, 1993.

[ELW15] J. Elisenda Grigsby, A. M. Licata, and S. M. Wehrli. Annular Khovanov homology and knotted Schur-Weyl representations. *ArXiv e-prints*, May 2015, 1505.04386.

[EW09] J. Elisenda Grigsby and S. M. Wehrli. Khovanov homology, sutured Floer homology, and annular links. *ArXiv e-prints*, July 2009, 0907.4375.

[FH91] W. Fulton and J. Harris. *Representation Theory: A First Course*. Graduate Texts in Mathematics. Springer New York, 1991.

[Gab86] David Gabai. Detecting fibred links in S^3. *Comment. Math. Helv.*, 61(4):519–555, 1986.

[Hed08] Matthew Hedden. Some remarks on cabling, contact structures, and complex curves. In *Proceedings of Gökova Geometry-Topology Conference 2007*, pages 49–59. Gökova Geometry/Topology Conference (GGT), Gökova, 2008.

[Hed10] Matthew Hedden. Notions of positivity and the Ozsváth-Szabó concordance invariant. *J. Knot Theory Ramifications*, 19(5):617–629, 2010.

[HKLM15] H. Hunt, H. Keese, A. Licata, and S. Morrison. Computing annular Khovanov homology. *ArXiv e-prints*, May 2015, 1505.04484.

[Hut] Michael Hutchings. Introduction to spectral sequences. `https://web.archive.org/web/20150326140125/https://math.berkeley.edu/~hutching/teach/215b-2011/ss.pdf`. Accessed: 2017-07-01.

[IW15] Tetsuya Ito and Bert Wiest. Lawrence-Krammer-Bigelow representations and dual Garside length of braids. *Geom. Topol.*, 19(3):1361–1381, 2015.

[Kau83] Louis H. Kauffman. *Formal knot theory*, volume 30 of *Mathematical Notes*. Princeton University Press, Princeton, NJ, 1983.

[Kaw96] A. Kawauchi. *A Survey of Knot Theory*. Birkhäuser Verlag, 1996.

[Kho00] Mikhail Khovanov. A categorification of the Jones polynomial. *Duke Math. J.*, 101(3):359–426, 2000.

[KM93] P. B. Kronheimer and T. S. Mrowka. Gauge theory for embedded surfaces. I. *Topology*, 32(4):773–826, 1993.

[Lee05] Eun Soo Lee. An endomorphism of the Khovanov invariant. *Adv. Math.*, 197(2):554–586, 2005.

[MM86] P. M. Melvin and H. R. Morton. Fibred knots of genus 2 formed by plumbing Hopf bands. *J. London Math. Soc. (2)*, 34(1):159–168, 1986.

[Ni07] Yi Ni. Knot Floer homology detects fibred knots. *Invent. Math.*, 170(3):577–608, 2007.

[Ni09] Yi Ni. Closed 3-braids are nearly fibred. *J. Knot Theory Ramifications*, 18(12):1637–1649, 2009.

[Pla06] Olga Plamenevskaya. Transverse knots and Khovanov homology. *Math. Res. Lett.*, 13(4):571–586, 2006.

[Rob13] Lawrence P. Roberts. On knot Floer homology in double branched covers. *Geom. Topol.*, 17(1):413–467, 2013.

[Rud93] Lee Rudolph. Quasipositivity as an obstruction to sliceness. *Bull. Amer. Math. Soc. (N.S.)*, 29(1):51–59, 1993.

[Rud98] Lee Rudolph. Quasipositive plumbing (constructions of quasipositive knots and links. V). *Proc. Amer. Math. Soc.*, 126(1):257–267, 1998.

[Rud01] Lee Rudolph. Hopf plumbing, arborescent Seifert surfaces, baskets, espaliers, and homogeneous braids. *Topology Appl.*, 116(3):255–277, 2001.

[Rud05] Lee Rudolph. Knot theory of complex plane curves. In *Handbook of knot theory*, pages 349–427. Elsevier B. V., Amsterdam, 2005.

[Shu07] Alexander N. Shumakovitch. Rasmussen invariant, slice-Bennequin inequality, and sliceness of knots. *J. Knot Theory Ramifications*, 16(10):1403–1412, 2007.

[Sta78] John R. Stallings. Constructions of fibred knots and links. In *Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 2*, Proc. Sympos. Pure Math., XXXII, pages 55–60. Amer. Math. Soc., Providence, R.I., 1978.

[Sto06] A. Stoimenow. Properties of closed 3-braids, June 2006, math/0606435.

Department of Mathematics, Boston College, Chestnut Hill, MA-02467

E-mail address: ian.banfield@bc.edu