Second moments of incomplete Eisenstein series and applications

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We prove a second moment formula for incomplete Eisenstein series on the homogeneous space $\Gamma\backslash G$ with G the orientation preserving isometry group of the real $(n+1)$-dimensional hyperbolic space and $\Gamma \subset G$ a non-uniform lattice. This result generalizes the classical Rogers' second moment formula for Siegel transform on the space of unimodular lattices. We give two applications of this moment formula. In Chapter 5 we prove a logarithm law for unipotent flows making cusp excursions in a non-compact finite-volume hyperbolic manifold. In Chapter 6 we study the counting problem counting the number of orbits of Γ-translates in an increasing family of generalized sectors in the light cone, and prove a power saving estimate for the error term for a generic Γ-translate with the exponent determined by the largest exceptional pole of corresponding Eisenstein series. When Γ is taken to be the lattice of integral points, we give applications to the primitive lattice points counting problem on the light cone for a generic unimodular lattice coming from $\mathrm{SO}_0(n+1,1)(\mathbb{Z})\backslash\mathrm{SO}_0(n+1,1)$.

Contents

Citations to Previous Work

Most of this thesis (Chpater 2 to Chapter 5) is based on [Yu17]. We also borrow some ideas from [KY18] (joint with Dubi Kelmer) to clarify some of the computations in Chapter 4.

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In memory of my grandparents

Chapter 1

Introduction

Let us first fix some notation. For two quantities x and y, we write $x \asymp y$ to indicate that there is some constant $c > 1$ such that $c^{-1}x \leq y \leq cx$. And we write $x \lesssim y$ or $x = O\left(y\right)$ to indicate that $x \leq cy$ for some positive constant c. We will use subscripts to indicate the dependence of the bounding constant on certain parameters.

1.1 Second moments of incomplete Eisenstein series

For any positive integer n, an *unimodular lattice of rank* n is a discrete subgroup of $\mathbb{R}^n$ with covolume one. The space of rank n unimodular lattices can be parameterized by the homogeneous space $\mathcal{X}_n := \mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R})$ via the map sending $\mathrm{SL}_n(\mathbb{Z})g \in \mathcal{X}_n$ to the lattice $\Lambda = \mathbb{Z}^n g$. For any bounded and compactly supported function $f : \mathbb{R}^n \to \mathbb{C}$, its Siegel transform is a function on $\mathcal{X}_n$ defined by

$$\widehat{f}(\Lambda) = \sum_{\vec{v} \in \Lambda\backslash\{0\}} f(\vec{v})$$

with $\Lambda \in \mathcal{X}_n$. There is a natural probability measure μ_n on $\mathcal{X}_n$ coming from the Haar measure of $\mathrm{SL}_n(\mathbb{R})$. In [Sie45] Siegel proved a mean value theorem (first moment formula) stating that

$$\int_{\mathcal{X}_n} \widehat{f}(\Lambda) d\mu_n(\Lambda) = \int_{\mathbb{R}^n} f(\vec{x}) d\vec{x},$$

where $d\vec{x}$ is the usual Lebesgue measure on $\mathbb{R}^n$. Later Rogers [Rog55] proved a k^{th} moment formula for $\widehat{f}$ for any $k < n$. At the heart of his proof is the following identity ([Rog55, Theorem 2 and Equation 8]) that for any $k < n$ and for any bounded and compactly supported functions $f_1, \ldots, f_k$ on $\mathbb{R}^n$,

$$\int_{\mathcal{X}_n} \sum_{\substack{\vec{v}_1,\ldots,\vec{v}_k \in \Lambda \\ \text{linearly independent}}} f_1(\vec{v}_1) \cdots f_k(\vec{v}_k) d\mu_n(\Lambda) = \int_{\mathbb{R}^n} f_1(\vec{x}) d\vec{x} \cdots \int_{\mathbb{R}^n} f_k(\vec{x}) d\vec{x}. \quad (1.1.1)$$

Note that when $k = 1$ this is just Siegel's mean value theorem. Although these higher moment formulas are explicitly written down in [Rog55], they are very complicated in general. It is usually more convenient to work with the modified Siegel transform when dealing with these moment formulas. Explicitly, for any f as above, its modified Siegel transform is defined by

$$\widetilde{f}(\Lambda) = \sum_{\vec{v} \in \Lambda_{\mathrm{pr}}} f(\vec{v})$$

with $\Lambda \in \mathcal{X}_n$ and Λ_{pr} denoting the set of primitive vectors in Λ. Using the fact that any lattice point can be uniquely written as positive multiple of a primitive lattice point, one can deduce that (1.1.1) is equivalent to

$$\int_{\mathcal{X}_n} \sum_{\substack{\vec{v}_1,\ldots,\vec{v}_k \in \Lambda_{\mathrm{pr}} \\ \text{linearly independent}}} f_1(\vec{v}_1) \cdots f_k(\vec{v}_k) d\mu_n(\Lambda) = \frac{\int_{\mathbb{R}^n} f_1(\vec{x}) d\vec{x} \cdots \int_{\mathbb{R}^n} f_k(\vec{x}) d\vec{x}}{\zeta(n)^k}, \quad (1.1.2)$$

where $\zeta(s)$ is the Riemann zeta function. The benefit of restricting to primitive lattice points is the simple fact that two primitive vectors $\vec{v}_1, \vec{v}_2 \in \Lambda$ are linearly dependent if and only if $\vec{v}_1 = \pm\vec{v}_2$. With this observation and (1.1.2) one can get relatively simpler moment formulas for $\widetilde{f}$. For example, for any bounded, compactly supported and even function f, we have the second moment formula for $n \geq 3$

$$\int_{\mathcal{X}_n} \left(\widetilde{f}(\Lambda)\right)^2 d\mu_n(\Lambda) = \left(\frac{\int_{\mathbb{R}^n} f(\vec{x})d\vec{x}}{\zeta(n)}\right)^2 + \frac{2\int_{\mathbb{R}^n} f^2(\vec{x})d\vec{x}}{\zeta(n)}, \tag{1.1.3}$$

and the third moment formula for $n \geq 4$

$$\int_{\mathcal{X}_n} \left(\widetilde{f}(\Lambda)\right)^3 d\mu_n(\Lambda) = \left(\frac{\int_{\mathbb{R}^n} f(\vec{x})d\vec{x}}{\zeta(n)}\right)^3 + \frac{6\int_{\mathbb{R}^n} f^2(\vec{x})d\vec{x}\int_{\mathbb{R}^n} f(\vec{x})d\vec{x}}{\zeta(n)^2} + \frac{4\int_{\mathbb{R}^n} f^3(\vec{x})d\vec{x}}{\zeta(n)}. \tag{1.1.4}$$

Remark 1.1.5. The restriction to even functions is harmless since we can always write $f = f_{\text{odd}} + f_{\text{even}}$ with $f_{\text{odd}}(\vec{x}) = \frac{f(\vec{x})-f(-\vec{x})}{2}$ and $f_{\text{even}}(\vec{x}) = \frac{f(\vec{x})+f(-\vec{x})}{2}$, and Λ_{pr} is invariant under inversion. We thus have $\widetilde{f} = \widetilde{f}_{\text{even}}$.

Rogers' formulas, especially the second moment formula, were since used in many applications of metric number theory. In fact, if we take $f = \chi_B$ to be the indicator function of some finite-volume Borel set $B \subset \mathbb{R}^n \backslash \{0\}$, then for any $\Lambda \in \mathcal{X}_n$,

$$\widehat{f}(\Lambda) = \sum_{\vec{v} \in \Lambda \backslash \{0\}} \chi_B(\vec{v}) = \#(\Lambda \cap B)$$

counts the number of lattice points of Λ in B, and

$$\widetilde{f}(\Lambda) = \sum_{\vec{v} \in \Lambda_{\text{pr}}} f(\vec{v}) = \#(\Lambda_{\text{pr}} \cap B)$$

counts the number of primitive lattice points of Λ in B. With this interpretation, Schmidt [Sch60] studied the lattice point counting problem counting the number of (primitive) lattice points in a Borel set $B \subset \mathbb{R}^n \backslash \{0\}$. Using the second moment formula (1.1.3), he proved an optimal bound for the discrepancy for a generic lattice. In [AM09] Athreya and Margulis used (1.1.3) to prove a random version of Minkowski theorem studying the set of lattices missing a large set, from which they deduced a logarithm law (see section 1.2 for more details) for unipotent flows making cusp excursions on $\mathcal{X}_n$ for $n \geq 2$. We note that while the second moment formula fails when $n = 2$, using spectral results from [Lan85, Ran70], they were able to prove a variant of (1.1.3). More recently Athreya and Margulis [AM18] proved an effective and quantitative version of Oppenheim conjecture for generic forms using (1.1.3). The second moment formula was also used in [SS06, SÏ3] to study values of Epstein zeta functions. Rogers' higher moment formulas were used in [SS16, Kim16] to study random lattices in large dimension.

The starting point of this thesis work is an observation that modified Siegel transform can be viewed as an incomplete Eisenstein series on $\mathcal{X}_n$ with respect to some maximal parabolic subgroup of $\mathrm{SL}_n(\mathbb{R})$. The advantage of this point of view is that it can be generalized to any other non-compact homogeneous spaces, and it is thus possible to use the spectral theory to prove many other similar moment formulas. For example, one can generalize Siegel's mean value theorem (first moment formula) easily for any incomplete Eisenstein series by the standard unfolding trick (see remark 4.1.3). Moreover, with these moment formulas, one can also hope to give applications to analogous problems mentioned in previous paragraph on different homogeneous spaces.

In this thesis, we give a second moment formula of the incomplete Eisenstein series on the frame bundle of a non-compact hyperbolic manifold generalizing a result of [KM12] on two and three dimensional hyperbolic manifolds. With this moment formula, we prove a logarithm law concerning unipotent flows making cusp excursions on a non-compact finite-volume hyperbolic manifold. We also give applications to a counting problem on the corresponding quadratic variety, namely the light cone.

To describe our main result properly, we first introduce some notation. From now on we fix an integer $n \geq 2$. Let $\mathbb{H}^{n+1}$ be the $(n+1)$-dimensional real hyperbolic space and $G = \mathrm{Iso}^+(\mathbb{H}^{n+1})$ be the orientation preserving isometry group of $\mathbb{H}^{n+1}$. Since G is of real rank one (that is, its maximal $\mathbb{R}$-split tours is of rank one), it has only one parabolic subgroup up to conjugacy. Fix an Iwasawa decomposition $G = NAK$ with N a maximal unipotent subgroup of G, A a maximal $\mathbb{R}$-split torus and K a maximal compact subgroup. There is a unique parabolic subgroup P attached to this Iwasawa decomposition containing N as its unipotent radical. This parabolic has a Langlands decomposition $P = NAM$ with N and A as above, and $M \subset K$ being the centralizer of A in K. Finally we denote by $Q = NM$.

For a discrete subgroup Γ of G, we say Γ is a *lattice* if the homogeneous space $\Gamma\backslash G$ is of finite covolume with respect to the Haar measure of G. We say a lattice Γ is *non-uniform* (resp. *co-compact*) if $\Gamma\backslash G$ is non-compact (resp. compact). Given a non-uniform lattice $\Gamma \subset G$, two parabolic subgroups are said to be Γ-*equivalent* (resp. Γ-*inequivalent*) if they are conjugate (resp. not conjugate) under Γ. The *cusps* of Γ are the Γ-equivalent classes of parabolic subgroups whose unipotent radicals intersect Γ nontrivially. Let $P_1, \ldots, P_h$ be a complete set of representatives of cusps of Γ, then there exists a subset $\{\xi_1, \ldots, \xi_h\} \subset K$ such that $\xi_j^{-1} P_j \xi_j = P$ for each $1 \le j \le h$ (see section 2.3 for more details).

Keep the notation as above. For any bounded and compactly supported function f on the homogeneous space $Q\backslash G$ (that we view as a left Q-invariant function on G), the *incomplete Eisenstein series at* P_j attached to f, denoted by Θ_f^j, is a function on $\Gamma\backslash G$ defined as

$$\Theta_f^j(g) = \sum_{\gamma \in \Gamma_{P_j}\backslash\Gamma} f\left(\xi_j^{-1}\gamma g\right),$$

where $\Gamma_{P_j} := \Gamma \cap P_j$. We can now state our main result.

Theorem 1. *Fix $n \ge 2$. Let $G = Iso^+(\mathbb{H}^{n+1})$ and $P \subset G$ be a fixed parabolic subgroup. Let $\Gamma \subset G$ be a non-uniform lattice with $P_1, \ldots, P_h$ a complete set of* *representatives of cusps. Then for each* $1 \leq j \leq h$*, there exists a finite set of numbers* $\frac{n}{2} < s_{j\ell_j} < \cdots < s_{j1} < n$*,* $c_{j0} > 0, c_{j1}, \ldots, c_{j\ell_j}$*, and a bounded linear operator* $\mathcal{T}_j : L^2(Q\backslash G) \to L^2(Q\backslash G)$ *with* $\|\mathcal{T}_j\|_2 \leq 1$ *such that for any bounded and compactly supported* f *on* $Q\backslash G$*,*

$$\int_{\Gamma\backslash G} \left|\Theta_f^j(g)\right|^2 d\mu(g) = |c_{j0}\langle f, 1\rangle|^2 + c_{j0}\langle f + \mathcal{T}_j(f), f\rangle + c_{j0}\sum_{r=1}^{\ell_j} c_{jr} M_f(s_{jr}), \quad (1.1.6)$$

where $\mu = \mu_{\Gamma\backslash G}$ *is the probability measure on* $\Gamma\backslash G$ *coming from the Haar measure of* G*, the inner product* $\langle , \rangle$ *is defined that for any two* f_1, f_2 *on* $Q\backslash G$*,* $\langle f_1, f_2\rangle := \int_{Q\backslash G} f_1 \overline{f_2} d\mu_{Q\backslash G}$ *with* $\mu_{Q\backslash G}$ *a right* G*-invariant measure on* $Q\backslash G$ *normalized as in* (2.2.3)*,* $M_f(s)$ *is defined in* (4.2.2) *and*

$$\|\mathcal{T}_j\|_2 := \inf\{c > 0 \mid \|\mathcal{T}_j(f)\|_2 \leq c\|f\|_2 \text{ for any } f \in L^2(Q\backslash G)\},$$

where this L^2*-norm is with respect to* $\mu_{Q\backslash G}$*.*

Remark 1.1.7. We will make all these constants explicit when we prove Theorem 1 in chapter 4. We note here that these finitely many s_{jr}'s are the exceptional poles of the Eisenstein series at the cusp P_j. More precisely, these exceptional poles come from the constant term of this Eisenstein series along P_j with c_{jr} its residue at s_{jr}. In fact, knowing this constant term is a crucial step for this spectral approach computing the second moment. We also note that the first term of the right-hand side of (1.1.6) (the mean value square term) comes from the trivial pole of this constant term. For applications it is interesting to know the exact values of these exceptional poles. For example, the largest exceptional pole controls the magnitude of the error term of certain counting functions (see Theorem 4). When n is small, there are many cases ([Iwa02, EGM98, Gri87]) where one can compute these constant terms explicitly (and thus know these exceptional poles explicitly). However, for general n not much is known about these exceptional poles. We also note that similar calculations can also be carried out for other groups. In fact in a recent joint work with Kelmer [KY18], with an explicit constant term computation we proved a second moment formula for the modified Siegel transform restricted to the space of symplectic lattices. With this second moment formula we extended Schmidt's counting result [Sch60] to a generic symplectic lattice.

Remark 1.1.8. Another main ingredient of our approach is an explicit computation of certain raising operator to bootstrap (1.1.6) from spherical functions to all other functions (see section 3.2). We note that this raising operator generalizes the classical ladder operators of $\mathrm{SL}_2(\mathbb{R})$ (see [Lan85, p.102]).

Similar to Rogers' formulas, for applications one needs to bound the right-hand side of (1.1.6) from above by the L^1- and L^2-norms of f. For the second term we have the trivial bound $\langle f + \mathcal{T}_j(f), f\rangle \leq 2\|f\|_2^2$ by Cauchy-Schwartz. We thus need to bound these $M_f(s_{jr})$ terms by the norms of f. Unfortunately we are not able to prove such an estimate for any functions on $Q\backslash G$. Instead, we will define a family of functions $\mathscr{A}_\lambda \subset L^2(Q\backslash G)$ (see section 4.3) for each parameter $\lambda > 0$ and prove an optimal bound for these functions.

Theorem 2. *Keep the notation as in Theorem 1. For any parameter $\lambda > 0$ there exists a positive constant C (depending on λ) such that for any $f \in \mathscr{A}_\lambda$ and for any $1 \leq j \leq h$ we have*

$$\int_{\Gamma\backslash G} \left|\Theta_f^j(g)\right|^2 d\mu(g) \leq c_{j0}^2\|f\|_1^2 + 2c_{j0}\|f\|_2^2 + C\sum_{r=1}^{\ell_j}\|f\|_1^{\frac{2(2s_{jr}-n)}{n}}\|f\|_2^{\frac{4(n-s_{jr})}{n}},$$

where the norms on the right-hand side is with respect to $\mu_{Q\backslash G}$.

Remark 1.1.9. The bounding constant C here depends on the parameter λ. For applications we need to have a uniform bounding constant for all the functions that we consider. In fact for applications we will only consider indicator functions of certain sets in $Q\backslash G$ and we will show that these functions are always contained in $\mathscr{A}_2$ (see Lemma 4.3.1).

Remark 1.1.10. Using the facts that $0 < \frac{2s-n}{n}, \frac{2(n-s)}{n} < 1$ and $\frac{2s-n}{n} + \frac{2(n-s)}{n} = 1$ for $s \in (\frac{n}{2}, n)$, it is easy to deduce from Theorem 2 that for any $f \in \mathscr{A}_\lambda$,

$$\|\Theta_f^j\|_2^2 \lesssim_{\lambda,\Gamma} \|f\|_1^2 + \|f\|_2^2.$$

We note that for modified Siegel transform this estimate follows trivially from Rogers' second moment formula, and if Γ has no exceptional poles, it also follows trivially from our moment formula (1.1.6). It is an interesting question whether one can extend this bound to any bounded and compactly supported function on $Q\backslash G$ when Γ has exceptional poles.

1.2 Logarithm laws

Let G denote a connected semisimple Lie group with no compact factors and $\Gamma \subset G$ be a non-uniform irreducible lattice. Let ν be the probability measure on $\Gamma\backslash G$ coming from the Haar measure of G. Any unbounded one-parameter subgroup $\{g_t\}_{t\in\mathbb{R}} \subset G$ acts on $\Gamma\backslash G$ by right multiplication. By Moore's Ergodicity Theorem [Zim84, Theorem 2.2.6] this action is ergodic with respect to ν, hence for ν-a.e. $x \in \Gamma\backslash G$ the orbit $\{xg_t\}_{t\in\mathbb{R}}$ is dense. In particular, since $\Gamma\backslash G$ is non-compact, these orbits will make excursions into the cusp(s) of $\Gamma\backslash G$. The asymptotics of cusp excursions of these orbits is an interesting object in homogeneous dynamics due to its rich connections with metric number theory.

One way to characterize cusp excursions is to use a distance function. Fix a maximal compact subgroup K of G, then there is a naturally defined distance function, dist, on $\Gamma\backslash G$ induced from a left G-invariant and bi-K-invariant Riemannian metric on G. Fix a base point $o \in \Gamma\backslash G$ and for any $r > 0$ define the cusp neighborhoods by

$$B_r := \{x \in \Gamma\backslash G \mid \text{dist}(o, x) > r\}.$$

By [KM99] there exists a constant $\varkappa > 0$ such that

$$\nu(B_r) \asymp e^{-\varkappa r}. \tag{1.2.1}$$

Note that the orbit xg_t makes excursions into the cusp neighborhood B_r if and only if $\text{dist}(o, xg_t) > r$. We are thus interested in studying how fast the distance function $\text{dist}(o, xg_t)$ can grow in t for a generic x. There is a natural upper bound for this distance function coming from the first half of Borel-Cantelli lemma asserting that $\overline{\lim}_{t\to\infty} \frac{\text{dist}(o,xg_t)}{\log(t)} \leq \frac{1}{\varkappa}$ for a generic x, where $\varkappa$ is the exponent as in (1.2.1) (see section 5.2.1). If this upper bound is sharp, that is,

$$\overline{\lim_{t\to\infty}} \frac{\text{dist}(o, xg_t)}{\log(t)} = \frac{1}{\varkappa},$$

for ν-a.e. $x \in \Gamma\backslash G$, we say that the flow $\{g_t\}_{t\in\mathbb{R}}$ satisfies the *logarithm law* for cusp excursion.

The problem of logarithm law (for cusp excursions) in the context of homogeneous space was first studied by Sullivan [Sul82] where he proved logarithm laws for geodesic flows on non-compact finite-volume hyperbolic manifolds. The general case of one-parameter diagonalizable flows on non-compact finite-volume homogeneous spaces was proved by Kleinbock and Margulis [KM99]. The main ingredient of their proof is the exponential decay of matrix coefficients of diagonalizable flows which enables them to apply a quantitative Borel-Cantelli lemma ([KM99, Lemma 2.6]) from which the logarithm law follows easily.

The problem of logarithm laws for unipotent flows is more subtle since the matrix coefficients of unipotent flows only decay polynomially, and it is not clear whether their decay rates would be enough for the quantitative Borel-Cantelli lemma used in [KM99]. Nevertheless, as mentioned in the previous section, using Rogers' second moment formula Athreya and Margulis [AM09] proved logarithm laws for one-parameter unipotent flows on the space of unimodular lattices. Later Kelmer and Mohanmmadi [KM12] proved the case when G is a product of copies of $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{C})$ and Γ is any irreducible non-uniform lattice. Their proof also relies on an estimate of second moments of certain incomplete Eisenstein series.

For a general homogeneous space $\Gamma\backslash G$ with an unipotent flow $\{g_t\}_{t\in\mathbb{R}}$, Athreya and Margulis [AM17] proved that there exists some $0<\beta\leq 1$ such that

$$\overline{\lim_{t\to\infty}}\frac{\mathrm{dist}\left(o, xg_t\right)}{\log\left(t\right)}=\frac{\beta}{\varkappa}$$

for ν-a.e. $x\in\Gamma\backslash G$. Moreover, they asked whether such β can always attain 1, which is the upper bound coming from the first half of Borel-Cantelli lemma

With our second moment formula (1.1.6) and using similar arguments as in [KM12], we give a positive answer to this question when $G=\mathrm{Iso}^{+}\left(\mathbb{H}^{n+1}\right)$ and $\Gamma\subset G$ is a non-uniform lattice. We note that this result extends Sullivan's original logarithm law for geodesic flows to unipotent (horocycle) flows.

Theorem 3. *Fix $n\geq 2$. Let $G=Iso^{+}\left(\mathbb{H}^{n+1}\right)$, $\Gamma\subset G$ a non-uniform lattice and $\{g_t\}_{t\in\mathbb{R}}$ a one-parameter unipotent subgroup of G. Let $dist(\cdot,\cdot)$ denote the distance function obtained from hyperbolic metric on the hyperbolic manifold $\Gamma\backslash\mathbb{H}^{n+1}$. Then for any fixed $o\in\Gamma\backslash G$,*

$$\overline{\lim_{t\to\infty}}\frac{dist\left(o, xg_t\right)}{\log t}=\frac{1}{n}, \tag{1.2.2}$$

for μ-a.e. $x\in\Gamma\backslash G$[1], where μ is the probability measure on $\Gamma\backslash G$ as in Theorem 1.

[1]Here by slight abuse of notation, for $o, xg_t\in\Gamma\backslash G$, we write $\mathrm{dist}\left(o, xg_t\right)$ for the distance between their projections to $\Gamma\backslash\mathbb{H}^{n+1}$.

Remark 1.2.3. Shortly after Theorem 3 ([Yu17, Theorem 1.1]) was proved, Kelmer [Kel17] proved the same result using an effective mean ergodic theorem. In fact he proved a much more general result, namely a logarithm law for discrete time unipotent flows making visits to any monotone shrinking family of sets in $\Gamma\backslash\mathbb{H}^{n+1}$ (not just cusp neighborhoods). We note that in the cusp excursion case, the logarithm laws for continuous and discrete time flows are equivalent. Later in a joint work with Kelmer [KY17], we studied more general shrinking target problems on homogeneous spaces, which in particular, implies logarithm laws for cusp excursions for unipotent flows on a non-compact finite-volume homogeneous space. Finally we also note that this logarithm law problem is also closed related to the classical first hitting time problem in dynamical systems (see [GP10]).

1.3 Counting on the light cone

Keep the notation as in Theorem 1. When taking the hyperboloid model for $\mathbb{H}^{n+1}$, $G = \mathrm{Iso}^+(\mathbb{H}^{n+1})$ can realized as $\mathrm{SO}_0(n+1,1)$, the identity component of the special orthogonal group preserving the standard quadratic form $\mathcal{Q}_0(v_0,\dots,v_{n+1}) = v_0^2 + \cdots + v_n^2 - v_{n+1}^2$ of signature $(n+1,1)$. In this setting we can identify the homogeneous space $Q\backslash G$ with the positive light cone

$$\mathcal{V}^+ := \{\vec{v} = (v_0,\dots,v_n,v_{n+2}) \in \mathbb{R}^{n+1} \mid \mathcal{Q}_0(\vec{v}) = 0, v_{n+1} > 0\}$$

by identifying Qg with $\vec{e}_0 g$ with $\vec{e}_0 := (0,\dots,0,-1,1) \in \mathcal{V}^+$ fixed by Q under right multiplication.

For the classical primitive lattice point counting problem, we can view the set of primitive vectors of a unimodular lattice as orbits of a translate of $\mathrm{SL}_m(\mathbb{Z})$. That is, given a unimodular lattice $\Lambda = \mathbb{Z}^m g \in \mathcal{X}_m$, we have $\Lambda_{\mathrm{pr}} = \vec{v}\,\mathrm{SL}_m(\mathbb{Z})g$ with $\vec{v} \in \mathbb{Z}^m_{\mathrm{pr}}$

some fixed base vector. With this interpretation we can generalize this primitive lattice point counting problem to our setting. More precisely, fix a non-uniform lattice $\Gamma \subset G$ with a cusp at P and we take $P = P_1$ to be one of the representatives of cusps of Γ. For any finite-volume (with respect to $\mu_{Q\backslash G}$) Borel set $B \subset \mathcal{V}^+$ and any Γ-translate Γg we would like to study the counting function

$$\mathcal{L}(B, \Gamma g) := \#\left(B \cap \vec{e}_0 \Gamma g\right)$$

counting the number of orbits of Γg in B. Similar to the modified Siegel transform, we can interpret this counting function as an incomplete Eisenstein series. In fact for $f = \chi_B$ and $g \in G$ we will show that $\Theta_f^1(g) = \mathcal{L}(B, \Gamma g)$. From a mean value theorem for this incomplete Eisenstein series that we will prove later on (remark 4.1.3), we see that the expected term for this counting function is $c_{10}\mu_{Q\backslash G}(B)$ with c_{10} a positive constant depending on Γ as in Theorem 1. We thus define the remainder function

$$\mathcal{E}(B, \Gamma g) := \left|\mathcal{L}(B, \Gamma g) - c_{10}\mu_{Q\backslash G}(B)\right|.$$

In spirt of Schmidt's classical results [Sch60], given a linearly ordered (with respect to inclusion) family, $\mathcal{B}$, of finite-volume Borel sets in $\mathcal{V}^+$, we would like to study the asymptotics of the remainder function $\mathcal{E}(B, \Gamma g)$ for a generic lattice translate Γg with B coming from $\mathcal{B}$. Following Schmidt's arguments, we can use our second moment formula (1.1.6) to get the following bound for $\mathcal{E}(B, \Gamma g)$ for a generic Γg.

Theorem 4. *Keep notation as in Theorem 1. Let $G = \mathrm{SO}_0(n+1, 1)$ and $\Gamma \subset G$ a non-uniform lattice with a cusp at P. For any linearly ordered family, $\mathcal{B}$, of generalized sectors in $\mathcal{V}^+$, for μ-a.e. $\Gamma g \in \Gamma\backslash G$, there exists $C_{\Gamma g}$ such that for any $B \in \mathcal{B}$ with $\mu_{Q\backslash G}(B) > C_{\Gamma g}$*

$$\mathcal{E}(B, \Gamma g) \leq \mu_{Q\backslash G}(B)^{\frac{s_{11}}{n}} \log^{3/2}\left(\mu_{Q\backslash G}(B)\right),$$

where $s_{11} \in (\frac{n}{2}, n)$ is some constant depending on Γ as in Theorem 1.

Remark 1.3.1. We will make it precise in chapter 6 for what we mean by a generalized sector. Our main tool for Theorem 4 is a mean square bound for the remainder function deduced from Theorem 2. Unlike Rogers' second moment formula, our moment formula has a third term coming form exceptional poles. The reader should note that we state our result (and Theorem 5) implicitly assuming that Γ has exceptional poles at P. If Γ has no such exceptional poles, then the exponent $\frac{s_{11}}{n}$ can be replaced by $\frac{1}{2}$ following Schmidt's original argument. Moreover, in this case the geometric assumption on $\mathcal{B}$ can also be removed. On the other hand, although having these exceptional poles is problematic for our analysis, we can actually show that the exponent $\frac{s_{11}}{n}$ (if it exists) in our bound is optimal in the sense that one can choose $\mathcal{B}$ such that the mean square bound we get from Theorem 2 is optimal for sets from $\mathcal{B}$ (see remark 6.1.4).

When $\Gamma = \mathrm{SO}_0(n+1,1)(\mathbb{Z})$ is taken to be the lattice of integral points, the homogeneous space $\Gamma\backslash G$ naturally embeds into the space of rank $n+2$ unimodular lattices, $\mathcal{X}_{n+2}$, as a null set (with respect to the Haar measure of $\mathrm{SL}_{n+2}(\mathbb{R})$). We can thus view elements in $\Gamma\backslash G$ as rank $n+2$ unimodular lattices. This interpretation gives another (perhaps more natural) way of generalizing the classical primitive lattice point counting problem. To be precise, for any finite-volume Borel set $B \subset \mathcal{V}^+$ and any $\Lambda \in \Gamma\backslash G$ (that we view as a rank $n+2$ unimodular lattice) we define

$$\mathcal{N}_{\mathrm{pr}}(B,\Lambda) := \#(B \cap \Lambda_{\mathrm{pr}})$$

to the the counitng function counting the number of primitive vectors of Λ in B. We note that this problem makes sense since for Λ coming from $\Gamma\backslash G$, $\Lambda_{\mathrm{pr}} \cap \mathcal{V}^+$ is an infinite set. We also note that for the classical primitive lattice point counting problem, the problem of counting orbits of $\mathrm{SL}_m(\mathbb{Z})$-translates and the problem of counting primitive lattice points are the same since the action of $\mathrm{SL}_m(\mathbb{Z})$ on $\mathbb{Z}^m_{\mathrm{pr}}$ is transitive. However, this is not the case in our setting. The action of $\mathrm{SO}_0(n+1,1)(\mathbb{Z})$ on $\mathcal{V}^+(\mathbb{Z})_{\mathrm{pr}} := \mathbb{Z}^{n+2}_{\mathrm{pr}} \cap \mathcal{V}^+$, the set of primitive integral points of $\mathcal{V}^+$, is not transitive in general. In fact, we will show that the number of orbits of the $\mathrm{SO}_0(n+1,1)(\mathbb{Z})$-action on $\mathcal{V}^+(\mathbb{Z})_{\mathrm{pr}}$ equals exactly the number of cusps of $\mathrm{SO}_0(n+1,1)(\mathbb{Z})$. This way we can interpret this counting function $\mathcal{N}_{\mathrm{pr}}(B,\Lambda)$ as a sum of certain incomplete Eisenstein series at all cusps. More precisely, we will show that for $f = \chi_B$ and $\Lambda = \mathbb{Z}^{n+2}g \in \Gamma\backslash G$

$$\mathcal{N}_{\mathrm{pr}}(B,\Lambda) = \sum_{j=1}^{h} \Theta^j_{f_j}(g), \tag{1.3.2}$$

with f_j indicator function of the dilation $\lambda_j B$ of B for some $\lambda_j > 0$. Again by the mean value theorem for $\Theta^j_{f_j}$, we see that the expected term for $\mathcal{N}_{\mathrm{pr}}(B,\Lambda)$ is $\sum_{j=1}^{h} c_{j0}\mu_{Q\backslash G}(\lambda_j B)$. We thus define

$$\mathcal{R}_{\mathrm{pr}}(B,\Lambda) := \left| \mathcal{N}_{\mathrm{pr}}(B,\Lambda) - \sum_{j=1}^{h} c_{j0}\mu_{Q\backslash G}(\lambda_j B) \right|$$

to be the corresponding remainder function. With (1.3.2) following similar arguments as for Theorem 4 we have the following bound for $\mathcal{R}_{\mathrm{pr}}(B,\Lambda)$ for a generic Λ coming from $\Gamma\backslash G$.

Theorem 5. *Keep the notation as in Theorem 1. Let* $G = \mathrm{SO}_0(n+1,1)$ *and* $\Gamma = \mathrm{SO}_0(n+1,1)(\mathbb{Z})$ *be the lattice of integral points. For any linearly ordered family,* $\mathcal{B}$*, of generalized sectors in* $\mathcal{V}^+$*, for* μ*-a.e.* $\Lambda \in \Gamma\backslash G$*, there exists* C_Λ *such that for any* $B \in \mathcal{B}$ *with* $\mu_{Q\backslash G}(B) > C_\Lambda$

$$\mathcal{R}_{\mathrm{pr}}(B,\Lambda) \leq \mu_{Q\backslash G}(B)^{\frac{s_\Gamma}{n}} \log^{3/2}\left(\mu_{Q\backslash G}(B)\right),$$

where $s_\Gamma = \max\{s_{j1} \mid 1 \leq j \leq h\}$ *with* s_{j1} *as in Theorem 1.*

Chapter 2

Preliminaries and notation

2.1 Two different hyperbolic models

Fix $n \geq 2$. Let $\mathbb{H}^{n+1}$ be the $(n+1)$-dimensional real hyperbolic space and $G = \text{Iso}^+(\mathbb{H}^{n+1})$ be its orientation preserving isometry group. There are various hyperbolic models of $\mathbb{H}^{n+1}$ and each model gives an explicit description of G. While the upper half space model suits for logarithm laws, the hyperboloid model is a more natural choice for applications to counting. We will thus use both models in this thesis. We will prove our main result (Theorem 1) using the upper half space model. One should note that the same calculation can be translated into the hyperboloid model without difficulty.

2.1.1 The upper half space model

When choosing the upper half space model for $\mathbb{H}^{n+1}$, the isometry group G can be realized via the Vahlen group which is defined as certain two by two matrices over the Clifford algebra (see[Ahl85, EGM87, EGM90] for more details about Vahlen group).

We note that this naturally generalizes the classical $\mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{PSL}_2(\mathbb{C})$-actions on the two and three-dimensional upper half spaces.

We first briefly recall some facts about Clifford algebra. The *Clifford algebra* $C\ell_n$ is an associative algebra over $\mathbb{R}$ with n generators $e_1, \ldots, e_n$ satisfying relations $e_j^2 = -1, e_j e_l = -e_l e_j, j \neq l$. Let $\mathcal{P}_n$ be the set of subsets of $\{1, \cdots, n\}$. For $J = \{j_1, \cdots, j_r\} \in \mathcal{P}_n$ with $j_1 < \cdots < j_r$ we define $e_J := e_{j_1} \cdots e_{j_r}$ and $e_\emptyset = 1$. These 2^n elements $e_J\,(J \in \mathcal{P}_n)$ form a basis of $C\ell_n$. The Clifford algebra $C\ell_n$ has a main anti-involution * and a main involution $'$. Explicitly, their actions on the basis elements are given by $(e_{j_1} \cdots e_{j_r})^* = e_{j_r} \cdots e_{j_1}$ and $(e_{j_1} \cdots e_{j_r})' = (-1)^r e_{j_1} \cdots e_{j_r}$. Their composition $\overline{e_J} := (e_J')^*$ gives the conjugation map on $C\ell_n$.

For any $1 \leq j \leq n$, let $\mathbb{V}^j$ denote the real vector space spanned by $1, e_1, \ldots, e_j$. Note that $\dim \mathbb{V}^j = j + 1$. The Clifford group T_j is defined to be the collection of all finite products of non-zero elements from $\mathbb{V}^j$ with group operation given by multiplication. There is a well-defined norm on $\mathbb{V}^n$ given by $|v| = \sqrt{v\bar{v}}$ and it extends multiplicatively to a norm on T_n.

In this setting, the $(n+1)$-dimensional hyperbolic space model is the upper half space

$$\mathbb{H}^{n+1} := \{x_0 + x_1 e_1 + \cdots + x_n e_n \in \mathbb{V}^n \mid (x_0, \ldots, x_{n-1}) \in \mathbb{R}^n, x_n > 0\} \tag{2.1.1}$$

endowed with the Riemannian metric

$$ds^2 = \frac{dx_0^2 + \cdots + dx_n^2}{x_n^2}. \tag{2.1.2}$$

Let $M_2\,(C\ell_n)$ be the set of 2×2 matrices over $C\ell_n$. The Vahlen group $\mathrm{SL}_2(T_{n-1})$ is defined by

$$\mathrm{SL}_2(T_{n-1}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2\left(C\ell_n\right) \;\middle|\; \begin{array}{l} a,b,c,d \in T_{n-1} \cup \{0\}, \\ ab^*, cd^* \in \mathbb{V}^{n-1}, \\ ad^* - bc^* = 1 \end{array} \right\}.$$

Elements in $\mathrm{SL}_2(T_{n-1})$ act on $\mathbb{H}^{n+1}$ as isometries via the Möbius transformation: for any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(T_{n-1})$ and $v \in \mathbb{H}^{n+1}$

$$g \cdot v = (av + b)\,(cv + d)^{-1}. \tag{2.1.3}$$

This gives a surjective homomorphism from $\mathrm{SL}_2(T_{n-1})$ to $\mathrm{Iso}^+\left(\mathbb{H}^{n+1}\right)$ with kernel $\pm I_2$, where I_2 is the 2 by 2 identity matrix. Hence G is realized as $\mathrm{PSL}_2(T_{n-1}) := \mathrm{SL}_2(T_{n-1})/\{\pm I_2\}$.

We fix an Iwasawa decomposition

$$\mathrm{PSL}_2(T_{n-1}) = NAK,$$

with

$$N = \left\{ u_{\mathbf{x}} = \begin{pmatrix} 1 & \mathbf{x} \\ 0 & 1 \end{pmatrix} \mid \mathbf{x} = x_0 + x_1 e_1 + \cdots + x_{n-1} e_{n-1} \in \mathbb{V}^{n-1} \right\},$$

$$A = \left\{ a_t = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix} \mid t \in \mathbb{R} \right\},$$

and

$$K = \left\{ g \in \mathrm{SL}_2(T_{n-1}) \mid g \cdot e_n = e_n \right\} / \{\pm I_2\}$$

is the stabilizer of e_n. From this we can identify G/K with $\mathbb{H}^{n+1}$ by sending gK to $g \cdot e_n$. Finally we note that an element in K is of the form $\begin{pmatrix} q_2' & -q_1' \\ q_1 & q_2 \end{pmatrix}$ with $|q_1|^2 + |q_2|^2 = 1$ and $q_1 {q_2}^* \in \mathbb{V}^{n-1}$.

With the above fixed Iwasawa decomposition, we fix the parabolic subgroup $P = NAM$ with M the centralizer of A in K. Explicitly, M is the subgroup of K consisting of diagonal matrices, and P is the upper triangular subgroup of G. For later use we note that K is isomorphic to $\mathrm{SO}(n+1)$, M is isomorphic to $\mathrm{SO}(n)$ and $M\backslash K$ can be identified with the n-sphere S^n via the map

$$\begin{aligned} M\backslash K &\longrightarrow S^n := \left\{x_0 + x_1e_1 + \cdots x_ne_n \mid x_0^2 + x_1^2 + \cdots x_n^2 = 1\right\} \\ \begin{pmatrix} q_2' & -q_1' \\ q_1 & q_2 \end{pmatrix} &\longmapsto 2\overline{q_1}q_2 + \left(|q_2|^2 - |q_1|^2\right) e_n. \end{aligned}$$

We note that this map is well-defined since $q_1q_2^* \in \mathbb{V}^{n-1}$ if and only if $\overline{q_1}q_2 \in \mathbb{V}^{n-1}$ ([Par, Corollary 7.15]).

2.1.2 The hyperboloid model

When choosing the hyperboloid model

$$\left\{\vec{v} = (v_0, \ldots, v_{n+1}) \in \mathbb{R}^{n+2} \mid \mathcal{Q}_0(\vec{v}) = -1, v_{n+1} > 0\right\},$$

for $\mathbb{H}^{n+1}$, where $\mathcal{Q}_0(\vec{v}) = v_0^2 + \cdots + v_n^2 - v_{n+1}^2$ is the standard quadratic form of signature $(n+1, 1)$, $G = \mathrm{Iso}^+(\mathbb{H}^{n+1})$ is realized as $\mathrm{SO}_0(n+1, 1)$, the identity component of the special orthogonal group

$$\mathrm{SO}(n+1,1) = \{X \in \mathrm{SL}_{n+2}(\mathbb{R}) \mid X^tJX = J\}$$

preserving $\mathcal{Q}_0$, where $J = \mathrm{diag}(1, \ldots, 1, -1)$. In this case $G = \mathrm{SO}_0(n+1, 1)$ acts on $\mathbb{H}^{n+1}$ as isometry via right multiplication: for any $\vec{v} \in \mathbb{H}^{n+1}$ and $g \in G$, $g \cdot \vec{v} = \vec{v}g^{-1}$. We fix an Iwasawa decomposition

$$G = NAK$$

with

$$N = \left\{ u_{\mathbf{x}} = \begin{pmatrix} I_n & -\mathbf{x}^t & \mathbf{x}^t \\ \mathbf{x} & 1-\frac{1}{2}\|\mathbf{x}\|^2 & \frac{1}{2}\|\mathbf{x}\|^2 \\ \mathbf{x} & -\frac{1}{2}\|\mathbf{x}\|^2 & 1+\frac{1}{2}\|\mathbf{x}\|^2 \end{pmatrix} \;\middle|\; \mathbf{x} = (x_0, \dots, x_{n-1}) \in \mathbb{R}^n \right\}, \tag{2.1.4}$$

$$A = \left\{ a_t = \begin{pmatrix} I_n & & \\ & \cosh t & \sinh t \\ & \sinh t & \cosh t \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\} \text{ and } K = \begin{pmatrix} \mathrm{SO}(n+1) & \\ & 1 \end{pmatrix}.$$

Here $\|\mathbf{x}\|^2 = x_0^2 + \cdots + x_{n-1}^2$ is the usual Euclidean norm and I_n is the n by n identity matrix. We note that K is the stabilizer of the vector $(0, \dots, 0, 1)$ and we can thus identify G/K with $\mathbb{H}^{n+1}$ via the map sending gK to $(0, \dots, 0, 1)g^{-1}$.

We fix the parabolic subgroup $P = NAM$ with N, A as above and M the centralizer of A in K. Explicitly, we have $M = \begin{pmatrix} \mathrm{SO}(n) & \\ & I_2 \end{pmatrix} \subset K$.

In the remaining sections of this chapter we will recall some more backgrounds on hyperbolic geometry. We will take G to be either $\mathrm{PSL}_2(T_{n-1})$ or $\mathrm{SO}_0(n+1,1)$ with all other groups fixed as in this section.

2.2 Coordinates and normalization

Keep the notation as in previous section. Let $\Gamma \subset G$ be a non-uniform lattice. In this section, we will normalize the measures on various spaces. First recall that K is isomorphic to $\mathrm{SO}(n+1)$ and $M \subset K$ is isomorphic to $\mathrm{SO}(n)$. Thus we denote by σ_{n+1} and σ_n to be the probability Haar measures on K and M respectively. Next, we denote by $\widetilde{\sigma}_{n+1}$ to be the right K-invariant probability measure on $M\backslash K$. Explicitly, for any function ϕ on K we have

$$\int_K \phi(k)d\sigma_{n+1}(k) = \int_{M\backslash K}\int_M \phi(m\tilde{k})d\sigma_n(m)d\widetilde{\sigma}_{n+1}(\tilde{k}).$$

For the probability measure $\mu = \mu_{\Gamma\backslash G}$ on $\Gamma\backslash G$, we note that under the coordinates $g = u_{\mathbf{x}}a_t k$, the Haar measure (up to scalars) of G is given by

$$dg = e^{-nt}d\mathbf{x}dtd\sigma_{n+1}(k),$$

where $d\mathbf{x}$ is the usual Lebesgue measure on N (identified with $\mathbb{R}^n$). Hence the probability measure μ on $\Gamma\backslash G$ is given by

$$d\mu\,(g) = \frac{1}{\nu_\Gamma}e^{-nt}d\mathbf{x}dtd\sigma_{n+1}(k) \tag{2.2.1}$$

with $\nu_\Gamma = \int_{\Gamma\backslash G} dg$.

For the subgroup $Q = NM$, we note that it is unimodular and we normalize its Haar measure, denoted by μ_Q, such that

$$d\mu_Q(q) = d\mathbf{x}d\sigma_n(m), \tag{2.2.2}$$

where $q = u_{\mathbf{x}}m$ with $u_{\mathbf{x}} \in N$ and $m \in M$. Finally we normalize the measure on $Q\backslash G$. We note that since both G and Q are unimodular, there exists some right G-invariant measure $\mu_{Q\backslash G}$ on $Q\backslash G$ satisfying

$$\int_G f(g)d\mu(g) = \lambda \int_{Q\backslash G}\int_Q f(qg)d\mu_Q(q)d\mu_{Q\backslash G}(g) \tag{2.2.3}$$

for any $f \in \mathcal{C}_c^\infty(G)$ and for some $\lambda > 0$. We then normalize the $\mu_{Q\backslash G}$ such that $\lambda = \frac{1}{\nu_\Gamma}$. Explicitly, if identifying $Q\backslash G$ with $A \times M\backslash K$, for any $a_t \in A$ and $\tilde{k} \in M\backslash K$ we have

$$d\mu_{Q\backslash G}(a_t\tilde{k}) = e^{-nt}dtd\widetilde{\sigma}_{n+1}(\tilde{k}). \tag{2.2.4}$$

Throughout this thesis, for any functions F on $\Gamma\backslash G$, f on $Q\backslash G$ and ϕ on $M\backslash K$, we denote by $\|F\|_2$, $\|f\|_2$ and $\|\phi\|_2$ for their L^2-norms with respect to μ, $\mu_{Q\backslash G}$ and $\widetilde{\sigma}_{n+1}$ respectively.

2.3 Cusps and reduction theory

Let $\Gamma \subset G$ be a non-uniform lattice. Recall that two parabolic subgroups are said to be Γ-equivalent if they are conjugate under Γ. The *cusps* of Γ are the Γ-equivalent classes of parabolic subgroups whose unipotent radicals intersects Γ nontrivially. Let $P_1, \ldots, P_h$ be a complete set of representatives for these classes. We note that G acts on the space of parabolic subgroups naturally via conjugation and it induces a map from G to the space of parabolic subgroups sending $g \in G$ to the parabolic subgroup $g^{-1}Pg$. Moreover, since P is self-normalizing, we can parameterize the space of parabolic subgroups by the homogeneous space $P\backslash G$ which can be identified with $M\backslash K$. Thus we can take a subset $\{\xi_1, \ldots, \xi_h\} \subset K$ such that $\xi_j^{-1} P_j \xi_j = P$ for any $1 \leq j \leq h$. Following [Iwa02] we call these ξ_j's *scaling elements*. Each P_j has a Langlands decomposition $P_j = N_j A_j M_j$ with $N_j = \xi_j N \xi_j^{-1}$, $A_j = \xi_j A \xi_j^{-1}$ and $M_j = \xi_j M \xi_j^{-1}$. For each $1 \leq j \leq h$ denote by $Q_j = N_j M_j$, $\Gamma_{P_j} := \Gamma \cap P_j$ and $\Gamma_{N_j} := \Gamma \cap N_j$. By definition, each Γ_{N_j} is nontrivial.

We note that $\Gamma\backslash G$ having finite covolume implies that Γ_{N_j} is a lattice in N_j (see [GR70, Definition 0.5 and Theorem 0.7]). Moreover, since Γ is discrete and Γ_{N_j} is nontrivial, we have $\Gamma_{P_j} \subset Q_j = N_j M_j$ since otherwise there will be a sequence of non-identity elements in Γ converging to the identity element. Identify N_j with $\mathbb{R}^n$ and Γ_{N_j} with a lattice Λ in $\mathbb{R}^n$. The conjugation action of Γ_{P_j} on N_j and Γ_{N_j} induces an injection

$$\Gamma_{N_j}\backslash\Gamma_{P_j} \hookrightarrow \mathrm{SO}(\mathbb{R}^n) \cap \mathrm{GL}(\Lambda),$$

where $\mathrm{SO}(\mathbb{R}^n)$ denote the special orthogonal group of $\mathbb{R}^n (= N_j)$. Hence the index $[\Gamma_{P_j} : \Gamma_{N_j}]$ is finite.

For later use, we record some quantities here. For each $1 \leq j \leq h$, let

$$\operatorname{vol}(\Gamma_{N_j}\backslash N_j) = \int_{\xi_j^{-1}\Gamma_{N_j}\xi_j\backslash N} d\mathbf{x} \tag{2.3.1}$$

be the covolume of Γ_{N_j} in N_j. We note that here we define the covolume indirectly as the covolume of the lattice $\xi_j^{-1}\Gamma_{N_j}\xi_j$ in N. The advantage of this definition is that we can work on the same coordinates (coming from P) at each cusp P_j via the scaling element ξ_j. We also note that while the lattice $\xi_j^{-1}\Gamma_{N_j}\xi_j \subset N$ depends on the choice of ξ_j, its covolume is independent of this choice (since ξ_j is taken from the maximal compact subgroup K). Thus $\operatorname{vol}(\Gamma_{N_j}\backslash N_j)$ is well-defined. Next, note that $\Gamma_{P_j} \subset Q_j$ is co-compact since Γ_{N_j} is a lattice in N_j and M_j is compact. Let

$$\omega_j = \int_{\xi_j^{-1}\Gamma_{P_j}\xi_j\backslash Q} d\mu_Q(q) \tag{2.3.2}$$

be the covolume of Γ_{P_j} in Q_j. We note that $\Gamma_{N_j}\backslash Q_j = \Gamma_{N_j}\backslash N_j \times M_j$ contains $[\Gamma_{P_j} : \Gamma_{N_j}]$ copies of $\Gamma_{P_j}\backslash Q_j$. Thus by the normalization (2.2.2) for μ_Q we have

$$\omega_j = \frac{1}{[\Gamma_{P_j} : \Gamma_{N_j}]} \int_{\xi_j^{-1}\Gamma_{N_j}\xi_j\backslash Q} d\mu_Q(q) = \frac{\operatorname{vol}(\Gamma_{N_j}\backslash N_j)}{[\Gamma_{P_j} : \Gamma_{N_j}]}. \tag{2.3.3}$$

Fix a fundamental domain for $\xi_j^{-1}\Gamma_{N_j}\xi_j$ in N which, with slight abuse of notation, we simply denote by $\xi_j^{-1}\Gamma_{N_j}\xi_j\backslash N$. Then the set

$$\{u_{\mathbf{x}}a_t k \mid u_{\mathbf{x}} \in \xi_j^{-1}\Gamma_{N_j}\xi_j\backslash N, t \in \mathbb{R}, k \in K\} \tag{2.3.4}$$

forms a fundamental domain for $\xi_j^{-1}\Gamma_{N_j}\xi_j\backslash G$, and it contains $[\Gamma_{P_j} : \Gamma_{N_j}]$ copies of $\xi_j^{-1}\Gamma_{P_j}\xi_j\backslash G$.

For any $\tau \in \mathbb{R}$, let us denote $A(\tau) = \{a_t \in A \mid t \geq \tau\}$. A *Siegel set* is a subset of G of the form $\Omega_{\tau,U} = UA(\tau)K$ where U is an open, relatively compact subset of N. Since G is of real rank one, we can apply the reduction theory of Garland and Raghunathan ([GR70, Theorem 0.6]). That is, there exists $\tau_0 \in \mathbb{R}$, an open, relatively compact subset $U_0 \subset N$, an open, relatively compact subset $\mathfrak{C}$ of G and the finite set $\Xi = \{\xi_1, \cdots, \xi_h\} \subset G$ as above such that the Siegel fundamental domain

$$\mathcal{F}_{\Gamma,\tau_0,U_0} = \mathfrak{C} \bigcup \left(\bigcup_{\xi_j \in \Xi} \xi_j \Omega_{\tau_0,U_0} \right), \tag{2.3.5}$$

satisfies the following properties:

(1) $\Gamma \mathcal{F}_{\Gamma,\tau_0,U_0} = G$;

(2) the set $\{\gamma \in \Gamma \mid \gamma \mathcal{F}_{\Gamma,\tau_0,U_0} \cap \mathcal{F}_{\Gamma,\tau_0,U_0} \neq \emptyset\}$ is finite;

(3) $\gamma \xi_j \Omega_{\tau_0,U_0} \cap \xi_l \Omega_{\tau_0,U_0} = \emptyset$ for all $\gamma \in \Gamma$ whenever $\xi_j \neq \xi_l \in \Xi$.

In other words, the restriction to $\mathcal{F}_{\Gamma,\tau_0,U_0}$ of the natural projection of G onto $\Gamma \backslash G$ is surjective, at most finite-to-one and the cusp neighborhood of each cusp of $\Gamma \backslash G$ can be taken to be disjoint. We will fix this Siegel fundamental domain $\mathcal{F}_{\Gamma,\tau_0,U_0}$ throughout this thesis. For later use, we note that U_0 contains a fundamental domain of $\xi_j^{-1} \Gamma_{N_j} \xi_j \backslash N$ for each $1 \leq j \leq h$.

2.4 Hyperbolic distance function

Keep the notation as before. Let dist_G and $\mathrm{dist} = \mathrm{dist}_\Gamma$ denote the hyperbolic distance functions on $G/K = \mathbb{H}^{n+1}$ and $\Gamma \backslash G/K = \Gamma \backslash \mathbb{H}^{n+1}$ respectively. By slightly abuse of notation, we also denote dist_G and dist to be their lifts to G and $\Gamma \backslash G$ respectively. The lift, dist_G, on G is left G-invariant and satisfies $\mathrm{dist}_G(\mathrm{id}, a_t k) = t$ for any $t \geq 0$ and $k \in K$, where $\mathrm{id} \in G$ is the identity element. The lift, dist, on $\Gamma \backslash G$ is defined that for any $g_1, g_2 \in G$,

$$\mathrm{dist}(\Gamma g_1, \Gamma g_2) = \inf_{\gamma \in \Gamma} \mathrm{dist}_G(g_1, \gamma g_2).$$

Clearly, $\text{dist}(\Gamma g_1, \Gamma g_2) \leq \text{dist}_G(g_1, g_2)$. Conversely, if g_1, g_2 are from the Siegel set Ω_{τ_0,U_0}, then there exists a constant D such that $\text{dist}_G(\xi_j g_1, \gamma \xi_l g_2) \geq \text{dist}_G(g_1, g_2) - D$ for any $\xi_j, \xi_l \in \Xi$ and any $\gamma \in \Gamma$ (see [Bor72, Theorem C]). In particular, this implies

$$\text{dist}(\Gamma \xi_j g_1, \Gamma \xi_j g_2) \geq \text{dist}_G(g_1, g_2) - D$$

for any $\xi_j \in \Xi$ and any $g_1, g_2 \in \Omega_{\tau_0,U_0}$. We then have

Lemma 2.4.1. *For $o \in \mathcal{F}_{\Gamma,\tau_0,U_0}$ fixed, there exists a constant D' such that*

$$\text{dist}_G(o, \xi_j g) - D' \leq \text{dist}(o, \xi_j g) \leq \text{dist}_G(o, \xi_j g) \tag{2.4.1}$$

for any $\xi_j \in \Xi$ and any $g \in \Omega_{\tau_0,U_0}$.

Remark 2.4.2. We view $o, \xi_j g$ as elements in $\Gamma \backslash G$ when we write $\text{dist}(o, \xi_j g)$, and as elements in G when we write $\text{dist}_G(o, \xi_j g)$.

Proof. The second inequality is trivial. For the first inequality fix an arbitrary $g' \in \Omega_{\tau_0,U_0}$, we have

$$\begin{aligned}
\text{dist}(o, \xi_j g) &\geq \text{dist}(\xi_j g', \xi_j g) - \text{dist}(o, \xi_j g') \\
&\geq \text{dist}_G(g', g) - D - \text{dist}_G(o, \xi_j g') \\
&= \text{dist}_G(\xi_j g', \xi_j g) - D - \text{dist}_G(o, \xi_j g') \\
&\geq \text{dist}_G(o, \xi_j g) - 2\text{dist}_G(o, \xi_j g') - D.
\end{aligned}$$

Then $D' = 2 \sup_{\xi_j \in \Xi} \text{dist}_G(o, \xi_j g') + D$ satisfies (2.4.1). □

Note that any $g \in \Omega_{\tau_0,U_0}$ can be written as $g = u a_t k$ with $u \in U_0, t \geq \tau_0, k \in K$. Since U_0 is relatively compact, Ξ is finite and dist is right K-invariant, in view of Lemma 2.4.1 we have

$$\text{dist}(o, \xi_j g) = \text{dist}_G(o, a_t) + O(1) = t + O(1). \tag{2.4.3}$$

We note that we will only use this section to prove logarithm laws and for that we can assume Γ has a cusp at P (see chapter 5). Thus Ξ can be taken such that it contains the identity element. In this case we have

$$\mathrm{dist}\,(o,g) = t + O(1) \tag{2.4.4}$$

for any $g = ua_t k \in \Omega_{\tau_0,U_0}$. Finally, we note that when r is sufficiently large, the set

$$B_r := \{x \in \Gamma\backslash G \mid \mathrm{dist}(o,x) > r\}$$

is a collection of neighborhoods at all cusps. In view of the above reduction theory and the Haar measure (2.2.1) (see below) we have

$$\mu\,(B_r) \asymp e^{-nr} \tag{2.4.5}$$

for any $r > 0$.

2.5 Fourier transform

In chapter 4 we will relate incomplete Eisenstein series with the corresponding Eisenstein series via the Fourier inversion formula. We thus recall some backgrounds of Fourier transform here. For any $v \in L^1(\mathbb{R})$, its Fourier transform $\widehat{v}$ is is a function on $\mathbb{R}$ defined by

$$\widehat{v}\,(r) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} v\,(t)\, e^{-irt} dt.$$

The Plancherel Theorem ([Fol99, Theorem 8.29]) states that if $v \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ then $\widehat{v} \in L^2(\mathbb{R})$ and the Fourier transform on $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ extends uniquely to an isometry to $L^2(\mathbb{R})$. In particular, for such v we have

$$\int_{\mathbb{R}} |\widehat{v}(r)|^2 dr = \int_{\mathbb{R}} |v(t)|^2 dt.$$

On the other hand, if $\widehat{v}$ is also in $L^1(\mathbb{R})$, then it satisfies the Fourier inversion formula ([Fol99, Theorem 8.26])

$$v(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{v}(r)\, e^{itr} dr.$$

Moreover, if $v \in \mathcal{C}_c^\infty(\mathbb{R})$ then $\widehat{v}$ can be extended to an entire function (by viewing r as a complex variable) and $|\widehat{v}(r)|$ decays super polynomially in r as $|r| \to \infty$. Making the substitution $s = ir$ we get

$$v(t) = \frac{1}{\sqrt{2\pi} i} \int_{\mathfrak{Re}(s)=0} \widehat{v}(-is)\, e^{st} ds.$$

For any $\sigma \in \mathbb{R}$ we can shift the contour of integration from $\mathfrak{Re}(s) = 0$ to $\mathfrak{Re}(s) = \sigma$ to get

$$v(t) = \frac{1}{\sqrt{2\pi} i} \int_{\mathfrak{Re}(s)=\sigma} \widehat{v}(-is)\, e^{st} ds. \tag{2.5.1}$$

Chapter 3

Constant terms of Eisenstein series

As mentioned in the introduction, one of the key steps of this spectral approach computing the second moment of incomplete Eisenstein series is a calculation of the constant term of the corresponding Eisenstein series. In this chapter we compute such constant terms. We will choose the upper half space model for $\mathbb{H}^{n+1}$, that is, we take $G = \mathrm{PSL}_2(T_{n-1})$. But we will only assume this in section 3.2 when computing the raising operator and we note that all the computations there can be done similarly for $\mathrm{SO}_0(n+1,1)$.

Let us first define the Eisenstein series. Let $\Gamma \subset G$ be a non-uniform lattice. Let $P_1, \ldots, P_h$ be a complete set of representatives for cusps of Γ and $\Xi = \{\xi_1, \ldots, \xi_h\} \subset K$ such that for each $1 \leq j \leq h$, $\xi_j^{-1} P_j \xi_j = P$. Recall the Iwasawa decomposition $G = NAK$ that for any $g \in G$ we can write $g = u a_t k$ uniquely with $u \in N$, $a_t \in A$ and $k \in K$. For each parameter $s \in \mathbb{C}$, we can define a left N-invariant and right K-invariant function φ_s on G by

$$\varphi_s\left(u a_t k\right) = e^{st}. \tag{3.0.1}$$

We note that since $M \subset K$ commutes with A, φ_s is also left Q-invariant. Given a function ϕ on $M\backslash K$ that we think of as a left P-invariant function on G, the Eisenstein series at P_j is defined by

$$E_j(\phi, s, g) = \sum_{\gamma \in \Gamma_{P_j}\backslash\Gamma} \varphi_s(\xi_j^{-1}\gamma g)\phi(\xi_j^{-1}\gamma g). \tag{3.0.2}$$

Note that this is well-defined since $\varphi_s\phi$ being left Q-invariant implies $\varphi_s(\xi_j^{-1}-)\phi(\xi_j^{-1}-)$ is left Q_j-invariant and $\Gamma_{P_j} \subset Q_j$. We also note that this definition is independent of the choice of ξ_j: suppose $\xi_j' \in K$ is another such element satisfying $(\xi_j')^{-1}P_j\xi_j' = P$, then using the fact that P is self-normalizing, we have $\xi_j^{-1}\xi_j' \in P \cap K = M$. Moreover, since $\varphi_s\phi$ is left M-invariant, we have for any $\gamma \in \Gamma$ and $g \in G$

$$\varphi_s\phi(\xi_j^{-1}\gamma g) = \varphi_s\phi\left(\xi_j^{-1}\xi_j'(\xi_j')^{-1}\gamma g\right) = \varphi_s\phi\left((\xi_j')^{-1}\gamma g\right).$$

Thus

$$\sum_{\gamma \in \Gamma_{P_j}\backslash\Gamma} \varphi_s(\xi_j^{-1}\gamma g)\phi(\xi_j^{-1}\gamma g) = \sum_{\gamma \in \Gamma_{P_j}\backslash\Gamma} \varphi_s\left((\xi_j')^{-1}\gamma g\right)\phi\left((\xi_j')^{-1}\gamma g\right).$$

On the other hand, since we always choose the scaling elements ξ_j to be in K, our definition of Eisenstein series (and incomplete Eisenstein series defined in the introduction) actually depends on the choice of representatives in a cusp (Γ-conjugacy class of parabolic subgroups). In fact, they give the same Eisenstein series (and incomplete Eisenstein series) up to a dilation.

This defining series (3.0.2) of Eisenstein series converges for $\mathfrak{Re}(s) > n$ and has a meromorphic continuation to the whole complex plane. Moreover, it satisfies a functional equation relating s and $n-s$. When $\phi = 1$, $E_j(1, s, g)$ is right K-invariant (called *spherical*) and we abbreviate it by $E_j(s, g)$.

The constant term of the Eisenstein series along P_j is defined by

$$E_j^0(\phi, s, g) = \frac{1}{\mathrm{vol}(\Gamma_{N_j}\backslash N_j)} \int_{\xi_j^{-1}\Gamma_{N_j}\xi_j\backslash N} E_j(\phi, s, \xi_j u_{\mathbf{x}} g)\, d\mathbf{x},$$

where $\mathrm{vol}(\Gamma_{N_j}\backslash N_j)$ is as in (2.3.1). Again for spherical Eisenstein series $E_j(s,g)$, we abbreviate its constant term along P_j by $E_j^0(s,g)$.

While the spherical constant term $E_j^0(s,g)$ is well-known, for a general non-spherical Eisenstein series $E_j(\phi,s,g)$, its constant term $E_j^0(\phi,s,g)$ is not so well-known. We will first review some facts about the spherical constant terms and use them to deduce a constant term formula for any non-spherical Eisenstein series via certain raising operator.

3.1 Constant terms for spherical Eisenstein series

We recall some facts about spherical Eisenstein series in this section. The general theory was developed in [Lan76]. See also [HC68, War79, Kel15] for more precise statements for the special case when G is of real rank one.

The spherical constant term $E_j^0(s,g)$ has the form

$$E_j^0(s,g) = \varphi_s(g) + \mathcal{C}_j(s)\varphi_{n-s}(g) \tag{3.1.1}$$

with $\mathcal{C}_j(s)$ a meromorphic function. Moreover, $\mathcal{C}_j(s)$ is analytic on the half plane $\mathfrak{Re}(s) \geq \frac{n}{2}$ except with a simple pole at $s = n$ (called the *trivial pole*) and possibly finitely many simple poles on the interval $(\frac{n}{2}, n)$ (called *exceptional poles*). It also satisfies a functional equation relating s and $n-s$. More precisely, $\mathcal{C}_1(s), \dots, \mathcal{C}_h(s)$ are the diagonal entries of the scattering matrix $\Psi(s)$ (a h by h matrix with entries meromorphic functions in s) which is symmetric and satisfies the functional equation $\Psi(s)\Psi(n-s) = I_h$, where I_h is the h by h identity matrix. In particular, for $\mathfrak{Re}(s) = \frac{n}{2}$, we have

$$\Psi(s)\overline{\Psi(s)^t} = \Psi(s)\Psi(\bar{s}) = \Psi(s)\Psi(n-s) = I_h.$$

This implies that for each $1 \leq j \leq h$ and for $\mathfrak{Re}(s) = \frac{n}{2}$

$$|\mathcal{C}_j(s)| \leq 1. \tag{3.1.2}$$

Finally, let $\frac{n}{2} < s_{j\ell_j} < \cdots < s_{j1} < s_{j0} = n$ denote the finitely many poles of $\mathcal{C}_j(s)$ on the half plane $\mathfrak{Re}(s) \geq \frac{n}{2}$ and for each $0 \leq r \leq \ell_j$ let

$$c_{jr} = \mathrm{Res}_{s=s_{jr}} \mathcal{C}_j(s)$$

be the residue of $\mathcal{C}_j(s)$ at s_{jr}. We note that the residue at the trivial pole can be computed explicitly and is related to the volumes of the fundamental domains as following

$$c_{j0} = \frac{\omega_j}{\nu_\Gamma}, \tag{3.1.3}$$

where $\omega_j = \int_{\xi_j^{-1}\Gamma_{P_j}\xi\backslash Q} d\mu_Q(q)$ and $\nu_\Gamma = \int_{\Gamma\backslash G} dg$ as before. We note that while this was only proved in [Sar83] for $n = 2$ case, but the same argument works in general.

3.2 The raising operator

Keep the notation as in section 2.1.1. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K respectively. Let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathfrak{k}_{\mathbb{C}} = \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C}$ be their complexifications. As a real vector space, $\mathfrak{k}$ is spanned by the matrices

$$-\frac{1}{2}\begin{pmatrix} e_j e_l & 0 \\ 0 & e_j e_l \end{pmatrix} (1 \leq j < l \leq n-1), \quad -\frac{1}{2}\begin{pmatrix} e_j & 0 \\ 0 & -e_j \end{pmatrix} (1 \leq j \leq n-1),$$

$$\frac{1}{2}\begin{pmatrix} 0 & e_j \\ e_j & 0 \end{pmatrix} (1 \leq j \leq n-1), \quad \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where $e_1, \ldots, e_{n-1}$ are the generators of the Clifford algebra $C\ell_n$ given as in section 2.1.1. The Lie algebra $\mathfrak{g}$ is spanned by

$$\begin{pmatrix} 0 & e_j \\ 0 & 0 \end{pmatrix} (1 \leq j \leq n-1), \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

together with the basis elements of $\mathfrak{k}$ given as above.

3.2.1 Root-space decomposition of $\mathfrak{k}_{\mathbb{C}}$

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{k}_{\mathbb{C}}$. Since $\mathfrak{k}_{\mathbb{C}}$ is a complex semisimple Lie algebra, it has a root-space decomposition with respect to $\mathfrak{h}$:

$$\mathfrak{k}_{\mathbb{C}} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi(\mathfrak{k}_{\mathbb{C}}, \mathfrak{h})} \mathfrak{k}_{\alpha},$$

where $\Phi = \Phi(\mathfrak{k}_{\mathbb{C}}, \mathfrak{h})$ is the corresponding set of roots, and for each $\alpha \in \Phi$ the root-space $\mathfrak{k}_{\alpha}$ is given by

$$\mathfrak{k}_{\alpha} := \{X \in \mathfrak{k}_{\mathbb{C}} \mid [H, X] = \alpha(H) X \text{ for any } H \in \mathfrak{h}\}.$$

Each root-space is one-dimensional and satisfies commutator relations $[\mathfrak{k}_{\alpha}, \mathfrak{k}_{\beta}] \subset \mathfrak{k}_{\alpha+\beta}$ for any $\alpha, \beta \in \Phi$. Fix a set of simple roots Δ and let Φ^+ denote the corresponding set of positive roots. Then $\Phi = \Phi^+ \cup (-\Phi^+)$. For backgrounds on complex semisimple Lie algebra, see [Kna02, Chaper II]. In this section, we first give an explicit isomorphism between K and $\mathrm{SO}(n+1)$, then use this isomorphism and the classical root-space decomposition of $\mathfrak{so}(n+1, \mathbb{C})$ to get an explicit root-space decomposition of $\mathfrak{k}_{\mathbb{C}}$.

Recall $\mathbb{V}^n = \{x_0 + x_1 e_1 + \cdots + x_n e_n \mid (x_0, \ldots, x_n) \in \mathbb{R}^{n+1}\}$ and the identification

$$\begin{aligned} M \backslash K &\longrightarrow S^n := \left\{x_0 + x_1 e_1 + \cdots + x_n e_n \mid x_0^2 + x_1^2 + \cdots + x_n^2 = 1\right\} \\ \begin{pmatrix} q_2' & -q_1' \\ q_1 & q_2 \end{pmatrix} &\longmapsto \qquad 2\overline{q_1} q_2 + \left(|q_2|^2 - |q_1|^2\right) e_n. \end{aligned}$$

Embed S^n in $\mathbb{V}^n$ and fix an inner product on $\mathbb{V}^n$ such that $\{1, e_1, \cdots, e_n\}$ forms an orthonormal basis of $\mathbb{V}^n$. Thus we can identify $\mathrm{SO}(n+1)$ with $\mathrm{SO}(\mathbb{V}^n)$. Then the right regular action of K on $M\backslash K = S^n$ (that is, for any $g \in K$ and $\tilde{k} \in M\backslash K$, $g \cdot \tilde{k} = \tilde{k}g^{-1}$) induces an isomorphism from K to $\mathrm{SO}\,(n+1)$. In particular, it induces a Lie algebra isomorphism between $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{so}\,(n+1, \mathbb{C})$. Explicitly, for any $0 \leq j < l \leq n$ define $L_{j,l} \in \mathfrak{k}_{\mathbb{C}}$ as following:

$$L_{j,l} = \begin{cases} -\frac{1}{2}\begin{pmatrix} e_j e_l & 0 \\ 0 & e_j e_l \end{pmatrix} & \text{if } \ 1 \leq j < l \leq n-1 \\ -\frac{1}{2}\begin{pmatrix} e_l & 0 \\ 0 & -e_l \end{pmatrix} & \text{if } \ j = 0, 1 \leq l \leq n-1 \\ \frac{1}{2}\begin{pmatrix} 0 & e_j \\ e_j & 0 \end{pmatrix} & \text{if } \ 1 \leq j \leq n-1, l = n \\ \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{if } \ j = 0, l = n. \end{cases} \tag{3.2.1}$$

By direct computation, the induced isomorphism from $\mathfrak{k}_{\mathbb{C}}$ to $\mathfrak{so}\,(n+1, \mathbb{C})$ is given by sending $L_{j,l}$ to $E_{j,l}$ for any $0 \leq j < l \leq n$, where $E_{j,l}$ is the anti-symmetric $(n+1) \times (n+1)$ matrix with $(j,l)^{\text{th}}$ entry equalling one, $(l,j)^{\text{th}}$ entry equalling negative one and zero elsewhere. Using the classical commutator relations of $E_{j,l}$ we get the commutator relations of $L_{j,l}$. To ease the notation, we let $L_{j,j} = 0$ for $0 \leq j \leq n$ and $L_{j,l} = -L_{l,j}$ for $0 \leq l < j \leq n$. Explicitly, $L_{j,l}$ satisfy the following commutator relations

$$[L_{j,l}, L_{s,t}] = \delta_{ls} L_{j,t} - \delta_{js} L_{l,t} - \delta_{lt} L_{j,s} + \delta_{jt} L_{l,s} \tag{3.2.2}$$

for any $0 \leq j, l, s, t \leq n$, where δ_{jl} is the Kronecker symbol. Moreover, using the root-space decomposition of $\mathfrak{so}\,(n+1, \mathbb{C})$ (see [Kna02, p. 127-129]) we get the following root-space decomposition of $\mathfrak{k}_{\mathbb{C}}$ depending on the parity of $n+1$.

Case I: $n+1=2k+1$ **is odd**

When $n+1=2k+1$ is odd, Φ is of type B_k. For each $0\le j\le k-1$, let

$$H_j = iL_{2j,2j+1}$$

with $L_{2j,2j+1}$ defined as in (3.2.1). Let $\mathfrak{h}$ be the complex vector space spanned by the set $\{H_j \mid 0\le j\le k-1\}$. For each $0\le j\le k-1$, let $\varepsilon_j:\mathfrak{h}\to\mathbb{C}$ be the linear functional on $\mathfrak{h}$ characterised by $\varepsilon_j(H_l)=\delta_{jl}$. Using the above isomorphism between $\mathfrak{k}_{\mathbb{C}}$ and the root-space decomposition of $\mathfrak{so}(2k+1,\mathbb{C})$, we know $\mathfrak{h}$ is a Cartan subalgebra and we can choose the set of simple roots to be

$$\Delta = \{\varepsilon_0-\varepsilon_1, \varepsilon_1-\varepsilon_2, \dots, \varepsilon_{k-2}-\varepsilon_{k-1}, \varepsilon_{k-1}\}.$$

The corresponding positive roots are given by

$$\Phi^+ = \{\varepsilon_j \pm \varepsilon_l \mid 0\le j<l\le k-1\} \cup \{\varepsilon_j \mid 0\le j\le k-1\}.$$

Moreover, the positive root-spaces are explicitly given as following. For any $0\le j< l\le k-1$

$$\mathfrak{k}_{\varepsilon_j\pm\varepsilon_l} = \mathbb{C}\left\langle (L_{2j,2l}-iL_{2j+1,2l}) \mp (L_{2j+1,2l+1}+iL_{2j,2l+1})\right\rangle, \tag{3.2.3}$$

and for any $0\le j\le k-1$

$$\mathfrak{k}_{\varepsilon_j} = \mathbb{C}\left\langle L_{2j,2k}-iL_{2j+1,2k}\right\rangle. \tag{3.2.4}$$

Case II: $n+1=2k$ **is even**

When $n+1=2k$ is even, Φ is of type D_k. Similar to the odd case, for each $0\le j\le k-1$, let

$$H_j = iL_{2j,2j+1}$$

and let $\mathfrak{h}$ be the complex vector space spanned by $\{H_j \mid 0 \leq j \leq k-1\}$. For each $0 \leq j \leq k-1$, denote $\varepsilon_j : \mathfrak{h} \to \mathbb{C}$ the linear functional on $\mathfrak{h}$ characterised by $\varepsilon_j(H_l) = \delta_{jl}$. The set of simple roots can be chosen to be

$$\Delta = \{\varepsilon_0 - \varepsilon_1, \varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{k-2} - \varepsilon_{k-1}, \varepsilon_{k-2} + \varepsilon_{k-1}\}.$$

The corresponding positive roots are given by

$$\Phi^+ = \{\varepsilon_j \pm \varepsilon_l \mid 0 \leq j < l \leq k-1\},$$

with $\mathfrak{k}_{\varepsilon_j \pm \varepsilon_l}$ spanned by the same elements as in (3.2.3).

Remark 3.2.5. The commutator relations (3.2.2) and root-space decomposition above can both be checked directly using the relations

$$e_j e_l + e_l e_j = -2\delta_{jl} \text{ for any } 1 \leq j, l \leq n$$

defining the Clifford algebra $C\ell_n$.

3.2.2 Parabolic induced representations

Let $G = NAK$ be the fixed Iwasawa decomposition and M be the centralizer of A in K as before. For each parameter $s \in \mathbb{C}$, we can view the function φ_s defined in (3.0.1) as a character of A that extends trivially to N. Thus φ_s can be viewed as a one-dimensional representation of NA. Let 1_M be the trivial representation of M. Then tensor product $\varphi_s \otimes 1_M$ is a one-dimensional representation of $P = NAM$. We can thus induce a representation from the parabolic subgroup P to the whole group G. This procedure is a special case of a more general process of constructing representations of G called *parabolic induction*. Explicitly, the induced representation $I^s = \mathrm{Ind}_P^G(\varphi_s \otimes 1_M)$ consists of measurable functions $f : G \to \mathbb{C}$ satisfying

$$f(uamg) = \varphi_s(a) f(g) \text{ for } \mu\text{-a.e. } g \in G, \text{ with } u \in N, a \in A \text{ and } m \in M. \quad (3.2.6)$$

with G acting on I^s via the right regular action. We note that (3.2.6) implies that elements in I^s are left Q-invariant. Thus $f \in I^s$ can be viewed a function on $Q\backslash G$ which can be identified with $A \times M\backslash K$. By the identification between $M\backslash K$ and S^n, $f \in I^s$ can be viewed as a function in coordinates $(t, x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+2}$ with the restriction $x_0^2 + x_1^2 \cdots + x_n^2 = 1$.

Let $I_\infty^s \subset I^s$ be the subspace of smooth functions. We note that I_∞^s is a dense subspace of I^s and the right regular action of G on I^s induces a $\mathfrak{g}$-module structure on I_∞^s by differentiating the group action: For any $X \in \mathfrak{g}$ and any $f \in I_\infty^s$, define the Lie derivative, $\pi(X)$, by

$$(\pi(X) f)(g) = \frac{d}{dy} f(g \exp(yX))\bigg|_{y=0}.$$

We note that the Lie derivative respects the Lie bracket, that is, $[\pi(X), \pi(Y)] = \pi([X, Y])$ for any $X, Y \in \mathfrak{g}$, where the first Lie bracket is the Lie bracket of endomorphisms. Since functions in I_∞^s are complex-valued, we can complexify the Lie derivative by defining

$$\pi(X + iY) := \pi(X) + i\pi(Y)$$

for any $X, Y \in \mathfrak{g}$. Thus I_∞^s becomes a $\mathfrak{g}_\mathbb{C}$-module. In particular, I_∞^s is also a $\mathfrak{k}_\mathbb{C}$-module. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{k}_\mathbb{C}$ and Φ^+ be the corresponding positive root system as in the previous section. Let $\mathfrak{h}^*$ denote the complex dual of $\mathfrak{h}$. Given a $\mathfrak{k}_\mathbb{C}$-module V and $\rho \in \mathfrak{h}^*$, we say $v \in V$ is of K-weight ρ if $H \cdot v = \rho(H) v$ for any $H \in \mathfrak{h}$. We say $v \in V$ is a highest weight vector if v is of K-weight ρ for some $\rho \in \mathfrak{h}^*$ and $X \cdot v = 0$ for any $\alpha \in \Phi^+$ and any $X \in \mathfrak{k}_\alpha$. We note that every irreducible representation of K is a finite-dimensional irreducible $\mathfrak{k}_\mathbb{C}$-module by differentiating the group action at the identity, and every finite-dimensional irreducible $\mathfrak{k}_\mathbb{C}$-module admits a unique (up to scalars) highest weight vector (see [Kna02, Theorem 5.5 (b)]).

Again by condition (3.2.6), a function $f \in I^s$ is totally determined (up to a null set) by its values on $M\backslash K$. This induces an isomorphism as K-representations from I^s to $L^2(M\backslash K)$ sending f to $f|_K$. Identify $M\backslash K$ with S^n as above, we have the following decomposition of $L^2(M\backslash K)$ as K-representations:

$$L^2(M\backslash K) = \widehat{\bigoplus_{p\geq 0}} L^2(M\backslash K, p),$$

where $L^2(M\backslash K, p)$ is the space of degree p harmonic polynomials in $n+1$ variables restricted to S^n (see [Gar, Corollary 5.0.3]) and $\widehat{\bigoplus}$ denotes the Hilbert direct sum. Moreover, let $\mathcal{H}^p$ be the space of degree p harmonic polynomials in coordinates $(x_0, x_1, \cdots, x_n) \in \mathbb{V}^n$. Then $\mathcal{H}^p$ is an irreducible K-representation and is isomorphic to $L^2(M\backslash K, p)$ via the map $\phi \mapsto \phi|_{S^n}$ ([T.76, Theorem 0.3 and 0.4]). Finally, we note that $(x_0 - ix_1)^p \in \mathcal{H}^p$ is of K-weight $p\varepsilon_0$ ([Kna02, p.277-278]) and $\mathcal{H}^p$ is of highest weight $p\varepsilon_0$ ([Kna02, p.339 Problem 9.2]). Hence $(x_0 - ix_1)^p$ is the unique (up to scalars) highest weight vector in $\mathcal{H}^p$.

Correspondingly, let $I^s_\infty(K,p) := \{f \in I^s_\infty \mid f|_K \in L^2(M\backslash K, p)\}$. Then we have a decomposition of I^s_∞

$$I^s_\infty = \bigoplus_{p=0}^{\infty} I^s_\infty(K,p).$$

Moreover, $I^s_\infty(K,p)$ is an irreducible $\mathfrak{k}_{\mathbb{C}}$-module of highest weight $p\varepsilon_0$, and the highest weight vector is given by

$$\varphi_{s,p}(t, x_0, x_1, \ldots, x_n) := e^{st}(x_0 - ix_1)^p.$$

Now we define the raising operator $R^+ \in \mathfrak{g}_{\mathbb{C}}$ by

$$R^+ = \frac{1}{2}\pi\left(\begin{pmatrix} 0 & -1+ie_1 \\ -1-ie_1 & 0 \end{pmatrix}\right) = -\frac{1}{2}\pi\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) + \frac{i}{2}\pi\left(\begin{pmatrix} 0 & e_1 \\ -e_1 & 0 \end{pmatrix}\right).$$

To compute R^+ explicitly, we use the spherical coordinates on S^n: Let $(x_0, x_1, \ldots, x_n)$ be the coordinates on S^n as above, define $(\theta_0, \theta_1, \ldots, \theta_{n-1}) \in [0, 2\pi]^{n-1} \times [0, \pi)$ such that

$$\begin{aligned} x_0 &= \cos\theta_0, \\ x_1 &= \sin\theta_0\cos\theta_1, \\ &\vdots \\ x_{n-1} &= \sin\theta_0\cdots\sin\theta_{n-2}\cos\theta_{n-1}, \\ x_n &= \sin\theta_0\cdots\sin\theta_{n-2}\sin\theta_{n-1}. \end{aligned}$$

Hence under the coordinates (t, θ_i), $\varphi_{s,p}$ is given by

$$\varphi_{s,p}(t, \theta_i) = e^{st}(\cos\theta_0 - i\sin\theta_0\cos\theta_1)^p.$$

Moreover, under these coordinates, for any $X \in \mathfrak{g}_{\mathbb{C}}$, the Lie derivative $\pi(X)$ is a first order differential operator of the form

$$\pi(X) = F\frac{\partial}{\partial t} + \sum_{i=0}^{n-1} F_i \frac{\partial}{\partial \theta_i},$$

where F, F_i are functions in (t, θ_i). For our purpose, we define

$$\widetilde{\pi(X)} := F\frac{\partial}{\partial t} + F_0\frac{\partial}{\partial\theta_0} + F_1\frac{\partial}{\partial\theta_1}.$$

Since $\varphi_{s,m}$ only depends on the variables (t, θ_0, θ_1), $\pi(X)\varphi_{s,d} = \widetilde{\pi(X)}\varphi_{s,d}$ for any $X \in \mathfrak{g}_{\mathbb{C}}$.

Now we describe the strategy to compute Lie derivatives. We first show how to extract the coordinates (t, θ_i) from a given element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$. Write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix} \begin{pmatrix} q_2' & -q_1' \\ q_1 & q_2 \end{pmatrix} \tag{3.2.7}$$

by Iwasawa decomposition. Comparing the second row of the matrices on both sides and recalling the identification between $M\backslash K$ and S^n we get

$$e^{-t} = |c|^2+|d|^2 \tag{3.2.8}$$

and

$$x_0 + x_1 e_1 + \cdots + x_n e_n = \frac{2\bar{c}d + \left(|d|^2-|c|^2\right) e_n}{|c|^2+|d|^2}, \tag{3.2.9}$$

where x_i are expressed by θ_i as above. Fix an element $g \in G$. For any $X \in \mathfrak{g}$, the coordinates (t, θ_i) of $g \exp(yX)$ can be viewed as functions in y as y varies. Denote $(t(y), \theta(y))$ to indicate this dependence on y. Then the Lie derivative $\widetilde{\pi(X)}$ is exactly given by

$$\widetilde{\pi(X)} = t'(0)\frac{\partial}{\partial t} + \theta_0'(0)\frac{\partial}{\partial \theta_0} + \theta_1'(0)\frac{\partial}{\partial \theta_1}.$$

Lemma 3.2.1. *Let* $B_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *and* $B_2 = \begin{pmatrix} 0 & e_1 \\ -e_1 & 0 \end{pmatrix}$. *Then*

$$\widetilde{\pi(B_1)} = -2\cos\theta_0 \frac{\partial}{\partial t} - 2\sin\theta_0 \frac{\partial}{\partial \theta_0}$$

and

$$\widetilde{\pi(B_2)} = -2\sin\theta_0\cos\theta_1 \frac{\partial}{\partial t} + 2\cos\theta_0\cos\theta_1 \frac{\partial}{\partial \theta_0} - 2\frac{\sin\theta_1}{\sin\theta_0}\frac{\partial}{\partial \theta_1}.$$

Proof. Using the formula $\exp(yB_1) = \sum_{i=0}^{\infty} \frac{(yB_1)^i}{i!}$ we get

$$\exp(yB_1) = \begin{pmatrix} \cosh y & \sinh y \\ \sinh y & \cosh y \end{pmatrix}.$$

Thus

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \cosh y & \sinh y \\ \sinh y & \cosh y \end{pmatrix} = \begin{pmatrix} \star & \star \\ c\cosh y + d\sinh y & c\sinh y + d\cosh y \end{pmatrix}.$$

Using (3.2.8) we get

$$e^{-t(y)} = |\cosh y + d\sinh y|^2 + |c\sinh y + d\cosh y|^2 = e^{-t}\left(\cosh(2y) + \sinh(2y)\cos\theta_0\right).$$

Taking derivatives with respect to y and evaluating at 0 on both sides we get

$$t'(0) = -2\cos\theta_0.$$

Similarly, using (3.2.9) and comparing the constant term and coefficient of e_1, we get

$$\cos(\theta_0(y)) = \frac{\sinh(2y) + \cosh(2y)\cos\theta_0}{\cosh(2y) + \sinh(2y)\cos\theta_0}$$

and

$$\sin(\theta_0(y))\cos(\theta_1(y)) = \frac{\sin\theta_0\cos\theta_1}{\cosh(2y) + \sinh(2y)\cos\theta_0}.$$

Taking derivatives with respect to y and evaluating at 0 we get

$$\theta_0'(0) = -2\sin\theta_0 \quad \text{and} \quad \theta_1'(0) = 0.$$

Thus $\widetilde{\pi(B_1)} = -2\cos\theta_0\frac{\partial}{\partial t} - 2\sin\theta_0\frac{\partial}{\partial\theta_0}$.

Similarly, for B_2 we have $\exp(yB_2) = \begin{pmatrix} \cosh y & \sinh y e_1 \\ -\sinh y e_1 & \cosh y \end{pmatrix}$ and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \cosh y & \sinh y e_1 \\ -\sinh y e_1 & \cosh y \end{pmatrix} = \begin{pmatrix} \star & \star \\ c\cosh y - de_1\sinh y & ce_1\sinh y + d\cosh y \end{pmatrix}.$$

Using (3.2.8) and (3.2.9), after some tedious but straightforward computations we get

$$e^{-t(y)} = e^{-t}\left(\cosh(2y) + \sinh(2y)\sin\theta_0\cos\theta_1\right),$$

$$\cos(\theta_0(y)) = \frac{\cos\theta_0}{\cosh(2y) + \sinh(2y)\sin\theta_0\cos\theta_1},$$

and

$$\sin(\theta_0(y))\cos(\theta_1(y)) = \frac{\sinh(2y) + \cosh(2y)\sin\theta_0\cos\theta_1}{\cosh(2y) + \sinh(2y)\sin\theta_0\cos\theta_1}.$$

Hence by taking derivatives with respect to y and evaluating at 0 we get

$$t'(0) = -2\sin\theta_0\cos\theta_1, \quad \theta_0'(0) = 2\cos\theta_0\cos\theta_1 \quad \text{and} \quad \theta_1'(0) = -2\frac{\sin\theta_1}{\sin\theta_0}.$$

Hence $\widetilde{\pi(B_2)} = -2\sin\theta_0\cos\theta_1\frac{\partial}{\partial t} + 2\cos\theta_0\cos\theta_1\frac{\partial}{\partial\theta_0} - 2\frac{\sin\theta_1}{\sin\theta_0}\frac{\partial}{\partial\theta_1}$. □

In view of Lemma 3.2.1 we get

$$\widetilde{R^+} = (\cos\theta_0 - i\sin\theta_0\cos\theta_1)\frac{\partial}{\partial t} + (\sin\theta_0 + i\cos\theta_0\cos\theta_1)\frac{\partial}{\partial\theta_0} - i\frac{\sin\theta_1}{\sin\theta_0}\frac{\partial}{\partial\theta_1}.$$

Since $R^+\varphi_{s,p} = \widetilde{R^+}\varphi_{s,p}$, applying $\widetilde{R^+}$ to $\varphi_{s,p}$ we get

$$R^+\varphi_{s,p} = (s+p)\,\varphi_{s,p+1}. \tag{3.2.10}$$

Remark 3.2.11. We note that using the explicit root-space decomposition described as above, one can directly check that the raising operator R^+ (more explicitly, the matrix representing R^+) satisfies

$$[H, R^+] = \varepsilon_0(H)\,R^+ \quad \text{and} \quad [R^+, \mathfrak{k}_\alpha] = 0 \tag{3.2.12}$$

for any $H \in \mathfrak{h}$ and any $\alpha \in \Phi^+$. The first part of (3.2.12) implies that R^+ sends a vector of K-weight ρ to a vector of K-weight $\rho + \varepsilon_0$ and the second part of (3.2.12) implies that R^+ sends a highest weight vector to either zero or another highest weight vector. Since $\varphi_{s,p}$ is a highest weight vector of K-weight $p\varepsilon_0$, $R^+\varphi_{s,p}$ is either zero or a highest weight vector of K-weight $(p+1)\,\varepsilon_0$. But since $I_\infty^s = \oplus_{p=0}^\infty I_\infty^s(K,p)$ and each $I_\infty^s(K,p)$ has a unique (up to scalars) highest weight vector $\varphi_{s,p}$, the set of highest weight vectors in I_∞^s is exactly $\{\varphi_{s,p} \mid p \geq 0\}$. Thus $R^+\varphi_{s,p}$ is a multiple of $\varphi_{s,p+1}$. In fact, (3.2.12) is the characterization we used to find R^+. However, once we have found R^+, (3.2.12) is no longer essential for our proof, since we get (3.2.10) (which trivially implies that $R^+\varphi_{s,p}$ is a multiple of $\varphi_{s,p+1}$) by explicit computation.

3.3 Constant terms for non-spherical Eisenstein series

With (3.2.10) we can finally give a constant term formula for non-spherical Eisenstein series. For any non-negative integer p we define

$$Z_p(s) := \begin{cases} 1 & \text{if } p = 0 \\ \prod_{k=0}^{p-1} \frac{n-s+k}{s+k} & \text{if } p > 0. \end{cases}$$

Proposition 3.3.1. *For any $\phi \in L^2(M\backslash K, p)$ that we view as a left P-invariant function on G,*

$$E_j^0(\phi, s, g) = (\varphi_s(g) + Z_p(s)\mathcal{C}_j(s)\varphi_{n-s}(g))\,\phi(g), \tag{3.3.1}$$

where $\mathcal{C}_j(s)$ is as in (3.1.1).

Proof. For any $p \geq 0$, let h_p be the highest weight vector in $L^2(M\backslash K, p)$. We first prove (3.3.1) for h_p. We prove by induction. If $p = 0$, this is the spherical constant formula (3.1.1). Assume now that (3.3.1) holds for h_p for some $p \geq 0$, we want to show that it also holds for h_{p+1}. We apply the raising operator R^+ to the constant term $E^0(h_p, s, g)$. On the one hand, by induction hypothesis and noting that $\varphi_s h_p = \varphi_{s,p}$ is the highest weight vector in $I_\infty^s(K, p)$ we have

$$E_j^0(h_p, s, g) = \varphi_{s,p}(g) + Z_p(s)\mathcal{C}_j(s)\varphi_{n-s,p}(g).$$

Hence by (3.2.10) applying R^+ to both sides of the above equation we get

$$R^+E_j^0(h_p, s, g) = (s+p)\,\varphi_{s,p+1}(g) + (n-s+p)\,Z_p(s)\mathcal{C}_j(s)\varphi_{n-s,p+1}(g).$$

On the other hand, since R^+ commutes with the left regular action, we have

$$\begin{aligned} R^+E_j^0\left(h_p, s, g\right) &= \frac{1}{\mathrm{vol}(\Gamma_{N_j}\backslash N_j)} \int_{\xi_j^{-1}\Gamma_{N_j}\xi_j\backslash N} \sum_{\gamma\in\Gamma_{P_j}\backslash\Gamma} R^+\varphi_{s,p}\left(\xi_j^{-1}\gamma\xi_j u_{\mathbf{x}} g\right) d\mathbf{x} \\ &= \frac{s+p}{\mathrm{vol}(\Gamma_{N_j}\backslash N_j)} \int_{\xi_j^{-1}\Gamma_{N_j}\xi_j\backslash N} \sum_{\gamma\in\Gamma_{P_j}\backslash\Gamma} \varphi_{s,p+1}\left(\xi_j^{-1}\gamma\xi_j u_{\mathbf{x}} g\right) d\mathbf{x} \\ &= \left(s+p\right) E_j^0\left(h_{p+1}, s, g\right). \end{aligned}$$

Comparing the right-hand sides of the above two equations we get

$$\begin{aligned} E_j^0\left(h_{p+1}, s, g\right) &= \frac{1}{s+p}\left(\left(s+p\right)\varphi_{s,p+1}\left(g\right) + \left(n-s+p\right) Z_p\left(s\right)\mathcal{C}_j\left(s\right)\varphi_{n-s,p+1}\left(g\right)\right) \\ &= \varphi_{s,p+1}\left(g\right) + \frac{n-s+p}{s+p} Z_p\left(s\right)\mathcal{C}_j\left(s\right)\varphi_{n-s,p+1}\left(g\right) \\ &= \varphi_{s,p+1}\left(g\right) + Z_{p+1}\left(s\right)\mathcal{C}_j\left(s\right)\varphi_{n-s,p+1}\left(g\right). \end{aligned}$$

Now for general $\phi \in L^2\left(M\backslash K, p\right)$, since $L^2\left(M\backslash K, p\right)$ is an irreducible $\mathfrak{k}_{\mathbb{C}}$-module, ϕ can be written as $\phi = \mathcal{D}h_p$ with $\mathcal{D}$ some differential operator on $L^2\left(M\backslash K, p\right)$ generated by $\pi\left(\mathfrak{k}_{\mathbb{C}}\right)$. Since $\pi\left(\mathfrak{k}_{\mathbb{C}}\right)$ acts trivially on the character φ_s, we have $\mathcal{D}\varphi_{s,p} = \varphi_s\mathcal{D}h_p = \varphi_s\phi$. Hence on the one hand,

$$\begin{aligned} \mathcal{D}E_j^0\left(h_p, s, g\right) &= \frac{1}{\mathrm{vol}(\Gamma_{N_j}\backslash N_j)} \int_{\xi_j^{-1}\Gamma_{N_j}\xi_j\backslash N} \sum_{\gamma\in\Gamma_{P_j}\backslash\Gamma} \mathcal{D}\varphi_{s,p}\left(\xi_j^{-1}\gamma\xi_j u_{\mathbf{x}} g\right) d\mathbf{x} \\ &= \frac{1}{\mathrm{vol}(\Gamma_{N_j}\backslash N_j)} \int_{\xi_j^{-1}\Gamma_{N_j}\xi_j\backslash N} \sum_{\gamma\in\Gamma_{P_j}\backslash\Gamma} \varphi_s\left(\xi_j^{-1}\gamma\xi_j u_{\mathbf{x}} g\right)\phi\left(\xi_j^{-1}\gamma\xi_j u_{\mathbf{x}} g\right) d\mathbf{x} \\ &= E_j^0\left(\phi, s, g\right). \end{aligned}$$

On the other hand, using (3.3.1) for h_p we get

$$\begin{aligned} \mathcal{D}E_j^0\left(h_p, s, g\right) &= \mathcal{D}\left(\left(\varphi_s\left(g\right) + Z_p\left(s\right)\mathcal{C}_j\left(s\right)\varphi_{n-s}\left(g\right)\right) h_p\left(g\right)\right) \\ &= \left(\varphi_s\left(g\right) + Z_p\left(s\right)\mathcal{C}_j\left(s\right)\varphi_{n-s}\left(g\right)\right)\phi\left(g\right). \end{aligned}$$

The proof finishes by comparing both equations. □

Remark 3.3.2. For later use we remark here that for any $p \geq 0$, $Z_p(s)$ satisfies the functional equation $Z_p(s)Z_p(n-s) = 1$. In particular, for $\mathfrak{Re}(s) = \frac{n}{2}$

$$|Z_p(s)|^2 = Z_p(s)\overline{Z_p(s)} = Z_p(s)Z_p(\bar{s}) = Z_p(s)Z_p(n-s) = 1. \tag{3.3.3}$$

Chapter 4

Proofs of main results

With all the preparations in chapter 3, we can now give the proofs of Theorem 1 and Theorem 2. Let us first recall the definition of incomplete Eisenstein series. Given a bounded and compactly supported function f on $Q\backslash G$ (that we view as a left Q-invariant function), the incomplete Eisenstein series at P_j attached to f is defined by

$$\Theta_f^j(g) = \sum_{\gamma \in \Gamma_{P_j} \backslash \Gamma} f\left(\xi_j^{-1} \gamma g\right).$$

Similar to Eisenstein series, Θ_f^j is well-defined due to the left Q-invariance of f. Moreover, since f is bounded and compactly supported, we can bound $|f(a_t \tilde{k})| \lesssim_\sigma e^{\sigma t} = \varphi_\sigma(a_t)$ for any $\sigma > n$. Thus the defining series for Θ_f^j is absolutely convergent and gives a well-defined function on $\Gamma\backslash G$. Our goal of this chapter is to compute $\|\Theta_f^j\|_2^2$ explicitly in terms of f.

4.1 Two preliminary identities

We first give a preliminary formula for $\|\Theta_f^j\|_2^2$ using the standard unfolding trick.

Lemma 4.1.1. *For any bounded and compactly supported function f on $Q\backslash G$ and for any $F \in L^2(\Gamma\backslash G)$*

$$\int_{\Gamma\backslash G} \Theta_f^j(g) F(g)\, d\mu(g) = \int_{\Gamma_{P_j}\backslash G} f\left(\xi_j^{-1} g\right) F(g)\, d\mu(g).$$

In particular, we have

$$\|\Theta_f^j\|_2^2 = \frac{\omega_j}{\nu_\Gamma} \int_{Q\backslash G} \overline{f(a_t\tilde{k})} \mathcal{P}_j(f)(a_t\tilde{k}) d\mu_{Q\backslash G}(a_t\tilde{k}), \tag{4.1.1}$$

where

$$\mathcal{P}_j(f)(a_t\tilde{k}) := \frac{1}{vol(\Gamma_{N_j}\backslash N_j)} \int_M \int_{\xi_j^{-1}\Gamma_{N_j}\xi_j\backslash N} \Theta_f^j(\xi_j u_{\boldsymbol{x}} m a_t \tilde{k}) d\boldsymbol{x}\, d\sigma_n(m) \tag{4.1.2}$$

with $vol(\Gamma_{N_j}\backslash N_j)$ given as in (2.3.1).

Proof. Let $\mathcal{F}_\Gamma$ be a fundamental domain for $\Gamma\backslash G$. Noting that $\mathcal{F}_{P_j} := \bigcup_{\gamma\in\Gamma_{P_j}\backslash\Gamma} \gamma\mathcal{F}_\Gamma$ forms a fundamental domain for $\Gamma_{P_j}\backslash G$, and for bounded and compactly supported f the defining series for Θ_f^j is absolutely convergent, we thus have

$$\begin{aligned}
\int_{\mathcal{F}_\Gamma} \Theta_f^j(g) F(g)\, d\mu(g) &= \sum_{\gamma\in\Gamma_{P_j}\backslash\Gamma} \int_{\mathcal{F}_\Gamma} f\left(\xi_j^{-1}\gamma g\right) F(g)\, d\mu(g) \\
&= \sum_{\gamma\in\Gamma_{P_j}\backslash\Gamma} \int_{\gamma\mathcal{F}_\Gamma} f\left(\xi_j^{-1} g\right) F(g)\, d\mu(g) \\
&= \int_{\bigcup_{\gamma\in\Gamma_{P_j}\backslash\Gamma} \gamma\mathcal{F}_\Gamma} f\left(\xi_j^{-1} g\right) F(g)\, d\mu(g) \\
&= \int_{\Gamma_{P_j}\backslash G} f\left(\xi_j^{-1} g\right) F(g)\, d\mu(g).
\end{aligned}$$

For the special case when $F = \overline{\Theta_f^j} = \Theta_{\overline{f}}^j$ we get

$$\|\Theta_f^j\|_2^2 = \int_{\Gamma_{P_j}\backslash G} \overline{f\left(\xi_j^{-1} g\right)} \Theta_f^j(g)\, d\mu(g) = \int_{\xi_j^{-1}\Gamma_{P_j}\xi_j\backslash G} \overline{f(g)} \Theta_f^j(\xi_j g) d\mu(g),$$

where for the second equality we used the fact that if $\mathcal{F}_{P_j}$ is a fundamental domain for $\Gamma_{P_j}\backslash G$, then $\xi_j^{-1}\mathcal{F}_{P_j}$ forms a fundamental domain for $\xi_j^{-1}\Gamma_{P_j}\xi_j\backslash G$. Moreover, by the normalization (2.2.3) and the right Q-invariance of f we have

$$\begin{aligned}\|\Theta_f^j\|_2^2 &= \frac{1}{\nu_\Gamma}\int_{Q\backslash G}\overline{f(a_t\tilde{k})}\int_{\xi_j^{-1}\Gamma_{P_j}\xi_j\backslash Q}\Theta_f^j(\xi_j q a_t\tilde{k})d\mu_Q(q)d\mu_{Q\backslash G}(a_t\tilde{k})\\ &= \frac{1}{\nu_\Gamma[\Gamma_{P_j}:\Gamma_{N_j}]}\int_{Q\backslash G}\overline{f(a_t\tilde{k})}\int_{\xi_j^{-1}\Gamma_{N_j}\xi_j\backslash Q}\Theta_f^j(\xi_j q a_t\tilde{k})d\mu_Q(q)d\mu_{Q\backslash G}(a_t\tilde{k}),\end{aligned}$$

where for the second equality we used the fact that $\xi_j^{-1}\Gamma_{N_j}\xi_j\backslash Q$ contains $[\Gamma_{P_j}:\Gamma_{N_j}]$ copies of $\xi_j^{-1}\Gamma_{P_j}\xi_j\backslash Q$. Finally by the description (2.3.4) of $\xi_j^{-1}\Gamma_{N_j}\xi_j\backslash G$, the normalization (2.2.2) and the relation (2.3.3) we have

$$\begin{aligned}\|\Theta_f^j\|_2^2 &= \frac{1}{\nu_\Gamma[\Gamma_{P_j}:\Gamma_{N_j}]}\int_{Q\backslash G}\overline{f(a_t\tilde{k})}\int_M\int_{\xi_j^{-1}\Gamma_{N_j}\xi_j\backslash N}\Theta_f^j(\xi_j u_{\mathbf{x}} m a_t\tilde{k})d\mathbf{x}d\sigma_n(m)d\mu_{Q\backslash G}(a_t\tilde{k})\\ &= \frac{\omega_j}{\nu_\Gamma}\int_{Q\backslash G}\overline{f(a_t\tilde{k})}\mathcal{P}_j(f)(a_t\tilde{k})d\mu_{Q\backslash G}(a_t\tilde{k}).\end{aligned}$$

□

Remark 4.1.3. When taking $F=1$ in Lemma 4.1.1, and using the left Q-invariance of f, the normalization (2.2.3) and (2.3.2) we have

$$\int_{\Gamma\backslash G}\Theta_f^j(g)d\mu(g) = \frac{\omega_j}{\nu_\Gamma}\int_{Q\backslash G}f(a_t\tilde{k})d\mu_{Q\backslash G}(a_t\tilde{k}). \tag{4.1.4}$$

We note that this is the first moment formula for Θ_f^j.

In view of Lemma 4.1.1 we are interested in computing $\mathcal{P}_j(f)(a_t\tilde{k})$ explicitly in terms of f. We will first do this by relating the incomplete Eisenstein series Θ_f^j with the Eisenstein series $E_j(\phi,s,g)$ when f is of the form $f(a_t\tilde{k}) = v(t)\phi(\tilde{k})$ with $v\in\mathcal{C}_c^\infty(\mathbb{R})$. This relation is stated in the following lemma.

Lemma 4.1.2. *For any f on $Q\backslash G$ of the form $f(a_t\tilde{k}) = v(t)\phi(\tilde{k})$ with $v\in\mathcal{C}_c^\infty(\mathbb{R})$ and for any $\sigma>n$ we have*

$$\Theta_f^j(g) = \frac{1}{\sqrt{2\pi}i}\int_{\mathfrak{Re}(s)=\sigma}\widehat{v}(-is)E_j(\phi,s,g)ds.$$

Proof. By the Fourier inversion formula (2.5.1) we have

$$f(a_t\tilde{k}) = \frac{1}{\sqrt{2\pi}i}\int_{\mathfrak{Re}(s)=\sigma} \widehat{v}(-is)e^{st}\phi(\tilde{k})ds$$
$$= \frac{1}{\sqrt{2\pi}i}\int_{\mathfrak{Re}(s)=\sigma} \widehat{v}(-is)\varphi_s(a_t)\phi(\tilde{k})ds.$$

Now view f, φ_s and ϕ as corresponding functions on G to get for any $g \in G$

$$f(g) = \frac{1}{\sqrt{2\pi}i}\int_{\mathfrak{Re}(s)=\sigma} \widehat{v}(-is)\varphi_s(g)\phi(g)ds.$$

Thus

$$\Theta_f^j(g) = \sum_{\gamma\in\Gamma_{P_j}\backslash\Gamma} f(\xi_j^{-1}\gamma g) = \frac{1}{\sqrt{2\pi}i}\int_{\mathfrak{Re}(s)=\sigma} \widehat{v}(-is)\sum_{\gamma\in\Gamma_{P_j}\backslash\Gamma}\varphi_s(\xi_j^{-1}\gamma g)\phi(\xi_j^{-1}\gamma g)ds$$
$$= \frac{1}{\sqrt{2\pi}i}\int_{\mathfrak{Re}(s)=\sigma} \widehat{v}(-is)E_j(\phi, s, g)ds.$$

We note that for the last equality we used the defining series (3.0.2) to represent $E_j(\phi, s, g)$ and it only holds for $\sigma > n$. □

In view of Lemma 4.1.2 for $f(a_t\tilde{k}) = v(t)\phi(\tilde{k})$ we have

$$\mathcal{P}_j(f)(a_t\tilde{k}) = \frac{1}{\sqrt{2\pi}i}\int_{\mathfrak{Re}(s)=\sigma} \widehat{v}(-is)\int_M E_j^0(\phi, s, ma_t\tilde{k})d\sigma_n(m)ds, \tag{4.1.5}$$

where $E_j^0(\phi, s, g)$ is the constant term of $E_j(\phi, s, g)$ along P_j defined as before.

4.2 Proof of Theorem 1

In this section we give the proof of Theorem 1. We first recall some notation here. Recall that $\mathcal{C}_j(s)$ is the meromorphic function in the constant term formula (3.1.1). It has simple poles at $\frac{n}{2} < s_{j\ell_j} < \cdots < s_{j1} < s_{j0} = n$ on the half plane $\mathfrak{Re}(s) \geq \frac{n}{2}$ with residues $c_{jr} = \mathrm{Res}_{s=s_{jr}}\mathcal{C}_j(s)$ for $0 \leq r \leq \ell_j$. We also recall that $c_{j0} = \frac{\omega_j}{\nu_\Gamma}$ with $\omega_j = \int_{\xi_j^{-1}\Gamma_{P_j}\xi_j\backslash Q} d\mu_Q(q)$ and $\nu_\Gamma = \int_{\Gamma\backslash G} dg$ (see (3.1.3)).

For any integer $p \geq 0$ let d_p be the dimension of $L^2(M\backslash K, p)$. Fix an orthonormal basis

$$\left\{\psi_{p,l} \in L^2(M\backslash K, p) \mid 1 \leq l \leq d_p\right\} \tag{4.2.1}$$

for $L^2\left(M\backslash K, p\right)$. Recall $\widetilde{\sigma}_{n+1}$ is the right K-invariant probability measure on $M\backslash K$. For any bounded and compactly supported function f on $Q\backslash G$ and any $s \in (\frac{n}{2}, n)$ we define

$$M_f\left(s\right) := \sum_{p=0}^{\infty} Z_p(s) \sum_{1\leq l\leq d_p} \left|\int_{\mathbb{R}} v_{f,p,l}\left(t\right) e^{-st} dt\right|^2, \tag{4.2.2}$$

where $Z_p\left(s\right)$ is as in Proposition 3.3.1 and

$$v_{f,p,l}\left(t\right) = \int_{M\backslash K} f(a_t\tilde{k})\overline{\psi_{p,l}(\tilde{k})} d\widetilde{\sigma}_{n+1}(\tilde{k}).$$

We note that $M_f(s)$ is not linear in f. For any $f_1, \ldots, f_l$ on $Q\backslash G$, it is usually not easy to get good estimates for $M_{f_1+\cdots+f_l}(s)$ in terms of $M_{f_1}(s), \ldots, M_{f_l}(s)$. Here for later use we record a very crude such estimate: using the inequality $|z_1 + \cdots + z_l|^2 \leq l(|z_1|^2 + \cdots + |z_l|^2)$ we can bound for any $s \in (\frac{n}{2}, n)$

$$M_{f_1+\cdots+f_l}(s) \leq l\left(M_{f_1}(s) + \cdots M_{f_l}(s)\right). \tag{4.2.3}$$

4.2.1 The linear operator $\mathcal{T}_j$

In this section we define the bounded linear operator $\mathcal{T}_j$ in Theorem 1. Let f be a bounded and compactly supported function on $Q\backslash G$. Recall the preliminary identity (4.1.1) that

$$\|\Theta_f^j\|_2^2 = \frac{\omega_j}{\nu_\Gamma} \int_{Q\backslash G} \overline{f(a_t\tilde{k})} \mathcal{P}_j(f)(a_t\tilde{k}) d\mu_{Q\backslash G}(a_t\tilde{k})$$

with the linear operator $\mathcal{P}_j(f)$ given by

$$\mathcal{P}_j(f)(a_t\tilde{k}) = \frac{1}{\mathrm{vol}(\Gamma_{N_j}\backslash N_j)} \int_M \int_{\xi_j^{-1}\Gamma_{N_j}\xi_j\backslash N} \Theta_f^j(\xi_j u_{\mathbf{x}} m a_t\tilde{k}) d\mathbf{x}\, d\sigma_n(m).$$

Write $f = \sum_{p,l} f_{p,l}$ with $f_{p,l}(a_t\tilde{k}) = v_{f,p,l}(t)\psi_{p,l}(\tilde{k})$. The linear operator $\mathcal{T}_j$ is defined by

$$\mathcal{T}_j(f)(a_t\tilde{k}) = \mathcal{P}_j(f)(a_t\tilde{k}) - f(a_t\tilde{k}) - \sum_{r=0}^{\ell_j}\sum_{p,l}\sqrt{2\pi}\widehat{v}_{f,p,l}(-is_{jr})c_{jr}Z_p(s_{jr})\varphi_{n-s_{jr}}(a_t)\psi_{p,l}(\tilde{k}), \tag{4.2.4}$$

where $\widehat{v}_{f,p,l}(-is_{jr}) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} v_{f,p,l}(t)e^{-s_{jr}t}dt$ is the Fourier transform of $v_{f,p,l}$ evaluated at $-is_{jr}$.

While the definition (4.2.4) of $\mathcal{T}_j$ looks complicated, we will show that when f is of certain form it has a very simple integral expression from which we will deduce that $\|\mathcal{T}_j\|_2 \leq 1$.

Lemma 4.2.1. *If f is of the form $f(a_t\tilde{k}) = v(t)\psi_{p,l}(\tilde{k})$ with $v \in C_c^\infty(\mathbb{R})$, then*

$$\mathcal{T}_j(f)(a_t\tilde{k}) = \frac{1}{\sqrt{2\pi}i}\int_{\mathfrak{Re}(s)=\frac{n}{2}} \widehat{v}(-is)Z_p(s)\mathcal{C}_j(s)\varphi_{n-s}(a_t)ds\psi_{p,l}(\tilde{k}). \tag{4.2.5}$$

Moreover, for such f we have $\|\mathcal{T}_j(f)\|_2 \leq \|f\|_2$.

Proof. First we note that for f of the above form, (4.2.4) is equivalent to

$$\mathcal{P}_j(f)(a_t\tilde{k}) = f(a_t\tilde{k}) + \mathcal{T}_j(f)(a_t\tilde{k}) + \sum_{r=0}^{\ell_j}\sqrt{2\pi}\widehat{v}(-is_{jr})c_{jr}Z_p(s_{jr})\varphi_{n-s_{jr}}(a_t)\psi_{p,l}(\tilde{k}). \tag{4.2.6}$$

Next, by equation (4.1.5) and Proposition 3.3.1 we have for any $\sigma > n$

$$\begin{aligned}\mathcal{P}_j(f)(a_t\tilde{k}) &= \frac{1}{\sqrt{2\pi}i}\int_{\mathfrak{Re}(s)=\sigma}\widehat{v}(-is)\int_M E_j^0(\psi_{p,l}, s, ma_t\tilde{k})d\sigma_n(m)ds \\ &= \frac{1}{\sqrt{2\pi}i}\int_{\mathfrak{Re}(s)=\sigma}\widehat{v}(-is)\int_M\left(\varphi_s(a_t) + Z_p(s)\mathcal{C}_j(s)\varphi_{n-s}(a_t)\right)\psi_{p,l}(\tilde{k})d\sigma_n(m)ds \\ &= v(t)\psi_{p,l}(\tilde{k}) + \frac{1}{\sqrt{2\pi}i}\int_{\mathfrak{Re}(s)=\sigma}\widehat{v}(-is)Z_p(s)\mathcal{C}_j(s)\varphi_{n-s}(a_t)\psi_{p,l}(\tilde{k})ds,\end{aligned}$$

where for the last equality we used the fact that $E_j^0(\psi_{p,l}, s, ma_t\tilde{k})$ is independent of $m \in M$ and the Fourier inversion formula (2.5.1). Now shifting the contour from

$\mathfrak{Re}(s) = \sigma$ to $\mathfrak{Re}(s) = \frac{n}{2}$ and picking up the poles at s_{jr} we get

$$\mathcal{P}_j(f)(a_t\tilde{k}) = f(a_t\tilde{k}) + \frac{1}{\sqrt{2\pi}i}\int_{\mathfrak{Re}(s)=\frac{n}{2}} \widehat{v}(-is)Z_p(s)\mathcal{C}_j(s)\varphi_{n-s}(a_t)\psi_{p,l}(\tilde{k})ds$$
$$+\sum_{r=0}^{\ell_j}\sqrt{2\pi}\widehat{v}(-is_{jr})c_{jr}Z_p(s_{jr})\varphi_{n-s_{jr}}(a_t)\psi_{p,l}(\tilde{k}).$$

Thus (4.2.5) follows by comparing this equation with (4.2.6).

For the second statement, first recalling the normalization (2.2.4) and applying Plancherel's theorem to $v(t)e^{-\frac{nt}{2}}$ and recalling that $\psi_{p,l} \in L^2(M\backslash K)$ is of norm one (with respect to $\widetilde{\sigma}_{n+1}$) we can compute

$$\|f\|_2^2 = \int_{\mathbb{R}} |v(t)|^2 e^{-nt}dt\|\psi_{p,l}\|_2^2 = \int_{\mathbb{R}} \left|\widehat{v}(r - i\frac{n}{2})\right|^2 dr.$$

Next, to compute $\|\mathcal{T}_j(f)\|_2^2$ we define w, a function on $\mathbb{R}$ such that

$$w(r) = \widehat{v}(r - i\frac{n}{2})Z_p(\frac{n}{2} + ir)\mathcal{C}_j(\frac{n}{2} + ir).$$

By (3.1.2), (3.3.3) and the fact that $|\widehat{v}(r - i\frac{n}{2})|$ decays super polynomially in r as $|r| \to \infty$, we have that

$$|w(r)| = \left|\widehat{v}(r - i\frac{n}{2})Z_p(\frac{n}{2} + ir)\mathcal{C}_j(\frac{n}{2} + ir)\right| \leq \left|\widehat{v}(r - i\frac{n}{2})\right| \tag{4.2.7}$$

decays super polynomially in r as well. Thus $w \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ has Fourier transform and satisfies Plancherel's theorem. Now making substitution $s = \frac{n}{2} + ir$ we can rewrite (4.2.5) to get

$$\mathcal{T}_j(f)(a_t\tilde{k}) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \widehat{v}(r - i\frac{n}{2})Z_p(\frac{n}{2} + ir)\mathcal{C}_j(\frac{n}{2} + ir)e^{(\frac{n}{2} - ir)t}dr\psi_{p,l}(\tilde{k}) = \widehat{w}(t)e^{\frac{n}{2}t}\psi_{p,l}(\tilde{k}).$$

Thus similarly we can compute

$$\begin{aligned}\|\mathcal{T}_j(f)\|_2^2 &= \int_{\mathbb{R}} \left|\widehat{w}(t)e^{\frac{n}{2}t}\right|^2 e^{-nt}dt\|\psi_{p,l}\|_2^2 \\ &= \int_{\mathbb{R}} |\widehat{w}(t)|^2 dt = \int_{\mathbb{R}} |w(r)|^2 dr \\ &\leq \int_{\mathbb{R}} \left|\widehat{v}(r - i\frac{n}{2})\right| dr = \|f\|_2^2,\end{aligned}$$

where for the third equality we used Plancherel's theorem for w and for the inequality we used (4.2.7). $\square$

We can now give the

Proof of Theorem 1. For any bounded and compacted supported function f on $Q\backslash G$, write $f = \sum_{p,l} v_{f,p,l}\psi_{p,l}$ as before. Rewrite (4.2.4) as

$$\mathcal{P}_j(f)(a_t\tilde{k}) = \mathcal{T}_j(f)(a_t\tilde{k}) + f(a_t\tilde{k}) + \sum_{r=0}^{\ell_j}\sum_{p,l}\sqrt{2\pi}\widehat{v}_{f,p,l}(-is_{jr})c_{jr}Z_p(s_{jr})\varphi_{n-s_{jr}}(a_t)\psi_{p,l}(\tilde{k}).$$

In view of (4.1.1), the above expression for $\mathcal{P}_j(f)$ and the identity $c_{j0} = \frac{\omega_j}{\nu_\Gamma}$, to prove formula (1.1.6), it suffices to show that for each $1 \le r \le \ell_j$

$$\int_{Q\backslash G}\overline{f(a_t\tilde{k})}\left(\sum_{p,l}\sqrt{2\pi}\widehat{v}_{f,p,l}(-is_{jr})Z_p(s_{jr})\varphi_{n-s_{jr}}(a_t)\psi_{p,l}(\tilde{k})\right)d\mu_{Q\backslash G}(a_t\tilde{k}) = M_f(s_{jr}) \tag{4.2.8}$$

and for $r = 0$ (noting that $s_{j0} = n$)

$$\int_{Q\backslash G}\overline{f(a_t\tilde{k})}\left(\sum_{p,l}\sqrt{2\pi}\widehat{v}_{f,p,l}(-in)Z_p(n)\psi_{p,l}(\tilde{k})\right)d\mu_{Q\backslash G}(a_t\tilde{k}) = \left|\langle f, 1\rangle\right|^2. \tag{4.2.9}$$

For (4.2.8), by orthogonality and recall normalization (2.2.4) we have

$$\begin{aligned}
&\int_{Q\backslash G}\overline{f(a_t\tilde{k})}\left(\sum_{p,l}\sqrt{2\pi}\widehat{v}_{f,p,l}(-is_{jr})Z_p(s_{jr})\varphi_{n-s_{jr}}(a_t)\psi_{p,l}(\tilde{k})\right)d\mu_{Q\backslash G}(a_t\tilde{k})\\
&= \sum_{p,l}\int_{\mathbb{R}}\overline{v_{f,p,l}(t)}e^{-s_{jr}t}dt\sqrt{2\pi}\widehat{v}_{f,p,l}(-is_{jr})Z_p(s_{jr})\\
&= \sum_{p,l}Z_p(s_{jr})\left|\int_{\mathbb{R}}v_{f,p,l}(t)e^{-s_{jr}t}dt\right|^2 = M_f(s_{jr}),
\end{aligned}$$

where for the last equality we used the fact that $s_{jr} \in (\frac{n}{2}, n)$ is real. For (4.2.9) we first note that $Z_p(n) = 0$ if $p > 0$ and $Z_0(n) = 1$. Moreover, $L^2(M\backslash K, 0)$ is one dimensional and spanned by the constant function $\psi_{0,1} = 1$. Thus we have

$v_{f,0,1}(t) = \int_{M\backslash K} f(a_t\tilde{k})d\widetilde{\sigma}_{n+1}(\tilde{k})$ and

$$\begin{aligned}\langle f, 1\rangle &= \int_{Q\backslash G} f(a_t\tilde{k})d\mu_{Q\backslash G}(a_t\tilde{k})\\ &= \int_{\mathbb{R}}\int_{M\backslash K} f(a_t\tilde{k})e^{-nt}d\widetilde{\sigma}_{n+1}(\tilde{k})dt = \sqrt{2\pi}\widehat{v}_{f,0,1}(-in).\end{aligned}$$

Hence we have

$$\begin{aligned}&\int_{Q\backslash G}\overline{f(a_t\tilde{k})}\left(\sum_{p,l}\sqrt{2\pi}\widehat{v}_{f,p,l}(-in)Z_p(n)\psi_{p,l}(\tilde{k})\right)d\mu_{Q\backslash G}(a_t\tilde{k})\\ &= \sqrt{2\pi}\widehat{v}_{f,0,1}(-in)\int_{Q\backslash G}\overline{f(a_t\tilde{k})}d\mu_{Q\backslash G}(a_t\tilde{k})\\ &= \langle f, 1\rangle\overline{\langle f, 1\rangle} = |\langle f, 1\rangle|^2.\end{aligned}$$

This finishes the proof of formula (1.1.6). It now remains to show that $\|\mathcal{T}_j\|_2 \leq 1$, that is, to show $\|\mathcal{T}_j(f)\|_2 \leq \|f\|_2$ for any $f \in L^2(Q\backslash G)$. We first prove this for smooth compactly supported functions. For f on $Q\backslash G$ smooth and compactly supported, write $f = \sum_{p,l} f_{p,l} = \sum_{p,l} v_{f,p,l}\psi_{p,l}$ as above with $v_{f,p,l} \in \mathcal{C}_c^\infty(\mathbb{R})$. It is clear from the definition (4.2.6) of $\mathcal{T}$ that $\mathcal{T}(f_{p,l})$ and $\mathcal{T}(f_{p',l'})$ are orthogonal to each other whenever $(p,l) \neq (p',l')$. Thus by Lemma 4.2.1 and orthogonality we have

$$\|\mathcal{T}_j(f)\|_2^2 = \sum_{p,l}\|\mathcal{T}_j(f_{p,l})\|_2^2 \leq \sum_{p,l}\|f_{p,l}\|_2^2 = \|f\|_2^2.$$

Finally we note that since the space of smooth compactly supported functions is dense in $L^2(Q\backslash G)$ and $\mathcal{T}_j$ is bounded, we can extend $\mathcal{T}_j$ uniquely to $L^2(Q\backslash G)$ with the same operator norm. □

Remark 4.2.10. We note that by choosing different testing functions, it is not hard to see that these residues c_{jr} are real. In fact, combining the first moment formula (4.1.4) and the second moment formula (1.1.6) we can bound

$$\|\Theta_f^j - c_{j0}\langle f, 1\rangle\|_2^2 \leq 2c_{j0}\|f\|_2^2 + c_{j0}\sum_{r=1}^{\ell_j} c_{jr}M_f(s_{jr}).$$

Moreover, we can take suitable testing functions f such that $M_f(s_{j1})$ is arbitrarily larger than all other terms in the right-hand side of this above inequality. Thus by the positivity of the left-hand side, we know that c_{j1}, the residue at the largest exceptional pole s_{j1}, is positive. We suspect that all these residues are positive and it is probably a well-known result, but we can not find a reference explicitly stating it. We note that for $\mathrm{PSL}_2(\mathbb{R})$ this is proved in [Iwa02, Theorem 6.9] using Maass-Selberg relations.

4.3 Proof of Theorem 2

In this section we give the proof of Theorem 2. Let us first define the family of functions $\mathscr{A}_\lambda$ occurring in this theorem. Fix a parameter $\lambda > 0$, define $\mathscr{A}_\lambda \subset L^2(Q\backslash G)$ to be the set of bounded functions of the form

$$f(a_t k) = v(t)\,\phi(k)$$

with v non-negative and satisfying

$$\frac{\int_{\mathbb{R}} v(t)\,e^{-st}dt}{\left(\int_{\mathbb{R}} v(t)\,e^{-nt}dt\right)^{\frac{2s-n}{n}}\left(\int_{\mathbb{R}} v^2(t)\,e^{-nt}dt\right)^{\frac{n-s}{n}}} \leq \lambda \tag{4.3.1}$$

for any $s \in \left(\frac{n}{2}, n\right)$, and $\phi \in L^2(M\backslash K)$ non-negative if v is not compactly supported.

Remark 4.3.2. We will prove Theorem 2 by showing that for any $s \in (\frac{n}{2}, n)$,

$$M_f(s) \lesssim_{s,\lambda} \|f\|_1^{\frac{2(2s-n)}{n}} \|f\|_2^{\frac{4(n-s)}{n}}$$

under the assumption of (4.3.1). We note that when ϕ is also non-negative, we do not require v to be compactly supported. So a priori for such f we can not apply formula (1.1.6) directly. Instead, we will prove a variant of (1.1.6) for such functions from which we deduce Theorem 2 (see Corollary 4.3.3).

Before proving Theorem 2, we show here that when v is an indicator function (which is the case for applications), we can always bound the left-hand side of (4.3.1) by 2.

Lemma 4.3.1. *For any function f on $Q\backslash G$ of the form $f = v\phi$ with v an indicator function of any Borel set in $\mathbb{R}$ away from $-\infty$ and ϕ any non-negative function on $M\backslash K$, then $f \in \mathscr{A}_2$.*

Proof. First we note that since v is bounded and supported away from $-\infty$, all three integrals on the left-hand side of (4.3.1) are finite. Moreover, since $v^2 = v$ it suffices to show that

$$\int_{\mathbb{R}} v(t)\, e^{-st} dt \leq 2 \left(\int_{\mathbb{R}} v(t)\, e^{-nt} dt \right)^{\frac{s}{n}}.$$

Take $\alpha = \frac{n-s}{2}$ and let T be some real number to be determined, then we can bound

$$\begin{aligned}
\int_{\mathbb{R}} v(t)e^{-st}dt &= \int_{-\infty}^{T} v(t)e^{-st}dt + \int_{T}^{\infty} v(t)e^{-st}dt \\
&\leq \int_{-\infty}^{T} v(t)e^{-(s+\alpha)t}e^{\alpha t}dt + \frac{e^{-sT}}{s} \\
&\leq \left(\int_{-\infty}^{T} \left(v(t)e^{-(s+\alpha)t}\right)^{\frac{n}{s+\alpha}} dt \right)^{\frac{s+\alpha}{n}} \left(\int_{-\infty}^{T} \left(e^{\alpha t}\right)^{\frac{n}{n-s-\alpha}} dt \right)^{\frac{n-s-\alpha}{n}} + \frac{e^{-sT}}{s} \\
&\leq \left(\int_{\mathbb{R}} v(t)e^{-nt}dt \right)^{\frac{s+\alpha}{n}} \left(\frac{n-s-\alpha}{n\alpha} e^{\frac{n\alpha T}{n-s-\alpha}} \right)^{\frac{n-s-\alpha}{n}} + \frac{e^{-sT}}{s} \\
&= \left(\frac{n-s-\alpha}{n\alpha} \right)^{\frac{n-s-\alpha}{n}} e^{\alpha T} \left(\int_{\mathbb{R}} v(t)e^{-nt}dt \right)^{\frac{s+\alpha}{n}} + \frac{e^{-sT}}{s},
\end{aligned}$$

where for the first inequality we used the bound $v \leq 1$, for the second inequality we used Hölder's inequality and the facts that $v^\beta = v$ for any $\beta > 0$ and $n - s - \alpha > 0$ (recalling $\alpha = \frac{n-s}{2}$) and for the third inequality we used the bound $\int_{-\infty}^{T} v(t)e^{-nt}dt \leq \int_{\mathbb{R}} v(t)e^{-nt}dt$. Now plugging in $\alpha = \frac{n-s}{2}$ and taking T such that

$$\left(\frac{n-s-\alpha}{n\alpha} \right)^{\frac{n-s-\alpha}{n}} e^{\alpha T} \left(\int_{\mathbb{R}} v(t)e^{-nt}dt \right)^{\frac{s+\alpha}{n}} = \frac{e^{-sT}}{s}$$

we get $e^{-T} = s^{\frac{2}{n+s}} n^{-\frac{n-s}{n(n+s)}} \left(\int_{\mathbb{R}} v(t) e^{-nt} dt \right)^{\frac{1}{n}}$ and

$$\int_{\mathbb{R}} v(t)e^{-st}dt \leq 2s^{-\frac{n-s}{n+s}} n^{-\frac{(n-s)s}{(n+s)n}} \left(\int_{\mathbb{R}} v(t)e^{-nt}dt \right)^{\frac{s}{n}} < 2 \left(\int_{\mathbb{R}} v(t)e^{-nt}dt \right)^{\frac{s}{n}},$$

where for the last inequality we used the inequality that $s^{-\frac{n-s}{n+s}} n^{-\frac{(n-s)s}{(n+s)n}} < 1$ for $s \in (\frac{n}{2}, n)$ and $n \geq 2$. □

4.3.1 Two preliminary estimates

As mentioned above we will prove Theorem 2 by bounding $M_f(s)$ by the norms of f. We first show that for f of the form $f(a_t\tilde{k}) = v(t)\phi(\tilde{k})$, we can have a simpler expression for $M_f(s)$.

Lemma 4.3.2. *If $f \in L^2(Q \backslash G)$ is of the form $f(a_t k) = v(t)\phi(k)$, then $M_f(s)$ can be written as*

$$M_f(s) = \left(\sum_{p=0}^{\infty} Z_p(s) \, ||\phi_p||_2^2 \right) \left| \int_{\mathbb{R}} v(t) \, e^{-st} dt \right|^2, \tag{4.3.3}$$

where ϕ_p denotes the projection of ϕ into $L^2(M \backslash K, p)$.

Proof. For each $p \geq 0$, let $\{\psi_{p,l} \mid 1 \leq l \leq d_p\}$ be the fixed orthonormal basis of $L^2(M \backslash K, p)$ as before. Then we have

$$\phi_p(\tilde{k}) = \sum_{1 \leq l \leq d_p} C_{p,l} \psi_{p,l}(\tilde{k})$$

with $C_{p,l} = \int_{M \backslash K} \phi(\tilde{k}) \overline{\psi_{p,l}(\tilde{k})} d\widetilde{\sigma}_{n+1}(\tilde{k})$. Moreover, we have

$$\|\phi_p\|_2^2 = \sum_{1 \leq l \leq d_p} |C_{p,l}|^2$$

and

$$v_{f,p,l}(t) = \int_{M \backslash K} v(t) \, \phi(\tilde{k}) \overline{\psi_{p,l}(\tilde{k})} d\widetilde{\sigma}_{n+1}(\tilde{k}) = C_{p,l} v(t).$$

Hence

$$\begin{aligned} M_f(s) &= \sum_{p,l} Z_p(s) \left| \int_{\mathbb{R}} v_{f,p,l}(t) e^{-st} dt \right|^2 \\ &= \sum_{p=0}^{\infty} Z_p(s) \sum_{1 \le l \le d_p} |C_{p,l}|^2 \left| \int_{\mathbb{R}} v(t) e^{-st} dt \right|^2 \\ &= \left(\sum_{p=0}^{\infty} Z_p(s) \|\phi_p\|_2^2 \right) \left| \int_{\mathbb{R}} v(t) e^{-st} dt \right|^2 . \end{aligned}$$

□

Corollary 4.3.3. *Keep the notation as in Theorem 1. For any bounded function of the form $f(a_t\tilde{k}) = v(t)\phi(\tilde{k})$ with both v and ϕ non-negative and for any $1 \le j \le h$, we have*

$$c_{j0}^2\|f\|_1^2 + \sum_{r=1}^{\ell_j} c_{jr} M_f(s_{jr}) \le \|\Theta_f^j\|_2^2 \le c_{j0}^2\|f\|_1^2 + 2c_{j0}\|f\|_2^2 + c_{j0}\sum_{r=1}^{\ell_j} c_{jr} M_f(s_{jr}).$$

Proof. For any $l \ge 1$ define $f_l = v_l\phi$ with $v_l(t) = v(t)\chi_{[-l,l]}(t)$, then f_l converges to f from below pointwise. Similarly $|\Theta_{f_l}^j|^2$ converges to $|\Theta_f^j|^2$ from below pointwise. Thus by the monotone convergence theorem, applying (1.1.6) to $\Theta_{f_l}^j$ (noting that f_l is bounded and compactly supported) and using the estimate $\langle \mathcal{T}_j(f_l), f_l\rangle \le \|f_l\|_2^2$ we have

$$\|\Theta_f^j\|_2^2 = \lim_{l\to\infty}\|\Theta_{f_l}^j\|_2^2 \le \lim_{l\to\infty}\left(c_{j0}^2\|f_l\|_1^2 + 2c_{j0}\|f_l\|_2^2 + c_{j0}\sum_{r=1}^{\ell_j} c_{jr} M_{f_l}(s_{jr}) \right).$$

Since f_l (resp. f_l^2) converge to f (resp. f^2) from below pointwise, again by monotone convergence theorem we have $\lim_{l\to\infty}\|f_l\|_1^2 = \|f\|_1^2$ and $\lim_{l\to\infty}\|f_l\|_2^2 = \|f\|_2^2$. For the third term, by the expression (4.3.3) and the fact that $v_l(t)e^{st}$ converges to $v(t)e^{-st}$ for any $s \in (\frac{n}{2}, n)$, we get $\lim_{l\to\infty} M_{f_l}(s) = M_f(s)$ for any $s \in (\frac{n}{2}, n)$. This finishes the proof for the upper bound. The other inequality follows similarly with the estimate $\langle \mathcal{T}_j(f_l), f_l\rangle \le \|f_l\|_2^2$ replaced by the estimate $\langle f_l + \mathcal{T}_j(f_l), f_l\rangle \ge 0$. □

Lemma 4.3.4. *For any $s \in \left(\frac{n}{2}, n\right)$, $Z_p(s) \asymp_s (p+1)^{(n-2s)}$.*

Proof. Since $Z_p(s) = \prod_{k=0}^{p-1} \frac{n-s+k}{s+k}$ we have

$$\log(Z_p(s)) = \log\left(\frac{n-s}{s}\right) + \sum_{k=1}^{p-1}\left(\log\left(1+\frac{n-s}{k}\right) - \log\left(1+\frac{s}{k}\right)\right)$$
$$= (n-2s)\sum_{k=1}^{p-1}\frac{1}{k} + O_s(1) = (n-2s)\log(p+1) + O_s(1). \qquad \square$$

We can now give the

Proof of Theorem 2. For any $f(a_t k) = v(t)\phi(k) \in \mathscr{A}_\lambda$, define

$$\widetilde{M}_f(s) := \left(\int_{\mathbb{R}} v(t) e^{-st} dt\right)^2 \sum_{p=0}^{\infty} \frac{\|\phi_p\|_2^2}{(p+1)^{(2s-n)}},$$

where ϕ_p is the projection of ϕ into $L^2(M\backslash K, p)$ as above. In view of Lemma 4.3.2 and Lemma 4.3.4, we have $\widetilde{M}_f(s) \asymp_s M_f(s)$. Thus in view of Theorem 1 and Corollary 4.3.3 (when f is non-negative but not compactly supproted), it suffices to show that

$$\widetilde{M}_f(s) \lesssim_{s,\lambda} \|f\|_1^{\frac{2(2s-n)}{n}} \|f\|_2^{\frac{4(n-s)}{n}}$$

for any $s \in (\frac{n}{2}, n)$. Let us first estimate the series occurring in $\widetilde{M}_f(s)$. We claim that

$$\sum_{p=0}^{\infty} \frac{\|\phi_p\|_2^2}{(p+1)^{(2s-n)}} \lesssim_s \|\phi\|_1^{\frac{2(2s-n)}{n}} \|\phi\|_2^{\frac{4(n-s)}{n}}.$$

First if $\|\phi\|_2 \leq \|\phi\|_1$, we can bound

$$\sum_{p=0}^{\infty} \frac{\|\phi_p\|_2^2}{(p+1)^{(2s-n)}} \leq \sum_{p=0}^{\infty} \|\phi_p\|_2^2 = \|\phi\|_2^2 \leq \|\phi\|_1^{\frac{2(2s-n)}{n}} \|\phi\|_2^{\frac{4(n-s)}{n}}.$$

Second if $\|\phi\|_2 > \|\phi\|_1$, then let $\iota := \frac{\|\phi\|_2}{\|\phi\|_1} > 1$. We separate the summation into two parts:

$$\sum_{p=0}^{\infty} \frac{\|\phi_p\|_2^2}{(p+1)^{(2s-n)}} = \left(\sum_{p=0}^{\lfloor \iota^{\frac{2}{n}} \rfloor} + \sum_{p=\lfloor \iota^{\frac{2}{n}} \rfloor + 1}^{\infty}\right) \frac{\|\phi_p\|_2^2}{(p+1)^{(2s-n)}}.$$

For the first part, we invoke an estimate from spherical harmonic analysis ([Sog86, inequality (4.4)]).[1] Namely, for any $\phi \in L^2(M\backslash K)$ and $p \geq 0$,

$$\|\phi_p\|_2^2 \lesssim (p+1)^{n-1} \|\phi\|_1^2$$

with the bounding constant only depends on n. Thus we have

$$\begin{aligned}\sum_{p=0}^{\lfloor \iota^{\frac{2}{n}} \rfloor} \frac{\|\phi_p\|_2^2}{(p+1)^{(2s-n)}} &\lesssim \sum_{p=0}^{\lfloor \iota^{\frac{2}{n}} \rfloor} \frac{\|\phi\|_1^2}{(p+1)^{1-2(n-s)}} \\ &\asymp_s \left(\iota^{\frac{2}{n}}\right)^{2(n-s)} \|\phi\|_1^2 \quad (\text{since } 1-2(n-s)<1) \\ &= \|\phi\|_1^{\frac{2(2s-n)}{n}} \|\phi\|_2^{\frac{4(n-s)}{n}}.\end{aligned}$$

For the second part, we have

$$\begin{aligned}\sum_{p=\lfloor \iota^{\frac{2}{n}} \rfloor+1}^{\infty} \frac{\|\phi_p\|_2^2}{(p+1)^{(2s-n)}} &\le \frac{1}{\left(\iota^{\frac{2}{n}}\right)^{(2s-n)}} \sum_{p=\lfloor \iota^{\frac{2}{n}} \rfloor+1}^{\infty} \|\phi_p\|_2^2 \\ &\le \iota^{-2\left(\frac{2s}{n}-1\right)} \|\phi\|_2^2 \\ &= \|\phi\|_1^{\frac{2(2s-n)}{n}} \|\phi\|_2^{\frac{4(n-s)}{n}}.\end{aligned}$$

This finishes the proof of the claim. Thus by the assumption (4.3.1) we have

$$\begin{aligned}\widetilde{M}_f(s) &\le \lambda^2 \left(\int_{\mathbb{R}} v(t)\, e^{-nt} dt\right)^{\frac{2(2s-n)}{n}} \left(\int_{\mathbb{R}} v^2(t)\, e^{-nt} dt\right)^{\frac{2(n-s)}{n}} \sum_{p=0}^{\infty} \frac{\|\phi_p\|_2^2}{(p+1)^{(2s-n)}} \\ &\lesssim_{s,\lambda} \left(\int_{\mathbb{R}} v(t)\, e^{-nt} dt\right)^{\frac{2(2s-n)}{n}} \left(\int_{\mathbb{R}} v^2(t)\, e^{-nt} dt\right)^{\frac{2(n-s)}{n}} \|\phi\|_1^{\frac{2(2s-n)}{n}} \|\phi\|_2^{\frac{4(n-s)}{n}} \\ &= \|f\|_1^{\frac{2(2s-n)}{n}} \|f\|_2^{\frac{4(n-s)}{n}},\end{aligned}$$

where for the last equality we note that for this f, $\|f\|_1^2 = \left(\int_{\mathbb{R}} v(t)e^{-nt}dt\right)^2 \|\phi\|_1^2$ and $\|f\|_2^2 = \int_{\mathbb{R}} v^2(t)e^{-nt}dt \|\phi\|_2^2$. Finally we note that the dependence of the bounding constant on s is absorbed into the dependence on Γ since Γ determines the positions of these exceptional poles $s_{j1}, \ldots, s_{j\ell_j}$. □

[1]The exact form of inequality (4.4) in [Sog86] is $||\phi_m||_2 \lesssim m^{\frac{n-1}{2}} ||\phi||_1$. Here we square both sides and replace m by $m+1$ to cover the case $m = 0$.

Chapter 5

Applications to logarithm laws

In this chapter we prove Theorem 3 using the estimate induced from Theorem 2. We choose the upper half space model for $\mathbb{H}^{n+1}$ and take $G = \mathrm{PSL}_2(T_{n-1})$ as defined in section 2.1.1.

Let $\Gamma \subset G$ be a non-uniform lattice. Fix $o \in \Gamma\backslash G$ and let $\{g_t\} \subset G$ be a one-parameter unipotent subgroup in G. Let dist_G and dist_Γ be the hyperbolic distance functions on G/K and $\Gamma\backslash G/K$ respectively as fixed in section 2.4. We will prove that for μ-a.e. $x \in \Gamma\backslash G$

$$\varlimsup_{t\to\infty} \frac{\mathrm{dist}_\Gamma\left(o, xg_t\right)}{\log t} = \frac{1}{n}. \tag{5.0.1}$$

First we note that if (5.0.1) holds for Γ, then it also holds for any of its conjugate $\Gamma' = \xi^{-1}\Gamma\xi$. This follows from the following identity that for any $g, g' \in G$

$$\mathrm{dist}_\Gamma\left(\Gamma g, \Gamma g'\right) = \mathrm{dist}_{\Gamma'}\left(\Gamma'\xi^{-1}g, \Gamma'\xi^{-1}g'\right),$$

where $\mathrm{dist}_{\Gamma'}$ is the induced hyperbolic distance function on $\Gamma'\backslash G/K$. Hence we can assume that Γ has a cusp at P, the upper triangular subgroup of G that we fix in section 2.1.1. Fix this Γ and we denote $\mathrm{dist} = \mathrm{dist}_\Gamma$ without ambiguity.

Next note that $\{g_t\}_{t\in\mathbb{R}}$ can be replaced by a new flow $\{\tilde{g}_t\}_{t\in\mathbb{R}}$ with $\tilde{g}_t = k^{-1}g_{\eta t}k$

for some $k \in K$ and $\eta > 0$. This is because

$$\varlimsup_{t\to\infty} \frac{\operatorname{dist}(o, x\tilde{g}_t)}{\log t} = \varlimsup_{t\to\infty} \frac{\operatorname{dist}(o, x'g_t)}{\log t}$$

with $x' = xk^{-1}$. For any $\mathbf{x} \in \mathbb{V}^{n-1}$, denote $u_{\mathbf{x}}^{-} = \begin{pmatrix} 1 & 0 \\ \mathbf{x} & 1 \end{pmatrix}$ to be the corresponding lower triangular unipotent matrix.

Lemma 5.0.1. *Every unipotent element in G is K-conjugate to u_x^- for some $x > 0$.*

Proof. Let g be an unipotent element in G. We first note that it suffices to show g is K-conjugate to u_{-x} for some $x > 0$. This is because u_x^- is conjugate to u_{-x} via $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in K$. Next we note that since g is unipotent, g is conjugate to some element in N. By Iwasawa decomposition and the fact that NA normalizes N, g is K-conjugate to some element in N. Hence we can assume g is contained in N. Finally, we note that any element in N is conjugate to some u_{-x} with $x > 0$ via the group M since the conjugation action of M on N realizes M as the rotation group of N. □

Since we can conjugate $\{g_t\}_{t\in\mathbb{R}}$ by some element in K and rescale it by a positive number, in view of Lemma 5.0.1 we can assume the unipotent flow is given by $\{g_t = u_t^-\}_{t\in\mathbb{R}}$.

5.1 The construction

As mentioned in the introduction, the hard part of Theorem 3 is to prove the lower bound that for μ-a.e. $x \in \Gamma\backslash G$

$$\varlimsup_{t\to\infty} \frac{\operatorname{dist}_\Gamma(o, xg_t)}{\log t} \geq \frac{1}{n}. \tag{5.1.1}$$

We will prove this by constructing a sequence of sets in $\Gamma\backslash G$ whose limit superior set only consists of points satisfying (5.1.1). Then by ergodicity of $\{g_t\}_{t\in\mathbb{R}}$, to show that (5.1.1) is satisfied for a full-measure set, it suffices to show that this limit superior set is of positive measure. We note that this is where we will use Theorem 2 (or more precisely, a direct consequence of Theorem 2). In this section, we construct these sets explicitly. Before we give the construction, we note that since we assume Γ has a cusp at P, we can take $P = P_1$ (and $\xi_1 = \mathrm{id}$) to be one of the representatives for cusps of Γ. In this case, for any function f on $\Gamma\backslash G$ and $g \in G$ we have

$$\Theta_f^1(g) = \sum_{\gamma\in\Gamma_P\backslash\Gamma} f(\gamma g).$$

Since we will only use this incomplete Eisenstein series in this chapter, we abbreviate it by Θ_f for simplicity.

For any $\mathfrak{D} \subset Q\backslash G$, we define the set $Y_{\mathfrak{D}} \subset \Gamma\backslash G$ corresponding to $\mathfrak{D}$ by

$$Y_{\mathfrak{D}} := \{\Gamma g \in \Gamma\backslash G \mid Q\gamma g \in \mathfrak{D} \text{ for some } \gamma \in \Gamma\}.$$

Note that the relation between $\mathfrak{D}$ and $Y_{\mathfrak{D}}$ is that if $\mathfrak{D}$ is the support of a function f on $Q\backslash G$, then $Y_{\mathfrak{D}}$ is the support of Θ_f.

Let $\{r_\ell\}$ be any sequence of positive numbers such that $r_\ell \to \infty$ and $\sum_{\ell=1}^{\infty} e^{-nr_\ell} = \infty$. For any integer $m \geq 1$, since $\sum_{\ell=1}^{\infty} e^{-nr_\ell} = \infty$ there exists some $p(m) > m$ such that $\sum_{\ell=m}^{p(m)} e^{-nr_\ell} \geq 1$. Let $N^- \subset G$ be the subgroup of lower triangular unipotent matrices and

$$B^- = \{u_{\mathbf{x}}^- \mid \mathbf{x} \in \mathbb{V}^{n-1}, |\mathbf{x}| < \frac{1}{2}\}$$

be the open ball in N^- with radius $\frac{1}{2}$, centered at the identity element, where the norm $|\mathbf{x}| = \sqrt{\mathbf{x}\overline{\mathbf{x}}}$ is as defined in section 2.1.1. For any integer $m \geq 1$ we define

$$\mathfrak{D}_m := Q\backslash \bigcup_{\ell=m}^{p(m)} QA(r_\ell)B^- g_{-\ell} \subset Q\backslash G,$$

where $A(\tau) = \{a_t \mid t \geq \tau\}$.

Lemma 5.1.1. $\left\{\mu_{Q\backslash G}(\mathfrak{D}_m)\right\}_{m\geq 1}$ *is uniformly bounded from below.*

Proof. Note that every matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ with $d \neq 0$ can be written uniquely as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & bd^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{d'}{|d|} & 0 \\ 0 & \frac{d}{|d|} \end{pmatrix} \begin{pmatrix} |d|^{-1} & 0 \\ 0 & |d| \end{pmatrix} \begin{pmatrix} 1 & 0 \\ d^{-1}c & 1 \end{pmatrix}.$$

Hence $NMAN^- = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \mid d \neq 0\right\}$ is a Zariski open dense subset of G. Thus there is a Zariski open dense subset in $Q\backslash G$ which can be expressed by the coordinates $Qg = Qa_t u_{\mathbf{x}}^-$. We note that under these coordinates, the right G-invariant measure on $Q\backslash G$ (up to scalars) is given by $e^{-nt}dtd\mathbf{x}$ since this is the right Haar measure for the group AN^-. Moreover, $\mathfrak{D}_m$ is a disjoint union of the sets $Q\backslash QA(r_\ell)B^- g_{-\ell}$ since $B^- g_{-\ell_1} \cap B^- g_{-\ell_2} = \emptyset$ whenever $\ell_1 \neq \ell_2$. Hence one can compute

$$\begin{aligned} \mu_{Q\backslash G}(\mathfrak{D}_m) &\asymp \sum_{\ell=m}^{p(m)} \int_{r_\ell}^{\infty} e^{-nt}dt \int_{B^- g_{-\ell}} 1 d\mathbf{x} \\ &\asymp \sum_{\ell=m}^{p(m)} e^{-nr_\ell} \geq 1. \end{aligned}$$

□

Lemma 5.1.2. *There is some sufficiently large integer L such that for any $m \geq L$ and for any $x \in Y_{\mathfrak{D}_m}$ there exists $m \leq \ell \leq p(m)$ such that*

$$dist(o, xg_\ell) \geq r_\ell + O(1).$$

Proof. First recall the Siegel fundamental domain $\mathcal{F}_{\Gamma,\tau_0,U_0}$ that we fixed in (2.3.5), take L such that $r_\ell - \log 2 \geq \tau_0$ for all $\ell \geq L$. Next using (3.2.8) we have that for any $\tau \in \mathbb{R}$

$$QA(\tau)B^- \subset QA(\tau - \log 2)K = NA(\tau - \log 2)K. \tag{5.1.2}$$

Hence for $m \geq L$, $x \in Y_{\mathfrak{D}_m}$ can be written as $x = \Gamma g g_{-\ell}$ for some $m \leq \ell \leq p(m)$ with $g \in QA(r_\ell) B^- \subset NA(r_\ell - \log 2) K$. After left multiplying by some $\gamma \in \Gamma_N$ we can assume that $g = ua_t k$ is contained in the Siegel set $U_0 A(r_\ell - \log 2) K$ (we can do this since U_0 contains a fundamental domain of $\Gamma_N \backslash N$). Since $\ell \geq m \geq L$, $r_\ell - \log 2 \geq \tau_0$. By (2.4.4) we have

$$\operatorname{dist}(o, xg_\ell) = \operatorname{dist}(o, ua_t k) = t + O(1) \geq r_\ell - \log 2 + O(1) = r_\ell + O(1). \qquad \square$$

The next lemma shows that there exists a nice set sitting inside $\mathfrak{D}_m$. We first identify $Q\backslash G$ with $A \times M\backslash K$ and let $\Pr : Q\backslash G \to M\backslash K$ be the natural projection map. Define

$$K(\ell) := \Pr\left(Q\backslash QA(r_\ell) B^- g_{-\ell}\right)$$

be the K-part of $Q\backslash QA(r_\ell) B^- g_{-\ell}$. We note that $K(\ell)$ is independent of r_ℓ, that is, $K(\ell) = \Pr(Q\backslash QA(\tau) B^- g_{-\ell})$ for any $\tau \in \mathbb{R}$.

Lemma 5.1.3. *For any $\ell \geq 1$, let $\tau_\ell = r_\ell - 2\log(\ell) + \log 2$. Then we have*

$$A(\tau_\ell) \times K(\ell) \subset Q\backslash QA(r_\ell) B^- g_{-\ell},$$

and

$$\mu_{Q\backslash G}(A(\tau_\ell) \times K(\ell)) \asymp \mu_{Q\backslash G}\left(Q\backslash QA(r_\ell) B^- g_{-\ell}\right)$$

with the implicit constant independent of ℓ.

Proof. For each $k \in K(\ell)$, define

$$I(k) := \{t \in \mathbb{R} \mid Qa_t k \in Q\backslash QA(r_\ell) B^- g_{-\ell}\}$$

and

$$t(k) := \inf I(k).$$

We first note that if $t \in I(k)$ (that is, $a_t k = q a_{t_0} u^-_{\mathbf{x}-\ell}$ for some $q \in Q$, $t_0 \geq r_\ell$ and $|\mathbf{x}| < \frac{1}{2}$), then $[t, \infty) \subset I(k)$. This is because for any $t' > t$ we have

$$a_{t'} k = a_{t'-t} a_t k = a_{t'-t} q a_{t_0} u^-_{\mathbf{x}-\ell} = q' a_{t'-t+t_0} u^-_{\mathbf{x}-\ell},$$

and $t' - t + t_0 > t_0 \geq r_\ell$. Here $q' = a_{t'-t} q a_{t-t'} \in Q$. This implies that

$$Q \backslash Q A(r_\ell) B^- g_{-\ell} = \bigcup_{k \in K(\ell)} A(t(k)) \times \{k\}. \tag{5.1.3}$$

Moreover, by (3.2.8), the relation $a_t k = q a_{t_0} u^-_{\mathbf{x}-\ell}$ implies

$$t = t_0 - \log\left(1 + |\mathbf{x} - \ell|^2\right). \tag{5.1.4}$$

In particular, the minimality of $t(k)$ implies that

$$t(k) = r_\ell - \log\left(1 + |\mathbf{x} - \ell|^2\right)$$

for some $\mathbf{x}$ (determined by k). As k ranges over $K(\ell)$ (that is, $\mathbf{x}$ ranges over B^-), $t(k)$ attains the maximal value when $\mathbf{x} = \frac{1}{2}$ and the minimal value when $\mathbf{x} = -\frac{1}{2}$. Let $t_{\ell, \pm\frac{1}{2}} = r_\ell - \log\left(1 + |\ell \mp \frac{1}{2}|^2\right)$, then in view of (5.1.3) we have

$$A\left(t_{\ell,\frac{1}{2}}\right) \times K(\ell) \subset Q \backslash Q A(r_\ell) B^- g_{-\ell} \subset A\left(t_{\ell,-\frac{1}{2}}\right) \times K(\ell).$$

Next, note that $e^{-nt_{\ell,\frac{1}{2}}} \asymp e^{-nt_{\ell,-\frac{1}{2}}} \asymp e^{-nr_\ell} \ell^{2n}$, hence

$$\mu_{Q\backslash G}\left(A\left(t_{\ell,\frac{1}{2}}\right) \times K(\ell)\right) \asymp \mu_{Q\backslash G}\left(Q \backslash Q A(r_\ell) B^- g_{-\ell}\right) \asymp \mu_{Q\backslash G}\left(A\left(t_{\ell,-\frac{1}{2}}\right) \times K(\ell)\right).$$

Finally, note that $t_{\ell,\frac{1}{2}} \leq \tau_\ell$ and $e^{-nt_{\ell,\frac{1}{2}}} \asymp e^{-n\tau_\ell}$, hence

$$A(\tau_\ell) \times K(\ell) \subset A\left(t_{\ell,\frac{1}{2}}\right) \times K(\ell) \subset Q \backslash Q A(r_\ell) B^- g_{-\ell}$$

and

$$\mu_{Q\backslash G}\left(A(\tau_\ell) \times K(\ell)\right) \asymp \mu_{Q\backslash G}\left(A\left(t_{\ell,\frac{1}{2}}\right) \times K(\ell)\right) \asymp \mu_{Q\backslash G}\left(Q \backslash Q A(r_\ell) B^- g_{-\ell}\right). \quad \square$$

Remark 5.1.5. Later we will take $r_\ell = \frac{1-\epsilon}{n} \log \ell$ with ϵ some fixed small positive number. We note that in this case we can take $p(m) = 2m$. Moreover, since $\tau_m \geq \tau_\ell$ and $e^{-n\tau_m} \asymp e^{-n\tau_\ell} \asymp m^{(2n-1+\epsilon)}$ for all $m \leq \ell \leq 2m$, in view of Lemma 5.2.1 we have

$$A(\tau_m) \times K_m \subset \mathfrak{D}_m \quad \text{and} \quad \mu_{Q\backslash G}(A(\tau_m) \times K_m) \asymp \mu_{Q\backslash G}(\mathfrak{D}_m),$$

where $K_m := \cup_{\ell=m}^{2m} K(\ell)$.

5.2 Proof of Theorem 3

Now we can give the proof of logarithm laws. For any $r > 0$ let

$$B_r = \{x \in \Gamma\backslash G \mid \text{dist}(o, x) > r\}$$

be the corresponding cusp neighborhood as before.

5.2.1 Upper bound

Fix $\epsilon > 0$ and let $r_\ell = \frac{1+\epsilon}{n} \log(\ell)$. By (2.4.5) the sets

$$\{x \in \Gamma\backslash G \mid x g_\ell \in B_{r_\ell}\} = B_{r_\ell} g_{-\ell}$$

satisfy

$$\sum_{\ell=1}^{\infty} \mu(B_{r_\ell} g_{-\ell}) = \sum_{\ell=1}^{\infty} \mu(B_{r_\ell}) \asymp \sum_{\ell=1}^{\infty} \frac{1}{\ell^{1+\epsilon}} < \infty.$$

Hence by Borel-Cantelli lemma the set

$$\{x \in \Gamma\backslash G \mid x g_\ell \in B_{r_\ell} \text{ for finitely many } \ell\}$$

has full measure. This implies that

$$\overline{\lim_{\ell\to\infty}} \frac{\text{dist}(o, x g_\ell)}{\log \ell} \leq \frac{1+\epsilon}{n}$$

for μ-a.e. $x \in \Gamma\backslash G$. Moreover, for all $t \in \mathbb{R}$ let $\ell = \lfloor t \rfloor$, we have

$$|\mathrm{dist}\,(o, xg_t) - \mathrm{dist}\,(o, xg_\ell)\,| \leq \mathrm{dist}\,(xg_t, xg_\ell) \leq \mathrm{dist}_G\,(e, g_{\ell-t}) = O\,(1)\,,$$

hence we can replace the discrete limit over $\ell \in \mathbb{N}$ with a continuous limit over $t \in \mathbb{R}$. Finally, letting $\epsilon \to 0$ we get

$$\overline{\lim_{t\to\infty}} \frac{\mathrm{dist}\,(o, xg_t)}{\log t} \leq 1$$

for μ-a.e. $x \in \Gamma\backslash G$.

5.2.2 Lower bound

Fix $\epsilon > 0$ and let $r_\ell = \frac{1-\epsilon}{n} \log \ell$. Let $\mathfrak{D}_m$ and $Y_{\mathfrak{D}_m}$ be as above. Note that in this case, for the definition of $\mathfrak{D}_m$ we can take $p\,(m) = 2m$. We first prove the following

Lemma 5.2.1. *There is a constant $\kappa_\Gamma > 0$ depending only on Γ such that $\sigma\,(Y_{\mathfrak{D}_m}) \geq \kappa_\Gamma$ for all $m \geq 1$.*

Proof. For any $m \geq 1$, define the set

$$\mathfrak{D}'_m := A(\tau_m) \times K_m,$$

where $\tau_m = r_m - 2\log\,(m) + \log 2$ and $K_m = \bigcup_{\ell=m}^{2m} K\,(\ell)$ as above. By remark 5.1.5 we have $\mathfrak{D}'_m \subset \mathfrak{D}_m$ and $\mu_{Q\backslash G}\,(\mathfrak{D}'_m) \asymp \mu_{Q\backslash G}\,(\mathfrak{D}_m)$. Hence by Lemma 5.1.1, $\mu_{Q\backslash G}\,(\mathfrak{D}'_m) \gtrsim 1$ are uniformly bounded from below for all $m \geq 1$. Take $f_m = \chi_{\mathfrak{D}'_m}$ to be the indicator function of $\mathfrak{D}'_m$. We note that $f_m(a_t\tilde{k}) = v_m(t)\chi_{K_m}(\tilde{k})$ with $v_m(t) = \chi_{[\tau_m,\infty)}(t)$ and by Lemma 4.3.1 $f_m \in \mathscr{A}_2$ with $\mathscr{A}_\lambda$ as defined in section 4.3. Apply the first moment formula (4.1.4) for Θ_{f_m} we get

$$\int_{\Gamma\backslash G} \Theta_{f_m}\,(g)\, d\mu\,(g) = \frac{\omega_\Gamma}{\nu_\Gamma} \mu_{Q\backslash G}\,(\mathfrak{D}'_m)\,,$$

where $\omega_\Gamma := \int_{\Gamma_P\backslash Q} d\mu_Q(q)$ and $\nu_\Gamma = \int_{\Gamma\backslash G} dg$. Recall that Θ_{f_m} is supported on $Y_{\mathfrak{D}'_m}$, thus by Cauchy-Schwartz we have

$$\left(\frac{\omega_\Gamma}{\nu_\Gamma}\mu_{Q\backslash G}\left(\mathfrak{D}'_m\right)\right)^2 = \left(\int_{Y_{\mathfrak{D}'_m}} \Theta_{f_m}(g)\, d\mu(g)\right)^2 \leq \mu\left(Y_{\mathfrak{D}'_m}\right) \|\Theta_{f_m}\|_2^2. \tag{5.2.1}$$

Since $f_m \in \mathscr{A}_2$, by Theorem 2 (and remark 1.1.10) we can bound

$$\|\Theta_{f_m}\|_2^2 \lesssim_\Gamma \|f_m\|_1^2 + \|f_m\|_2^2 = \mu_{Q\backslash G}\left(\mathfrak{D}'_m\right)^2 + \mu_{Q\backslash G}\left(\mathfrak{D}'_m\right) \lesssim \mu_{Q\backslash G}\left(\mathfrak{D}'_m\right)^2, \tag{5.2.2}$$

where for the last inequality we used the fact that $\mu_{Q\backslash G}\left(\mathfrak{D}'_m\right) \gtrsim 1$ are uniformly bounded from below for all $m \geq 1$. Combining (5.2.1) and (5.2.2) we get

$$\mu_{Q\backslash G}\left(\mathfrak{D}'_m\right)^2 \lesssim_\Gamma \mu(Y_{\mathfrak{D}'_m})\mu_{Q\backslash G}\left(\mathfrak{D}'_m\right)^2.$$

This implies that $\mu(Y_{\mathfrak{D}'_m})$ are uniformly bounded from below for all $m \geq 1$. That is, there exists some $\kappa_\Gamma > 0$ such that $\mu(Y_{\mathfrak{D}'_m}) \geq \kappa_\Gamma$ for all $m \geq 1$. Finally, since $Y_{\mathfrak{D}'_m} \subset Y_{\mathfrak{D}_m}$, we get $\mu(Y_{\mathfrak{D}_m}) \geq \mu(Y_{\mathfrak{D}'_m}) \geq \kappa_\Gamma$ for all $m \geq 1$ as claimed. □

Now consider the limit superior set $\mathcal{B}_\epsilon := \cap_{\ell=L}^\infty \cup_{m=\ell}^\infty Y_{\mathfrak{D}_m}$, where L is as in Lemma 5.1.2. Then $\mu(\mathcal{B}_\epsilon) \geq \kappa_\Gamma > 0$ by Lemma 5.2.1. Moreover, by Lemma 5.1.2, for any $m \geq L, x \in Y_{\mathfrak{D}_m}$ there is some $\ell \geq m$ such that $\operatorname{dist}(o, xg_\ell) \geq r_\ell + O(1)$. Hence for any $x \in \mathcal{B}_\epsilon$ there is a sequence $\ell_m \to \infty$ such that $\operatorname{dist}(o, xg_{\ell_m}) \geq r_{\ell_m} + O(1)$. Consequently, we have

$$\mathcal{B}_\epsilon \subset \left\{x \in \Gamma\backslash G \mid \varlimsup_{t\to\infty} \frac{\operatorname{dist}(o, xg_t)}{\log t} \geq \frac{1-\epsilon}{n}\right\}.$$

Since the latter set is invariant under the action of $\{g_t\}_{t\in\mathbb{R}}$ and contains a set of positive measure, by ergodicity it has full measure. Letting $\epsilon \to 0$ we get

$$\varlimsup_{t\to\infty} \frac{\operatorname{dist}(o, xg_t)}{\log t} \geq \frac{1}{n}$$

for μ-a.e. $x \in \Gamma\backslash G$.

Chapter 6

Applications to counting

In this chapter we prove Theorem 4 and Theorem 5. We choose the hyperboloid model for $\mathbb{H}^{n+1}$ and take $G = \mathrm{SO}_0(n+1, 1)$ as defined in section 2.1.2. Let $P = NAM$ be the fixed parabolic subgroup and $\mathcal{Q}_0(v_0, \ldots, v_n, v_{n+1}) = v_0^2 + \cdots + v_n^2 - v_{n+1}^2$ be the fixed quadratic form as in section 2.1.2. Let

$$\mathcal{V}^+ := \{\vec{v} = (v_0, \ldots, v_n, v_{n+1}) \in \mathbb{R}^{n+2} \mid \mathcal{Q}_0(\vec{v}) = 0, v_{n+1} > 0\}$$

be the positive light cone corresponding to $\mathcal{Q}_0$. Since $\mathrm{SO}(n+1, 1)$ preserves $\mathcal{Q}_0$ and $G \subset \mathrm{SO}(n+1, 1)$ is the identity component, the right multiplication action of G on $\mathbb{R}^{n+2}$ preserves $\mathcal{V}^+$. Moreover, it is well known that G acts on $\mathcal{V}^+$ transitively, and each parabolic subgroup is the stabilizer of some ray (starting from the origin) in $\mathcal{V}^+$ (see [HT93, p.4]). We note that our fixed parabolic subgroup P is exactly the stabilizer of the ray spanned by $\vec{e}_0$ with $\vec{e}_0 = (0, \ldots, 0, -1, 1) \in \mathcal{V}^+$. More precisely, the subgroup A acts as scalars and $Q = NM$ fixes every vector on this ray. Thus the map $G \to \mathcal{V}^+$ sending $g \in G$ to $\vec{e}_0 g$ induces an identification between the homogeneous space $Q \backslash G$ and $\mathcal{V}^+$.

6.1 Counting orbits of lattice translates

Fix a non-uniform lattice $\Gamma \subset G$ with a cusp at P. As in the introduction we take $P = P_1$ (and $\xi_1 = \mathrm{id}$) to be one of the representatives of cusps of Γ. Hence for any bounded and compactly supported function f on $Q\backslash G$ and for any $g \in G$ we have

$$\Theta_f^1(g) = \sum_{\gamma \in \Gamma_P \backslash \Gamma} f(\gamma g).$$

For any finite-volume Borel set $B \subset \mathcal{V}^+(= Q\backslash G)$, recall the counting function

$$\mathcal{L}(B, \Gamma g) = \#(B \cap \vec{e}_0 \Gamma g)$$

counting the number of orbits of Γg in B. The starting point of applying our second moment formula to counting problem is the following lemma relating the counting function $\mathcal{L}(B, \Gamma g)$ with certain incomplete Eisenstein series.

Lemma 6.1.1. *For any finite-volume Borel set $B \subset \mathcal{V}^+$ let $f = \chi_B$ and we view it as a left Q-invariant function on G. Then for any $g \in G$ we have*

$$\mathcal{L}(B, \Gamma g) = \Theta_f^1(g).$$

Proof. Consider the map $\Gamma \to \mathcal{V}^+$ sending $\gamma \in \Gamma$ to $\vec{e}_0\gamma$. Clearly, its image is $\vec{e}_0\Gamma$. Since Q is the stabilizer of $\vec{e}_0$, it has kernel $\Gamma \cap Q$. Recall that the discrete subgroup $\Gamma_P = \Gamma \cap P$ is contained in Q, thus $\Gamma_P = \Gamma \cap Q$ is exactly the kernel of this map. We thus get an identification between $\Gamma_P\backslash\Gamma$ and $\vec{e}_0\Gamma$ identifying $\Gamma_P\gamma$ with $\vec{e}_0\gamma$. For $f = \chi_B$, when viewed as a left Q-invariant function on G, we have, for any $g \in G$, $f(g) = \chi_B(\vec{e}_0 g)$. We thus get

$$\mathcal{L}(B, \Gamma g) = \sum_{\vec{v} \in \vec{e}_0\Gamma} \chi_B(\vec{v} g) = \sum_{\gamma \in \Gamma_P\backslash\Gamma} \chi_B(\vec{e}_0\gamma g) = \sum_{\gamma \in \Gamma_P\backslash\Gamma} f(\gamma g) = \Theta_f^1(g). \qquad \square$$

With this identification and the first moment formula (4.1.4) we have

$$\int_{\Gamma\backslash G} \mathcal{L}(B,\Gamma g)d\mu(g) = c_{10}\mu_{Q\backslash G}(B), \tag{6.1.1}$$

where $c_{10} = \frac{\omega_1}{\nu_\Gamma}$ the residue of $\mathcal{C}_1(s)$ at the trivial pole. Thus the expected term for $\mathcal{L}(B,\Gamma g)$ is $c_{10}\mu_{Q\backslash G}(B)$ and we define the remainder function

$$\mathcal{E}(B,\Gamma g) := \left|\mathcal{L}(B,\Gamma g) - c_{10}\mu_{Q\backslash G}(B)\right|.$$

6.1.1 Mean square bound

The key input of Schmidt's arguments on lattice point counting problem is a mean square bound for the discrepancy function coming from Rogers' second moment. In this section, we prove an analogous mean square bound for $\mathcal{E}(B,\Gamma g)$ using our moment formula. As mentioned in the introduction, we are not able to prove such a bound for any finite-volume Borel set. More precisely, we only prove such a bound for (disjoint unions of) what we call generalized sectors. Let us first define here precisely what we mean by a generalized sector.

Definition 6.1.2. A finite-volume Borel subset $B \subset \mathcal{V}^+$ is called a *generalized sector* if it is of the form $B = \mathcal{U} \times \mathcal{K}$ with $\mathcal{U} \subset \mathbb{R}$ any Borel set in $\mathbb{R}$ away from $-\infty$ and $\mathcal{K} \subset M\backslash K$ any Borel set in $M\backslash K$.

We note that by Lemma 4.3.1 for any generalized sector, its indicator function is always contained in $\mathscr{A}_2$.

Theorem 6.1.2. *Keep the notation as in Theorem 1. Fix an integer $m \geq 1$. There exists a constant C_m (depending on m and Γ) such that for any $B \subset Q\backslash G$ with $\mu_{Q\backslash G}(B) \geq 1$ and of the form $B = \bigsqcup_{l=1}^m B_l$ with each $B_l \subset Q\backslash G$ a generalized sector*

, *we have*

$$\int_{\Gamma\backslash G} |\mathcal{E}(B,\Gamma g)|^2 \, d\mu(g) \leq C_m \mu_{Q\backslash G}(B)^{\frac{2s_{11}}{n}},$$

where $s_{11} \in (\frac{n}{2}, n)$ *is the largest exceptional pole of the function* $\mathcal{C}_1(s)$*.*

Proof. Let $f = \chi_B$ and $f_l = \chi_{B_l}$ for $1 \leq l \leq m$. For simplicity of notation, we denote by $V = \mu_{Q\backslash G}(B)$ and $V_l = \mu_{Q\backslash G}(B_l)$. Thus $V = \sum_{l=1}^m V_l \geq 1$. By (6.1.1) and Corollary 4.3.3 we have

$$\begin{aligned}\int_{\Gamma\backslash G} |\mathcal{E}(B,\Gamma g)|^2 \, d\mu(g) &= \int_{\Gamma\backslash G} \left|\Theta_f^1(g)\right|^2 d\mu(g) - (c_{10}V)^2 \\ &\leq 2c_{10}V + c_{10}\sum_{r=1}^{\ell_1} c_{1r} M_f(s_{1r}),\end{aligned}$$

where $\frac{n}{2} < s_{1\ell_1} < \cdots < s_{11} < n$ are the exceptional poles of $\mathcal{C}_1(s)$ with c_{1r} ($1 \leq r \leq \ell_1$) the corresponding residues. Since B is the disjoint union of B_l's, we have $f = f_1 + \cdots f_m$. By (4.2.3) we can bound

$$M_f(s) \leq m\left(M_{f_1}(s) + \cdots + M_{f_m}(s)\right)$$

for any $s \in (\frac{n}{2}, n)$. Moreover, since each B_l is a generalized sector, by Lemma 4.3.1 we know that $f_l \in \mathscr{A}_2$. We can thus use the estimate obtained in the proof of Theorem 2 to get for any $s \in (\frac{n}{2}, n)$

$$M_f(s) \lesssim_m \sum_{l=1}^m M_{f_l}(s) \lesssim_s \sum_{l=1}^m \|f_l\|_1^{\frac{2(2s-n)}{n}} \|f_l\|_2^{\frac{4(n-s)}{n}} = \sum_{l=1}^m V_l^{\frac{2s}{n}},$$

where for the last equality we used the identity that $\|f_l\|_1 = \|f_l\|_2^2 = V_l$. Since $\frac{2s}{n} > 1$ we have

$$\sum_{l=1}^m V_l^{\frac{2s}{n}} = \sum_{l=1}^m V_l V_l^{\frac{2s}{n}-1} < \sum_{l=1}^m V_l (V_1 + \cdots + V_l)^{\frac{2s}{n}-1} = (V_1 + \cdots + V_m)^{\frac{2s}{n}} = V^{\frac{2s}{n}}.$$

We thus get

$$\int_{\Gamma\backslash G} |\mathcal{E}(B,\Gamma g)|^2 \, d\mu(g) \lesssim_{m,\Gamma} V + \sum_{r=1}^{\ell_1} V^{\frac{2s_{1r}}{n}} \lesssim V^{\frac{2s_{11}}{n}},$$

where for the last inequality we used the assumption that $V \geq 1$. □

Remark 6.1.3. For applications (Theorem 4) we will only consider generalized sectors and differences of two generalized sectors with one contained in the other. We note that for the later case, that is, $B = B_1 \backslash B_2$ with $B_2 = \mathcal{U}_2 \times \mathcal{K}_2 \subset B_1 = \mathcal{U}_1 \times \mathcal{K}_1$ both generalized sectors, then

$$B = (\mathcal{U}_1 \times \mathcal{K}_1) \backslash (\mathcal{U}_2 \times \mathcal{K}_2) = (\mathcal{U}_1 \backslash \mathcal{U}_2) \times \mathcal{K}_1 \bigsqcup \mathcal{U}_2 \times (\mathcal{K}_1 \backslash \mathcal{K}_2)$$

can be expressed as a disjoint union of at most two generalized sectors. In other words, we can have a uniform bounding constant for all the sets we consider for applications.

Remark 6.1.4. When B is a norm ball with respect to the usual Euclidean norm on $\mathbb{R}^{n+2}$, that is, $f(a_t\tilde{k}) = \chi_B(a_t\tilde{k}) = \chi_{(r,\infty)}(t)$ for some $r \in \mathbb{R}$, then we have $M_f(s) \asymp \mu_{Q\backslash G}(B)^{\frac{2s}{n}}$ for any $s \in (\frac{n}{2}, n)$. Combining this with the lower bound in Corollary 4.3.3 (and remark 4.2.10) we get $\int_{\Gamma\backslash G} |\mathcal{E}(B, \Gamma g)|^2 \, d\mu(g) \asymp \mu_{Q\backslash G}(B)^{\frac{2s_{11}}{n}}$. Thus for such B this mean square bound we get in Theorem 6.1.2 is optimal.

6.1.2 Schmidt's argument

With the above mean square bound, together with Schmidt's arguments we can now prove Theorem 4. In fact, we will prove the following slightly more general theorem which implies Theorem 4 by taking $\psi(t) = \frac{1}{t^2}$.

Theorem 6.1.3. *Let $G = \mathrm{SO}_0(n+1, 1)$ and $\Gamma \subset G$ a non-uniform lattice with a cusp at P. Let $\mathcal{B}$ be a linearly ordered family of generalized sectors in $\mathcal{V}^+$. Let ψ be a positive, non-increasing function defined on positive real numbers such that $e^t\psi(t)$ is eventually non-decreasing and $\int_1^\infty \psi(t)dt < \infty$. Then for μ-a.e. lattice translate $\Gamma g \in \Gamma\backslash G$ there is $C_{\Gamma g}$ such that for all $B \in \mathcal{B}$ with $\mu_{Q\backslash G}(B) > C_{\Gamma g}$*

$$\mathcal{E}(B, \Gamma g) \leq \mu_{Q\backslash G}(B)^{\frac{s_{11}}{n}} \frac{\log^{1/2}\left(\mu_{Q\backslash G}(B)\right)}{\psi^{1/2}\left(\log(\mu_{Q\backslash G}(B)\right)},$$

where $s_{11} \in (\frac{n}{2}, n)$ *is the largest exceptional pole of* $\mathcal{C}_1(s)$ *as in Theorem 6.1.2.*

Proof. Let $\psi_c = c\psi$ with some positive constant c to be determined. Note that ψ_c satisfies all the conditions of ψ that we assume in the theorem. Moreover, for simplicity of notation, we denote by $\alpha = \frac{2s_{11}}{n}$ and note that $1 < \alpha < 2$.

We first note that if the set of volumes of sets in $\mathcal{B}$ is bounded, then the statement holds vacuously by taking $C_{\Gamma g}$ larger than the volume of any set in $\mathcal{B}$, so we can assume there are arbitrarily large volumes. With this assumption, by [Sch60, Lemma 1], we can assume without loss of generality (after perhaps adding more sets to $\mathcal{B}$) that $\{\mu_{Q\backslash G}(B) \mid B \in \mathcal{B}\} = \mathbb{R}^+$, where $\mathbb{R}^+$ denotes the set of positive real numbers. Thus for any positive integer L, there exists some $B_L \in \mathcal{B}$ with $\mu_{Q\backslash G}(B_L) = L$. For any $\Gamma g \in \Gamma\backslash G$, and $L \geq 1$ we denote

$$S_L(\Gamma g) = \#(B_L \cap \vec{e}_0\Gamma g) - c_{10}L$$

and for any $1 \leq L_1 < L_2$

$${}_{L_1}S_{L_2}(\Gamma g) = \#\left((B_{L_2}\backslash B_{L_1}) \cap \vec{e}_0\Gamma g\right) - c_{10}(L_2 - L_1).$$

For any integer $T \geq 3$ we denote by $\mathcal{W}_T$ the set of all pairs of integers L_1, L_2 of the form $0 \leq L_1 < L_2 \leq 2^T, L_1 = \ell 2^t$ and $L_2 = (\ell+1)2^t$, for integers ℓ and $t \geq 0$. For any such pair $(L_1, L_2) \in \mathcal{W}_T$, applying Theorem 6.1.2 to the sets $B_{L_2}\backslash B_{L_1}$ (noting that $B_{L_2}\backslash B_{L_1}$ can be expressed as a disjoint union of at most two generalized sectors and $\mu_{Q\backslash G}(B_{L_2}\backslash B_{L_1}) = L_2 - L_1 \geq 1$) we get

$$\int_{\Gamma\backslash G} \left|{}_{L_1}S_{L_2}(\Gamma g)\right|^2 d\mu(g) \leq C_2(L_2 - L_1)^\alpha,$$

where $\alpha = \frac{2s_{11}}{n}$ and C_2 is as in Theorem 6.1.2. As (L_1, L_2) runs over all the pairs in $\mathcal{W}_T$, for each $0 \leq t \leq T$, $L_2 - L_1 = 2^t$ occurs exactly 2^{T-t} times. Thus

$$\sum_{(L_1,L_2)\in\mathcal{W}_T} \int_{\Gamma\backslash G} |{}_{L_1}S_{L_2}(\Gamma g)|^2 d\mu_n(g) \leq C_2 \sum_{t=0}^{T} 2^{\alpha t}2^{T-t} \leq C_2' 2^{\alpha T}, \tag{6.1.5}$$

where $C_2' = \frac{C_2}{2^{\alpha-1}-1}$. Next, let $\mathcal{E}_T \subseteq \Gamma\backslash G$ denote the set of all elements $\Gamma g \in \Gamma\backslash G$ for which

$$\sum_{(L_1,L_2)\in\mathcal{W}_T} |{}_{L_1}S_{L_2}(\Gamma g)|^2 > \frac{2^{\alpha T}}{\psi_c(\log(2)(T-1))}. \tag{6.1.6}$$

Then (6.1.5) implies that

$$\mu(\mathcal{E}_T) < C_2'\psi_c(\log(2)(T-1)). \tag{6.1.7}$$

Consider the limit superior set

$$\mathcal{E}_\infty = \overline{\lim_{T\to\infty}}\,\mathcal{E}_T := \bigcap_{j\geq 3}\bigcup_{T\geq j}\mathcal{E}_T.$$

Since the right-hand side of (6.1.7) is summable we have that $\mu(\mathcal{E}_\infty) = 0$, and we will take its complement to be the full-measure set of lattice translates for which the remainder term is small.

Now, note that for $L < 2^T$, the interval $[0, L)$ can be expressed as a disjoint union of at most T intervals of the form $[L_1, L_2)$ with $(L_1, L_2) \in \mathcal{W}_T$. We can thus write

$$S_L(\Gamma g) = \sum_{[L_1,L_2)\in\mathcal{I}} {}_{L_1}S_{L_2}(\Gamma g),$$

where $\mathcal{I}$ is a set consisting of at most T intervals of the form $[L_1, L_2)$ with $(L_1, L_2) \in \mathcal{W}_T$. Using Cauchy-Schwartz and (6.1.6) we have for any $\Gamma g \notin \mathcal{E}_T$ and any $L < 2^T$

$$|S_L(\Gamma g)|^2 \leq \frac{T2^{\alpha T}}{\psi_c(\log 2(T-1))}.$$

Now, for any $\Gamma g \notin \mathcal{E}_\infty$ there is some T_g such that for all $T \geq T_g$ we have that $\Gamma g \notin \mathcal{E}_T$ and hence $|S_L(\Gamma g)|^2 \leq \frac{T2^{\alpha T}}{\psi_c((T-1)\log 2)}$ for all $L < 2^T$.

Now, for any $\Gamma g \notin \mathcal{E}_\infty$ let $C_{\Gamma g} = \max\{2^{T_g} + 1, L_0\}$ with L_0 sufficiently large that for all $L \geq L_0$ we have

$$\frac{6(L+1)^{\alpha/2}\log^{1/2}(L+1)}{\psi_c^{1/2}(\log(L+1))} + 1 \leq \frac{12L^{\alpha/2}\log^{1/2}(L)}{\psi_c^{1/2}(\log(L))}, \tag{6.1.8}$$

where we used the assumption that $L\psi(\log(L))$ is eventually non-decreasing to ensure that such L_0 exists. Then, for any integer $L > C_{\Gamma g} - 1$, choose integer T such that $2^{T-1} \leq L < 2^T$. In particular we have that $T \geq T_g$ and $L < 2^T$ so,

$$\begin{aligned}|S_L(\Gamma g)|^2 &\leq \frac{T2^{\alpha T}}{\psi_c(\log 2(T-1))} \\ &\leq \left(\frac{\log L}{\log 2} + 1\right) \frac{(2L)^\alpha}{\psi_c(\log(N))} < \frac{36L^\alpha \log L}{\psi_c(\log(L))},\end{aligned}$$

where we used that $\left(\frac{\log L}{\log 2} + 1\right) \leq 9\log(L)$ for all $L \geq 2$ and $2^\alpha < 4$ (since $\alpha < 2$). We have thus verified that for all $L > C_{\Gamma g} - 1$ we have $|S_L(\Gamma g)| \leq \frac{6L^{\alpha/2}\log^{1/2}(L)}{\psi_c^{1/2}(\log(L))}$.

Next, for any set $B \in \mathcal{B}$ with $\mathrm{vol}(B) > C_{\Gamma g}$, there exists an integer $L > C_{\Gamma g} - 1$ such that $B_L \subseteq B \subseteq B_{L+1}$. We can interpolate, to bound

$$\mathcal{E}(B, \Gamma g) \leq \max\left\{|S_L(\Gamma g)|, |S_{L+1}(\Gamma g)|\right\} + 1,$$

and since $L, L+1 \geq C_{\Gamma g} - 1$ we can bound

$$\mathcal{E}(B, \Gamma g) \leq \frac{6(L+1)^{\alpha/2}\log^{1/2}(L+1)}{\psi_c^{1/2}(\log(L+1))} + 1 \leq \frac{12\mu_{Q\backslash G}(B)^{\alpha/2}\log^{1/2}\left(\mu_{Q\backslash G}(B)\right)}{\psi_c^{1/2}\left(\log\left(\mu_{Q\backslash G}(B)\right)\right)},$$

where we used (6.1.8) recalling that $L \geq L_0$. Finally we finish the proof by taking $c = 144$ and recalling $\frac{\alpha}{2} = \frac{s_{11}}{n}$. □

6.2 Counting primitive lattice points

In this section we give the proof of Theorem 5. One key ingredient of our proof is an observation relating the cusps of $\mathrm{SO}_0(n+1,1)(\mathbb{Z})$ with the orbits of $\mathrm{SO}_0(n+1,1)(\mathbb{Z})$-action on $\mathcal{V}^+(\mathbb{Z})_{\mathrm{pr}} := \mathbb{Z}^{n+2}_{\mathrm{pr}} \cap \mathcal{V}^+$, the set of primitive integral points in $\mathcal{V}^+$. For that we first give a geometric description of cusps for a non-uniform lattice.

6.2.1 A geometric description of cusps

Let $\mathbb{P}(\mathcal{V}^+) := \mathcal{V}^+/\sim$ be the projectivization of $\mathcal{V}^+$ with the natural projection $\mathcal{V}^+ \to \mathbb{P}(\mathcal{V}^+)$, where the equivalence relation is defined that $\vec{y}_1 \sim \vec{y}_2$ if and only if there exists $\lambda > 0$ such that $\vec{y}_1 = \lambda \vec{y}_2$. It naturally parameterizes the space of rays (starting from the origin) in $\mathcal{V}^+$. For any $\vec{v} \in \mathcal{V}^+$ we denote by $[\vec{v}] \in \mathbb{P}(\mathcal{V}^+)$ to be the ray spanned by $\vec{v}$.

On the other hand, recall that the homogeneous space $P\backslash G$ parameterizes the space of parabolic subgroups of G by identifying $Pg \in P\backslash G$ with the parabolic subgroup $g^{-1}Pg$. Thus with the above identification between $Q\backslash G$ and $\mathcal{V}^+$ and the natural projection $Q\backslash G \to P\backslash G$, we can parametrize $P\backslash G$, the space of parabolic subgroups, by the projective variety $\mathbb{P}(\mathcal{V}^+)$. More precisely, if writing down all the above maps explicitly we see that for any $[\vec{v}] \in \mathbb{P}(\mathcal{V}^+)$, the corresponding parabolic subgroup $P_{[\vec{v}]}$ is given by $P_{[\vec{v}]} = g^{-1}Pg$ with $g \in G$ such that $\vec{v} = \vec{e}_0 g$ (this g exists since G acts on $\mathcal{V}^+$ transitively). We note that $P_{[\vec{v}]}$ is exactly the parabolic subgroup fixing $[\vec{v}]$ (with its unipotent radical fixing $\vec{v}$) (see Lemma 6.2.1).

The following lemma gives a geometric description of cusps for non-uniform lattices.

Lemma 6.2.1. *Keep the notation as above. Let $\Gamma \subset G$ be a non-uniform lattice of G. For any $\vec{v} \in \mathcal{V}^+$, then Γ has a cusp at $P_{[\vec{v}]}$ if any only if there exists some unipotent element $\gamma \in \Gamma$ fixing $\vec{v}$. Two parabolic subgroups $P_{[\vec{v}_1]}$ and $P_{[\vec{v}_2]}$ is Γ-equivalent if any only if there exists some $\gamma \in \Gamma$ such that $[\vec{v}_1] = [\vec{v}_2\gamma]$.*

Proof. For any $\vec{v} \in \mathcal{V}^+$, $P_{[\vec{v}]} = g^{-1}Pg$ with some $g \in G$ such that $\vec{v} = \vec{e}_0 g$. By definition, Γ has a cusp at $P_{[\vec{v}]}$ if and only if Γ intersects its unipotent radical, $g^{-1}Ng$, nontrivially. It thus suffices to show that $\Gamma \cap g^{-1}Ng$ is nontrivial if and only if there exists unipotent element in Γ fixing $\vec{v}$. If $\Gamma \cap g^{-1}Ng$ is nontrivial, then there exists $\gamma = g^{-1}ug \in \Gamma$ with $u \in N$ nontrivial. Clearly, γ is unipotent and recalling that N fixes $\vec{e}_0$ we have $\vec{v}\gamma = \vec{e}_0 gg^{-1}ug = \vec{e}_0 ug = \vec{e}_0 g = \vec{v}$. For the other direction, suppose there exists some unipotent $\gamma \in \Gamma$ fixing $\vec{v}$, then we have $\vec{e}_0 g\gamma = \vec{e}_0 g$. Or equivalently, $\vec{e}_0 g\gamma g^{-1} = \vec{e}_0$. Since the stabilizer of $\vec{e}_0$ is $Q = NM$ and the only unipotent elements in Q are elements in N, we have $g\gamma g^{-1} = u$ for some $u \in N$ nontrivial. This implies that $\Gamma \cap g^{-1}Ng$ containing γ is nontrivial.

For the second statement, choose $g_1, g_2 \in G$ such that $\vec{v}_1 = \vec{e}_0 g_1$ and $\vec{v}_2 = \vec{e}_0 g_2$. Then $P_{[\vec{v}_1]} = g_1^{-1}Pg_1$ and $P_{[\vec{v}_2]} = g_2^{-1}Pg_2$. Suppose $P_{[\vec{v}_1]}$ and $P_{[\vec{v}_2]}$ are Γ-equivalent, then by definition, there exists some $\gamma \in \Gamma$ such that $P_{[\vec{v}_1]} = \gamma^{-1}P_{[\vec{v}_2]}\gamma$. This implies that $g_2\gamma g_1^{-1}Pg_1\gamma^{-1}g_2^{-1} = P$. Since P is self-normalizing, we get $g_1\gamma^{-1}g_2^{-1} \in P$. Then using the fact that P fixes the ray $[\vec{e}_0]$ we have $[\vec{v}_1] = [\vec{e}_0 g_1] = [\vec{e}_0 g_1\gamma^{-1}g_2^{-1}g_2\gamma] = [\vec{e}_0 g_2\gamma] = [\vec{v}_2\gamma]$. For the other direction, suppose there exists $\gamma \in \Gamma$ such that $[\vec{v}_1] = [\vec{v}_2\gamma]$, then this implies that $g_1\gamma^{-1}g_2^{-1}$ fixes $[\vec{e}_0]$. Since P is the stabilizer of $[\vec{e}_0]$ and P is self-normalizing we have $g_1\gamma^{-1}g_2^{-1} \in P$ and $g_2\gamma g_1^{-1}Pg_1\gamma^{-1}g_2^{-1} = P$. This implies $P_{[\vec{v}_1]} = \gamma^{-1}P_{[\vec{v}_2]}\gamma$, that is, $P_{[\vec{v}_1]}$ and $P_{[\vec{v}_2]}$ are Γ-equivalent. □

In view of this lemma, we say that Γ has a cusp at $[\vec{v}]$ if there exists some unipotent element $\gamma \in \Gamma$ fixing $\vec{v}$. Moreover, we say that $[\vec{v}_1]$ and $[\vec{v}_2]$ are Γ-equivalent if there exists some $\gamma \in \Gamma$ such that $[\vec{v}_1] = [\vec{v}_2\gamma]$.

6.2.2 Cusps of $\mathrm{SO}_0(n+1,1)(\mathbb{Z})$ and its orbits on $\mathcal{V}^+(\mathbb{Z})_{\mathrm{pr}}$

Now we take $\Gamma = \mathrm{SO}_0(n+1,1)(\mathbb{Z})$ to be the lattice of integral points. It is clear that the right action of Γ on $\mathcal{V}^+$ preserves $\mathcal{V}^+(\mathbb{Z})_{\mathrm{pr}}$, the set of primitive integral points in $\mathcal{V}^+$. To count primitive lattice points on $\mathcal{V}^+$ we would like to study the Γ-action on $\mathcal{V}^+(\mathbb{Z})_{\mathrm{pr}}$. A priori we do not know much about this action. However, it turns out that this action is closely related to the cusps of Γ, and we will show, in particular, that the number of orbits of this action on $\mathcal{V}^+(\mathbb{Z})_{\mathrm{pr}}$ is exactly the number of cusps of Γ.

We first give an explicit description of cusps of Γ. Namely, we determine the set of parabolic subgroups whose unipotent radicals intersect Γ nontrivially. Or equivalently, we find the set of rays of $\mathcal{V}^+$ which are fixed by unipotent elements in Γ. Before stating our result, let us first give a definition.

Definition 6.2.1. We say that a ray $[\vec{v}] \in \mathbb{P}(\mathcal{V}^+)$ is *rational* if there exists some $\lambda > 0$ such that $\lambda\vec{v} \in \mathbb{Q}^{n+2}$. We denote by $\mathbb{P}(\mathcal{V}^+)(\mathbb{Q}) \subset \mathbb{P}(\mathcal{V}^+)$ to be the set of rational rays of $\mathcal{V}^+$.

We note that for any $[\vec{v}] \in \mathbb{P}(\mathcal{V}^+)(\mathbb{Q})$, there exists a unique $\lambda > 0$ such that $\lambda\vec{v} \in \mathcal{V}^+(\mathbb{Z})_{\mathrm{pr}}$. Conversely, every primitive integral point lies in a unique rational ray. Thus $\mathbb{P}(\mathcal{V}^+)(\mathbb{Q})$ is naturally in bijection with $\mathcal{V}^+(\mathbb{Z})_{\mathrm{pr}}$, and it induces a bijection between the Γ-orbits

$$\mathcal{V}^+(\mathbb{Z})_{\mathrm{pr}}/\Gamma \longleftrightarrow \mathbb{P}(\mathcal{V}^+)(\mathbb{Q})/\Gamma, \tag{6.2.2}$$

where Γ acts on both spaces from right via the right multiplication.

Not surprisingly, the cusps of Γ are exactly the Γ-equivalent classes of rational rays.

Proposition 6.2.2. *Let* $\Gamma = \mathrm{SO}_0(n+1,1)(\mathbb{Z})$ *be the lattice of integral points. Then* Γ *has at cusp at* $[\vec{v}]$ *if and only if* $[\vec{v}]$ *is rational.*

Proof. Keep the notation as before. Let

$$N^- := \{u_{\mathbf{x}}^t \mid u_{\mathbf{x}} \in N\}$$

be the transpose of N and let $\vec{v}_0 = (0, \dots, 0, 1, 1) \in \mathcal{V}^+$. We first note that the ray $[\vec{v}_0]$ is rational and it is clear that Γ has a cusp at $[\vec{v}_0]$ since $\vec{v}_0$ is fixed by the maximal unipotent subgroup N^- and N^- intersects Γ nontrivially. Next we note that for any $[\vec{v}] \neq [\vec{v}_0]$, there exists a unique $\mathbf{y} = (y_0, \ldots, y_{n-1}) \in \mathbb{R}^n$ such that $[\vec{v}] = [\vec{v}_{\mathbf{y}}]$ with

$$\vec{v}_{\mathbf{y}} = (2y_0, \ldots, 2y_{n-1}, \|\mathbf{y}\|^2-1, \|\mathbf{y}\|^2+1) \in \mathcal{V}^+.$$

Moreover, we note that $[\vec{v}_{\mathbf{y}}]$ is rational if and only if $\mathbf{y}$ is a rational vector, that is $\mathbf{y} \in \mathbb{Q}^n$. Thus we need to show that Γ has a cusp at $[\vec{v}_{\mathbf{y}}]$ if and only if $\mathbf{y}$ is a rational vector. When $\mathbf{y} = \mathbf{0}$, $[\vec{v}_{\mathbf{y}}] = [\vec{e}_0]$ is rational and Γ again clearly has a cusp at $[\vec{e}_0]$. We thus assume $\mathbf{y} \neq \mathbf{0}$.

By direct computation we see that $\vec{v}_{\mathbf{y}} = \vec{e}_0 u^t_{\mathbf{y}}$ with $u_{\mathbf{y}}$ as in (2.1.4). Thus as in the proof of Lemma 6.2.1, Γ has a cusp at $[\vec{v}_{\mathbf{y}}]$ if and only if Γ intersects $N_{\mathbf{y}} := u^t_{-\mathbf{y}} N u^t_{\mathbf{y}}$ nontrivially. Hence it suffices to show that for $\mathbf{y} \neq \mathbf{0}$, $\Gamma \cap N_{\mathbf{y}}$ is nontrivial if and only if $\mathbf{y}$ is a rational vector. By direct computation we see that a general element in $N_{\mathbf{y}}$ is of the form

$$u^t_{-\mathbf{y}} u_{\mathbf{x}} u^t_{\mathbf{y}} = \begin{pmatrix} S & \mathbf{u}_1^t & \mathbf{u}_2^t \\ \mathbf{w}_1 & 1 - \frac{\|\mathbf{x}\|^2(\|\mathbf{y}\|^2-1)^2}{2} & 2\mathbf{x}\mathbf{y}^t + \frac{\|\mathbf{x}\|^2 - \|\mathbf{x}\|^2\|\mathbf{y}\|^4}{2} \\ \mathbf{w}_2 & 2\mathbf{x}\mathbf{y}^t - \frac{\|\mathbf{x}\|^2 - \|\mathbf{x}\|^2\|\mathbf{y}\|^4}{2} & 1 + \frac{\|\mathbf{x}\|^2(\|\mathbf{y}\|^2+1)^2}{2} \end{pmatrix}, \tag{6.2.3}$$

where

$$S = I_n - 2\mathbf{y}^t\mathbf{x} + 2\mathbf{x}^t\mathbf{y} - 2\|\mathbf{x}\|^2\mathbf{y}^t\mathbf{y} \in M_n(\mathbb{R}),$$

$$\mathbf{u}_1^t = -2\mathbf{y}^t\mathbf{x}\mathbf{y}^t - \mathbf{x}^t + \|\mathbf{x}\|^2\mathbf{y}^t + \|\mathbf{y}\|^2\mathbf{x}^t - \|\mathbf{x}\|^2\|\mathbf{y}\|^2\mathbf{y}^t,$$

$$\mathbf{u}_2^t = -2\mathbf{y}^t\mathbf{x}\mathbf{y}^t + \mathbf{x}^t - \|\mathbf{x}\|^2\mathbf{y}^t + \|\mathbf{y}\|^2\mathbf{x}^t - \|\mathbf{x}\|^2\|\mathbf{y}\|^2\mathbf{y}^t,$$

$$\mathbf{w}_1 = (1 - \|\mathbf{y}\|^2)\mathbf{x} + (2\mathbf{y}\mathbf{x}^t + \|\mathbf{x}\|^2 - \|\mathbf{x}\|^2\|\mathbf{y}\|^2)\mathbf{y}$$

and

$$\mathbf{w}_2 = (1 + \|\mathbf{y}\|^2)\mathbf{x} + (-2\mathbf{y}\mathbf{x}^t + \|\mathbf{x}\|^2 + \|\mathbf{x}\|^2\|\mathbf{y}\|^2)\mathbf{y}.$$

For later use we note that

$$\frac{\mathbf{w}_1+\mathbf{w}_2}{2}=\mathbf{x}+\|\mathbf{x}\|^2\mathbf{y} \quad \text{and} \quad \frac{\mathbf{w}_1-\mathbf{w}_2}{2}=-\|\mathbf{y}\|^2(\mathbf{x}+\|\mathbf{x}\|^2\mathbf{y})+2\mathbf{y}\mathbf{x}^t\mathbf{y}. \tag{6.2.4}$$

Suppose $\Gamma\cap N_{\mathbf{y}}$ is nontrivial, we need to show that $\mathbf{y}$ is a rational vector. By definition, there exists some $\mathbf{x}=(x_0,\ldots,x_{n-1})\neq\mathbf{0}$ such that all the entries of $u^t_{-\mathbf{y}}u_{\mathbf{x}}u^t_{\mathbf{y}}$ are integers. In particular, we have

$$\frac{I_n-S}{2}=\|\mathbf{x}\|^2\mathbf{y}^t\mathbf{y}+\mathbf{y}^t\mathbf{x}-\mathbf{x}^t\mathbf{y}\in M_n(\mathbb{Q}).$$

Since $\|\mathbf{x}\|^2\mathbf{y}^t\mathbf{y}$ is symmetric and $\mathbf{y}^t\mathbf{x}-\mathbf{x}^t\mathbf{y}$ is anti-symmetric, we can conclude

$$\|\mathbf{x}\|^2\mathbf{y}^t\mathbf{y}=\Big(\|\mathbf{x}\|^2y_jy_l\Big)\in M_n(\mathbb{Q}) \quad \text{and} \quad \mathbf{y}^t\mathbf{x}-\mathbf{x}^t\mathbf{y}=\Big(y_jx_l-x_jy_l\Big)\in M_n(\mathbb{Q}).$$

Since $\mathbf{y}\neq\mathbf{0}$, there exists some $0\leq j\leq n-1$ such that $y_j\neq 0$. Then for any $l\neq j$,

$$\frac{y_l}{y_j}=\frac{\|\mathbf{x}\|^2y_jy_l}{\|\mathbf{x}\|^2y_j^2}\in\mathbb{Q}.$$

Thus we can find some $\lambda_1\neq 0$ such that $\mathbf{y}=\lambda_1\mathbf{r}$ for some nonzero $\mathbf{r}\in\mathbb{Q}^n$ (In fact, y_j would work). Thus to show $\mathbf{y}$ is a rational vector, it suffices to show that λ_1 is rational. Multiplying the first equation of (6.2.4) by $\mathbf{y}^t=\lambda_1\mathbf{r}^t$ from left we get

$$S_1:=\lambda_1\mathbf{r}^t(\frac{\mathbf{w}_1+\mathbf{w}_2}{2})=\mathbf{y}^t\mathbf{x}+\|\mathbf{x}\|^2\mathbf{y}^t\mathbf{y}\in\lambda_1M_n(\mathbb{Q}).$$

Combining the fact that $\mathbf{y}^t\mathbf{x}-\mathbf{x}^t\mathbf{y}\in M_n(\mathbb{Q})$ this implies

$$S_1-S_1^t=\mathbf{y}^t\mathbf{x}-\mathbf{x}^t\mathbf{y}\in\lambda_1M_n(\mathbb{Q})\cap M_n(\mathbb{Q}).$$

Thus λ_1 is rational unless $\mathbf{y}^t\mathbf{x}-\mathbf{x}^t\mathbf{y}$ is the zero matrix. If $\mathbf{y}^t\mathbf{x}-\mathbf{x}^t\mathbf{y}$ is the zero matrix, then $\mathbf{x}$ and $\mathbf{y}$ are parallel and there exists some $\lambda_2\neq 0$ such that $\mathbf{x}=\lambda_2\mathbf{r}$. Adding the $(n+1,n+2)^{\text{th}}$ and $(n+2,n+1)^{\text{th}}$ entry of $u^t_{-\mathbf{y}}u_{\mathbf{x}}u^t_{\mathbf{y}}$ in (6.2.3) we see that

$\mathbf{y}\mathbf{x}^t = \lambda_1\lambda_2\|\mathbf{r}\|^2$ is rational. Thus $\lambda_1\lambda_2 \in \mathbb{Q}$. Similarly, plugging $\mathbf{y} = \lambda_1\mathbf{r}$ and $\mathbf{x} = \lambda_2\mathbf{r}$ into the first equation of (6.2.4) we get

$$\frac{\mathbf{w}_1 + \mathbf{w}_2}{2} = \left(\lambda_2 + \lambda_1\lambda_2^2\|\mathbf{r}\|^2\right)\mathbf{r} \in \mathbb{Q}^n.$$

This implies that $\lambda_2 + \lambda_1\lambda_2^2\|\mathbf{r}\|^2 = \lambda_2(1 + \lambda_1\lambda_2\|\mathbf{r}\|^2) \in \mathbb{Q}$. But since $\lambda_1\lambda_2 \in \mathbb{Q}$ we have either $\lambda_2 \in \mathbb{Q}$ or $1 + \lambda_1\lambda_2\|\mathbf{r}\|^2 = 0$. If λ_2 is rational, then so is λ_1 since $\lambda_1\lambda_2$ is rational and both λ_1 and λ_2 are nonzero. If $1 + \lambda_1\lambda_2\|\mathbf{r}\|^2 = 0$, then $\mathbf{x} + \|\mathbf{x}\|^2\mathbf{y} = \mathbf{0}$. Now plugging $\mathbf{y} = \lambda_1\mathbf{r}$ and $\mathbf{x} = \lambda_2\mathbf{r}$ into the second equation of (6.2.4) we get

$$\frac{\mathbf{w}_1 - \mathbf{w}_2}{2} = 2\mathbf{y}\mathbf{x}^t\mathbf{y} = 2\lambda_1^2\lambda_2\|\mathbf{r}\|^2\mathbf{r} \in \mathbb{Q}^n.$$

This implies that $\lambda_1^2\lambda_2 \in \mathbb{Q}$. But again since $\lambda_1\lambda_2 \in \mathbb{Q}$ we have $\lambda_1 \in \mathbb{Q}$. To conclude, we have proved that for $\mathbf{y} \neq \mathbf{0}$, if $\Gamma \cap N_{\mathbf{y}}$ is nontrivial, then $\mathbf{y}$ is a rational vector.

For the other direction, we need to show that if $\mathbf{y} \neq \mathbf{0}$ is a rational vector, then there exists some $\mathbf{x} \in \mathbb{R}^n$ such that $u^t_{-\mathbf{y}}u_{\mathbf{x}}u^t_{\mathbf{y}} \in \Gamma$, or equivalently, all the entries of $u^t_{-\mathbf{y}}u_{\mathbf{x}}u^t_{\mathbf{y}}$ are integers. It is clear from the matrix expression (6.2.3) that if L is a positive integer such that $L\mathbf{y} \in \mathbb{Z}^n$, then $\mathbf{x} = (2L^2, \ldots, 2L^2)$ satisfies the prescribed property. □

The cusps of Γ are thus parameterized by the orbit space $\mathbb{P}(\mathcal{V}^+)(\mathbb{Q})/\Gamma$. In view of the bijection (6.2.2) we have the following immediate corollary.

Corollary 6.2.3. *Let h be the number of cusps of $\Gamma = \mathrm{SO}_0(n+1,1)(\mathbb{Z})$. Then*

$$h = \#(\mathcal{V}^+(\mathbb{Z})_{\mathrm{pr}}/\Gamma).$$

Remark 6.2.5. By elementary matrix operation one can show that for $n \leq 8$, the Γ-action on $\mathcal{V}^+(\mathbb{Z})_{\mathrm{pr}}$ is in fact transitive. Thus for $n \leq 8$, $\Gamma = \mathrm{SO}_0(n+1,1)(\mathbb{Z})$ has only one cusp.

6.2.3 Proof of Theorem 5

For any $\Lambda \in \Gamma\backslash G$ (that we view as a rank $n+2$ unimodular lattice) and any finite-volume Borel set $B \subset \mathcal{V}^+$, recall the counting function

$$\mathcal{N}_{\mathrm{pr}}(B,\Lambda) := \#(B \cap \Lambda_{\mathrm{pr}}).$$

As for Theorem 4 we first relate this counting function with certain incomplete Eisenstein series.

Lemma 6.2.4. *For any $\Lambda = \mathbb{Z}_{\mathrm{pr}}^{n+2} g \in \Gamma\backslash G$ and any finite-volume Borel set $B \subset \mathcal{V}^+$, there exists some positive constants $\lambda_1 = 1, \lambda_2, \ldots, \lambda_h$ such that*

$$\mathcal{N}_{pr}(B,\Lambda) = \sum_{j=1}^{h} \Theta_{f_j}^{j}(g)$$

with $f_j = \chi_{\lambda_j B}$ the indicator function of the dilation $\lambda_j B := \{\lambda_j \vec{v} \mid \vec{v} \in B\}$ of B.

Proof. In view of Corollary 6.2.3 we can take $\vec{v}_1 = \vec{e}_0, \vec{v}_2, \ldots, \vec{v}_h \in \mathcal{V}^+(\mathbb{Z})_{\mathrm{pr}}$ such that

$$\mathcal{V}^+(\mathbb{Z})_{\mathrm{pr}} = \bigsqcup_{j=1}^{h} \vec{v}_j \Gamma.$$

Since G acts on $\mathcal{V}^+$ transitively, there exist $g_1 = \mathrm{id}, g_2, \ldots, g_h \in G$ such that $\vec{v}_j = \vec{e}_0 g_j$. By the Iwasawa decomposition fixed in section 2.1.2 and the fact that $Q = NM$ fixes $\vec{e}_0$, we can take $g_j = a_{t_j}\xi_j^{-1}$ for some $a_{t_j} \in A$ and $\xi_j \in K$. Using similar arguments as in Lemma 6.1.1 we can see that for each $1 \le j \le h$, the map $\Gamma \to \vec{v}_j\Gamma$ sending γ to $\vec{v}_j\gamma$ induces a bijection between the coset $\Gamma_{P_j}\backslash\Gamma$ and the orbit $\vec{v}_j\Gamma$ identifying $\Gamma_{P_j}\gamma$ with $\vec{v}_j\gamma$, where $P_j := P_{[\vec{v}_j]} = g_j^{-1}Pg_j = \xi_j P \xi_j^{-1}$ is the parabolic subgroup fixing the ray $[\vec{v}_j]$ and $\Gamma_{P_j} = \Gamma \cap P_j$ as before. Take $f = \chi_B$ to be the indicator function of B and view it as a left Q-invariant function on G. Noting that for $\Lambda = \mathbb{Z}^{n+2}g$,

$\Lambda_{\mathrm{pr}} = \mathbb{Z}_{\mathrm{pr}}^{n+2} g$ and $\Lambda_{\mathrm{pr}} \cap \mathcal{V}^+ = (\mathbb{Z}_{\mathrm{pr}}^{n+2} \cap \mathcal{V}^+) g = \mathcal{V}^+(\mathbb{Z})_{\mathrm{pr}} g = \bigsqcup_{j=1}^h \vec{v}_j \Gamma g$, we thus have

$$\begin{aligned} \mathcal{N}_{\mathrm{pr}}(B,\Lambda) = \#(B \cap \mathcal{V}^+(\mathbb{Z})_{\mathrm{pr}} g) &= \sum_{j=1}^h \sum_{\vec{v} \in \vec{v}_j \Gamma} \chi_B(\vec{v} g) \\ &= \sum_{j=1}^h \sum_{\gamma \in \Gamma_{P_j} \backslash \Gamma} \chi_B(\vec{e}_0 g_j \gamma g) \\ &= \sum_{j=1}^h \sum_{\gamma \in \Gamma_{P_j} \backslash \Gamma} f(a_{t_j} \xi_j^{-1} \gamma g) = \sum_{j=1}^h \Theta_{f_j}^j(g), \end{aligned}$$

where $f_j(g) := f(a_{t_j} g)$ for any $g \in G$. Finally we note that since A normalizes Q, f_j is still left Q-invariant, and thus a function on $\mathcal{V}^+ (= Q \backslash G)$. In fact, f_j is the indicator function of the dilation $\lambda_j B$ with $\lambda_j = e^{t_j}$. □

In view of Lemma 6.2.4 and the mean value theorem for $\Theta_{f_j}^j$, we see that the expected term for $\mathcal{N}_{\mathrm{pr}}(B,\Lambda)$ is $\sum_{j=1}^h c_{j0} \mu_{Q\backslash G}(\lambda_j B)$. We thus define the remainder function

$$\mathcal{R}_{\mathrm{pr}}(B,\Lambda) = \left| \mathcal{N}_{\mathrm{pr}}(B,\Lambda) - \sum_{j=1}^h c_{j0} \mu_{Q\backslash G}(\lambda_j B) \right|$$

as in the introduction. We note that here both λ_j and $c_{j0} = \frac{\omega_j}{\nu_\Gamma}$ depend on the choice of orbit representatives $\vec{v}_j$, but the product $c_{j0}\mu_{Q\backslash G}(\lambda_j B)$ does not. We can now give the

Proof of Theorem 5. For any $\Lambda = \mathbb{Z}^{n+2} g \in \Gamma \backslash G$ and any finite-volume Borel set B by Lemma 6.2.4 we have

$$\mathcal{R}_{\mathrm{pr}}(B,\Lambda) = \left| \mathcal{N}_{\mathrm{pr}}(B,\Lambda) - \sum_{j=1}^h c_{j0} \mu_{Q\backslash G}(\lambda_j B) \right| \leq \sum_{j=1}^h \left| \Theta_{f_j}^j(g) - c_{j0} \mu_{Q\backslash G}(\lambda_j B) \right|.$$

For a given linearly ordered family of generalized sectors $\mathcal{B}$, repeating the arguments in section 6.1.1 and 6.1.2 (with Θ_f^1 replaced by $\Theta_{f_j}^j$ and $\vec{e}_0$ replaced by $\vec{v}_j$ and some other slight modifications) we see that for μ-a.e. $\Lambda \in \Gamma \backslash G$ there exists $C_{j,\Lambda}$ such that for any $\mu_{Q\backslash G}(B) \geq C_{j,\Lambda}$ we have

$$\left|\Theta^j_{f_j}(g) - c_{j0}\mu_{Q\backslash G}(\lambda_j B)\right| \leq \frac{1}{h}\mu_{Q\backslash G}(B)^{\frac{s_{j1}}{n}} \log^{3/2}\left(\mu_{Q\backslash G}(B)\right),$$

where s_{j1} is the largest exceptional pole of Γ at P_j. Taking $C_\Lambda = \max\{C_{j,\Lambda} \mid 1 \leq j \leq h\}$ and noting that the intersection of finitely many full-measure set is still of full measure, we can conclude that for μ-a.e. $\Lambda \in \Gamma\backslash G$ whenever $\mu_{Q\backslash G}(B) \geq C_\Lambda$ we have

$$\begin{aligned}\mathcal{R}_{\text{pr}}(B, \Lambda) &\leq \sum_{j=1}^{h} \left|\Theta^j_{f_j}(g) - c_{j0}\mu_{Q\backslash G}(\lambda_j B)\right| \\ &\leq \sum_{j=1}^{h} \frac{1}{h}\mu_{Q\backslash G}(B)^{\frac{s_{j1}}{n}} \log^{3/2}\left(\mu_{Q\backslash G}(B)\right) \leq \mu_{Q\backslash G}(B)^{\frac{s_\Gamma}{n}} \log^{3/2}\left(\mu_{Q\backslash G}(B)\right),\end{aligned}$$

where $s_\Gamma := \max\{s_{j1} \mid 1 \leq j \leq h\}$ is the largest exceptional pole of Γ at all cusps. □

Remark 6.2.6. We end this thesis by noting that one can get similar estimates for the regular lattice point counting problem counting all lattice points in a Borel set. Using the fact that all the lattice point in a unimodular lattice can be written uniquely as a positive multiple of a primitive lattice point, one can get a similar mean square bound as in Theorem (6.1.2) for the corresponding remainder function. Then one can apply Schmidt's arguments to get similar estimates. See [KY18, Theorem 2] for more details about this translation.

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