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A Lot of Ambiguity*

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Abstract

We consider a risk averse decision maker who dislikes ambiguity as in the Ellsberg urns and compare the certainty equivalent of this gamble with the certainty equivalent of the anchoring probabilistic lottery. We deal first with the Choquet EU model and show that under some conditions on the capacity ν, when independent ambiguous gambles are repeated and the expected value of the anchoring lottery is zero, the difference between the average ambiguous and risky certainty equivalents converges to zero. When the parallel expected value is positive, we show that if the average certainty equivalent of the risky lottery is non-negative, then so is the limit of the average value for the ambiguous model. These results do not extend to the maxmin model or to the smooth recursive model.

Keywords: Ellsberg urns, repeated ambiguity, repeated risk, Choquet expected utility, maxmin

1 Introduction

A patient sees his doctor and it is clear to both of them that a treatment may improve his health. The doctor offers him two possible treatments. A

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standard, well investigated one, which with probability p leads to a good outcome and with probability $1-p$ leads to an outcome that is worse than the no-treatment outcome. Alternatively, she offers him a new treatment with somewhat ambiguous probabilities of success. It is however known that whatever the outcome, it improves over that of the standard treatment. Moreover, although the probabilities are not known for sure, they are believed to be somewhere around $p : 1-p$. The patient is ambiguity averse, and as the improvement in the outcomes of the new treatment is not much, he prefers the old treatment with the known probability of success. In particular, he may prefer the standard treatment to no treatment, which in turn he prefers to the ambiguous one.

The doctor does not have any information she did not share with the patient. Moreover, although she knows that she will see many patients like him, she believes that she won't gain any information about the probability of success of the new treatment, as this probability depends entirely on unobservable charactristics of the patients. Her preferences over risk and uncertain prospects are the same as the patient's (alternatively, she adopts the patient's preferences). Does it follow that she too will prefer the old treatment to the new one?

Although they have exactly the same information and preferences, there is one dimension in which the patient and the doctor are different, and this is the number of cases they face. The patient sees only one case, his. Ambiguity aversion can be explained as fear of the unknown. Many people believe that they are unlucky and therefore, if they choose the ambiguous prospect, they'll find out that the winning probabilities took a bad turn and are on the lower side of the expectations. But can people really believe that they are always unlucky? The doctor is ambiguity averse, but as she is facing many similar cases, her aversion to each case is probably diminishing. Our aim in this paper is to formalize this intuition. Our main results show that under some assumptions, within the Choquet expected utility model (Schmeidler [24]), the following results hold. If the expected value of the gamble with the known probabilities is zero, then the average certainty equivalents of the repeated ambiguous and the repeated probabilistic gambles converge to each other (Theorem 1). When the average certainty equivalent of the probabilistic gamble is non-negative (by risk aversion, so is its expected value), then in the limit, the average certainty equivalent of the ambiguous lottery becomes non-negative (Theorem 2).

These results lead to immediate policy-making questions. Suppose that the risky and ambiguous treatment have zero expected value. Suppose further that the ambiguous treatment is less expensive than the probabilistic one, but the difference is less than the difference between the two certainty equivalents of the patient. Should society encourage, maybe even force, the use of the ambiguous treatment? Patients may be willing to pay the extra price for the unambiguous treatment, but if we adopt the point of view of care takers (who don't have any better information) we may opt out for the ambiguous treatment. Answers to such questions are beyond the scope of the current paper, but our aim here is to show that they are meaningful and real.

Section 2 presents the structure of our gambles and the Choquet model. The main results are given in section 3. Our results do not extend to some other models, for example, the maxmin expected utility model (Gilboa and Schmeidler [12]) or to the smooth recursive utility model (Klibanoff, Marinacci, and Mukerji [15]). We show this in section 4. We discuss some further issues and the literature in section 5. All claims are proved in the appendix.

2 Setup

An urn contains C balls of c colors. One ball is picked at random, and state of nature s_i is that color i is picked . Denote $S = \{s_1, \ldots, s_c\}$, and define $\Sigma = 2^S$. The number of balls of some colors may be known to be C/c, making the corresponding states of nature probabilistic with probability $\frac{1}{c}$.[1] This ratio also serves as an anchor for non prbabilistic states and events. For example, in the 3-color Ellsberg [3] urn which contains 90 balls, of which 30 are red and each of the other 60 is either black or yellow, the anchoring probabilities are $\frac{1}{3}$ for each of the three colors and $\frac{2}{3}$ for each of complementing events.[2] For more on the anchoring probabilities, see Fox and Tversky [10], Nau [19], Chew and Sagi [2] and Ergin and Gul [6].

Assume now the existence of a sequence of such urns. Let $S_i = S$ be the set of states in urn i with the corresponding algebra $\Sigma_i = \Sigma$. The information regarding each of these urns is the same, but the urns are independent in

[1]There are always at least two probabilistic events, $\varnothing$ and S.

[2]More complicated urns are also possible, for example, an urn containing 100 balls. Twenty of which are yellow, and each of the others is either red or green. The anchoring probabilities for (Y,R,G) are $(\frac{1}{5}, \frac{2}{5}, \frac{2}{5})$, but this situation can easily be described as an urn containing balls of five colors.

the sense that the outcome or even the mere existence of any urn doesn't change the decision maker's information regarding any other urn. Finally, let $S^n = S_1 \times \ldots \times S_n$ and $\Omega^n = 2^{S^n}$.

Consider a non-degenerate act $L = (x_1, E_1; \ldots; x_m, E_m)$ where $x_1 \leqslant \ldots \leqslant x_m$, $x_1 < x_m$, and $E_1, \ldots, E_m$ is a partitiion of Σ, with the anchor lottery $X = (x_1, p_1; \ldots; x_m, p_m)$. Denote the expected value of X by μ. The gamble L^n is the sequence of gamble L played once on each of the n urns. We assume that the decision maker is interested in the total outcomes he wins but not in the order or the composition of colors leading to these wins and will therefore view L^n as $(x_1^n, E_1^n; \ldots; x_{k_n}^n, E_{k_n}^n)$, where $x_1^n \leqslant \ldots \leqslant x_{k_n}^n$ and E_i^n is the collection of sequences of events from $\Sigma^1, \ldots, \Sigma^n$ such that the sum of their corresponding outcomes is x_i^n. The lottery X^n is a sequence of n independent lotteries of type X and it serves as a natural anchor for L^n. According to CEU theory (Schmeidler [24]), there is a utility function $u^n : \Re \to \Re$ and a capacity $\nu^n : \Omega^n \to [0,1]$ such that $\nu^n(\varnothing) = 0$, $\nu^n(S^n) = 1$, and the value of L^n is

$$\mathrm{CEU}^n(L^n) = u^n(x_{k_n}^n)\nu^n(E_{k_n}^n) + \sum_{i=1}^{k_n-1} u^n(x_i^n)\left[\nu^n\left(\bigcup_{j=i}^{k_n} E_j^n\right) - \nu^n\left(\bigcup_{j=i+1}^{k_n} E_j^n\right)\right]$$

We assume that all the utility functions $u^1, \ldots, u^n$ are the same and denote them u. Also, we assume that the decision maker is risk averse (hence his vNM utility u is concave) and ambiguity averse in the sense that he prefers to play X^n to playing L^n. To ensure ambiguity aversion we assume that $\nu^n(E) \leqslant \Pr(E)$ for all $E \in \Omega^n$ (which is equivalent to $p \in \mathrm{Core}(\nu^n)$. Note that we do not require the capacities ν^n to be convex). For exact definitions and analysis of these concepts, see Ghirardato and Marinacci [11] and Chateauneuf and Tallon [1].

Given the anchor lottery $X^n = (x_1^n, p_1^n; \ldots; x_{k_n}^n; p_{k_n}^n)$ we define $f^n : [0,1] \to [0,1]$ such that $f^n(0) = 0$, for $i = 1, \ldots, k_n$,

$$f^n\left(\sum_{j=i}^{k_n} p_j^n\right) = \nu^n\left(\bigcup_{j=i}^{k_n} E_j^n\right) \tag{1}$$

and let f^n be piecewise linear on the segment $[0, p_{k_n}^n]$ and on the segments $[\sum_{j=i+1}^{k_n} p_j^n, \sum_{j=i}^{k_n} p_j^n]$, $i = 1, \ldots, k_n - 1$. Note that we have $f(p) \leqslant p$.

There is no reason to assume that the decision maker considers the act "win \$100 if a red ball is sampled from a single urn containing red and black balls" the same as "win \$100 if two or three red balls are sampled from three such urns," since even though the anchor probability of both events equals $\frac{1}{2}$, ν^1 of the former event may be different from ν^3 of the latter. We cannot therefore assume that f^{n+m} agrees with f^n on the points $\sum_{j=i}^{k_n} p_j^n$, $i = 1, \ldots, k_n - 1$, even when for some $1 \leqslant i < n$ and $1 \leqslant i' < k_{n+m}$,

$$\sum_{j=i}^{k_n} p_j^n = \sum_{j=i'}^{k_{n+m}} p_j^{n+m}$$

Instead, we assume that $f := \lim_{n\to\infty} f^n$ exists (the convergence is in the supremum norm topology), and that $f, f^1, \ldots$ are all Lipschitz with (the same) constant K.

We assume that the decision maker evaluates the lottery X^n using expected utility theory with the vNM function u. Denote by c^n the certainty equivalent of X^n, that is, the number satisfiying $u(c^n) = \mathrm{E}[u(X^n)]$. One can view $\mathrm{E}[X^n] - c^n = n\mu - c^n$ as the risk premium the decision maker is willing to pay for trading the lottery X^n for its expected value.

For $z_1 \leqslant \ldots \leqslant z_\ell$, the rank dependent (RD, Quiggin [21]) value of the lottery $Z = (z_1, p_1; \ldots; z_\ell, p_\ell)$ is given by

$$\mathrm{RD}(Z) = u(z_\ell)\varphi(p_\ell) + \sum_{i=1}^{\ell-1} u(z_i)\left[\varphi\left(\sum_{j=i}^{\ell} p_j\right) - \varphi\left(\sum_{j=i+1}^{\ell} p_j\right)\right]$$

The value CEU^n of L^n can thus be viewed as the RD value of the lottery X^n using the utility function u and the transformation function f^n. By ambiguity aversion, the certainty equivalent of L^n, denoted d^n, is even less than c^n. Formally, d^n is given by $u(d^n) = \mathrm{RD}^n(X^n)$. We view $c^n - d^n$ as the ambiguity premium the decision maker is willing to pay for trading the ambiguous act L^n for the risky lottery X^n.

In a famous article, Samuelson [22] showed that the risk premium-per-gamble, $\mu - \frac{c^n}{n}$, does not necessarily go down to zero as n increases to infinity. Since the ambiguity premium is positive, it is obvious that the total premium paid to trade an ambiguous lottery for its expected value too does not have to go to zero as n increases. But we are interested in a different question: What happens to the ambiguity premium-per-urn the decision maker is willing to pay to avoid the ambiguous act L^n, beyond what he is willing to pay to avoid the non-ambiguous, probabilistic lottery X^n? Formally, we show what can be said about the connection between $\lim_{n\to\infty} \frac{d^n}{n}$ and $\lim_{n\to\infty} \frac{c^n}{n}$.

3 Choquet Expected Utility

Following the discussion of the last section, consider a given non-degenerate random variable L with the anchoring lottery X, and suppose that the decision maker is using the CEU model for ambiguous random variables. Our analysis yields different results when the expected value of X is zero and when it is positive. We assume throughout that the vNM utility function u (used to assess monetary payoffs) is normalized such that $u(0) = 0$ and $u'(0) = 1$.

Consider first the case $\mathrm{E}[X] = 0$. A risk averse decision maker will reject it. And if he dislikes ambiguity and ambiguity is added to the risk, then such a decision maker will certainly reject an ambigous random variable L with the anchoring lottery X. The next theorem shows that when the expected value of X is zero, the average risk premium and the average ambiguity premium converge to the same limit (which may be strictly negative or zero).

Theorem 1 Let u be concave and let f be Lipschitz with $K \geqslant 1$ and such that for all p, $f(p) \leqslant p$. Then for random variable L with anchoring lottery X such that $\mathrm{E}(X) = 0$, $\lim\limits_{n\to\infty} \left[\frac{c^n}{n} - \frac{d^n}{n}\right] = 0$.

Consider now a different case, where $\mathrm{E}[X] > 0$. This of course doesn't mean that the decision maker would like to play X, or even that if he would like to play it once he would like to play it n times. And it may certainly happen that he would like to play X, but will decline the corresponding random variable L. For example, the decision maker may accept the lottery $(-100, \frac{1}{2}; 110, \frac{1}{2})$, yet decline the gamble where in the two-color Ellsberg urn he wins 110 if he correctly guesses the color of the drawn ball, but loses 100 if he does not. Nevertheless, an implication of the next theorem is that $\lim\limits_{n\to\infty} \frac{d^n}{n}$ and $\lim\limits_{n\to\infty} \frac{c^n}{n}$ cannot be on strictly opposite sides of zero.

Theorem 2 Let u be concave and f Lipschitz with $K \geqslant 1$, and let L be a random variable with anchoring lottery X such that $\mathrm{E}[X] > 0$. If for all n $c^n \geqslant 0$, then $\lim\limits_{n\to\infty} \frac{d^n}{n} \geqslant 0$.

The theorem does not require that for all p, $f(p) \leqslant p$. If we add this requirement than by ambiguity aversion $d^n \leqslant c^n$ for all n, hence $\lim\limits_{n\to\infty} \frac{d^n}{n} \leqslant \lim\limits_{n\to\infty} \frac{c^n}{n}$. Theorem 2 thus implies the following result:

Conclusion 1 Let u be concave and f Lipschitz with $K \geqslant 1$ such that for all p, $f(p) \leqslant p$. Then $\lim_{n\to\infty} \frac{d^n}{n} \times \lim_{n\to\infty} \frac{c^n}{n} \geqslant 0$

Stricker results can be obtained if further assumptions are made regarding the boundedness of the utility function u. Assuming that u is bounded from above is used to avoid phenomena in the spirit of the famous St. Petersburg paradox. As was shown by Fishburn [9, Section 14.1], Savage's [23] axioms imply that the utility function has to be bounded, both from above and from below. The next theorem covers several such restrictions. The first part shows that if u is bounded from above, then not only is the average certainty equivalent of L^n asymptotically non-negative, but from a certain point on the certainty equivalent itself is positive. The second and third parts extend the main result of Theorem 1, first to the case where $\mathrm{E}[X] = 0$ and u is bounded from above and from below, and then to general lotteries X where u is exponential.

Theorem 3 Let f be Lipshcitz with $K \geqslant 1$. If u is bounded from above and $\lim_{n\to\infty} \frac{c^n}{n} > 0$, then there exists n_δ such that $\forall n > n_\delta$, $L^n \succ 0$. If $\mathrm{E}[X] = 0$ and u is also bounded from below, then $\lim_{n\to\infty} \frac{c^n}{n} = \lim_{n\to\infty} \frac{d^n}{n} = 0$. Finally, if u is exponential and concave and for all p, $f(p) \leqslant p$, then $\lim_{n\to\infty} \frac{c^n}{n} = \lim_{n\to\infty} \frac{d^n}{n}$.[3]

The requirement that f is Lipschitz is sufficient for theorem 1, but not necessary. The function $f(p) = p - (1-p)\ln(1-p)$ is not Lipschitz since its derivative goes to infinity when p approaches 1. Yet numerical analysis shows that when $u(x) = x$, for the anchor lottery $X = (-1, \frac{1}{2}; 1, \frac{1}{2})$, $\lim_{n\to\infty} \left[\frac{c^n}{n} - \frac{d^n}{n}\right] = 0$. But the theorems do not hold for general non-Lipschitz functions f. For Theorem 1, consider for example a CEU decision maker with the utility function $u(x) = 1 - e^{-x}$ and the non-Lipschitz transformation function $f(p) = 1 - \sqrt{1-p}$, who is facing the risky lottery $X = (-0.5, \frac{1}{2}; 0.5, \frac{1}{2})$ and the corresponding ambiguous lottery $L = (-0.5, E_1; 0.5, E_2)$. Calculating c^n and d^n yields $\lim_{n\to\infty} \frac{c^n}{n} = -0.1201$ while, $\lim_{n\to\infty} \frac{d^n}{n} < -0.21$. The same functions applied to the lottery $X = (-0.35, \frac{1}{2}; 0.65, \frac{1}{2})$ yield $\lim_{n\to\infty} \frac{d^n}{n} =$

[3] A sufficient condition for boundedness from above is that the Arrow-Pratt measure of absolute risk aversion is bounded away from 0. That is, that there exists $\delta > 0$ such that for all z, $r_u(z) = -u''(z)/u'(z) > \delta$. To see it, let $v(z) = -e^{-\delta z}$. Then $r_u(z) > r_v(z)$ and, by Pratt [20], there exists a concave h such that $u = h \circ v$. The boundedness of u follows from that of v.

$-0.068 < 0$ while for all n, $\frac{c^n}{n} = 0.030 > 0$ show that Theorem 2 does not hold for all non-Lipschitz functions.

The boundedness of u is required for the first part of Theorem 3 as without it it is possible to have $\lim\limits_{n\to\infty} \frac{c^n}{n} > 0$ while for every n^* there is $n \geqslant n^*$ such that $d^n = 0$. See Example 1 in the Appendix.

4 Maxmin EU and Smooth Recursive Utility

In this section we show that the conclusions of Theorems 1 and 2 do not necessarily hold in other models of ambiguity. It will also offer an explanation for our assumption that the functions $f, f^1, \ldots$ are Lipschitz.

Gilboa and Schmeidler [12] suggested the folowing maxmin expected utility (MEU) theory. Under ambiguity, the decision maker behaves as if he has a (convex) set of possible probability distributions as well as a utility function u. For each gamble he computes the values of the expected utility of u with respect to the different possible probability distributions, and evaluates the gamble as the minimum of all these values.

Consider the following example, where each of n urns contains the same number of balls and each of them contains two colours, red and green. The ambiguous act L is ($-\frac{1}{2}$ if green; $\frac{1}{2}$ if red), with the corresponding probabilistic lottery $X = (-\frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2})$. Let $s < \frac{1}{2}$, and assume that there are two possible priors for the proportion of green and red balls: $(1-s, s)$ (the bad urn) and $(s, 1-s)$ (the good urn). The compositions of the n urns are statistically independent. Following the notation of Section 2, the decision maker wins $i - \frac{n}{2} - 1$ if event E_i^n happens, which is the event "$i-1$ red balls and $n-i+1$ green balls were drawn from the n urns," $i = 1, \ldots, n+1$.

A profile of the n urns is an ordered list of the 'good' and 'bad' urns. There are 2^n such profiles, and each of them will induce a probability distribution over $\mathcal{E}^n = \{E_1^n, \ldots, E_{n+1}^n\}$. Observe that two profiles with the same number of 'good' urns induce the same probability distribution. In other words, the decision maker has $n+1$ possible priors over $\mathcal{E}^n$, one for each possible number of 'good' urns, which is an integer between zero and n. Denote this set of priors $\mathcal{Q}^n$.[4]

[4]The results of this section hold even when the basic set of priors $\mathcal{Q}$ is the entire interval $[(1-s, s), (s, 1-s)]$, and not just its end points.

Unlike the results of Theorems 1 and 2 with respect to the CEU model, in the MEU model the average ambiguity premium $\frac{d^n}{n}$ does not necessarily converge to the average risk premium $\frac{c^n}{n}$ when $\mathrm{E}[X] = 0$, nor does it become non-negative when $\frac{c^n}{n} \geqslant 0$. First, consider Theorem 1. If the decision maker is risk neutral (that is, $u(x) = x$), then $c^n \equiv 0 \equiv \frac{c^n}{n}$. But as the worst possible prior results from a profile in which all n urns are 'bad' (that is, in each one of them the proportion of red balls is $s < \frac{1}{2}$), the MEU value of playing the n urns is $n[\frac{1}{2}s - \frac{1}{2}(1-s)] = n[s - \frac{1}{2}]$, and since u is linear this is also the value of d^n. It follows that $\lim\limits_{n\to\infty} \left[\frac{c^n}{n} - \frac{d^n}{n}\right] = \frac{1}{2} - s > 0$. To see that Theorem 2 does not hold, consider the basic risky lottery $X = (-\frac{1}{2}+a, \frac{1}{2};\ \frac{1}{2}+a, \frac{1}{2})$, with the corresponding ambiguous lottery, and assume that $0 < a < \frac{1}{2} - s$. Then $\frac{c^n}{n} = a > 0$ while $\frac{d^n}{n} = s - \frac{1}{2} + a < 0$.

There is a connection between the MEU and CEU models. A capacity ν is convex if for all E, E', $\nu(E) + \nu(E') \leqslant \nu(E \cup E') + \nu(E \cap E')$. For a convex capacity ν, the core $\mathcal{C}_\nu$ of ν is the set of all distributions such that for every E, $\Pr(E) \geqslant \nu(E)$. The CEU preferences with the capacity ν are the same as the MEU preferences when the set of possible priors is the core of ν (see Schmeidler [24]). Moreover, for every E, $\nu(E) = \min_{p\in\mathcal{C}_\nu}\{p(E)\}$.

Define a capacity ν^n by $\nu^n(E_i^n) = \min_{q\in\mathcal{Q}^n}\{q(E_i^n)\}$. Since MEU and CEU are equivalent when the capacity is convex, it follows by the above inequality that if ν is convex, then one of the assumptions of Theorems 1 and 2 must be violated. We show that the assumption that all the functions f^n are Lipschitz with the same constant K cannot be satisfied. Following the analysis of the previous section, observe that

$$\lim_{n\to\infty} \frac{f^n(1) - f^n(1 - 0.5^n)}{1 - (1 - 0.5^n)} = \lim_{n\to\infty} \frac{1 - \nu(\neg E_0^n)}{0.5^n} = \frac{1 - [1 - (1-s)^n]}{0.5^n} = \infty$$

The event $\neg E_0^n$ is "having at least one red ball." The probability of this event is minimized when all urns are 'bad' (that is, in all of them there are more green than red balls). In this case, the probability of E_0^n is $(1-s)^n$, and the probability of $\neg E_0^n$ is $1 - (1-s)^n$. The limit is ∞ since $s < 0.5$.

Remark Checking for convexity of ν is not trivial. But consider an extreme case — each urn contains one ball, either red or green. In this case $s = 0$ and if the decision maker is willing to consider all possible scenarios then the derived capacity from the multiple priors is given by $\nu(E) = 0$ for all $E \neq S$ and $\nu(S) = 1$ (S is the sure event). This capacity is trivially convex.

This analysis implies that the results of the previous section do not hold under this set of multiple priors. But are these beliefs reasonable? A decision maker is uncertain about the composition of a single urn and therefore, being pessimistic, is considering unfavorable compositions. This may be true when he is facing two or three urns. But when he is facing many urns, is it reasonable for him to believe that *all* of them are 'bad?' Such beliefs require an extreme degree of pessimism, and seem less and less reasonable when more urns are involved. This is indeed the attitude revealed by a decision maker who believes that all urns are independent and it is indeed not surprising that his ambiguity premium will not disappear.

It is more reasonable to assume that at the presence of many urns, the cautious decision maker may fear that he is unlucky, but not to the extreme level of facing a sequence of urns that are all 'bad.' Suppose for example that he believes that at least $t < \frac{1}{2}$ proportion of the urns are 'good.' It turns out that this will not solve the problem. We show that even if these beliefs lead to CEU preferences, the derived functions $f, f^1, \ldots$ are not Lipschitz with the same constant K, and $\lim\limits_{n\to\infty}[\frac{c^n}{n} - \frac{d^n}{n}]$ may be strictly negative.

Consider the events E_0^n and $\neg E_0^n$. The corresponding objective lottery is where exactly half of the urns are 'good' and half are 'bad.' In this case, the probability of E_0^n, that is, of drawing no red balls, is

$$\lim_{n\to\infty} \frac{f^n(1) - f^n(1 - [s(1-s)]^{n/2})}{1 - (1 - [s(1-s)]^{n/2})} = \lim_{n\to\infty} \frac{1 - \nu(\neg E_0^n)}{[s(1-s)]^{n/2}} =$$

$$\lim_{n\to\infty} \frac{s^{tn}(1-s)^{(1-t)n}}{[s(1-s)]^{n/2}} = \lim_{n\to\infty} \left(\frac{1-s}{s}\right)^{(\frac{1}{2}-t)n}$$

Since $s < \frac{1}{2}$, this last expression is unbounded. To calculate $\nu(\neg E_0^n)$, note that the worst possible scenario for the decision maker is that only t proportion of the urns are 'good.' The probability of winning zero is therefore the probability that green balls will be drawn from all the tn 'good' urns and from all the $(1-t)n$ 'bad' urns. The probability of this event is $s^{tn}(1-s)^{(1-t)n}$.

Next we show that $\lim\limits_{n\to\infty} \frac{c^n}{n} = 0$ while $\lim\limits_{n\to\infty} \frac{d^n}{n} < 0$, thus proving that the results of Theorems 1 and 2 do not hold. It is indeed easy to verify that when u is linear, $c^n \equiv \frac{c^n}{n} \equiv 0$. The ambiguity premium is computed with respect to the worst possible scenario, when only t proportion of the urns are 'good.' Using the Binomial distribution we obtain that the expected value of the ambiguous lottery is $d^n = n[t(\frac{1}{2} - s) + (1-t)(s - \frac{1}{2})] = n(2t-1)(\frac{1}{2} - s)$, hence $\lim_{n\to\infty} \frac{d^n}{n} = (2t-1)(\frac{1}{2}-s)$. Since $t < \frac{1}{2}$, this limit is negative.

Klibanoff, Marinacci, and Mukerji [15] Suggested the following smooth case of the recursive model. According to this model, the decision maker has a set of possible probability distributions, and he attach a probability to each of them. He computes the certainty equivalent of the uncertain act using expected utility with the vNM function u for each of the possible distributions, and then evaluates the lottery over these values using the vNM function ϕ. We show that in this model one can have $\lim_{n\to\infty} \frac{d^n}{n} < \lim_{n\to\infty} \frac{c^n}{n} = 0$.

Let $u(x) = x$ and $\phi(x) = -e^{-x}$. As before, there are n urns, each containing 100 balls of two colors, red and green. For each of these urns the decision maker believes that with probability $\frac{1}{2}$ the composition is 75:25 and with probability $\frac{1}{2}$ it is 25:75. He believes that the compositions of the urns are statistically independent. One ball is to be drawn at random from each urn, and the decision maker's payoff is the number or red balls minus the number of green balls. The anchoring probabilities are $\frac{1}{2} : \frac{1}{2}$ for each of the boxes, and as his vNM utility is linear, $c^n = \lim_{n\to\infty} \frac{c^n}{n} = 0$ for all n.

With probability $\binom{n}{i}2^{-n}$, the composition of i urns is 75:25 and that of $n-i$ urns is 25:75. His expected win in such a case is $(0.75-0.25)i + (0.25-0.75)(n-i) = i - \frac{n}{2}$. We thus get

$$\begin{aligned}\phi(d^n) &= \sum_{i=0}^{n} \binom{n}{i} \frac{1}{2^n} \phi\left(i - \frac{n}{2}\right) \Longrightarrow e^{-d^n} = \sum_{i=0}^{n} \binom{n}{i} \frac{1}{2^n} e^{\frac{n}{2}-i} \\ \Longrightarrow & \quad \frac{d^n}{n} = -\frac{1}{n} \ln\left(\sum_{i=0}^{n} \binom{n}{i} \frac{1}{2^n} e^{\frac{n}{2}-i} \right) \approx -0.12011\end{aligned}$$

5 Discussion

As early as 1961 did William Fellner [7, pp. 678–9] ask: "there is the question whether, if we observe in him [the decision maker] the trait of nonadditivity, he is or is not likely gradually to lose this trait *as he gets used to the uncertainty with which he is faced*." Fellner pointed out a fundamental problem in answering this question empirically: In an experiment, decision makers may understand that the ambiguity is generated by a randomization mechanism and is therefore not ambiguous, but this is not necessarily the case with processes of nature or social life.

Our analysis shows that a lot depends on the way we choose to model ambiguity. But at least within CEU, ambiguity aversion disappears if the decision maker is faced with many similar ambiguous situations. The term "similar" is of course not well defined, but loosely speaking, our analysis shows that even though decision makers don't learn anything new about the world as they face repeated ambiguity, they may still learn not to fear this lack of knowledge. So the doctor of the introduction may learn after seeing many patients to look at the anchoring probability as a guideline for her repeated medical decisions, but it may still be the case that she'll avoid ambiguity and prefer to take risk with known probabilities on her first trip out of the country, even if the anchoring probabilities of the ambiguous option are better than the risky one.

Theorem 1 does not claim that overall ambiguity aversion disappears. It doesn't even rule out the possibility that as the number of incidents n grows, the difference between the certainty equivalents of the anchoring probabilistic lottery and the ambiguous gamble may be unbounded. Similarly, Theorem 2 permits the certainty equivalent of the ambiguous gamble to be unboundedly negative. Both theorems deal with the certainty equivalents per case. An alternative way to analyze attitudes per case is to divide the gamble L^n and the anchoring lottery X^n by n. The probabilistic lottery will then converge to its average. Maccheroni and Marinacci [17] proved that as $n \to \infty$, the capacity of the event "the average outcome of the ambiguous act L is between its CEU (with the linear utility $u(x) = x$) and minus the CEU value of $-L$" is one. Similarly to this extension of the law of large numbers, the central limit theorem of the classical probability was also extended to the uncertainty framework. This was done by Marinacci [18], who used a certain set of capacities, and by Epstein, Kaido, and Seo [5], who made use of belief functions. The latter authors also studies confidence regions.

Very few experiments checked attitudes to repeated ambiguity (although it seems that several more are currently being conducted). Liu and Colman [16] report that participants chose ambiguous options significantly more frequently in repeated-choice than in single-choice. This suggests that repetition diminishes the effect of ambiguity aversion. Filiz-Ozbay, Gulen, Masatlioglu, and Ozbay [8] report that ambiguity aversion diminishes with the size of the urn. The intuition behind their result agrees with our finding, since both are based on the idea that the more options there are (number of balls to draw from an urn/number of urns) the less plausible is the extreme pessimistic view that Nature always acts against the decision-maker. Halevy and Feltkamp [13] and Epstein and Halevy [4] conducted experiments that involve drawing from two urns and report that when no information regarding the dependence between the urns is provided, individuals display ambiguity aversion with respect to it. Since we assume that urns are independent, this type of ambiguity is not relevant to the current work.

Other models imply a connection between the CEU and the EU models. Klibanoff [14] studied the relation between stochastic independence and convexity of the capacity in the CEU model and found that together they imply EU (hence the capacity must be additive). His results are not related to ours since we do not assume stochastic independence and, furthermore, the capacities we analyse are not required to be convex.

Appendix: Proofs

Denote by F_Z the distribution of lottery Z. In the sequal we use the integral versions of the expected utility and the rank-dependent models. Also, we use the cummulative (rather than the decummulative) version of the RD model, defining $\psi(p) = 1 - \varphi(1-p)$, to obtain for a lottery X

$$\mathrm{EU}(Z) = \int u(z)dF_Z(z)$$

$$\mathrm{RD}(Z) = \int u(z)d\psi(F_Z(z)) \tag{2}$$

Observe that as φ is Lipschitz with K, so is ψ. Also, if for all p, $\varphi(p) \leqslant p$ then for all p, $\psi(p) \geqslant p$. We use below the notation $\mathrm{CEU}^n(L^n)$ for the Choquet expected utility value of L^n, which is given by $\mathrm{RD}(X^n)$ using eq. (2) with $g^n(p) := 1 - f^n(1-p)$. Clearly, if $f = \lim_{n\to\infty} f^n$ then $g := 1 - f(1-p) = \lim_{n\to\infty} g^n$.

Fact 1 Let $u(x) = -e^{-ax}$. Then for all n, $\frac{c^n}{n} = c^1$.

Proof: Note first that $\int u(z)\mathrm{d}(F_{X^n}(z)) = -\left|\int u(z)\mathrm{d}(F_X(z))\right|^n$. Indeed, using induction, we get

$$\int u(z)\mathrm{d}(F_{X^n}(z)) = \sum_{i=1}^{m} p_i \int u(z)\mathrm{d}(F_{x_i+X^{n-1}}(z))$$

$$= \sum_{i=1}^{m} p_i e^{-ax_i} \int u(z)\mathrm{d}(F_{X^{n-1}}(z)) = \int u(z)\mathrm{d}(F_{X^{n-1}}(z)) \left(\sum_{i=1}^{m} p_i e^{-ax_i}\right)$$

$$= -\left|\int u(z)\mathrm{d}(F_X(z))\right|^{n-1} \times \left|\int u(z)\mathrm{d}(F_X(z))\right| = -\left|\int u(z)\mathrm{d}(F_X(z))\right|^n$$

Then, using $u(c^1) = \int u(z)\mathrm{d}(F_X(z))$,

$$\begin{aligned} u(c^n) = \int u(z)\mathrm{d}(F_{X^n}(z)) &= -\left|u(c^1)\right|^n = -\left(e^{-ac^1}\right)^n \\ &= -e^{-anc^1} = u\left(nc^1\right) \end{aligned}$$

Hence $c^n = nc^1$. □

Proof of Theorem 1: We prove the theorem through a sequence of claims.

Claim 1 If $\lim_{x\to-\infty} u'(x) = \infty$, then

$$\lim_{n\to\infty} \int_{x>0} u(x)\, dF_{X^n}(x) \Big/ \int_{x<0} u(x)\, dF_{X^n}(x) = 0$$

Proof: Let $y(\mu) = \sup\{y \leqslant 0 : u(y) < \mu y\}$. Since $\lim_{x\to-\infty} u'(x) = \infty$, it follows that $y(\mu)$ is finite. By the Central Limit Theorem, as $n \to \infty$, the probability that X^n will be in any finite segment goes to 0 yet the probability that it is negative goes to $\frac{1}{2}$. hence $\Pr\{X^n < u^{-1}(\mu x)\} \underset{n\to\infty}{\longrightarrow} \frac{1}{2}$.

Since for positive x, $u'(x) \leqslant 1$, it follows that for such x, $u(x) \leqslant x$. Therefore

$$\frac{\int_{x>0} u(x)\, dF_{X^n}(0)}{\int_{x<0} u(x)\, dF_{X^n}(x)} \geqslant \frac{\int_{x>0} x dF_{X^n}(x)}{\int_{x<0} u(x)\, dF_{X^n}(x)}$$

Since $\mathrm{E}(X^n) = 0$, it follows that $\int_{x<0} x\, dF_{X^n}(x) = -\int_{x>0} x\, dF_{X^n}(x)$. Therefore

$$\frac{\displaystyle\int_{x>0} x\, dF_{X^n}(x)}{\displaystyle\int_{x<0} u(x)\, dF_{X^n}(x)} = \frac{\displaystyle -\int_{y(\mu)}^{0} x\, dF_{X^n}(x) - \int_{x<y(\mu)} x\, dF_{X^n}(x)}{\displaystyle\int_{y(\mu)}^{0} u(x)\, dF_{X^n}(x) + \int_{x<y(\mu)} u(x)\, dF_{X^n}(x)} >$$

$$\frac{\displaystyle -\int_{y(\mu)}^{0} x\, dF_{X^n}(x) - \int_{x<y(\mu)} x\, dF_{X^n}(x)}{\displaystyle\int_{y(\mu)}^{0} u(x)\, dF_{X^n}(x) + \mu \times \int_{x<y(\mu)} x\, dF_{X^n}(x)} \underset{n\to\infty}{\longrightarrow} -\frac{1}{\mu}$$

This is true for every $\mu > 1$, hence the claim. □

Claim 2 Suppose that $\lim\limits_{x\to-\infty} u'(x) = \infty$. Then for EU with u and CEU^n with u and f^n, $\lim\limits_{n\to\infty} \frac{\mathrm{CEU}^n(L^n)}{\mathrm{EU}(X^n)} \leqslant K$.

Proof: We obtain by claim 1 that

$$\frac{\mathrm{CEU}^n(L^n)}{\mathrm{EU}(X^n)} \leqslant$$

$$\frac{\int_{x<0} u(x)(g^n)'(F_{X^n}(x))\, dF_{X^n}(x)}{\mathrm{EU}(X^n)} \leqslant$$

$$\frac{K\int_{x<0} u(x)\, dF_{X^n}(x)}{\mathrm{EU}(X^n)} \to K$$

□

Claim 3 If $\lim\limits_{x\to-\infty} u'(x)/u(x) < -\ell < 0$, then $\lim\limits_{n\to\infty} \left[\frac{c^n}{n} - \frac{d^n}{n}\right] = 0$.

Proof: Since u is concave, $u(c^n) < 0$. It follows by claim 2 that for $n \geqslant n_0$, $u(d^n) \geqslant (K+1)u(c^n)$. It follows by the concavity of u and by the fact that $d^n \leqslant c^n$ that

$$\frac{u(c^n) - u(d^n)}{c^n - d^n} \geqslant u'(c^n)$$

hence for $n \geqslant n_0$,

$$c^n - d^n \leqslant \frac{u(c^n) - u(d^n)}{u'(c^n)} \leqslant -\frac{Ku(c^n)}{u'(c^n)}$$

Since u is concave, $\lim_{x\to-\infty} u(x) = -\infty$, and as $\lim_{x\to-\infty} u'(x)/u(x) < -\ell < 0$, it follows that $\lim_{x\to-\infty} u'(x) = \infty$. By Fact 2 below, $\lim_{n\to\infty} c^n = -\infty$, hence for a sufficiently large n,

$$-\frac{Ku(c^n)}{u'(c^n)} \leqslant \frac{K}{\ell}$$

Therefore

$$0 \leqslant \frac{c^n}{n} - \frac{d^n}{n} \leqslant \frac{K}{\ell n} \xrightarrow[n\to\infty]{} 0$$

It thus follows that $\lim_{n\to\infty} \left[\frac{c^n}{n} - \frac{d^n}{n}\right] = 0$, which is the claim. $\square$

Fact 2 If $\lim_{x\to-\infty} u'(x) = \infty$, then $\lim_{n\to\infty} c^n = -\infty$.

Proof: We show that for every integer $m < 0$, $\lim_{n\to\infty} \mathrm{EU}(X^n) \leqslant u(m-1)$. The value of $\mathrm{EU}(X^n)$ equals

$$\int\limits_{x\leqslant 2(m-1)} u(x)\, dF_{X^n}(x) \left[1 + \frac{\int\limits_{2(m-1)}^{0} u(x)\, dF_{X^n}(x)}{\int\limits_{x\leqslant 2(m-1)} u(x)\, dF_{X^n}(x)} + \frac{\int\limits_{x>0} u(x)\, dF_{X^n}(x)}{\int\limits_{x\leqslant 2(m-1)} u(x)\, dF_{X^n}(x)}\right]$$

Again by the central limit theorem, $\lim_{n\to\infty} \int_{2(m-1)}^{0} u(x)\, dF_{X^n}(x) = 0$. Also, by the same argument,

$$\lim_{n\to\infty} \frac{\int_{x>0} u(x)\, dF_{X^n}(x)}{\int_{x\leqslant 2(m-1)} u(x)\, dF_{X^n}(x)} = \lim_{n\to\infty} \frac{\int_{x>0} u(x)\, dF_{X^n}(x)}{\int_{x\leqslant 0} u(x)\, dF_{X^n}(x)}$$

By Claim 1 the last limit is zero. By the Central Limit Theorem, the probability of receiving an outcome between $2(m-1)$ and 0 converges to zero and the probability of receiving a negative outcome is $\frac{1}{2}$. It thus follows that

$$\lim_{n\to\infty} \int u(x)\, dF_{X^n}(x) = \lim_{n\to\infty} \int\limits_{x\leqslant 2(m-1)} u(x)\, dF_{X^n}(x) \leqslant \frac{u(2(m-1))}{2} \leqslant u(m-1)$$

It thus follows that $\lim_{n\to\infty} c^n \leqslant m-1 < m$. $\square$

The next two claims deal with the case $\lim_{x\to-\infty} u'(x)/u(x) \to 0$.

Claim 4 If $\lim_{x\to-\infty} u'(x) = H < \infty$, then $\lim_{n\to\infty} \frac{c^n}{n} = \lim_{n\to\infty} \frac{d^n}{n} = 0$.

Proof: Since for all n, $d^n \leqslant c^n$, it is enough to prove that $\lim_{n\to\infty} \frac{d^n}{n} = 0$. Define $v(x) = \min\{Hx, 0\}$. By assumption, $u(x) \geqslant v(x)$ for all x. Let CEU_v^n denote the CEU^n functional with respect to v. Then $\mathrm{CEU}^n(L^n) \geqslant \mathrm{CEU}_v^n(L^n)$ and, as above,

$$\begin{aligned}
\mathrm{CEU}_v^n(L^n) &= \int v(z)\mathrm{d}g^n(F_{X^n}(z)) \\
&= H\int_{z\leqslant 0} z\mathrm{d}g^n(F_{X^n}(z)) \geqslant KH\int_{z\leqslant 0} z\mathrm{d}F_{X^n}(z)
\end{aligned}$$

Let σ^2 be the variance of X and $n\sigma^2$ the variance of X^n. Note that $\mathrm{E}(X^n) = 0$ and choose $\frac{1}{2} < \alpha < 1$. By Chebyshev's inequality,

$$\Pr(X^n < -n^\alpha) \leqslant \frac{n\sigma^2}{n^{2\alpha}} = \frac{\sigma^2}{n^{2\alpha-1}}$$

Assume that n is sufficiently large to satisfy $nx_1 < -n^\alpha$. Then

$$\begin{aligned}
KH\int_{z\leqslant 0} z\mathrm{d}F_{X^n}(z) &\geqslant KH\left(nx_1 \times \frac{\sigma^2}{n^{2\alpha-1}} - n^\alpha \times 1\right) \\
&= KH\left(\frac{x_1\sigma^2}{n^{2(\alpha-1)}} - n^\alpha\right) \implies \\
u(d^n) &= \mathrm{CEU}^n(L^n) \geqslant KH\left(\frac{x_1\sigma^2}{n^{2(\alpha-)}} - n^\alpha\right)
\end{aligned}$$

and, since u is concave and $u'(0) = 1$,

$$d^n \geqslant KH\left(\frac{x_1\sigma^2}{n^{2(\alpha-1)}} - n^\alpha\right)$$

Therefore,

$$\lim_{n\to\infty} \frac{d^n}{n} \geqslant KH \lim_{n\to\infty}\left(\frac{x_1\sigma^2}{n^{2\alpha-1}} - \frac{1}{n^{1-\alpha}}\right) = 0$$ $\square$

Claim 5 If $\lim_{x\to-\infty} u'(x) = \infty$ but $\lim_{x\to-\infty} \frac{u'(x)}{u(x)} = 0$, then $\lim_{n\to\infty} \frac{c^n}{n} = \lim_{n\to\infty} \frac{d^n}{n} = 0$.

Proof: By l'Hospital's rule, $\lim_{x\to-\infty} u''(x)/u'(x) = \lim_{x\to-\infty} u'(x)/u(x) = 0$. Consider the exponential utility $v_\varepsilon(x) = -e^{-\varepsilon x}$ for which $-v''_\varepsilon/v'_\varepsilon \equiv \varepsilon$. Denote by c^n_ε the value of c^n obtained for the function v_ε. By Fact 1, $\lim_{n\to\infty} c^n_\varepsilon/n = c^1_\varepsilon < 0$ where c^1_ε, the certainty equivalent X, satisfies

$$-e^{-\varepsilon c^1_\varepsilon} = \int -e^{-\varepsilon z}\, dF_X(z) \Longrightarrow c^1_\varepsilon = -\frac{1}{\varepsilon} \ln \left[\int e^{-\varepsilon z}\, dF_X(z) \right]$$

Note that, using l'Hospital's rule and $\mathrm{E}(X) = 0$, $\lim_{\varepsilon\to 0} c^1_\varepsilon = 0$.

As $\lim_{x\to-\infty} u''(x)/u'(x) = 0$, it follows that for every $\varepsilon > 0$ there is $x(\varepsilon)$ such that for all $x < x(\varepsilon)$, $-u''(x)/u'(x) < \varepsilon$. Define a function u_ε as follows.

$$u_\varepsilon = \begin{cases} u(x) & x \leqslant x(\varepsilon) \\ \\ a v_\varepsilon(x) + b & x > x(\varepsilon) \end{cases}$$

where $a = \frac{u'(x(\varepsilon))}{v'_\varepsilon(x(\varepsilon))}$ and $b = u(x(\varepsilon)) - a v_\varepsilon(x(\varepsilon))$. Clearly u_ε is less risk averse than v_ε, hence $c^1_{u_\varepsilon} \geqslant c^1_\varepsilon$. By Fact 2 below, $\lim_{n\to\infty} c^n_{u_\varepsilon}/n = \lim_{n\to\infty} c^n/n$. We saw that $\lim_{n\to\infty} c^n_\varepsilon/n = c^1_\varepsilon$, hence $\lim_{n\to\infty} c^n/n \geqslant c^1_\varepsilon$. The claim now follows by the fact that $\lim_{\varepsilon\to 0} c^1_\varepsilon = 0$.

Since u is concave, the fact that (for sufficiently large n) $u(d^n) \geqslant (K+1)u(c^n)$ implies that $d^n \geqslant (K+1)c^n$. As $\lim_{n\to\infty} \frac{c^n}{n} \to 0$, it follows that $\lim_{n\to\infty} \frac{d^n}{n} \to 0$. □

Fact 3 If for $x < M$, $u(x) = v(x)$, then $\lim_{n\to\infty} \frac{c^n_u}{n} = \lim_{n\to\infty} \frac{c^n_v}{n}$.

Proof: For $M \geqslant 0$, the fact follows from Claim 1. For $M < 0$, observe that the probability that X^n is between M and 0 goes to zero with n. □

Claims 3–5 cover all possible cases of $\lim_{x\to-\infty} u(x)$, hence the theorem. ■

Proof of Theorem 2: Note that, by assumption and by the monotonicity of u, $c^n \geqslant 0$ for all n and $d^1 < 0$.

First, assume $\lim_{x\to-\infty} u'(x) = \infty$. Define $u^n(x) = u(x) - u(nx_m)$ and note that $u^n(nx_m) = 0$ and $u^n(x) < 0$, for all outcomes of X^n. These inequalities and $g'(p) \leqslant K$ (the latter follows from g being Lipschitz with K) imply that for the CEU^n, the CEU functional with respect to u^n,

$$\begin{aligned} \mathrm{CEU}^n(L^n) &= \int u^n(z)\mathrm{d}g^n(F_{X^n}(z)) \\ &\geqslant K\int u^n(z)\mathrm{d}F_{X^n}(z) \geqslant Ku^n(c^n) \end{aligned}$$

The inequality $u^n(c^n) \geqslant u^n(0)$ yields

$$u^n(d^n) = \mathrm{CEU}^n(L^n) \geqslant Ku^n(c^n) \geqslant Ku^n(0)$$

Going back to u, noting that $1-K \leqslant 0$ and that, by concavity, $u(nx_m) \leqslant nu(x_m)$,

$$\begin{aligned} u(d^n) &= u^n(d^n) + u(nx_m) \geqslant Ku^n(0) + u(nx_m) \\ &= -Ku(nx_m) + u(nx_m) = (1-K)u(nx_m) \\ &\geqslant n(1-K)u(x_m) \end{aligned}$$

Denote $A = (1-K)u(x_m)$. By assumption, $A \leqslant 0$. Note that the concavity of u and $\lim_{x\to-\infty} u'(x) = \infty$ imply $\lim_{y\to-\infty} u^{-1}(y)/y = 0$. Then, $d^n \geqslant u^{-1}(nA)$ implies

$$\lim_{n\to\infty} \frac{d^n}{n} \geqslant \lim_{n\to\infty} \frac{u^{-1}(nA)}{nA} A = 0$$

Finally, if $\lim_{x\to-\infty} u'(x) = H < \infty$ ($u'' < 0$ implies that $\lim_{x\to-\infty} u'(x)$ exists), then the proof follows that of Claim 4. ■

Proof of Conclusion 1: If $\lim_{n\to\infty} \frac{c^n}{n} > 0$, then for sufficiently large n, $c^n > 0$. By Theorem 2, $\lim_{n\to\infty} \frac{d^n}{n} \geqslant 0$. And if $\lim_{n\to\infty} \frac{c^n}{n} < 0$, then for sufficiently large n, $c^n < 0$ and by ambiguity aversion $\lim_{n\to\infty} \frac{d^n}{n} < 0$. ■

Proof of Theorem 3: We prove the three parts of the proposition separately.

1. If u is bounded from above and $\lim\limits_{n\to\infty} \frac{c^n}{n} > 0$, then there exists n_δ such that $\forall n > n_\delta$, $L^n \succ 0$: Without loss of generality, assume that $u(x) < 0$ for all x and that $\lim\limits_{x\to\infty} u(x) = 0$. Similarly to the proof of the first part of Theorem 2,

$$\mathrm{CEU}^n(L^n) = \int u(z)\mathrm{d}g^n(F_{X^n}(z)) \geqslant K \int u(z)\mathrm{d}(F_{X^n}(z)) \geqslant Ku(c^n)$$

Choose δ that satisfies $\lim\limits_{n\to\infty} \frac{c^n}{n} > \delta > 0$ and note that for a sufficiently large n, $c^n > n\delta$. As $n\delta$ goes to infinity, $\lim\limits_{n\to\infty} u(c^n) = 0$ and, by the above argument, $\lim\limits_{n\to\infty} \mathrm{CEU}^n(L^n) = 0$. This implies the existence of n_δ such that for all $n > n_\delta$, $\mathrm{CEU}^n(L^n) > u(0)$. For these n, $L^n \succ 0$. $\square$

2. If $\mathrm{E}[X] = 0$ and u is bounded from above and from below, then $\lim\limits_{n\to\infty} \frac{c^n}{n} = \lim\limits_{n\to\infty} \frac{d^n}{n} = 0$: As in the proof of Theorem 1, we assume that $u(0) = 0$ and $u'(0) = 1$. Let $\hat{u} = \lim\limits_{x\to\infty} u(x)$ and $\check{u} = \lim\limits_{x\to-\infty} u(x)$. First we show that $\lim\limits_{n\to\infty} \frac{c^n}{n} = 0$. Choose $\varepsilon > 0$. Let $\check{x} = u^{-1}(\check{u} + \varepsilon)$ and $\hat{x} = u^{-1}(\hat{u} - \varepsilon)$. Then

$$\mathrm{EU}(X^n) \leqslant \Pr(X^n \leqslant \check{x})(\check{u} + \varepsilon) + \Pr(\check{x} < X^n < \hat{x})(\hat{u} - \varepsilon) + \Pr(X^n \geqslant \hat{x})\hat{u}$$

and

$$\mathrm{EU}(X^n) \geqslant \Pr(X^n \leqslant \check{x})\check{u} + \Pr(\check{x} < X^n < \hat{x})(\check{u} + \varepsilon) + \Pr(X^n \geqslant \hat{x})(\hat{u} - \varepsilon)$$

Since $\lim\limits_{n\to\infty} \Pr(\check{x} < X^n < \hat{x}) = 0$ and by the Central Limit Theorem, $\lim\limits_{n\to\infty} \Pr(X^n \leqslant \check{x}) = \lim\limits_{n\to\infty} \Pr(X^n \geqslant \hat{x}) = \frac{1}{2}$ (here we use the fact that $\mathrm{E}[X] = 0$),

$$\frac{\check{u} + \hat{u} - \varepsilon}{2} \leqslant \lim_{n\to\infty} \mathrm{EU}(X^n) \leqslant \frac{\check{u} + \hat{u} + \varepsilon}{2}$$

and since the above holds for all ε, we get

$$\lim_{n\to\infty} \mathrm{EU}\left(X^n\right) = \frac{\check{u} + \hat{u}}{2}$$

Therefore, $\lim\limits_{n\to\infty} c^n = u^{-1}\left(\frac{\check{u}+\hat{u}}{2}\right)$ and $\lim\limits_{n\to\infty} \frac{c^n}{n} = 0$.

To prove that $\lim\limits_{n\to\infty} \frac{d^n}{n} = 0$, note that the boundedness of u guarantees the existence of $0 < L < \infty$ satisfying $Lx < u(x)$ for all $x < 0$. Then, define

$$v\left(x\right) = \begin{cases} Lx & x \leqslant 0 \\ 0 & x > 0 \end{cases}$$

and proceed as in the proof of Claim 4. $\square$

3. If u is exponential and concave and for all p, $f(p) \leqslant p$, then $\lim\limits_{n\to\infty} \frac{c^n}{n} = \lim\limits_{n\to\infty} \frac{d^n}{n}$: Let $u(x) = -e^{-ax}$, with $a > 0$. By Fact 1, $c^n = nc^1$ and hence $\lim\limits_{n\to\infty} \frac{c^n}{n} = c^1$. By the definitions of c^1 and d^n we have

$$\begin{aligned} \mathrm{EU}(X - c^1) &= \int -e^{-az}\mathrm{d}F_{X-c^1}(z) = \int -e^{-a(z-c^1)}\mathrm{d}F_X(z) \\ &= e^{ac^1}\int -e^{-az}\mathrm{d}F_X(z) = e^{ac^1}(-e^{-ac^1}) = -1 \end{aligned} \tag{3}$$

and

$$\begin{aligned} &\mathrm{CEU}^n\left(\left(L - \tfrac{d^n}{n}\right)^n\right) = \int -e^{-az}\mathrm{d}g^n\left(F_{\left(X-\frac{d^n}{n}\right)^n}(z)\right) = \\ &\int -e^{-az}\mathrm{d}g^n(F_{X^n-d^n}(z)) = \int -e^{-a(z-d^n)}\mathrm{d}g^n\left(F_{X^n}(z)\right) = \\ &e^{ad^n}\int -e^{-az}\mathrm{d}g^n\left(F_{X^n}(z)\right) = e^{ad^n}\left(-e^{-ad^n}\right) = -1 \end{aligned} \tag{4}$$

The sequence $\left\{\frac{d^n}{n}\right\}_{n=1}^{\infty}$ is bounded (since the support of X is) and, by $g(p) \geqslant p$, $\frac{d^n}{n} \leqslant \frac{c^n}{n} = c^1$. Assume, by way of negation, that the sequence does not converge to c^1. Then, there exists $\varepsilon > 0$ and a subsequence $\left\{\frac{d^{n_j}}{n_j}\right\}_{j=1}^{\infty}$ satisfying $\lim\limits_{j\to\infty} \frac{d^{n_j}}{n_j} < c^1 - \varepsilon$. Without loss of generality, assume that for all j, $\frac{d^{n_j}}{n_j} < c^1 - \varepsilon$. Hence,

$$\begin{aligned} &\mathrm{CEU}\left(\left(L - \tfrac{d^{n_j}}{n_j}\right)^{n_j}\right) = \int -e^{-az}\mathrm{d}g^n\left(F_{(X-d^{n_j}/n_j)^{n_j}}(z)\right) \\ > &\int -e^{-az}\mathrm{d}g^n\left(F_{(X-c^1+\varepsilon)^{n_j}}\right)(z) \geqslant -K\int e^{-az}\mathrm{d}F_{(X-c^1+\varepsilon)^{n_j}}(z) \\ = &-K\left[\int e^{-az}\mathrm{d}F_{X-c^1+\varepsilon}(z)\right]^{n_j} = -Ke^{-an_j\varepsilon}\left[\int e^{-az}\mathrm{d}F_{X-c^1}(z)\right]^{n_j} \\ = &-Ke^{-an_j\varepsilon} \underset{j\to\infty}{\longrightarrow} 0 \end{aligned}$$

where the last equality follows by eq. (3). Therefore, for sufficiently large j,

$$\mathrm{CEU}\left(\left(L - \tfrac{d^{n_j}}{n_j}\right)^{n_j}\right) > -1$$

in contradiction with eq. (4). To conclude, $\lim_{n\to\infty} \frac{d^n}{n} = c^1 = \lim_{n\to\infty} \frac{c^n}{n}$. ■

Example 1 Let $X = (-\frac{1}{4}, \frac{1}{2}; \frac{3}{4}, \frac{1}{2})$. Define $g = g^1 = \ldots$ by

$$g(p) = \begin{cases} 2p & 0 \leqslant p \leqslant \frac{1}{2} \\ 1 & \frac{1}{2} < p \leqslant 1 \end{cases}$$

which is Lipschitz with $K = 2$. We get

$$\text{EU}(X^{4n}) = \sum_{i=-n}^{3n} \binom{4n}{i+n} \frac{1}{2^{4n}} u(i) \tag{5}$$

$$\text{CEU}(L^{4n}) = 2\sum_{i=-n}^{n-1} \binom{4n}{i+n} \frac{1}{2^{4n}} u(i) + \binom{4n}{2n} \frac{1}{2^{4n}} u(n) \tag{6}$$

Let $u(x) = x$ for $x \geqslant 0$. We define $u(-n)$ inductively. Let

$$\begin{aligned} v_n &= -\sum_{i=-n+1}^{-1} \binom{4n}{i+n} u(i) - \sum_{i=1}^{n-1} \binom{4n}{i+n} i - \binom{4n}{2n} \frac{n}{2} \qquad (7) \\ w_n &= 2u(-n+1) - u(-n+2) \end{aligned}$$

and define u for $x < 0$ as follows. For $n = 1, \ldots,$ let $u(-n) = \min\{v_n, w_n\}$, and for $x \in (-n, -n+1)$ let $u(x) = u(-n) + (x+n)[u(-n+1) - u(-n)]$. The function u is strictly increasing and weakly concave.

Claim 6 $\lim_{n\to\infty} u(-n)/n = -\infty$.

Proof: Suppose not. Then there exists $A > 0$ such that for all n, $-u(-n)/n \leqslant A$, and since between $-n$ and $-n+1$ the function u is linear, it follows that for all n, $-u(-n)/n \leqslant A$.

By definition, $u(-n) \leqslant v_n$, hence it follows by eqs. (6) and (7) that for all n, $\text{CEU}(X^{4n}) \leqslant 0$. On the other hand, by eq. (6),

$$\begin{aligned} \text{CEU}(X^{4n}) &= 2\sum_{i=-n}^{-1} \binom{4n}{i+n} \frac{u(i)}{2^{4n}} + 2\sum_{i=1}^{n-1} \binom{4n}{i+n} \frac{i}{2^{4n}} + \binom{4n}{2n} \frac{n}{2^{4n}} \\ &\geqslant -\frac{(n-1)nA}{2^{4n-1}} \binom{4n}{n-1} + 1 \times \left[\frac{1}{2} - \Pr(X^{4n} \leqslant 0)\right] \qquad (8) \end{aligned}$$

Let $\beta_n = \frac{(n-1)nA}{2^{4n-1}}\binom{4n}{n-1}$. Clearly

$$\begin{aligned}\frac{\beta_{n+1}}{\beta_n} &= \frac{n(n+1)A2^{4n-1}\binom{4n+4}{n}}{(n-1)nA2^{4n+3}\binom{4n}{n-1}} \\ &= \frac{(n+1)(4n+4)(4n+3)(4n+2)(4n+1)}{16(n-1)n(3n+4)(3n+3)(3n+2)} \to \frac{4^4}{16 \times 3^3} = \frac{16}{27}\end{aligned}$$

Hence $\lim_{n\to\infty}\beta_n = 0$. Likewise, $\Pr(X^{4n} \leqslant 0) \leqslant \frac{n}{2^{4n}}\binom{4n}{n} \to 0$, hence the expression of eq. (8) converges to $\frac{1}{2}$; a contradiction. $\square$

Define $n_0 = 0$, and let n_i satisfy

1. $u(-n_i) = v_{n_i}$
2. For $n_{i-1} < j < n_i$, $u(-j) < v_j$

It follows by Claim 6 that $\{n_i\}$ is not a finite sequence, as otherwise the function u would become linear from a certain point on to the left and will never intersect the line Ax for sufficiently high A.

By definition, $\mathrm{RD}(X^{4n_i}) = 0$. It thus follows by eq. (5) that

$$\begin{aligned}c^{4n_i}(\tfrac{1}{4}) = \mathrm{EU}(X^{4n_i}) &= \left[\binom{4n_i}{2n_i}\frac{n_i}{2} + \sum_{i=n_i+1}^{3n_i}\binom{4n_i}{i+n_i}i\right]\frac{1}{2^{4n_i}} \\ &> \frac{n_i}{2} \times \Pr\left(X^{4n_i} \geqslant n_i\right) = \frac{n_i}{4}\end{aligned}$$

Hence $\lim_{i\to\infty} c^{4n_i}/4n_i > \frac{1}{16}$ while $d^{4n_i}/4n_i \equiv 0$. $\blacksquare$

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