Theta liftings on higher covers of symplectic groups

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We study a new lifting of automorphic representations using the theta representation Θ on the 4-fold cover of the symplectic group, $\overline{\mathrm{Sp}}_{2r}(\mathbb{A})$. This lifting produces the first examples of CAP representations on higher degree metaplectic covering groups. Central to our analysis is the identification of the maximal nilpotent orbit associated to Θ.

We conjecture a natural extension of Arthur's parameterization of the discrete spectrum to $\overline{\mathrm{Sp}}_{2r}(\mathbb{A})$. Assuming this, we compute the effect of our lift on Arthur parameters and show that the parameter of a representation in the image of the lift is non-tempered. We conclude by relating the lifting to the dimension equation of Ginzburg to predict the first non-trivial lift of a generic cuspidal representation of $\overline{\mathrm{Sp}}_{2r}(\mathbb{A})$.

Contents

Citations to Previous Work

This thesis is based on [Les17].

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To my best friend Casey.

Chapter 1

Introduction

1.1 Theta functions and liftings

One of the first modular functions ever studied was the classical *theta function* of Jacobi, which may be written as

$$\Theta(\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau}, \qquad \text{where } \tau \in \mathbb{C},\ \ \mathrm{Im}(\tau) > 0.$$

While variations of this function were studied as far back as Euler, one of the most important early applications of the theta function comes in Riemann's second proof of the functional equation of the eponymous zeta function. The key property of this function is the functional equation

$$\Theta(\tau) = \frac{1}{\sqrt{\tau}} \Theta\left(\frac{-1}{\tau}\right),$$

which implies, along with the obvious invariance $\Theta(\tau + 2) = \Theta(\tau)$, that Θ is a modular form of weight $\frac{1}{2}$. Today, we understand, thanks to Weil [Wei64], that the theory of half-integral weight modular forms may be understood as studying certain automorphic representations of the *metaplectic* double cover $\mathrm{Mp}_2(\mathbb{R})$ of the

group $\mathrm{SL}_2(\mathbb{R})$. Moreover, the Jacobi theta function corresponds to a particularly important automorphic representation ω known as the Segal-Shale-Weil(-oscillator) representation. In particular, the theory of automorphic forms on the non-linear covers of reductive groups is as old as the theory of modular forms itself.

Let K be a number field, and let G be a reductive group over K. Motivated by the classical theory of theta functions and other applications to number theory ([KP84], [Kub69]), the theory of automorphic forms on covers of the group $\mathrm{G}(\mathbb{A})$, where $\mathbb{A}$ is the ring of adeles of K, has enjoyed much recent attention (for example, [KP84, Wei03, Wei14, GG14, FG17, BBF11, Gao18b]). While many people have developed the theory of topological covering groups of $\mathrm{G}(\mathbb{A})$, we will use the theory of Brylinski and Deligne [BD01] to describe the groups in which we are interested. Assuming K contain the n^{th} roots of unity, Brylinski and Deligne classify degree n topological extensions $\overline{\mathrm{G}}_{\mathbb{A}}$ of $\mathrm{G}(\mathbb{A})$ by $\mu_n(K)$ which arise from the algebraic K-theory of the field K. More precisely, viewing G as a sheaf of groups on the big Zariski site over $\mathrm{Spec}(K)$, Brylinski-Deligne characterize the objects in the Picard category $\mathrm{CExt}(\mathrm{G}, \mathbb{K}_2)$ of central extensions (of sheaves) of G by Quillen's algebraic K-theory sheaf $\mathbb{K}_2$. That is, an object $\overline{G} \in \mathrm{CExt}(\mathrm{G}, \mathbb{K}_2)$ is a sheaf of groups such that there is a short exact sequence

$$1 \longrightarrow \mathbb{K}_2 \longrightarrow \overline{\mathrm{G}} \longrightarrow \mathrm{G} \longrightarrow 1 \tag{1.1}$$

with the image of $\mathbb{K}_2$ central in $\overline{G}$. The classification is particularly simple when G is split, and for G semi-simple and simply connected, $\mathrm{CExt}(\mathrm{G}, \mathbb{K}_2)$ is discrete with a single generator, essentially recovering Steinberg's universal extension. This classification of (BD-)covering groups works over a general base scheme S; in particular, we have an analogous theory over local fields F.

Given a BD-extension $\overline{G}$, we obtain a degree n topological covering group

$$1 \longrightarrow \mu_n(K) \longrightarrow \overline{\mathrm{G}}(\mathbb{A}) \xrightarrow{\pi} \mathrm{G}(\mathbb{A}) \longrightarrow 1$$

by taking $\mathbb{A}$-points of (1.1) and pushing out by the n^{th}-Hilbert symbol $\mathbb{K}_2(\mathbb{A}) \xrightarrow{(\cdot,\cdot)_n} \mu_n(\mathbb{A}) = \mu_n(K)$. The reason for studying central extensions by n^{th}-roots of unity is that, as noticed by Weil [Wei64], the n^{th}-order reciprocity theorem of class field theory implies that the cover π splits over the discrete subgroup $\mathrm{G}(K) \subset \mathrm{G}(\mathbb{A})$. Thus, upon fixing a section over the rational points, we may meaningfully study automorphic forms on $\overline{\mathrm{G}}(\mathbb{A})$. This construction recovers essentially all topological covering groups that have arisen in automorphic representation theory.

One reason for working with this formalism is that there is a reasonable notion of an L-group for such covering groups. Given an n-fold topological BD-covering group $\overline{\mathrm{G}}$, we may associate to it a natural complex dual group $\overline{\mathrm{G}}^{\vee}$, which can be distinct from the dual group of G. Recently, Weissman has put forward a promising candidate, ${}^L\overline{G}$, for the L-group of such covering groups (see [Wei14] and [Wei15]) as an extension

$$1 \longrightarrow \overline{\mathrm{G}}^{\vee} \longrightarrow {}^L\overline{G} \longrightarrow W_K \longrightarrow 1. \tag{1.2}$$

It is thus important to test if this L-group allows us to extend Langlands functoriality and Arthur's conjectures to the case of BD-covering groups. For example, the construction of the metaplectic tensor product has been shown to be functorial by Gan [Gan16].

Historically, the metaplectic 2-fold cover of the symplectic group, $\mathrm{Mp}(W)$, where $(W, \langle \cdot, \cdot \rangle)$ is a symplectic vector space over K, is the first example of a BD-covering group to be studied. Much is known about the representation theory of $\mathrm{Mp}(W)$; in

particular, the local Langlands correspondence[1] has been extended to this group via work of Adams-Barbasch [AB95], [AB98] in the real case, and Gan-Savin [GS12] in the nonarchimedean case. A primary tool in the study of the representation theory of $\mathrm{Mp}(W)$ has been the theta correspondence, which is an integral lifting construction utilizing the oscillator representation ω_ψ[2], the generalization of the classical theta representation mentioned above.

We pause here to sketch what we mean by an "integral lifting construction". Stated imprecisely, given two reductive groups (G, H) such that $G \times H \subset \mathrm{Sp}(W)$ and a kernel representation Θ of $\mathrm{Mp}(W)_{\mathbb{A}}$, we consider the restriction $\Theta\big|_{G_{\mathbb{A}} \times H_{\mathbb{A}}}$. In good settings, this restriction decomposes as a Hilbert direct sum

$$\Theta\big|_{G_{\mathbb{A}} \times H_{\mathbb{A}}} \cong \hat{\bigoplus_{\pi}} \pi \boxtimes \Theta(\pi)$$

where the sum ranges over a suitable subset of the automorphic spectrum of $\mathrm{G}_{\mathbb{A}}$. For each such π, the space $\Theta(\pi)$ is an automorphic representation of $H_{\mathbb{A}}$, which may be viewed as having been "lifted from π by Θ."

More explicitly, given an irreducible cuspidal automorphic representation (π, V) of $G_{\mathbb{A}}$, consider the space $\Theta(\pi)$, defined as the Hilbert space closure of the space of functions on $H_{\mathbb{A}}$ of the form

$$F(h) = \int_{[G]} \overline{\varphi(g)} \theta(g, h)\, dg,$$

where $\varphi \in V$ and $\theta \in \Theta$. The map $\pi \mapsto \Theta(\pi)$ is precisely the one indicated above. This general setup is referred to as the "small representation method" of constructing liftings of automorphic representations between reductive group (for example, see

[1] Anachronistically, this correspondence is compatible with the L-group á la Weissman. Historically, the Shimura correspondence allowed for an *ad hoc* notion of local Langlands parameter with which a correspondence may be proved.

[2] Here we need to fix an additive character $\psi : K\backslash\mathbb{A} \to \mathbb{C}^\times$.

[Gin14a] for this terminology); we explain what is meant by small in terms of Gelfand-Kirillov dimension below.

When the kernel representation Θ is the Weil representation ω_ψ, the lifting is the theta correspondence, which has played a central role in the automorphic theory of classical groups. For example, such liftings were recently used by Gan-Ichino to extend the notion of Arthur parameter to $\mathrm{Mp}(W)_{\mathbb{A}}$ [GI17]. Motivated by the work of Bump-Friedberg-Ginzburg [BFG06], one may attempt to study generalizations of theta liftings on BD-covering groups in the hope of gaining insight into the representation theory of higher degree covers.

In this thesis, we consider higher degree covers $\overline{\mathrm{Sp}}(W)_{\mathbb{A}}$ of $\mathrm{Sp}(W)_{\mathbb{A}}$ and study theta-type liftings of automorphic representations of certain dual reductive pairs inside $\overline{\mathrm{Sp}}(W)_{\mathbb{A}}$. In our lifting constructions, we replace the oscillator representation ω_ψ with a *(generalized) theta representation* Θ. As outlined in Chapter 3, this representation is a distinguished residual representation obtained from a Borel Eisenstein series, a process which also gives a construction of ω_ψ [Shi85]. Such theta representations have been studied on several different covering groups (see [KP84, BFG03, FG17, Gao16, Cai16]) and may be defined for a large class of BD-covering groups.

One of the most important properties of ω_ψ is that it has Schrödinger-type models. This both implies that ω_ψ is a minimal representation (in the sense of minimal Gelfand-Kirillov dimension; see discussion below) and enables explicit computations. A crucial distinction in our case is the lack of such a model for Θ or its local components. Moreover, Θ is not even a minimal representation, and the Gelfand-Kirillov dimension of theta representations grows with the degree of the cover. Nevertheless, one of our main results is that *in the case of the* 4*-fold cover of* $\mathrm{Sp}(W)_{\mathbb{A}}$, Θ is

sufficiently small in a sense to be made precise below. This allows it to effectively function as a kernel to construct a lifting of automorphic representations between certain (reducible) dual reductive pairs in $\mathrm{Sp}(W)$.

1.2 Main Results

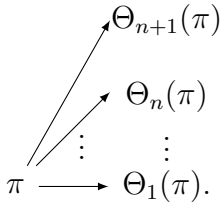
More specifically, to any decomposition $W = W_1 \oplus W_2$ into the sum of two lower dimensional symplectic spaces, we obtain an embedding

$$\mathrm{Sp}(W_1) \times \mathrm{Sp}(W_2) \hookrightarrow \mathrm{Sp}(W).$$

We study the problem (both locally and globally) of restricting the theta representation Θ_W to the subgroup $\overline{\mathrm{Sp}}(W_1) \times_{\mu_4} \overline{\mathrm{Sp}}(W_2)$. We study properties of this representation from another perspective. Using ideas of Rallis in the theory of the classical theta correspondence [Ral84], given a symplectic space $W = W_0$, one can form the larger symplectic space

$$W_n = X_n \oplus W_0 \oplus X_n^*,$$

where X_n is totally isotropic and $\dim_K(X_n) = n$, thus giving an embedding $\mathrm{Sp}(W_0) \hookrightarrow \mathrm{Sp}(W_n)$ for each $n \geq 0$. Setting $V_n = X_n \oplus X_n^*$, we may then consider $\Theta_n(\pi)$, the representation of $\overline{\mathrm{Sp}}(V_n)_{\mathbb{A}}$ formed by the pairing vectors of π with vectors in Θ_{W_n}, and study properties of this lifting as n varies:



In this spirit, we show the following:

Theorem 1.2.1. *Suppose that π is a cuspidal automorphic representation of $\widetilde{\mathrm{Sp}}(W_0)_{\mathbb{A}}$. Then*

1. *If $\Theta_k(\pi) = 0$, then $\Theta_{k-1}(\pi) = 0$;*
2. *There exists an $n \gg 0$ such that $\Theta_n(\pi) \neq 0$;*
3. *If $\Theta_{k-1}(\pi) = 0$, then $\Theta_k(\pi)$ is cuspidal.*

Note that these results are analogous to those of Rallis [Ral84] in the context of liftings between dual reductive pairs of symplectic-orthogonal type, and of those in [BFG06] for the double cover of odd orthogonal groups. This is the first example of such results for higher degree covers.

Remark 1.2.2. The restriction to the 4-fold cover in the theorem is not for convenience, but is necessary for the construction to be well-behaved. For example, while parts (1) and (2) hold for higher degree covers, part (3) of this theorem intimately relies on Theorem 1.2.4 stated below, which does not hold for the $2n$-th degree cover when $n > 2$. See the comments after Theorem 1.2.4 below. □

In [Art89], Arthur conjectures for a connected reductive group G over K a precise decomposition of the discrete spectrum $L^2_{disc}(\mathrm{G}(K)\backslash \mathrm{G}(\mathbb{A}))$ in terms of Arthur parameters

$$\Psi : L_K \times \mathrm{SL}_2(\mathbb{C}) \to {}^L\mathrm{G}\,.$$

Here L_K is the conjectural Langlands group attached to K, and ${}^L\mathrm{G} = \mathrm{G}^\vee \rtimes W_F$ is the Langlands L-group associated to G. Understanding parameters Ψ, called non-tempered parameters, which are nontrivial when restricted to $\mathrm{SL}_2(\mathbb{C})$ is related to the classification of non-tempered cuspidal representations of $\mathrm{G}(\mathbb{A})$. These representations, which are called CAP representations, provide counterexamples to the naive generalization of Ramanujan's conjecture to groups other than GL_n.

As a corollary of Theorem 1.2.1, for a given cuspidal representation π there is a first index $n(\pi)$, where $\Theta(\pi) := \Theta_{n(\pi)}(\pi)$ is a non-zero cuspidal representation. One of the main results of the paper is that the representation $\Theta(\pi)$ is a CAP representation.

Theorem 1.2.3. *Suppose π is a cuspidal representation of $\overline{\mathrm{Sp}}(W_0)$, and suppose $\Theta(\pi) := \Theta_{n(\pi)}(\pi)$ is the first nontrivial lift. Then each irreducible summand of $\Theta(\pi)$ is CAP with respect to the triple $(Q_{t(\pi),n}, \pi, \chi_\Theta)$.*

The precise notations and results are stated in Section 8. While CAP representations have previously been studied for $\mathrm{Mp}(W_0)$ (see [Yam16], for example), this constitutes the first construction of CAP representations on higher degree covering groups. Given the connection between Arthur's parametrization of the discrete automorphic spectrum and CAP representations, it is natural to ask if our construction may be understood in terms of non-tempered A-parameters, which may be conjecturally defined in terms of Weissman's L-group.

Motivated by a natural unramified correspondence between $\overline{\mathrm{Sp}}_{2r}$ and SO_{2r+1}, we conjecture in Section 8.2 an extension of Arthur's parameterization of the discrete spectrum to $\overline{\mathrm{Sp}}_{2r}$. Such a result has only recently been achieved in the case of Mp_{2r} [GI17], and relies heavily on the theta lifting associated to ω_ψ. Assuming such a parameterization, in Section 8.3 we compute the effect of our lift on Arthur parameters. This may be viewed simultaneously as evidence of our conjectural parameterization as well as an extension of Arthur's conjectures to BD-covers using the definition of L-group provided by Weissman. We remark here that this computation must be understood in light of the fact that local Arthur packets have yet to be defined for any covering group aside from $\mathrm{Mp}(W)$ [Li11]. Moreover, we do not know how to formulate the conjectural multiplicity formula for covering groups, so in a strong sense, we are not yet able to fully generalize Arthur's conjectures to include covering groups.

As indicated above, the properties of this theta lifting rely heavily on the "smallness" of Θ. More precisely, we need to compute the maximal nilpotent orbit $\mathcal{O}(\Theta)$ associated to Θ. Recall that nilpotent orbits of $\mathfrak{sp}_{2r}(\mathbb{C})$ may be parametrized by certain partitions $(p_1^{e_1} \cdots p_k^{e_k})$ of $2r$, and form a partially ordered set under $\mathcal{O} \leq \mathcal{O}'$ if $\mathcal{O} \subset \overline{\mathcal{O}'}$. We say that a nilpotent orbit $\mathcal{O}$ supports an automorphic representation π if certain Fourier coefficients attached to $\mathcal{O}$ do not vanish on π. For details, see Chapter 4.

Utilizing the notation from [Gin14b], we define $\mathcal{O}(\pi)$ to be the set of orbits maximal with respect to the partial order supporting π. Assuming $\mathcal{O}(\pi)$ is a single nilpotent orbit, the Gelfand-Kirillov dimension of π is given by

$$\dim(\pi) = \frac{1}{2} \dim_{\mathbb{C}}(\mathcal{O}(\pi)).$$

For example, π is generic if and only if the *regular* nilpotent orbit supports π if and only if $\dim(\pi) = r^2$. The representation ω_ψ on $\mathrm{Mp}(W)_{\mathbb{A}}$ is *minimal* in the sense that only the minimal nilpotent orbit supports it; this is equivalent to $\dim(\omega_\psi) = r$.

The results of Sections 5.1 and 5.2 may be stated as follows:

Theorem 1.2.4. *Let Θ_{2r} denote a theta representation of $\overline{\mathrm{Sp}}_{2r}(\mathbb{A})$. Let $\mathcal{O}_{\Theta,r} = (2^r)$.*

1. *$\mathcal{O}_{\Theta,r}$ supports the representation Θ_{2r}.*
2. *If $\mathcal{O}'$ is any nilpotent orbit greater than or not comparable to $\mathcal{O}_{\Theta,r}$, then $\mathcal{O}'$ does not support the representation Θ_{2r}.*

In particular, we have that

$$\dim(\Theta_{2r}) = \frac{r(r+1)}{2}.$$

This result is interesting in its own right as it is an important question in the non-generic case to understand the nilpotent orbits associated to theta representations [BFG03, Cai16]. In our context, this identification is crucial to the proofs of

Theorems 1.2.1 and 1.2.3, and the proof of this theorem occupies Chapter 5. We note in Appendix A.1 that if one sought to study the corresponding lift in the 2-fold covering case, the minimality of ω_ψ forces $\Theta_m(\pi) = 0$ for all m. In this sense, the oscillator representation is "too small" for this lift.

Remark 1.2.5. It is natural to hope that this result may be extended to the $2n$-th degree cover. It is an open question what the nilpotent orbit associated to the theta representation $\Theta_{2r}^{(n)}$ on higher degree covers of the symplectic group ought to be. While Friedberg and Ginzburg have presented a conjecture in the case of odd-degree covers [FG17], this is still open and there are fundamental obstructions to verifying this conjecture. See the comments after Proposition 5.2.3. In any case, producing an analogous conjecture for even degree covers does not seem straightforward. We remark that the proof of Theorem 1.2.4 (1) shows that for $n > 2$, we have

$$\mathcal{O}\left(\Theta_{2r}^{(2n)}\right) \geq (2^r),$$

and in general the inequality is strict. In particular, these representations are too large for the integral lift considered here, so one must consider different constructions if one hopes to develop a generalization of this lift to higher degree covers. We expect a variation on the descent constructions of Friedberg-Ginzburg [FG17] should replace the theta kernel used in our construction, though this will change the nature of the local analysis and still requires knowing $\mathcal{O}(\Theta_{2r}^{(2n)})$. □

Our construction, as well as the value of $\mathcal{O}(\Theta_{2r})$, is motivated by Ginzburg's dimension equation, which heuristically relates first occurrences of liftings defined in terms of integral pairings such as ours to a certain equality relating the dimensions of the groups being integrated over to the Gelfand-Kirillov dimension of certain representations. For more details, see [Gin14b] and Chapter 9. In our case, the di-

mension equation suggests that in the case that if π is cuspidal and generic, then $n(\pi) = \frac{1}{2}\dim(W) + 1$. In Chapter 9, we provide evidence for this conjecture by relating the Whittaker coefficients of the lift to those of π.

The remainder of this thesis is organized as follows:

After fixing notation in Chapter 2, we recall the definition of BD-covering groups. Chapter 3 develops the local and global theory of theta representations in this language, with emphasis on the role of the dual group $\overline{\mathrm{G}}^{\vee}$ as we will be interested in an explicit Satake map in Chapter 6. Chapter 4 recalls the theory of Fourier coefficients and Jacquet modules associated to a nilpotent orbit $\mathcal{O}$, and in Section 4.2 we define the local Fourier-Jacobi modules associated to specific orbits. Importantly, Corollary 4.2.4 allows us to relate vanishing of local Fourier-Jacobi modules to the associated Jacquet module. The study of these modules is our main tool to prove vanishing of certain global Fourier coefficients. In Section 4.3, we recall the tool (both locally and globally) of root exchange.

As stated above, Chapter 5 is dedicated to the proof of Theorem 1.2.4. In Section 5.1, we prove Theorem 1.2.4 (2). The method of proof combines in a new way global results of [GRS03] and [JL15] with a detailed study of local Fourier-Jacobi modules to force vanishing of the relevant Fourier coefficients. We prove Theorem 1.2.4 (1) in Section 5.2 in two stages. In the even rank case, we show that the non-vanishing of the certain Fourier coefficients of Θ_{4r} associated to $\mathcal{O}_{\Theta,2r} = (2^{2r})$ is equivalent to the non-vanishing of certain semi-Whittaker coefficients of theta functions on the double cover of $\mathrm{GL}_{2r}(\mathbb{A})$, which are non-zero by results of Bump-Ginzburg [BG92]. We then show that this forces non-vanishing in the odd rank case by inspecting the restriction of certain coefficients of theta functions to the root subgroup of the highest root.

In Chapter 6, we study the corresponding local lifting in the case of principal series

representations. Despite the lack of a Schrödinger model, we establish the existence of and analyze a filtration of certain Jacquet modules of the local theta representation Θ (see Appendix A.1). Using the local results obtained in the proof of Theorem 1.2.4, we explicate the effect of the theta lift on Satake parameters. See Theorem 6.1.1. This is the technical backbone of Theorem 1.2.3.

Having established the relevant global and local results, we turn to the proofs of Theorem 1.2.1 and Theorem 1.2.3 in Chapter 7. More specifically, Theorem 7.1.1 of Sections 7.1 establishes Theorem 1.2.1 (2). The proof utilizes ideas of Rallis and Ikeda showing that for $n \gg 0$, Fourier coefficients of Θ may be approximated with arbitrary precision by certain coefficients of the lifted representation $\Theta_n(\pi)$. Our knowledge of the Fourier coefficients of Θ thus concludes that the lifted representation must have been non-trivial. In 7.2, we prove Theorem 1.2.1, parts (1) and (3).

In Section 8, we prove Theorem 1.2.3 establishing that the lift produces CAP representations, and conjecture a natural extension of Arthur's parametrization of the discrete spectrum to the case of $\overline{\mathrm{Sp}}_{2r}(\mathbb{A})$. We then relate these results to Arthur's conjectures by computing the effect of the generalized theta lift on Arthur parameters. In particular, we show that when one combines the natural lifting of unramified data from $\overline{\mathrm{Sp}}_{2r}$ to SO_{2r+1} with Theorem 6.1.1, we may determine the (non-tempered) Arthur parameter of the lift.

In Chapter 9, we review the notion of Gelfand-Kirillov dimension and discuss Ginzburg's philosophy of the dimension equation [Gin14b] and use it to study the case of a generic cuspidal representation π of $\overline{\mathrm{Sp}}_{2r}$. We show that the Whittaker coefficient of the lift of such a representation to $\overline{\mathrm{Sp}}_{2r+2}$ may be expressed in terms of the Whittaker coefficients of π as well as a certain integral over the theta function, supporting the conjecture that $n(\pi) = r + 1$ in this case.

Chapter 2

Notations and Preliminaries

We will work both locally and globally. To distinguish, we let K stand for a number field which contains the full set of 4^{th} roots of unity, and we use F to denote a non-archimedean local field. Often F will arise as the completion of our number field K with respect to some finite place, $F = K_\nu$. Fix once and for all an injective character $\epsilon : \mu_4 \to \mathbb{C}^\times$. Let $\mathbb{A}$ be the associated ring of adeles for the number field K, and fix a non-trivial additive character $\psi : K\backslash\mathbb{A} \to \mathbb{C}^\times$. Locally, we focus mainly on the case that F is a local field of residue characteristic $p > 2$. In this setting, we fix a nontrivial additive character $\psi : F \to \mathbb{C}^\times$ of level 0.

2.1 Subgroups and characters

Let $(W, \langle\cdot,\cdot\rangle)$ be a $2r$-dimensional symplectic space over F, and let $\mathrm{Sp}(W)_F$ be the group of isometries. Upon fixing a choice of basis for $W \cong F^{2r}$, we may identify $\mathrm{Sp}(W)_F \cong \mathrm{Sp}_{2r}(F)$. When we do so, we always use the symplectic form given by

$$\begin{pmatrix} & -J_r \\ J_r & \end{pmatrix}, \tag{2.1}$$

where

$$J_r = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}. \tag{2.2}$$

Note that J_r corresponds to a split quadratic form Q_J on F^r. We denote the corresponding set of symmetric matrices by

$$\mathrm{Sym}_J^r(F) = \{l \in \mathrm{Mat}_{r\times r} : J_r l - {}^T l J_r = 0\}.$$

With this form, we have the maximal torus T_r of diagonal elements, and Borel subgroup $B_r = T_r U_r$ of upper triangular matrices. Here, U_r is the unipotent radical of B_r. This choice gives us the (standard) based root datum

$$(X_r, \Phi_r, \Delta_r, Y_r, \Phi_r^\vee, \Delta_r^\vee),$$

where $X_r = X^*(T_r)$ is the character lattice and $Y_r = X_*(T_r)$ is the cocharacter lattice. We label the associated simple roots as in Bourbaki: $\Delta = \{\alpha_1, \cdots, \alpha_r\}$, with α_r the unique long simple root. Also, let $W(\mathrm{Sp}_{2r})$ denote the Weyl group of Sp_{2r}, with w_i the standard representative of the Weyl reflection for the simple root α_i.

We need to also set notations for various parabolic subgroups and characters: Let $\underline{s} = (p_1, \ldots, p_k, r')$ be a partition of r. Set $P_{\underline{s}} = M_{\underline{s}} U_{\underline{s}}$ to be the standard parabolic subgroup of Sp_{2r} with Levi factor

$$M_{\underline{s}} \cong \mathrm{GL}_{p_1} \times \cdots \times \mathrm{GL}_{p_k} \times \mathrm{Sp}_{2r'}\,.$$

To ease the notation, we make a separate notation for certain parabolic subgroups. Namely, set $Q_{k,r-k} = P_{(1,\cdots,1,r-k)}$ to be the parabolic subgroups $Q_{k,r-k} = L_{k,r-k} V_{k,r-k}$ such that

$$L_{k,r-k} \cong \mathrm{GL}_1^k \times \mathrm{Sp}_{2(r-k)}\,.$$

We will have cause to consider two main types of character of $V_{k,r-k}$. We set $\psi_k : V_{k,r-k} \to \mathbb{C}^\times$ to be the Gelfand-Graev character

$$\psi_k(v) = \psi(v_{1,2} + v_{2,3} + \cdots + v_{k,k+1}),$$

and we set $\psi_k^0 : V_{k,r-k} \to \mathbb{C}^\times$ to be the character

$$\psi_k^0(v) = \psi(v_{1,2} + v_{3,4} + \cdots v_{k_0,k_0+1}),$$

where k_0 is the largest odd integer $\leq k$.

For each $k \leq r$, one has the subgroup

$$H_k \hookrightarrow U_r$$

given by

$$h(x:y:z) := \begin{pmatrix} I_{k-1} & & & & & & \\ & 1 & x & y & z & & \\ & & I_{r-k} & & y^* & & \\ & & & I_{r-k} & x^* & & \\ & & & & 1 & & \\ & & & & & I_{k-1} & \end{pmatrix} \qquad x, y \in \mathbb{G}_a^{r-k},\ z \in \mathbb{G}_a. \quad (2.3)$$

We have an isomorphism $H_k \cong \mathcal{H}_{2(r-k)+1}$, the Heisenberg group on $2(r-k)+1$ variables. Note that the center of H_k, $Z(H_k)$, is the one dimensional root subgroup corresponding to the long root μ_k (see the next subsection).

We will often need to integrate over various spaces of the form $H(K)\backslash H(\mathbb{A})$ for some K-subgroup $H \subset \mathrm{Sp}_{2r}$. We adopt the shorthand $[H] = H(K)\backslash H(\mathbb{A})$.

2.2 Formalism on roots

We work extensively with the root system $\Phi = \mathrm{C}_r$, so we take a moment to introduce a simple notation for the 3 types of root in this system. Recall that we label the simple roots $\Delta = \{\alpha_1, \cdots, \alpha_r\}$, with α_r the unique long simple root.

For $1 \leq i \leq k \leq r-1$, set

$$\gamma_{i,k} = \alpha_i + \cdots + \alpha_k.$$

These parametrize our short roots within the sub-root system of type A_{r-1}, corresponding to the Siegel parabolic. For the long roots, we set

$$\mu_k = 2\alpha_k \cdots + 2\alpha_{r-1} + \alpha_r, \quad \mu_r = \alpha_r.$$

The positive roots not listed above are of the form

$$\eta_{i,k} = \gamma_{i,k} + \mu_{k+1}.$$

Chapter 3

Theta Representations: Local and Global Theory

In this chapter, we develop the local and global theory of theta representations. In the local setting we choose to work in the language of Brylinski-Deligne covers. The primary reason for this is to connect in Chapter 7 with the L-group of Weissman. After recalling the notions of exceptional characters and the associated theta representations, we specialize to the case of primary interest: the four-fold cover of symplectic groups. We end our local discussion by stating properties about the Jacquet modules of these representations.

We then recall the definition of global theta representations as residues of certain Borel Eisenstein series, and recall results of Friedberg and Ginzburg [FG17] on the constant terms of global theta functions.

3.1 Theta Representations: local theory

Let F be a local field equipped with a discrete valuation. In this section, we will develop the theory of local theta representations for $\overline{\mathrm{Sp}}_{2r}(F)$.

3.1.1 Brylinski-Deligne Extensions

We develop the notion of local theta (or exceptional) representations to facilitate later comparison with the work of Weissman.

Assume for now that F contains the full set of n^{th} roots of unity, and that $|n|_F = 1$ (this is what Gan-Gao refer to as the tame case [GG14]). Suppose that $\mathbb{G}$ is a connected reductive group scheme over F, which for simplicity we assume is split, semi-simple, and simply-connected. For the more general set up, we refer the reader to [Gao16].

We fix a pinning $(\mathbb{G}, \mathbb{T}, \mathbb{B}, \{x_\alpha\}_{\alpha\in\Delta})$ of $\mathbb{G}$, and consider the corresponding to the based root datum

$$(X, \Phi, \Delta, Y, \Phi^\vee, \Delta^\vee).$$

Here, $Y = \mathrm{Hom}(\mathbb{G}_m, \mathbb{T})$ is the cocharacter lattice and $X = \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ is the character lattice. Recall that $W = N(\mathbb{T})/\mathbb{T}$ is the Weyl group of $\mathbb{G}$.

Let $\mathbb{K}_2$ be Quillen K_2-sheaf on the big Zariski site over $\mathrm{Spec}(F)$, and set $\mathrm{CExt}(\mathbb{G}, \mathbb{K}_2)$ to be the category of central extensions of $\mathbb{G}$ by $\mathbb{K}_2$ as sheaves over $\mathrm{Spec}(F)$. With our restriction on $\mathbb{G}$, then Brylinski and Deligne give a combinatorial description of $\mathrm{CExt}(\mathbb{G}, \mathbb{K}_2)$ (reminiscent of the classification of split reductive groups in terms of root data) in terms of triples $(Q, \mathcal{E}, f)$. Here, Q is a W-invariant $\mathbb{Z}$-valued quadratic form on Y, $\mathcal{E}$ is a central extension of Y by $F^\times$, and f is a certain morphism of

abelian groups which is more subtle. See the exposition in [GG14] for more details, along with the parallel parameterization in terms of pairs (D, η) where D developed by Weissman [Wei14]. Note that if $\mathbb{G}$ is semi-simple and simply connected, then we may ignore the more subtle data of f (or η) in the classification.

If we further assume that $\mathbb{G}$ is simple and simply connected, then any W-invariant quadratic form $Q : Y \to \mathbb{Z}$ is completely determined by its value on $\alpha^\vee$, where $\alpha \in \Delta$ is any long simple root. In other words,

$$\mathrm{CExt}(\mathbb{G}, \mathbb{K}_2) \cong \mathbb{Z},$$

where we view $\mathbb{Z}$ as a discrete category. In this case, we will always choose the quadratic form such that $Q(\alpha^\vee) = -1$ for any short coroot $\alpha^\vee$.

For simplicity, we assume $\mathbb{G}$ is semi-simple and simply connected for the remainder of the section and fix a central extension $\overline{\mathbb{G}} \in \mathrm{CExt}(\mathbb{G}, \mathbb{K}_2)$ corresponding to the quadratic form Q. By taking F-points, we have a short exact sequence of F-groups

$$1 \longrightarrow K_2(F) \longrightarrow \overline{\mathbb{G}}(F) \longrightarrow \mathrm{G} \longrightarrow 1,$$

where $\mathrm{G} = \mathbb{G}(F)$. One of the main properties of classical algebraic K_2 is that the n^{th} Hilbert symbol $(\cdot, \cdot)_n$ gives a homomorphism

$$(\cdot, \cdot)_n : K_2(F) \to \mu_n(F).$$

Pushing out the above short exact sequence by this morphism gives rise to the exact sequence of topological groups

$$1 \longrightarrow \mu_n(F) \longrightarrow \overline{\mathrm{G}}^{(n)} \longrightarrow \mathrm{G} \longrightarrow 1,$$

which by the assumptions on F realizes the covering group $\overline{\mathrm{G}}^{(n)}$ as a degree n covering group. This covering group is determined by the data (Q, n), though not uniquely so. For any subset $S \subset \mathrm{G}$, we denote its full inverse image under the covering map by $\overline{S}$.

To the covering group $\overline{\mathrm{G}}^{(n)}$, one may associate a complex dual group which may be distinct from the dual of G (see [McN12], [FL10], and [Rei12]). Let $B_Q : Y \times Y \to \mathbb{Z}$ be the symmetric bilinear form associated to the quadratic form Q. To the data (Q, n), we may associate the sublattice

$$Y_{Q,n} = \{y \in Y : B_Q(y, y') \in n\mathbb{Z} \text{ for all } y' \in Y\} \subset Y.$$

For each $\alpha^\vee \in \Psi^\vee$, we define $n_\alpha = n/\gcd(n, Q(\alpha^\vee))$. Set

$$\alpha^\vee_{Q,n} = n_\alpha \alpha^\vee, \quad \alpha_{Q,n} = \frac{1}{n_\alpha}\alpha.$$

Set $Y^{sc}_{Q,n} = \mathrm{span}_{\mathbb{Z}}\{\alpha^\vee_{Q,n} : \alpha \in \Phi\}$. Then the complex dual group $\overline{\mathrm{G}}^\vee$ of $\overline{\mathrm{G}}^{(n)}$ is the split group associated to the root datum

$$\left(Y_{Q,n}, \{\alpha^\vee_{Q,n}\}, \mathrm{Hom}(Y_{Q,n}, \mathbb{Z}), \{\alpha_{Q,n}\}\right).$$

As $\overline{\mathrm{Sp}_{2r}}(F) := \overline{\mathrm{Sp}}^{(4)}_{2r}(F)$ is the covering group of primary interest for this paper, we remark that in this case $\overline{\mathrm{G}}^\vee = \mathrm{Sp}_{2r}(\mathbb{C})$.

Finally, we remark that the assumption $|n|_F = 1$ implies that the cover $\overline{K}$ over a fixed hyperspecial maximal compact subgroup $K = \mathbb{G}(\mathfrak{O}_F)$ is split; that is, there exists a homomorphism $\kappa : K \to \overline{\mathrm{G}}^{(n)}$ splitting the cover. Such a splitting need not be unique (in fact, splittings form a $\mathrm{Hom}(K, \mu_n)$-torsor). That being said, we fix a splitting from here on, and nothing shall depend upon this choice.

3.1.2 Principal series and Exceptional representations

We turn now to the theory of principal series representations for the covering group $\overline{\mathrm{G}}^{(n)}$. We follow the setup and notation in [Gao16] closely.

Fix an embedding $\epsilon : \mu_n(F) \to \mathbb{C}^\times$. For any subgroup $H \subset \mathrm{G}$, we say that a representation (ρ, W) of $\overline{H}$ is (ϵ-)genuine if $\rho|_{\mu_n(F)} \equiv \epsilon$.

Let $T = \mathbb{T}(F)$ be the split maximal torus in G contained in $B = \mathbb{B}(F)$, and let $\overline{T}$ be its full inverse image in $\overline{\mathrm{G}}^{(n)}$. In general, $\overline{T}$ is no longer abelian, but rather a 2-step nilpotent group. One can show that the center $Z(\overline{T})$ is given by the full inverse image of the isogeny

$$i : T_{Q,n} = Y_{Q,n} \otimes F^{\times} \to T.$$

Let $\chi : Z(\overline{T}) \to \mathbb{C}^{\times}$ be a genuine character of $Z(\overline{T})$. For any maximal abelian subgroup $A \supset Z(\overline{T})$, we set

$$i(\chi) = \mathrm{ind}_{A}^{\overline{T}}(\chi'),$$

where χ' is any extension of χ to A. A version of the Stone-Von Neumann theorem tells us that $i(\chi)$ is independent of A or the extension χ', and that the map

$$\chi \mapsto i(\chi)$$

induces a bijection between the irreducible genuine representations of $Z(\overline{T})$ and $\overline{T}$. In particular, we see that any irreducible representation of $\overline{T}$ is determined up to isomorphism by its central character. Despite the independence of $i(\chi)$ from A, it is useful to point out that in our setting, we may take $A = Z(\overline{T})(\overline{T} \cap K)$.

Define $\mathrm{Ind}(\chi) := \mathrm{Ind}_{\overline{B}}^{\overline{\mathrm{G}}^{(n)}}(i(\delta_B^{1/2}\chi))$, where δ_B is the modular character of the Borel $B = \mathbb{B}(F)$, which we view as a non-genuine character of $Z(\overline{T})$. These are the principal series representations of $\overline{\mathrm{G}}^{(n)}$. As in the reductive setting, $\mathrm{Ind}(\chi)$ is unramified ($\mathrm{Ind}(\chi)^K \neq \{0\}$) if and only if χ is unramified (that is $\chi|_{Z(\overline{T})\cap K} \equiv 1$). Note that unramified characters are the same as characters of $\overline{Y_{Q,n}}$ where

$$1 \longrightarrow \mu_n \longrightarrow \overline{Y_{Q,n}} \longrightarrow Y_{Q,n} \longrightarrow 1$$

is the induced (abelian) extension of the lattice.

We now turn to the definition of an exceptional character. Following the notation in [Gao16], we have for each root α the natural map $\overline{h}_\alpha : \mathbb{G}_m \to \overline{\mathrm{G}}^{(n)}$, covering the embedding into G induced by our choice of pinning.

Definition 3.1.1. An unramified genuine character $\overline{\chi} : \overline{T} \to \mathbb{C}^\times$ is called *exceptional* if $\overline{\chi}(\overline{h}_\alpha(\varpi^{n_\alpha})) = q^{-1}$ for each simple root $\alpha \in \Delta$.

The set of unramified exceptional characters of $\overline{\mathrm{G}}^{(n)}$ is a torsor over

$$Z(\overline{\mathrm{G}}^\vee) = \mathrm{Hom}(Y_{Q,n}/Y_{Q,n}^{sc}, \mathbb{C}^\times).$$

To each such character, we may associate a theta representation:

Definition 3.1.2. For a exceptional character $\overline{\chi}$, the theta representation $\Theta(\overline{\mathrm{G}}^{(n)}, \overline{\chi})$ associated to $\overline{\chi}$ is the unique Langlands quotient of $\mathrm{Ind}(\overline{\chi})$.

For properties of theta representations, see [Gao16] and [BFG03].

3.1.3 The Case at hand: $G = \mathrm{Sp}_{2r}$ $n = 4$

In this section, we inspect further the case of primary interest in this paper: the 4-fold cover $\overline{\mathrm{Sp}_{2r}}(F)$ of $\mathrm{Sp}_{2r}(F)$. Thus, from this point forward we assume that F is a local field containing the 4^{th} roots of unity.

As before, we choose our quadratic form such that $Q(\alpha_r^\vee) = -1$, where α_r is the unique long simple root. In this case, if we identify $Y = Y^{sc} = \mathrm{span}_{\mathbb{Z}}\{\alpha_i^\vee\} \to \mathbb{Z}^r$ via

$$\sum_i c_i \alpha_i^\vee \mapsto (c_1, c_2 - c_1, \ldots, c_r - c_{r-1}),$$

we obtain (setting $n = 4$)

$$Y_4 := Y_{Q,4} = \{(x_1, \ldots, x_r) : 2|x_i\}, \;\; Y_4^{sc} = \{(x_1, \ldots, x_r) \in Y_4 : 4|\sum_i x_i\}.$$

This shows that $|Z(\overline{G}^\vee)| = 2$, so there are two distinct unramified exceptional characters.

For the sake of explicit computations, we find it useful to work with an explicit cocycle. To this end, we will utilize the bisector $D : Y \otimes Y \to \mathbb{Z}$ given by

$$D(\alpha_i^\vee, \alpha_j^\vee) = \begin{cases} B_Q(\alpha_i^\vee, \alpha_j^\vee) & : i < j \\ Q(\alpha_i^\vee) & : i = j \\ 0 & : i > j \end{cases}.$$

With this choice of bisector, we may compute explicitly the cocycle on the maximal torus. Since the map $H^2(\mathrm{Sp}_{2r}(F), \mu_4) \to H^2(T_r, \mu_4)$ is injective, we see that this does in fact determine the cover completely.

Let us identify $\overline{\mathrm{Sp}}_{2r}(F) = \mathrm{Sp}_{2r}(F) \times \mu_4$, and parametrize T_r via

$$M(t_1, \dots, t_r) = \mathrm{diag}(t_1, \dots, t_r, t_r^{-1}, \dots, t_1^{-1}).$$

Then, our bisector induces the cocycle σ_D given by

$$\sigma_D(M(t_1, \dots, t_r), M(s_1, \dots, s_r)) = \prod_{k=1}^{r} (t_k, s_k)_4^{-1},$$

where $(\cdot, \cdot)_n$ is the n^{th} power Hilbert symbol.

Remark 3.1.3. Note that this choice of bisector (since Sp_{2r} is simply connected, we may set $\eta = 1$; see [GG14]), determines our 4-fold covering group up to isomorphism and identifies it as the pullback along the standard embedding $\mathrm{Sp}_{2r} \hookrightarrow \mathrm{SL}_{2r}$ of the 4-fold cover of $\mathrm{SL}_{2r}(F)$ with the BLS block compatible cocycle (see [BLS99]).

Recall the isogeny of tori

$$i : T_{Q,4} \to T_r.$$

An observation made clear by our choice of bisector is that if $t, t' \in i\,(T_{Q,4})$, then $\sigma_D(t, t') = 1$. This implies that there is a bijection between characters of $i(T_{Q,4})$

and genuine characters of $Z(\overline{T})$. In particular, the genuine unramified characters of $Z(\overline{T})$ are in bijection with characters of the lattice Y_4. The bijection is not, however, canonical.

In [GG14], Gan-Gao introduce the idea of a genuine distinguished character $\overline{\chi}_0$ of $Z(\overline{T})$. We do not recall the full definition, but wish to clarify the case of $\overline{\mathrm{Sp}}_{2r}(F)$ as the choice of distinguished character affects the Satake parameter attached to an unramified principal series representation. While there are 4 distinct distinguished characters for $\overline{\mathrm{Sp}}_{2r}(F)$, only two of them, $\overline{\chi}^0_+$ and $\overline{\chi}^0_-$, are unramified. Here, setting $y_i = 2\alpha_i^\vee$, these characters are determined by

$$\overline{\chi}^0_\pm(y_i(a),\zeta) = \begin{cases} \zeta & : \quad i < r \\ \\ \pm\zeta(a,a)_2 & : \ i = r. \end{cases}$$

Here $(\cdot,\cdot)_2$ is the quadratic Hilbert symbol. and $\psi : F \to \mathbb{C}^\times$ is our fixed additive character. Note that the distinguished character $\overline{\chi}^0_+$ equals the unitary distinguished character $\overline{\chi}^0_\psi$ constructed in [GG14]. One can show that in the $4k$-fold covering case, the dependence on the choice of character $\psi : F \to \mathbb{C}^\times$ disappears. We note that if we were to further assume $\mu_8 \subset F^\times$, then $(a,a)_2 = 1$, and $\overline{\chi}^0_+$ becomes essentially trivial. In any case, precisely one of these characters agrees with the choice of distinguished character implicit in [FG17].

We fix a choice $\overline{\chi}^0$ of unramified distinguished character to obtain our bijection,

$$\chi \mapsto \overline{\chi}^0\chi,$$

for any unramified character $\chi : i(T_{Q,4}) \to \mathbb{C}^\times$, viewed on the right hand side as a non-genuine character of $Z(\overline{T})$. This choice also gives us our exceptional character as follows. Consider the character

$$\rho_4 = \sum_{i\in\Delta} \omega_i^\vee \in \mathrm{Hom}(Y_{Q,n},\mathbb{Z}) = X_*(\overline{T}^\vee),$$

the sum of fundamental coweights of the dual group $\overline{G}^{\vee}$. Setting $\chi_{\rho_4} : Y_4 \otimes F^{\times} \to \mathbb{C}^{\times}$ as the corresponding unramified character, we have our chosen genuine unramified exceptional character

$$\overline{\chi} = \overline{\chi}^0 \chi_{\rho_4}.$$

This choice will become important in Chapter 6 as it fixes a notion of Satake parameters relative to a distinguished splitting of the L-group ${}^L\overline{\mathrm{Sp}}_{2r}$ (see [Wei14], [Wei15]). Set $\Theta_{2r} := \Theta(\overline{\mathrm{Sp}}_{2r}, \overline{\chi})$ to be the corresponding theta representation.

3.1.4 Jacquet modules of local theta representations

Consider the maximal parabolic subgroup $P_{(k,r-k)}(F) = M_{(k,r-k)}(F)U_{(k,r-k)}(F)$, and note that, by a simple computation using the block compatibility of the cocycle, the induced cover

$$\overline{M}_{(k,r-k)}(F) = \overline{\mathrm{GL}}_k^{(2)}(F) \times_{\mu_4} \overline{\mathrm{Sp}}_{2(r-k)}(F)$$

is given by the direct product amalgamated at μ_4 of a metaplectic cover $\overline{\mathrm{GL}}_k^{(2)}(F)$ corresponding to the BD-covering group $\widetilde{\mathbb{GL}}_k[-1,0]$ (see the notation in [Gao18a, Section 2.1]) and the 4-fold metaplectic cover $\overline{\mathrm{Sp}}_{2(r-k)}(F)$.

Remark 3.1.4. We denote the corresponding cover of the general linear group by $\overline{\mathrm{GL}}_k^{(2)}(F)$ despite the fact that it is a four-fold cover since it is the determinantal twist of the four-fold cover associated to the BD-covering group $\widetilde{\mathbb{GL}}_k[0,2]$. The associated four-fold covering group $\overline{\mathrm{GL}}_k^{(4)}(F)_S$[1] is the pushout of the double cover $\overline{\mathrm{GL}}_k^{(2)}(F)_{KP}$

[1]The S stands for Savin, per [Gao18a].

of the Kazhdan-Patterson cover $\widetilde{\mathbb{GL}_k}[0,1]$; that is, we have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_2 & \longrightarrow & \overline{\mathrm{GL}}_k^{(2)}(F)_{KP} & \longrightarrow & \mathrm{GL}_k(F) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow{=} & & \\ 1 & \longrightarrow & \mu_4 & \longrightarrow & \overline{\mathrm{GL}}_k^{(4)}(F)_S & \longrightarrow & \mathrm{GL}_k(F) & \longrightarrow & 1. \end{array}$$

It follows that the category of genuine smooth admissable representations of $\overline{\mathrm{GL}}_k^{(2)}(F)_{KP}$ is equivalent to the category of ϵ-genuine smooth admissable representations of $\overline{\mathrm{GL}}_k^{(4)}(F)_S$, and for our purposes the determinantal twist does not complicate things.[2] This is useful for translating results about theta representations on the double cover to this covering group.

Under the inclusion of covering tori

$$Z(\overline{T}_{\mathrm{GL}_k}) \times_{\mu_4} Z(\overline{T}_{r-k}) \hookrightarrow Z(\overline{T}_r),$$

our choice of distinguished character $\overline{\chi}^0$ induces a distinguished character on both factors, and either choice of character gives the trivial distinguished character for $Z(\overline{T_{\mathrm{GL}_k}})$. Let $\Theta_{2(r-k)}$ be the corresponding theta representation of $\overline{\mathrm{Sp}}_{2(r-k)}(F)$. Additionally, define the unramified exceptional character of $Z(\overline{T}_{\mathrm{GL}_k})$:

$$\chi_{\mathrm{GL},k}(\mathrm{diag}(t_1,\dots,t_k),\zeta) = \zeta \prod_{i=1}^{k} |t_i|^{(2(r-i)+3)/4} \tag{3.1}$$

Of central importance to the study of theta representations on symplectic groups is the following version of the "periodicity theorem" of [KP84] and its corollary (see also [BFG03] for a version in the odd orthogonal case):

Theorem 3.1.5. *Let F be a non-archimedean local field, and let Θ_{2r} be the theta representation on $\overline{\mathrm{Sp}}_{2r}(F)$. Let $\Theta^{(2)}_{\mathrm{GL}_k}$ denote the local theta representation on $\overline{\mathrm{GL}}_a^{(2)}(F)$*

[2] This is essentially contained in the cocycle computation of Friedberg-Ginzburg [FG17, Section 2].

associated to the exceptional character $\chi_{\mathrm{GL},k}$. *Then for any maximal parabolic subgroup* $P_{(k,r-k)}$, *we have an isomorphism of* $\mathrm{GL}_k^{(2)}(F) \times_{\mu_4} \overline{\mathrm{Sp}}_{2(r-k)}(F)$ *representations*

$$J_{U_k}(\Theta_{2r}) \cong \Theta^{(2)}_{GL_k} \otimes \Theta_{2(r-k)}.$$

Proof. This follows as in [BFG03], Theorem 2.3. Note, however, that the exceptional character for the GL-component of the Levi subgroup differs here from the character in the odd orthogonal case by multiplication by the character

$$g \mapsto |\det(g)|^{3/4}.$$

This is allowed as the definition of exceptional character depends only on its values on the image of the root subgroups. □

Corollary 3.1.6. *There exists a* $\overline{\mathrm{Sp}}_{2r}(F)$*-equivariant embedding*

$$\Theta_{2r} \hookrightarrow \mathrm{Ind}\left(\Theta^{(2)}_{GL_k} \otimes \Theta_{2(r-k)}\right).$$

Additionally, there exists a value $s_0 \in \mathbb{C}$ *such that there is a* $\overline{\mathrm{Sp}}_{2r}(F)$*-equivariant surjection*

$$\mathrm{Ind}\left(\Theta^{(2)}_{GL_k} \otimes \Theta_{2(r-k)} \otimes \delta_{P_k}^{s_0}\right) \twoheadrightarrow \Theta_{2r}.$$

3.2 Theta Representations: global theory

Let K be a number field containing the n^{th} roots of unity. In this section, we develop the necessary global theory of theta representations following the notation of [Gao17]. The picture is similar to the local one discussed in Section 3.1.2.

Let G be a simple and simply-connected group over K. We fix a choice of maximal split torus T, Borel subgroup B containing T, and pinning as before. The theory of

Brylinski and Deligne works similarly to the case of a local field. Fix a K_2-covering $\overline{\mathrm{G}}$, and let $\overline{\mathrm{G}}_{\mathbb{A}}^{(n)}$ be the induced topological covering group

$$1 \longrightarrow \mu_n(K) \longrightarrow \overline{\mathrm{G}}_{\mathbb{A}}^{(n)} \longrightarrow \mathrm{G}_{\mathbb{A}} \longrightarrow 1.$$

By [Wei16], $T_K \cdot Z(\overline{T}_{\mathbb{A}})$ is a maximal abelian subgroup of the covering torus $\overline{T}_{\mathbb{A}}$. Letting

$$\chi = \prod_{\nu} \chi_\nu : Z(\overline{T}_{\mathbb{A}}) \longrightarrow \mathbb{C}^\times$$

be a genuine quasi-character trivial on $T_K \cap Z(\overline{T}_{\mathbb{A}})$, we may inflate χ to an automorphic character on $T_K \cdot Z(\overline{T}_{\mathbb{A}})$. For each root $\alpha \in \Phi$, we obtain a character $\chi_\alpha : \mathbb{A}^\times \longrightarrow \mathbb{C}^\times$ given by

$$\chi_\alpha((t_\nu)) = \prod_{\nu} \chi_\nu(\overline{h}_\alpha(t_\nu^{n_\alpha})).$$

This is trivial on $K^\times$. Similarly to the local picture, we define the notion of a global exceptional character

Definition 3.2.1. An automorphic character $\chi : T_K \cdot Z(\overline{T}_{\mathbb{A}}) \longrightarrow \mathbb{C}^\times$ is called *exceptional* if for each $\alpha \in \Phi$ we have $\chi_\alpha = |\cdot|_{\mathbb{A}}$.

Such characters will correspond to reducibility points for certain Eisenstein series, as we now explain.

Fix our automorphic character χ. As in the local setting, we may form the induced representation $I(\chi)$ be inducing in stages to $\overline{T}_{\mathbb{A}}$ and then parabolically inducing to $\overline{\mathrm{G}}_{\mathbb{A}}^{(n)}$. For $f \in I(\chi)$, define the Eisenstein series associated to f,

$$E(f, \chi, g) = \sum_{\gamma \in B(K)\backslash G(K)} f(\gamma g), \quad \text{for } g \in \overline{\mathrm{G}}_{\mathbb{A}}^{(n)}$$

To make sense of this sum, we remark that the space of genuine automorphic characters χ possesses an analytic structure. The Eisenstein series is convergent in a

particular cone, and possesses meromorphic continuation to the entire space of such characters. We remark that exceptional characters are not contained in the cone of characters on which the above sum converges.

The Gindikin-Karpelevich formula for covering groups ([McN16] and [Gao18b]), along with the theory of global intertwining integrals, tells us that the Eisenstein series has a multi-pole at exceptional characters. For such an exceptional character, we define theta functions as the corresponding residues

$$\theta(f,\chi,g) := \lim_{\chi' \longrightarrow \chi} \prod_{\alpha \in \Phi} L(\chi'_\alpha)^{-1} E(f,\chi',g),$$

where $L(\chi'_\alpha)$ denotes the Hecke L-function associated to the character χ. Setting $\Theta(\overline{\mathrm{G}}_{\mathbb{A}}^{(n)},\chi)$ to be the automorphic representation of $\overline{\mathrm{G}}_{\mathbb{A}}^{(n)}$ generated by these residues. Then we have $\Theta(\overline{\mathrm{G}}_{\mathbb{A}}^{(n)},\chi) \cong \bigotimes'_\nu \Theta(\overline{\mathrm{G}}_{\nu}^{(n)},\chi_\nu)$, where for almost all finite places $\Theta(\overline{\mathrm{G}}_{\nu}^{(n)},\chi_\nu)$ is a local theta representation associated to the local exceptional character χ_ν, as defined in the previous section. We call $\Theta(\overline{\mathrm{G}}_{\mathbb{A}}^{(n)},\chi)$ a global theta representation.

3.2.1 Restriction to $\overline{\mathrm{Sp}}_{2r}(\mathbb{A})$

We now specialize to the case of the four-fold cover $\overline{\mathrm{Sp}}_{2r}(\mathbb{A})$. For the sake of explicit computations and comparison to much of the literature, we shall fix a choice of global distinguished character $\overline{\chi}^0 : Z(\overline{T}_r)_K \backslash Z(\overline{T}_r)_{\mathbb{A}} \to \mathbb{C}^\times$. This enables us to make explicit a corresponding automorphic exceptional character

$$\overline{\chi}_{\Theta,2r} := \overline{\chi}^0 \chi_\Theta,$$

where

$$\chi_\Theta(M(t_1,\ldots,t_r)) = \prod_{i=1}^{r} |t_i|^{(2(r-i)+1)/4}$$

Let $\Theta_{2r} := \Theta(\overline{\mathrm{Sp}}_{2r}^{(4)}(\mathbb{A}))$ denote the global theta representation with respect to this choice of character. Thus, Θ_{2r} is an irreducible genuine automorphic representation

and if $\overline{\chi}^0 = \prod_\nu \overline{\chi}^0_\nu$,[3] we have

$$\Theta_{2r} = \bigotimes_\nu {}' \Theta_{2r,\nu},$$

where for almost all places $\Theta_{2r,\nu}$ is the local theta representation discussed in Section 3.1.3. Note that in [FG17], Friedberg and Ginzburg work with the case $\overline{\chi}^0_\nu$ is the trivial distinguished character for each finite place ν. However, the set-up is the same for a general $\overline{\chi}^0$.

We take note of the following proposition of [FG17, Prop. 1]: Let $\overline{\chi}_{\mathrm{GL},k}$ denote the global analogue of the exceptional character (3.1) of $\overline{\mathrm{GL}}^{(2)}_k$ from the previous section, and let $\Theta^{(2)}_{\mathrm{GL}_k}$ be the corresponding global theta representation.

Theorem 3.2.2. *Let $\theta_{2r} \in \Theta_{2r}$. Then there exists $\theta_{\mathrm{GL}_k} \in \Theta^{(2)}_{\mathrm{GL}_k}$ and $\theta_{2(r-k)} \in \Theta_{2(r-k)}$ such that for any diagonal $g \in \mathrm{GL}^{(2)}_k(\mathbb{A})$ which lies in the center of the Levi of the parabolic subgroup $\overline{P}_{(k,r-k)}(\mathbb{A})$ and for all unipotent $h \in \overline{\mathrm{Sp}}_{2(r-k)}$ and $v \in \mathrm{GL}^{(2)}_k(\mathbb{A})$, we have*

$$\int_{[U_{(k,r-k)}]} \theta_{2r}(u(gv,h))\,du = \overline{\chi}_{\Theta,k}(g)\theta_{\mathrm{GL}_k}(v)\theta_{2(r-k)}(h).$$

The global analogue of Corollary 3.1.6 also holds. We refer the reader to [FG17] for the details.

[3] For each finite place, $\overline{\chi}^0_\nu$ is automatically distinguished.

Chapter 4

Fourier Coefficients Associated to a Nilpotent Orbit

In this chapter, we outline the process of associating harmonic-analytic data to a nilpotent orbit $\mathcal{O}$ in a complex Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$[1] and a representation π of G. To simplify the exposition, we will discuss only $\mathrm{G} = \mathrm{Sp}_{2r}$. However, as these coefficients are defined globally by integration over certain unipotent subgroups and locally in terms of certain Jacquet functors, all the definitions and statements in this section apply without change to the case of a BD-covering group. These concepts are discussed in more detail in [Gin06], [Mœg96], and [MW87].

We will then discuss the notion of Fourier-Jacobi coefficients, and end with a discussion of the technical tool of global and local root exchange. In Section 4.3.3, we develop the notation we use when applying root exchange in subsequence chapters.

[1] or equivalently, a unipotent orbit in any connected Lie group G such that $\mathrm{Lie}(G) = \mathfrak{g}$.

4.1 Algebraic Setup

Let F be any field, and fix an algebraic closure $\overline{F}$. Also, let $\mathcal{O}$ be a nilpotent orbit in the $\overline{F}$-Lie algebra $\mathfrak{sp}_{2m}$. These orbits are finite in number and are parametrized by symplectic partitions of $2r$; that is, partitions where each odd number occurs with even multiplicity. If the orbit $\mathcal{O}$ corresponds to the partition $(p_1^{e_1}p_2^{e_2}\cdots p_r^{e_r})$ where $p_i \leq p_{i+1}$, we simply write

$$\mathcal{O} = (p_1^{e_1}p_2^{e_2}\cdots p_r^{e_r}).$$

There is a natural partial order on both nilpotent orbits (via closure relations) and partitions (via dominance). The above correspondence identifies these partial orders.

Let $\mathcal{O} = (p_1^{e_1}p_2^{e_2}\cdots p_r^{e_r})$ be such an orbit. To each entry p_i which appears in the partition associated to $\mathcal{O}$ (including mutiplicities), we associate the torus element

$$h_{p_i}(t) = \mathrm{diag}(t^{p_i-1}, t^{p_i-3}, \ldots, t^{3-p_i}, t^{1-p_i}).$$

From, these we may produce a semisimple element $h_{\mathcal{O}}(t) \in \mathrm{Sp}_{2r}(\overline{F}(t))$ by concatenating all the $h_{p_i}(t)$ and then conjugating by a Weyl group element until the entries are such that the powers of t are decreasing. It is a consequence of the fact that $\mathcal{O}$ is a symplectic partition of $2r$ that the resulting semisimple element lies in Sp_{2r}. See [CM93] for more background.

Example 4.1.1. When $n = 3$, and $\mathcal{O} = (41^2)$. In this case

$$h_{(41^2)}(t) = \mathrm{diag}(t^3, t, 1, 1, t^{-1}, t^{-3}).$$

One can check directly that this matrix is symplectic with respect to our chosen symplectic form. □

This data induces a filtration on the unipotent radical of B

$$I \subset \cdots \subset V_2(\mathcal{O}) \subset V_1(\mathcal{O}) \subset V_0(\mathcal{O}) = U,$$

where the terms in the filtration are defined in terms of root subgroups

$$V_l(\mathcal{O}) = \langle x_\alpha(s) \in U : h_\mathcal{O}(t) x_\alpha(s) h_\mathcal{O}(t)^{-1} = x_\alpha(t^k s) \text{ for some } k \geq l \rangle.$$

Also, we define

$$M(\mathcal{O}) = T \cdot \langle x_{\pm\alpha}(s) : h_\mathcal{O}(t) x_\alpha(s) h_\mathcal{O}(t)^{-1} = x_\alpha(s) \rangle.$$

Note that all of these subgroups are defined over $\mathbb{Z}$, and that $P_\mathcal{O} = M(\mathcal{O})V_1(\mathcal{O})$ is a parabolic subgroup of $\mathrm{Sp}_{2r}(F)$.

Example 4.1.2. Continuing with the example of $\mathcal{O} = (41^2)$, we find that

$$V_1((41^2)) = \left\{ \begin{pmatrix} 1 & * & * & * & * & * \\ & 1 & * & * & * & * \\ & & 1 & & * & * \\ & & & 1 & * & * \\ & & & & 1 & * \\ & & & & & 1. \end{pmatrix} \right\}$$

and

$$V_2((41^2)) = \left\{ \begin{pmatrix} 1 & * & * & * & * & * \\ & 1 & & & * & * \\ & & 1 & & & * \\ & & & 1 & & * \\ & & & & 1 & * \\ & & & & & 1. \end{pmatrix} \right\}.$$

We will see below that the Fourier coefficients associated to the orbit $\mathcal{O}$ will be defined as integration against $V_2(\mathcal{O})$. This example shows that $V_2(\mathcal{O})$ is in general not the unipotent radical of a parabolic subgroup. □

Since we are working with an algebraically closed field, $M(\mathcal{O})_{\overline{F}}$ acts via conjugation on the quotient

$$Z_2(\mathcal{O}) := V_2(\mathcal{O})/[V_2(\mathcal{O}), V_2(\mathcal{O})]$$

with a dense orbit. Let $u_{\mathcal{O}} \in V_2(\mathcal{O})$ be a representative of this orbit. Set $L(\mathcal{O})^{alg} \subset M(\mathcal{O})_{\overline{F}}$ to be the stabilizer of $u_{\mathcal{O}}$. While the exact group $L(\mathcal{O})^{alg}$ depends on the choice of representative, its Cartan type does not.

4.1.1 Global Setting and Fourier coefficients

Let K be a number field with adele ring $\mathbb{A}$, and let (π, V) be an automorphic representation of $\mathrm{Sp}_{2r}(\mathbb{A})$. Since $M(\mathcal{O})_{\mathbb{A}}$ acts on $V_2(\mathcal{O})_{\mathbb{A}}$ by conjugation, one obtains an action of $M(\mathcal{O})_K$ on the character group

$$(Z_2(\mathcal{O})_K \backslash Z_2(\mathcal{O})_{\mathbb{A}})^{\vee} \cong Z_2(\mathcal{O})_K.$$

We consider only characters $\psi_{\mathcal{O}} : V_2(\mathcal{O})_K \backslash V_2(\mathcal{O})_{\mathbb{A}} \to \mathbb{C}^{\times}$ such that the connected component of the stabilizer $L(\mathcal{O})_K := \mathrm{Stab}_{M(\mathcal{O}),K}(\psi_{\mathcal{O}})$ has the same Cartan type as $L(\mathcal{O})^{alg}$ after base changing to $\overline{K}$. Such characters are called *generic characters* associated to $\mathcal{O}$. There can exist infinitely many $M(\mathcal{O})_K$-orbits of generic characters.

Definition 4.1.3. Let $\psi_{\mathcal{O}} : V_2(\mathcal{O})_K \backslash V_2(\mathcal{O})_{\mathbb{A}} \to \mathbb{C}^{\times}$ be a generic character associated to $\mathcal{O}$. For a vector $\varphi \in \pi$, we consider the coefficient

$$F_{\psi_{\mathcal{O}}}(\varphi)(g) = \int_{[V_2(\mathcal{O})]} \varpi(vg)\psi_{\mathcal{O}}(v)\, dv.$$

We say that the orbit $\mathcal{O}$ supports π if there exists a choice of data $(\varphi,\ \psi_{\mathcal{O}})$ such that the above integral is not identically zero. Otherwise, we say that $\mathcal{O}$ does not support π. □

Note that $F_{\psi_{\mathcal{O}}}(\varphi)(g)$ is an automorphic function of $L(\mathcal{O})_{\mathbb{A}}$.

4.1.2 Local Setting and Jacquet modules

Now suppose that F is a non-archimedean local field. Let U be a unipotent subgroup of an l-group G (in the terminology of [BZ77]), let $\chi : U \to \mathbb{C}^{\times}$ be a character of U. Suppose that $M \subset G$ is a subgroup of G normalizing U and leaving χ invariant.

Then we have the exact functor

$$J_{U,\chi} : Rep(G) \to Rep(M),$$

the details of which are discussed in [BZ77], and we refer there for properties of Jacquet modules. We now define the local analogue of the Fourier coefficients associated to a nilpotent orbit $\mathcal{O}$. We have the action of $M(\mathcal{O})_F$ on $Z_2(\mathcal{O})_F$ by conjugation, and hence an action on the Pontryagin dual $Z_2(\mathcal{O})_F^{\vee} \cong Z_2(\mathcal{O})_F$. As in the global setup, we consider only characters $\psi_{\mathcal{O}}$ such that the stabilizer $L(\mathcal{O})_F \subset M(\mathcal{O})_F$ is of the same absolute Cartan type as $L(\mathcal{O})^{alg}$. We call such characters *generic* characters associated to $\mathcal{O}$.

Definition 4.1.4. Let (π, V) be a smooth admissible representation of finite length for $\overline{\mathrm{Sp}}_{2r}(F)$. We define the **twisted Jacquet modules** associated to the nilpotent orbit $\mathcal{O}$ and the generic character $\psi_{\mathcal{O}}$

$$J_{\psi_{\mathcal{O}}}(\pi) := J_{V_2(\mathcal{O}),\psi_{\mathcal{O}}}(\pi).$$

We say that a nilpotent orbit $\mathcal{O}$ supports π if there exists a generic character such that $J_{\psi_{\mathcal{O}}}(\pi) \neq 0$. $\square$

It is perhaps worth pausing here to meditate on the relationship between the local and global data we have attached to a given unipotent orbit $\mathcal{O}$. Suppose that $\pi \cong \otimes'_\nu \pi_\nu$ is an automorphic representation of $\mathrm{Sp}_{2r}(\mathbb{A})$ and suppose that there exists a vector φ and a generic character $\psi_{\mathcal{O}} = \prod_\nu \psi_{\mathcal{O},\nu}$ such that

$$F_{\psi_{\mathcal{O}}}(\varphi)(g) = \int_{[V_2(\mathcal{O})]} \varpi(vg)\psi_{\mathcal{O}}(v)dv \neq 0.$$

We may assume that under the isomorphism $\pi \cong \otimes_\nu \pi_\nu$,

$$\varphi \mapsto \otimes_\nu \varphi_\nu.$$

Now, choosing any finite place λ, one easily sees that the local character $\psi_{\mathcal{O},\lambda}$ is generic as well. Consider the functional

$$v \in \pi_\lambda \mapsto F_{\psi_{\mathcal{O}}}(v \otimes \left(\bigotimes_{\nu \neq \lambda} \varphi_\nu\right))(1).$$

We see that this gives a nonzero element in

$$\mathrm{Hom}_{V_2(\mathcal{O})}(\pi_\lambda, \mathbb{C}(\psi_{\mathcal{O},\lambda})) = \mathrm{Hom}_{V_2(\mathcal{O})}(J_{\psi_{\mathcal{O},\lambda}}(\pi_\lambda), \mathbb{C}(\psi_{\mathcal{O},\lambda})).$$

Thus, we see that $J_{\psi_{\mathcal{O},\lambda}}(\pi_\lambda) \neq 0$.

We record this as the following proposition:

Proposition 4.1.5. *Suppose that $\pi \cong \otimes'_\nu \pi_\nu$ is an automorphic representation of $\overline{\mathrm{Sp}}_{2r}(\mathbb{A})$ and suppose that the nilpotent orbit $\mathcal{O}$ supports π. Then for any finite place λ, $\mathcal{O}$ also supports π_v.*

4.2 Local Fourier-Jacobi Coefficients

Our main technique for proving the vanishing of Fourier coefficients associated to certain small nilpotent orbits is to study associated local Fourier-Jacobi models. These are local analogues of the Fourier-Jacobi coefficients studied in [GRS03], and the functor we study is intimately related to the Fourier-Jacobi map studied by Weissman [Wei03] as well as the generalized Whittaker models of [GGS16], though this is the first occurrence of this technique in the context of higher degree covering groups.

Letting F be a non-archimedean local field, we note the natural compatibility between the various covers of $\mathrm{Sp}_{2r}(F)$ arising from a fixed BD-cover induced by the compatibility between the various Hilbert symbols:

$$(x,y)_{nm}^{n} = (x,y)_m \qquad \text{for any } x,y \in F^{\times}.$$

Thus, there is a canonical covering group morphsim

$$\overline{\mathrm{Sp}}_{2r}(F) \twoheadrightarrow \mathrm{Mp}_{2r}(F), \tag{4.1}$$

where $\mathrm{Mp}_{2r}(F)$ is the classical metaplectic group, viewed as a 2-fold BD-cover of $\mathrm{Sp}_{2r}(F)$.

Let (π, V) be a smooth admissible representation of $\overline{\mathrm{Sp}}_{2r}(F)$, and let $\mathcal{O}$ be a nontrivial nilpotent orbit. To $\mathcal{O}$ we have attached the unipotent subgroups $V_1(\mathcal{O})$ and $V_2(\mathcal{O})$. It is shown in [GRS03] that the quotient

$$V_1(\mathcal{O})/\ker(\psi_{\mathcal{O}}) = (X \oplus Y) \oplus Z,$$

is a generalized Heisenberg group $\mathcal{H}_{2m+1} \oplus F^k$, for some $m, k \in \mathbb{Z}_{\geq 0}$. Here $Z = V_2(\mathcal{O})/\ker(\psi_{\mathcal{O}})$ forms the center and

$$V_1(\mathcal{O})/V_2(\mathcal{O}) = X \oplus Y$$

gives the symplectic space of dimension $2m$ over F. One sees that there always exists a choice of polarization $X \oplus Y$ such that both isotropic subspaces embed as subgroups of U.

Consider the nilpotent orbits $\mathcal{O}_m = ((2m)1^{2(r-m)})$ for some $1 \leq m \leq r$.[2] We fix the following choice of generic character $\psi_{m,\alpha} := \psi_{\mathcal{O}_m}$: for $v \in V_2(\mathcal{O}_m)_F$ and $\alpha \in F^\times$,

$$\psi_{m,\alpha}(v) = \psi(v_{1,2} + \cdots + v_{m-1,m} + \alpha v_{m,2r-m+1}).$$

We also define the characters $\psi_m(v) = \psi(v_{1,2}+\cdots+v_{m-1,m})$ and $\psi_\alpha(v) = \psi(\alpha v_{m,2r-m+1})$.

In terms of matrices, the Heisenberg group $H_m = V_1(\mathcal{O}_m)/\ker(\psi_\alpha) \cong \mathcal{H}_{2(r-m)+1}$ associated to the orbit $\mathcal{O}_m$ may be represented by the natural Heisenberg group occurring as a subgroup of U by matrices of the form $h(x : y : z)$ as in equation (2.3)

In this way, we may identify ψ_α as a character of the center

$$Z(H_m) = \{h(0 : 0 : z) : z \in F\},$$

and form the corresponding oscillator representation

$$\omega_\alpha = \mathrm{ind}_{YZ(H_m)}^{H_m}(\psi_\alpha^{-1}),$$

where $Y = \{h(0 : y : 0) : y \in F^{r-m}\}$ is a Lagrangian subspace. We may identify this as a representation of $V_1(\mathcal{O}_m)$, and note that twisting by the character ψ_m^{-1} preserves the induction:

$$\omega_{m,\alpha} = \mathrm{ind}_{YV_2(\mathcal{O}_m)}^{V_1(\mathcal{O}_m)}(\psi_\alpha^{-1}) \otimes \psi_m^{-1} \cong \mathrm{ind}_{YV_2(\mathcal{O}_m)}^{V_1(\mathcal{O}_m)}(\psi_{m,\alpha}^{-1})$$

Note that the subgroup $\mathrm{Mp}_{2(r-m)}(F) \subset \mathrm{Mp}_{2r}(F)$, corresponding to the full inverse image of $L(\mathcal{O}_m)_F \cong \mathrm{Sp}_{2(r-m)}(F)$ acts on this oscillator representation. Precomposing with the projection (4.1), we may view $\omega_{m,\alpha}$ as a representation of $\overline{L(\mathcal{O}_m)}_F \cong \overline{\mathrm{Sp}}_{2(r-m)}(F)$.

[2] The following construction works in full generality, but we need to only consider orbits of this form.

Consider the diagonal action of $V_1(\mathcal{O}) \rtimes \overline{L(\mathcal{O}_m)}_F$ on the tensor product $\pi \otimes \omega_{m,\alpha}$.

Definition 4.2.1. The **Fourier-Jacobi module** of π with respect to orbit $\mathcal{O}_m$ and character $\psi_{m,\alpha}$ is the Jacquet module

$$FJ_{m,\alpha}(\pi) := J_{V_1(\mathcal{O})}(\pi \otimes \omega_{m,\alpha}).$$

It is naturally a $\bar{\epsilon}$-genuine smooth representation of $\overline{L(\mathcal{O})}_F \cong \overline{\mathrm{Sp}}_{2(r-m)}(F)$.

The purpose for introducing Fourier-Jacobi modules rather than working with the Jacquet modules directly stems largely from the following properties.

Proposition 4.2.2. *Set* $H = \mathrm{Stab}_{N_{L(\mathcal{O}_m)}(YV_2(\mathcal{O}_m))}(\psi_{m,\alpha})$. *As representations of* $YV_2(\mathcal{O}_m) \rtimes H$, *we have an isomorphism*

$$FJ_{m,\alpha}(\pi) \cong J_{YV_2(\mathcal{O}_m),\psi_{m,\alpha}}(\pi).$$

Proof. Restricting $FJ_{m,\alpha}(\pi)$ to $YV_2(\mathcal{O}_m) \rtimes H$, the action is preserved under Frobenius reciprocity. Therefore, applying Frobenius reciprocity [GGS16, Lemma 2.3.6], we obtain

$$\begin{aligned} FJ_{m,\alpha}(\pi) &= J_{V_1(\mathcal{O}_m)}\left(\pi \otimes \mathrm{ind}_{YV_2(\mathcal{O}_m)}^{V_1(\mathcal{O}_m)}(\psi_{m,\alpha}^{-1})\right) \\ &\cong J_{YV_2(\mathcal{O}_m)}\left(\pi \otimes \psi_{m,\alpha}^{-1}\right). \end{aligned}$$

We are done, since as H-representations,

$$J_{YV_2(\mathcal{O}_m)}\left(\pi \otimes \psi_{m,\alpha}^{-1}\right) \cong J_{YV_2(\mathcal{O}_m),\psi_{m,\alpha}}(\pi). \quad \square$$

$\square$

In particular, we have that the standard Borel subgroup $B_{\mathcal{O}_m} \subset L(\mathcal{O}_m)_F$ is contained in $H = \mathrm{Stab}_{N_{L(\mathcal{O}_m)}(YV_2(\mathcal{O}_m))}(\psi_{m,\alpha})$, allowing us to conclude the following.

Proposition 4.2.3. *Let $V \subset B_{\mathcal{O}_m}$ be a unipotent subgroup of $\overline{L(\mathcal{O}_m)}_F$, and suppose that ψ_V is a character of V. Then we have*

$$J_{V,\psi_V}(FJ_{m,\alpha}(\pi)) \cong J_{V,\psi_V}(J_{YV_2(\mathcal{O}_m),\psi_{m,\alpha}}(\pi)).$$

Note that the case that $V = \{I\}$ along with local root exchange implies the following local version of [GRS03, Lemma 1.1]:

Corollary 4.2.4. *As representations of $V_2(\mathcal{O}_m)_F$, we have isomorphisms*

$$FJ_{m,\alpha}(\pi) \cong J_{YV_2(\mathcal{O}_m),\psi_{m,\alpha}}(\pi) \cong J_{V_2(\mathcal{O}_m),\psi_{\mathcal{O}_m}}(\pi)$$

Thus, Fourier-Jacobi modules are isomorphic as $V_2(\mathcal{O})$-modules to the Jacquet module associated to $\mathcal{O}_m$, implying that we may study one to ascertain vanishing of the other. The benefit of the Fourier-Jacobi modules is the smooth admissible $\overline{L(\mathcal{O}_m)}_F$-action. This action allows us to apply the following observation:

Proposition 4.2.5. *Suppose that $\mathbb{G}(F) = G$ is a reductive group over a non-archimedean local field F containing a full set of n^{th} roots of unity $\mu_n(F)$, and suppose that $\overline{\mathrm{G}}^{(n)}$ is a central extension of G by $\mu_n(F)$ arising from $\overline{\mathbb{G}} \in CExt(\mathbb{G}, \mathbb{K}_2)$. Let (π, V) be a non-zero smooth admissible representation of $\overline{G}$. If π is unramified, then π is not supercuspidal.*

Proof. The Satake isomorphism holds in this context and, as in [McN12] and [GG14], one still has that if (τ, W) is an irreducible unramified representation, then there exists an unramified character $\chi : Z(\tilde{T}) \to \mathbb{C}^\times$ such that

$$\mathrm{Hom}_{\overline{G}}(\tau, \mathrm{Ind}_{\overline{B}}^{\overline{G}}(\rho(\chi))) \neq 0.$$

In particular, τ cannot be cuspidal. By [BZ77, Theorem 2.4.a], which is proven for reductive groups, but the proof goes through without change for metaplectic covers

once we replace parabolic subgroups with their covers, we have that our representation decomposes

$$\pi = \pi_c \oplus \pi_c^{\perp},$$

where π_c is a cuspidal representation, and $\pi_c^{\perp}$ has no nontrivial cuspidal sub-quotients. Since(π, V) is unramified, it has at least one spherical composition factor τ_π. This forces $\pi_c^{\perp} \neq 0$, and hence π is not itself cuspidal. □

4.3 Root Exchange

4.3.1 Global Root Exchange

In this section we will introduce the statement of the general lemma on (global) root exchange. The main reference for this is [GRS11]. Let K be a number field with adele ring $\mathbb{A}$. Fix a unipotent subgroup $C \subset \mathrm{G}_m$, and let ψ_C be a non-trivial character of $C(K)\backslash C(\mathbb{A})$.

Suppose there exist two unipotent F-subgroups $X, Y \subset \mathrm{G}$ satisfying the following six properties:

1. X and Y normalize C,
2. $X \cap C$ and $Y \cap C$ are normal in X and Y, respectively, and $X \cap C\backslash X$, $Y \cap C\backslash Y$ are both abelian,
3. $X(\mathbb{A})$ and $Y(\mathbb{A})$, acting via conjugation, preserve ψ_C,
4. ψ_C is trivial on $(X \cap C)(\mathbb{A})$ and $(Y \cap C)(\mathbb{A})$,
5. $[X, Y] \subset C$,

6. The pairing $(X \cap C)(\mathbb{A})\backslash X(\mathbb{A}) \times (Y \cap C)(\mathbb{A})\backslash Y(\mathbb{A}) \to \mathbb{C}^\times$, given by

$$(x, y) \mapsto \psi_C([x, y]),$$

is multiplicative in each coordinate , non-degenerate, and identifies $(Y \cap C)(K)\backslash Y(F)$ with the dual of $X(K)(X \cap C)(\mathbb{A})\backslash X(\mathbb{A})$, and vice versa.

With this set up, set $B = YC$, $D = XC$ and extend ψ_C trivially to both $B(K)\backslash B(\mathbb{A})$ and $D(K)\backslash D(\mathbb{A})$, referring to each separately as ψ_B and ψ_D. Finally, set $A = BX = DY$.

Lemma 4.3.1. *Suppose that the quadruple (C, ψ_C, X, Y) satisfy the above criteria. If f is an automorphic form on G_m of uniform moderate growth, then*

$$\int_{[C]} f(cg)\psi_C^{-1}(c)dc \equiv 0, \quad \forall g \in A(\mathbb{A}) \tag{4.2}$$

if and only if

$$\int_{[B]} f(vg)\psi_B^{-1}(v)\, dv \equiv 0, \quad \forall g \in A(\mathbb{A}) \tag{4.3}$$

if and only if

$$\int_{[D]} f(ug)\psi_D^{-1}(u)\, du \equiv 0, \quad \forall g \in A(\mathbb{A}), \tag{4.4}$$

Proof. The main reference is [GRS11][Sec.7.1]. For completeness, we reproduce the proof here. We show the eqivalence between the vanishing of integral (4.2) and of integral (4.3). By the symmetry in X and Y in our set up, an identical argument works for integral (4.4).

By our assumptions (1), (2), and (3), it is immediate that we may decompose the integral over $[B]$ as

$$\int_{[B]} f(vg)\psi_B^{-1}(v)dv = \int_{Y(F)(Y\cap C)(\mathbb{A})\backslash Y(\mathbb{A})} \int_{[C]} f(ycg)\psi_C^{-1}(c)dcdy,$$

showing the vanishing of integral (4.2) implying the vanishing of integral (4.3). To show the other direction, consider the function, for $g \in A(\mathbb{A})$

$$\phi(g) = \int_{[C]} f(cg)\psi_C^{-1}(c)dc.$$

By assumptions (3) and (4) above, we see that $\varphi(g)$ is invariant under $Y(F)(Y\cap C)(\mathbb{A})$, and so viewing it as a function on $Y(F)(Y\cap C)(\mathbb{A})\backslash Y(\mathbb{A})$, we expand along this abelian subgroup (assumption (2)) to get

$$\varphi(g) = \sum_{x\in(X\cap C)(F)\backslash X(F)} \int_{Y(F)(Y\cap C)(\mathbb{A})\backslash Y(\mathbb{A})} \varphi(yg)\psi_C^{-1}([y,x])dy,$$

where we have used assumption (6) to identity $(X \cap C)(F)\backslash X(F)$ as the unitary dual of the compact group $Y(F)(Y \cap C)(\mathbb{A})\backslash Y(\mathbb{A})$. Note that this sum converges absolutely and uniformly in compact sets due to the assumption that f is smooth and of uniform moderate growth.

Thus, we may now identify integral (4.2) with

$$\int_{[C]} f(cg)\psi_C^{-1}(c)dc = \sum_{x\in(X\cap C)(F)\backslash X(F)} \int_{Y(F)(Y\cap C)(\mathbb{A})\backslash Y(\mathbb{A})} \int_{[C]} f(cyg)\psi_C^{-1}(c[y,x])dcdy.$$

Consider for the moment the individual summand corresponding to x above. Using assumption (5), we see that $[y,x] \in C(\mathbb{A})$, and so making the change of variables $c \mapsto c[y,x]^{-1}$, we see that this summand equals

$$\begin{aligned}&\int_{Y(F)(Y\cap C)(\mathbb{A})\backslash Y(\mathbb{A})} \int_{[C]} f(cxyx^{-1}g)\psi_C^{-1}(c)dcdy \\ &= \int_{Y(F)(Y\cap C)(\mathbb{A})\backslash Y(\mathbb{A})} \int_{[C]} f(x^{-1}cxyx^{-1}g)\psi_C^{-1}(c)dcdy,\end{aligned}$$

where we use the automorphy of f. Now using assumption (3), we make another change of variables to arrive at

$$\int_{Y(F)(Y\cap C)(\mathbb{A})\backslash Y(\mathbb{A})} \int_{[C]} f(xyx^{-1}g)\psi_C^{-1}(c)dcdy = \int_{[B]} f(vx^{-1}g)\psi_B^{-1}(v)dv$$

so that we have the equality

$$\int_{[C]} f(cg)\psi_C^{-1}(c)dc = \sum_{x\in(X\cap C)(F)\backslash X(F)} \int_{[B]} f(vxg)\psi_B^{-1}(v)dv,$$

thus showing that if integral (4.3) vanishes for all $g' \in A(\mathbb{A})$, so does integral (4.2), and we are done.

□

4.3.2 Local Root Exchange

Suppose now that F is a non-archimedean local field, and let G be the F-rational points of a split algebraic group over F or finite cover thereof. As in the global setting, we consider unipotent subgroups $C, X, Y \subset \mathrm{G}$, and a nontrivial character

$$\psi_C : C \to \mathbb{C}^\times.$$

As before, set $B = YC$, $D = XC$, and $A = BX = DY$. We also assume that these subgroups satisfy the local analogue of the six properties preceding Lemma 4.3.1 (For a precise statement and proof, see [Cai16, Section 6.1]).

Lemma 4.3.2. *Suppose that π is a smooth representation of A, and extend the character ψ_C trivially to ψ_B on B and ψ_D on D. Then we have an isomorphism of C-modules*

$$J_{B,\psi_B}(\pi) \cong J_{D,\psi_D}(\pi).$$

Moreover,

$$J_{C,\psi_C}(\pi) = 0 \iff J_{B,\psi_B}(\pi) = 0 \iff J_{D,\psi_D}(\pi) = 0.$$

4.3.3 How one uses root exchange

There are many contexts in which the above lemmas are useful, but we pause here to introduce a formalism for a particularly common application. We illustrate this

application globally. Let C and ψ_C be as above, only now we assume that $C \subset U_r$. For a positive root β, let $U_\beta \subset U_r$ be the corresponding root subgroup.

Assume that $x_\beta(t) \in C(\mathbb{A})$ for all t such that the induced character

$$\mathbb{G}_a(K)\backslash\mathbb{G}_a(\mathbb{A}) \xrightarrow{x_\beta} C(K)\backslash C(\mathbb{A}) \xrightarrow{\psi_C} \mathbb{C}^\times$$

is nontrivial. Let $\alpha, \gamma \in \Phi$ such that the following properties hold:

1. Set $\Phi_C = \{\gamma : x_\gamma(t) \subset C\}$. Then for all $i, j \in \mathbb{Z}_{>0}$ such that $i\alpha + j\gamma \in \Phi$, we have
$$i\alpha + j\gamma \in \Phi_C,$$
2. For all $\mu \in \Phi_C$ and all $i, j \in \mathbb{Z}_{>0}$ such that $i\alpha + j\mu \in \Phi$, we have
$$i\alpha + j\mu \in \Phi_C,$$
and likewise for γ and μ.
3. We have
$$\beta = \alpha + \gamma,$$
4. Suppose $\beta' \in \Phi_C$, with $\beta' \neq \beta$, such that the induced character on $x_{\beta'}(t)$ is also non-trivial. Then there do not exist i, j such that
$$\beta' = i\alpha + j\gamma.$$

Set $B = U_\alpha \cdot C$, $D = U_\gamma \cdot C$.

When the group C and character ψ_C are clear from the context, we shall refer to the ordered triple of roots

$$(\alpha, \gamma, \beta)$$

as above as an **exchange triple**.

Suppose then that we have the integral (4.3), and want to apply Lemma 4.3.1 to obtain the integral (4.4). In this setting, we say that we apply root exchange to the exchange triple (α, γ, β), meaning that we apply Lemma 4.3.1 to the quadruple $(C, \psi_C, \{x_\alpha(t)\}, \{x_\gamma(t)\})$.

Thus, the equivalence

$$\int_{[U_\alpha \cdot C]} f(vg)\psi_C^{-1}(v)\, dv \equiv 0, \quad \forall g \in \mathrm{G}_m$$

if and only if

$$\int_{[U_\gamma \cdot C]} f(ug)\psi_D^{-1}(u)\, du \equiv 0, \quad \forall g \in \mathrm{G}_m,$$

from Lemma 4.3.1 tells us that, holding all else constant, we can exchange the integration over the root group of α with integration over the root group of γ. When we apply root exchange to the triple (α, β, γ), we say that we exchange the root α for the root γ.

Chapter 5

Fourier-Whittaker Coefficients: Identification of the Orbit

In this chapter, we return to $\overline{\mathrm{G}}^{(n)} = \overline{\mathrm{Sp}}_{2r}$ with $n = 4$ and compute the nilpotent orbit associated to Θ_{2r}. As usual for such statements, the proof comes in two steps: in Section 5.1, we prove Theorem 5.1.1 which gives an upper bound for $\mathcal{O}(\Theta_{2r})$ by showing the vanishing of local Fourier-Whittaker models. This argument is inductive in the rank of the group, the base case relying on the fact that Θ_4 is non-generic.

In Section 5.2, we establish Theorem 5.2.1, which completes the identification of $\mathcal{O}(\Theta_{2r})$. The argument relies on the parity of the rank: in the even rank case, we view Θ_{2r} as a residue of the Siegel Eisenstein series induced from a theta representation on a certain cover of GL_r. This enables us to take advantage of the work of [BG92] to establish global non-vanishing of the appropriate coefficient. The odd rank case is then deduced from the even rank case. We remark after Proposition 5.2.3 how this step requires more to use the same technique for higher degree covers of the symplectic group.

5.1 Vanishing Statements

In this section we seek to prove part (2) of Theorem 1.2.4:

Theorem 5.1.1. *Suppose that $\mathcal{O}'$ is a nilpotent orbit such that either $\mathcal{O}' > \mathcal{O}_\Theta = (2^r)$ or $\mathcal{O}'$ and $\mathcal{O}_\Theta$ are not comparable. Then $\mathcal{O}'$ does not support Θ_{2r}.*

The proof relies on global results of Ginzburg-Rallis-Soudry [GRS03] and Jiang-Liu [JL15]. If we can show that nilpotent orbits of the form

$$\mathcal{O}' = ((2k)1^{2r-2k})$$

for $2 \leq k \leq n$ do not support Θ_{2r}, then by [GRS03, Lemma 2.6] , it follows that all orbits of the form

$$\mathcal{O}' = ((2k)p_1^{e_1} \cdots p_l^{e_l})$$

with $2 \leq k \leq r$ also fail to support Θ_{2r}. Combining this vanishing with Propositions 3.2 and 3.3 of Jiang-Liu [JL15], we now see that nilpotent orbits of type

$$\mathcal{O}' = ((2k+1)^2 p_1^{e_1} \cdots p_l^{e_l})$$

for $2 \leq k \leq r$ also fail to support Θ_{2r}. This exhausts those orbits $\mathcal{O}'$ such that either $\mathcal{O}' > \mathcal{O}_\Theta$ or $\mathcal{O}'$ and $\mathcal{O}_\Theta$ are not comparable. We are thus reduced to showing that the orbits of the form $\mathcal{O}_k = ((2k)1^{2r-2k})$, as well as $\mathcal{O}' = (3^2 1^{2r-6})$, do not support Θ_{2r}. In fact, it suffices to see that the single orbit $\mathcal{O}_2 = (41^{2r-4})$ fails to support Θ_{2r}.

We study the orbit $\mathcal{O}_2$ by considering the corresponding local Fourier-Jacobi module $FJ_{2,\alpha}(\Theta_{2r,\nu})$. In particular, subject to mild restrictions on the local field $F = K_\nu$, we show that if this representation is non-zero, it must be both unramified and supercuspidal. This contradicts Proposition 4.2.5. Therefore, by Corollary 4.2.4, we see that $\mathcal{O}_2$ does not support Θ_{2r}. The sufficiency of this result to prove Theorem 5.1.1 will become clear as the argument progresses.

5.1.1 Local Vanishing results

Consider the nilpotent orbit $\mathcal{O}_2 = (41^{2r-4})$. The unipotent subgroup $V_1(\mathcal{O})$ is the unipotent radical of the standard parabolic subgroup of Sp_{2r} with Levi subgroup $L_{2,2r-4} \cong \mathrm{GL}_1^2 \times \mathrm{Sp}_{2r-4}$, and the subgroup $V_1(\mathcal{O})$ is as follows:

$$\left\{ \begin{pmatrix} u & X & Y \\ & I_{2r-4} & X^* \\ & & u^{-1} \end{pmatrix} : u \in U_{\mathrm{GL}_2}; Y \in \mathrm{Sym}_J^2(F); X \in \mathrm{Mat}_{2\times(2r-4)}, X_{2,i} = 0 \right\}.$$

Here U_{GL_2} denotes the subgroup of upper triangular unipotent elements of GL_2. The character $\psi_{\mathcal{O}}$ may be chosen to be of the form

$$\psi_{\mathcal{O},\alpha}(v) = \psi(v_{1,2} + \alpha v_{2,2r-1}),$$

where $\alpha \in F^\times/(F^\times)^2$.

Let ν be a finite place of K with with odd residue characteristic. In this case, both Θ_{2m} and $\omega_{2,\alpha}$ are unramified representations of $\overline{\mathrm{Sp}}_{2m-4}(K_\nu)$, so that the local Fourier-Jacobi module $FJ_{2,\alpha}(\Theta_{2m})$ is unramified as a representation of $\overline{\mathrm{Sp}}_{2m-4}(F)$ for all such places. We will work only with $F = K_\nu$ satisfying this property.

Proposition 5.1.2. *Let Θ_6 be the theta representation of $\overline{\mathrm{Sp}}_6(F)$. Then the Fourier-Jacobi module*

$$FJ_{2,\alpha}(\Theta_6) = J_{V_1(\mathcal{O}_2)}(\Theta_6 \otimes \omega_{2,\alpha})$$

is trivial.

Proof. In this case, one easily checks that $L(\mathcal{O})(F)$ is the embedded copy of $\mathrm{SL}_2(F)$ corresponding to the long simple root of Sp_6. Thus, $FJ_{2,\alpha}(\Theta_6)$ is a smooth $\bar{\epsilon}$-genuine representation of $\overline{\mathrm{SL}}_2(F)$. Let U_{α_3} be the root subgroup of $\mathrm{Sp}_6(F)$ corresponding to the unipotent radical in $\overline{\mathrm{SL}}_2(F)$.

By our assumption on ν, it suffices to show that the representation is supercuspidal. This will follow if we can show that

$$J_{U_{\alpha_3}}(J_{V_1(\mathcal{O}_2)}(\Theta_6 \otimes \omega_{2,\alpha})) = 0.$$

By Proposition 4.2.3, we see that this is equivalent to showing that

$$J_{U_{\alpha_3}}(J_{YV_2(\mathcal{O}_2),\psi_{2,\alpha}}(\Theta_6)) \cong J_{V,\psi_V}(\Theta_6) = 0.$$

Here, we have $V = \{v \in U_6 : v_{2,3} = v_{4,5} = 0\}$, and $\psi_V(v) = \psi(v_{1,2} + \alpha v_{2,5})$.

Conjugating by the Weyl group element $w = w_1w_2$ and applying Lemma 4.3.2 to the exchange triple

$$(-\gamma_{1,1}, \gamma_{1,2}, \gamma_{2,2}) = (-\alpha_1, \alpha_1 + \alpha_2, \alpha_2),$$

we see that this module is isomorphic to $J_{U^0,\psi^0}(\Theta_6)$, where U^0 is the unipotent radical of the parabolic subgroup associated to the simple coroot $\alpha_1^\vee$ and $\psi^0(v) = \psi(v_{2,3} + \alpha v_{3,4})$.

Since Θ_6 is not generic (see [Gao16, Section 5]), we see that this Jacquet module is isomorphic to $J_{U_6,\psi'}(\Theta_6)$, where ψ' is the trivial extension of ψ^0 to the full unipotent radical.[1] Note that this Jacquet module factors through the constant term of type $\mathrm{GL}_1 \times \mathrm{Sp}_4$, and by Theorem 3.1.5, we see that

$$J_{U_6,\psi'}(\Theta_6) \cong J_{U_4,\psi_{(4),\alpha}}(\Theta_4) = 0,$$

as Θ_4 is also not generic. □

It follows from Corollary 4.2.4 that the Jacquet module associated to $\mathcal{O}_2$ vanishes.

[1] This follows from noting that the SL_2-factor of the Levi subgroup corresponding to the simple coroot $\alpha_1^\vee$ preserves the character ψ^0, and thus we may view this Jacquet module as an $\overline{\mathrm{SL}}_2(F)$-representation. As Θ_6 is not generic, this $\overline{\mathrm{SL}}_2(F)$-representation has no non-trivial twisted Jacquet module. Therefore, upon restricting to the unipotent radical, this representation is isomorphic to its Jacquet module.

Theorem 5.1.3. *Let $r \geq 3$ and let Θ_{2r} be the theta representation of $\overline{\mathrm{Sp}}_{2r}(F)$. Then the Fourier-Jacobi module*

$$FJ_{2,\alpha}(\Theta_{2r}) = J_{V_1(\mathcal{O}_2)}(\Theta_{2r} \otimes \omega_{2,\alpha})$$

vanishes.

We shall prove this by induction, noting that Proposition 5.1.2 forms the base case. Thus, we assume the theorem holds for r. In light of this assumption, we are able to say much about certain Jacquet modules on Θ_{2r+2}:

Proposition 5.1.4. *Suppose that Theorem 5.1.3 holds for $m \leq r$. Then for all $3 \leq k \leq m \leq r+1$, the nilpotent orbits $\mathcal{O}_{m,k} = ((2k)1^{2(m-k)})$ do not support Θ_{2m}.*

Proof. The case $k = m = 3$ is proven so by induction we may assume that it is proven up to m.

By Corollary 3.1.6, we know that there is a value $s_0 \in \mathbb{C}$ such that there is a surjection

$$\mathrm{Ind}(\delta^{s_0} \otimes \Theta_{2m}) \twoheadrightarrow \Theta_{2m+2}.$$

Therefore, it suffices to show that

$$\mathrm{Hom}_{V_2(\mathcal{O}_{m+1,k})}(\mathrm{Ind}(\delta^{s_0} \otimes \Theta_{2m}), \psi_{\mathcal{O}_{m+1,k}}) = 0.$$

For this task, we apply [BZ77, Theorem 5.2], and study the double cosets

$$P_{(1,m)}(F) \backslash \mathrm{Sp}_{2m+2}(F) / \mathrm{Sp}_{2(m-k-2)}(F) V_2(\mathcal{O}_{m+1,k})$$

One can check, using the Bruhat decomposition, that the coset representatives may be chosen to be Weyl group elements or Weyl group elements times certain unipotent elements.

As in the proof of [BFG03, Lemma 7], most cases do not contribute as the Weyl group element conjugates a root of $V_2(\mathcal{O}_{m+1,k})$ where the character is nontrivial into the unipotent radical of $P_{(1,m)}$. There are only two cases that need to be considered:

1. The representatives of the form $w = w_1 w_2 \cdots w_{m+1} u$ give rise to characters corresponding to orbits $\mathcal{O}_{m',k'}$ where $m' < m+1$. By induction, these also vanish.

2. The relative long element $w = w_1 w_2 \cdots w_m w_{m+1} w_m \cdots w_1$ gives rise to the character corresponding to (41^{2m-4}) on Θ_{2m}. This vanishes by the assumption that Theorem 5.1.3 holds for m, and we are done.

□

Corollary 5.1.5. *The orbit* $(3^2 1^{2r-4})$ *does not support* Θ_{2r+2}.

Proof. Set $\mathcal{O} = (3^2 1^{2r-4})$. Then we have $V_2(\mathcal{O}) = U_{(2,r-1)}$ and we may choose the generic character

$$\psi_{\mathcal{O}}(v) = \psi(v_{1,3} + v_{2,2r}).$$

Supposing that $J_{\psi_{\mathcal{O}}}(\Theta_{2r+2})$ does not vanish, it follows that it gives to an unramified representation of $\overline{L(\mathcal{O})}_F = \overline{\mathrm{SL}}_2(F)^{\Delta} \times \overline{\mathrm{Sp}}_{2r-4}(F)$. Restricting this representation to the diagonally embedded SL_2, we see there exists a character χ of $Z(\widetilde{T}_1) \subset \overline{\mathrm{SL}}_2(F)$ such that by Frobenius reciprocity,

$$0 \neq \mathrm{Hom}_{\overline{\mathrm{SL}}_2(F)}(J_{V_2(\mathcal{O}),\psi_{\mathcal{O}}}(\Theta_{2r+2}), \mathrm{Ind}(\chi)) \cong \mathrm{Hom}_{\widetilde{T}_1}(J_{V_2(\mathcal{O})N,\psi_{\mathcal{O}}}(\Theta_{2r+2}), \iota(\chi)).$$

We study the Jacquet module $J_{V_2(\mathcal{O})N,\psi_{\mathcal{O}}}(\Theta_{2r+2})$ by conjugating the Weyl group element w_2.

Applying the root exchanges associated to the exchange triples

$$(\gamma_{3,i}, \eta_{2,i}, \eta_{2,2})$$

for $3 \leq i \leq r+1$ in increasing values of i followed by the exchange triples

$$(\eta_{3,j}, \gamma_{2,j}, \eta_{2,2})$$

for $3 \leq j \leq r$ in decreasing values for j, one finds that this constant term is isomorphic to $J_{U',\psi'}(\Theta_{2r+2})$ where

$$U' = \{u \in V_2(421^{2r-4}) : u_{1,2} = 0\},$$

and $\psi'(u) = \psi(u_{2,3}+u_{3,2r})$. Conjugating by the standard lift of the simple reflection associated to the long root μ_3, and applying root exchange to the triple $(-\mu_3, \eta_{2,2}, \gamma_{2,2})$, we obtain an isomorphism with the Jacquet module $J_{L_{2,r-1},\psi_2}(\Theta_{2r+2})$.

However, by the vanishing of of the Jacquet modules associated to $((2k)1^{2(r+1-k)})$ for $k > 2$ from the proposition forces this final Jacquet module to be trivial. To see this, one applies an inductive argument identical to the global proof of Lemma A.2.1. This is not circular as the argument in the proof of Lemma A.2.1 requires only the vanishing of the coefficients associated to the aforementioned orbits. By way of a contradiction, we find that

$$J_{V_2(\mathcal{O}),\psi_{\mathcal{O}}}(\Theta_{2r+2}) = 0.$$

□

Having established that the inductive hypothesis forces vanishing of the Jacquet modules associated to $\mathcal{O}_k$ for $k > 2$, as well as for the orbit $\mathcal{O}' = (3^2 1^{2r-4})$, we may now consider $\mathcal{O}_2$.

Proposition 5.1.6. *As a representation of $\overline{\mathrm{Sp}}_{2r-2}(F)$, the Fourier-Jacobi module $FJ_{2,\alpha}(\Theta_{2r+2})$ is supercuspidal.*

Proof. The argument mirrors the one in the base case. Let $\alpha_0 = \mu_3 = 2\alpha_3 + \cdots + 2\alpha_n + \alpha_{r+1}$ be the long root corresponding to the highest root in the embedded

copy of Sp_{2r-2}. As the root group U_{α_0} corresponding to this root is contained in the unipotent radical of each maximal parabolic of Sp_{2r-2}, to show that the Jacquet module is supercuspidal, it suffices to see that

$$J_{U_{\alpha_0}}(FJ_{2,\alpha}(\Theta_{2r+2})) = 0.$$

To demonstrate this, we note that by Proposition 4.2.3 it suffices to show that

$$J_{U_{\alpha_0}}(J_{YV_1(\mathcal{O}_2),\psi_{2,\alpha}}(\Theta_{2r+2})) \cong J_{V,\psi_V}(\Theta_{2r+2}) = 0.$$

In the above isomorphism, we have used root exchange to remove the roots $\eta_{2,k}$ for $3 \leq k \leq n$, which fit into the exchange triples $(\gamma_{2,k}, \eta_{2,k}, \mu_2)$.

Conjugating the subgroup $V = V_2(421^{2r-4})$ by the Weyl group element $w = w_1w_2$, and then applying root exchange to the exchange triple $(-\alpha_1, \alpha_1+\alpha_2, \alpha_2)$, we see that this Jacquet module is isomorphic to $J_{V^w,\psi_{V^w}}(\Theta_{2r+2})$, where

$$V^w = \left\{ v = \begin{pmatrix} Z & X & Y \\ & I_{2r-4} & X^* \\ & & Z^* \end{pmatrix} \right\},$$

where $Y \in \mathrm{Sym}_J^3(F)$, $X \in \mathrm{Mat}_{3\times 2r-4}(F)$ such that $X_{ij} \neq 0$ implies $i = 2$, and $Z \in U(\mathrm{GL}_3)$ is of the form

$$Z = \begin{pmatrix} 1 & & * \\ & 1 & * \\ & & 1 \end{pmatrix}.$$

Here, $\psi_{V^w}(v) = \psi(v_{2,3} + \alpha v_{3,2r})$. It should be noted that $J_{V^w,\psi_{V^w}}(\Theta_{2r+2})$ is a representation of $\overline{\mathrm{Sp}}_{2r-4}(F)$.

Consider now the abelian subgroup of U_{2r+2}

$$Z = \left\{ z = \begin{pmatrix} 1 & X & 0 \\ & I_{2r} & X^* \\ & & 1 \end{pmatrix} : X \in F^{2r} \text{ such that } x_1 = x_2 = x_{2r-1} = x_{2r} = 0 \right\}.$$

The subgroup $\overline{\mathrm{Sp}}_{2r-4}(F)$ acts by conjugation on the characters of Z with two orbits. Expanding $J_{V^w,\psi_{V^w}}(\Theta_{2r+2})$ along Z, we see that all representatives of the nontrivial orbit vanish as they are isomorphic to a quotient of the Jacquet module associated to the nilpotent orbit $(3^2 1^{2r-4})$, which were shown to vanish on Θ_{2r+2} in Corollary 5.1.5.

Therefore, the only term which remains is the trivial orbit, so that our Jacquet module vanishes if and only if $J_{ZV^w,\psi^0}(\Theta_{2r+2})$ vanishes, where

$$\psi^0(v) = \psi(v_{2,3} + \alpha v_{3,2r})$$

is the trivial extension of ψ_{V^w} to this larger unipotent subgroup.

Finally, the vanishing of (61^{2r-4}) forces this Jacquet module to be isomorphic to $J_{R,\psi^0}(\Theta_{2r+2})$, where $R = U_{\alpha_1} Z V^w = V_2(61^{2r-4})$. Factoring this Jacquet module through the unipotent radical of the maximal parabolic of type $\mathrm{GL}_1 \times \mathrm{Sp}_{2r}$, and applying Theorem 3.1.5, we see that

$$J_{R,\psi^0}(\Theta_{2r+2}) \cong J_{V_2(41^{2r-4}),\psi_{(41^{2r-4})}}(\Theta_{2r}),$$

which vanishes by induction.

Thus, we have shown that

$$FJ_{2,\alpha}(\Theta_{2r+2})$$

is supercuspidal as a $\overline{\mathrm{Sp}}_{2r-2}(F)$ representation. □

As remarked above, $FJ_{2,\alpha}(\Theta_{2r}) = 0$ by Proposition 4.2.5. By Corollary 4.2.4, we have proven the Theorem 5.1.3. Combining this with Proposition 5.1.4, Corollary 5.1.5, and the argument above, we have proven Theorem 5.1.1. □

5.2 Non-Vanishing Statement

We turn now to part (1) of Theorem 1.2.4. For the orbit $\mathcal{O}_\Theta := \mathcal{O}_{\Theta,r} = (2^r)$ on Sp_{2r}, we have

$$V_2(\mathcal{O}_\Theta) = \left\{ \begin{pmatrix} I_r & Y \\ & I_r \end{pmatrix} : Y \in \mathrm{Sym}_J^r(\mathbb{A}) \right\}.$$

For $v \in V_2(\mathcal{O}_\Theta)$, we define the character $\psi_{\mathcal{O}_\Theta,\underline{\epsilon}} : V \to \mathbb{C}^\times$ by

$$\psi_{\mathcal{O}_\Theta,\underline{\epsilon}}(v) = \psi(\epsilon_1 v_{1,2r} + \cdots + \epsilon_r v_{r,r+1}),$$

where $\epsilon_i \in K^\times$. With this notation, the Fourier coefficients of Θ_{2r} associated to the nilpotent orbit $\mathcal{O}$ are given by

$$\int_{[V_2(\mathcal{O}_\Theta)]} \theta(vg)\psi_{\mathcal{O}_\Theta,\underline{\epsilon}}(v)\, dv. \tag{5.1}$$

By the automorphy of θ, we have the freedom to conjugate the above integral by various elements $t \in T_r(K)$ which, after a change of variables, implies that our coefficient only depends on $\epsilon_i \pmod{(K^\times)^2}$. Additionally, $M(\mathcal{O}_\Theta) \cong \mathrm{GL}_r$ is the Siegel Levi subgroup of Sp_{2r} and one can show that for each choice $\underline{\epsilon}$, $L(\mathcal{O}_\Theta)_K = \mathrm{Stab}_{M(\mathcal{O}_\Theta),K}(\psi_{\mathcal{O}_\Theta,\underline{\epsilon}})$ is isomorphic to $\mathrm{O}_{\underline{\epsilon}}(K)$, the orthogonal group on K^r with respect to the quadratic form

$$q_{\underline{\epsilon}}(v) = \epsilon_1 v_1^2 + \cdots + \epsilon_r v_r^2.$$

Theorem 5.2.1. *The nilpotent orbit $\mathcal{O}_\Theta = (2^r)$ supports Θ_{2r}.*

This will be proven in two stages: first proving non-vanishing for $\overline{\mathrm{Sp}}_{4r}$, then using this to obtain the result for $\overline{\mathrm{Sp}}_{4r+2}$.

Proposition 5.2.2. *The nilpotent orbit $\mathcal{O}_\Theta = (2^{2r})$ supports Θ_{4r}.*

Proof. Set $\mathcal{O} = \mathcal{O}_\Theta$. To prove the proposition, it is sufficient to show that there is at least one $M(\mathcal{O})_K$-orbit of characters such that (5.1) is non-vanishing. To this end, suppose $\underline{\epsilon}$ is such that $\mathrm{O}_{\underline{\epsilon}}(K)$ is split. This is equivalent to the assumption that, up to conjugation by a Weyl group element, for each $1 \leq i \leq 2r-1$ odd,

$$\epsilon_i \epsilon_{i+1} = -\alpha_i^2$$

for some $\alpha_i \in K^\times$. With this choice, we refer to $\psi_{\mathcal{O},\underline{\epsilon}}$ as the split character. With this choice, there exists an $l \in M(\mathcal{O})_K$ such that, conjugating the integral (5.1) by l, we see that non-vanishing of our Fourier coefficient for the choice of split character is equivalent to the non-vanishing of

$$\int_{[V_2(\mathcal{O})]} \theta(vg)\hat{\psi}_{\mathcal{O}}(v)\, dv, \tag{5.2}$$

where

$$\hat{\psi}_{\mathcal{O}}(v) = \psi(v_{2r-1,2r+1} + v_{2r-3,2r+3} + \cdots + v_{1,4r-1}).$$

Note that the roots with a nontrivial character on their root subgroups are

$$\beta_k = \eta_{2r-2k+1,2r-2k+1} = \alpha_{2r-2k+1} + 2\alpha_{2r-2k+2} + \cdots + \alpha_{2r},$$

for $1 \leq k \leq 2r$.

Thus, to get to an integral we can work with, for each $1 \leq i \leq 2r$ set r_i to be the

standard lift of the simple reflection for the long root μ_i; namely

$$r_i = \begin{pmatrix} I_{i-1} & & & & \\ & & & -1 & \\ & & I_{4r-2i} & & \\ & 1 & & & \\ & & & & I_{i-1} \end{pmatrix}.$$

The idea is to iteratively move the character to the appropriate simple roots. We begin by conjugating (5.2) by r_{2r}. This moves the character from the root β_1 to the simple root α_{2r-1}, but also moves α_{2r} to $-\alpha_{2r}$. We then need to exchange the integration over this negative root with an integration over β_1, which is permitted by root exchange. That is, we apply Lemma 4.3.1 to the exchange triple

$$(-\alpha_{2r}, \beta_1, \alpha_{2r-1}).$$

After this first step, we obtain the integral

$$\int_{[R_1]} \theta(vg)\hat{\psi}_1(v)\, dv,$$

where

$$R_1 = \left\{ \begin{pmatrix} I_{2r-2} & & X_1 & & X_2 & Y \\ & 1 & t & y & z & X_2^* \\ & & 1 & 0 & y & \\ & & & 1 & -t & X_1^* \\ & & & & 1 & \\ & & & & & I_{2r-2} \end{pmatrix} : Y \in \mathrm{Mat}^0_{2r-2}; X_i \in \mathbb{A}^{2r-2};\ t, y, z \in \mathbb{A} \right\},$$

and

$$\hat{\psi}_1(v) = \psi(v_{2r-1,2r} + v_{2r-3,2r+3} + \cdots + v_{1,4r-1}).$$

We proceed inductively and assume that we have shown that, for $1 \leq k \leq r-1$, the non-vanishing of (5.1) is equivalent to the non-vanishing of

$$\int_{[R_k]} \theta(vg)\hat{\psi}_k(v)\,dv, \tag{5.3}$$

where

$$R_k = \left\{ \begin{pmatrix} I_{2r-2k} & X & Y \\ & M_k & X^* \\ & & I_{2r-2k} \end{pmatrix} : Y \in \mathrm{Sym}_J^{2r-2k}; X_i \in \mathrm{Mat}^E_{(2r-2k)\times(4k)}\ M_k \in N_k \right\},$$

$$\mathrm{Mat}^E_{l\times r} = \{X \in \mathrm{Mat}_{l\times r} : x_{i,j} \neq 0 \Rightarrow j \text{ even}\},$$

and

$$N_k = \{M \in U(\mathrm{Sp}_{4k}) : \text{for } i < j \leq 4k-i+1, m_{i,j} \neq 0 \Rightarrow i \text{ odd}\}.$$

Here we are considering the natural embedding $\mathrm{Sp}_{4k} \hookrightarrow \mathrm{Sp}_{4r}$ into the Levi subgroup of $P_{(2r-2k,2k)}$, and the character is given by

$$\hat{\psi}_k(v) = \psi(v_{2r-1,2r} + v_{2r-3,2r-2} + \cdots + v_{2r-2k+1,2r-2k+2})$$
$$\times\, \psi(v_{2r-2k-1,2r+2k+1} + \cdots + v_{1,4r-1}).$$

We need to show that the non-vanishing of (5.3) is equivalent to the non-vanishing of

$$\int_{[R_{k+1}]} \theta(vg)\hat{\psi}_{k+1}(v)\,dv.$$

To this end, the next root we need to move is

$$\beta_{k+1} = \alpha_{2r-2(k+1)+1} + 2\alpha_{2r-2k} + \cdots + \alpha_{2r}.$$

Before conjugating by r_{2r-2k}, we apply root exchange to all the root groups in V_k which would be conjugated to negative roots. Thus, we apply root exchange to the triples of type

$$(\gamma_{2(r-k),2(r-i)},\ \eta_{2(r-k)-1,2(r-i)},\ \beta_{k+1})$$

in the order of descending values of i, as i ranges over $1 \leq i \leq r-1$. These correspond to the roots contained in the bottom row of the "even columns" of X which are contained in the Siegel Levi subgroup.

Next, we apply root exchange on the triples of type

$$(\eta_{2(r-k),2(r-i)},\ \gamma_{2(r-k)-1,2(r-i)},\ \beta_{k+1})$$

in order of descending values of j, where here $1 \leq j \leq r$. These roots correspond to those even columns not contained in the Siegel Levi.

Finally, we apply root exchange to the triple

$$(\mu_{2(r-k)},\ \alpha_{2r-2k-1},\ \beta_{k+1})$$

to obtain the integral

$$\int_{[R_k]} \theta(vg)\hat{\psi}_k(v)\,dv,$$

where

$$R_k = \left\{ \begin{pmatrix} I_{2r-2(k+1)} & X & Y \\ & M_{k+1} & X^* \\ & & I_{2r-2(k+1)} \end{pmatrix} : Y \in \mathrm{Sym}_J^{2r-2k};\ M_{k+1} \in N_{k+1} \right\},$$

where

$$X \in \mathrm{Mat}^*_{l\times m} = \{X \in \mathrm{Mat}_{l\times m} : x_{i,j} \neq 0 \text{ and } j < m-1 \Rightarrow j \text{ even}\},$$

and where the character has the same formula as before. Conjugating this integral by r_{2r-2k} we see that the resulting integral is

$$\int_{[R_{k+1}]} \theta(vg)\hat{\psi}_{k+1}(v)\,dv,$$

thus finishing our induction. Thus, we have shown that the non-vanishing of (5.1) is equivalent to the non-vanishing of

$$\int_{[R_r]} \theta(vg)\hat{\psi}_r(v)\,dv, \tag{5.4}$$

where, in this final form, we recall that

$$R_r = N_r = \{m \in U_{2r} : \text{for } i < j \leq 4r - i + 1, m_{i,j} \neq 0 \Rightarrow i \text{ odd}\},$$

and

$$\hat{\psi}_r(v) = \psi(v_{1,2} + v_{3,4} + \cdots + v_{2r-1,2r}).$$

We claim that this integral is not trivial for some choice of data. If not, setting $U_{(2^r)}$ to be the unipotent radical of the standard parabolic $P_{(2^r)}$ which has Levi subgroup $L \cong \mathrm{GL}_2^r$, one may show (using the results in Appendix A.2) that (5.4) is equal to the constant term

$$\int_{[R_r]} \theta(vg)\hat{\psi}_r(v)\,dv = \int_{[U_{(2^r)}]} \int_{[R_r]} \theta(uvg)\hat{\psi}_r(v)\,dvdu = \int_{[U_{2r}]} \theta(vg)\psi_{2r}^0(v)\,dv, \tag{5.5}$$

which is also zero for all choices of data. Here ψ_{2r}^0 is as in Section 2.1. In particular, the integral vanishes when $g = 1$. Applying Theorem 3.2.2, we see that this integral equals, using the notation in [Cai16],

$$\int_{[U_{\mathrm{GL}_{2r}}]} \theta_{\mathrm{GL}_{2r}}(u)\psi_\lambda(u)du,$$

where $\theta_{\mathrm{GL}_{2r}} \in \Theta^{(2)}_{\mathrm{GL}_{2r}}$ is a theta function on the double cover of GL_{2r}[2], and $U_{\mathrm{GL}_{2r}}$ is the unipotent radical of the Borel subgroup of GL_{2r}.

This integral is the λ-semi-Whittaker coefficient of θ_{GL} corresponding to the partition $\lambda = (2^r)$. This coefficient is non-vanishing on $\Theta^{(2)}_{\mathrm{GL}_{2r}}$ as seen in [BG92], so that our assumption gives a contradiction. Thus, there exists a choice of data such that the integral (5.4) is non-vanishing, and the proposition is proved. □

To finish the proof of Theorem 5.2.1, we consider the odd rank case:

[2]As noted in Section 3.1.4, we are working with a twist of a pushout of this double cover. The analysis of Bump and Ginzburg in [BG92] on the global theta representation applies to this cover without essential change.

Proposition 5.2.3. *The orbit* $\mathcal{O}_\Theta = (2^{2r+1})$ *supports* Θ_{4r+2}.

Remark 5.2.4. If one sought to prove analogous results for higher degree covers, a critical obstruction occurs at this stage. The non-vanishing of the Fourier coefficient (5.7) below follows from standard results for automorphic forms on SL(2). In the case of studying Fourier coefficients of $\Theta_{2r}^{(2n)}$ with $n > 2$, one needs to prove genericity results for residues of certain Eisenstein series to obtain the analogous non-vanishing result. Accomplishing this is a fundamental obstacle to considering generalizations to higher degree covers.

Proof. Now set $\mathcal{O} = (2^{2r+1})$, and consider the integral

$$\int_{[V_2(\mathcal{O})^0]} \theta(vg)\psi_0(v)\,dv, \tag{5.6}$$

where

$$V_2(\mathcal{O})^0 = \left\{ \begin{pmatrix} I_{2r+1} & Y \\ & I_{2r+1} \end{pmatrix} : Y \in \mathrm{Sym}_J^{2r+1}(\mathbb{A}),\ \text{and } y_{2r+1,1} = 0 \right\},$$

and

$$\psi_0(v) = \psi(\epsilon_1 v_{2,4r+1} + \cdots + \epsilon_m v_{2r+1,2r+2}).$$

Here we make the same assumption on ϵ_i as before, so that the corresponding orthogonal group is split. We claim that this integral is non-vanishing for some choice of data. If not, then the integral

$$\int_{[V_{1,2r}]} \int_{[V_2(\mathcal{O})^0]} \theta(uvg)\psi_0(v)\,dvdu,$$

where $V_{1,2r}$ is the unipotent radical of the maximal parabolic in Sp_{4r+2} with Levi $L_{1,2r} \cong \mathrm{GL}_1 \times \mathrm{Sp}_{4r}$, is zero for all choices of data. However, Theorem 3.2.2 tells us that this integral is equal to

$$\int_{[V(2^{2r})]} \theta_{4r}(v)\psi_{\underline{\epsilon}}(v)\,dv,$$

which is non-vanishing for some choice of data by Proposition 5.2.2. Thus, the integral (5.6) is non-vanishing for some choice of data.

As before, our assumption that the stabilizer $\mathrm{O}_{\underline{\epsilon}}(K)$ is a split group tells us that there exists an $h \in M(\mathcal{O})_K$ such that if we conjugate (5.6) by h we obtain

$$\int_{[V_2(\mathcal{O})^0]} \theta(vg')\hat{\psi}_0(v)\,dv,$$

where

$$\hat{\psi}_0(v) = \psi(v_{2r,2r+1} + v_{2r-2,2r+3} + \cdots + v_{3,4m-3}).$$

Note that the roots with a nontrivial character on the corresponding root group are

$$\eta_{2k,2k} = \alpha_{2k} + 2\alpha_{2k+1} + \cdots + \alpha_{2r},$$

for $1 \leq k \leq r$.

Consider the following r root triples

$$(\eta_{1,2k}, -\gamma_{1,2k-1}, \eta_{2k,2k}).$$

One can check that these are in fact exchange triples, to which we apply Lemma 4.3.1. Thus, we have that the function

$$f(g) = \int_{[V']} \theta(vg)\hat{\psi}_0(v)\,dv,$$

is a non-zero function in g. Here, we have

$$V' = \left\{ \begin{pmatrix} 1 & & & \\ u & I_{2r} & & \\ & & I_{2r} & \\ & & u^* & 1 \end{pmatrix} \begin{pmatrix} I_{2r+1} & Y \\ & I_{2r+1} \end{pmatrix} \right\},$$

with

$$Y \in \mathrm{Sym}_J^{2r+1}(\mathbb{A}), \text{ and } \qquad y_{i,1} = 0 \text{ for } i \text{ odd},$$

$$u \in \mathbb{A}^{2r} \qquad \text{with} \qquad u_j = 0 \text{ for } j \text{ even}.$$

The point of this last root exchange is that when we restrict $g \in \overline{\mathrm{Sp}}_{4r+2}(\mathbb{A})$ to the copy of $\overline{\mathrm{SL}}_2(\mathbb{A})$ corresponding to the highest root, f is a nonzero automorphic function on this $\overline{\mathrm{SL}}_2$. That is, f is invariant under the K-points of

$$\iota\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & & b \\ & I_{4r} & \\ c & & d \end{pmatrix}.$$

As a nonzero automorphic function of $\overline{\mathrm{SL}}_2(\mathbb{A})$, $f \circ \iota$ has a non-trivial Fourier coefficient; thus, there exists an $\xi \in K^\times$ such that the integral

$$\int_{[\mathbb{A}]} f\left(\iota\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} g\right) \psi(\xi t)\, dt \tag{5.7}$$

is not zero.

Reversing the previous root exchanges, and conjugating by an appropriate element in $M(\mathcal{O})_K$, we see that the non-vanishing of (5.7) is equivalent to the non-vanishing of

$$\int_{[V_2(\mathcal{O})]} \theta(vg)\psi_{\mathcal{O},\underline{\epsilon}}(v)\, dv.$$

□

As a final note, we remark that the equality analogous to (5.5) holds in the odd rank case. We record this as follows.

Proposition 5.2.5. *Set $\mathcal{O} = (2^r)$. Then the integral*

$$\int_{[V_2(\mathcal{O})]} \theta_r(vg)\psi_{\mathcal{O},\epsilon}(u)\, du \neq 0$$

if and only if

$$\theta^{U_r,\psi_r^0}(g) = \int_{[U_r]} \theta(vg)\psi_r^0(v)\,dv \neq 0.$$

We refer to the function $\theta^{U_r,\psi_r^0}(g)$ as the *split coefficient* of θ. It arises in the computation of certain Whittaker coefficients of lifting representations in Chapter 9.

Chapter 6

Unramified Local Lifting and Satake Parameters

In this chapter, we study the local restriction problem in the case that π and τ are both principal series representations. More precisely, let W_0 be a symplectic space over our local field F. We require the same restrictions on F imposed in Section 5.1. For example, F may be the completion of a number field K at a finite place of odd residue characteristic. This ensures that the results of that section hold for theta representations over F.

6.1 Statement of the local correspondence

Let $W_n = X_n \oplus W_0 \oplus X_n^*$ be the $\dim(W) + 2n$ dimensional symplectic space as in the introduction. Setting $V_n = X_n \oplus X_n^*$, we have the embedding of metaplectic covers

$$\overline{\mathrm{Sp}}(W_0)_F \times_{\mu_4} \overline{\mathrm{Sp}}(V_n)_F \to \overline{\mathrm{Sp}}(W_n)_F.$$

Setting $\dim(W_0) = 2m$, this embedding restricts to an embedding

$$Z(\overline{T}_m) \times_{\mu_4} Z(\overline{T}_n) \to Z(\overline{T}_{m+n}).$$

As noted in Section 3.1, there are two unramified theta representations, $\Theta_{2(m+n)}$, for $\overline{\mathrm{Sp}}(W_n)_F$ corresponding to the two choices of genuine unramified exceptional characters, corresponding to the two genuine unramified distinguished characters: setting $y_i = 2\alpha_i^\vee$, these characters are determined by

$$\overline{\chi}^0_{\pm}(y_i(a), \zeta) = \begin{cases} \zeta & : \quad i < r \\ \\ \pm\zeta(a,a)_2 : & i = r \end{cases}.$$

Here $(\cdot,\cdot)_2$ is the quadratic Hilbert symbol.

The importance of this choice is discussed in [GG14], where it is shown that the choice of such a character is equivalent to choosing a distinguished splitting of the L-group short exact sequence

$$1 \longrightarrow \overline{\mathrm{Sp}}(W_n)^\vee \longrightarrow {}^L\overline{\mathrm{Sp}}(W_n) \longrightarrow WD_F \longrightarrow 1,$$

where $WD_F = W_F \times \mathrm{SL}_2(\mathbb{C})$ denotes the Weil-Deligne group of F. See [Wei14] and [Wei15]. We fix a such choice, $\overline{\chi}^0$, and note that the restriction to either subtorus $Z(\overline{T}_m)$ or $Z(\overline{T}_n)$ is also a distinguished character for the smaller rank group. In particular, if we choose $\overline{\chi}^0_{\pm}$ on $Z(\overline{T}_{m+n})$, then the restriction to $Z(\overline{T}_*)$ for $* \in \{m, n\}$ is also the unitary distinguished character $\overline{\chi}^0_{\pm}$. Therefore, our choice of unramified genuine distinguished character in the construction of $\Theta_{2(m+n)}$ uniquely determines an unramified distinguished character for both smaller symplectic groups. Changing the choice multiplies the Satake parameters by the nontrivial element of $Z(\overline{\mathrm{G}}^\vee)$.

Without loss, assume $m \leq n$. Fix a polarization $W_n = X \oplus Y$ and choose a basis of X, $\{e_1\}_{i=1}^{m+n}$. Let $\{f_i\}_{i=1}^{m+n}$ be the dual basis:

$$\langle e_i, f_j \rangle = \delta_{i,m+n-j+1}.$$

We do this so that by restriction, we have induced polarizations

$$W_0 = \mathrm{span}_F\{e_i : n < i\} \oplus \mathrm{span}_F\{f_j : j \leq n\}$$

and

$$V_n = \mathrm{span}_F\{e_i : i \leq n\} \oplus \mathrm{span}_F\{f_j : n < j\}.$$

Acting by the appropriate Weyl group element $\hat{w}$ (which respects the polarization of W_n), we have

$$\hat{w}(W_0) = \mathrm{span}_F\{e_i : 2 \leq i \leq 2n \text{ even}\} \oplus Y(\hat{w}(W))$$

and

$$\hat{w}(V_n) = \mathrm{span}_F\{e_i : 1 \leq i < 2n+1 \text{ odd or } i > 2n\} \oplus Y(\hat{w}(V_n)),$$

where $Y(V) \subset Y$ indicates the appropriate dual to the subspace $V \cap X$. In this section, we use the induced embedding of $\overline{\mathrm{Sp}}(W_0)_F \times_{\mu_4} \overline{\mathrm{Sp}}(V_n)_F$ into $\overline{\mathrm{Sp}}(W_n)_F$.

The main reason for this choice is to facilitate easier comparison with our results on the Jacquet modules of $\Theta_{2(m+n)}$.

Having fixed a basis of W_0, we now identify $\mathrm{Sp}(W_0)_F = \mathrm{Sp}_{2m}(F)$ and recall the parametrization of our maximal torus T_m

$$M(t_1, \ldots, t_m) = \mathrm{diag}(t_1, \ldots, t_m, t_m^{-1}, \ldots, t_1^{-1}).$$

With this parametrization, it is easy to parametrize genuine characters of $Z(\overline{T_m})$: $\overline{\chi} = \overline{\chi}^0\chi$, with

$$\chi(M(t_1, \ldots, t_m), \epsilon) = \prod_i \chi_i(t_i),$$

where $\chi_i : F^\times \to \mathbb{C}^\times$ are multiplicative characters. Since we are discussing characters of $Z(\overline{T_m})$, it is implicit in the above formulas that $t_i \in (F^\times)^2$.

Theorem 6.1.1. *Suppose $\chi : Z(\overline{T_m}) \to \mathbb{C}^\times$ and $\mu : Z(\overline{T_n}) \to \mathbb{C}^\times$ are characters such that*

$$\mathrm{Hom}_{\overline{\mathrm{Sp}}_{2m}(F)\times_{\mu_4}\overline{\mathrm{Sp}}_{2n}(F)}(\Theta_{2(m+n)}, \mathrm{Ind}(\chi)\otimes\mathrm{Ind}(\mu)) \neq 0.$$

Set $k = n - m$. Then there exist k indices $\{j_1, j_2, \ldots, j_k\} \subset \{1, \ldots, n\}$ such that

$$\mu_{j_i} = |\cdot|^{-(2(k-i)+1)/4}.$$

Furthermore, we have

$$\left\{\chi_i^{\pm 1}\right\}_{1\leq i\leq m} = \left\{\mu_j^{\pm 1}\right\}_{j\notin\{j_1,j_2,\ldots,j_k\}}$$

Remark 6.1.2. In the above notation, we have made explicit use of our choice of distinguished character $\overline{\chi}^0$. Therefore, the characters $\left\{\chi_i^{\pm 1}\right\}$ encode the Satake parameters relative to this choice in the case χ is unramified, and likewise with $\{\mu_j^{\pm 1}\}$.

Proof. We argue by induction on m. The case of $\mathrm{SL}_2 \times \mathrm{SL}_2 \hookrightarrow \mathrm{Sp}_4$ is trivial to work out and we leave the details to the reader. The argument is of the same form as the general case.

By transitivity of induction, we identify

$$\mathrm{Ind}(\mu) \cong \mathrm{Ind}_{\overline{P}_{(1,n-1)}}^{\overline{\mathrm{Sp}}_{2n}(F)}\left(\mu_1\delta_{(1,n-1)}^{1/2}\otimes\mathrm{Ind}(\mu')\right),$$

where $P_{(1,n-1)} = M_{(1,n-1)}U_{(1,n-1)}$ is the maximal parabolic of $\mathrm{Sp}_{2n}(F)$ with Levi subgroup $M_{(1,n-1)} \cong \mathrm{GL}_1 \times \mathrm{Sp}_{2n-2}$, and $\mu' : Z(\overline{T_{n-1}}) \to \mathbb{C}^\times$ is the character induced by deleting μ_1. Additionally, $\delta_{(1,n-1)}$ is the modular character of $P_{(1,n-1)}$.

Therefore by Frobenius reciprocity,

$$\mathrm{Hom}_{\overline{\mathrm{Sp}}_{2m}(F)\times_{\mu_4}\overline{M}_{(1,n-1)}}\left(J_{U_{1,n-1}}(\Theta_{2(m+n)}), \mu_1\delta_{(1,n-1)}^{1/2}\otimes\mathrm{Ind}(\chi)\otimes\mathrm{Ind}(\mu')\right) \neq 0 \quad (6.1)$$

At this point, we note the following proposition:

Proposition 6.1.3. *There exists a* $\overline{\mathrm{Sp}}_{2m}(F) \times_{\mu_4} \overline{M}_{(1,n-1)}$*-equivariant filtration*

$$0 \subset V(\Theta, \psi_1) \subset J_{U_{1,n-1}}(\Theta_{2(m+n)})$$

such that

$$V(\Theta, \psi_1) \cong J_{U_{1,n-1}}\left(\mathrm{ind}_{\overline{Q}'_{2,n+m-2}}^{\overline{Q}_{1,n+m-1}}\left(J_{U_{1,n+m-1},\psi_1}(\Theta_{2(m+n)})\right)\right), \tag{6.2}$$

and

$$J_{U_{1,n-1}}(\Theta_{2(m+n)})/V(\Theta, \psi_1) \cong J_{U_{1,n+m-1}}(\Theta_{2(m+n)}). \quad \square$$

For sake of continuity, the proof of this proposition is postponed to Appendix A.1.

Thus, we have that

$$\begin{aligned} 0 &< \dim_{\mathbb{C}}\left(\mathrm{Hom}\left(J_{U_{1,n-1}}(\Theta_{2(m+n)}), \mu_1\delta_{(1,n-1)}^{1/2} \otimes \mathrm{Ind}(\chi) \otimes \mathrm{Ind}(\mu')\right)\right) \\ &\leq \dim_{\mathbb{C}}\left(\mathrm{Hom}\left(V(\Theta, \psi_1), \mu_1\delta_{(1,n-1)}^{1/2} \otimes \mathrm{Ind}(\chi) \otimes \mathrm{Ind}(\mu')\right)\right) \\ &\quad + \dim_{\mathbb{C}}\left(\mathrm{Hom}\left(J_{U_{1,n+m-1}}(\Theta_{2(m+n)}), \mu_1\delta_{(1,n-1)}^{1/2} \otimes \mathrm{Ind}(\chi) \otimes \mathrm{Ind}(\mu')\right)\right). \end{aligned}$$

We have two cases to consider:

Case 1: Suppose that

$$\mathrm{Hom}\left(V(\Theta, \psi_1), \mu_1\delta_{(1,n-1)}^{1/2} \otimes \mathrm{Ind}(\chi) \otimes \mathrm{Ind}(\mu')\right) \neq 0. \tag{6.3}$$

According to the isomorphism (6.2), we may apply [BZ77, Theorem 5.2] and, setting

$$H_\gamma = \left(\mathrm{Sp}_{2m}(F) \times P_{(1,n-1)}\right) \cap \gamma^{-1}Q'_{2,n+m-2}\gamma,$$

we see that this Hom-space is glued together from the vector spaces

$$\mathrm{Hom}_{H_\gamma}\left(^\gamma\left(J_{U_{1,n+m-1},\psi_1}(\Theta_{2(m+n)})\right), \delta_{(1,n-1)}^{-1}\delta_{H_\gamma} \otimes \mu_1\delta_{(1,n-1)}^{1/2} \otimes \mathrm{Ind}(\chi) \otimes \mathrm{Ind}(\mu')\right),$$

as γ ranges over double coset representatives

$$\gamma \in Q'_{2,n+m-2} \backslash \mathrm{Sp}_{2(n+m)}(F) / \mathrm{Sp}_{2m}(F) \times P_{(1,n-1)},$$

where we are viewing the group in the right quotient as a subgroup of $\mathrm{Sp}_{2(n+m)}(F)$ It is an enjoyable exercise (of a similar spirit to the proof of [Kud86, Prop. 3.4]) to see that there are only three double cosets. By applying [BZ77, Theorem 5.2], it is immediate that the only coset which contributes to (6.3) corresponds to $\gamma = 1$. We thus obtain a non-zero $H = H_1 = (\mathrm{Sp}_{2m}(F) \times P_{(1,n-1)}) \cap Q'_{1,n+m-1}$- equivariant morphism

$$J_{U_{1,n+m-1},\psi_1}(\Theta_{2(m+n)}) \longrightarrow \delta_{(1,n-1)}^{-1}\delta_H \otimes \mu_1\delta_{(1,n-1)}^{1/2} \otimes \mathrm{Ind}(\chi) \otimes \mathrm{Ind}(\mu').$$

We have the following proposition:

Proposition 6.1.4. *Suppose $r > 1$ and consider the local theta representation Θ_{2r} on $\overline{\mathrm{Sp}}_{2r}(F)$. Consider the unipotent radical $U_{(2,r-2)}$ of the maximal parabolic subgroup with Levi subgroup $M_{(2,r-2)} \cong \mathrm{GL}_2 \times \mathrm{Sp}_{2r-4}$. Then there is a surjection of $\overline{L}'_{2,r-2}(F)$-modules*

$$J_{U_{(2,r-2)}}(\Theta_{2r}) \twoheadrightarrow J_{U_{1,r-1},\psi_1}(\Theta_{2r}),$$

where $L'_{k,r-k} = \mathrm{GL}_1^{\Delta} \times \mathrm{Sp}_{2(r-k)} \subset L_{k,r-k}$ for all $1 \leq k \leq r$.

Proof. Our assumptions on the field F ensure that Theorem 5.1.3 holds. If we let V denote the subgroup $V = U_{1,r-1}U_{\mu_2}$, where U_{μ_2} is the one dimensional root subgroup associated to the long positive root

$$\mu_2 = 2\alpha_2 + \cdots + 2\alpha_{r-1} + \alpha_r,$$

then there is a $\overline{M}'_{(2,r-2)}(F)$-isomorphism

$$J_{U_{1,r-1},\psi_1}(\Theta_{2r}) \cong J_{V,\psi_1}(\Theta_{2r}),$$

where by a slight abuse of notation, we denote by ψ_1 the character of V which trivially extends ψ_1. This is equivalent to the statement that the nilpotent orbit $\mathcal{O} = (41^{2r-4})$ does not support Θ_{2r}.

The proof can proceed in similar fashion to the global argument in Lemma A.2.1 and Lemma A.2.2. It is clear that the result follows if we show that the unipotent subgroup $U_{1,r-2} \subset \mathrm{Sp}_{2(r-1)}(F) \subset \mathrm{Sp}_{2r}(F)$ acts trivially on $J_{V,\psi_1}(\Theta_{2r})$. If not, then there is some nontrivial character of $\psi : U_{1,r-2}/U_{\mu_2} \to \mathbb{C}^\times$ such that

$$J_{U_{1,r-2}/U_{\mu_2},\psi'}(J_{V,\psi_1}(\Theta_{2r})) \cong J_{V_{2,r-2},\psi_2}(\Theta_{2r}) \neq 0.$$

We have made use of the fact that $\mathrm{Sp}_{2(r-1)}(F) \subset \mathrm{Sp}_{2r}(F)$ acts transitively on such characters, so that we obtain the twisted Jacquet module with respect to the Gelfand-Graev character

$$\psi_2 : V_{2,r-2} \to \mathbb{C}^\times, \qquad v \mapsto \psi(v_{1,2} + v_{2,3}).$$

Again applying Theorem 5.1.3, we have an isomorphism of $\overline{L}'_{3,r-3}(F)$-modules

$$J_{V_{2,r-2},\psi_2}(\Theta_{2r}) \cong J_{V_{2,r-2}U_{\mu_3},\psi_2}(\Theta_{2r}),$$

where $\mu_3 = 2\alpha_3 \cdots + 2\alpha_{r-1} + \alpha_r$ is the next long root. This is equivalent to the statement that the nilpotent orbit $\mathcal{O} = (61^{2r-6})$ does not support Θ_{2r}.

We know, as in the proof of Lemma A.2.1, that $U_{1,r-3}/U_{\mu_3}$ cannot act trivially on this Jacquet module, for if it did then we could apply Theorem 3.1.5 to obtain a non-zero Whittaker module on $\Theta^{(2)}_{GL_3}$, a contradiction as this representation is non-generic. Hence, as before we conclude the non-vanishing of $J_{V_{3,r-3},\psi_3}(\Theta_{2r})$.

Continuing in this way, we finally arrive at the non-vanishing of $J_{U_r,\psi_r}(\Theta_{2r})$, where we recall that U_r denotes the full unipotent radical of the Borel subgroup of $\mathrm{Sp}_{2r}(F)$ and ψ_r is a generic Gelfand-Graev character of U_r. This contradicts the fact that Θ_{2r} is non-generic, completing the proof. $\square$

It follows from the Proposition along with Theorem 3.1.5 that we in fact have a non-zero $H \cap \overline{L}'_{2,n+m-2}(F)U_{1,n+m-2}$- equivariant morphism

$$J_{N_1,\psi'}(\Theta^{(2)}_{\mathrm{GL}_2}) \otimes \Theta_{2(m+n-2)} \longrightarrow \delta^{-1}_{(1,n-1)}\delta_H \otimes \mu_1\delta^{1/2}_{(1,n-1)} \otimes \mathrm{Ind}(\chi) \otimes \mathrm{Ind}(\mu').$$

Here $N_1 \subset \mathrm{GL}_2(F)$ is the subgroup of upper triangular unipotent matrices, and

$$\psi' \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} = \psi(x).$$

To simplify the notation, we collect all the characters on the right hand side as

$$\lambda = \delta^{-1}_{(1,n-1)}\delta_H \otimes \mu_1\delta^{1/2}_{(1,n-1)}$$

Note that

$$\begin{aligned} H \cap \overline{L}'_{2,n+m-2}(F)U_{1,n+m-2} &= \mathrm{Sp}_{2(n-1)} \times(\mathrm{GL}_1^{\Delta} \times \mathrm{Sp}_{2(m-1)})U_{1,m-1} \\ &\cong \mathrm{Sp}_{2(n-1)} \times Q_{1,m-1}, \end{aligned}$$

By the properties of induction, this implies that

$$\mathrm{Hom}\left(\mathrm{Ind}^{\overline{\mathrm{Sp}}_{2(n-1)}\times\overline{\mathrm{Sp}}_{2m}}_{\overline{\mathrm{Sp}}_{2(n-1)}\times\overline{Q}_{1,m-1}}\left(J_{N_1,\psi'}(\Theta^{(2)}_{\mathrm{GL}_2}) \otimes \Theta_{2(m+n-2)}\right), \lambda \otimes \mathrm{Ind}(\chi) \otimes \mathrm{Ind}(\mu')\right) \neq 0. \tag{6.4}$$

We make use of a result of Bernstein (see [Bus01] for the proof):

Proposition 6.1.5. *If $P = MU$ is a parabolic subgroup of G. Let $P_- = MU_-$ denote the opposite parabolic of P. If W is a smooth representation of the Levi subgroup M and V is a smooth representation of G, there there is an isomorphism*

$$\mathrm{Hom}_G(\mathrm{Ind}^G_P(W), V) \cong \mathrm{Hom}_M(W, J_{U_-}(V)). \qquad \square$$

Applying this proposition to (6.4), we find that

$$\mathrm{Hom}\left(J_{N_1,\psi'}(\Theta^{(2)}_{\mathrm{GL}_2}) \otimes \Theta_{2(m+n-2)}, \lambda \otimes J_{\overline{U}_{1,m-1}}(\mathrm{Ind}(\chi)) \otimes \mathrm{Ind}(\mu')\right) \neq 0$$

Applying the geometric lemma of [BZ77] to this Jacquet module, we have that there is a $\overline{M}_{(1,m-1)}$-equivariant filtration of $J_{\overline{U}_{1,m-1}}(\mathrm{Ind}(\chi))$ such that the corresponding factors are representations of the form

$$\chi_i^{\pm 1}\delta_{(1,m-1)}^{-1/2} \otimes \mathrm{Ind}(\chi'),$$

with $k \in \{2, 3, \ldots, m\}$, and where $\chi' : Z(\overline{T_{m-1}}) \to \mathbb{C}^\times$ is a character such that

$$\chi'_j \in \{\chi_k^{\pm 1} : k \in \{2, 3, \ldots, m\}, k \neq i\}.$$

Therefore, there exists some k such that our morphism induces a non-zero morphism (without loss we can assume χ_k^{-1} appears):

$$J_{N_1,\psi'}(\Theta^{(2)}_{\mathrm{GL}_2}) \otimes \Theta_{2(m+n-2)} \longrightarrow \lambda \otimes \chi_k^{-1}\delta_{(1,m-1)}^{-1/2} \otimes \mathrm{Ind}(\chi') \otimes \mathrm{Ind}(\mu').$$

We now conclude: the diagonal GL_1 acts on the right hand side by the accumulated character

$$\lambda\chi_k^{-1}\delta_{(1,m-1)}^{-1/2} = \mu_1\chi_k^{-1}\delta_{(1,n-1)}^{-1}\delta_H\delta_{(1,n-1)}^{1/2}\delta_{(1,m-1)}^{-1/2} = \mu_1\chi_k^{-1}|\cdot|^{m+n}$$

where the modular characters are easy to compute, and we note $\delta_H = \delta_{(1,n-1)}\delta_{(1,m-1)}$. On the other hand, it acts on the left hand side by the appropriate exceptional character on GL_2 as indicated in Theorem 3.1.5:

$$\chi_{\mathrm{GL},2}\begin{pmatrix} a & \\ & a \end{pmatrix} = |a|^{m+n}.$$

Therefore $\mu_1 = \chi_k$, and we are left with

$$\mathrm{Hom}_{\overline{\mathrm{Sp}}_{2(m-1)}(F)\times_{\mu_4}\overline{\mathrm{Sp}}_{2(n-1)}(F)}\left(\Theta_{2(m+n-2)}, \mathrm{Ind}(\chi') \otimes \mathrm{Ind}(\mu')\right) \neq 0.$$

By induction, this case is proved. We note that in this case the difference in the rank of the smaller symplectic groups k remains unchanged.

Case 2: Suppose instead that

$$\mathrm{Hom}\left(J_{U_{1,n+m-1}}(\Theta_{2(m+n)}), \mu_1\delta_{(1,n-1)}^{1/2}\otimes \mathrm{Ind}(\chi)\otimes \mathrm{Ind}(\mu')\right)\neq 0.$$

Applying Theorem 3.1.5, we have a non-zero $\overline{\mathrm{Sp}}_{2m}(F)\times_{\mu_4}\overline{M}_{(1,n-1)}$-equivariant map

$$\chi_{\mathrm{GL},1}\otimes\Theta_{2(m+n-1)}\longrightarrow \mu_1\delta_{(1,n-1)}^{1/2}\otimes \mathrm{Ind}(\chi)\otimes \mathrm{Ind}(\mu').$$

Here, $\chi_{\mathrm{GL},1}$ is the character on $\overline{\mathrm{GL}_1}(F)$ arising in Theorem 3.1.5. Comparing the action of this GL_1-component of $\overline{M}_{(1,n-1)}$, we find that

$$\chi_{\mathrm{GL},1}=\mu_1\delta_{(1,n-1)}^{1/2}$$

so that $\mu_1=|\cdot|^{-(2k-1)/4}$, where recall that $k=n-m$. We therefore obtain

$$\mathrm{Hom}_{\overline{\mathrm{Sp}}_{2m}(F)\times_{\mu_4}\overline{\mathrm{Sp}}_{2(n-1)}(F)}(\Theta_{2(m+n-1)}, \mathrm{Ind}(\chi)\otimes\mathrm{Ind}(\mu'))\neq 0.$$

Since our analysis has utilized the choice of embedding

$$\mathrm{Sp}_{2m}\times\mathrm{Sp}_{2(n-1)}\hookrightarrow \mathrm{Sp}_{2(m+n-1)},$$

continuing the same argument would lead us to now apply transitivity of induction to $\mathrm{Ind}(\chi)$ and study the Jacquet module $J_{V_{1,m-1}}(\Theta_{2(m+n-1)})$. As above, this Jacquet module has a filtration and there are two cases to consider. If Case 1 occurs, then the analysis is the same and we reduce to

$$\mathrm{Hom}_{\overline{\mathrm{Sp}}_{2(m-1)}(F)\times_{\mu_4}\overline{\mathrm{Sp}}_{2(n-2)}(F)}(\Theta_{2(m+n-3)}, \mathrm{Ind}(\chi'')\otimes\mathrm{Ind}(\mu''))\neq 0.$$

Repeat this (i times, say) until Case 2 occurs. Thus, we have a non-zero map

$$\overline{\chi}_1\otimes\Theta_{2(m+n-2i-1)}\longrightarrow \chi_1^{(i)}\delta_{(1,m-i-1)}^{1/2}\otimes\mathrm{Ind}(\chi^{(i)})\otimes\mathrm{Ind}(\mu^{(i+1)})),$$

where $\chi^{(i)}: Z(\overline{T}_{m-i})\to\mathbb{C}^\times$ and $\mu^{(i+1)}: Z(\overline{T}_{n-i-1})\to\mathbb{C}^\times$ are the characters arising from the argument in Case 1 applied i times. An identical computation to the one above now shows that

$$\chi_1^{(i)}=|\cdot|^{-(2(1-k)-1)/4}=\mu_1^{-1}.$$

This shows that if Case 2 occurs when analyzing $\mathrm{Ind}(\chi)$, we still obtain $\chi_1 \in \{\mu_j^{\pm 1}\}$. This is the only case not immediate from induction. Applying the case-by-case argument above, the theorem is proven.

In Section 8, we will interpret this result in terms of Arthur's conjectures, subject to a natural generalization of Shimura's correspondence.

Chapter 7

The Global Lifting and Cuspidality

In this chapter, we turn to the global theta lifting and prove Theorem 7.2.1.

7.1 Global Lifting

Suppose now that K is a number field such that $|\mu_4(K)| = 4$. As in the introduction, let W_0 be a $2m$-dimensional symplectic space over K, and let

$$W_n = X_n \oplus W_0 \oplus X_n^*, \quad V_n = X_n \oplus X_n^*$$

be symplectic spaces of dimension $2(m+n)$ (resp. $2n$). Then there is an inclusion of symplectic groups

$$\mathrm{Sp}(W_0)_{\mathbb{A}} \times \mathrm{Sp}(V_n)_{\mathbb{A}} \hookrightarrow \mathrm{Sp}(W_n)_{\mathbb{A}}.$$

Here $\mathbb{A} = \mathbb{A}_K$ is the adele ring associated to K. Then the above embedding is covered by an embedding of metaplectic groups

$$\overline{\mathrm{Sp}}(W_0)_{\mathbb{A}} \times_{\mu_4} \overline{\mathrm{Sp}}(V_n)_{\mathbb{A}} \hookrightarrow \overline{\mathrm{Sp}}(W_n)_{\mathbb{A}}$$

Let π be an ϵ-genuine cuspidal automorphic representation of $\overline{\mathrm{Sp}}(W_0)_{\mathbb{A}}$, and let

$\Theta_{W_n} := \Theta_{2(m+n)}$ be the theta representation of $\overline{\mathrm{Sp}}(W_n)_{\mathbb{A}}$. Consider the integral pairing

$$\int_{[\mathrm{Sp}(W_0)]} \overline{\varphi(g)}\theta_n((h,g))\,dg, \tag{7.1}$$

where $\varphi \in \pi$ and $\theta_n \in \Theta_{W_n}$. The vector space generated by such integrals as we vary over all vectors in π and Θ_{W_n} forms an ϵ-genuine automorphic representation of $\overline{\mathrm{Sp}}(V_n)_{\mathbb{A}}$, which we denote by $\Theta_n(\pi)$. Note that it is immediate from the spectral decomposition that if π is cuspidal then, $\Theta_0(\pi) = 0$, and for any π we set $\Theta_{-1}(\pi) = 0$.

We begin our global study by noting the following theorem:

Theorem 7.1.1. *Let π be as above. Then $\Theta_{4m}(\pi) \neq 0$, so that the π lifts nontrivially to $\overline{\mathrm{Sp}}(V_{4m})_{\mathbb{A}} \cong \overline{\mathrm{Sp}}_{8m}(\mathbb{A})$.*

We sketch the proof as this result is analogous to Theorem 2 in [BFG06], and is proved in a similar manner. Supposing toward a contradiction that the integral

$$\int_{[\mathrm{Sp}(W_0)]} \overline{\varphi(g)}\theta_{4m}((h,g))\,dg$$

vanishes for all data, it is clear that the integral

$$\int_{[\mathrm{Sp}(W_0)]} \int_{[V_2(\mathcal{O})]} \overline{\varphi(g)}\theta_{4m}((v,g))\psi_{\mathcal{O}}(v)\,dv\,dg, \tag{7.2}$$

where $\mathcal{O} = (2^{4m}1^{2m})$ and

$$\psi_{\mathcal{O}}(v) = \psi(v_{4m+1,6m+1} + \cdots v_{1,10m-1})$$

must also vanish for all data. Note that $V_1(\mathcal{O})$ has the structure of a generalized Heisenberg group with center equal to $V_2(\mathcal{O})$, so that there is a homomorphism $\lambda : V_1(\mathcal{O})_{\mathbb{A}} \to \mathcal{H}_{8m^2+1}(\mathbb{A})$ to the Heisenberg group on $8m^2+1$ variables.

Denote by ω_ψ the global oscillator representation of the dual pair $(\mathrm{SO}_{4m}, \mathrm{Mp}_{2m}) \subset \mathrm{Mp}_{8m^2}$ and let $\tilde{\theta}_\phi$ be the theta function associated to $\phi \in \omega_\psi$. Results of Ikeda on

Fourier-Jacobi coefficients [Ike94] tell us that functions of the form

$$\tilde{\theta}_{\phi_1}(1,g)\int_{[V_1(\mathcal{O})]}\theta_{4m}\left(v(1,g)\right)\tilde{\theta}_{\phi_2}\left(\lambda(v)(1,g)\right)\,dv \tag{7.3}$$

are dense in the space of functions of the form

$$\int_{[V_2(\mathcal{O})]}\theta_{4m}(v,g)\psi_{\mathcal{O}}(v)\,dv. \tag{7.4}$$

Thus, the vanishing (7.2) tells us that if we set

$$H(g)=\overline{\varphi}(g)\int_{[V_1(\mathcal{O})]}\theta_{4m}\left(v(1,g)\right)\tilde{\theta}_{\phi_2}\left(\lambda(v)(1,g)\right)\,dv,$$

then we have

$$\int_{[\mathrm{Sp}(W_0)]}H(g)\tilde{\theta}_{\phi}(1,g)\,dg=0$$

for all $\phi\in\omega_\psi$. As in [BFG06], since we are considering a high enough lift, the arguments used in the proof of [Ral84, Theorem I.2.2] allow us to conclude that $H(g)\equiv 0$. This forces all functions of the form (7.3) to vanish, and hence the density results force all functions of the form (7.4) to also vanish. However, it is a straightforward check to show that the non-vanishing of Fourier coefficients associated to $\mathcal{O}_{\Theta,5m}=(2^{5m})$ in Theorem 5.2.1 forces these Fourier coefficients associated to the orbit $\mathcal{O}=(2^{4m}1^m)$ to be non-vanishing for some choice of data. We therefore obtain a contradiction.

7.2 Towering property of the theta lift

In this section we prove the rest of Theorem 1.2.1.

Theorem 7.2.1. *Suppose that π is a cuspidal automorphic representation of $\overline{\mathrm{Sp}}(W_0)_{\mathbb{A}}$. Then*

1. *If* $\Theta_k(\pi) = 0$, *then* $\Theta_{k-1}(\pi) = 0$;

2. *If* $\Theta_{k-1}(\pi) = 0$, *then* $\Theta_k(\pi)$ *is cuspidal.*

Proof. We first prove the persistence property (1). By assumption we have that for all choices of data, the integral (7.1) vanishes. Thus, if $V_{1,n-1} \subset \mathrm{Sp}(V_n)$ is the unipotent radical of the maximal parabolic preserving an isotropic line, then

$$\int_{[\mathrm{Sp}(W)]} \int_{[V_{1,n-1}]} \overline{\varphi(g)} \theta_n((u,g)) du\, dg = 0. \tag{7.5}$$

We have that $V_{1,n-1} \subset V_{1,n+m-1}$ with an abelian complimentary subgroup $(V_{1,n+m-1}/V_{1,n-1}) \cong \mathbb{G}_a^{2m}$. Acting via conjugation, we identify this vector space with the standard representation of $\mathrm{Sp}(W)$. Expanding the inner integral of (7.5) against this subgroup, we have that $\mathrm{Sp}(W)_K \cong \mathrm{Sp}_{2m}(K)$ acts on the character group with two orbits. Applying Theorem 3.2.2 with $a = 1$, the trivial orbit gives the integral

$$\int_{[\mathrm{Sp}_{2n}]} \overline{\varphi(g)} \theta_{n-1}((1,g))\, dg,$$

which, as we range over all data, gives an arbitrary element in $\Theta_{n-1}(\pi)$. Thus, the proposition follows if we show that the contribution of the non-trivial orbit in the above expansion vanishes.

Let $Q_{1,m-1}$ be the maximal parabolic subgroup of Sp_{2m} preserving a line. We may choose a representative of the nontrivial $\mathrm{Sp}_{2m}(F)$-orbit to be the character $\tilde{\psi}(u) = \psi(u_{1,m+1})$, the stabilizer of which is $Q'_{1,m-1} \subset Q_{1,m-1}$, the subgroup obtained by omitting the $\mathrm{GL}_1(F)$ piece of the Levi. Unfolding, we find that the non-trivial orbit contributes

$$\int_{Q'_{1,m-1}(K)\backslash \mathrm{Sp}_{2m}(\mathbb{A})} \int_{[V_{1,n+m-1}]} \overline{\varphi(g)} \theta_n(u(1,g)) \tilde{\psi}(u)\, du\, dg.$$

Consider the Weyl group element $w \in W$ preserving U_1 such that $w \cdot \tilde{\psi} = \psi_1^0$. Using automorphy of θ_n, we may conjugate the inner integration by w and change

variables so that the above integral equals

$$\int_{Q'_{1,m-1}(K)\backslash \mathrm{Sp}_{2m}(\mathbb{A})} \overline{\varphi(g)}\theta^{V_{1,n+m-1},\psi_1^0}(w(1,g))\,dg, \tag{7.6}$$

where

$$\theta^{V_{k,m-k},\psi_k^0}(g) = \int_{[V_{k,m-k}]} \theta(vg)\psi_k^0(v)\,dv.$$

Recall that $V_{1,m-1}$ is the unipotent radical of $Q_{1,m-1}$. We may decompose the outer integral so that we obtain

$$\int_{[V_{1,m-1}]} \overline{\varphi(vg)}\theta^{V_{1,m+n-1},\psi_1^0}(w(1,vg))\,dv$$

as an inner integration of (7.6). Identifying $V_{1,m-1}$ with its image in $\mathrm{Sp}_{2(m+n)}$, we see that $wV_{1,m-1}w^{-1} \subset V_{2,m+n-2}$, so that by Lemma A.2.2 we may change variables in θ^{V_1,ψ_1^0} to remove the dependence on v, implying that the integral equals

$$c\int_{[V_{1,m-1}]} \overline{\varphi(vg)}\,dv,$$

for some constant $c \in \mathbb{C}$. By the cuspidality of π, this is trivial. Therefore, the non-trivial orbit does not contribute. This completes the proof of part (1).

We now turn to proving part (2). For the sake of this proof, let R_l denote the image of the unipotent radical $U_{(l,n-l)}$ of the maximal parabolic subgroups of Sp_{2n}. We need to show that the vanishing of $\Theta_{n-1}(\tau)$ implies the vanishing of the integral

$$\int_{[\mathrm{Sp}_{2m}]}\int_{[R_l]} \overline{\varphi(g)}\theta_n((v,g))\,dv\,dg. \tag{7.7}$$

We begin by noting that the root subgroup associated to the highest root $\alpha_0 = \mu_1$ is a subgroup of the images of R_l for all l. Thus, by identifying this root subgroup with $Z(H_1)$, we may expand the inner integration against $[H_1/Z(H_1)]$.

As before, $\mathrm{Sp}_{2(n+m-1)}(K)$ acts on the characters of this group with two orbits. By Theorem 3.2.2, the trivial orbit gives an element of the lifting $\Theta_{m-1}(\tau)$, and

so the corresponding contribution in the Fourier expansion vanishes. Left with the non-trivial orbit, we unfold to see that (7.7) is equal to

$$\sum_{\gamma\in Q_{1,n+m-2}(K)\backslash \mathrm{Sp}_{2(n+m-1)}(K)} \int_{[\mathrm{Sp}_{2m}]} \int_{[R_l]} \overline{\varphi(g)}\vartheta_n^{V_1,\psi_1^0}(\gamma(v,g))\,dv\,dg, \tag{7.8}$$

where $Q_{1,n+m-2}$ is the maximal parabolic subgroup of $\mathrm{Sp}_{2(n+m-1)}(K)$ preserving a line, and $V_i = V_{i,n+m-l}$ is the simplifying notation we use in this argument.

Note that we have replaced $\overline{\theta}_n^{V_1,\psi_1^0}(g)$ with

$$\vartheta_n^{V_1,\psi_1^0}(g) = \sum_{\xi\in K^\times} \theta_n^{V_1,\psi_1^0}(h_1(\xi)g),$$

where

$$h_1(\xi) = \mathrm{diag}(1,\xi,1,\ldots,1,\xi^{-1},1).$$

We do this to simplify the Mackey theory we need to move forward. We may further decompose the above sum by considering the double cosets

$$Q_{1,n+m-2}(K)\backslash \mathrm{Sp}_{2(n+m-1)}(K)/Q_{1,n+m-2}(K)$$

which has 3 elements with the representatives $1, w_2$, and $w_{0,2}$ where

$$w_{0,2} = w_2w_3\cdots w_n\cdots w_3w_2$$

is the relative long element corresponding to the parabolic subgroup $Q_{1,n+m-2}$.

We claim that the only double coset to contribute to (7.8) is the coset represented by w_2. In fact, to see that the case of $w_{0,2}$ does not contribute, we need only comment that the short root group U_α, where

$$\alpha = \eta_{1,2} = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n$$

is conjugated by $\tilde{w}_0$ to the root group U_{α_1}. Thus, since this root group is contained

in R_l for all l, we see that as an inner integration we have

$$\int_{U_\alpha(\mathbb{A})R_l(K)\backslash R_l(\mathbb{A})} \int_{[U_\alpha]} \vartheta_n^{V_1,\psi_1^0}(\tilde{w}_0 x_\alpha(t)(v,g))\,dt\;dv$$
$$= \int_{U_\alpha(\mathbb{A})R_l(K)\backslash R_l(\mathbb{A})} \sum_{\xi\in K^\times} \left(\int_{[\mathbb{A}]} \psi(\xi t)\,dt\right) \theta_n^{V_1,\psi_1^0}(h(\xi)\tilde{w}_0(v,g))\;dv.$$

This vanishes since $\xi \neq 0$, and thus

$$\int_{[\mathbb{A}]} \psi(\xi t)\,dt = 0.$$

The contribution of $w = 1$ also vanishes. In this case, the contribution equals

$$\sum_{\xi\in K^\times} \int_{[\mathrm{Sp}_{2m}]} \int_{[R_l]} \overline{\varphi(g)}\theta_n^{V_1,\psi_1^0}(h(\xi)(v,g))\;dv\,dg.$$

By Lemma A.2.2, we see that we may replace $\theta_n^{V_1,\psi_1^0} = \theta_n^{V_2,\psi_2^0}$. Factoring this integral along the unipotent radical $U_{(2,n+m-2)}$, and applying Theorem 3.2.2 with $a = 2$, we have an inner integration contained in the lifting $\Theta_{m-2}(\tau)$. By part (1) and the assumption that $\Theta_{n-1}(\pi) = 0$, this vanishes.

Therefore, the only double coset which contributes to (7.8) is the one represented by w_2. With this simplification, we may rewrite (7.8) as

$$\sum_{\gamma\in Q_{1,n+m-3}\backslash \mathrm{Sp}_{2(n+m-2)}(K)} \sum_{\epsilon\in K} \int_{[\mathrm{Sp}_{2m}]} \int_{[R_l]} \overline{\varphi(g)}\vartheta_n^{V_1,\psi_1^0}(w_2\gamma x_{\alpha_2}(\epsilon)(v,g))\,dv\,dg. \tag{7.9}$$

We now consider the set of double cosets $Q_{1,n+m-3}\backslash \mathrm{Sp}_{2(n+m-2)}(K)/Q_{1,n+m-3}$, which again has only 3 elements, represented by $1, w_3$, and $w_{0,3}$. The argument is the same as the previous case, and by induction we find that our integral reduces to

$$\int_{Q_{1,m-1}(K)\backslash \mathrm{Sp}_{2m}(\mathbb{A})} \int_{[R_l]} \overline{\varphi(g)} \sum_{\epsilon_2,\cdots,\epsilon_n} \vartheta_n^{V_1,\psi_1^0}(w_2 x_{\alpha_2}(\epsilon_2)\cdots w_n x_{\alpha_n}(\epsilon_n)(v,g))\,dv\,dg.$$

Setting $\tilde{w}_2 = w_2 w_3 \cdots w_n = \mathrm{diag}(w, I_{2m-2}, w^*)$, where

$$w = \begin{pmatrix} 1 & & \\ & & 1 \\ & I_{n-1} & \end{pmatrix},$$

and $r(\epsilon_2, \cdots, \epsilon_m) = \mathrm{diag}(u(\epsilon), I_{2m-2}, u(\epsilon)^*)$, where

$$u(\epsilon) = \begin{pmatrix} 1 & \cdots & 0 \\ & & \epsilon_2 \\ & \ddots & \vdots \\ & & \epsilon_n \\ & & 1 \end{pmatrix} \in \mathrm{GL}_{n+1}(F),$$

we may rewrite this as

$$\int_{Q_{1,m-1}(K)\backslash \mathrm{Sp}_{2m}(\mathbb{A})} \int_{[R_l]} \overline{\varphi(g)} \sum_{\epsilon_2,\cdots,\epsilon_n} \vartheta_n^{V_1,\psi_1^0}(\tilde{w}_2 r(\epsilon_1,\ldots,\epsilon_n)(v,g))\, dv\, dg.$$

We may immediately remove several of the above summands as follows: decompose our unipotent radical $R_l = L \cdot R_l^0$ where

$$L_1' = \langle x_\alpha(t) : \alpha \in \{\gamma_{1,l}, \ldots, \gamma_{1,n-1}\}\rangle \cong \mathbb{G}_a^{n-l}.$$

Decomposing the integration over R_l and conjugating

$$\lambda = x_{\gamma_{1,l}}(t_l) \cdots x_{\gamma_{1,n-1}}(t_{n-1}) \in L_{\mathbb{A}}$$

past $\tilde{w}_2 r(\epsilon_1, \ldots, \epsilon_n)$, we change variables an obtain an inner integration of the form

$$\int_{[\mathbb{A}^{n-l+1}]} \psi(\epsilon_{l+1} t_l + \cdots + \epsilon_n t_{n-1})\, dt_i,$$

which vanishes unless $\epsilon_{l+1} = \cdots = \epsilon_n = 0$. Therefore, we have reduced the integral to the case

$$\int_{Q_{1,m-1}(K)\backslash \mathrm{Sp}_{2m}(\mathbb{A})} \int_{[R_l^1]} \overline{\varphi(g)} \sum_{\epsilon_2,\cdots,\epsilon_l} \vartheta_n^{V_1,\psi_1^0}(\tilde{w}_2 r(\epsilon_2,\ldots,\epsilon_l,0,\cdots,0)(v,g))\, dv\, dg, \quad (7.10)$$

where here R_l^1 is the subgroup of elements of R such the projection to the root groups U_α with $\alpha \in \{\gamma_{1,k}, \eta_{1,k} : 1 \leq k \leq n+m\}$ are trivial. The reason we have this integral is, as before, decomposing $R_l^0 = L_1'' \cdot R_l^1$ where L' is the rest of the top row, we can conjugate L' past $\tilde{w}_2$ and absorb the integration over L' into the integration over V_1 in $\vartheta_n^{V_1,\psi_1^0}(g)$.

If $l = 1$ or $\epsilon_i = 0$ for all i, then the above integral vanishes. To see this, note that we may decompose the integration over $Q_{1,m-1}(K)\backslash \mathrm{Sp}_{2m}(\mathbb{A})$ to obtain as an inner integration over $[L_{1,m-1}]$. As noted in the proof of part (1), $\vartheta_n^{V_1,\psi_1^0}$ is invariant under $\tilde{w}_2 L_{1,m-1} \tilde{w}_2^{-1}$, so that upon changing variables we find the constant term

$$\int_{[L_{1,m-1}]} \varphi(lg)\, dl = 0$$

as an inner integration. In the case $l > 1$, we have thus reduced to the case $\epsilon_i \neq 0$.

Consider, within the Levi subgroup $M_{(l,n-l)} \cong \mathrm{GL}_l \times \mathrm{Sp}_{2(n-l)}$ corresponding to our unipotent radical, the subgroup GL_{l-1} embedded as the (lower) mirabolic subgroup of the GL_l factor. Thus, if $\zeta \in \mathrm{GL}_{l-1}(K)$, then ζ embeds into Sp_{2n} as

$$\underline{\zeta} = \mathrm{diag}(1, \zeta, I_{2(m+n-l)}, \zeta^*, 1).$$

This subgroup acts by conjugation on the set $\{r(\epsilon_1, \ldots, \epsilon_l, 0, \ldots, 0) : \epsilon_i \neq 0\}$ with one orbit. One can check that this GL_{l-1} commutes with embedded $\overline{\mathrm{Sp}}_{2m}(\mathbb{A})$. As it is a subgroup of the embedded parabolic subgroup

$$M_{(l,n-l)} \subset \mathrm{Sp}_{2n} \subset \mathrm{Sp}_{2(m+n)},$$

it clearly normalizes R_l. Using these two observations, we may rewrite (7.10) as

$$\sum_{\zeta \in H_1 \backslash \mathrm{GL}_{l-1}(K)} \int_{Q_{1,m-1}(K)\backslash \mathrm{Sp}_{2m}(\mathbb{A})} \int_{[R_l^1]} \overline{\varphi(g)} \vartheta_n^{V_1,\psi_1^0}(\tilde{w}_2 r_1(v,g)\underline{\zeta})\, dv\, dg.$$

Here, $r_1 := r(1, 0, \ldots, 0) = x_{\gamma_{2,n}}(1)$ and H_1 is the stabilizer in GL_{l-1} of r_1. Note that we have used the fact that $\tilde{w}_2 \underline{\zeta}^{-1} \tilde{w}_2^{-1}$ stabilizes the integral $\vartheta_n^{V_1,\psi_1^0}(g)$.

Thus, by replacing θ_n with a translate, we see that it suffices to show that

$$\int_{Q^0_{1,m-1}(K)\backslash \mathrm{Sp}_{2m}(\mathbb{A})} \int_{[R^1_l]} \overline{\varphi(g)}\theta_n^{V_1,\psi_1^0}(\tilde{w}_2 r_1(v,g))\,dv\,dg = 0,$$

where we have unraveled the definition of ϑ_n and let $Q'_{1,m-1}$ denote the same parabolic with the GL_1-factor omitted. By Lemma A.2.2, we may replace $\theta_n^{V_1,\psi_1^0} = \theta_{2n}^{V_2,\psi_2^0}$.

If $l = 2$, then we may conclude that the above integral vanishes. To see this, one can show that

$$Z(H_3)V_{2,m+n-2} \subset \tilde{w}_2 r(1) R_2^1 (\tilde{w}_2 r(1))^{-1} \subset V_{3,m+n-3}.$$

In fact, after conjugating and changing variables, our integral becomes

$$\int_{Q^0_{1,m-1}(K)\backslash \mathrm{Sp}_{2m}(\mathbb{A})} \int_{[V_3^1]} \overline{\varphi(g)}\theta_n(\tilde{w}_2 r_1(v,g))\psi_3^1(v)\,dv\,dg.$$

Here we have $V_3^1 \subset V_{3,m+n-3}$ defined by

$$V_3^1 = \{v \in V_{3,m+n-3} : v_{3,j} \neq 0 \implies j \leq m+1 \text{ or } j \geq 2(m+n)-n\},$$

against the character $\psi_3^1(v) = \psi(v_{1,2})$.

Thus, we may expand along the complimentary subgroup $(V_3 \setminus V_3^1) \cong \mathbb{G}_a^{2m-2} \subset H_3$ [1]. The subgroup $\mathrm{Sp}_{2m-2}(K) \subset \overline{\mathrm{Sp}}_{2m}(\mathbb{A})$ acts via conjugation with two orbits on the characters of this subgroup.

The trivial orbit does not contribute since, similarly to the proof of part (1), we note that $\tilde{w}_2 r_1 V_{1,m-1}(\tilde{w}_2 r_1)^{-1} \subset V_{3,n+m-3}$, and so conjugating and changing variables, we obtain

$$\int_{[V_{1,m-1}]} \overline{\varphi(u'g)}du'$$

as an inner integration. This vanishes by the cuspidality of π. For the nontrivial orbit, define the character $\psi_3^2 : V_{3,m+n-3} \to \mathbb{C}^\times$ to be

$$\psi_3^2(v) = \psi(v_{1,2} + v_{3,m+2}).$$

[1]Note that the center of H_3 is already being integrated against.

Then, we see that our integral becomes

$$\sum_{\gamma\in Q^0_{1,m-2}\backslash \mathrm{Sp}_{2m-2}(K)} \int_{Q^0_{1,m-1}(K)\backslash \mathrm{Sp}_{2m}(\mathbb{A})} \overline{\varphi(g)}\theta_n^{V_3,\psi_3^2}(\gamma\tilde{w}_2 r_1(1,g))\,dv\,dg$$
$$= \int_{Q^0_{2,m-2}(K)\backslash \mathrm{Sp}_{2m}(\mathbb{A})} \overline{\varphi(g)}\theta_n^{V_3,\psi_3^2}(\tilde{w}_2 r_1(1,g))\,dv\,dg,$$

where we used the fact that γ commutes with $\tilde{w}_2 r_1$ to allow the unfolding. Let $\tilde{w}_4 = w_4\cdots w_{n+1}$ be the Weyl group element preserving $V_{3,m+n-3}$ such that $\tilde{w}_4\cdot\psi_3^2 = \psi_3^0$. Conjugating by this element, we see that our integral becomes

$$\int_{Q^0_{2,m-2}(K)\backslash \mathrm{Sp}_{2m}(\mathbb{A})} \overline{\varphi(g)}\theta_n^{V_3,\psi_3^0}(\tilde{w}_4\tilde{w}_2 r_1(1,g))\,dv\,dg.$$

We claim that this integral vanishes. To see this, note that we may factor the unipotent radical of $Q_{2,m-2}$

$$V_{2,m-2} = N\cdot U_{(2,m-2)},$$

where $U_{(2,m-2)}$ is the unipotent radical of the maximal parabolic subgroup $P_{(2,m-2)}$ of Sp_{2m}. Conjugating the integration along $U_{(2,m-2)}$ across $\tilde{w}_4\tilde{w}_2 r_1$, we see that $\theta_{2n}^{V_3,\psi_3^0}$ is invariant under this subgroup. Thus, we obtain an inner integration of the form

$$\int_{[U_{(2,m-2)}]} \overline{\varphi(ug)}\,dg,$$

which vanishes by cuspidality. This completes the case of $l = 2$.

For $l \geq 3$, note that the root subgroup associated to the long root μ_2 is contained in R_l^1. Doubling the integration along this root group, we may write our integral as

$$\int_{Q^0_{1,m-1}(K)\backslash \mathrm{Sp}_{2m}(\mathbb{A})}\int_{[R_l^1]}\int_{[\mathbb{A}]} \overline{\varphi(g)}\theta_n^{V_2,\psi_2^0}(\tilde{w}_2 r_1 x_{\mu_2}(t)(v,g))\,dt\,dv\,dg,$$

and conjugating past $\tilde{w}_2 r_1$, we obtain

$$\int_{Q^0_{1,m-1}(K)\backslash \mathrm{Sp}_{2m}(\mathbb{A})}\int_{[R_l^1]}\int_{[Z(H_3)]} \overline{\varphi(g)}\theta_n^{V_2,\psi_2^0}(z\tilde{w}_2 r_1(v,g))dz\,dv\,dg.$$

Therefore, we may expand $\int_{[Z(H_3)]} \theta^{V_2,\psi_2^0}(z\tilde{w}_2 r_1(v,g))dz$ along $[H_3/Z(H_3)]$. As before, we obtain two orbits under the action of the subgroup $\mathrm{Sp}_{2(m+n-3)}(K)$. Again, we see that cuspidality kills off the trivial orbit, and we are left to analyze the nontrivial orbit.

This process is inductive, and the final case will depend on the relation between m and n. In both cases, one is able to decompose the integration so that we have an inner integration along an appropriate unipotent radical of Sp_{2m}. After conjugating across the accrued Weyl group elements and unipotent elements, the period along θ_n is invariant under the resulting subgroup. We obtain the constant term of φ as an inner integration, and this vanishes by the cuspidality assumption.

Specifically, if $m < n$, then in the finale we obtain the integral

$$\int_{U_m(F)\backslash \mathrm{Sp}_{2m}(\mathbb{A})} \int_{[R_l^t]} \overline{\varphi(g)}\theta^{V_{2m},\psi_{2m}^0}(\tilde{w}r(v,g))\,dv\,dg.$$

Here U_m is the unipotent radical of the Borel subgroup of Sp_{2m}, $\tilde{w} \in W(\mathrm{Sp}(W_n))$ is an explicit Weyl group element, r is an explicit unipotent element, and R_l^t is a subgroup of R_l. To see that this integral vanishes, we decompose the integration to obtain an inner integration over $U_m(F)\backslash U_m(\mathbb{A})$. One may then show that $\theta^{V_{2m},\psi_{2m}^0}$ is invariant under $(\tilde{w}r)U_m(\mathbb{A})(\tilde{w}r)^{-1}$, so that after a change of variables we obtain

$$\int_{[U_m]} \overline{\varphi(ug)}du$$

as an inner integral. This vanishes by cuspidality.

The case of $m > n$ is similar. This final analysis is similar to the considerations in [BFG06]. We omit the final details, as the form of the argument above deals with all potential issues. This completes the proof of Theorem 7.2.1. □

Chapter 8

CAP Representations and Arthur Parameters

In this chapter, we draw certain conclusions from the previous sections. We also relate our results to Arthur's conjectures.

8.1 CAP representations

We say that two irreducible automorphic representations $\pi = \otimes'_\nu \pi_\nu$ and $\sigma = \otimes'_\nu \sigma_\nu$ are nearly equivalent if $\pi_\nu \cong \sigma_\nu$ for almost all places ν. Recall the definition of a CAP representation of a reductive group G over K.

Definition 8.1.1. A cuspidal representation $\pi = \otimes'_\nu \pi_\nu$ of $\mathrm{G}(\mathbb{A})$ is said to be **CAP** (or, cuspidal associated to parabolic) if there exists

1. a parabolic subgroup $P = MN$ of G,

2. a cuspidal unitary representation τ of $M(\mathbb{A})$,

3. an unramified character χ of $M(\mathbb{A})$,

such that π is nearly equivalent to a irreducible constituent of $\mathrm{Ind}_P^G(\tau \otimes \chi)$. Equivalently, π_ν is isomorphic to the Langlands quotient of $\mathrm{Ind}_P^G(\tau_\nu \otimes \chi_\nu)$ for almost all ν. In this case, we say that π is CAP with respect to (P, τ, χ).

In the case of G quasi-split, Arthur's conjecture on square-integrable automorphic forms produces a precise determination of those triples (P, τ, χ) for which CAP representations exist.

For covering groups, there is an obvious generalization of the above definition, where the parabolic $P = MN$ is replaced with the full inverse image $\overline{P} = \overline{M}N$. This concept has been studied previously in the case of Mp_{2n}, the two-fold metaplectic cover of Sp_{2n} (see, for example, [Yam16]).

Recall that if π is a cuspidal representation of $\overline{\mathrm{Sp}}(W_0)$, then we denote by $n(\pi)$ the index such that $\Theta_{n(\pi)}(\pi) \neq 0$, while $\Theta_{n(\pi)-1}(\pi) = 0$. By Theorem 7.1.1 such an index exists and is unique.Set $t(\pi) = n(\pi) - n$.

Theorem 8.1.2. *Suppose π is a genuine cuspidal automorphic representation of $\overline{\mathrm{Sp}}(W_0)$, and suppose $\Theta(\pi) := \Theta_{n(\pi)}(\pi)$ is the first nontrivial lift. Then*

$$\Theta(\pi) = \hat{\bigoplus_{\lambda \in \Lambda}} \tau_\lambda$$

is semi-simple, and the irreducible summands all lie in the same near equivalence class.

1. *If $t(\pi) \geq 0$, then each τ_λ is CAP with respect to the triple $(Q_{t(\pi),n}, \pi, \overline{\chi}_{\Theta,2t(\pi)})$. Here, we have $\overline{\chi}_{\Theta,2t(\pi)} = \overline{\chi}^0 \chi_{\Theta,2t(\pi)}$ where*

$$\chi_{\Theta,2t(\pi)}((M(t_1, \ldots, t_{t(\pi)}), m), \zeta) = \prod_{i=1}^{t(\pi)} |t_i|^{(2(t(\pi)-i)+1)/4}.$$

2. *If $t(\pi) < 0$, then π is CAP with respect to $(Q_{-t(\pi),n(\pi)}, \tau_\lambda, \overline{\chi}_{\theta,-2t(\pi)})$ for any $\lambda \in \Lambda$.*

Proof. By Theorem 7.2.1, we know that $\Theta(\pi) \hookrightarrow L^2_{cusp}(\mathrm{Sp}(V_{n(\pi)})_F \backslash \overline{\mathrm{Sp}}(V_{n(\pi)})_{\mathbb{A}})$ is cuspidal. It is therefore semi-simple, and we may write

$$\Theta(\pi) = \hat{\bigoplus_{\lambda \in \Lambda}} \tau_\lambda,$$

where for each $\lambda \in \Lambda$, τ_λ is an irreducible cuspidal automorphic representation of $\overline{\mathrm{Sp}}(V_{n(\pi)})_{\mathbb{A}}$. Fix an arbitrary λ. Using the Petersson inner product, we see then that

$$\mathrm{Hom}_{\overline{\mathrm{Sp}}(V_{n(\pi)})_{\mathbb{A}}}(\tau_\lambda^\vee \otimes \Theta(\pi), \mathbb{C}) \neq 0,$$

so that we obtain a nontrivial global trilinear form

$$\mathrm{Hom}_{\overline{\mathrm{Sp}}(W_0)_{\mathbb{A}} \times_{\mu_4} \overline{\mathrm{Sp}}(V_n)_{\mathbb{A}}}(\tau_\lambda^\vee \otimes \pi \otimes \Theta_{n(\pi)}, \mathbb{C}) \neq 0.$$

Since all three representations are restricted tensor products, we obtain at each place ν a non-zero local trilinear form. That is,

$$\mathrm{Hom}_{\overline{\mathrm{Sp}}(W_0)_{\mathbb{A}} \times_{\mu_4} \overline{\mathrm{Sp}}(V_n)_{\mathbb{A}}}(\tau_{\lambda,\nu}^\vee \otimes \pi_\nu \otimes \Theta_{n(\pi),\nu}, \mathbb{C}) \neq 0.$$

This implies, for any place ν, the corresponding local hom-space is non-zero:

$$\mathrm{Hom}_{\overline{\mathrm{Sp}}(W_0)_{K_\nu} \times_{\mu_4} \overline{\mathrm{Sp}}(V_n)_{K_\nu}}(\Theta_{n(\pi),\nu}, \tau_{\lambda,\nu} \otimes \pi_\nu^\vee) \neq 0,$$

Suppose ν is a finite place such that all three local representations are unramified. This holds for all but finitely many places. Thus, we may apply Theorem 6.1.1 to conclude the appropriate relationship between the local factors. □

This constitutes the first construction of CAP representations on higher degree covering groups. In Chapter 9, we provide evidence based on the dimension equation that $t(\pi) > 0$ holds if π is generic.

8.2 Conjectural Shimura Lift

Consider a finite place ν of K such that the completion $F = K_\nu$ has odd residue characteristic. Recall that to the 4-fold cover $\overline{\mathrm{Sp}}_{2r}$ one may associate the dual root datum

$$(Y_{Q,4}, \{\alpha_{Q,4}^\vee\}, \mathrm{Hom}(Y_{Q,4}, \mathbb{Z}), \{\alpha_{Q,4}\}),$$

which corresponds to the complex dual group $\overline{\mathrm{Sp}}_{2r}^\vee = \mathrm{Sp}_{2r}(\mathbb{C})$. To this root datum, we may also attach the split linear algebraic group over F, $\mathbb{G}_{Q,4} = \mathrm{SO}_{2r+1}$. Gan-Gao [GG14] speculate that $\mathbb{G}_{Q,4}$ should be viewed as the principal endoscopic group of $\overline{\mathrm{Sp}}_{2r}$. Set $\mathbb{G}_{Q,4}(F) = \mathrm{G}_{Q,4}$.

Let $T_{Q,4} = Y_{Q,4} \otimes F^\times \subset \mathrm{G}_{Q,4}$ be the induced split maximal torus. Then the inclusion $Y_{Q,4} \hookrightarrow Y$ induces an isogeny

$$i : T_{Q,4} \to T_r.$$

Let $\chi : i(T_{Q,4}) \to \mathbb{C}^\times$ be a character of the subtorus $i(T_{Q,4})$ of T_r. For example, χ could be an unramified character arising from a character of the lattice $Y_{Q,4}$. Upon choosing a distinguished character $\overline{\chi}^0$, we previously saw how to obtain the principal series representation

$$\mathrm{Ind}_{\overline{B_r}}^{\overline{\mathrm{Sp}}_{2r}(F)}(\overline{\chi}^0\chi).$$

As noted in [GG14, Section 15], this choice induces a well-defined W-equivariant lifting

$$\mathrm{Ind}_{\overline{B_r}}^{\overline{\mathrm{Sp}}_{2r}(F)}(\overline{\chi}^0\chi) \mapsto \mathrm{Ind}_{B_{Q,4}}^{G_{Q,4}}(\chi \circ i). \tag{8.1}$$

This lifting provides a bijection between principal series representations of $\overline{\mathrm{Sp}}_{2r}(F)$ and principal series representations of $\mathrm{G}_{Q,4}$ induced from characters χ' such that $\ker(i) \subset \ker(\chi')$.

One may check that in the case of χ unramified, this lifting has the effect of squaring the Satake parameter. More explicitly, if

$$\chi(M(t_1,\dots,t_r)) = \prod_{i=1}^{r} \chi_i(t_i),$$

and if we parametrize $T_{Q,4} \subset \mathrm{G}_{Q,4}$ by $D(s_1,\dots,s_r) = \mathrm{diag}(s_1,\dots,s_r,1,s_r^{-1},\dots,s_1^{-1})$, then

$$\chi \circ i(D(s_1,\dots,s_r)) = \prod_{i=1}^{r} \chi_i^2(s_i).$$

A similar unramified correspondence exists for Mp_{2r} and SO_{2r+1}, as follows from [Kud86], Theorem 2.5. In the classical Shimura correspondence between Mp_2 and PGL_2, this is made explicit in [Gan, Sec. 2.17]. With these results in mind, one is led to the (naive) conjecture of a generalized Shimura-type correspondence (dependent on our distinguished character $\overline{\chi}^0$):

$$Shim_{\overline{\chi}^0} : \mathrm{Rep}_\epsilon(\overline{\mathrm{Sp}}_{2r}(F)) \longrightarrow \mathrm{Rep}(\mathrm{G}_{Q,4}),$$

of which this is the restriction to the principal series. One might also hope for an analogous global correspondence

$$Shim_{\overline{\chi}^0} : \mathcal{A}_\epsilon(\overline{\mathrm{Sp}}_{2r}(\mathbb{A})) \longrightarrow \mathcal{A}(\mathrm{G}_{Q,4}(\mathbb{A})) \tag{8.2}$$

of (genuine) automorphic representations.[1] Further evidence for such a conjecture may be seen in the metaplectic correspondences known for covers of general linear groups (see [Fli80] and [FK86]).

[1]Note that the classical dependence upon a choice of additive character ψ is here encoded in the choice of distinguished character $\overline{\chi}^0$.

8.3 Arthur's conjectures

For a reductive linear algebraic group G over a number field K, Arthur has conjectured a decomposition

$$L^2_{disc}(\mathrm{G}(K)\backslash \mathrm{G}(\mathbb{A})) = \hat{\bigoplus_{\Psi}} L^2_{\Psi},$$

where the Hilbert space direct sum runs over equivalence classes of global Arthur parameters Ψ. These are continuous maps

$$\Psi : L_K \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{G}^\vee,$$

satisfying certain axioms. Here L_K is the conjectural Langlands group of K. For each Ψ, L^2_Ψ is a direct sum of near equivalent representations. For a summary of Arthur's conjecture, see [Art89].

The primary motivation for considering the conjectural lifting (8.2) is the hope of extending the notion of Arthur parameters from $\mathrm{G}_{Q,4}$ to $\overline{\mathrm{Sp}}_{2r}$. This has only recently been accomplished by Gan-Ichino [GI17] in the case of Mp_{2r}. The proof uses the local Shimura correspondence between Mp_{2r} and SO_{2r+1} along with the global theta liftings from Mp_{2r} to $\mathrm{SO}_{2(r+k)+1}$ for $k > r$. This indicates that the conjectural correspondence (8.2), even if it exists, is likely insufficient to prove a transfer of definition to $\overline{\mathrm{Sp}}_{2r}$. Likely further development of the trace formula in the context of BD-covering groups is necessary. Such investigations are underway in the work of Li.

In any case, given an Arthur parameter Ψ for SO_{2r+1}, the bijection above enables us to define a near equivalence class $\mathcal{A}_{\Psi,\overline{\chi}^0}$ of genuine representations for $\overline{\mathrm{Sp}}_{2r}(\mathbb{A})$ as follows. To the global parameter Ψ, we have a family of local parameters

$$\Psi_\nu : L_{K_\nu} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_{2r}(\mathbb{C}),$$

where if ν is a finite place, then $L_{K_\nu} = WD_{K_\nu}$ is the Weil-Deligne group of K_ν. For almost all finite places, we obtain a Satake parameter

$$s_{\Psi_\nu} = \Psi_\nu \left(Fr_\nu, \begin{pmatrix} q_\nu^{1/2} & \\ & q_\nu^{-1/2} \end{pmatrix} \right),$$

where Fr_ν is a Frobenius element and q_ν is the size of the residue field of K_ν. This corresponds to an irreducible spherical representation π_{Ψ_ν} of $\mathrm{G}_{Q,4}(K_\nu) = \mathrm{SO}_{2r+1}(K_\nu)$. Using the unramified lifting (8.1), we obtain an unramified representation of $\overline{\mathrm{Sp}}_{2r}(K_\nu)$, $\pi_{\Psi_\nu,\overline{\chi}^0}$. Thus, Ψ determines the near equivalence class

$$\mathcal{A}_{\Psi,\overline{\chi}^0} := \left\{ \pi = \otimes'_\nu \pi_\nu : \pi_\nu \cong \pi_{\Psi_\nu,\overline{\chi}^0} \text{ for almost all } \nu \right\}.$$

We are led to the following:

Conjecture 8.3.1. *For a choice of distinguished character $\overline{\chi}^0$,*

$$\mathcal{A}_\epsilon(\overline{\mathrm{Sp}}_{2r}(\mathbb{A})) = \hat{\bigoplus_\Psi} \mathcal{A}_{\Psi,\overline{\chi}^0}. \tag{8.3}$$

This is analogous to Theorem 2.3 in [GL17].

In the remainder of this section, we assume the existence of the decomposition (8.3) and use it to interpret our construction in terms of Arthur parameters of $\mathrm{SO}_{2r+1}(\mathbb{A})$. Let π be a genuine cuspidal representation of $\overline{\mathrm{Sp}}_{2r}(\mathbb{A})$, and suppose $\Theta(\pi) := \Theta_{r+k}(\pi)$ is the first nontrivial lift. Suppose that $\pi \in \mathcal{A}_{\Psi,\overline{\chi}^0}$, and consider the parameter

$$\Psi_\pi : L_K \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_{2r}(\mathbb{C}).$$

If $\{\chi_{\nu,i}^{\pm 1}(\varpi_\nu)\}$ are the Satake parameters of π at ν relative to $\overline{\chi}^0$, then Theorem 6.1.1 tells us that the corresponding parameters for an irreducible summand τ_π of $\Theta(\pi)$ are given by

$$\{\chi_{\nu,i}^{\pm 1}(\varpi_\nu)\} \cup \{q_\nu{}^{(2(k-i)+1)/4} : 1 \le i \le k\}.$$

Passing through the unramified lifting (8.1), we see that, up to action by the Weyl group,

$$s_{\Psi_{\pi,\nu}} = M(\chi^2_{\nu,1}(\varpi_\nu), \dots, \chi^2_{\nu,r}(\varpi_\nu)) \in T_r(\mathbb{C}).$$

Thus, the parameter of the lift $\Psi_{\Theta(\pi)}$ must satisfy

$$s_{\Psi_{\Theta(\pi),\nu}} = M({q_\nu}^{(2k-1)/2}, \dots, {q_\nu}^{1/2}, \chi^2_{\nu,1}(\varpi_\nu), \dots, \chi^2_{\nu,r}(\varpi_\nu)) \in T_{r+k}(\mathbb{C}),$$

for almost all ν.

Consider the unipotent orbit $\mathcal{O}_k = ((2k)1^{2r})$ of $\mathrm{Sp}_{2(r+k)}(\mathbb{C})$. This corresponds to a conjugacy class of morphisms $\phi_k : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_{2(r+k)}(\mathbb{C})$. Then a simple computation based on the above requirements shows that the Arthur parameter $\Psi_{\Theta(\pi)}$ associated to the near equivalence class in which $\Theta(\pi)$ sits should be the composition

$$L_K \times \mathrm{SL}_2(\mathbb{C}) \xrightarrow{Id\times\Delta} L_K \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \xrightarrow{\Psi_\pi \times Id} \mathrm{Sp}_{2r}(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \xrightarrow{\iota\times\phi_k} \mathrm{Sp}_{2(r+k)}(\mathbb{C}),$$

where $\iota : \mathrm{Sp}_{2r}(\mathbb{C}) \to \mathrm{Sp}_{2(r+k)}(\mathbb{C})$ has image in the centralizer of the image of ϕ_k. In the notation of [Art13], we have

$$\Psi_{\Theta(\pi)} = \Psi_\pi \boxplus S_{2k},$$

where S_{2k} is the unique $2k$-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$. In particular, the Arthur parameter Ψ_Θ is non-tempered, in agreement with the representation $\Theta(\pi)$ being a CAP representation.

Chapter 9

Generic Lifts and the Dimension Equation

In this final section, we investigate the case that π is assumed to be a generic cuspidal representation of $\overline{\mathrm{Sp}}(W_0)_{\mathbb{A}} \cong \overline{\mathrm{Sp}}_{2m}(\mathbb{A})$; that is, we assume π has a non-trivial Whittaker model. With this assumption, one expects that $n(\pi) = m + 1$. Moreover, we anticipate that $\Theta_{m+1}(\pi)$ is a generic representation of $\overline{\mathrm{Sp}}(V_{m+1})_{\mathbb{A}} \cong \overline{\mathrm{Sp}}_{2m+2}(\mathbb{A})$.

The reason is the dimension equation of Ginzburg (see [Gin14b]), which we recall now. Suppose π is an automorphic representation of a reductive group or finite degree cover there of G, and assume $\mathcal{O}(\pi)$ is a single nilpotent orbit. Recall that the Gelfand-Kirillov dimension of π is given by

$$\dim(\pi) = \frac{1}{2}\dim_{\mathbb{C}}(\mathcal{O}(\pi)).$$

The general philosophy of Ginzburg is that one may utilize the dimension equation to anticipate when a lifting such as the one under consideration is non-vanishing. In our case, the equation amounts to

$$\dim(\pi) + \dim(\Theta_{W_n}) = \dim_{\mathbb{C}}\left(\mathrm{Sp}_{2m}(\mathbb{C})\right) + \dim(\Theta_n(\pi)). \tag{9.1}$$

Our computation of $\mathcal{O}(\Theta_n)$ in Theorem 1.2.4 tells us that

$$\dim(\Theta_{W_n}) = \frac{(n+m+1)(n+m)}{2}.$$

If we assume π is generic, so that $\dim(\pi) = m^2$, we find equality when $n = m+1$ and $\Theta_{m+1}(\pi)$ is generic.

For the remainder of this section, we compute the Whittaker coefficient of a vector in $\Theta_{m+1}(\pi)$ and showing that it may be expressed in terms of an integral pairing of the Whittaker coefficients of π and the split coefficient (see Proposition 5.2.5) of Θ_{m+1}. Let $\psi_{n,\alpha} : U_n(K)\backslash U_n(\mathbb{A}) \to \mathbb{C}^\times$ be the Gelfand-Graev character of the full unipotent radical of the Borel subgroup of Sp_{2n} corresponding to the square class $\alpha \in K^\times/(K^\times)^2$. Thus,

$$\psi_{n,\alpha}(u) = \psi(u_{1,2} + u_{2,3} + \cdots + u_{n-1,n} + \alpha u_{n,n+1}).$$

For an automorphic form φ of $\overline{\mathrm{Sp}}_{2n}(\mathbb{A})$, we set

$$W_\alpha(\varphi)(g) = \int_{[U_n]} \varphi(ug)\psi_{n,\alpha}(u)du$$

to be the Whittaker coefficient of φ with respect to this character.

Proposition 9.0.2. *Let π be an irreducible cuspidal generic representation of $\overline{\mathrm{Sp}}_{2m}(\mathbb{A})$, and let $\varphi \in \pi$. For a vector $\theta_{m+1} \in \Theta_{W_{m+1}}$, consider the pairing*

$$F_{\varphi,\theta}(h) = \int_{[\mathrm{Sp}_{2m}]} \overline{\varphi(g)}\theta_{m+1}((h,g))\,dg,$$

where $h \in \overline{\mathrm{Sp}}_{2m+2}(\mathbb{A})$. Then we have

$$W_\alpha(F_{\varphi,\theta})(1) = \int_{U_m(\mathbb{A})\backslash \mathrm{Sp}_{2m}(\mathbb{A})} W_\alpha(\overline{\varphi})(g)\theta_{m+1}^{U_{2m+1},\psi^0_{2m+1,\alpha}}(g)\,dg. \tag{9.2}$$

Proof. We seek to compute the integral

$$\int_{[\mathrm{Sp}_{2n}]}\int_{[U_{n+1}]} \varphi_\tau(g)\overline{\theta_{4n+2}(u,g)}\psi_{n+1,\alpha}(u)dudg. \tag{9.3}$$

Many of the steps in this proof are structurally similar to the argument in proving Theorem 7.2.1, except that at certain steps we are able to simplify for different reasons. We will carry out the first few steps in order to illustrate what goes into the computation before inducting.

We begin by noting that the highest root group U_{α_0} is contained in the image of U_{n+1}, so that we may expand θ_{2n} along the subgroup $[H_1/Z(H_1)]$. As before, there are two orbits of characters of this group under the conjugation action of $\mathrm{Sp}_{4n}(F)$.

The trivial orbit vanishes since the root group $U_{\alpha_1} \subset V_1$ is being integrated over twice: once with in the constant term, and again in the Whittaker coefficient. Thus, we may obtain as an inner integration

$$\int_{U_{\alpha_1}(\mathbb{A})U_{n+1,K}\backslash U_{n+1,\mathbb{A}}} \int_{[\mathbb{A}]} \int_{[V_1]} \theta_{4n+2}(vx_{\alpha_1}(t)(u',g))\psi_{n+1,\alpha}(u')\psi(t)dvdtdu' \tag{9.4}$$

$$= \left(\int_{[\mathbb{A}]} \psi(t)dt\right) \int_{U_{\alpha_1}(\mathbb{A})U_{n+1,K}\backslash U_{n+1,\mathbb{A}}} \int_{[V_1]} \theta_{4n+2}(v(u',g))\psi_{n+1,\alpha}(u')dvdu' = 0. \tag{9.5}$$

This leaves the non-trivial orbit, so that we may write (9.3) as

$$\sum_{\gamma \in Q_{1,2n-1}\backslash \mathrm{Sp}_{4n}(K)} \int_{[\mathrm{Sp}_{2n}]} \int_{[U_{n+1}]} \varphi_\tau(g)\vartheta_{4n+2}^{V_1,\psi_1^0}(\gamma(u,g))\psi_{n+1,\alpha}(u)dudg, \tag{9.6}$$

where ϑ_{4n+2} is defined as in the proof of Theorem 7.2.1. We now must consider the double cosets

$$Q_{1,2n-1}\backslash \mathrm{Sp}_{4n}(K)/Q_{1,2n-1} = \{1, w_2, w_{0,1}\}.$$

Again we claim that the only orbit which contribute to (9.6) is the orbit associated to w_2. The proof of vanishing of the contribution associated to the relative long element $w_{0,1}$ is identical to the same in the proof of Theorem 7.2.1. To see that the trivial orbit also does not contribute, we utilize Lemma A.2.2 to replace $\vartheta_{4n+2}^{V_1,\psi_1^0}$ with $\vartheta_{4n+2}^{V_2,\psi_2^0}$.

Now we again find a situation where the root group U_{α_2} is integrated over twice, once against a nontrivial character and once without. Hence, this contribution van-

ishes as with (9.4). We are reduced to the contribution of the coset represented by w_2, and obtain

$$\sum_{\gamma\in Q_{1,2n-2}\backslash \mathrm{Sp}_{4n-2}(K)}\sum_{\epsilon_2\in F}\int_{[\mathrm{Sp}_{2n}]}\int_{[U_{n+1}]}\varphi_\tau(g)\vartheta^{V_1,\psi_1^0}_{4n+2}(w_2x_{\alpha_2}(\epsilon_2)\gamma(u,g))\psi_{n+1,\alpha}(u)dudg, \tag{9.7}$$

analogous to the reduction to (7.9) in the proof of Theorem 7.2.1.

We continue in this way until we obtain

$$\int_{Q_{1,n-1}\backslash \mathrm{Sp}_{2n}(\mathbb{A})}\int_{[U_{n+1}]}\varphi_\tau(g)\sum_{\epsilon_2,\cdots,\epsilon_{n+1}}\vartheta^{V_1,\psi_1^0}_{4n+2}(\tilde{w}_2r(\epsilon_2,\ldots,\epsilon_{n+1})(u,g))\psi_{n+1,\alpha}(u)dudg. \tag{9.8}$$

Since the integration over U_{n+1} contains the unipotent radicals R_l, we are able now to carry out the same reduction steps. The only difference is the presence of a character on the integration over U_{α_1}, so that when conjugating the root group associated to the root $\alpha_1+\cdots+\alpha_{n+1}$, we obtain an inner integration of

$$\int_{[\mathbb{A}]}\psi((1-\epsilon_2\xi^{-1})t)dt,$$

which vanishes unless $\xi=\epsilon_2$. Therefore, we are reduced to the expression

$$\int_{Q_{1,n-1}\backslash \mathrm{Sp}_{2n}(\mathbb{A})}\int_{[U^1_{n+1}]}\varphi_\tau(g)\sum_{\xi\in K^\times}\theta^{V_1,\psi_1^0}_{4n+2}(h_1(\xi)\tilde{w}_2r(\xi,0,\ldots,0)(u_1,g))\psi_{n+1,\alpha}(u_1)du_1dg.$$

Here, the subgroup U^1_{n+1} indicates that the integration is over only those elements of $U(n+1)$ with zeroes in the first row.

Collapsing the summation with the integration we have

$$\int_{Q^0_{1,n-1}\backslash \mathrm{Sp}_{2n}(\mathbb{A})}\int_{[U^1_{n+1}]}\varphi_\tau(g)\theta^{V_1,\psi_1^0}_{4n+2}(\tilde{w}_2r(1,0,\ldots,0)(u_1,g))\psi_{n+1,\alpha}(u_1)du_1dg.$$

Lemma A.2.2 allows us to now replace $\theta^{V_1,\psi_1^0}_{4n+2}$ with $\theta^{V_2,\psi_2^0}_{4n+2}$, and since $Z(H_3)$ is contained in the image of U^1_{n+1}, we may further expand along $[H_3/Z(H_3)]$. Continuing

in this way, we eventually obtain

$$\int_{U_{n,K}\backslash \mathrm{Sp}_{2n}(\mathbb{A})} \int_{[U^n_{n+1}]} \varphi_\tau(g)\theta^{V_{2n-1},\psi^0_{2n-1}}(\tilde{w}r(u_n,g))\psi_{n+1,\alpha}(u_n)du_n dg.$$

We need to explain the notation. Here $U^n_{n+1} = U_{\mu_{n+1}}$ is the one dimensional subgroup corresponding to the long simple root of Sp_{2n+2} ; we have the Weyl group element

$$\tilde{w} = \tilde{w}_{2n}\cdots\tilde{w}_2, \quad \text{where} \quad \tilde{w}_{2i} = w_{2i}\cdots w_{n+i};$$

and finally, we have the unipotent element $r = \mathrm{diag}(1, u, u*, 1)$, where

$$u = \begin{pmatrix} I_n & I_n \\ & I_n \end{pmatrix}.$$

Replacing $\theta^{V_{2n-1},\psi^0_{2n-1}}$ with $\theta^{V_{2n},\psi^0_{2n}}$ by Lemma A.2.2, we come now to the final step.

Note that the root group $x_{\mu_{n+1}}(t)$ commutes with r, and furthermore,

$$\tilde{w}x_{\mu_{n+1}}(\mathbb{A})\tilde{w}^{-1} = x_{\alpha_{2n+1}}(\mathbb{A})$$

so that we find the above integral becomes

$$\int_{U_{n,K}\backslash \mathrm{Sp}_{2n}(\mathbb{A})} \varphi_\tau(g)\theta^{U_{2n+1},\psi^0_{2n+1,\alpha}}_{4n+2}(\tilde{w}r(1,g))dg.$$

At this point, we are essentially done, as we may decompose the outer integration

$$[U_{n,K}\backslash \mathrm{Sp}_{2n}(\mathbb{A})] = [U_{n,\mathbb{A}}\backslash \mathrm{Sp}_{2n}(\mathbb{A})][U_{n,K}\backslash U_{n,\mathbb{A}}],$$

so that we have

$$\int_{[U_{n,\mathbb{A}}\backslash \mathrm{Sp}_{2n}(\mathbb{A})]} \int_{[U_{n,K}\backslash U_{n,\mathbb{A}}]} \varphi_\tau(ng)\theta^{U_{2n+1},\psi^0_{2n+1,\alpha}}_{4n+2}(\tilde{w}r(1,ng))\ dndg.$$

Conjugating n past $\tilde{w}r$, and changing variables, we obtain

$$\int_{[U_{n,\mathbb{A}}\backslash \mathrm{Sp}_{2n}(\mathbb{A})]} \left(\int_{[U_n]} \varphi_\tau(ng)\psi_{n,\alpha}(n)dn \right) \theta^{U_{2n+1},\psi^0_{2n+1,\alpha}}_{4n+2}(\tilde{w}r(1,g))\ dg$$
$$= \int_{U_{n,\mathbb{A}}\backslash \mathrm{Sp}_{2n}(\mathbb{A})} W_\alpha(\varphi_\tau)(g)\theta^{U_{2n+1},\psi^0_{2n+1,\alpha}}_{4n+2}(g)dg,$$

as was to be shown. □

We remark here that the adelic nature of this period motivates analysis of the corresponding local functional. We intend to return to this question at some point in the future. In any case, the utter failure of local multiplicity one of Whittaker models for covering groups means that we cannot conclude that this Whittaker period is Eulerian.

Combining this proposition with Proposition 5.2.5, we end by concluding the following.

Corollary 9.0.3. *Fix $\alpha \in K^\times/(K^\times)^2$. If $\Theta_{m+1}(\pi)$ is generic with respect to the α-coefficient, then so is π.*

Chapter 10

Future Directions

In this final chapter, we consider possible directions for future work related to this thesis. Three main directions of inquiry present themselves: further investigating the index of first occurrence in the theta lift, the local lifting beyond the unramified setting, and generalizing the construction of CAP representations to higher degree covers of the symplectic group.

By Theorem 1.2.1, for any cuspidal representation π there is a first index $m(\pi)$ where $\Theta(\pi) := \Theta_{m(\pi)}(\pi)$ is a non-zero cuspidal representation. It is important to understand the possible values of $m(\pi)$. While Theorem 7.1.1 provides an upper bound for $m(\pi)$ in terms of the rank of the initial symplectic group, it is important to provide a more precise characterization of $m(\pi)$ in terms of π.

For example, we consider in Chapter 9 the case when π is generic, where it is expected that $m(\pi)$ is either r or $r+1$. Along with heuristics coming from the dimension equation, this conjecture arises from an analogy with the theta correspondence on the two-fold metaplectic cover of the symplectic group, in which the first occurrence is related to both the nilpotent orbit of the representation π but also with arithmetic invariants such as the global root number and the vanishing or non-vanishing of the

central L-value $L(\frac{1}{2}, \pi)$, where $L(s, \pi)$ is the automorphic L function associated to the standard representation of the L-group of $\overline{\mathrm{Sp}}_{2r}^{(4)}(\mathbb{A})$. While this L-function has been defined (see [GG14] for a thorough discussion of this definition), essentially nothing is currently known about its analytic properties.

Towards our conjecture, we prove a formula relating the Whittaker coefficients of the $\Theta_{r+1}(\pi)$ to those of π. The final integral is adelic: both the domain of integration and the integrand may be written as a product over local components. In the reductive setting (that is, when the degree of the cover is $n = 1$), one typically applies global and local uniqueness of Whittaker models to rewrite this as a product over local integrals followed by an analysis of the unramified local case. Unfortunately, uniqueness of Whittaker models fails miserably for the four-fold cover, so such a direct approach is not sufficient to analyze the non-vanishing of (9.2). Nevertheless, we may pose the natural question

Question 10.0.4. *Considering the local unramified version of (9.2) at a finite place ν, can one recover the local L-factor associated to π_ν? Further, can one recover the conjectural (partial) L-function associated to π by [GG14]?*

Additional information on the index of first occurrence will require more understanding of the local theta lift. While we have undertaken this study for principal series representations in Chapter 6, it is important to study the lift on discrete series representations of covers of p-adic groups. Natural questions include the following

Question 10.0.5. *Suppose that F is a p-adic local field and consider $\overline{\mathrm{Sp}}_{2r}(F)$. Suppose π is a discrete series representation of $\overline{\mathrm{Sp}}_{2r}(F)$.*

1. *Does the local theta lift satisfy a towering property similar to Theorem 7.2.1?*

2. *Assume such a towering property exists, and let $\Theta(\pi)$ be the first non-zero lift. Is $\Theta(\pi)$ a discrete series representation? If π is supercuspidal, is $\Theta(\pi)$?*

While the general theory of discrete series representations is not as developed as for reductive groups, there are notable exceptions (see [HW09] and [Li17] for examples of depth zero and epipelagic supercuspidal representations, respectively).

Question 10.0.6. *How does the local theta lift interact with depth on supercuspidal representations of $\overline{\mathrm{Sp}}_{2r}(F)$?*

As discussed in the introduction, it is interesting to consider how one might generalize this construction to higher degree covers of symplectic groups. In general, we expect there exist generalizations of this lifting to the $4n$-fold cover of Sp_{2r}[1]. When $n > 1$, this will not be a Θ-type lifting, and one ought to replace Θ by a certain Fourier coefficient. As discussed after Proposition 5.2.3, the most challenging step is to prove the genericity of certain residual representations. The representations which arise may be related to theta representations on components of Levi subgroups. In particular, we would need to prove the following conjecture (or perhaps a more nuanced version thereof):

Conjecture 10.0.7. *Let $\Theta_{2r}^{(2r+2)}$ be a theta representation of $\overline{\mathrm{Sp}}_{2r}^{(2r+2)}(\mathbb{A})$. Then $\Theta_{2r}^{(2r+2)}$ is generic.*

Note that the corresponding local statement is known [Gao16]. Moreover this conjecture is consistent with the general philosophy that the GK dimension of theta representations increases with the degree of the cover and the conjecture of Friedberg and Ginzburg in the odd degree covers, which we recall now. In [FG17], Friedberg

[1]Perhaps also form more general covers.

and Ginzburg state a conjecture for the nilpotent orbits associated to $\Theta(\mathrm{Sp}_{2r}, n)$ for n odd. Write $2r = a(r,n)n + b(r,n)$ where $a(r,n)$ and $b(r,n)$ are both nonnegative numbers such that $0 \leq b(r,n) \leq n-1$. We recall that a partition λ of $2r$ is a symplectic partition if any odd number in the partition occurs with even multiplicity. Given a partition λ of $2r$, define the Sp-collapse of λ to be the greatest symplectic partition which is smaller than λ.

Conjecture 10.0.8. *Let n be an odd number. If $n < 2r$, then the set $\mathcal{O}(\Theta_{2r}^{(n)})$ consists of the partition which is the Sp-collapse of the partition $(na(r,n)b(r,n))$. If $n > 2r$, then the representation $\Theta_{2r}^{(n)}$ is generic.*

Establishing these conjectures is a necessary first step toward generalizing the theta lift to higher degree covers of symplectic groups. The methods developed in Chapter 5, in particular the study of local Fourier-Jacobi models, will play a similar roll in the verification of the vanishing component of these conjectures for small n. For genericity, we hope to use the methods of Banks, Bump, and Lieman [BBL03] as recently generalized by Gao [Gao17] to study the poles of Whittaker coefficients of certain metaplectic Eisenstein series having $\Theta_{2r}^{(n)}$ as a certain residue.

Appendix A

Appendices to Part 1

A.1 Local filtration

In this appendix, we sketch the proof of Proposition 6.1.3.

Lemma A.1.1. *Suppose π is a smooth representation of $\mathrm{Sp}_{2r}(F)$ or a finite central extension thereof. If U_{α_0} denotes the root subgroup for the highest root α_0, then there is a short exact sequence*

$$0 \longrightarrow \mathrm{ind}_{\overline{Q}'_{2,r-2}}^{\overline{Q}_{1,r-1}} \left(J_{U_{1,r-1},\psi_1}(\pi) \right) \longrightarrow J_{U_{\alpha_0}}(\pi) \longrightarrow J_{U_{1,r-1}}(\pi) \longrightarrow 0. \tag{A.1}$$

Here $Q'_{2,r-2}$ indicates the subgroup of $Q_{2,r-2}$ containing only

$$\mathrm{GL}_1^{\Delta} \times \mathrm{Sp}_{2(r-2)} \subset \mathrm{GL}_1^2 \times \mathrm{Sp}_{2(r-2)}\,.$$

Proof. This follows in a similar fashion to [BZ76, Prop. 5.12 (d)] in which the case of $\mathrm{GL}_n(F)$ is handled. The only difference is the need in the symplectic case to study the Jacquet module $J_{U_{\alpha_0}}(\pi)$ due to the fact that the unipotent radical of $Q_{1,r-1}$ is a Heisenberg group with center U_{α_0} rather than abelian.

After quotienting out by the center of the unipotent radical, the analysis is similar to the general linear case. We omit the details. □

We now apply the Jacquet functor $J_{U_{1,n-1}}(\cdot)$ to the short exact sequence (A.1) in the case that $\pi = \Theta_{2(m+n)}$.[1] Using the exactnesses of the Jacquet functor, we have the short exact sequence

$$0 \longrightarrow V(\Theta, \psi_1) \longrightarrow J_{U_{1,n-1}}(\Theta_{2(m+n)}) \longrightarrow J_{U_{1,n+m-1}}(\Theta_{2(m+n)}) \longrightarrow 0 \tag{A.2}$$

where

$$V(\Theta, \psi_1) = J_{U_{1,n-1}}\left(\mathrm{ind}_{\overline{Q}'_{2,n+m-2}}^{\overline{Q}_{1,n+m-1}}\left(J_{U_{1,n+m-1},\psi_1}(\Theta_{2(m+n)})\right)\right).$$

This completes the proof of Proposition 6.1.3. This lemma also shows us why attempting to study the analogous restriction question in the case of the metaplectic group Mp_{2r} with the classical theta representation $\omega_{2r,\psi}$ will not work: the twisted Jacquet modules

$$J_{U_{1,r-1},\psi_1}(\omega_{2r,\psi})$$

which arise in the short exact sequence vanish due to the minimality of $\omega_{2r,\psi}$. This implies that the analogous local restriction question forces unique parameters for the characters χ and μ. Of course, this is due to the isomorphism

$$\omega_{2r,\psi}\Big|_{\mathrm{Mp}_{2k}\times\mathrm{Mp}_{2r-2k}} \cong \omega_{2k,\psi}\otimes\omega_{2r-2k,\psi}.$$

A.2 Technical Lemmas

In this section, we gather a few technical lemmas which are used in Section 7.2. Note that while everything we state here is global, the arguments in Section 5.1.1 allow us to conclude the analogous local results at almost all places. Let Θ_{2r} be the theta representation on $\overline{\mathrm{Sp}}_{2r}(\mathbb{A})$.

[1] Note that, under the our chosen embedding, $U_{1,n-1}$ contains the root group for the highest root of $\mathrm{Sp}_{2(n+m)}$.

Lemma A.2.1. *Let $\theta \in \Theta_{2r}$ and let $2 \leq k \leq r$. Then the integrals*

$$\int_{[V_{k,r-k}]} \theta(vg)\psi_k(v)\,dv$$

vanish for all $g \in \widetilde{\mathrm{Sp}}_{2r}(\mathbb{A})$. Here, recall that

$$\psi_k(v) = \psi(v_{1,2} + v_{2,3} + \cdots + v_{k,k+1}).$$

Proof. Note that the case of $k = r$ is clear as the above integral corresponds to a Whittaker coefficient of Θ_{2r} (corresponding to the nilpotent orbit $(2r)$), which vanishes by Theorem 5.1.1.

We shall induct in $m = r - k$, the above giving the base case. Suppose now that the lemma is true for m, and consider $m + 1 = r - k$.

Expanding along the root subgroup corresponding to the long root μ_{k+1}, we see that the nontrivial Fourier coefficients correspond to the nilpotent orbit $((2k)1^{2(r-k)})$, which vanishes by Theorem 5.1.1. Thus, we have that

$$\int_{[V_{k,r-k}]} \theta(vg)\psi_k(v)\,dv = \int_{[U_{\mu_{k+1}}V_{k,r-k}]} \theta(vg)\psi_k(v)\,dv.$$

We have integrated over the center of the Heisenberg group H_{k+1}, which enables us to expand the above integral along $[H_{k+1}/Z(H_{k+1})] \cong (F\backslash\mathbb{A})^{2(r-k+1)}$. The subgroup $\mathrm{Sp}_{2(r-k)}(F)$ acts by conjugation on the dual of this quotient with two orbits. A representative of the nontrivial orbit is of the form

$$\int_{[H_{k+1}/Z(H_{k+1})]} \int_{[U_{\mu_{k+1}}V_{k,r-k}]} \theta(vg)\psi_k(v)\psi(v_{k+1,k+2})\,dv = \int_{[V_{k+1,r-k-1}]} \theta(v'g)\psi_{k+1}(v')\,dv',$$

which vanishes by induction. Therefore, setting $\psi^1_{k+1} : V_{k+1} \to \mathbb{C}^\times$ to be the trivial extension of ψ_k to $V_{k+1,r-k-1}$, we have that

$$\int_{[V_k]} \theta(vg)\psi_k(v)\,dv = \int_{[V_{k+1,r-k-1}]} \theta(v'g)\psi^1_{k+1}(v')\,dv'.$$

This last integral may be decomposed with inner integration the constant term along $U_{(k,r-k)}$. By Theorem 3.2.2, this constant term is a theta function on a smaller symplectic group times an integral of the form

$$\int_{[U_{\mathrm{GL}_{k+1}}]} \theta_{\mathrm{GL}_{k+1}}(uh)\psi_{gen}(u)du,$$

where $U_{\mathrm{GL}_{k+1}}$ is the unipotent radical of the Borel subgroup of GL_{k+1}, $\theta_{\mathrm{GL}_{k+1}} \in \Theta^{(2)}_{\mathrm{GL}_{k+1}}$, and

$$\psi_{gen}(u) = \psi(u_{1,2} + \cdots + u_{k,k+1}).$$

We see that this integral is a Whittaker coefficient of $\theta_{\mathrm{GL}_{k+1}}$, which vanishes as $k \geq 2$ (see [KP84]). Thus,

$$\int_{[V_{k,r-k}]} \theta(vg)\psi_k(v)\,dv = 0. \qquad \square$$

For the next lemma, we simplify the notation by setting $V_{2k-1} = V_{2k-1,r-2k+1}$.

Lemma A.2.2. *Let $\theta \in \Theta_{2r}$ and let $1 \leq k \leq r$. The function*

$$\theta^{V_{2k-1},\psi^0_{2k-1}}(g) = \int_{[V_{2k-1}]} \theta(vg)\psi^0_{2k-1}(v)\,dv$$

is invariant under H_{2k}.

Proof. We expand the function $f_k(g) = \theta^{V_{2k-1},\psi^0_{2k-1}}(g)$ along the long root group $U_{\mu_{2k}}$:

$$f_k(g) = \sum_{\xi \in F} \int_{[\mathbb{A}]} f_1(x_{\mu_{2k}}(t)g)\psi(\xi t)\,dt.$$

For $\xi \neq 0$, we see that this integral may be decomposed into the constant term with respect to $U_{(2k-2,r-2k+2)}$ followed by a Fourier coefficient of type $(41^{2(r-2k)-2})$ on $\Theta_{2(r-2k+2)}$, which vanishes by Theorem 5.1.1. Thus, we have that

$$f_k(g) = \int_{[U_{\mu_{2k}}V_{2k-1}]} \theta(vg)\psi^0_{2k-1}(v)\,dv.$$

We now expand along $[H_{2k}/Z(H_{2k})] \cong (F\backslash\mathbb{A})^{2(r-2k)}$. The subgroup $\mathrm{Sp}_{2(r-2k)}(F)$ acts by conjugation on the dual of this quotient with two orbits. A representative of the nontrivial orbit is of the form

$$\int_{[V_{2k}]} \theta(vg)\psi^0_{2k-1}(v)\psi(v_{2k,2k+1})\,dv,$$

which may also be decomposed as a constant term along $U_{(2k-2,r-2k+2)}$. Applying Theorem 3.2.2, we obtain a constant times an integral of the form

$$\int_{[V_{2,(r-2k)-2}]} \theta_{2(r-2k)}(v'g')\psi_2(v')\,dv',$$

where $\theta_{2(r-2k)} \in \Theta_{2(r-2k)}$. This vanishes by the previous lemma. Thus, we are left with the constant term and we obtain

$$f_k(g) = \int_{[H_{2k}]} f(hg)dh. \qquad \square$$

Bibliography

[AB95] Jeffrey Adams and Dan Barbasch. Reductive dual pair correspondence for complex groups. *J. Funct. Anal.*, 132(1):1–42, 1995.

[AB98] Jeffrey Adams and Dan Barbasch. Genuine representations of the metaplectic group. *Compositio Math.*, 113(1):23–66, 1998.

[Art89] James Arthur. Unipotent automorphic representations: conjectures. *Astérisque*, (171-172):13–71, 1989. Orbites unipotentes et représentations, II.

[Art13] James Arthur. *The endoscopic classification of representations*, volume 61 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2013. Orthogonal and symplectic groups.

[BBF11] Ben Brubaker, Daniel Bump, and Solomon Friedberg. Weyl group multiple Dirichlet series, Eisenstein series and crystal bases. *Ann. of Math. (2)*, 173(2):1081–1120, 2011.

[BBL03] William Banks, Daniel Bump, and Daniel Lieman. Whittaker-Fourier coefficients of metaplectic Eisenstein series. *Compositio Math.*, 135(2):153–178, 2003.

[BD01] Jean-Luc Brylinski and Pierre Deligne. Central extensions of reductive groups by $\mathbf{K}_2$. *Publ. Math. Inst. Hautes Études Sci.*, (94):5–85, 2001.

[BFG03] Daniel Bump, Solomon Friedberg, and David Ginzburg. Small representations for odd orthogonal groups. *Int. Math. Res. Not.*, (25):1363–1393, 2003.

[BFG06] Daniel Bump, Solomon Friedberg, and David Ginzburg. Lifting automorphic representations on the double covers of orthogonal groups. *Duke Math. J.*, 131(2):363–396, 2006.

[BG92] Daniel Bump and David Ginzburg. Symmetric square L-functions on $\mathrm{GL}(r)$. *Ann. of Math. (2)*, 136(1):137–205, 1992.

[BLS99] William D. Banks, Jason Levy, and Mark R. Sepanski. Block-compatible metaplectic cocycles. *J. Reine Angew. Math.*, 507:131–163, 1999.

[Bus01] Colin J. Bushnell. Representations of reductive p-adic groups: localization of Hecke algebras and applications. *J. London Math. Soc. (2)*, 63(2):364–386, 2001.

[BZ76] I.N. Bernstein and A.V. Zelevinsky. Representations of the group $GL(n, F)$, where F is a local non-Archimedean field. *Uspehi Mat. Nauk*, 31(3(189)):5–70, 1976.

[BZ77] I. N. Bernstein and A. V. Zelevinsky. Induced representations of reductive $\mathfrak{p}$-adic groups. I. *Ann. Sci. École Norm. Sup. (4)*, 10(4):441–472, 1977.

[Cai16] Yuanqing Cai. Fourier coefficients for Theta representations on covers of general linear groups. *arXiv:1602.06614*, 2016.

[CM93] David H. Collingwood and William M. McGovern. *Nilpotent orbits in semisimple Lie algebras*. Van Nostrand Reinhold Mathematics Series. Van Nostrand Reinhold Co., New York, 1993.

[FG17] S. Friedberg and D. Ginzburg. Theta functions on covers of symplectic groups. *Bull. Iranian Math. Soc.*, 43(4):89–116, 2017.

[FK86] Yuval Z. Flicker and David A. Kazhdan. Metaplectic correspondence. *Inst. Hautes Études Sci. Publ. Math.*, (64):53–110, 1986.

[FL10] Michael Finkelberg and Sergey Lysenko. Twisted geometric Satake equivalence. *J. Inst. Math. Jussieu*, 9(4):719–739, 2010.

[Fli80] Yuval Z. Flicker. Automorphic forms on covering groups of GL(2). *Invent. Math.*, 57(2):119–182, 1980.

[Gan] Wee Teck Gan. The Shimura correspondence á la Waldspurger.

[Gan16] Wee Teck Gan. The metaplectic tensor product as an instance of Langlands functoriality. *preprint*, 2016.

[Gao16] Fan Gao. Distinguished theta representations for certain covering groups. *arXiv preprint arXiv:1602.01880*, 2016.

[Gao17] Fan Gao. Hecke L-functions and Fourier coefficients of covering Eisenstein series. *preprint*, 2017.

[Gao18a] Fan Gao. Generalized Bump–Hoffstein conjecture for coverings of the general linear groups. *Journal of Algebra*, 499:183–228, 2018.

[Gao18b] Fan Gao. The Langlands-Shahidi L-functions for Brylinski-Deligne extensions. *Amer. J. Math.*, 140:83–137, 2018.

[GG14] Wee Teck Gan and Fan Gao. The Langlands-Weissman program for Brylinski-Deligne extensions. *arXiv preprint arXiv:1409.4039*, 2014.

[GGS16] Raul Gomez, Dmitry Gourevitch, and Siddhartha Sahi. Whittaker supports for representations of reductive groups. *arXiv preprint arXiv:1610.00284*, 2016.

[GI17] Wee Teck Gan and Atsushi Ichino. The Shimura-Waldspurger correspondence for $\mathrm{Mp}(2n)$. *arXiv preprint arXiv:1705.10106*, 2017.

[Gin06] David Ginzburg. Certain conjectures relating unipotent orbits to automorphic representations. *Israel J. Math.*, 151:323–355, 2006.

[Gin14a] David Ginzburg. Towards a classification of global integral constructions and functorial liftings using the small representations method. *Advances in Mathematics*, 254:157–186, 2014.

[Gin14b] David Ginzburg. Towards a classification of global integral constructions and functorial liftings using the small representations method. *Adv. Math.*, 254:157–186, 2014.

[GL17] Wee Teck Gan and Wen-Wei Li. The Shimura-Waldspurger correspondence for $\mathrm{Mp}(2n)$. *arXiv preprint arXiv:1705.10106*, 2017.

[GRS03] D. Ginzburg, S. Rallis, and D. Soudry. On Fourier coefficients of automorphic forms of symplectic groups. *Manuscripta Math.*, 111(1):1–16, 2003.

[GRS11] David Ginzburg, Stephen Rallis, and David Soudry. *The descent map from automorphic representations of* $\mathrm{GL}(n)$ *to classical groups*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011.

[GS12] Wee Teck Gan and Gordan Savin. Representations of metaplectic groups I: epsilon dichotomy and local Langlands correspondence. *Compositio Mathematica*, 148(6):1655–1694, 2012.

[HW09] Tatiana K. Howard and Martin H. Weissman. Depth-zero representations of nonlinear covers of p-adic groups. *Int. Math. Res. Not. IMRN*, (21):3979–3995, 2009.

[Ike94] Tamotsu Ikeda. On the theory of Jacobi forms and Fourier-Jacobi coefficients of Eisenstein series. *J. Math. Kyoto Univ.*, 34(3):615–636, 1994.

[JL15] Dihua Jiang and Baiying Liu. On special unipotent orbits and Fourier coefficients for automorphic forms on symplectic groups. *Journal of Number Theory*, 146:343–389, 2015.

[KP84] D. A. Kazhdan and S. J. Patterson. Metaplectic forms. *Inst. Hautes Études Sci. Publ. Math.*, (59):35–142, 1984.

[Kub69] Tomio Kubota. *On automorphic functions and the reciprocity law in a number field*. Lectures in Mathematics, Department of Mathematics, Kyoto University, No. 2. Kinokuniya Book-Store Co., Ltd., Tokyo, 1969.

[Kud86] Stephen S. Kudla. On the local theta-correspondence. *Invent. Math.*, 83(2):229–255, 1986.

[Les17] Spencer Leslie. A generalized theta lifting, CAP representations, and Arthur parameters. *arXiv preprint arXiv:1703.02597*, 2017.

[Li11] Wen-Wei Li. Transfert d'intégrales orbitales pour le groupe métaplectique. *Compos. Math.*, 147(2):524–590, 2011.

[Li17] Wen-Wei Li. Stable conjugacy and epipelagic l-packets for brylinski-deligne covers of sp (2n). *arXiv preprint arXiv:1703.04365*, 2017.

[McN12] Peter J. McNamara. Principal series representations of metaplectic groups over local fields. In *Multiple Dirichlet series, L-functions and automorphic forms*, volume 300 of *Progr. Math.*, pages 299–327. Birkhäuser/Springer, New York, 2012.

[McN16] Peter J. McNamara. The metaplectic Casselman-Shalika formula. *Trans. Amer. Math. Soc.*, 368(4):2913–2937, 2016.

[Mœg96] C. Mœglin. Front d'onde des représentations des groupes classiques p-adiques. *Amer. J. Math.*, 118(6):1313–1346, 1996.

[MW87] C. Mœglin and J.-L. Waldspurger. Modèles de Whittaker dégénérés pour des groupes p-adiques. *Math. Z.*, 196(3):427–452, 1987.

[Ral84] S. Rallis. On the Howe duality conjecture. *Compositio Math.*, 51(3):333–399, 1984.

[Rei12] Ryan Cohen Reich. Twisted geometric Satake equivalence via gerbes on the factorizable Grassmannian. *Represent. Theory*, 16:345–449, 2012.

[Shi85] Goro Shimura. On Eisenstein series of half-integral weight. *Duke Math. J.*, 52(2):281–314, 1985.

[Wei64] André Weil. Sur certains groupes d'opérateurs unitaires. *Acta Math.*, 111:143–211, 1964.

[Wei03] Martin H. Weissman. The Fourier-Jacobi map and small representations. *Represent. Theory*, 7:275–299, 2003.

[Wei14] Martin H. Weissman. Split metaplectic groups and their L-groups. *J. Reine Angew. Math.*, 696:89–141, 2014.

[Wei15] Martin H Weissman. L-groups and parameters for covering groups. *arXiv preprint arXiv:1507.01042*, 2015.

[Wei16] Martin H. Weissman. Covers of tori over local and global fields. *Amer. J. Math.*, 138(6):1533–1573, 2016.

[Yam16] Shunsuke Yamana. The CAP representations indexed by Hilbert cusp forms. *arXiv preprint arXiv:1609.07879*, 2016.