# Special Cycles on GSpin Shimura Varieties:

Author: Cihan Soylu

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# SPECIAL CYCLES ON GSPIN SHIMURA VARIETIES

Cihan Soylu

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Boston College Morrissey College of Arts and Sciences Graduate School

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#### SPECIAL CYCLES ON GSPIN SHIMURA VARIETIES

Cihan Soylu

Advisor: Prof. Ben Howard

The results in this dissertation are on the intersection behavior of certain special cycles on  $\operatorname{GSpin}(n,2)$  Shimura varieties for  $n \geq 1$ . In particular, we will determine when the intersection of the special cycles defined by a collection of special endomorphisms consists of isolated points in terms of the fundamental matrix of this collection. These generalize the corresponding results in the lower dimensional cases proved by Kudla and Rapoport.

# Contents

1	Intr	roduction	1
<b>2</b>	2 Moduli spaces of p-divisible groups		8
	2.1	Formal Schemes	8
	2.2	p-divisible groups	12
	2.3	Rapoport-Zink Spaces	14
૧	Sne	cial Cycles on CSnin Shimura variatias	16
J	Special Cycles on GSpin Similar a varieties		
	3.1	Hodge Type Shimura Varieties	16
	3.2	GSpin Shimura Variety	19
	3.3	Special Cycles	24
	a		•
4	Spe	cial Cycles on the Rapoport-Zink Space	28
	4.1	Rapoport-Zink Space	28
	4.2	Special Cycles	37
		4.2.1 The case $n$ even	42
		4.2.2 The case $n$ odd $\ldots$ $\ldots$ $\ldots$ $\ldots$ $\ldots$ $\ldots$ $\ldots$	50

## 1 Introduction

Illustrated by the works of Gross-Zagier [6], Hirzebruch-Zagier [7], Gross-Keating [5] and Kudla-Millson [16], there are relations between certain arithmetic cycles on Shimura varieties and Fourier coefficients of modular forms and special values of L-functions. Motivated by these results Kudla formulated a program [15, 14] generalizing such relations between specific classes of arithmetic cycles on Shimura varieties and derivatives of incoherent Siegel-Eisenstein series at their central point and Rankin-Selberg L-functions.

Special cases of these relations have been proved. In [20] Kudla-Rapoport-Yang prove these relations for the intersections of cycles of CM points on Shimura curves and Siegel Eisenstein series of genus 2 and weight 3/2. Here Shimura curve can be viewed as a GSpin Shimura variety of signature (1, 2) and the CM points can be viewed as arising from embeddings  $GSpin(0, 2) \rightarrow GSpin(1, 2)$ . Similarly, in [17] Kudla and Rapoport study the triple intersections of cycles on a GSpin(2, 2)Shimura variety arising from embeddings  $GSpin(1, 2) \rightarrow GSpin(2, 2)$ , and relate the intersection multiplicities at isolated points to the Fourier coefficients of Siegel Eisenstein series of genus 3. In [18], the same authors study the intersections of cycles on GSpin(3, 2) Shimura variety arising from embeddings  $GSpin(2, 2) \rightarrow$ GSpin(3, 2), and relate the intersection numbers at the isolated points to a Siegel Eisenstein series of genus 4. Finally more recent work of Kudla and Rapoport [12, 13], considers the n-fold intersections of divisors on GU(n - 1, 1) Shimura varieties arising from embeddings  $GU(n - 2, 1) \rightarrow GU(n - 1, 1)$  and relating these intersections to Fourier coefficients of Eisenstein series on U(n, n).

The Shimura varieties in all the works mentioned above have 'nice' moduli interpretations, meaning they can be given as moduli spaces of abelian varieties with extra structures such as, endomorphism, polarization and level structures. The GU(n-1,1) Shimura varieties are of PEL type and in the cases of GSpin(2,2)and GSpin(3,2), there are exceptional isomorphisms which can be used to identify the Shimura varieties with Hilbert-Blumenthal surfaces and Siegel threefolds, respectively. Other than exceptional cases, for general n this interpretation of the GSpin(n, 2) Shimura varieties as moduli spaces of abelian varieties breaks down. One of the main advantages of having a 'nice' moduli interpretation is that in these cases the integral models of the Shimura varieties are well understood [11]. Another advantage of having a PEL Shimura variety is that the local analogue of the Shimura variety as a formal moduli space of p-divisible groups, the Rapoport-Zink spaces, is well understood [23]. The relation between local and global situations is given by the uniformization of the supersingular locus by the Rapoport-Zink space. For general n the integral model of the Shimura variety is constructed separately by Kisin [10] and Vasiu [26]. The Rapoport-Zink space for these Shimura varieties are constructed separately by Howard-Pappas [8] and Kim [9]. We will briefly recall these constructions in the following sections. Now we will describe the content of this thesis in more detail.

In chapter 2 we will recall the basic definitions and facts on formal schemes, *p*-divisible groups and moduli spaces of *p*-divisible groups, Rapoport-Zink spaces.

In chapter 3, we start with summarizing the results of [10] on the integral model of the Shimura varieties of Hodge type. Then we will define the GSpin Shimura variety. Let (V, Q) be a quadratic space over  $\mathbb{Q}$  of signature (n - 2, 2)with  $n \geq 3$ . The Spinor similitude group  $G = \operatorname{GSpin}(V)$  is a reductive group over  $\mathbb{Q}$ . Let  $\mathcal{D}$  be the space of negative definite oriented 2-planes in  $V_{\mathbb{R}}$ . Then the pair  $(G, \mathcal{D})$  is a Shimura datum. Let  $L \subset V$  be a maximal lattice (i.e.  $[L, L] \subset \mathbb{Z}$  and is maximal among such lattices) that is self-dual at  $\mathbb{Z}_{(p)}$ . Throughout the prime p will be assumed to be > 2. Let  $L_{(p)} = L_{\mathbb{Z}_{(p)}}$  and  $G_{(p)} = \operatorname{GSpin}(L_{(p)})$ . Then G is the generic fiber of  $G_{(p)}$ . We denote both groups by G. Set  $K = K_p K^p \subset G(\mathbb{A}_f)$ where  $K_p = G(\mathbb{Z}_p) \subset G(\mathbb{Q}_p)$  and  $K^p \subset G(\mathbb{A}_f^p)$  is a sufficiently small compact open subgroup. Thus  $K_p$  is hyperspecial. We get an associated Shimura variety with canonical model M over  $\mathbb{Q}$ . The Shimura variety M is of Hodge type and in [10] Kisin shows that there is a smooth integral model  $\mathcal{M} = \mathcal{M}_{(p)}$  over  $\mathbb{Z}_{(p)}$ . By construction of  $\mathcal{M}_{(p)}$ , the Kuga-Satake abelian scheme over M extends to a polarized abelian scheme  $A \to \mathcal{M}_{(p)}$ .

Now I will explain the notion of special endomorphisms, see [21, 1] for details. Let  $H_{(p)} = C(L_{(p)})$  viewed as a  $\mathbb{Z}_{(p)}$ -representation of G via left multiplication. Associated to  $H_{(p)}$ , there is a  $\mathbb{Z}_{(p)}$ -local system  $\mathbf{H}_B$  on  $M(\mathbb{C})$ . Similarly there is a  $\mathbb{Z}_{(p)}$ -local system  $\mathbf{V}_B$  on  $M(\mathbb{C})$  associated to  $L_{(p)}$ . These induce vector bundles with filtrations  $\mathbf{H}_{dR}$  and  $\mathbf{V}_{dR}$  on  $M(\mathbb{C})$  and these descend over the canonical model M. The  $\mathbb{Q}_\ell$ -local systems  $\mathbf{H}_B \otimes \mathbb{Q}_\ell$  and  $\mathbf{V}_B \otimes \mathbb{Q}_\ell$  over  $M(\mathbb{C})$  have descends over M. Similarly the  $\mathbb{Z}_p$ -local systems  $\mathbf{H}_B \otimes \mathbb{Z}_p$  and  $\mathbf{V}_B \otimes \mathbb{Z}_p$  on  $M(\mathbb{C})$  descend over M. The sheaves  $\mathbf{H}_\ell, \mathbf{V}_\ell$  for  $\ell \neq p$  and  $\mathbf{H}_{dR}, \mathbf{V}_{dR}$  extend over the integral model  $\mathcal{M}_{(p)}$ . Finally associated to the  $\mathbb{Z}_{(p)}$ -representations of G, there are F-crystals  $\mathbf{H}_{cris}$  and  $\mathbf{V}_{cris}$  over the special fiber  $\mathcal{M}_{\mathbb{F}_p}$ . The sheaves  $\mathbf{H}_\ell$  recovers the relative cohomology of the universal abelian scheme A over  $\mathcal{M}_{(p)}$ .

The action of  $L_{(p)}$  on  $H_{(p)}$  by left multiplication induces embeddings on cohomological realizations

$$\mathbf{V}_{\bullet} \subset \underline{\operatorname{End}}_{C(L)}(\mathbf{H}_{\bullet})$$

where  $\bullet = B, \ell, dR, cris.$  The sheaves  $\mathbf{H}_{\bullet}$  have interpretations in terms of the cohomology of the Kuga-Satake abelian scheme: B for Betti homology, dR for first relative de Rham homology,  $\ell$  for first relative etale homology and cris for the first relative crystalline homology of A. Now for any  $\mathcal{M}_{(p)}$ -scheme S, an endomorphism  $f \in \operatorname{End}(A_S)_{(p)}$  is called *special* if all of its homological realizations lie in the image of the above embedding. Write  $V(A_S) \subset \operatorname{End}(A_S)_{(p)}$  for the space of special endomorphisms. If s is a geometric point of  $\mathcal{M}_{(p)}$  valued in a field of characteristic p, then  $x \in \operatorname{End}(A_s)_{(p)}$  is special if and only if the crystalline realization  $x_{cris}$  lies in  $\mathbf{V}_{cris,s}$ . For each  $x \in V(A_S)$  we have  $x \circ x = Q(x) \cdot \operatorname{id}_{A_S}$  for some  $Q(x) \in \mathbb{Z}_{(p)}$ . The map  $x \mapsto Q(x)$  is a positive definite  $\mathbb{Z}_{(p)}$ -valued quadratic form on  $V(A_S)$ . For details of this definition see chapter 3.2.

For  $m \in \mathbb{Q}$ , define the special cycle  $\mathcal{Z}(m) \to \mathcal{M}_{(p)}$  as the stack over  $\mathcal{M}_{(p)}$  with

functor of points

$$\mathcal{Z}(m)(S) = \{ x \in V(A_S) : Q(x) = m, \}$$

for any scheme  $S \to \mathcal{M}_{(p)}$ . The special cycle is empty unless  $m \in \mathbb{Z}_{(p)}$  and positive. Similarly for  $T \in \text{Sym}_k(\mathbb{Z}_{(p)})$  with  $\det(T) \neq 0$ , the special cycle  $\mathcal{Z}(T) \to \mathcal{M}_{(p)}$  is defined as the stack over  $\mathcal{M}_{(p)}$  with functor of points

$$\mathcal{Z}(T)(S) = \{ (x_1, \dots, x_k) \in V(A_S)^k : Q(x) = T \quad \eta^p \circ x_i \circ (\eta^p)^{-1} \in L \otimes \widehat{\mathbb{Z}}^{(p)} \}.$$

for each  $\mathcal{M}_{(p)}$ -scheme S. Here  $\eta^p : V^p(A_S) \xrightarrow{\sim} C(\mathbb{A}_f^p)$  is the  $K^p$ -level structure induced from the level structure on the universal abelian scheme over the Siegel Shimura variety  $\mathscr{A}_g$ .

Similar to the results of [17, 18], we proved the following

**Theorem 1.1.** If k = n-1 and T is nonsingular positive definite, then the special cycle  $\mathcal{Z}(T)$  lies over the supersingular locus in the special fiber  $\mathcal{M}_{\mathbb{F}_p}$ . In particular, the generic fiber of  $\mathcal{Z}(T)$  is empty.

This suggests that we can use the uniformization of the supersingular locus via the Rapoport-Zink space constructed by Howard-Pappas [8], in order to relate these special cycles to the local cycles that will be defined on the associated Rapoport-Zink space.

In chapter 4, we define the special cycles on the Rapoport-Zink space and prove our main results on their dimensions. Let RZ be the Rapoport-Zink space associated to a supersingular point in the GSpin Shimura variety. Let  $k = \overline{\mathbb{F}}_p$ , W = W(k) be the Witt vectors and K = W[1/p]. The space of special quasiendomorphisms is a  $\mathbb{Q}_p$ -quadratic space V' of the same dimension and determinant as  $V_{\mathbb{Q}_p}$  but different Hasse invariant. We have  $V' \subset \operatorname{End}(X_0)_{\mathbb{Q}}$  where  $X_0$  is the base point which is used to define the Rapoport-Zink space. The vertex lattices are certain  $\mathbb{Z}_p$ -lattices  $\Lambda \subset V'$ . In [8], these lattices parametrize certain closed formal subschemes  $RZ_{\Lambda}$  of the Rapoport-Zink space RZ. For a special endomorphism  $j \in V'$  we define the special cycle Z(j) associated to j to be the closed formal subscheme of RZ consisting of all points  $(X, \rho)$  such that

$$\rho \circ j \circ \rho^{-1} \in \operatorname{End}(X).$$

If  $(j_1, \ldots, j_{n-1}) \in (V')^{n-1}$  then Z(J) is defined similarly, where J is the  $\mathbb{Z}_p$ -span in V' of the endomorphisms  $\{j_1, \ldots, j_{n-1}\}$ . Here Z(J) is the intersection of the cycles  $Z(j_1), \ldots, Z(j_{n-1})$ . The fundamental matrix is defined as  $T = Q(J) \in$  $\operatorname{Sym}_{n-1}(\mathbb{Z}_p)$ . By the structure of the quadratic space V', the rank of the reduction of T modulo p can not be n-1, see Proposition 4.4. Our main results are determining the dimension of the intersection  $Z(J)^{\operatorname{red}}$  in terms of this matrix T where  $Z(J)^{\operatorname{red}}$  is the underlying reduced scheme of Z(J). The key observation proving these results is writing these special cycles as a union of closed formal subschemes  $\operatorname{RZ}_{\Lambda}$  which are well understood by [8]. We will always assume that Z(J)is nonempty. Our results depends on parity of n and the determinant  $\det(V_{\mathbb{Q}_p})$ and there are three cases. The reason for these different cases is that in each of these cases the dimension of the irreducible components of the Rapoport-Zink space is different. Throughout all the determinants of quadratic spaces are taken to be modulo squares.

**Theorem 1.2.** Let n be even and  $\det(V_{\mathbb{Q}_p}) \neq (-1)^{n/2}$ . Let  $m = \operatorname{rank}(\bar{T})$  and  $d_T = \det(\bar{T}/\operatorname{Rad}(\bar{T}))$  where  $\bar{T}$  is reduction of T modulo p. Then

- (i) If m = 0, then  $Z(J)^{\text{red}}$  is  $\frac{n-2}{2}$  dimensional.
- (ii) If m is n-2 or n-3, then  $Z(J)^{\text{red}}$  is 0-dimensional.
- (iii) Suppose  $1 \le m \le n-4$ . Then
  - (a) If m is odd, then  $\dim Z(J)^{\operatorname{red}} = \frac{n-m-3}{2}$
  - (b) If m is even, then

dim 
$$Z(J)^{\text{red}} = \begin{cases} \frac{n-m-2}{2} & \text{if } d_T = (-1)^{m/2} \\ \frac{n-m-2}{2} - 1 & \text{if } d_T \neq (-1)^{m/2} \end{cases}$$

In particular, when m = n - 4 we have

dim 
$$Z(J)^{\text{red}} = \begin{cases} 1 & \text{if } d_T = (-1)^{n/2} \\ 0 & \text{if } d_T \neq (-1)^{n/2} \end{cases}$$

This gives us explicit conditions on the fundamental matrix T so that  $Z(J)^{\text{red}}$  is zero dimensional.

In the other cases, similar to the above theorem, we have

**Theorem 1.3.** Let n be even and  $\det(V_{\mathbb{Q}_p}) = (-1)^{n/2}$ . Let  $m = \operatorname{rank}(\bar{T})$  and  $d_T = \det(\bar{T}/\operatorname{Rad}(\bar{T}))$  where  $\bar{T}$  is reduction of T modulo p. Then

- (i) If m = 0, then  $Z(J)^{\text{red}}$  is  $\frac{n-4}{2}$  dimensional.
- (ii) If m is n-2 or n-3, then  $Z(J)^{\text{red}}$  is 0 dimensional.
- (iii) Suppose  $1 \le m \le n-4$ . Then
  - (a) If *m* is odd, then  $\dim Z(J)^{\text{red}} = \frac{n-m-1}{2} 1$ .
  - (b) If m is even, then

dim 
$$Z(J)^{\text{red}} = \begin{cases} \frac{n-m-2}{2} & \text{if } d_T \neq (-1)^{m/2} \\ \frac{n-m-2}{2} - 1 & \text{if } d_T = (-1)^{m/2} \end{cases}$$

These two theorems cover the cases when n is even. For odd n, we have the following theorem

**Theorem 1.4.** Let n be odd and  $m = \operatorname{rank}(\bar{T})$ . Let  $d_T = \det(\bar{T}/\operatorname{Rad}(\bar{T}))$ . Then

- (i) If m = 0, then  $Z(J)^{\text{red}}$  is  $\frac{n-3}{2}$  dimensional.
- (ii) If m is n-2 or n-3, then  $Z(J)^{\text{red}}$  is 0 dimensional.
- (iii) Suppose  $1 \le m \le n-4$ . Then
  - (a) If m is even, then  $\dim Z(J)^{\operatorname{red}} = \frac{n-m-3}{2}$ .

(b) If m is odd, then

$$\dim Z(J)^{red} = \begin{cases} \frac{n-m-2}{2} & \text{if } d_T \neq \epsilon(-1)^{\frac{m+4-n}{2}} \\ \frac{n-m-2}{2} - 1 & \text{otherwise} \end{cases}$$

where  $\epsilon \in \{\pm 1\}$  is the modulo p square class of  $\det(V_{\mathbb{Q}_p})$ .

The above results tell us exactly when the special cycles Z(J) are 0-dimensional generalizing the results [17, Theorem 6.1] and [18, Corollary 5.15]. The above theorems form a crucial part of a general program mentioned in the introduction. The main goal of this program for GSpin Shimura varieties is to obtain a relation between the degrees of the special cycles and Fourier coefficients of Eisenstein series. As noted before our results tell us explicitly when the intersection of special cycles consists of isolated points which provides a step towards this goal.

Furthermore the above results completely determine the degenerate intersections where the dimension of the intersection is not zero. One can try to follow the program mentioned above for the degenerate intersections. In this case the intersection multiplicities are defined by using derived tensor products. For signature (2, 2) case, this is done by Terstiege [25]. They generalize the results in [17] to the case where the intersection of special cycles are one dimensional. The above theorems show that for general n these degenerate intersections can be higher dimensional.

# 2 Moduli spaces of p-divisible groups

#### 2.1 Formal Schemes

In this section we briefly recall some preliminary facts about formal schemes. The references for formal schemes are [4, 2].

Schemes are locally ringed spaces which are locally affine. We will define formal schemes similarly to be locally ringed spaces built from 'affine pieces'. In order to define what these 'affine pieces' are, we need the notion of adic rings.

A ring R with a topology is called a topological ring if the multiplication and addition defined by continuous maps. Given any ring R and an ideal  $\mathfrak{a} \subset R$ , we can consider the topology defined by the ideals  $\mathfrak{a}^n, n \in \mathbb{N}$  viewed as a basis of neighborhoods of  $0 \in R$ . Hence in this topology  $U \subset R$  is open if every element  $x \in U$  is contained in an open subset of R contained in U, i.e. there exists n such that  $x + \mathfrak{a}^n \subset U$  for some n. This topology on R is called  $\mathfrak{a}$ -adic topology and the ideal  $\mathfrak{a}$  is called an ideal of definition. An adic ring R is a topological ring Rwhose topology is  $\mathfrak{a}$ -adic topology for some ideal  $\mathfrak{a} \subset R$ .

Given an adic ring R with an ideal of definition  $\mathfrak{a}$ . Then the separated completion  $\hat{R}$  of R is defined as

$$\widehat{R} = \varprojlim R/\mathfrak{a}^n$$

We say that R is complete with respect to the  $\mathfrak{a}$ -adic topology if the canonical map

$$R \to \underline{\lim} R / \mathfrak{a}^n$$

is an isomorphism. If  $\mathfrak{a}$  is a finitely generated ideal, then the topology on  $\widehat{R}$  is the same as  $\mathfrak{a}\widehat{R}$ -adic topology.

Let A be a complete and separated adic ring with an ideal of definition  $\mathfrak{a} \subset A$ . Define Spf A to be the set of open prime ideals of A. If  $\mathfrak{p} \subset A$  is an open prime ideal, then by definition of the  $\mathfrak{a}$ -adic topology,  $\mathfrak{a}^n \subset \mathfrak{p}$  for some n and since  $\mathfrak{p}$  is prime, we have  $\mathfrak{a} \subset \mathfrak{p}$ . This shows that as a set Spf A is identified with the closed subset Spec  $A/\mathfrak{a} \subset$  Spec A. This does not depend on the choice of an ideal of definition. Thus Spf A admits the Zariski topology induced from Spec A. Now we will describe a sheaf of topological rings on Spf A. For any  $f \in A$ , define an open subset D(f) of Spf A as,

$$D(f) = \{ \mathfrak{p} \in \mathrm{Spf} \ A : f \notin \mathfrak{p} \}$$

in other words, viewing f as a function on Spf A, D(f) is the locus on which f does not vanish. Note that D(f) is the intersection of the basic open subset of Spec A defined by f and Spf A, hence it is open with respect to the induced topology. Consider the presheaf  $\mathcal{O}$ 

$$\mathcal{O}(D(f)) = \underline{\lim}(A/\mathfrak{a}^n[f^{-1}])$$

The presheaf  $\mathcal{O}$  is in fact a sheaf with respect to the basis consisting of open subsets of the form  $D(f) \subset \text{Spf } A$ . Now let  $U \subset \text{Spf } A$  be an arbitrary Zariski open subset and consider an open cover  $U = \bigcup_i D(f_i)$  by basic open subsets. Consider for each n the exact sequence

$$\mathcal{O}_{\text{Spec } A/\mathfrak{a}^n}(U) \to \prod_i A/\mathfrak{a}^n[f_i^{-1}] \rightrightarrows \prod_{i,j} A/\mathfrak{a}^n[(f_i f_j)^{-1}]$$

and take the projective limit over n to get an exact sequence

$$\mathcal{O}(U) \to \prod_{i} \varprojlim (A/\mathfrak{a}^{n}[f_{i}^{-1}]) \Longrightarrow \prod_{i,j} \varprojlim (A/\mathfrak{a}^{n}[(f_{i}f_{j})^{-1}])$$

Hence  $\mathcal{O}$  is defined as  $\varprojlim \mathcal{O}_{\text{Spec } A/\mathfrak{a}^n}$ . Thus we have defined a sheaf of topological rings  $\mathcal{O}$  on Spf A. This also shows that as a locally topologically ringed space, Spf A is the direct limit

$$\varinjlim \operatorname{Spec} A/\mathfrak{a}^n = (\varinjlim \operatorname{Spec} A/\mathfrak{a}^n, \varprojlim \mathcal{O}_{\operatorname{Spec} A/\mathfrak{a}^n})$$

**Definition 2.1.** Let A be an adic ring with an ideal of definition  $\mathfrak{a}$ . Then the locally ringed space  $(X, \mathcal{O}_X) = (\text{Spf } A, \mathcal{O})$  is called an affine formal scheme.

Remark 2.2. Consider an open subset  $D(f) \subset \text{Spf } A$ . Since the completion  $\varprojlim(A/\mathfrak{a}^n[f^{-1}])$  is not necessarily an adic ring, there is a problem with viewing the locally ringed space  $(U, \mathcal{O}|_U)$  as the affine formal scheme  $\text{Spf } \varprojlim(A/\mathfrak{a}^n[f^{-1}])$ . We can overcome this issue by considering admissible rings in place of adic rings. This problem does not occur if A has an ideal of definition which is finitely generated.

**Definition 2.3.** A locally topologically ringed space  $(X, \mathcal{O}_X)$  is called a formal scheme if for each point  $x \in X$  there is an open neighborhood U of x such that  $(U, \mathcal{O}_X|_U)$  is isomorphic to an affine formal scheme Spf A.

The most important example of formal schemes is the formal completion of a scheme along a closed subscheme. Now we will explain this. Let X be a scheme and  $Y \subset X$  be a closed subscheme with ideal sheaf  $\mathcal{I} \subset \mathcal{O}_X$ . Each  $\mathcal{O}_X/\mathcal{I}^n$  restricts to a sheaf on Y and hence we can consider the projective limit  $\varprojlim \mathcal{O}_X/\mathcal{I}^n$  on Y. This makes  $(Y, \varprojlim \mathcal{O}_X/\mathcal{I}^n)$  into a formal scheme  $(\widehat{X})_{/Y}$  called formal completion of X along Y. Locally it looks as follows: Let X = Spec A and assume that Y is defined by an ideal  $\mathfrak{a} \subset A$ . Then

$$(Y, \varprojlim \mathcal{O}_X/\mathcal{I}^n) = \operatorname{Spf}(\varprojlim A/\mathfrak{a}^n) = \operatorname{Spf} \widehat{A}$$

*Example* 2.4. Let X be a scheme. Take Y = X. Then  $(\widehat{X})_{/Y} = X$  and so the category of schemes is a full subcategory of the category of formal schemes.

Example 2.5. Let  $Y = x \in X$  be a closed point. Then  $(\widehat{X})_{/Y}$  is the one point space  $\{x\}$  with the structure sheaf  $\widehat{\mathcal{O}_{X,x}}$ , or with the above notation  $\operatorname{Spf} \widehat{\mathcal{O}_{X,x}}$ . Intiutively, this is the point x together with all the 'differential data' at x.

Example 2.6. As a special case of the previous example, let  $X = \text{Spec } \mathbb{Z}$  and take  $x = (p) \in X$ . Then  $(\widehat{X})_{/Y}$  is  $\text{Spf } \mathbb{Z}_p$ .

Now we will define the notion of the underlying reduced scheme of a formal scheme. First consider an formal affine scheme Spf A for an adic ring A with an ideal of definition  $\mathfrak{a}$ . Then the ideal of topologically nilpotent elements  $A^{\circ\circ}$  is simply  $\sqrt{\mathfrak{a}}$  and the scheme Spec  $A/A^{\circ\circ}$  is a reduced scheme. In general, if X is a formal scheme, define the ideal sheaf  $\mathcal{I}^{\mathrm{red}} \subset \mathcal{O}_X$  as follows: for affine open formal subscheme Spf  $A \subset X$ ,

$$\mathcal{I}^{\mathrm{red}}(\mathrm{Spf}\;A) = A^{\circ\circ}$$

Then the locally ringed space  $(X, \mathcal{O}_X/\mathcal{I}^{\text{red}})$  is a reduced scheme denoted as  $X^{\text{red}}$ . There is a natural morphism  $X^{\text{red}} \to X$ . The association  $X \mapsto X^{\text{red}}$  is functorial: if  $f: X \to Y$  is a morphism of formal schemes, then there is an induced morphism  $f^{\text{red}}: X^{\text{red}} \to Y^{\text{red}}$ .

Example 2.7. Let X be a scheme and Y be an reduced closed subscheme. Let us denote  $(\widehat{X}_{/Y})$  by  $\mathcal{X}$ . Then  $\mathcal{X}^{\text{red}} = Y$ . For each  $n \geq 1$ , consider the scheme

$$X_n = (\mathcal{X}, \mathcal{O}_{\mathcal{X}}/(\mathcal{I}^{\mathrm{red}})^n)$$

and so  $X_n$  is an infinitesimal thickening of  $\mathcal{X}^{\text{red}} = Y$  and set  $X_0 = \mathcal{X}^{\text{red}} = Y$ . Then  $\mathcal{X} = \varinjlim X_n$  in the category of locally topologically ringed spaces. This way  $\mathcal{X}$  can be viewed as Y together with the infinitesimal thickenings of Y.

#### 2.2 p-divisible groups

In this section we will briefly recall the definition and basic properties of p-divisible groups. References for these are [3, 24, 22]

We will define p-divisible groups as directed systems of locally free group schemes. The appropriate category they live in is the category of fppf sheaves of groups. Fix a prime number p and let S be a scheme. All schemes over S will be identified with the corresponding fppf sheaf. By S-group we will mean an fppf sheaf of groups on the site Sch(S).

**Definition 2.8.** A p-divisible group over S is an S-group X satisfying

- 1. The morphism  $p: X \to X$  is an epimorphism, i.e. X is p-divisible
- 2.  $X = \varinjlim X(n)$ , where  $X(n) = \ker(p^n : X \to X)$ , i.e. X is p-power torsion
- 3. Each X(n) is a finite locally free group scheme over S.

If S = Spec R for some commutative ring, then a locally free group scheme of rank m over S is of the form Spec A for some locally free algebra A of rank mover R.

There is an equivalent definition given by Tate.

**Definition 2.9.** A *p*-divisible group over *S* is an inductive system  $\{X_n\}$  of finite locally free group schemes over *S* satisfying

1. For each n, we have the following exact sequence

$$0 \to X_n \to X_{n+1} \xrightarrow{p^n} X_{n+1}$$

2. There is a locally constant function h on S such that the rank of  $X_n$  is  $p^{nh}$ . The function h is called the height of X.

Given a *p*-divisible group  $\{X_n\}$  as in Tate's definition, inductive limit  $X = \lim_{n \to \infty} X_n$  defines a *p*-divisible group as in definition 2.8.

The morphisms between *p*-divisible groups are simply morphisms of fppf sheaves. In terms of Tate's definition, a morphism  $f : X \to Y$  is given by a compatible family  $f = \{f_n : X_n \to Y_n\}.$ 

Given a finite locally free group scheme G over S, we have a Cartier dual defined as the group scheme  $G^{\vee}$  over S, representing the functor  $G^{\vee}(T) = \operatorname{Hom}_{T}(G_{T}, \mathbb{G}_{mT})$ . Now given a p-divisible group  $X_{n}$ , the morphisms  $p : X_{n+1} \to X_{n}$  induces dual morphisms  $p^{\vee} : X_{n}^{\vee} \to X_{n+1}^{\vee}$  and these morphisms yield an inductive system  $\{X_{n}^{\vee}\}$ which defines a p-divisible group  $X^{\vee}$  called the Cartier dual of X. Thus  $G \mapsto G^{\vee}$ gives a duality in the category of p-divisible groups.

Examples 2.10. (1) Consider the constant group schemes  $G_n = \frac{1}{p^n} \mathbb{Z}/\mathbb{Z}$ . Then  $G = \varinjlim G_n = \mathbb{Q}_p/\mathbb{Z}_p$  is a p-divisible group of height 1.

(2) Consider  $G_n \mu_{p^n} = \mathbb{G}_m[p^n]$ . Then  $G = \varinjlim G_n = \mu_{p^{\infty}}$  is a p-divisible group of height 1.

(3) Let A/S be an abelian scheme of relative dimension g and  $G_n = A[p^n]$ . Then  $G_n$  is a finite group scheme of order  $p^{2gn}$  and

$$G = \lim G_n = A[p^\infty]$$

is a p-divisible group of height 2g.

A morphism of p-divisible groups is called an isogeny if it is an epimorphism with finite kernel. A quasi-isogeny  $\rho: G \to G'$  of p-divisible groups is defined to be a global section of the Zariski sheaf  $\underline{\operatorname{Hom}}_{S}(G, G') \otimes_{\mathbb{Z}} \mathbb{Q}$  such that, Zariski locally  $p^{n}\rho$  is an isogeny for some n. Denote the set of quasi-isogenies over S between G and G' by  $\operatorname{Qisg}_{S}(G, G')$ . There is the following rigidity property. Let S be a scheme on which p is locally nilpotent and  $S' \subset S$  be a closed subscheme defined by a locally nilpotent ideal sheaf. Then the natural map obtained by pullbacks

$$\operatorname{Qisg}_{S}(G, G') \to \operatorname{Qisg}_{S'}(G, G')$$
 (2.2.1)

is bijective. We will refer this as the 'rigidity property'. We will need the following

result later.

**Proposition 2.11.** [23, Proposition 2.9] Let  $f : G \to G'$  be a quasi-isogeny of p-divisible groups over a scheme S. Then the functor  $F : \operatorname{Sch}/S \to \operatorname{Sets}$  defined as

$$F(T) = \{\varphi : T \to S \mid \varphi^* f \text{ is an isogeny}\}$$

is represented by a closed subscheme of S.

#### 2.3 Rapoport-Zink Spaces

In this section we will briefly recall the definitions and facts on Rapoport-Zink spaces following [23].

Fix a prime p. Let  $k = \overline{\mathbb{F}}_p$  and W = W(k) be the Witt ring over k. Let  $K_0$  be the fraction field of W and  $\sigma$  be the Frobenius automorphism on W which also induces a Frobenius on  $K_0$ . Let  $\mathcal{O}$  be a complete DVR of mixed characteristic (0, p) and Nilp<sub> $\mathcal{O}$ </sub> be the category of locally noetherian schemes S over  $\mathcal{O}$  such that p is locally nilpotent. Define  $\overline{S} = S \times_{\text{Spec } \mathcal{O}} \text{Spec } \mathcal{O}/p\mathcal{O}$ .

Let  $X_0$  be a fixed *p*-divisible group over k and consider the moduli problem  $\operatorname{RZ}(X_0)$  on  $\operatorname{Nilp}_W$  assigning to each  $S \in \operatorname{Nilp}_W$ , the pairs  $(X, \rho)$  where

- X is a p-divisible group over S,
- $\rho: X_0 \times_k \overline{S} \to X \times_S \overline{S}$  is a quasi-isogeny.

Two pairs  $(X, \rho)$  and  $(X', \rho')$  over S are identified if the quasi-isogeny  $\rho \circ \rho'^{-1}$ :  $X' \times_S \overline{S} \to X \times_S \overline{S}$  lifts to an isomorphism  $X' \to X$ .

**Theorem 2.12.** [23, Theorem 2.16] The functor  $\mathcal{M}$  is represented by a locally formally of finite type formal scheme over Spf W

An alternative definition of the above moduli problem can be given as follows: First by Grothendieck-Messing theory the deformation functor  $\text{Def}(X_0/W)$  is formally smooth and so one can choose a lift  $\widetilde{X}_0$  of  $X_0$  over W. And by rigidity a quasi-isogeny of the form  $\rho: X_0 \times_k \overline{S} = \widetilde{X}_{0S} \times_S \overline{S} \to X \times_S \overline{S}$  lifts to a quasi-isogeny  $\tilde{\rho}: \widetilde{X}_{0S} \to X$ . Thus an S-point of  $\mathcal{M}$  can be given by  $(X, \tilde{\rho})$ 

- X is a p-divisible group over S,
- $\widetilde{\rho}: \widetilde{X}_{0S} \to X$  is a quasi-isogeny.

Similarly, if  $X_0$  has a principal polarization  $\lambda_0 : X_0 \xrightarrow{\sim} X_0^{\vee}$ , consider the moduli problem  $\operatorname{RZ}(X_0, \lambda_0)$  on  $\operatorname{Nilp}_W$  assigning to each S, isomorphism classes of triples  $(X, \lambda, \rho)$  where

- X is a p-divisible group over S,
- $\lambda: X \xrightarrow{\sim} X^{\vee}$  is a principal polarization,
- $\widetilde{\rho}: \widetilde{X}_{0S} \to X$  is a quasi-isogeny such that, Zariski locally on  $\overline{S}$ ,  $\rho^{\vee} \circ \lambda \circ \rho = c\lambda_0$ where  $c \in \mathbb{Q}_p^{\times}$ , i.e.  $\rho$  respects polarizations up to a scalar.

By [23], this functor is representable by a locally formally of finite type formal scheme over Spf W. Forgetting the polarization gives a closed embedding  $RZ(X_0, \lambda_0) \rightarrow RZ(X_0).$ 

*Examples* 2.13. (1) Let  $X_0$  be a p-divisible group over k of dimension 1 and height h. Assume that the isocrystal of  $X_0$  is isoclinic of slope 1/h. Then

$$\operatorname{RZ}(X_0) \simeq \bigsqcup_{n \in \mathbb{Z}} \operatorname{Spf} W[[T_1, \dots, T_{h-1}]]$$

For a proof of this, see [23, Proposition 3.79].

(2) Now consider the case  $X_0 = \mu_{p^{\infty}}^n \times (\mathbb{Q}_p/\mathbb{Z}_p)^n$  then

$$\operatorname{RZ}(X_0) \simeq \bigsqcup_{(\operatorname{GL}_n(\mathbb{Q}_p)/\operatorname{GL}_n(\mathbb{Z}_p))^2} \operatorname{Spf} W[[T_{11}, \dots, T_{nn}]]$$

For a proof of this, see [23, Proposition 3.81].

## 3 Special Cycles on GSpin Shimura varieties

#### 3.1 Hodge Type Shimura Varieties

In this section we will give a review of the Shimura varieties of Hodge type following [10].

Fix a Q-vector space V equipped with a symplectic form  $\psi : V \times V \to \mathbb{Q}$ . Let  $G = \operatorname{GSp}(V, \psi)$  be the group of symplectic similitudes. Let  $S^{\pm}$  be the Siegel double space defined as the set of maps  $h : \mathbb{S} \to G_{\mathbb{R}}$  satisfying

• The  $\mathbb{C}^{\times}$  action on  $V_{\mathbb{R}}$  induced by  $\mathbb{R}$ -points of h gives a Hodge structure

$$V_{\mathbb{C}} \xrightarrow{\sim} V^{-1,0} \oplus V^{0,-1}$$

• The pairing  $(u, v) \mapsto \psi(u, h(i)v)$  is positive or negative definite on  $V_{\mathbb{R}}$ .

The pair  $(G, S^{\pm})$  is a Shimura datum with reflex field  $\mathbb{Q}$ . We will assume that there exists a  $\mathbb{Z}$ -lattice  $V_{\mathbb{Z}} \subset V$  such that  $V_{\mathbb{Z}_{(p)}}$  self dual with respect to  $\psi$ . Thus G admits a reductive model over  $\mathbb{Z}_{(p)}$ , which we will denote by G again. Let  $U = U_p U^p \subset G(\mathbb{A}_f)$  be a compact open subgroup such that  $U_p = G(\mathbb{Z}_p)$  is hyperspecial and  $U^p \subset G(\mathbb{A}_f^p)$  is sufficiently small. Now we will describe the integral model over  $\mathbb{Z}_{(p)}$  of the Shimura variety  $\mathrm{Sh}_U(G, S^{\pm})$ . Given a  $\mathbb{Z}_{(p)}$ -scheme S and an abelian scheme A over S, define an etale local system on S as

$$T^p(A) = \varprojlim_{pm} A[n]$$

and set  $V^p(A) = T^p(A) \otimes \mathbb{Q}$ . Now consider the category of abelian schemes over Sup to prime to p isogeny. The objects of this category are abelian schemes over Sand the morphisms between two objects A and B are  $\operatorname{Hom}_S(A, B) \otimes \mathbb{Z}_{(p)}$ . A weak polarization on A is an equivalence class of prime to p isogenies  $\lambda : A \xrightarrow{\sim} A^{\vee}$  such that  $c\lambda$  is a polarization for some  $c \in \mathbb{Z}_{(p)}^{\times}$ . Fix a pair  $(A, \lambda)$ , an abelian scheme over S up to prime to p isogeny and a weak polarization  $\lambda$  on A. Then a  $U^p$ -level structure on A is defined as a glocal section

$$\eta^p \in \Gamma(S, \underline{\operatorname{Isom}}(V \otimes \mathbb{A}_f^p, V^p(A))/U^p)$$

where  $\underline{\text{Isom}}(V \otimes \mathbb{A}_f^p, V^p(A))/U^p$  is the etale sheaf on S consisting of  $U^p$ -orbits of isomorphisms

$$V \otimes \mathbb{A}_f^p \xrightarrow{\sim} V^p(A)$$

mapping the symplectic pairing  $\psi$  to a  $(\mathbb{A}_f^p)^{\times}$  multiple of the symplectic pairing induced by  $\lambda$ . Now consider the functor on  $\operatorname{Sch}/\mathbb{Z}_{(p)}$  which assigns to each S the set of triples  $(A, \lambda, \eta^p)$  up to isomorphism. This functor is representable by a smooth scheme  $\mathscr{A}_{g,U}$  over  $\mathbb{Z}_{(p)}$  such that

$$\mathscr{A}_{g,U} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q} \xrightarrow{\sim} \operatorname{Sh}_U(G, S^{\pm})$$

Now let (G, X) be a Shimura data together with an embedding  $(G, X) \to (\operatorname{GSp}(V, \psi), S^{\pm})$ . Such (G, X) is called Hodge type Shimura datum. For simplicity we will assume that the reflex field of (G, X) is  $\mathbb{Q}$ . Let  $U = U_p U^p \subset G(\mathbb{A}_f)$  and  $U' = U'_p U'^p \subset \operatorname{GSp}(\mathbb{A}_f)$  be compact open subgroups such that  $U_p = G(\mathbb{Z}_p)$ ,  $U'_p \cap G(\mathbb{Q}_p) = U_p$  and  $U^p \subset U'^p$ . The embedding of Shimura data induces a map of the corresponding canonical models over  $\mathbb{Q}$ ,

$$\iota : \operatorname{Sh}_U(G, X) \to \operatorname{Sh}_{U'}(\operatorname{GSp}, S^{\pm})$$

By the above discussion there is a universal abelian scheme  $\mathcal{A} \to \operatorname{Sh}_{U'}(\operatorname{GSp}, S^{\pm})$ which induces an abelian scheme  $f : \mathcal{A} \to \operatorname{Sh}_U(G, X)$ .

In [10, Proposition 1.3.2], Kisin proves that there is a finite collection of tensors  $(s_{\alpha}) \in V_{\mathbb{Z}_{(p)}}^{\otimes}$  cutting out  $G \subset \operatorname{GL}(V_{\mathbb{Z}_{(p)}})$ . Let  $\mathbf{H}_B$  be the first Betti cohomology of  $A_{\mathbb{C}} \to \operatorname{Sh}_U(G, X)_{\mathbb{C}}^{\operatorname{an}}$  viewed as a  $\mathbb{Z}_{(p)}$ -local system. The tensors  $(s_{\alpha})$  induces tensors  $s_{\alpha,B}^{\operatorname{univ}} \in \mathbf{H}_B^{\otimes}$ . Let  $\mathbf{H}_{\mathrm{dR}}$  be the first relative de Rham cohomology of  $A \to \operatorname{Sh}_U(G, X)$  and  $\mathbf{H}_{\mathrm{dR},\mathbb{C}}$  be the pullback to  $\operatorname{Sh}_U(G, X)_{\mathbb{C}}$ , these are filtered vector bundles on M

and  $M(\mathbb{C})$ , respectively. Let  $\mathbf{H}_{\mathbb{A}_{f}^{p}} = V^{p}(\mathcal{A})$ . Then the tensors  $s_{\alpha,B}^{\text{univ}}$  induce, via de Rham comparison isomorphism, de Rham tensors  $s_{\alpha,dR}^{\text{univ}} \in \mathbf{H}_{\mathrm{dR},\mathbb{C}}^{\otimes}$ , which then descend to de Rham tensors  $s_{\alpha,dR}^{\text{univ}} \in \mathbf{H}_{\mathrm{dR}}^{\otimes}$ . The tensors  $s_{\alpha,B}^{\text{univ}}$  also induce etale tensors  $s_{\alpha,\mathrm{et}}^{\mathrm{univ},p} \in \mathbf{H}_{\mathbb{A}_{f}^{p}}^{\otimes}$ . For a suitable choice of  $U' \subset \mathrm{GSp}(\mathbb{A}_{f}^{p})$ , the integral model  $\mathcal{M}$  of the Shimura variety  $\mathrm{Sh}_{U}(G, X)$  is defined in [10] as the normalization of the closure of  $\mathrm{Sh}_{U}(G, X)$  in  $\mathscr{A}_{g,U'}$ :

$$\operatorname{Sh}_U(G, X) \to \operatorname{Sh}_U(G, X)^- \to \mathscr{A}_{q,U}$$

One of the main results of [10] shows that this integral model is smooth. By construction the universal abelian scheme over  $\mathcal{A} \to \mathscr{A}_{g,U'}$  induces an abelian scheme  $\mathcal{A} \to \mathcal{M}$ . The relative de Rham cohomology of  $\mathcal{A}$ , gives an extension of  $\mathbf{H}_{dR}$  to  $\mathcal{M}$  and the tensors  $s_{\alpha,dR}^{univ}$  extend over the integral model. Similarly the sheaf  $V_p(\mathcal{A})$  on  $\mathrm{Sh}_U(G, X)$  extends to a sheaf  $V^p(\mathcal{A})$  over  $\mathcal{M}$  and the tensors  $s_{\alpha,\mathrm{et}}^{univ,p}$ extend to the integral model. There are also crystalline tensors defined as follows: Let  $\hat{f} : \hat{\mathcal{A}} \to \hat{\mathcal{M}}$  be the completion along the special fiber. There is a natural isomorphism

$$H^1_{\mathrm{dR}}(\widehat{\mathcal{A}}/\widehat{\mathcal{M}}) \simeq (R^1 \widehat{f}_{\mathrm{cris},*} \mathcal{O}_{\widehat{\mathcal{A}}/\mathbb{Z}_p})_{\widehat{\mathcal{M}}}$$

where the righthand side is the restriction to the Zariski site. Let  $\mathbf{H}_{\text{cris}}$  is the first relative crystalline cohomology of  $\mathcal{A}_{\mathbb{F}_p}$  over  $\mathcal{M}_{\mathbb{F}_p}$ . Hence  $\mathbf{H}_{\text{cris}} = R^1 \hat{f}_{\text{cris},*} \mathcal{O}_{\hat{\mathcal{A}}/\mathbb{Z}_p} =$  $\mathbb{D}(X^{\text{univ}})$  where  $X^{\text{univ}}$  is the p-divisible group of  $\hat{\mathcal{A}}$  and  $\mathbb{D}(X^{\text{univ}})$  is the Dieudonne crystal of  $X^{\text{univ}}$ . Via the above isomorphism, the de Rham tensors  $s_{\alpha,\text{dR}}^{\text{univ}}$  induce tensors  $s_{\alpha}^{\text{univ}} \in \mathbb{D}(X^{\text{univ}})^{\otimes}$ . The point is that the universal object  $\mathcal{A}$  over the integral model comes equipped with de Rham, etale and crystalline tensors and these tensors are compatible with respect to comparison isomorphisms.

#### 3.2 GSpin Shimura Variety

Let (V, Q) be a quadratic space over  $\mathbb{Q}$  of signature (n - 2, 2) with  $n \ge 3$ . Let C(V) be the Clifford algebra of (V, Q) with its  $\mathbb{Z}/2$ -grading

$$C(V) = C^+(V) \oplus C^-(V)$$

The Spinor similitude group  $G = \operatorname{GSpin}(V)$  is a reductive group over  $\mathbb{Q}$  defined as

$$G(R) = \{ g \in C^+(V_R)^{\times} : gV_R g^{-1} = V_R \}$$

for any  $\mathbb{Q}$ -algebra R. Hence G acts on V by conjugation. Let  $\mathcal{D}$  be the space of negative definite oriented 2-planes in  $V_{\mathbb{R}}$ . We have an identification

$$\mathcal{D} = \{z \in V_{\mathbb{C}} : [z, z] = 0, [z, \overline{z}] < 0\} / \mathbb{C}^{\times} \subset \mathbb{P}(V_{\mathbb{C}})$$

as follows: Given  $z = u + iv \in \mathcal{D}$ ,  $\operatorname{Span}_{\mathbb{R}}\{u, v\} \subset V_{\mathbb{R}}$  is a negative definite 2-plane. We may assume  $u, v \in V_{\mathbb{R}}$  are orthogonal and Q(u) = Q(v) = -1. Then we get  $\mathbb{R}$ -algebra maps

$$\mathbb{C} \xrightarrow{\sim} C^+(\operatorname{Span}_{\mathbb{R}}\{u,v\}) \to C^+(V_{\mathbb{R}})$$

given by  $i \mapsto uv$ . This map restricts to an injection

$$h_z: \mathbb{C}^{\times} \to G(\mathbb{R})$$

which is induced from a morphism of algebraic groups  $h_z : \mathbb{S} \to G_{\mathbb{R}}$ , where  $\mathbb{S} = \operatorname{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$ . Thus  $z \mapsto h_z$  realizes  $\mathcal{D}$  as a  $G(\mathbb{R})$ -conjugacy class of homomorphisms in  $\operatorname{Hom}(\mathbb{S}, G_{\mathbb{R}})$  and the pair  $(G, \mathcal{D})$  is a Shimura datum with reflex field  $\mathbb{Q}$ .

Let  $L_{(p)} \subset V$  be a self-dual  $\mathbb{Z}_{(p)}$  lattice and  $G_{(p)} = \operatorname{GSpin}(L_{(p)})$ . Then G is the generic fiber of  $G_{(p)}$ . We denote both groups by G. We will also fix a  $\mathbb{Z}$ -lattice L such that  $L_{\mathbb{Z}_{(p)}} = L_{(p)}$ . Set  $K = K_p K^p \subset G(\mathbb{A}_f)$  where  $K_p = G(\mathbb{Z}_p) \subset G(\mathbb{Q}_p)$  and  $K^p \subset G(\mathbb{A}_f^p)$  is a sufficiently small compact open subgroup and we assume that  $L \otimes \widehat{\mathbb{Z}}$  is K stable. Thus  $K_p$  is hyperspecial. Associated to  $(G, \mathcal{D})$  we get a Shimura variety M over  $\mathbb{Q}$  whose complex points are

$$M_K(\mathbb{C}) = G(\mathbb{Z}_{(p)}) \setminus \mathcal{D} \times G(\mathbb{A}_f^p) / K^p$$

see [21, 3.2]. Given a  $\mathbb{Z}_{(p)}$ -representation N of G. There is a  $\mathbb{Z}_{(p)}$  local system  $\mathbb{N}_B$ on  $M_K(\mathbb{C})$  defined as

$$G(\mathbb{Z}_{(p)}) \setminus N \times \mathcal{D} \times G(\mathbb{A}_f^p) / K^p$$

Let  $\mathbf{N}_{\mathrm{dR},M(\mathbb{C})} = \mathbf{N}_B \otimes \mathcal{O}_{M(\mathbb{C})}$ . This is a vector bundle over  $M(\mathbb{C})$  which comes with a natural filtration  $F^{\bullet}\mathbf{N}_{\mathrm{dR},M(\mathbb{C})}$  such that at any point (z,g), the filtration on the fiber  $\mathbf{N}_{\mathrm{dR},M(\mathbb{C}),(z,g)}$  is induced by  $h_z : \mathbb{S} \to G_{\mathbb{R}}$ .

The above construction can be applied to the  $\mathbb{Z}_{(p)}$  representations  $L_{(p)}$  and  $H = C(L_{(p)})$  of G. In this way, one gets  $\mathbb{Z}_{(p)}$ -local systems and vector bundles with filtrations

$$(\mathbf{H}_B, \mathbf{H}_{\mathrm{dR}, M(\mathbb{C})})$$
 and  $(\mathbf{V}_B, \mathbf{V}_{\mathrm{dR}, M(\mathbb{C})})$ 

on the complex fiber  $M(\mathbb{C})$ . Now we will explain how these objects can be realized as cohomologies of the 'universal' object over the canonical model M of the Shimura variety obtained via the Hodge embedding. One can choose an element  $\delta \in C^+(L_{(p)})^{\times}$  such that  $\delta^* = -\delta$  and consider the symplectic pairing  $\psi_{\delta}$ on  $C(L_{(p)})$  defined as  $\psi_{\delta}(x_1, x_2) = \operatorname{Trd}(x_1 \delta x_2^*)$ , see [21, Lemma 3.6]. We have a faithful representation

$$G \to \mathrm{GSp}(C(L_{(p)}), \psi_{\delta})$$

which induces a morphism of Shimura data from  $(G, \mathcal{D})$  to the Siegel Shimura datum determined by  $(C(L_{(p)}), \psi_{\delta})$ 

$$(G, \mathcal{D}) \to (\operatorname{GSp}((C(L_{(p)}), \psi_{\delta})), S^{\pm}).$$
(3.2.1)

Let  $K' \subset \operatorname{GSp}(\mathbb{A}_f)$  be a compact open subgroup such that  $K \subset K'$ . This induces a morphism of Shimura varieties

$$M \to \operatorname{Sh}_{K'}(\operatorname{GSp}, S^{\pm}).$$

which is defined over  $\mathbb{Q}$ . Pulling back the universal object yields an abelian scheme  $A \to M$  which is called the Kuga-Satake abelian scheme. The fiber of this over a point  $[(z,g)] \in M(\mathbb{C})$  is the abelian variety up to prime to-p-isogeny  $A_{[(z,g)]}$  whose Betti homology is

$$H_1(A_{[(z,g)]}, \mathbb{Z}_{(p)}) = g \cdot C(L_{(p)}) \subset C(V)$$

with Hodge structure is given by  $h_z$ . The symplectic form  $\psi_{\delta}$  induces a polarization  $\lambda$  on  $A_{[(z,g)]}$ . The relative degree 1 Betti cohomology of  $A_{M(\mathbb{C})}$  over  $M(\mathbb{C})$  with coefficients in  $\mathbb{Z}_{(p)}$  is identified with  $\mathbf{H}_B$  as  $\mathbb{Z}_{(p)}$ -local systems over  $M(\mathbb{C})$ . Hence at a point [(z,g)], the fiber  $\mathbf{H}_{B,[(z,g)]} \simeq g \cdot C(L_{(p)})$ .

Similarly the relative degree 1 de Rham cohomology of  $A_{M(\mathbb{C})}$  over  $M(\mathbb{C})$  is identified with  $\mathbf{H}_{\mathrm{dR},M(\mathbb{C})}$ . Moreover the relative degree 1 de Rham cohomology of A over M gives a descend of  $\mathbf{H}_{\mathrm{dR},M(\mathbb{C})}$  to a vector bundle  $\mathbf{H}_{\mathrm{dR},M}$  over M equipped with a filtration.

Let  $\mathbf{H}_{\ell}$  be the relative degree 1 etale cohomology of A over M with coefficients on  $\mathbb{Q}_{\ell}$ . By the comparison of etale cohomology with singular cohomology, the restriction of  $\mathbf{H}_{\ell}$  to  $M(\mathbb{C})$  can be identified with  $\mathbf{H}_B \otimes \mathbb{Q}_{\ell}$ . Similarly, define  $\mathbf{V}_{\ell,M(\mathbb{C})} = \mathbf{V}_B \otimes \mathbb{Q}_{\ell}$  as a  $\mathbb{Q}_{\ell}$  local system over  $M(\mathbb{C})$  then there is a descend  $\mathbf{V}_{\ell}$  to a  $\mathbb{Q}_{\ell}$ -local system over M.

Let  $\mathbf{H}_p$  be the relative degree 1 cohomology of A over M with coefficients in  $\mathbb{Z}_p$ . Then the restriction  $\mathbf{H}_{p,M(\mathbb{C})}$  is identified with  $\mathbf{H}_B \otimes \mathbb{Z}_p$ . Similarly, let  $\mathbf{V}_{p,M(\mathbb{C})} = \mathbf{V}_B \otimes \mathbb{Z}_p$  a  $\mathbb{Z}_p$ -local system over  $M(\mathbb{C})$ . There is a descend  $\mathbf{V}_p$  of  $\mathbf{V}_{p,M(\mathbb{C})}$  over M.

Let  $\mathbf{H}_{\mathbb{A}_{f}^{p}}$  be the relative degree 1 et ale cohomology of A over M with coefficients in  $\mathbb{A}_{f}^{p}$ . This decends to M as a  $\mathbb{A}_{f}^{p}$ -local system  $\mathbf{H}_{B} \otimes \mathbb{A}_{f}^{p}$  over  $M(\mathbb{C})$ . Similar to before  $\mathbf{V}_{B} \otimes \mathbb{A}_{f}^{p}$  over  $M(\mathbb{C})$  has a descend to  $\mathbf{V}_{\mathbb{A}_{f}^{p}}$  over M.

The universal abelian scheme  $A \to M$  has a polarization  $\lambda$  and a  $K'^p$ -level structure  $\eta^p : H_{\mathbb{A}_f^p} \xrightarrow{\sim} H \otimes \mathbb{A}_f^p$  induced by the universal object  $(A, \lambda, \eta^p)$  over the integral model  $\mathscr{S}_{K'}(\mathrm{GSp}, S^{\pm})$  of the Siegel Shimura variety. Then [21, Proposition 3.14] is showing that there is a canonical such  $K^p$ -level structure  $\eta^p : H_{\mathbb{A}_f^p} \xrightarrow{\sim} H \otimes \mathbb{A}_f^p$  on A that maps  $\mathbf{V}_{\mathbb{A}_f^p}$  onto  $L_{(p)} \otimes \mathbb{A}_f^p$ .

The G-equivariant action of V on H by left multiplication induces an embedding of homological realizations

#### $\mathbf{V}_{\bullet} \subset \underline{\mathrm{End}}(\mathbf{H}_{\bullet})$

where  $\bullet = B, \ell, d\mathbf{R}$ .

In the previous section we described the integral models of Shimura varieties of Hodge type. By the Hodge embedding (3.2.1), M is of Hodge type and so it has a smooth integral model  $\mathcal{M}_{(p)}$  over  $\mathbb{Z}_{(p)}$ , for details see [10, 21]. Recall that by construction of  $\mathcal{M}_{(p)}$ , the Kuga-Satake abelian scheme  $(A, \lambda, \eta^p)$  over  $\mathscr{S}_{K'}(\mathrm{GSp}, S^{\pm})$  induces an abelian scheme  $(A, \lambda, \eta^p)$  over  $\mathcal{M}_{(p)}$ . This gives extensions of the sheaves  $\mathbf{H}_{\ell}, \mathbf{H}_{\mathbb{A}_f^p}, \mathbf{V}_{\ell}, \mathbf{V}_{\mathbb{A}_f^p}$  extends over the integral model  $\mathcal{M}_{(p)}$ . By [21, Prop 3.7],  $\mathbf{H}_{\mathrm{dR}}$  and  $\mathbf{V}_{\mathrm{dR}}$  also extend to filtered vector bundles over  $\mathcal{M}_{(p)}$ .

By [1, Prop 4.2.5], there is a canonical functor  $N \mapsto N_{\text{cris}}$  from the algebraic  $\mathbb{Z}_{(p)}$ -representations of  $G_{(p)}$  to F-crystals over  $\mathcal{M}_{(p),\mathbb{F}_p}$  which recovers  $\mathbf{H}_{\text{cris}}$ , the first relative crystalline cohomology of  $A_{\mathbb{F}_p}$  over  $\mathcal{M}_{(p),\mathbb{F}_p}$ , when applied to  $H_{(p)}$ . Let  $x \in \mathcal{M}_{(p)}(k)$  be a point in characteristic p. Then there is a natural isomorphism

$$\mathbf{H}_{\mathrm{cris},x} \xrightarrow{\sim} H \otimes W$$

and an isometry  $\mathbf{V}_{\mathrm{cris},x} \xrightarrow{\sim} L_{(p)} \otimes W$ . And we get a canonical embedding  $\mathbf{V}_{\mathrm{cris}} \subset$ 

 $End(\mathbf{H}_{cris})$ . We have described the embeddings

$$\mathbf{V}_{\bullet} \subset \underline{\mathrm{End}}(\mathbf{H}_{\bullet}) \tag{3.2.2}$$

where  $\bullet = B, \ell, dR, cris$ . Using these embeddings one can define the notion of a special endomorphism.

**Definition 3.1.** For any *M*-scheme *S*, an endomorphism  $f \in \text{End}(A_S)_{(p)}$  is called *l*-special if its *l*-adic realization  $f_l$  lies in the image of the embedding  $\mathbf{V}_l \rightarrow$  $\text{End}(\mathbf{H}_l)$ , and it is called *p*-special if it induces an endomorphism in the image of  $\mathbf{V}_p \rightarrow \text{End}(\mathbf{H}_p)$ .

The definition of  $\ell$ -specialness works for any  $\mathcal{M}_{(p)}$ -scheme S.

**Definition 3.2.** For a characteristic p point in the special fiber x: Spec  $k \to \mathcal{M}_{(p),\mathbb{F}_p}$ , an endomorphism  $f \in \operatorname{End}(A_x)_{(p)}$  is called p-special if the crystalline realization  $f_{\operatorname{cris}}$  lies in the image of  $\mathbf{V}_{\operatorname{cris},x} \to \operatorname{End}(\mathbf{H}_{\operatorname{cris},x})$ . For a scheme  $S \to \mathcal{M}_{(p)}$  on which p is locally nilpotent, an endomorphism  $f \in \operatorname{End}(A_S)_{(p)}$  is called p-special if at every k-valued point  $x \to S$ , the endomorphism  $f_x \in \operatorname{End}(A_x)_{(p)}$  is p-special. For a general scheme  $S \to \mathcal{M}_{(p)}$ , an endomorphism  $f \in \operatorname{End}(A_S)_{(p)}$  is p-special if the restrictions to  $S \otimes \mathbb{Q}$  and  $S \otimes \mathbb{F}_p$  are p-special.

In fact an endomorphism  $f \in \text{End}(A_S)_{(p)}$  is  $\ell$ -special (resp. *p*-special) if in every connected component of *S* there is a geometric point *x* in *S* such that  $f_x$  is  $\ell$ -special (resp. *p*-special). For all these see [21].

**Definition 3.3.** For a given  $S \to \mathcal{M}_{(p)}$ , an endomorphism  $f \in \operatorname{End}(A_S)_{(p)}$  is called special if it is  $\ell$  special for every prime  $\ell$ .

In fact [21, Corollary 5.21] shows that for  $S \to \mathcal{M}_{(p)}$ , an endomorphism  $f \in \operatorname{End}(A_S)_{(p)}$  is special if it is *p*-special. In particular for a geometric point  $x \in \mathcal{M}_{(p)}(k)$  valued in a perfect field of characteristic *p*, an endomorphism  $f \in \operatorname{End}(A_s)_{(p)}$  is special if and only if  $f_{\text{cris}}$  lies in  $\mathbf{V}_{\text{cris}}$ .

Using the comparisons between different cohomological relatizations, one can see that if an endomorphism  $f \in \text{End}(A_S)_{(p)}$  is special, then the Betti realization over  $S_{\mathbb{C}}$  lies in the image of  $\mathbf{V}_B$  under the embedding (3.2.2) and the de Rham realization lies in the image of  $\mathbf{V}_{dR}$ .

Write  $V(A_S) \subset \operatorname{End}(A_S)_{(p)}$  for the space of special endomorphisms. If s is a geometric point valued in a field of characteristic p, then  $x \in \operatorname{End}_{C(L)}(A_s)$  is special if and only if the crystalline realization  $x_{\operatorname{cris}}$  lies in  $\mathbf{V}_{\operatorname{cris},s}$ .

Let  $S \to \mathcal{M}_{(p)}$ . The polarization  $\lambda$  on the Kuga-Satake abelian scheme Ainduces the Rosati involution on  $V(A_S)$  and  $V(A_S)$  is point-wise fixed by this involution. For each  $x \in V(A_S)$  we have

$$x \circ x = Q(x) \cdot \mathrm{id}_{A_S}$$

for some  $Q(x) \in \mathbb{Z}_{(p)}$ . The map  $x \mapsto Q(x)$  is a positive definite  $\mathbb{Z}_{(p)}$ -quadratic form on  $V(A_S)$ , see [21, Lemma 5.12].

#### 3.3 Special Cycles

Using the special cycles defined in the previous section, we will define the special cycles on the Shimura variety.

Recall we have the fixed  $\mathbb{Z}$ -lattice in  $L_{(p)}$ . This gives a K-stable compact open subset  $\widehat{L}^{(p)} = L \otimes \widehat{\mathbb{Z}}^{(p)} \subset V(\mathbb{A}_f^p)$  where  $V = L_{\mathbb{Q}}$ . Recall also that for any  $\mathcal{M}_{(p)}$ -scheme S, the fiber  $A_S$  comes equipped with a  $K^p$ -level structure

$$\eta^p: \mathbf{H}_{\mathbb{A}_f^p} \xrightarrow{\sim} H \otimes \mathbb{A}_f^p$$

which is mapping  $\mathbf{V}_{\mathbb{A}_{f}^{p}}$  onto  $V \otimes \mathbb{A}_{f}^{p}$ . Given a special endomorphism  $f \in V(A_{S}) \subset$ End $(A_{S})_{(p)}$ , then by definition of specialness, the induced endomorphism of  $\mathbf{H}_{\mathbb{A}_{f}^{p}}$ via f lies in  $\mathbf{V}_{\mathbb{A}_{f}}$  and so via  $\eta^{p}$ , the endomorphism  $\eta^{p} \circ f \circ (\eta^{p})^{-1}$  of  $H \otimes \mathbb{A}_{f}^{p}$ lies in  $V \otimes \mathbb{A}_{f}^{p}$ . Hence it makes sense as an integrality condition to ask for this endomorphism to lie in  $\widehat{L}^{(p)}$ . **Definition 3.4.** Given  $T \in \text{Sym}_k(\mathbb{Q})$  with  $\det(T) \neq 0$ , the special cycle  $\mathcal{Z}(T) \rightarrow \mathcal{M}_{(p)}$  is defined as the stack over  $\mathcal{M}_{(p)}$  with functor of points

$$\mathcal{Z}(T)(S) = \{ (x_1, \dots, x_k) \in V(A_S)^k : Q(x) = T, \quad \eta^p \circ x_i \circ (\eta^p)^{-1} \in L \otimes \widehat{\mathbb{Z}}^{(p)} \}$$

for an  $\mathcal{M}_{(p)}$ -scheme S.

Remark 3.5. The definition of the special cycle in [19, 17, 18] depends on a choice of a compact open K-stable subset  $\omega \in V(\mathbb{A}_f^p)$ . Here we are making a similar choice  $\widehat{L}^{(p)} = L \otimes \widehat{\mathbb{Z}}^{(p)} \subset V(\mathbb{A}_f^p)$ . In [1, Chapter 4], the definition of the special cycle depends on a choice of  $\mu \in L^{\vee}/L$ . There is a subspace  $V_{\mu}(A_S) \subset V(A_S)$ defined by integrality conditions on  $\ell$ -adic realizations. Definition 3.4 corresponds to the choice  $\mu = 0$ .

The natural map  $\mathcal{Z}(T) \to \mathcal{M}_{(p)}$  is finite and unramified [1]. Let  $Z(T) = \mathcal{Z}(T) \times_{\text{Spec } \mathbb{Z}_{(p)}} \text{Spec } \mathbb{Q}$  be the generic fiber. Then we have

**Proposition 3.6.** Z(T) is empty unless  $k \leq n-2$  and T is positive definite in  $\operatorname{Sym}_k(\mathbb{Z}_{(p)})$ .

*Proof.* Suppose  $Z(T) \neq \emptyset$  and let  $\xi \in Z(T)(\mathbb{C})$ . Let [(z,g)] be the image of  $\xi$ under  $Z(T)(\mathbb{C}) \to M(\mathbb{C})$ . Then by construction we have an isomorphism

$$\mu: H_1(A_{[(z,g)]}, \mathbb{Q}) \xrightarrow{\sim} C(V)$$

with the complex structure given by  $j_z$ . The Betti realizations of  $x_i$  lie in  $gL_{(p)}$ via the identifications  $\mathbf{H}_{B,\xi} \xrightarrow{\sim} gC(L_{(p)})$  and  $\mathbf{V}_{B,\xi} \xrightarrow{\sim} gL_{(p)}$ , hence  $Q(x) = T \in$  $\operatorname{Sym}_k(\mathbb{Z}_{(p)})$ . Since each  $x_i$  commutes with  $j_z$ , by definition of the action of V on H, we have  $x_i \in z^{\perp}$ . Now  $z^{\perp}$  is a positive (n-2)-hyperplane in  $V_{\mathbb{R}}$ , so  $k \leq n-2$ since otherwise we would have  $\det(T) = 0$ . Finally since the quadratic form on  $V(A_{\xi})$  is positive definite, T is positive definite.

If  $k \ge n-1$ , then the generic fiber Z(T) is empty and so in this case the image of  $\mathcal{Z}(T)$  lies over the special fiber of  $\mathcal{M}_{(p)}$ . Fix  $d_1, \ldots, d_{n-1} \in \mathbb{Z}_{(p)}$  which are positive. And let

$$\mathcal{Z} = \mathcal{Z}(d_1) \times_{\mathcal{M}_{(p)}} \ldots \times_{\mathcal{M}_{(p)}} \mathcal{Z}(d_{n-1})$$

be the fiber product of corresponding special cycles. For a point  $\xi \in \mathbb{Z}$ , we have a matrix

$$T_{\xi} = Q(j_1, \dots, j_{n-1}) \in \operatorname{Sym}_{n-1}(\mathbb{Z}_{(p)})$$

where  $(j_1, \ldots, j_{n-1})$  is the (n-1)-tuple of special endomorphisms attached to  $\xi$ . The function  $\xi \mapsto T_{\xi}$  is locally constant so we can define  $\mathcal{Z}_T$  to be the union of connected components of  $\mathcal{Z}$  where the matrix  $T_{\xi} = T$ . Then

$$\mathcal{Z} = \prod_{T} \mathcal{Z}_{T} = \prod_{\text{diag}(T)=(d_{1},\dots,d_{n-1})} \mathcal{Z}(T)$$

Let  $\xi \in \mathcal{Z} = \mathcal{Z}(d_1) \times_{\mathcal{M}_{(p)}} \ldots \times_{\mathcal{M}_{(p)}} \mathcal{Z}(d_{n-1})$  where  $d_i \in \mathbb{Z}_{(p)}$  and positive. Then by Proposition 3.6,  $\xi$  lies in the special fiber of  $\mathcal{Z}$ . In fact next proposition shows that it lies in the supersingular locus.

**Proposition 3.7.**  $\mathcal{Z}(T)$  lies in the supersingular locus.

Proof. Let  $\xi \in \mathcal{Z}(T)(\overline{\mathbb{F}}_p)$  and  $x_1, \ldots, x_{n-1} \in V(A_{\xi})$  be special endomorphisms determined by  $\xi$ . Assume that  $A_{\xi}$  is not supersingular. Let  $W = W(\overline{\mathbb{F}}_p)$  and  $K = \operatorname{Frac}(W)$ . Associated to  $\xi$ , there is a  $b_{\xi} \in G(K)$  such that the Frobenius on the Dieudonne module of  $\xi$  which is isomorphic to  $D \otimes W$  is given by  $F = b_{\xi} \circ \sigma$ , where  $\sigma$  is the Frobenius on K. Here  $D = \operatorname{Hom}(C(\mathbb{Z}_{(p)}), \mathbb{Z}_{(p)})$ . The element  $b_{\xi}$ also induces an isocrystal structure  $(V_K, \Phi = b_{\xi} \circ \sigma)$ . By [8, Lemma 4.2.4], since  $\xi$ is not supersingular,  $V_K$  is not isoclinic of slope 0. Let  $r \neq 0$  be a slope of  $V_K$  and write r = m/n. Then the slope r isotypic component of  $V_K$  is

$$V_K^r = V_K^{p^{-m}\Phi^n} \otimes_E K$$

where  $E = K^{\sigma^n} = \mathbb{Q}_{p^n}$  and  $V_K^{p^{-m}\Phi^n} = \{x \in V_K : \Phi^n x = p^m x\}$ . Let  $x, y \in V_K^{p^{-m}\Phi^n}$ .

Then under the induced quadratic form on  $V_K$ , we have  $(x, y)^{\sigma^n} = (x, y)$  and so

$$(x,y)=(x,y)^{\sigma^n}=(\Phi^n x,\Phi^n y)=p^{2m}(x,y)$$

which shows that (x, y) = 0. Now let  $x \in V_K^{p^{-m}\Phi^n}$ . Then similarly (x, z) = 0for any z in the slope 0 isotypical component. Now since  $V_K$  is a nondegenerate quadratic space, there exists  $y \in V_K^{p^{-u}\Phi^v}$  with  $(x, y) \neq 0$  and  $s = u/v \neq 0$ . Then similar computation as above shows that s = -r which shows that the slopes of  $V_K$  come in pairs r, -r, i.e. if r is a slope then -r is also a slope. Hence the dimension of the slope 0 component of  $V_K$ , which is equal to  $V_K^{\Phi} \otimes_{\mathbb{Q}_p} K$ , drop by at least two and so  $\dim_{\mathbb{Q}_p} V_K^{\Phi} \leq n-2$ . The crystalline realizations  $x_{i,cris}$  of  $x_i$ , lie in  $\mathbf{V}_{cris,\xi} \xrightarrow{\sim} L_{(p)} \otimes W$  and in particular lie in  $\mathbf{V}_{cris,\xi} \otimes K \xrightarrow{\sim} V_K$ . Since  $x_{i,cris}$  commute with the Frobenius F on the Dieudonne module, they are fixed by  $\Phi$ . Hence  $x_{i,cris} \in V_K^{\Phi}$ . This is a contradiction since there are n-1 special endomorphisms on  $A_{\xi}$ .

# 4 Special Cycles on the Rapoport-Zink Space

#### 4.1 Rapoport-Zink Space

In this section we will describe the structure of the Rapoport-Zink spaces of Hodge type and in particular we will concentrate on the case of GSpin groups following [8]. A construction of these Rapoport-Zink spaces using different techniques is also given by Kim [9].

Let  $k = \overline{\mathbb{F}}_p$ . We will use the same notation as in section 3.2, so let  $L_{(p)}$  be a  $\mathbb{Z}_{(p)}$ quadratic space and  $C(L_{(p)})$  be its Clifford algebra. Also let  $L_{\mathbb{Z}_p} = L_{(p)} \otimes \mathbb{Z}_p$ . Recall that we have a smooth integral model  $\mathcal{M} = \mathcal{M}_{(p)}$  over  $\mathbb{Z}_{(p)}$  for the GSpin Shimura variety M. Associated to a point  $x_0 \in \mathcal{M}_{(p)}(k)$  and let  $X_0$  be the p-divisible group of the corresponding abelian variety. Then there is a local unramified Shimura-Hodge datum  $(G, b, \mu, C) = (G_{\mathbb{Z}_p}, b_{x_0}, \mu_{x_0}, C(L_{\mathbb{Z}_p}))$ , where

- $\mu_{x_0} : \mathbb{G}_{mW} \to G_W$  is a cocharacter, up to G(W)-conjugacy, such that the Hodge filtration  $\operatorname{Fil}^1(X_0) \subset \mathbb{D}(X_0)(k) \simeq H^1_{\mathrm{dR}}(A_{x_0})$  is induced by  $\mu_{x_0} : \mathbb{G}_{mk} \to G_W \otimes_W k$ ,
- $b_{x_0} \in G(K)$  up to  $G(W) \sigma$ -conjugation, such that  $b_{x_0} \in G(W)\mu_{x_0}^{\sigma}(p)G(W)$ and the Frobenius of the contravariant Dieudonne module  $\mathbb{D}(X_0)(W)$  is of the form  $F = b_{x_0} \circ (\mathrm{id} \otimes \sigma)$  after choosing an isomorphism

$$\beta_0: D \otimes_{\mathbb{Z}_p} W \xrightarrow{\sim} \mathbb{D}(X_0)(W)$$

where  $D = \text{Hom}(C, \mathbb{Z}_p)$  with contragradient action of G.

Define the algebraic group  $J_b$  over  $\mathbb{Q}_p$  as

$$J_b(R) = \{g \in G(R \otimes_{\mathbb{Q}_p} K) : gb\sigma(g)^{-1} = b\}$$

for any  $\mathbb{Q}_p$ -algebra R.

Associated to the datum  $(G, b, \mu, C)$  there exists a formal scheme  $RZ_G$  over Spf W formally smooth, locally of finite type satisfying the following

- It is a formal closed subscheme of the usual Rapoport-Zink formal scheme  $RZ(X_0)$  over Spf W from section 2.3. Recall that  $RZ(X_0)$  represents the functor associating to any  $S \in Nilp_W$  the isomorphism classes of pairs  $(X, \rho)$  where X is a p-divisible group over S and  $\rho : X_0 \times_k S \dashrightarrow X \times_S \overline{S}$  is a quasi-isogeny.
- There is a bijection

$$\operatorname{RZ}_G(k) \xrightarrow{\sim} X_{G,b,\mu^{\sigma}}(k)$$

where  $X_{G,b,\mu^{\sigma}}(k)$  is the affine Deligne-Lusztig set

$$\{g \in G(K) : g^{-1}b\sigma(g) \in G(W)\mu^{\sigma}(p)G(W)\}/G(W)$$

• If  $x_0$  is supersingular, or equivalently that b is basic, then there is an isomorphism of formal schemes

$$\Theta: I(\mathbb{Q}) \setminus \mathrm{RZ}_G \times G(\mathbb{A}_f^p) / K^p \xrightarrow{\sim} (\widehat{\mathcal{M}_W}) / \mathcal{M}_b$$
(4.1.1)

where  $(\widehat{\mathcal{M}}_W)/\mathcal{M}_b$  is the completion of  $\mathcal{M}_W$  along the basic locus of the special fiber and I is a reductive group over  $\mathbb{Q}$  with

$$I(\mathbb{Q}_{\ell}) = \begin{cases} J_b(\mathbb{Q}_p) & \text{if } \ell = p \\ \\ G(\mathbb{Q}_{\ell}) & \text{if } \ell \neq p \end{cases}$$

The description of the moduli interpretation of  $\operatorname{RZ}_G(k)$  is given as follows: There is a finite list of tensors  $s_{\alpha}$  in  $C^{\otimes}$  such that

$$G(R) = \{ g \in \operatorname{GL}(C \otimes_{\mathbb{Z}_p} R) : g \cdot (s_\alpha \otimes 1) = (s_\alpha \otimes 1) \ \forall \alpha \}$$

for any  $\mathbb{Z}_p$ -algebra R. We have a canonical isomorphism  $C^{\otimes} = D^{\otimes}$ . Hence we obtain tensors  $s_{\alpha} \otimes 1$  in  $D^{\otimes} \otimes_{\mathbb{Z}_p} W$  which induces, via the above identification  $\beta_0$ , tensors

$$t_{\alpha,0} = s_\alpha \otimes 1 \in D_W^{\otimes} = \mathbb{D}(X_0)(W)^{\otimes}$$

Now  $\operatorname{RZ}_G(k)$  is given by triples  $(X, \rho, (t_\alpha))$  where

- X is a p-divisible group over k
- $(t_{\alpha}) \subset \mathbb{D}(X)(W)^{\otimes}$  is a collection of tensors Frobenius invariant in  $\mathbb{D}(X)(W)^{\otimes}[1/p]$
- $\rho: X_0 \dashrightarrow X$  is a quasi-isogeny identifying  $t_\alpha$  with  $t_{\alpha,0}$

satisfying some extra properties [8, Definition 2.3.3]. The group  $J_b(\mathbb{Q}_p)$  acts on  $\mathrm{RZ}_G(k)$  on the left by

$$g \cdot (X, \rho, (t_{\alpha})) = (X, \rho \circ g^{-1}, (t_{\alpha}))$$
(4.1.2)

For a given local unramified Shimura-Hodge datum  $(G, b, \mu, C)$ , there is a unique, up to isomorphism, *p*-divisible group  $X_0 = X_0(G, b, \mu, C)$  over k with contravariant Dieudonne module  $\mathbb{D}(X_0)(W) = D_W$  and Frobenius  $F = b \circ \sigma$  where the Hodge filtration  $VD_k \subset D_k = \mathbb{D}(X_0)(k)$  is induced by a conjugate of the reduction  $\mu_k : \mathbb{G}_{mk} \to G_k$  of  $\mu$  mod p.

Let  $x_0 \in \mathcal{M}_{(p)}(k)$  be supersingular and let  $(G_{\mathbb{Z}_p}, b, \mu, C_{\mathbb{Z}_p})$  be the corresponding unramified local Shimura-Hodge datum. Let  $\mathrm{RZ} = \mathrm{RZ}_G$  be the associated formal scheme over W described as above. In [8], Howard-Pappas give an explicit description of the underlying reduced locally finite type k-scheme  $\mathrm{RZ}^{\mathrm{red}}$ . Now we will summarize this description.

We will work locally so by abusing the notation, we denote the self-dual  $\mathbb{Z}_p$ quadratic space  $L_{\mathbb{Z}_p}$  by V and  $G_{\mathbb{Z}_p} = \operatorname{GSpin}(V)$  by G. Consider the quadratic space  $V_K$  over K = W[1/p] with the natural action  $G_K \to \operatorname{SO}(V_K)$ . The operator  $\Phi = b \circ \sigma$  turns  $V_K$  into an isocrystal of slope 0, [8, Lemma 4.2.4] and the subspace of  $\Phi$ -invariant vectors  $V_K^{\Phi}$  is a  $\mathbb{Q}_p$ -quadratic space of the same dimension and determinant as  $V_{\mathbb{Q}_p}$  but with Hasse invariant  $\epsilon(V_K^{\Phi}) = -\epsilon(V_{\mathbb{Q}_p})$ . Recall  $D = \operatorname{Hom}_{\mathbb{Z}_p}(C(V), \mathbb{Z}_p)$  is the contragradient representation. Then there is an isomorphism

$$C(V)^{op} \otimes_{Z(V)} C(V) \xrightarrow{\sim} \operatorname{End}_{Z(V)}(D)$$

defined by  $((c_1 \otimes c_2)d)(x) = d(c_1xc_2)$  where  $Z(V) \subset C(V)$  is the center. The inclusion  $V \subset C(V)^{op}$  gives an embedding

$$V \subset \operatorname{End}_{\mathbb{Z}_p}(D)$$

and the action of  $v \in V$  on D is given by (vd)(x) = d(vx). The relation with the *G*-action on D is given by  $g \circ v \circ g^{-1} = g \bullet v$  for any  $g \in G(\mathbb{Z}_p)$ , where  $g \bullet v$  is the action of G on V via conjugation. The above embedding induces  $V_K \subset \operatorname{End}_K(D_K)$  and so

$$V_K^{\Phi} \subset \operatorname{End}_F(D_K)$$

where  $F = b \circ \sigma$ . Since  $D_K$  is the isocrystal for  $X_0$ , we have  $V_K^{\Phi} \subset \operatorname{End}(X_0)_{\mathbb{Q}}$  where  $X_0$  is the p-divisible group associated to  $x_0$ . The  $\mathbb{Q}_p$ -subspace  $V_K^{\Phi}$  is the space of special quasi-endomorphisms of  $X_0$ . The relation with the definition of special endomorphisms given in the previous section is as follows: Recall the integral model  $\mathcal{M}_{(p)}$  of the Shimura variety and consider a k-point  $s \in \mathcal{M}_{(p)}(k) = \mathcal{M}_{\mathbb{F}_p}(k)$ . Then by the construction in section 2, we have

$$H^1_{\operatorname{cris}}(A_s/W) = \mathbf{H}_{\operatorname{cris},s}$$

which is an *F*-crystal over *W* under  $F_s : \sigma^* \mathbf{H}_{\mathrm{cris},s} \to \mathbf{H}_{\mathrm{cris},s}$  induced by the Frobenius on  $A_s$ . This further induces via conjugation an *F*-isocrystal  $(\mathbf{H}_{\mathrm{cris},s}[1/p]^{\otimes(1,1)}, \Phi_s)$ . By [21, Proposition 4.7],  $H \otimes_{\mathbb{Z}_{(p)}} W \simeq \mathbf{H}_{\mathrm{cris},s}$  and  $\mathbf{V}_{\mathrm{cris},s} \subset \mathbf{H}_{\mathrm{cris},s}^{\otimes(1,1)}$ , as a self dual quadratic space over *W*, is isometric to  $V \otimes_{\mathbb{Z}_{(p)}} W$ . And so  $\mathbf{V}_{\mathrm{cris},s} \otimes_W K \simeq V_K \hookrightarrow$ End<sub>*K*</sub> $(\mathbf{H}_{\mathrm{cris},s}[1/p])$ .

The Rapoport-Zink formal scheme  $RZ = RZ_G$  is a closed formal subscheme

of  $\operatorname{RZ}(X_0, \lambda_0)$  where  $\lambda_0$  is a polarization on  $X_0$  induced by the perfect symplectic form  $\psi_{\delta}$  on C(V), see section 3.2. Hence by restricting the universal object via  $\operatorname{RZ} \hookrightarrow \operatorname{RZ}(X_0, \lambda_0)$  we get a pair  $(X, \rho)$ , where X is a p-divisible group over  $\operatorname{RZ}$ and

$$\rho: X_0 \times_{\mathrm{Spf}\,(k)} \overline{\mathrm{RZ}} \dashrightarrow X \times_{\mathrm{RZ}} \overline{\mathrm{RZ}}$$

is a quasi-isogeny. It respects the polarizations  $\lambda$  and  $\lambda_0$  up to scaling so we have

$$\rho^* \lambda = \rho^{\vee} \circ \lambda \circ \rho = c(\rho)^{-1} \lambda_0$$

for some  $c(\rho) \in \mathbb{Q}_p^{\times}$ . For each  $\ell \in \mathbb{Z}$  define  $\mathrm{RZ}^{(\ell)}$  to be the open and closed formal subscheme on which  $\mathrm{ord}_p(c(\rho)) = \ell$ . Then we have

$$RZ = \bigsqcup_{\ell \in \mathbb{Z}} RZ^{(\ell)}$$

The  $\mathbb{Q}_p$  points of the algebraic group  $J_b = \operatorname{GSpin}(V_K^{\Phi})$  is

$$J_b(\mathbb{Q}_p) = \{g \in G(K) : gb = b\sigma(g)\}$$

and so via  $G(K) \to \operatorname{GL}(D_K)$ , we have  $J_b(\mathbb{Q}_p) \subset \operatorname{End}(X_0)^{\times}_{\mathbb{Q}}$ . This gives an action on RZ defined as in 4.1.2. For every  $g \in J_b(\mathbb{Q}_p)$  we get an isomorphism

$$g : \mathrm{RZ}^{(\ell)} \to \mathrm{RZ}^{(\ell + \mathrm{ord}_p \eta_b(g))}$$

Consider the action of the subgroup  $p^{\mathbb{Z}} \subset J_b(\mathbb{Q}_p)$ . As  $\eta_b(p) = p^2$ , we have

$$p^{\mathbb{Z}} \setminus \mathrm{RZ} \xrightarrow{\sim} \mathrm{RZ}^{(0)} \sqcup \mathrm{RZ}^{(1)}$$

Since the spinor similitude  $\eta_b : J_b(\mathbb{Q}_p) \to \mathbb{Q}_p^{\times}$  is surjective, all  $\mathrm{RZ}^{(\ell)}$  are (noncanonically) isomorphic. Now we will describe the linear algebraic data which will be used to give a description of RZ. For details see [8, Section 5].

A vertex lattice is a  $\mathbb{Z}_p$ -lattice  $\Lambda \subset V_K^{\Phi}$  such that  $p\Lambda \subset \Lambda^{\vee} \subset \Lambda$ . The quadratic form pQ on  $V_K^{\Phi}$  induces a quadratic form on the  $\mathbb{F}_p$ -vector space  $\Omega_0 = \Lambda/\Lambda^{\vee}$ . The type of  $\Lambda$  is defined as  $t_{\Lambda} = \dim \Omega_0$ . It is even and  $2 \leq t_{\Lambda} \leq t_{\max}$  where

$$t_{\max} = \begin{cases} n-2 & \text{if } n \text{ is even and } \det(V_{\mathbb{Q}_p}) = (-1)^{n/2} \\ n-1 & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is even and } \det(V_{\mathbb{Q}_p}) \neq (-1)^{n/2} \end{cases}$$

A special lattice  $L \subset V_K$  is defined to be a self-dual W-lattice such that

$$(L + \Phi(L))/L \xrightarrow{\sim} W/pW$$

For every special lattice  $L \subset V_K$ , there is a minimal vertex lattice  $\Lambda = \Lambda(L) \subset V_K^{\Phi}$ with

$$\Lambda_W^{\vee} \subset L \subset \Lambda_W$$

and  $\Lambda^{\vee} = L^{\Phi} = \{x \in L : \Phi(x) = x\}$ , see [8, Proposition 5.2.2].

Consider the smooth projective k-variety  $S_{\Lambda}$ 

$$S_{\Lambda}(k) = \{ \text{Lagrangians } \mathcal{L} \subset \Omega_0 \otimes k : \dim(\mathcal{L} + \Phi(\mathcal{L})) = t_{\Lambda}/2 + 1 \}$$

where  $\Phi = \mathrm{id} \otimes \sigma$ . The k-variety  $S_{\Lambda} = S_{\Lambda}^+ \sqcup S_{\Lambda}^-$  has two connected components that are non-canonically isomorphic, and smooth of dimension  $(t_{\Lambda}/2) - 1$  and  $S_{\Lambda}(k)$ parametrizes certain special lattices:

$$S_{\Lambda}(k) \xrightarrow{\sim} \{ \text{special lattices } L \subset V_K : \Lambda_W^{\vee} \subset L \subset \Lambda_W \}$$

The reduced k-scheme  $\mathrm{RZ}^{\mathrm{red}}$  underlying the formal W-scheme can be expressed as a union of closed subschemes  $\mathrm{RZ}^{\mathrm{red}}_{\Lambda}$  indexed by vertex lattices and each  $\mathrm{RZ}^{\mathrm{red}}_{\Lambda}$  is related to  $S_{\Lambda}$ . For a vertex lattice  $\Lambda \subset V_K^{\Phi}$ , let  $\mathrm{RZ}_{\Lambda} \subset \mathrm{RZ}$  be the closed formal subscheme defined by the condition

$$\rho \circ \Lambda^{\vee} \circ \rho^{-1} \subset \operatorname{End}(X)$$

The closedness follows from 2.11. Set  $RZ_{\Lambda}^{(\ell)} = RZ^{(\ell)} \cap RZ_{\Lambda}$ . Then we have

$$p^{\mathbb{Z}} \setminus \mathrm{RZ}_{\Lambda} \simeq \mathrm{RZ}_{\Lambda}^{(0)} \sqcup \mathrm{RZ}_{\Lambda}^{(1)}$$

By [8, Prop 6.1.2], the reduced k-scheme underlying  $\mathrm{RZ}_{\Lambda}^{(\ell)}$  is projective. Now we explain the relation between  $\mathrm{RZ}_{\Lambda}$  and the special lattices. Given  $y \in \mathrm{RZ}(k)$ , via the universal quasi-isogeny, we obtain a quasi-isogeny  $\rho_y : X_0 \to X_y$  which induces an isomorphism of isocrystals

$$\mathbb{D}(X_y)(W)[1/p] \xrightarrow{\sim} \mathbb{D}(X_0)(W)[1/p] = D_K$$

and in this way, the Dieudonne module  $M_y = \mathbb{D}(X_y)(W)$  of  $X_y$  can be viewed as a W-lattice in  $D_K$ . The Hodge filtration  $\operatorname{Fil}_{X_y}^1 \subset \mathbb{D}(X_y)(k) = M_y/pM_y$  induces a submodule  $M_{1,y} \subset M_y$ . So associated to  $y \in \operatorname{RZ}(k)$  we have W-lattices  $M_{1,y} \subset$  $M_y \subset D_K$ . Recall the inclusion  $V_K \subset \operatorname{End}_K(D_K)$ . Define the following W-lattices in  $V_K$ 

$$L_y = \{x \in V_K : xM_{1,y} \subset M_{1,y}\}$$
$$L_y^{\sharp} = \{x \in V_K : xM_y \subset M_y\}$$
$$L_y^{\sharp\sharp} = \{x \in V_K : xM_{1,y} \subset M_y\}$$

In [8], Howard-Pappas prove that for every  $y \in RZ(k)$ , the lattice  $L_y$  is special and satisfies

$$\Phi(L_y) = L_y^{\sharp}$$
 and  $L_y + L_y^{\sharp} = L_y^{\sharp\sharp}$ 

and  $y \mapsto L_y$  induces the following bijections

$$p^{\mathbb{Z}} \setminus \mathrm{RZ}(k) \xrightarrow{\sim} \{ \text{special lattices } L \subset V_K \}$$
  
 $p^{\mathbb{Z}} \setminus \mathrm{RZ}_{\Lambda}(k) \xrightarrow{\sim} \{ \text{special lattices } L \subset V_K : \Lambda_W^{\vee} \subset L \subset \Lambda_W \}$ 

As a corollary of these bijections, there is a decomposition of RZ(k),

$$\mathrm{RZ}(k) = \bigcup_{\Lambda} \mathrm{RZ}_{\Lambda}(k)$$

where the union is over vertex lattices with  $t_{\Lambda} = t_{\text{max}}$ . We also have for any two vertex lattices  $\Lambda_1$  and  $\Lambda_2$ 

$$\mathrm{RZ}_{\Lambda_1}(k) \cap \mathrm{RZ}_{\Lambda_2}(k) = \begin{cases} \mathrm{RZ}_{\Lambda_1 \cap \Lambda_2}(k) & \text{if } \Lambda_1 \cap \Lambda_2 \text{ is a vertex lattice} \\ \emptyset & \text{otherwise} \end{cases}$$
(4.1.3)

By the above discussion, it follows that

$$p^{\mathbb{Z}} \setminus \mathrm{RZ}_{\Lambda}(k) \xrightarrow{\sim} \{ \text{special lattices } L \subset V_K : \Lambda_W^{\vee} \subset L \subset \Lambda_W \} \xrightarrow{\sim} S_{\Lambda}(k)$$

The relation is stronger than this, there is a unique isomorphism of k-schemes

$$p^{\mathbb{Z}} \setminus \mathrm{RZ}^{\mathrm{red}}_{\Lambda} \xrightarrow{\sim} S_{\Lambda}$$

inducing the above bijection on k-points [8, Theorem 6.3.1]. As a corollary of this, the reduced scheme  $\text{RZ}^{(\ell),\text{red}}_{\Lambda}$  is connected and nonempty, and is isomorphic to  $S^{\pm}_{\Lambda}$ , [8, Corollary 6.3.2]. The next theorem gives the explicit description of  $\text{RZ}^{\text{red}}$ .

**Theorem 4.1.** [8, Theorem 6.4.1] The k-schemes  $\mathrm{RZ}^{(\ell),red}$  are connected and the subschemes  $\mathrm{RZ}^{(\ell),red}_{\Lambda}$  are projective and smooth of dimension  $(t_{\Lambda}/2)-1$ . The closed subschemes  $\mathrm{RZ}^{(\ell),red}_{\Lambda}$  as  $\Lambda$  runs over the vertex lattices of type  $t_{max}$ , are the irre-

ducible components of  $\mathrm{RZ}^{(\ell),\mathrm{red}}.$  In particular, we have

$$\dim(\mathrm{RZ}^{red}) = \frac{1}{2} \begin{cases} n-2 & \text{if } n \text{ is even and } \det(V_{\mathbb{Q}_p}) = (-1)^{n/2} \\ n-1 & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is even and } \det(V_{\mathbb{Q}_p}) \neq (-1)^{n/2} \end{cases}$$

#### 4.2 Special Cycles

In this section, we will define special cycles on the Rapoport-Zink space associated to a supersingular point of the Shimura variety and determine the intersection behavior of these cycles, in particular determine when this intersection consists of isolated points.

Recall the  $\mathbb{Q}_p$ -quadratic space  $V_K^{\Phi}$  of special quasi-endomorphisms. Under the inclusion  $V_K \hookrightarrow \operatorname{End}_K(D_K)$  the space  $V_K^{\Phi}$  embeds into  $\operatorname{End}(X_0)_{\mathbb{Q}}$ . The elements of  $V_K^{\Phi}$  are called special quasi-endomorphisms of  $X_0$ . Let  $j \in V_K^{\Phi}$  be a special quasi-endomorphism. The special cycle Z(j) associated to j is defined to be the closed formal subscheme of RZ consisting of all points  $(X, \rho)$  such that

$$\rho \circ j \circ \rho^{-1} \in \operatorname{End}(X).$$

The fact that this defines a closed formal subscheme follows from Proposition 2.11. Let  $y \in Z(j)(k)$ . Then j induces an endomorphism of  $M_y^1$  and so  $j \in L_y^{\Phi} = L_y \cap V_K^{\Phi}$ . By definition of  $\mathrm{RZ}_{\Lambda}$ , if  $y \in \mathrm{RZ}_{\Lambda}(k)$ , then  $\Lambda^{\vee} \subset L_y^{\Phi} \subset \Lambda$ . Similarly, by construction, if  $j \in \Lambda^{\vee}$ , then  $\mathrm{RZ}_{\Lambda}(k) \subset Z(j)(k)$ . Let  $j \in V_K^{\Phi}$  and  $\Lambda \subset V_K^{\Phi}$  be a vertex lattice.

Suppose  $y \in \mathrm{RZ}_{\Lambda}(k) \cap Z(j)(k)$ . Then  $j \in L_y^{\Phi} \subset \Lambda$ . By the previous section,  $L_y^{\Phi} = \Lambda(L_y)^{\vee}$  for some vertex lattice  $\Lambda(L_y)$  and  $\Lambda(L_y) \subset \Lambda$ . Hence if  $j \in \Lambda(L_y)^{\vee} = L_y^{\Phi}$ , and so  $\mathrm{RZ}_{\Lambda(L_y)} \subset Z(j)$ . Let  $y \in Z(j)(k)$ , then  $j^2 = Q(j)$  is an endomorphism of the Dieudonne module  $M_y$  and so  $Q(j) \in \mathbb{Z}_p$ . This shows that if  $Q(j) \notin \mathbb{Z}_p$ , then  $Z(j)(k) = \emptyset$ .

The following proposition gives a description of the special cycle Z(j) in terms of the pieces  $RZ_{\Lambda}$  used in the description of the supersingular locus.

**Proposition 4.2.** Given  $j \in V_K^{\Phi}$ , we have

$$Z(j)^{red} = \bigcup_{j \in \Lambda^{\vee}} \mathrm{RZ}_{\Lambda}^{red}$$

*Proof.* By construction, if  $j \in \Lambda^{\vee}$ , then  $\operatorname{RZ}_{\Lambda} \subset Z(j)$ . Hence  $\bigcup_{j \in \Lambda^{\vee}} \operatorname{RZ}_{\Lambda} \subset Z(j)$ . For the other inclusion let  $y \in Z(j)(k)$ . If  $L_y$  is the corresponding special lattice, we have

$$\Lambda(L_y)_W^{\vee} \subset L_y \subset \Lambda(L_y)_W$$

where  $\Lambda(L_y)$  is minimal with this property. By the previous section  $\Lambda(L_y)^{\vee} = L_y^{\Phi}$ . The description of  $\mathrm{RZ}_{\Lambda(L_y)}(k)$  in terms of special lattices implies that  $y \in \mathrm{RZ}_{\Lambda(L_y)}(k)$ . Since j preserves the Hodge filtration  $M_y^1$ , by definition of  $L_y$ , we also have  $j \in L_y^{\Phi} = \Lambda(L_y)^{\vee}$ , thus for every  $y \in Z(j)(k)$ , we have a vertex lattice  $\Lambda(L_y)$  such that  $j \in \Lambda(L_y)^{\vee}$ . This shows that the k-rational points of the reduced schemes  $Z(j)^{\mathrm{red}}$  and  $\bigcup_{j \in \Lambda^{\vee}} \mathrm{RZ}_{\Lambda}^{\mathrm{red}}$  are the same. Recall the formal scheme RZ is locally formally of finite type. From this it is easy to see that the union  $\bigcup_{j \in \Lambda^{\vee}} \mathrm{RZ}_{\Lambda}^{(\ell),\mathrm{red}}$  is closed in  $\mathrm{RZ}^{\mathrm{red}}$  which implies that  $\bigcup_{j \in \Lambda^{\vee}} \mathrm{RZ}_{\Lambda}^{\mathrm{red}}$  is also closed in  $\mathrm{RZ}^{\mathrm{red}}$ . This shows the equality in the proposition.

Recall the disjoint union

$$RZ = \bigsqcup_{\ell \in \mathbb{Z}} RZ^{(\ell)}$$

with each  $\mathrm{RZ}^{(\ell)}$  is connected, open and closed formal subscheme. Then for a vertex lattice  $\Lambda \subset V_K^{\Phi}$ 

$$\mathrm{RZ}^{\mathrm{red}}_{\Lambda} = \bigsqcup_{\ell \in \mathbb{Z}} \mathrm{RZ}^{(\ell),\mathrm{red}}_{\Lambda}$$

and so by the above proposition

$$Z(j)^{\mathrm{red}} = \bigcup_{j \in \Lambda^{\vee}} \bigsqcup_{\ell \in \mathbb{Z}} \mathrm{RZ}_{\Lambda}^{(\ell),\mathrm{red}}$$

where  $\operatorname{RZ}_{\Lambda}^{(\ell),\operatorname{red}}$  for different  $\ell$  are isomorphic and they are smooth projective of dimension  $(t_{\Lambda}/2) - 1$ . Hence the dimension of  $Z(j)^{\operatorname{red}}$  is determined by the pieces  $\operatorname{RZ}_{\Lambda}^{(\ell),\operatorname{red}}$  that appears in the above union.

If  $(j_1, \ldots, j_{n-1}) \in (V_K^{\Phi})^{n-1}$ , then  $Z(j_1, \ldots, j_{n-1})$  is defined similarly and it only depends on the  $\mathbb{Z}_p$ -span J of  $j_1, \ldots, j_{n-1}$ . Hence we will denote it by Z(J). It is the intersection of the cycles  $Z(j_1), \ldots, Z(j_{n-1})$ . If  $y \in Z(J)(k)$ , then  $J \subset L_y^{\Phi}$  and so  $J \subset \Lambda(L_y)^{\vee}$ . Hence  $Q(J) \in \text{Sym}_k(\mathbb{Z}_p)$ . We also have

$$Z(J)^{\mathrm{red}} = \bigcup_{J \subset \Lambda^{\vee}} \mathrm{RZ}_{\Lambda}^{\mathrm{red}}$$

This follows from (4.1.3).

By Proposition 3.7, we know that if  $T \in \operatorname{Sym}_{n-1}(\mathbb{Z}_{(p)})$  is positive definite, then the special cycle  $\mathcal{Z}(T)$  lies in the supersingular locus. Hence from now on, we fix an (n-1)-tuple of special quasi-endomorphisms  $j_1, \ldots, j_{n-1} \in V_K^{\Phi}$  and assume that  $Z(J) \neq \emptyset$ . Let  $T = Q(J) \in \operatorname{Sym}_{n-1}(\mathbb{Z}_p)$ . In the rest of this section we will determine the dimension of  $Z(J)^{\operatorname{red}}$  in terms of T.

The main idea will be the following: Let  $y \in Z(J)(k)$ , then  $y \in \operatorname{RZ}_{\Lambda}(k)$  for some vertex lattice  $\Lambda$  such that  $J \subset \Lambda^{\vee}$ . This shows that for all vertex lattices  $\Lambda_0 \subset \Lambda$  of type 2, we have  $J \subset \Lambda_0^{\vee}$  and so  $\operatorname{RZ}_{\Lambda_0} \subset Z(J)$ . Hence for each point of the cycle Z(J), the pieces  $\operatorname{RZ}_{\Lambda}$  containing that point pass through  $\operatorname{RZ}_{\Lambda_0} \subset Z(J)$ for all vertex lattices  $\Lambda_0 \subset \Lambda$  of type 2. Thus in order to determine the pieces  $\operatorname{RZ}_{\Lambda}^{\text{red}}$  that appears in the above union, it is enough to consider the points of  $\operatorname{RZ}_{\Lambda_0} \subset Z(J)$  for type 2 vertex lattices  $\Lambda_0$  and find the pieces  $\operatorname{RZ}_{\Lambda}$  with  $J \subset \Lambda^{\vee}$ that contains  $\operatorname{RZ}_{\Lambda_0}$ .

Since we are assuming Z(J) is nonempty, by the above paragraph there is a vertex lattice  $\Lambda_0$  of type 2 such that  $\operatorname{RZ}_{\Lambda_0} \subset Z(J)$ . Then  $J \subset \Lambda_0^{\vee}$  and  $\overline{J} = J/(p\Lambda_0 \cap J) \subset \Lambda_0^{\vee}/p\Lambda_0$  is a subspace. The  $\mathbb{F}_p$ -vector space  $\Lambda_0^{\vee}/p\Lambda_0$  is an (n-2)dimensional nondegenerate quadratic space with the quadratic form is given by  $Q \mod p$ . The reduction modulo p of T is the matrix of the  $\mathbb{F}_p$ -quadratic space J/pJ.

**Lemma 4.3.** The nondegenerate parts of the  $\mathbb{F}_p$ -quadratic spaces J/pJ and  $\overline{J}$  are isomorphic.

*Proof.* Consider the following exact sequence

$$0 \to (p\Lambda_0 \cap J)/pJ \to J/pJ \to \bar{J} \to 0$$

and observe that  $(p\Lambda_0 \cap J)/pJ \subset \operatorname{Rad}(J/pJ)$ . We have the following decomposition

$$J/pJ = \operatorname{Rad}(J/pJ) \oplus W = (p\Lambda_0 \cap J)/pJ \oplus U \oplus W$$

where W is nondegenerate and  $\operatorname{Rad}(J/pJ) = (p\Lambda_0 \cap J)/pJ \oplus U$ . Hence we have  $\overline{J} \simeq U \oplus W$  and  $\operatorname{Rad}(\overline{J}) \simeq U$ . This implies that

$$\overline{J}/\operatorname{Rad}(\overline{J}) \simeq (J/pJ)/\operatorname{Rad}(J/pJ) \simeq W$$

Thus modulo radicals the two quadratic spaces J/pJ and  $\overline{J}$  are isomorphic to W and so have the same matrix.

By the above lemma, we have

$$\operatorname{rank}(\bar{T}) = \dim_{\mathbb{F}_p}((J/pJ)/\operatorname{Rad}(J/pJ)) = \dim_{\mathbb{F}_p}(\bar{J}/\operatorname{Rad}(\bar{J})).$$

The following proposition shows that  $\operatorname{rank}(\overline{T})$  can not be n-1.

**Proposition 4.4.** If  $j_1, \ldots, j_{n-1} \in V_K^{\Phi}$  such that  $(j_r, j_s) = T$ , then  $rank(\overline{T}) \neq n-1$ .

Proof. Assume that rank $(\overline{T}) = n-1$ . Then we may assume T is diagonal with unit entries. Now J is a subset of a maximal lattice in  $V_K^{\Phi}$ , say  $J \oplus \langle v \rangle$  as an orthogonal sum. If  $Q(v) \in \mathbb{Z}_p^{\times}$  then  $J \oplus \langle v \rangle$  would be a self-dual lattice contradicting the Hasse invariant of  $V_K^{\Phi}$ . If  $\operatorname{ord}_p(Q(v)) = 1$ , then  $\operatorname{ord}_p(\det V_K^{\Phi})$  would be odd, contradiction. Finally if  $\operatorname{ord}_p(Q(v)) = 2$ , then  $J \oplus \langle \frac{1}{p}v \rangle$  is a lattice contained in its dual and containing the maximal lattice  $J \oplus \langle v \rangle$ , contradicting the maximality of  $J \oplus \langle v \rangle$ . This finishes the proof of the proposition. In the rest of this section we will determine the dimension of Z(J) in terms of  $\operatorname{Rank}(\overline{T})$  and the determinant of  $\overline{T}$  modulo its radical. We will need the following lemmas.

**Lemma 4.5.** Let  $\Lambda_0 \subset V_K^{\Phi}$  be a vertex lattice of type 2. Then for any  $k \leq \frac{n-2}{2}$  we have a bijection

$$\{vertex \ lattices \ \Lambda \supset \Lambda_0 \ of \ type \ 2k+2\} \leftrightarrow \begin{cases} totally \ isotropic \ subspaces \ of \\ \Lambda_0^{\vee}/p\Lambda_0 \ of \ dimension \ k \end{cases} \}.$$

which is defined by  $\Lambda \mapsto p\Lambda/p\Lambda_0$ .

*Proof.* We will write the inverse. The quadratic space  $\Lambda_0^{\vee}/p\Lambda_0$  is n-2 dimensional. Let  $\ell/p\Lambda_0 \subset \Lambda_0^{\vee}/p\Lambda_0$  be a k-dimensional totally isotropic subspace. Hence  $p\Lambda_0 \subset \ell \subset \Lambda_0^{\vee}$ . Since  $(\ell, \ell) \subset p\mathbb{Z}_p$ , we have  $(p^{-1}\ell, \ell) \subset \mathbb{Z}_p$  and so  $p^{-1}\ell \subset \ell^{\vee}$ . We have the following inclusions

$$p\Lambda_0 \xrightarrow{\leftarrow} \ell \xrightarrow{n-k-2} \Lambda_0^{\vee} \xrightarrow{\sim} \Lambda_0 \xrightarrow{\leftarrow} p^{-1}\ell \xrightarrow{n-2k-2} \ell^{\vee}$$

As  $p\Lambda_0 \subset \ell$ , we have  $\ell^{\vee} \subset p^{-1}\Lambda_0^{\vee}$ . We also have  $p^{-1}\Lambda_0 \subset p^{-2}\ell$ , and so

$$p^{-1}\ell \underset{n-2k-2}{\hookrightarrow} \ell^{\vee} \underset{k}{\hookrightarrow} p^{-1}\Lambda_0^{\vee} \underset{2}{\hookrightarrow} p^{-1}\Lambda_0 \underset{k}{\hookrightarrow} p^{-2}\ell$$

Now set  $\Lambda = p^{-1}\ell$  and so  $\Lambda \subset p^{-1}\Lambda^{\vee} \subset p^{-1}\Lambda$  which shows that  $\Lambda$  is a vertex lattice of type 2k + 2.

**Lemma 4.6.** Let  $\Lambda_0 \subset V_K^{\Phi}$  be a vertex lattice of type 2. Let  $\ell/p\Lambda_0 \subset \Lambda_0^{\vee}/p\Lambda_0$  be a k-dimensional subspace with  $Rad(\ell/p\Lambda_0)$  is m-dimensional. Then  $\ell \subset \Lambda^{\vee}$  for some vertex lattice of type 2m + 2.

Proof. For m = 0, take  $\Lambda = \Lambda_0$ . Suppose  $m \ge 1$ . Then  $\operatorname{Rad}(\ell/p\Lambda_0)$  is a totally isotropic subspace of  $\Lambda_0^{\vee}/p\Lambda_0$  of dimension m. Hence by Lemma 4.5,  $\operatorname{Rad}(\ell/p\Lambda_0) = p\Lambda/p\Lambda_0$  for some vertex lattice  $\Lambda \supset \Lambda_0$  of type 2m + 2. Now we have  $(p\Lambda, \ell) \subset p\mathbb{Z}_p$ and so  $(\Lambda, \ell) \subset \mathbb{Z}_p$  which means  $\ell \subset \Lambda^{\vee}$ . Depending on the parity of n and the  $\det(V_{\mathbb{Q}_p})$  we have three cases for  $t_{\max}$ . Now we will consider each case separately.

#### 4.2.1 The case n even

In this case we have

$$t_{\max} = \begin{cases} n-2 & \text{if } \det(V_{\mathbb{Q}_p}) = (-1)^{n/2} \\ n & \text{if } \det(V_{\mathbb{Q}_p}) \neq (-1)^{n/2} \end{cases}$$

We start with the case  $t_{\max} = n$ , i.e.  $\det(V_{\mathbb{Q}_p}) \neq (-1)^{n/2}$ . Then the irreducible components of  $\mathrm{RZ}^{(\ell),\mathrm{red}}$  are  $\frac{n-2}{2}$  dimensional. Given full chain of vertex lattices  $\Lambda_0 \subset \Lambda_1 \subset \ldots \subset \Lambda_k$  where  $\Lambda_0$  is of type 2 and  $\Lambda_k$  is of type n. Then

$$p\Lambda_0 \xrightarrow{\sim} \dots \xrightarrow{\sim} p\Lambda_k = \Lambda_k^{\vee} \xrightarrow{\sim} \dots \xrightarrow{\sim} \Lambda_1^{\vee} \xrightarrow{\sim} \Lambda_0^{\vee} \xrightarrow{\sim} \Lambda_0 \xrightarrow{\sim} \Lambda_1 \xrightarrow{\sim} \dots \xrightarrow{\sim} \Lambda_k$$

and  $k = \frac{n-2}{2}$ . Now  $\Lambda_0^{\vee}/p\Lambda_0$  is a (n-2)-dimensional nondegenerate  $\mathbb{F}_p$ -quadratic space and since  $p\Lambda_k/p\Lambda_0$  is  $\frac{n-2}{2}$ -dimensional totally isotropic subspace

$$\Lambda_0^{\vee}/p\Lambda_0\simeq\mathbb{H}^k$$

where  $\mathbb{H}$  denotes the hyperbolic plane. In fact,  $\Lambda_i^{\vee}/p\Lambda_i \simeq \mathbb{H}^{k-i}$ .

By the proof of [8, Proposition 5.1.2],

$$\Lambda_k = \operatorname{Span}_{\mathbb{Z}_p} \{ e_1, f_1, \dots, e_r, f_r \} \oplus Z$$

where  $(Z_{\mathbb{Q}_p}, Q) \simeq (\mathbb{Q}_{p^2}, cx\bar{x})$  for  $c \in \mathbb{Q}_p^{\times}/\mathrm{Nm}(\mathbb{Q}_{p^2}^{\times})$  and  $\mathrm{ord}_p(c)$  is odd and

$$(e_i, e_j) = 0, \quad (f_i.f_j) = 0, \quad (e_i, f_j) = p^{-1}\delta_{i,j}.$$

Hence we can assume  $\operatorname{ord}_p(c) = 1$ . Then  $\det(Z_{\mathbb{Q}_p}) = -u$  for the unique nonsquare

unit  $u \in \mathbb{Z}_p^{\times} \setminus \mathbb{Z}_p^{\times,2}$  where  $\mathbb{Q}_{p^2} \simeq \mathbb{Q}_p(\sqrt{u})$ . Thus we have

$$\det(V_{\mathbb{Q}_p}) = (-1)^{k+1} u$$

Let  $j_1, \ldots, j_{n-1} \in V_K^{\Phi}$  be special endomorphisms such that  $T \in \operatorname{Sym}_{n-1}(\mathbb{Z}_p)$ . Assume that  $Z(J) \neq \emptyset$ . Hence  $\operatorname{RZ}_{\Lambda_0} \subset Z(J)$  for some vertex lattice  $\Lambda_0$  of type 2 and so  $J \subset \Lambda_0^{\vee}$  and define as before  $\overline{J} = J/(p\Lambda_0 \cap J)$  which is a subspace of  $\Lambda_0^{\vee}/p\Lambda$ and so

$$0 \leq \dim_{\mathbb{F}_p}(\bar{J}/\operatorname{Rad}(\bar{J})) = \operatorname{rank}(\bar{T}) \leq n-2$$

**Lemma 4.7.** Let W be a nondegenerate quadratic space of dimension n and V be a subspace with dim  $V = \ell$  and dim Rad(V) = m. Then  $\ell + m \leq n$ .

*Proof.* The Witt decomposition for V is  $V = U \oplus \text{Rad}(V)$  and U is a  $(\ell - m)$ -dimensional nondegenerate subspace of W. Hence

$$W = U \oplus U^{\perp}$$

where dim  $U^{\perp} = n - \ell + m$ . Now  $\operatorname{Rad}(V) \subset U^{\perp}$  is *m*-dimensional totally isotropic subspace of  $U^{\perp}$  and since  $U^{\perp}$  is nondegenerate,  $2m \leq n - \ell + m$  which proves the lemma.

**Proposition 4.8.**  $Z(J)^{red}$  is  $\frac{n-2}{2}$  dimensional if and only if  $rank(\bar{T}) = 0$ , i.e. p|T.

Proof. Assume that p|T. Then either  $\overline{J} = 0$  or  $\overline{J} = \operatorname{Rad}(\overline{J})$  and dim  $\overline{J} \ge 1$ . In the first case, we have  $J \subset p\Lambda_0$  and so  $J \subset \Lambda_k^{\vee}$  for all  $\Lambda_k \supset \Lambda_0$  of type n. In the second case, by Lemma 4.5,  $\overline{J} = p\Lambda/p\Lambda_0$  for some vertex lattice  $\Lambda \supset \Lambda_0$  of type  $\ge 4$  and so  $J \subset p\Lambda$ . Hence  $J \subset \Lambda_k^{\vee}$  for any vertex lattice  $\Lambda_k \supset \Lambda$  of type n. Thus in either case Z(J) is  $\frac{n-2}{2}$ -dimensional.

Conversely assume that  $Z(J)^{\text{red}}$  is  $\frac{n-2}{2}$ -dimensional. Hence  $J \subset \Lambda_k^{\vee}$  for some vertex lattice  $\Lambda_k \supset \Lambda_0$  of type n. Proposition follows from  $\Lambda_k^{\vee} = p\Lambda_k$ .

**Proposition 4.9.** If  $rank(\overline{T})$  is n-2 or n-3, then  $Z(J)^{red}$  is 0-dimensional.

Proof. First suppose rank $(\bar{T}) = n-2$ . Since  $\bar{J} \subset \Lambda_0^{\vee}/p\Lambda_0$  and  $\dim \Lambda_0^{\vee}/p\Lambda_0 = n-2$ , we have  $\bar{J} = \Lambda_0^{\vee}/p\Lambda_0$ . Now if  $J \subset \Lambda^{\vee}$  for some vertex lattice  $\Lambda \supset \Lambda_0$  of type  $t_{\Lambda} \ge 4$ , then  $\bar{J} \subset \Lambda^{\vee}/p\Lambda_0$  but  $\dim \Lambda^{\vee}/p\Lambda_0 < \dim \Lambda_0^{\vee}/p\Lambda_0$  and so  $J \not\subseteq \Lambda^{\vee}$  for any  $\Lambda \supset \Lambda_0$ of type  $\ge 4$ . This shows that in the union

$$Z(J)^{\mathrm{red}} = \bigcup_{J \subset \Lambda^{\vee}} \mathrm{RZ}_{\Lambda}^{\mathrm{red}}$$

 $\mathrm{RZ}^{\mathrm{red}}_{\Lambda}$  with  $t_{\Lambda} \geq 4$  do not appear. Thus  $Z(J)^{\mathrm{red}}$  is 0-dimensional.

Now suppose rank $(\bar{T}) = n - 3$ . Since  $\Lambda_0^{\vee}/p\Lambda_0$  is nondegenerate,  $\bar{J} \neq \Lambda_0^{\vee}/p\Lambda_0$ . Hence dim  $\bar{J} = n - 3$  and Rad $(\bar{J}) = 0$ . Assume that  $J \subset \Lambda_1^{\vee}$  for some vertex lattice  $\Lambda_1 \supset \Lambda_0$  of type 4. Then  $\bar{J} \subset \Lambda_1^{\vee}/p\Lambda_0$  and in fact we have  $\bar{J} = \Lambda_1^{\vee}/p\Lambda_0$  since both have the same dimension. But Rad $(\Lambda_1^{\vee}/p\Lambda_0) = p\Lambda_1/p\Lambda_0$ , contradiction. Hence  $J \not\subseteq \Lambda_1^{\vee}$  for any vertex lattice  $\Lambda_1$  of type 4, and so  $Z(J)^{\text{red}}$  is 0-dimensional.

The remaining possibilities are  $1 \leq \operatorname{rank}(\bar{T}) \leq n - 4$  and now we will cover these cases.

**Proposition 4.10.** Let  $m = rank(\bar{T})$  and suppose  $1 \le m \le n-4$ . Let  $d_T = det(\bar{T}/Rad(\bar{T}))$ . Then

- (a) If m is odd, then  $\dim Z(J)^{red} = \frac{n-m-3}{2}$
- (b) If m is even, then

dim 
$$Z(J)^{red} = \begin{cases} \frac{n-m-2}{2} & \text{if } d_T = (-1)^{m/2} \\ \frac{n-m-2}{2} - 1 & \text{if } d_T \neq (-1)^{m/2} \end{cases}$$

*Proof.* Recall that  $m = \dim_{\mathbb{F}_p}(\bar{J}/\operatorname{Rad}(\bar{J}))$  and  $\bar{J} \subset \Lambda_0^{\vee}/p\Lambda_0$ . Let  $\dim \operatorname{Rad}(\bar{J}) = \ell$ and so  $\dim \bar{J} = m + \ell$ . By Lemma 4.7, we have  $2\ell + m \leq n - 2$  and so

$$\ell \le \frac{n-2-m}{2}$$

Hence  $0 \leq \ell \leq \lfloor \frac{n-2-m}{2} \rfloor$ . Consider the Witt decomposition for  $\bar{J}$ ,

$$\bar{J} = \operatorname{Rad}(\bar{J}) \oplus W$$

where W is an m-dimensional nondegenerate subspace of  $\Lambda_0^{\vee}/p\Lambda_0$ . Hence we have an orthogonal decomposition

$$\Lambda_0^{\vee}/p\Lambda_0 = W \oplus W^{\perp}$$

where dim  $W^{\perp} = n - 2 - m$ . Note that  $\operatorname{Rad}(\overline{J})$  is a totally isotropic subspace of  $W^{\perp}$ .

Now assume that m is odd and so n - 2 - m is odd. Since any  $\mathbb{F}_p$ -quadratic space of dimension  $\geq 3$  is isotropic,  $W^{\perp}$  contains a totally isotropic subspace Uof dimension  $\frac{n-3-m}{2}$ . We may assume that  $\operatorname{Rad}(\bar{J}) \subset U$ . Then  $\bar{J} \subset W \oplus U \subset \Lambda_0^{\vee}/p\Lambda_0$  with  $\operatorname{Rad}(W \oplus U) = U$ . Hence, by Lemma 4.6,  $W \oplus U \subset \Lambda^{\vee}/p\Lambda_0$  for some vertex lattice  $\Lambda \supset \Lambda_0$  of type n - m - 1. This shows that  $\dim Z(J)^{\operatorname{red}}$  is at least  $\frac{n-m-1}{2} - 1 = \frac{n-m-3}{2}$ . If  $J \subset \Lambda^{\prime\vee}/p\Lambda_0$  and  $\dim \Lambda^{\prime\vee}/p\Lambda_0 = \frac{n+m-3}{2}$  and  $\operatorname{Rad}(\Lambda^{\prime\vee}/p\Lambda_0) = p\Lambda'/p\Lambda_0$ . Since  $\dim p\Lambda'/p\Lambda_0 = \frac{n-m-1}{2}$ , we have

$$\Lambda'^{\vee}/p\Lambda_0 = (p\Lambda'/p\Lambda_0) \oplus V'$$

where V' is the nondegenerate part with dim V' = m - 1. But our assumption on  $\overline{J} \subset \Lambda'^{\vee}/p\Lambda_0$  implies that W is an m-dimensional nondegenerate subspace of  $\Lambda'^{\vee}/p\Lambda_0$ , contradiction. This shows that dim  $Z(J)^{\text{red}} = \frac{n-m-3}{2}$ .

Now assume that m is even and so n-2-m is even. Since dim  $W^{\perp} = n-2-m$ ,  $W^{\perp}$  contains a totally isotropic subspace U of dimension at least  $\frac{n-m-2}{2} - 1$  and it contains a totally isotropic subspace of dimension  $\frac{n-m-2}{2}$  if and only if

$$\det W^{\perp} = (-1)^{\frac{n-m-2}{2}}$$

in which case  $W^{\perp} \simeq \mathbb{H}^{\frac{n-m-2}{2}}$ . Hence, if we let U to be a maximal totally isotropic subspace of  $W^{\perp}$ , then

dim 
$$U = \begin{cases} \frac{n-m-2}{2} & \text{if } \det W^{\perp} = (-1)^{\frac{n-m-2}{2}} \\ \frac{n-m-2}{2} - 1 & \text{otherwise} \end{cases}$$

As before, since  $\operatorname{Rad}(\overline{J}) \subset W^{\perp}$  is totally isotropic, we may assume  $\operatorname{Rad}(\overline{J}) \subset U$ . Hence  $\overline{J} \subset W \oplus U \subset \Lambda_0^{\vee}/p\Lambda_0$  and  $\operatorname{Rad}(W \oplus U) = U$ . Now by Lemma 4.6,  $U \oplus W \subset \Lambda^{\vee}/p\Lambda_0$  for some vertex lattice  $\Lambda$  of type  $t_{\Lambda} = 2 \dim U + 2$ .

Let's first assume that  $\det W^{\perp} = (-1)^{\frac{n-m-2}{2}}$  so that  $t_{\Lambda} = n - m$ . If  $J \subset \Lambda'^{\vee}$  for some vertex lattice  $\Lambda' \supset \Lambda$  of type  $t_{\Lambda'} = t_{\Lambda} + 2 = n - m + 2$ , then  $\bar{J} \subset \Lambda'^{\vee} / p\Lambda_0$  and

$$\Lambda^{\prime\prime}/p\Lambda_0 = \operatorname{Rad}(\Lambda^{\prime\prime}/p\Lambda_0) \oplus V' = (p\Lambda'/p\Lambda_0) \oplus V'$$

where V' is nondegenerate subspace with dim V' = m-2 since dim  $p\Lambda'/p\Lambda_0 = \frac{n-m}{2}$ . But W is an m-dimensional nondegenerate subspace of  $\Lambda'^{\vee}/p\Lambda_0$ , contradiction. Thus dim  $Z(J)^{\text{red}} = \frac{n-m-2}{2}$ .

Secondly, assume that det  $W^{\perp} \neq (-1)^{\frac{n-m-2}{2}}$  so  $t_{\Lambda} = n - m - 2$ . If  $J \subset \Lambda'^{\vee}$  for some vertex lattice  $\Lambda' \supset \Lambda$  of type  $t_{\Lambda'} = t_{\Lambda} + 2 = n - m$ , then  $\bar{J} \subset \Lambda'^{\vee}/p\Lambda_0$  and

$$\Lambda^{\prime\vee}/p\Lambda_0 = \operatorname{Rad}(\Lambda^{\prime\vee}/p\Lambda_0) \oplus V' = (p\Lambda'/p\Lambda_0) \oplus V'$$

where V' is nondegenerate subspace with dim V' = m since dim  $p\Lambda'/p\Lambda_0 = \frac{n-m-2}{2}$ . Then W is an m-dimensional nondegenerate subspace of  $\Lambda'^{\vee}/p\Lambda_0$  and since  $V' \simeq \Lambda'^{\vee}/p\Lambda'$ , we have

$$\det W = \det((\Lambda^{\prime\vee}/p\Lambda_0)/\operatorname{Rad}(\Lambda^{\prime\vee}/p\Lambda_0)) = \det V' = \det \Lambda^{\prime\vee}/p\Lambda' = (-1)^{\frac{m}{2}}$$

and so det  $W^{\perp} = (-1)^{\frac{n-m-2}{2}}$ , contradicting our assumption. Thus dim  $Z(J)^{\text{red}} = \frac{n-m-2}{2} - 1$ . This finishes the proof.

To summurize the above results, we have

**Theorem 4.11.** Let n be even and  $det(V_{\mathbb{Q}_p}) \neq (-1)^{n/2}$ . Let  $m = rank(\bar{T})$  and  $d_T = det(\bar{T}/Rad(\bar{T}))$ . Then

- (i)  $Z(J)^{red}$  is  $\frac{n-2}{2}$  dimensional if and only if  $rank(\bar{T}) = 0$ , i.e. p|T.
- (ii) If  $rank(\overline{T})$  is n-2 or n-3, then  $Z(J)^{red}$  is 0-dimensional.
- (iii) Suppose  $1 \le m \le n-4$ .
  - (a) If m is odd, then  $\dim Z(J)^{red} = \frac{n-m-3}{2}$
  - (b) If m is even, then

$$\dim Z(J)^{red} = \begin{cases} \frac{n-m-2}{2} & \text{if } d_T = (-1)^{m/2} \\ \frac{n-m-2}{2} - 1 & \text{if } d_T \neq (-1)^{m/2} \end{cases}$$

In particular, when m = n - 4 we have

dim 
$$Z(J)^{\text{red}} = \begin{cases} 1 & \text{if } d_T = (-1)^{n/2} \\ 0 & \text{if } d_T \neq (-1)^{n/2} \end{cases}$$

and note that  $d_T \neq (-1)^{n/2}$  if and only if  $d_T = \det V_{\mathbb{Q}_p} \mod p$ .

Remark 4.12. When n = 4, the above theorem implies that if Z(J) is nonempty, then it is 0-dimensional if and only if  $p \nmid T$ . Compare this result with [17, Theorem 6.1].

Now we will consider the case when det  $V_{\mathbb{Q}_p} = (-1)^{n/2}$  which implies  $t_{\max} = n-2$ . The irreducible components of  $\mathrm{RZ}^{(\ell),\mathrm{red}}$  are  $\frac{n-4}{2}$ -dimensional. Given a full chain of vertex lattices  $\Lambda_0 \subset \ldots \subset \Lambda_k$  where  $t_{\Lambda_0} = 2$  and  $t_{\Lambda_k} = n-2$ , we have

$$p\Lambda_0 \xrightarrow{\sim}_1 \dots \xrightarrow{\sim}_1 p\Lambda_k \xrightarrow{\sim}_2 \Lambda_k^{\vee} \xrightarrow{\sim}_1 \dots \xrightarrow{\sim}_1 \Lambda_1^{\vee} \xrightarrow{\sim}_1 \Lambda_0^{\vee} \xrightarrow{\sim}_2 \Lambda_0 \xrightarrow{\sim}_1 \Lambda_1 \xrightarrow{\sim}_1 \dots \xrightarrow{\sim}_1 \Lambda_k$$

and  $k = \frac{n-4}{2}$ . Now  $\Lambda_0^{\vee}/p\Lambda_0$  is an (n-2) dimensional nondegenerate  $\mathbb{F}_p$ -quadratic space. By [8, Proposition 5.1.2], we have

$$\Lambda_k = \operatorname{Span}_{\mathbb{Z}_n} \{ e_1, f_1, \dots, e_k, f_k \} \oplus Z$$

where  $(Z_{\mathbb{Q}_p}, Q) \simeq (B, \operatorname{Nrd})$  is 4-dimensional anisotropic quadratic space with Bis a quaternion division algebra over  $\mathbb{Q}_p$ . Then  $\Lambda_k^{\vee}/p\Lambda_k \simeq Z^{\vee}/pZ$  and a simple calculation shows that  $\det \Lambda_k^{\vee}/p\Lambda_k = -\bar{u}$  where  $u \in \mathbb{Z}_p^{\times} \setminus \mathbb{Z}_p^{\times,2}$  is the unique nonsquare unit. This shows that  $\Lambda_k^{\vee}/p\Lambda_k$  is anisotropic. Now consider the subspace  $\Lambda_k^{\vee}/p\Lambda_0 \subset \Lambda_0^{\vee}/p\Lambda_0$ . We have  $\operatorname{Rad}(\Lambda_k^{\vee}/p\Lambda_0) = p\Lambda_k/p\Lambda_0$  so we can write

$$\Lambda_k^{\vee}/p\Lambda_0 \simeq p\Lambda_k/p\Lambda_0 \oplus W$$

where W is the nondegenerate part. Consider the exact sequence of quadratic spaces

$$0 \to p\Lambda_k/p\Lambda_0 \to \Lambda_k^{\vee}/p\Lambda_0 \to \Lambda_k^{\vee}/p\Lambda_k \to 0$$

and choose a splitting so that  $W \simeq \Lambda_k^{\vee}/p\Lambda_k$  as quadratic spaces. Hence W is 2-dimensional anisotropic subspace and we have  $\Lambda_0^{\vee}/p\Lambda_0 = W \oplus W^{\perp}$  and  $p\Lambda_k/p\Lambda_0$ is an  $\frac{n-4}{2}$ -dimensional totally isotropic subspace of the (n-4)-dimensional nondegenerate subspace  $W^{\perp}$ . Thus  $W^{\perp} \simeq \mathbb{H}^k$  and so

$$\Lambda_0^{\vee}/p\Lambda_0 \simeq \mathbb{H}^k \oplus \Lambda_k^{\vee}/p\Lambda_k.$$

Similarly, we have

$$\Lambda_i^{\vee}/p\Lambda_i \simeq \mathbb{H}^{k-i} \oplus \Lambda_k^{\vee}/p\Lambda_k$$

As before let  $J = \operatorname{Span}_{\mathbb{Z}_p} \{ j_1, \ldots, j_{n-1} \} \subset V_K^{\Phi}$  and assume  $Z(J) \neq \emptyset$ . Let  $\Lambda_0$ be a vertex lattice of type 2 such that  $\operatorname{RZ}_{\Lambda_0} \subset Z(J)$ , i.e.  $J \subset \Lambda_0^{\vee}$ . Recall that  $\overline{J} = J/(p\Lambda_0 \cap J) \subset \Lambda_0^{\vee}/p\Lambda_0$  and  $\dim_{\mathbb{F}_p}(\overline{J}/\operatorname{Rad}(\overline{J})) = \operatorname{rank}(\overline{T})$ . The dimension of the special cycle  $Z(J)^{\operatorname{red}}$  in this case is described by the next theorem. **Theorem 4.13.** Let n be even and  $det(V_{\mathbb{Q}_p}) = (-1)^{n/2}$ . Let  $m = rank(\bar{T})$  and  $d_T = det(\bar{T}/Rad(\bar{T}))$ . Then

- (i) If m = 0, then  $Z(J)^{red}$  is  $\frac{n-4}{2}$  dimensional.
- (ii) If m is n-2 or n-3, then  $Z(J)^{red}$  is 0 dimensional.
- (iii) Suppose  $1 \le m \le n-4$ . Then
  - (a) If *m* is odd, then  $\dim Z(J)^{red} = \frac{n-m-1}{2} 1$ .
  - (b) If m is even, then

dim 
$$Z(J)^{red} = \begin{cases} \frac{n-m-2}{2} & \text{if } d_T \neq (-1)^{m/2} \\ \frac{n-m-2}{2} - 1 & \text{if } d_T = (-1)^{m/2} \end{cases}$$

*Proof.* The proofs of (i) and (ii) are identical to the proofs of the Propositions 4.8 and 4.9. So we will only prove (iii). Let dim  $\operatorname{Rad}(\overline{J}) = \ell$  and so dim  $\overline{J} = m + \ell$ . Hence

$$\ell \le \lfloor \frac{n-2-m}{2} \rfloor.$$

Consider the Witt decomposition for  $\bar{J}$ ,

$$\bar{J} = \operatorname{Rad}(\bar{J}) \oplus W$$

where W is an m-dimensional nondegenerate subsoace of  $\Lambda_0^{\vee}/p\Lambda_0$ . Then we have an orthogonal decomposition

$$\Lambda_0^{\vee}/p\Lambda_0 = W \oplus W^{\perp}$$

where  $W^{\perp}$  is (n-2-m)-dimensional nondegenerate subspace and note that  $\operatorname{Rad}(\bar{J})$  is a totally isotropic subspace of  $W^{\perp}$ .

Now assume that m is odd and so n - 2 - m is odd. Then  $W^{\perp}$  contains a maximal totally isotropic subspace U of dimension  $\frac{n-3-m}{2}$  that contains  $\operatorname{Rad}(\bar{J})$ .

Then  $\overline{J} \subset W \oplus U \subset \Lambda_0^{\vee}/p\Lambda_0$  with  $\operatorname{Rad}(W \oplus U) = U$ . By Lemma 4.6,  $W \oplus U \subset \Lambda^{\vee}/p\Lambda_0$  for some vertex lattice  $\Lambda \supset \Lambda_0$  of type n - m - 1. If  $J \subset \Lambda'^{\vee}$  for some vertex lattice  $\Lambda' \supset \Lambda$  of type  $t_{\Lambda'} = t_{\Lambda} + 2 = n - m + 1$ , then  $\overline{J} \subset \Lambda'^{\vee}/p\Lambda_0$  and since  $\operatorname{Rad}(\Lambda'^{\vee}/p\Lambda_0) = p\Lambda'/p\Lambda_0$ ,

$$\Lambda'^{\vee}/p\Lambda_0 = (p\Lambda'/p\Lambda_0) \oplus V'$$

where V' is the nondegenerate part which has dimension m-1. This contradicts with the fact that W is an m-dimensional nondegenerate subspace of  $\Lambda'^{\vee}/p\Lambda_0$ . Thus  $J \not\subseteq \Lambda'^{\vee}$  and dim  $Z(J)^{\text{red}} = \frac{n-m-1}{2} - 1$ .

Now assume that m is even and so n - 2 - m is even. Now the (n - 2 - m)-dimensional nondegenerate space  $W^{\perp}$  contains a totally isotropic subspace of dimension at least  $\frac{n-2-m}{2} - 1$  and it contains a totally isotropic subspace of dimension  $\frac{n-2-m}{2}$  if and only if

$$\det W^{\perp} = (-1)^{\frac{n-2-m}{2}} \Leftrightarrow \det W = (-1)^{m/2} \bar{u} \Leftrightarrow \det W \neq (-1)^{m/2}$$

where u is the unique nonsquare unit in  $\mathbb{Z}_p^{\times}$ . Here we used the fact that det  $\Lambda_0^{\vee}/p\Lambda_0 = (-1)^{k+1}\bar{u}$ . Let U be a maximal totally isotropic subspace of  $W^{\perp}$ . Then

dim 
$$U = \begin{cases} \frac{n-m-2}{2} & \text{if } \det W^{\perp} = (-1)^{\frac{n-m-2}{2}} \\ \frac{n-m-2}{2} - 1 & \text{otherwise} \end{cases}$$

This is why we have different conditions on the determinant in Theorem 4.11 and 4.13. The rest of the proof is the same as the proof of Proposition 4.10.  $\Box$ 

#### 4.2.2 The case n odd

In this case  $t_{\max} = n - 1$  and so the dimension of the irreducible components of  $RZ^{(\ell), red}$  are  $\frac{n-3}{2}$ -dimensional. Given a full chain of vertex lattices  $\Lambda_0 \subset \ldots \subset \Lambda_k$ 

where  $\Lambda_0$  is of type 2 and  $\Lambda_k$  is of type n-1, we have

$$p\Lambda_0 \xrightarrow{\sim}_1 \dots \xrightarrow{\sim}_1 p\Lambda_k \xrightarrow{\sim}_1 \Lambda_k^{\vee} \xrightarrow{\sim}_1 \dots \xrightarrow{\sim}_1 \Lambda_1^{\vee} \xrightarrow{\sim}_1 \Lambda_0^{\vee} \xrightarrow{\sim}_2 \Lambda_0 \xrightarrow{\sim}_1 \Lambda_1 \xrightarrow{\sim}_1 \dots \xrightarrow{\sim}_1 \Lambda_k$$

and  $k = \frac{n-3}{2}$  and as before  $\Lambda_0^{\vee}/p\Lambda_0$  is an (n-2) dimensional nondegenerate  $\mathbb{F}_p$ quadratic space with quadratic form  $Q \mod p$ . Similar to the case where n is even and  $t_{max} = n-2$ , we have

$$\Lambda_0^{\vee}/p\Lambda_0\simeq \mathbb{H}^k\oplus \Lambda_k^{\vee}/p\Lambda_k$$

and in fact  $\Lambda_i^{\vee}/p\Lambda_i \simeq \mathbb{H}^{k-i} \oplus \Lambda_k^{\vee}/p\Lambda_k$ . As before, for a vertex lattice of maximal type we have

$$\Lambda_k = \operatorname{Span}_{\mathbb{Z}_n} \{ e_1, f_1, \dots, e_k, f_k \} \oplus Z$$

where  $(Z_{\mathbb{Q}_p}, Q) \simeq (\mathbb{Q}_p^3, c(-ux_1^2 - px_2^2 + upx_3^2))$  for  $c \in \mathbb{Q}_p^{\times}/\mathbb{Q}_p^{\times,2}$  with  $\operatorname{ord}_p c$  is even and  $u \in \mathbb{Z}_p^{\times} \setminus \mathbb{Z}_p^{\times,2}$ . So we can take c = 1 or c = u. It follows that  $\det Z_{\mathbb{Q}_p} = c$  and so  $\det(V_{\mathbb{Q}_p}) = \det(V_K^{\Phi}) = (-1)^k c$ . We also have,  $\Lambda_k^{\vee}/p\Lambda_k \simeq Z^{\vee}/pZ$ , hence

$$\det \Lambda_k^{\vee} / p\Lambda_k = \det Z^{\vee} / pZ = -\bar{u}\bar{c}$$

and

$$\det \Lambda_i^{\vee} / p\Lambda_i = (-1)^{k-i+1} \bar{u}\bar{c}.$$

Assume that  $Z(J) \neq \emptyset$  and so  $\operatorname{RZ}_{\Lambda_0} \subset Z(J)$  for some vertex lattice  $\Lambda_0$  of type 2. As before let  $\overline{J} = J/p\Lambda_0 \cap J \subset \Lambda_0^{\vee}/p\Lambda_0$  and recall that  $\dim_{\mathbb{F}_p}(\overline{J}/\operatorname{Rad}(\overline{J})) = \operatorname{rank}(\overline{T})$ . Thus we have  $0 \leq \operatorname{rank}(\overline{T}) \leq n-2$ .

**Proposition 4.14.** If  $rank(\overline{T}) = 0$ , then  $\dim Z(J)^{red} = \frac{n-3}{2}$ .

Proof. We have  $\overline{J} = \operatorname{Rad}(\overline{J})$  and so  $\overline{J}$  is totally isotropic subspace of  $\Lambda_0^{\vee}/p\Lambda_0$  of dimension  $m = \dim \overline{J}$ . Hence by Lemma 4.5, we have  $\overline{J} = p\Lambda/p\Lambda_0$  for some vertex lattice  $\Lambda \supset \Lambda_0$  of type  $t_{\Lambda} = 2m + 2$ . Thus  $J \subset p\Lambda \subset \Lambda_k^{\vee}$  for some vertex lattice  $\Lambda_k$ of maximal type. This shows that at least one irreducible component of  $\operatorname{RZ}^{(\ell),\operatorname{red}}$  passing through the point  $\operatorname{RZ}_{\Lambda_0}^{(\ell),\operatorname{red}}$  lies in  $Z(J)^{\operatorname{red}}$  and so  $\dim Z(J)^{\operatorname{red}} = \frac{n-3}{2}$ .  $\Box$ 

**Proposition 4.15.** If  $rank(\overline{T}) = n - 2$  or n - 3, then  $Z(J)^{red}$  is 0-dimensional.

*Proof.* Similar to Proposition 4.9.

**Theorem 4.16.** Let  $m = rank(\bar{T})$  and suppose  $1 \leq m \leq n-4$ . Let  $d_T = det(\bar{T}/Rad(\bar{T}))$ . Then

- (a) If m is even, then  $\dim Z(J)^{red} = \frac{n-m-3}{2}$ .
- (b) If m is odd, then

$$\dim Z(J)^{red} = \begin{cases} \frac{n-m-2}{2} & \text{if } d_T \neq \epsilon(-1)^{\frac{m+4-n}{2}} \\ \frac{n-m-2}{2} - 1 & \text{otherwise} \end{cases}$$

where  $\epsilon \in \{\pm 1\}$  is the modulo p square class of  $\det(V_{\mathbb{Q}_p})$ .

*Proof.* Let dim Rad $(\bar{J}) = \ell$ . Then dim  $\bar{J} = \ell + m$  and by Lemma 4.7,  $\ell \leq \frac{n-2-m}{2}$ . Consider the Witt decomposition of  $\bar{J}$ ,

$$\bar{J} = \operatorname{Rad}(\bar{J}) \oplus W$$

where W is an m-dimensional nondegenerate subspace of  $\Lambda_0^{\vee}/p\Lambda_0$ . Hence we have an orthogonal decomposition

$$\Lambda_0^{\vee}/p\Lambda_0 = W \oplus W^{\perp}$$

and dim  $W^{\perp} = n - 2 - m$ . Also  $\operatorname{Rad}(\overline{J})$  is a totally isotropic subspace of  $W^{\perp}$ . Considering the cases m is even and m is odd separately and noting that  $d_T = (-1)^{(m+1)/2} \overline{u}\overline{c}$  if and only if  $d_T \neq (-1)^{\frac{m+4-n}{2}} \det V_{\mathbb{Q}_p} \mod p$  the result follows similar to Proposition 4.10.

In particular, when m = n - 4, by Theorem 4.16 we have

$$\dim Z(J)^{\mathrm{red}} = \begin{cases} 1 & \text{if } d_T \neq \epsilon \\ 0 & \text{otherwise} \end{cases}$$

Remark 4.17. For n = 5 we are in the case of signature (3, 2) which is studied in [18]. In their paper Kudla-Rapoport start with an indefinite quaternion algebra B with discriminant D(B) and define

$$V = \{x \in C : x' = x \text{ and } tr(x) = 0\}$$

where  $C = M_2(B)$ ,  $x \mapsto x' = {}^t x^{\iota}$  and  ${}^{\iota}$  is the main involution of B. Then V is a quadratic space of signature (3,2). By definition of V it follows that det V = 1. By the above theorem, we have

$$\dim Z(J)^{\mathrm{red}} = \begin{cases} 1 & \text{if } d_T \neq \epsilon \\ 0 & \text{otherwise} \end{cases}$$

Thus the condition for the cycle to be 0-dimensional can be stated as T represents 1, which is the condition in [18, Theorem 5.11].

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