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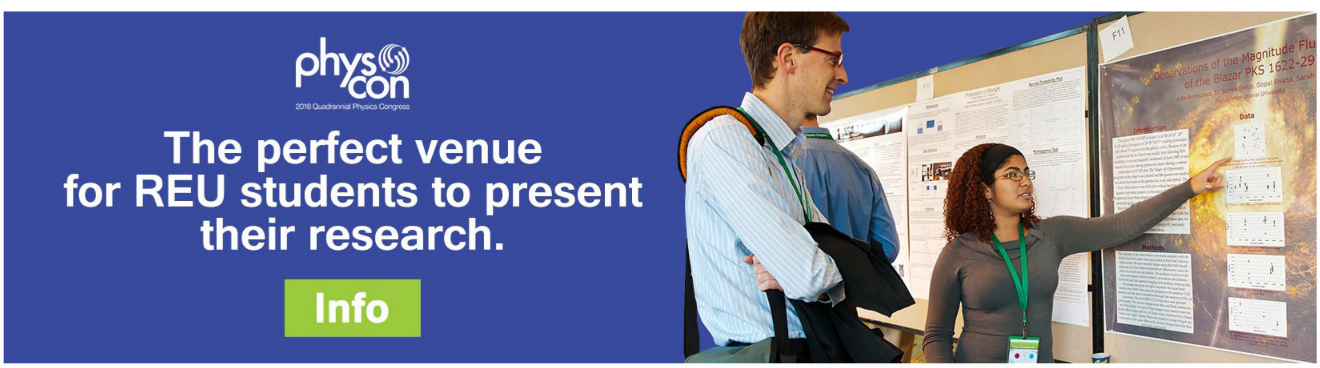

Bound charges and currents

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Bound charges and currents are among the conceptually challenging topics in advanced courses on electricity and magnetism. It may be tempting for students to believe that they are merely computational tools for calculating electric and magnetic fields in matter, particularly because they are usually introduced through abstract manipulation of integral identities, with the physical interpretation provided *a posteriori*. Yet these charges and currents are no less real than free charges and currents and can be measured experimentally. A simpler and more direct approach to introducing this topic, suggested by the ideas in the classic book by Purcell and emphasizing the physical origin of these phenomena, is proposed.
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I. INTRODUCTION

The term *bound charge* (or *bound current*) is somewhat misleading because it may seem to refer to charges (or currents) within isolated atoms and molecules, whereas in reality it is a collective phenomenon. Edward Purcell, in his textbook on electricity and magnetism,[1] proposed the term *structural charge* as an alternative, which accurately suggests that the material properties and the geometry of the device play a critical role in establishing these charges.[2] Nevertheless, he continued to adhere to the standard terminology.

Nomenclature aside, Purcell's way of introducing bound charges is original and has not been, by and large, adopted in modern textbooks. This is probably because his derivation assumes a linear response of the material, is written in the Gaussian system of units, and unfolds piece-wise within a much broader discussion (see chapter 10 of Ref. 1). Yet, as so much in his book, the elegant simplicity and the physical insights of Purcell's analysis more than compensate for the lack of generality. The present contribution, which arose from my experience of teaching a junior-level electricity and magnetism course based on the textbook by Griffiths,[3] offers an alternative discussion of bound charges and currents inspired by Purcell's treatment of the electric field in matter.

The essence of the proposed approach is to show that bound volumetric charges and currents arise directly from Gauss's and Ampère's Laws, requiring no abstract derivations or reference to the potentials. But the pedagogical benefits go far beyond the simplified mathematical analysis. First, these "bound" phenomena are introduced as a result of careful "book-keeping," of asking which charges account for which part of the field. This is instructive because bound charges and currents in fact serve to account for a vast number of individual electric or magnetic dipoles, too numerous to be treated one-by-one using Maxwell's equations. Second, deriving expressions for bound charges and currents directly from Maxwell's equations emphasizes that these are real physical phenomena—no less so than free charges—and can be measured experimentally; they are neither a mathematical artifact nor merely "an equivalent" of the polarization or magnetization. And third, the proposed analysis yields the relationship between free and bound volumetric charges (free and bound volumetric currents) without any additional effort, and without the need of defining the displacement field (*H*-field) or invoking Gauss's Law (Ampère's Law) again.

As a bonus, the constitutive relationships for linear media arise in this approach naturally and the standard definitions of susceptibilities appear as a means of simplifying notation. The usual approach, starting with the constitutive relations, has the disadvantage of requiring right away an "orgy of terminology" (see the footnote on page 180 of Ref. 3), some of which may seem peculiar (e.g., setting $\chi_e = \varepsilon_r - 1$).

The remainder of this paper is organized as follows. Section II provides a brief outline of the standard approach based on integral expressions for scalar and vector potentials. Purcell's approach to bound volumetric charges, streamlined and recast compactly in SI units, and the proposed new treatment of bound volumetric currents, are presented in Secs. III and IV, respectively (bound surface charges and currents are noted for completeness). Lastly, Sec. V provides some concluding remarks.

II. POLARIZATION AND MAGNETIZATION

To set the stage, we first outline the common way of introducing bound charges and currents at the advanced level (see, e.g., Sec. 4.2 of Ref. 3). The discussion of fields in matter typically begins by noting that when dielectrics are placed in an external electric field they become polarized; that is, they acquire a microscopically distributed electric dipole moment. This polarization arises via induced dipole moment in the molecules caused by charge separation (as in hydrogen), or by alignment of heretofore randomly oriented existing dipoles (as in water). The aggregate effect of these dipoles in the material is to produce a spatially dependent dipole moment per unit volume, or polarization, **P**.

The dipole moment in an infinitesimal volume element can be expressed as $d\mathbf{p} = \mathbf{P}dv$, and therefore the electric potential $V(\mathbf{r})$ of a polarized material at a point $\mathbf{r}$ can be written as

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o}\int_V \frac{\hat{\imath}\cdot\mathbf{P}(\mathbf{r}')}{\imath^2}dv', \quad (1)$$

where $\imath = \mathbf{r} - \mathbf{r}'$, $\mathbf{r}'$ denotes position within the volume, and the "hat" indicates a unit vector. Invoking $\nabla'(1/\imath) = \hat{\imath}/\imath^2$, using a product rule for the divergence and then applying the divergence theorem—effectively integrating by parts—renders Eq. (1) as the sum of two integrals, one over the surface S and the other over the volume it encloses,

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o}\oint_S \frac{1}{\imath}\mathbf{P}(\mathbf{r}')\cdot d\mathbf{s}' - \frac{1}{4\pi\varepsilon_o}\int_V \frac{1}{\imath}\nabla\cdot\mathbf{P}(\mathbf{r}')dv'. \tag{2}$$

Because the first term looks like the potential due to a surface charge and the second as the potential due to a volumetric charge, Eq. (2) suggests the presence of two charge distributions due to polarization,

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}},\ \rho_b = -\nabla\cdot\mathbf{P}, \tag{3}$$

where $\hat{\mathbf{n}}$ represents the outward pointing unit vector normal to the surface so that $d\mathbf{s} = \hat{\mathbf{n}}ds$. As required by conservation of charge, the net bound charge is zero because, by virtue of the divergence theorem, a volume integral of $\nabla\cdot\mathbf{P}$ leads to the corresponding surface integral of $\mathbf{P}\cdot\hat{\mathbf{n}}$. This argument can also be inverted to obtain the second relation of Eq. (3) from the first by invoking the conservation of charge and Gauss's law (see, e.g., Sec. 10-3 of *The Feynman Lectures on Physics*[4]).

A similar but more cumbersome derivation for magnetism (e.g., section 6.2 of Ref. 3) starts with the integral expression for the magnetic vector potential analogous to Eq. (1) in terms of the magnetization **M** (magnetic dipole moment per unit volume). This expression can be rendered in terms of two integrals suggesting the existence of surface and volumetric bound currents given by

$$\mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}} \quad \text{and} \quad \mathbf{J}_b = \nabla\times\mathbf{M}, \tag{4}$$

respectively. Notice the absence of the negative sign in the second of these equations—the electric and magnetic formulations are not perfectly symmetric for various pragmatic reasons (though they can be made so[5]).

It is mathematically appealing, and sometimes convenient, to introduce analogous expressions to Eqs. (3) and (4) with the roles of the fields **P** and **M** reversed. This leads to the construction of fictitious magnetization charges ($\sigma_M = \mathbf{M}\cdot\hat{\mathbf{n}}$, $\rho_M = -\nabla\cdot\mathbf{M}$) and fictitious polarization currents ($\mathbf{K}_P = \mathbf{P}\times\hat{\mathbf{n}}$, $\mathbf{J}_P = \nabla\times\mathbf{P}$), first introduced by Jefimenko[6] and later by Mata-Mendez.[7] However, unlike the quantities defined in Eqs. (3) and (4), these mathematical objects do not represent real physical quantities.

III. BOUND CHARGES IN LINEAR DIELECTRICS

Consider a parallel-plate capacitor constructed of two identical square plates of area A and separation d, such that $d \ll \sqrt{A}$. When a charge density σ is placed on the capacitor, the electric field $\mathbf{E}_0$ between the plates can be considered uniform because the fringing fields due to edge effects are negligible. If a dielectric material fills the gap between the plates, it will acquire a polarization **P**, effectively introducing additional charges on the capacitor plates due to alignment of the electric dipoles. This bound surface charge σ_b, will in turn result in an additional electric field between the plates $\mathbf{E}'$.

Let us now assume the material between the capacitor plates is an isotropic, linear dielectric so that the net electric field **E** is parallel to the external (applied) electric field $\mathbf{E}_0$. In this case, the field due to bound charges alone $\mathbf{E}'$ is antiparallel to $\mathbf{E}_0$ and therefore $\mathbf{E} = \mathbf{E}_0 - \mathbf{E}'$ (see Fig. 1). The ratio of the magnitudes of the external electric field (in vacuum)

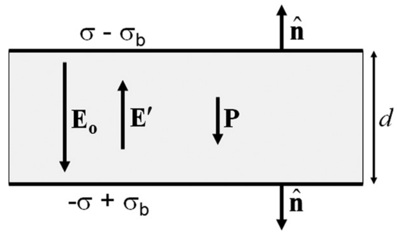

Fig. 1. Bound surface charges, the electric field, and polarization inside a parallel-plate capacitor filled with a dielectric. Here, $\mathbf{E}_0$ is the (external) field due to free charges alone and $\mathbf{E}'$ is the field due to the bound charges alone.

to the net electric field in the material is the relative permittivity (or the dielectric constant) ε_r, so that

$$\mathbf{E} = \frac{\mathbf{E}_0}{\varepsilon_r}, \tag{5}$$

where $\varepsilon_r > 1$. The relationship between the polarization **P** and the bound surface charge σ_b is easy to derive by considering a "stack" of infinitesimal electric dipoles, extending from one plate of the capacitor to the other; all charges associated with these dipoles cancel out inside the volume and it is seen that $P = \sigma_b$. For an arbitrary geometry, this relationship becomes $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}}$ (for details see Sec. 4.2.2 of Ref. 3).

To derive the second relation of Eq. (3), we note first that $\mathbf{E} = \mathbf{E}_0 - \mathbf{E}' = \varepsilon_r\mathbf{E} - \mathbf{E}' = \varepsilon_r\mathbf{E} - \sigma_b\hat{\mathbf{n}}/\varepsilon_0$, and therefore, using $\sigma_b\hat{\mathbf{n}} = (\mathbf{P}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} = \mathbf{P}$ (valid for both upper and lower plates), we find

$$\mathbf{P} = \varepsilon_o(\varepsilon_r - 1)\mathbf{E}. \tag{6}$$

Equation (6) states that the polarization field is parallel to the electric field inside the dielectric and is often taken as the definition of a linear medium. It has been derived here from Eq. (5) assuming a uniform external electric field, but it is valid in general for linear dielectrics. The positive factor $\chi_e = \varepsilon_r - 1$ appearing here is the electric susceptibility, which converts Eq. (6) to a more compact form $\mathbf{P} = \varepsilon_0\chi_e\mathbf{E}$.

The second relation of Eq. (3) can now be obtained for non-uniform electric fields by repeatedly invoking Gauss's Law. This is the key to Purcell's "magically" simple derivation. Let ρ_f denotes any free charge in the volume, so that the net charge within the dielectric is $\rho = \rho_f + \rho_b$. If $\mathbf{E}_0$ is the electric field due to ρ_f alone, then

$$\rho_f = \nabla\cdot\varepsilon_0\mathbf{E}_0 = \nabla\cdot\varepsilon_0\varepsilon_r\mathbf{E}. \tag{7}$$

But the total charge in the presence of the material is given by

$$\rho_f + \rho_b = \nabla\cdot\varepsilon_0\mathbf{E}, \tag{8}$$

so that, after substituting Eq. (7), the volumetric bound charge can be expressed as

$$\rho_b = -\nabla\cdot[\varepsilon_o(\varepsilon_r - 1)\mathbf{E}]. \tag{9}$$

However, according to Eq. (6), the vector in the square brackets in Eq. (9) is the polarization **P**, which then leads to $\rho_b = -\nabla\cdot\mathbf{P}$.

For linear, homogenous, isotropic materials, bound volumetric charges are related in a simple way to free charges.

This relationship is usually obtained using Gauss's Law formulated in terms of the displacement field[8] [note that $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} = \varepsilon_0\varepsilon_\mathbf{r}\mathbf{E}$ appears implicitly in Eq. (7)]. Here, it can be obtained immediately by substituting Eq. (7) into Eq. (9) to get

$$\rho_b = -\frac{\varepsilon_r - 1}{\varepsilon_r}\nabla\cdot(\varepsilon_0\varepsilon_r\mathbf{E}) = -\frac{\chi_e}{1+\chi_e}\rho_f. \tag{10}$$

Thus, within a linear, homogeneous, isotropic dielectric, in regions where there are no (free) volumetrically distributed charges, there are also no bound volumetric charges.

IV. BOUND CURRENTS IN LINEAR MEDIA

We limit the discussion of bound currents to linear, diamagnetic and paramagnetic materials (ferromagnetism is an intrinsically nonlinear quantum phenomenon). Analogous to Eq. (5), the magnetic field in linear, isotropic medium is assumed to satisfy

$$\mathbf{B} = \mu_r\mathbf{B}_0, \tag{11}$$

where $\mathbf{B}_0$ is the external (applied) magnetic field and μ_r is the relative permeability of the material ($\mu_r > 1$ for paramagnetic materials and $\mu_r < 1$ for diamagnetic materials).

Now consider an ideal, circular solenoid of cross-sectional area A and length $L \gg \sqrt{A}$, of infinitesimally thin coils and sufficiently long so that fringing effects near the ends can be ignored. Let the z-axis be the axis of symmetry of the solenoid. Then, when current i is flowing through the (empty) solenoid, the magnetic field inside is nearly uniform and given by $\mathbf{B}_0 = \mu_0 in\,\hat{\mathbf{z}}$, where n is the number of coils per unit length and $\hat{\mathbf{z}}$ is the unit vector along the z-axis. This magnetic field can be expressed in terms of the effective surface current flowing azimuthally $\mathbf{K} = in\,\hat{\boldsymbol{\phi}}$, by writing $\mathbf{B}_0 = -\mu_0\mathbf{K}\times\hat{\mathbf{n}}$, with $\hat{\mathbf{n}}$ denoting the outward pointing unit vector normal to the (curved) surface (see Fig. 2).

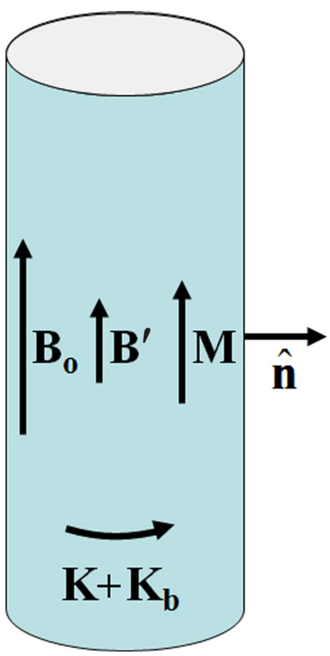

Fig. 2. Bound surface currents, the magnetic field, and magnetization inside a circular solenoid filled with a paramagnet. Here, $\mathbf{B}_0$ is the (external) field due to free currents alone and $\mathbf{B}'$ is the field due to bound currents alone.

When the solenoid is filled with a paramagnetic or diamagnetic material, the magnetic field inside becomes $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}'$, where $\mathbf{B}'$ is due to the surface current $\mathbf{K_b}$ resulting from the magnetization. In the geometry considered here, where $\mathbf{B}'$ is either parallel or anti-parallel to $\mathbf{B}_0$, it follows that $\mathbf{B}' = -\mu_0\mathbf{K_b}\times\hat{\mathbf{n}}$, where $\mathbf{K_b}$ is in the positive direction (i.e., $\hat{\boldsymbol{\phi}}$) for a paramagnet, and in the negative direction for a diamagnet.

The first of the relationships in Eq. (4) can be obtained by considering a collection of infinitesimal magnetic dipoles (current loops) arranged side-by-side and extending throughout the cross-section of the solenoid. In this case, all the internal currents will cancel and only the surface (bound) current remains (for details see Sec. 6.2.2 of Ref. 3). To derive the second expression in Eq. (4) we note that $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}' = \mathbf{B}/\mu_\mathbf{r} - \mu_0(\mathbf{M}\times\hat{\mathbf{n}})\times\hat{\mathbf{n}}$, and thus, using $(\mathbf{M}\times\hat{\mathbf{n}})\times\hat{\mathbf{n}} = -\mathbf{M}$, we find

$$\mathbf{M} = \frac{\mu_r - 1}{\mu_0\mu_r}\mathbf{B}. \tag{12}$$

The factor $\chi_m = \mu_r - 1$ is the magnetic susceptibility (which can be positive or negative). Identifying $\mu = \mu_0\mu_r$ as the magnetic permeability, Eq. (12) can now be rendered succinctly as $\mathbf{M} = \chi_m\mathbf{B}/\mu$. Although derived under the assumption of a uniform external magnetic field, this expression is valid in general for a linear medium.

For arbitrary external magnetic field and arbitrary geometry, Purcell's "magic" can now be recreated for the magnetic fields by repeated application of Ampère's Law. Set $\mathbf{J} = \mathbf{J}_f + \mathbf{J}_b$, where $\mathbf{J}_f$ and $\mathbf{J}_b$ are the free and bound volume current densities, respectively, and let $\mathbf{B}_0$ be the field due to $\mathbf{J}_f$ alone. We then have

$$\mathbf{J}_f = \nabla\times\frac{1}{\mu_0}\mathbf{B}_0 = \nabla\times\frac{1}{\mu_0\mu_r}\mathbf{B}, \tag{13}$$

and the total volumetric current in the material is

$$\mathbf{J}_f + \mathbf{J}_b = \nabla\times\frac{1}{\mu_0}\mathbf{B}. \tag{14}$$

Finally, using Eq. (13) in Eq. (14) yields

$$\mathbf{J}_b = \nabla\times\left(\frac{\mu_\mathbf{r} - 1}{\mu_0\mu_r}\mathbf{B}\right). \tag{15}$$

According to Eq. (12), the vector in parentheses on the right-hand-side of Eq. (15) is the magnetization $\mathbf{M}$, leading to $\mathbf{J}_b = \nabla\times\mathbf{M}$.

For linear, homogenous, isotropic materials, rewriting Eq. (15) with the help of Eq. (13) gives

$$\mathbf{J}_b = (\mu_\mathbf{r} - 1)\nabla\times\left(\frac{1}{\mu_0\mu_r}\mathbf{B}\right) = \chi_m\mathbf{J}_f. \tag{16}$$

In analogy to bound charges, we see that in regions of linear, homogenous, isotropic medium in which free volumetric currents are absent, bound volumetric currents are absent as well.

V. CONCLUDING REMARKS

Expressing the effect of electric and magnetic fields in matter using macroscopic quantities of relative permittivity

(ε_r) and relative permeability (μ_r) is equivalent to averaging contributions of a very large number of microscopic point charges (all the electrons and protons in the material). In principle, one could use Maxwell's equations for a vacuum with all such point charges taken into account explicitly; then the distinction between the free and the bound charges (or currents) would not arise. However, the number of the individual sources in a macroscopic volume would make such a computation prohibitively complicated and thus impractical (see the discussion of this point in the introductory chapter of Jackson's *Classical Electrodynamics*[9]).

The scheme presented here for introducing bound charges and currents may be used as an alternative or a supplement to the standard approach. The main innovation is the treatment of the volumetric effects based on a direct appeal to Gauss's and Ampère's Laws, first for free and then for all charges and currents. This "bookkeeping" leads immediately to the expressions for the bound charges and currents in terms of the polarization and magnetization without any mathematical "sleight-of-hand." Thus, the derivation is grounded in a physically transparent argument.

The symmetry between the discussion of bound charges and currents clearly reflects the broader symmetry between the two fields and the analogous role played by electric and magnetic dipoles. A moving electric dipole, after all, has a magnetic dipole moment and vice-versa.[10] It stands to reason, therefore, that Purcell's argument for bound volumetric charges can be recreated for bound volumetric currents.

While the relative ease of this approach, and the natural way in which the constitutive relations (6) and (12) arise, is pedagogically advantageous, the apparent cost is the loss of generality. However, most materials (excluding ferroelectric and ferromagnetic substances) respond in a linear fashion unless the external fields are very large, so that the discussion presented here is broadly applicable.

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[2] Less frequently used now are "polarization charge," or the somewhat problematic "induced charge." See L. H. Fisher, "On the representation of the static polarization of rigid dielectrics by equivalent charge distributions," Am. J. Phys. **19**, 73–78 (1951).

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[4] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman lectures on physics*, (Addison-Wesley, Reading, MA, 1965).

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[8] The term "displacement" was introduced by Maxwell for the polarization field **P** making its origin clear, but now that implied meaning is lost. See J. Bromberg, "Maxwell's Electrostatics," Am. J. Phys. **36**, 142–151 (1968).

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General Radio Strobotac

The 1935 General Radio catalogue showed the ancestor of this instrument, the Edgerton Stroboscope with its flashing mercury-vapor lamp. By 1939 the Type 631-B Strobotac was in the GR catalogue at a cost of $95; the replacement Strobotron lamp was $5. The flashing rate for the lamp could be varied from 600 to 14,400 flashes per minute, and each flash lasted from 5 to 10 µs. This instrument was widely used in industry to "freeze" the motion of rotating machinery, but in the teaching laboratory I used it for multiple-flash photography of falling bodies and projectiles, used with an open camera lens in a darkened room. The device is in the Greenslade Collection. (Notes and photograph by Thomas B. Greenslade, Jr., Kenyon College)