Properties and applications of the annular filtration on Khovanov homology

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PROPERTIES AND APPLICATIONS OF THE ANNULAR FILTRATION ON KHOVANOV HOMOLOGY

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Abstract

The first part of this thesis is on properties of annular Khovanov homology. We prove a connection between the Euler characteristic of annular Khovanov homology and the classical Burau representation for closed braids. This yields a straightforward method for distinguishing, in some cases, the annular Khovanov homologies of two closed braids. As a corollary, we obtain the main result of the first project: that annular Khovanov homology is not invariant under a certain type of mutation on closed braids that we call axis-preserving.

The second project is joint work with Adam Saltz. Plamenevskaya showed in 2006 that the homology class of a certain distinguished element in Khovanov homology is an invariant of transverse links. In this project we define an annular refinement of this element, κ, and show that while κ is not an invariant of transverse links, it is a conjugacy class invariant of braids. We first discuss examples that show that κ is non-trivial. We then prove applications of κ relating to braid stabilization and spectral sequences, and we prove that κ provides a new solution to the word problem in the braid group. Finally, we discuss definitions and properties of κ in the reduced setting.

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Acknowledgments

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Chapter 1

Introduction

This dissertation is made up of two projects, both of which involve *annular Khovanov homology* (for brevity referred to as AKh). It is an invariant of annular knots and links, that is, knots and links embedded in the thickened annulus $A \times I$. The first project describes the behavior of AKh under certain moves on annular knots. The second project (joint work with Adam Saltz) describes an annular refinement of the transverse invariant in Khovanov homology. In this chapter we introduce these projects and state the main results. Chapter 2 contains the necessary background and context. Chapters 3 and 4 contain more detailed discussion of the results along with their proofs.

1.1 Mutations on closed braids

Given a link L embedded in S^3, locate a sphere $C \subset S^3$, called a Conway sphere, such that C is transverse to L and $|C \cap L| = 4$.

Definition 1.1.1. A **mutation** of L is obtained as follows: cut along C, rotate 180 degrees about an axis disjoint from $C \cap L$ that preserves $C \cap L$ setwise, and reglue C to produce a new link L'.

In this scenario the links L and L' are said to be mutants. Mutation is of interest to knot theorists as it may or may not change the isotopy type of the link. Knot and link invariants often have difficulty distinguishing mutant knots and links. For example, the

Jones, Alexander, and HOMFLY polynomials are invariant under mutation (see [49]). Determining the behavior of an invariant under mutation is one way of measuring its strength and precision.

Khovanov homology (for simplicity denoted Kh), described in more detail in Section 2.2, is a homological invariant for knots or links embedded in S^3. It captures all of the information of the Jones polynomial, and furthermore it is known to be strictly stronger than the Jones polynomial [8]. Many results are now known about the behavior of Khovanov homology under mutation, though it is still not known whether Khovanov homology with $\mathbb{Z}$ coefficients distinguishes mutant knots. Khovanov homology with $\mathbb{Z}/2\mathbb{Z}$ coefficients was shown to be invariant under mutation by Bloom ([19]) and independently by Wehrli and Khovanov homology with $\mathbb{Z}$ coefficients was shown by Wehrli to generally detect mutations that switch components of a link ([89]).

In the annular setting, we will be most interested in a specific type of mutation that we call axis-preserving. Given a link $L \subset A \times I \subset S^3$, denote the point at the center of the annulus A as z.

Definition 1.1.2. An **axis-preserving mutation** is a mutation as in Definition 1.1.1 such that the axis of rotation contains the line segment $z \times I$.

In general, axis-preserving mutations may change the isotopy class of the link in $A \times I$ while preserving the isotopy class in S^3, and hence studying these types of mutations seemed likely to reveal interesting differences between AKh and Kh.

It is an observation of Wehrli that AKh is not invariant under axis-preserving mutation for knots. His example is shown in Figure 1.1, where the dotted circle denotes the intersection of the Conway sphere with the annulus. After undoing the trivial kink in both knots, we see that this mutation switches a negatively stabilized unknot with a positively stabilized unknot. These two knots are not isotopic in $A \times I$, and a quick calculation yields that they have distinct AKh. Indeed, the knot on the left has a generator of AKh in homological grading -1 and none in grading 1, while the knot on the right has a generator of AKh in homological grading 1 and none in grading -1.

This example shows that in the annular setting mutation can be quite a powerful move on knots, since adding trivial kinks allows us to switch crossings. With this in mind,

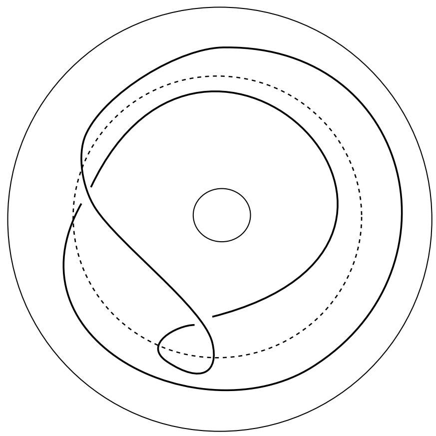

Figure 1.1: Wehrli's example

it is perhaps not surprising that AKh can distinguish annular knots or links under axis-preserving mutation.

It is natural to ask next how annular Khovanov homology behaves under axis-preserving mutation on braids, where such trivial kinks are not allowed. Indeed, AKh is a natural tool for studying braids: see Section 2.3. We assume that braids are embedded in the natural way in $A \times I$: that is, the axis from Definition 1.1.2 is precisely the braid axis. Axis-preserving mutations are particularly interesting in the braid setting. For instance, exchange moves and flypes - moves that were studied by Birman and Menasco in their work on the braid group and applications to contact geometry - are special cases of axis-preserving mutations on braids (see Figure 1.2: the x represents the braid axis, and the w stands for w strands).

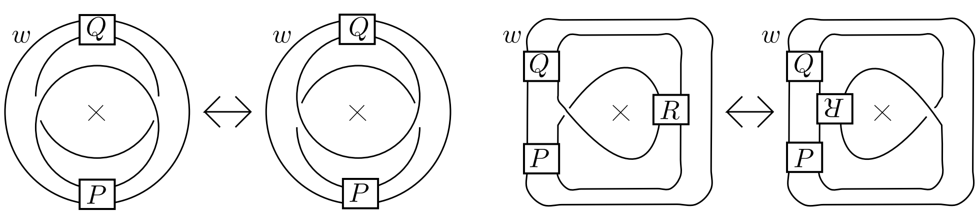

Figure 1.2: Exchange moves (left) and negative flypes (right)

The main result of this project is:

Theorem 1. *The annular Khovanov homology of a closed braid is not invariant under an axis-preserving mutation. Indeed, there exist infinite families of mutant 4-braid pairs and* *mutant 5-braid pairs, shown in Figures 1.3 and 1.4, whose annular Khovanov homologies differ.*

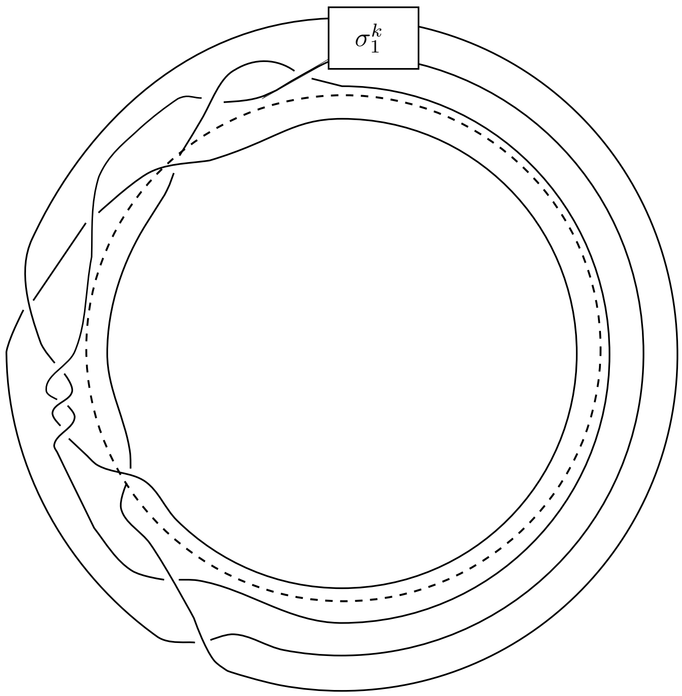

Figure 1.3: Infinite family of 4-braid mutants whose AKh differ; k is an integer ≥ 0

Again, the dotted circles in Figures 1.3 and 1.4 represent the intersection of the Conway sphere with the annulus.

Note: The mutant pairs in Theorem 1 are related by exchange moves. Annular Khovanov homology can also distinguish mutants that are related by negative flypes. In particular, the pair

$$\sigma_3^2\sigma_2^2\sigma_3^{-1}\sigma_1^2\sigma_2\sigma_1^{-1} \text{ and } \sigma_3^2\sigma_2^2\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_1^2$$

from [74], both representing the knot 7_2 and related by a negative flype, are distinguished by AKh. In addition, the pair

$$\sigma_3\sigma_2^{-2}\sigma_3^2\sigma_2\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_1^2 \text{ and } \sigma_3\sigma_2^{-2}\sigma_3^2\sigma_2\sigma_3^{-1}\sigma_1^2\sigma_2\sigma_1^{-1}$$

from [51] (see also [74]), both representing 10_{132} and related by a negative flype, are also

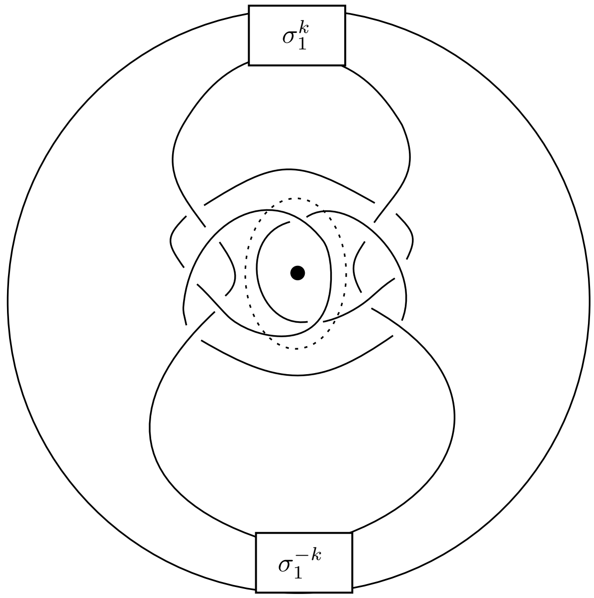

Figure 1.4: Infinite family of 5-braid mutants whose AKh differ; k is an integer ≥ 0

distinguished by AKh.

We emphasize here that AKh is *not* a transverse invariant since in general it is not preserved under positive stabilization. Indeed, Theorem 1 shows that AKh is not invariant under exchange moves, which are transverse isotopies. So in particular, we cannot conclude from this calculation that the negative flype pairs mentioned above are transversely non-isotopic (though this is true, see [51] and [74]).

The example shown in Figure 1.3 appears in [15] with $k = 0$ (building on work of Morton in [72]) as the intermediate and third braid in a series of three braids related by exchange moves, where the first does not admit a destabilization and the third does. The example shown in Figure 1.4 was suggested by Menasco.

Given a braid β and its associated closed braid $\bar{\beta}$, denote the classical unreduced Burau representation (as described in [14]) with variable t as $\Phi(\beta, t)$. We denote the graded Euler characteristic of $\mathrm{AKh}(\bar{\beta})$ as $\chi_{\mathrm{AKh}}(\bar{\beta})$. The relationship between the Burau representation of a braid and the $U_q(sl_2)$ Reshetikhin-Turaev invariant is well-known among experts (cf. [48], [46], [47]), and a relationship between the Reshetikhin-Turaev invariant and the graded Euler characteristic of annular Khovanov homology is described in [40], building on work in Khovanov's thesis [55] (see also [54]). In this work we recover the relationship between the Burau representation and annular Khovanov homology explicitly in Khovanov's diagrammatic language:

Theorem 2. *Given an n-braid β with n_+ positive crossings and n_- negative crossings,*

$$\chi_{\mathrm{AKh}}(\bar{\beta})|_{k=n-2} = (qt)^{n-2}(q)^{n_+-n_-}\,\mathrm{Tr}\left(\Phi(\beta, q^2)\right)$$

That is, the trace of the Burau representation of a braid can be recovered from the AKh of its closure.

Theorem 1 is a consequence of Theorem 2 along with calculations of the corresponding traces. Indeed, Theorem 2 gives a useful method for distinguishing the annular Khovanov homologies of some closed braids:

Corollary 3. *Suppose two n-braids β_1 and β_2 have the same exponent sum. If the traces of the Burau representations of β_1 and β_2 differ, then the annular Khovanov homologies of the two closed braids differ as well.*

We also note here that AKh cannot *always* distinguish mutant closed braids. For instance, in Corollary 2 of [6], Baldwin and Grigsby (using a result of Birman and Menasco in [17]) proved that there exist infinitely many pairs of non-conjugate mutant closed 3-braids with the same AKh.

1.2 An annular refinement of Plamenevskaya's invariant

The work on this project was joint with Adam Saltz. What follows in this section and in Chapter 4 will appear, with some small changes, in the journal *Algebraic & Geometric Topology.* The publication will be titled "An annular refinement of the transverse element in Khovanov homology".

Khovanov homology has proven to be a powerful tool for studying links and link cobordisms in S^3. Given a link L with diagram $\mathcal{D}$, the homology of the bigraded Khovanov chain complex $\mathrm{CKh}(\mathcal{D})$ is a link invariant denoted $\mathrm{Kh}(L)$. In [80], Plamenevskaya constructs from Khovanov homology an invariant of transverse links presented as braid closures. (Recall that Orevkov and Shevchishin [75], and independently Wrinkle [90], have shown that there is a one-to-one correspondence between transverse links in S^3 up to transverse isotopy and braids up to positive stabilization and isotopy. We review this correspondence in Section 2.1.4.) Recall that the Khovanov chain complex is constructed by assigning a vector space to each complete resolution of a diagram. This vector space over $\mathbb{Z}/2\mathbb{Z}$ is generated by labelings of the components in each resolution by the symbols v_+ and v_-. The diagram of an n-strand braid closure L has a unique resolution into an n-strand unlink, and the transverse element $\psi(L)$ is the labeling of each component of this unlink by v_-. It is easy to see that $\psi(L)$ is a cycle.

Theorem. *[80] Let L and L' be transversely isotopic transverse links with braid closure diagrams $\mathcal{D}$ and $\mathcal{D}'$, respectively. Then any sequence of transverse Reidemeister moves connecting the two diagrams induces a map* $\mathrm{CKh}(\mathcal{D}) \to \mathrm{CKh}(\mathcal{D}')$ *which sends $\psi(L)$ to $\psi(L')$.*

This implies that the homology class $[\psi(L)]$ (and indeed, the chain $\psi(L)$) is a transverse invariant which detects the classical self-linking number. An invariant of transverse links is called *effective* if it takes on different values for a pair of transverse links with the same self-linking number and smooth link type. A smooth link type is called *transversely non-simple* if it supports transversely non-isotopic links with the same self-linking number. It is not known if Plamenevskaya's invariant is effective.

Given a link L equipped with an embedding into a thickened annulus (i.e. $L \subset A \times I \subset S^3$), its Khovanov chain complex can be endowed with an additional grading which we call the k-grading, first studied by [5] and [85]. For a resolution with a single component, $k(v_\pm) = \pm 1$ if the component is not null-homotopic in $A \times I$, and $k(v_\pm) = 0$ otherwise. We extend the grading to tensor products by summation. The Khovanov differential is non-increasing in the k-grading, which induces a filtration on the Khovanov complex. The homology of the associated graded chain complex is *annular Khovanov homology*, denoted here as $\mathrm{AKh}(L)$ (elsewhere also called *sutured annular Khovanov homology* or *sutured Khovanov homology* and denoted $\mathrm{SKh}(L)$). As mentioned in Section 1.1, AKh is an invariant of annular links and not a transverse invariant. (See Section 2.3 for more details.) For a braid closure $\bar{\beta}$, the element $\psi(\bar{\beta}) \in \mathrm{CKh}(\bar{\beta})$ is the unique element with lowest k-grading.

Standard algebraic machinery (see [45] for an introduction and [70] for a thorough treatment) produces a spectral sequence from the associated graded object of a filtered complex to the homology of that complex and therefore from AKh to Kh. Our original goal in this work was to define a (perhaps effective) transverse invariant by exploring the behavior of Plamenevskaya's class in this spectral sequence. AKh is known to distinguish some braids whose closures are smoothly isotopic but not transversely isotopic (see Section 1.1), and so it is natural to suspect that the spectral sequence from AKh to Kh also captures non-classical information.

In this work we define a refinement of Plamenevskaya's invariant that measures how long $\psi(L)$ survives in the spectral sequence, or equivalently, the lowest filtration level at which the class of $\psi(L)$ vanishes. For a braid β with closure $\bar{\beta}$, write $\mathcal{F}_i(\bar{\beta}) = \{x \in \mathrm{CKh}(\bar{\beta}) : k(x) \leq i\}$.

Definition 1.2.1. Let β be an n-strand braid with closure $\bar{\beta}$ and suppose that $\psi(\bar{\beta})$ is a boundary in $\mathrm{CKh}(\bar{\beta})$. Define

$$\kappa(\beta) = n + \min\{i : [\psi(\bar{\beta})] = 0 \in H(\mathcal{F}_i)\}.$$

If $\psi(\bar{\beta})$ is not a boundary then define $\kappa(\beta) = \infty$.

However, $\kappa(\beta)$ is a conjugacy class invariant of braids rather than a transverse invariant.

Theorem 4. *κ is an invariant of conjugacy classes in the braid group B_n. It may increase by 2 under positive stabilization and is thus not a transverse invariant.*

Nevertheless, κ can distinguish conjugacy classes of some braids whose closures are transversely non-isotopic but have the same classical invariants.

Proposition 5. *For any $a, b \in \{0, 1, 2\}$, the pair of closed 4-braids*

$$A(a,b) = \sigma_3\sigma_2^{-2}\sigma_3^{2a+2}\sigma_2\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_1^{2b+2} \quad \textit{and}$$

$$B(a,b) = \sigma_3\sigma_2^{-2}\sigma_3^{2a+2}\sigma_2\sigma_3^{-1}\sigma_1^{2b+2}\sigma_2\sigma_1^{-1},$$

related by a negative flype, can be distinguished by κ: indeed, $\kappa(A(a,b)) = 4$ and $\kappa(B(a,b)) = 2$. For any pair (a,b), the braids $A(a,b)$ and $B(a,b)$ are transversely non-isotopic but have the same classical invariants [51].

Negative flypes are currently the primary move on braids that is known to sometimes produce transversely non-isotopic pairs that share the same classical invariants. Lipshitz, Ng, and Sarkar, using a filtered refinement of $\psi(L)$ valued in the Lee–Bar-Natan deformation of Khovanov homology, showed that Plamenevskaya's class is invariant under negative flypes [62]. The above proposition could be seen as evidence that κ carries non-classical information even if ψ does not.

κ has nice properties mirroring those of ψ, and our calculations have some interesting consequences. In Section 4.2 we collect these observations. In particular, we show using Proposition 5 that the spectral sequence from AKh to Kh does not necessarily collapse immediately, providing a counterexample to Conjecture 4.2 from [44]. In addition, our work together with that of Baldwin and Grigsby in [6] provides a solution (faster than that of [6]) to the word problem for braids.

Recall that the Khovanov chain complex has two reduced variants obtained by placing a basepoint on the link diagram [53]. The homologies of these complexes are isomorphic as bigraded objects up to a global grading shift. The behavior of κ under positive stabilization provided some promise that a reduced analogue of κ might be a transverse invariant. In Section 4.3 we define κ for both versions of reduced Khovanov homology. However, these constructions depend on the placement of the basepoint. We still have some hope that these reduced constructions will provide non-classical transverse information. In any case, the fact that the two reduced variants are largely independent demonstrates that the two reductions of Khovanov homology are quite different with respect to the k-grading.

This project was inspired by similar spectral sequence constructions in Floer homology. Let (Y, ξ) be a contact three-manifold. Recall that there are elements $c_\xi \in \widehat{HF}(Y)$ (Heegaard Floer homology) and $\emptyset_\xi \in ECH(Y)$ (embedded contact homology) which are invariants of ξ. It is known that each of these elements vanishes if (Y, ξ) is overtwisted ([78], [24]) or if (Y, ξ) contains *Giroux n-torsion* for any $n > 0$ ([35]) (both converses are false). In [60], Latschev and Wendl study *algebraic torsion* in symplectic field theory and show that it can obstruct fillability. Hutchings adapts this work to embedded contact homology by constructing a relative filtration on $ECH(Y)$. He defines the algebraic torsion of the contact element to be the lowest filtration level at which $\emptyset_\xi$ vanishes. As ECH is known to be isomorphic to $\widehat{HF}$ (see [61]) by an isomorphism carrying $\emptyset_\xi$ to c_ξ, it is reasonable to suspect that there is an analogous construction in Heegaard Floer homology . This is the subject of ongoing work by Baldwin and Vela-Vick and independent work by Kutluhan, Matić, Van-Horn Morris, and Wand in [59].

Now let L be a link with mirror $m(L)$, and let $\Sigma(L)$ be the double cover of S^3 branched over L. There is a spectral sequence $E^i(L)$ so that $E^2 \cong \widetilde{CKh}(m(L))$ and $E^\infty = \widehat{HF}(\Sigma(L))$ [79]. If L is a transverse link then $\Sigma(L)$ inherits a contact structure $\xi(L)$. Plamanevskaya conjectured [80] and Roberts proved [85] (see also [7]) that $\psi(L)$ "converges" to $c_{\xi(L)}$ in the sense that there is some $x \in E^0(L)$ so that $[x]_2 = \psi(L) \in E^2(L)$ and $[x]_\infty = c_{\xi(L)} \in E^\infty(L)$. This is a weak sort of convergence – in particular, the vanishing or non-vanishing of the two elements are independent – but it has been used fruitfully in e.g. [7]. We hope to use this connection to derive contact-theoretic information from κ.

Chapter 2

Background

2.1 Knots, braids, and contact geometry

2.1.1 Knots and their invariants

A *knot* is a smooth embedding of a circle S^1 into $\mathbb{S}^3$ considered up to smooth isotopy. The very simplest example of a knot is the *unknot*, that is, the unknotted circle. It is standard to represent a knot with a *knot diagram*: a projection of a knot onto a plane, with the over- and under- strands marked at double points (see Figure 2.1). A *link* is a collection of knots that may be linked or knotted together. For ease of exposition, the word "knot" will often be used in this dissertation to refer to both knots and links.

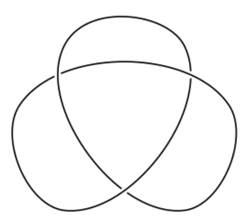

Figure 2.1: The trefoil knot

Historically, one of the central questions of knot theory has been recognition: that is, how to tell if two given knot diagrams represent the same knot within a reasonable computational time frame. A wide variety of *knot invariants* were originally defined to shed light on this question. A knot invariant is an assignment of some mathematical "object" (for example, an integer, a polynomial, a group) to a knot such that any two knot diagrams representing the same knot are given the same assignment. Knot invariants are an active area of study today not only because of their ability to shed light on the recognition problem, but also because of their potential to give information on which properties some knots may share. There are many knot invariants; some examples including tricolorability, the crossing number, the knot genus, the Alexander and Jones polynomials, Khovanov homology, and knot Floer homology, among others.

Most knot invariants fall into one of two groups: ones that can be calculated directly from any knot diagram for a knot, and ones that are defined to be a minimum value of some integer-valued function over all possible knot diagrams of a given knot. Invariance of knot invariants in the latter group is by definition, and the strategy for showing invariance of constructions in the former group is straightforward. Reidemeister [82] proved that any two knot diagrams of a given knot can be related by a combination of three local moves, called *Reidemeister moves*. In order to show that a candidate for a knot invariant is indeed invariant, it is sufficient to show that it is unchanged under these moves.

2.1.2 The braid group and closed braids

Over the past thirty years, it has been fruitful for mathematicians to view knots as closed braids. The *braid group on n strands*, first explicitly introduced by Emil Artin in [4], has the presentation

$$B_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{l} \sigma_i\sigma_j = \sigma_j\sigma_i : |i-j| \geq 2 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} : 1 \leq i \leq n-2 \end{array} \right\rangle$$

where σ_i corresponds to a positive half-twist between the i^{th} and $(i+1)^{\text{st}}$ strand. Braids have a natural geometric representation (see Figure 2.2). If we connect the ends of a braid as shown in Figure 2.2, the result is called the *closure* of the braid. The closure of a braid is a knot (or link); conversely, Alexander showed in [1] that every knot (and link) in S^3 can be represented as a closed braid. Given a standard diagram of a braid closure, we refer to the line that is orthogonal to the page and pierces the "center point" (labeled with an x on Figure 2.2) of the closed braid as the *braid axis*.

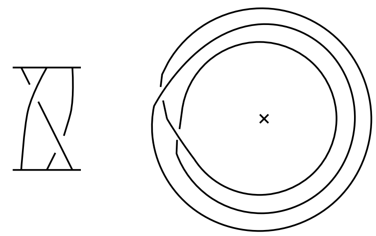

Figure 2.2: On the left, the 3-braid $\sigma_1\sigma_2^{-1}$. On the right, its closure.

Just as Reidemeister's theorem in Section 2.1.1 gave a method for relating two different knot diagrams for the same knot, Markov's theorem [68] gives a method for relating two braid words whose closures represent the same knot. The theorem states that the closures of two braid words $\beta \in B_n$ and $\beta' \in B_m$ are isotopic as knots in S^3 if they are related by some combination of the relations in the braid group, conjugation in a given braid group, and positive/negative (de)stabilization. *Positive stabilization* replaces $\beta \in B_n$ by $\beta\sigma_n^{+1} \in B_{n+1}$, and positive destabilization is the reverse (see Figure 2.3). *Negative stabilization* replaces $\beta \in B_n$ by $\beta\sigma_n^{-1} \in B_{n+1}$, and negative destabilization is the reverse.

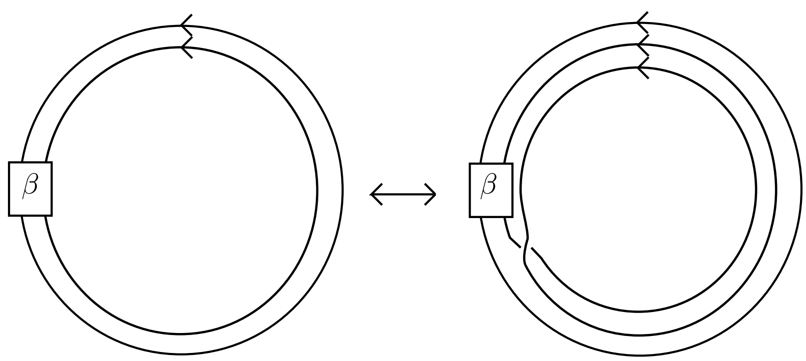

Figure 2.3: From left to right: positive stabilization. From right to left: positive destabilization.

It is certainly not always possible to destabilize a closed braid. Every knot has a minimum number of strands necessary to represent it as a closed braid; this integer is often called the *braid index*. However, it may be impossible to destabilize a closed braid even if it has more strands than the braid index for the corresponding knot. Indeed, Morton in [72] gave an example of a closed 4-strand representative of the unknot that could not be destabilized. Markov's theorem implies that in order to reduce the braid to the standard 1-strand representative of the unknot, it would be necessary to first stabilize it, thereby increasing the strand number.

Detecting whether a closed braid admits a destabilization (that is, detecting whether a braid $\beta \in B_{n+1}$ is conjugate to $\alpha\sigma_n^{\pm 1}$ for some $\alpha \in B_n$) is quite difficult. There exist two complete algorithms that solve this problem, one due to Menasco using the theory of braid foliations ([71]) and one due to Malyutin taking the perspective of mapping class groups ([67]). However, both are in some sense theoretical algorithms, since they are quite difficult to apply. (For example, the algorithm in [67] involves first determining whether a braid word corresponds to a periodic, reducible, or pseudo-Anosov mapping class - itself not an easy problem.)

In [16], Birman and Menasco proved what they referred to as the "Markov theorem without stabilization". In that work, they replaced the stabilization move in Markov's theorem with a collection of moves that instead preserve or reduce the strand number. Two examples of such moves which will be of interest in this work - exchange moves and flypes - are shown in Figure 2.4. The figure shows only negative flypes; for a positive flype, simply switch the negative crossing in the diagram to a positive crossing. The w in the figure represents w strands. Birman and Menasco showed in [15] that the Morton example from the previous paragraph could, by a series of two exchange moves, be changed to a braid that admits a destabilization.

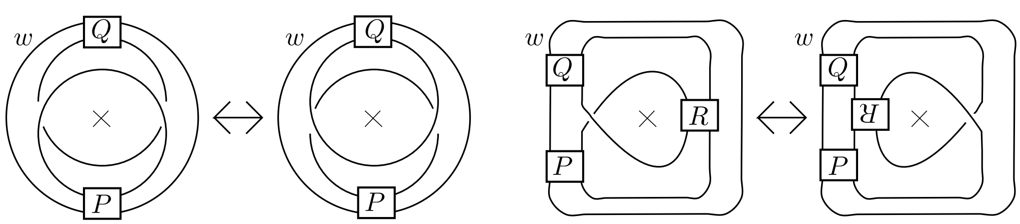

Figure 2.4: Exchange moves (left) and negative flypes (right)

Knot theorists find braids appealing partly because many computations within the braid group are possible. For example, the *word problem* and *conjugacy problem* are both solved in the braid group with efficient algorithms. The word problem is the following: given two words β, β' in the standard n-braid group generators, is there an algorithm to determine if $\beta = \beta' \in B_n$? The first solution to the word problem in the braid group was given by Artin in 1925 [4] (see also [3]). The approach commonly used today (with improvements by a variety of mathematicians) was originally given by Garside in 1969 [32]. More details on the history of the word problem in the braid group can be found in the survey paper by Birman and Brendle [14].

The conjugacy problem in the braid group is similar to the word problem: given two words β, β' in the standard n-braid group generators, is there an algorithm to determine if β is conjugate to β' in B_n? The conjugacy problem is of particular interest to knot theorists, since conjugate braids are isotopic as closed braids (and hence two conjugate braids close to the same knot). The conjugacy problem was also solved by Garside in 1969 [32], and his approach has since been sharpened by contributions of several other mathematicians. Again, see [14] for details.

2.1.3 Braid representations

The braid group has been extensively studied through the lens of representations to matrix groups. One classical such representation is the *Burau representation*, introduced by Burau in 1936 and described in detail in [14]. It is a homomorphism $\Phi : B_n \to GL_n(\mathbb{Z}[t, t^{-1}])$ defined as follows:

$$\sigma_i \mapsto I_{i-1} \oplus \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1}$$

Later on in this dissertation, when the choice of variable will be important, we will write this representation as $\Phi(\beta, t)$, taking as input both a braid β and the variable t. The Burau representation was historically of interest since it was thought to be a candidate for showing that braid groups are linear. However, it is now known that it is not faithful for $n \geq 5$ ([66], [64],[11]). For $n = 4$, the question is open, and showing that it fails faithfulness for $n = 4$ would likely show that the Jones polynomial does not detect the unknot [13].

Bigelow [12], using topological methods, and later Krammer [56], using algebraic methods, showed that the braid group is indeed linear for all n using a representation called the *Lawrence-Krammer representation*. It is a faithful homomorphism $\lambda : B_n \to GL_r(\mathbb{Z}[t^{\pm 1}, q^{\pm 1}])$.

2.1.4 Contact geometry and transverse knots

A *contact structure* on a 3-manifold M is a plane field ξ such that for any 1-form α with $\xi = \ker\alpha$, $\alpha \wedge d\alpha \neq 0$. This condition implies that there does not exist any surface in M whose tangent plane field agrees with ξ. The classic first example of a contact structure is $\xi_{std} = \ker(dz - ydx)$ on S^3, known as the *standard* contact structure on S^3. The contact structure of most interest in this dissertation is the *symmetric* contact structure on S^3, that is, $\xi_{sym} = \ker(dz + r^2 d\theta)$. In fact, these two contact structures are equivalent as there is a diffeomorphism of S^3 taking ξ_{sym} to ξ_{std}, and hence can both be called the "standard" contact structure; for our purposes, working with ξ_{sym} will be more convenient. See Figure 2.5 (this image was created by S. Schönenberger, and appears in Etnyre's survey [28]).

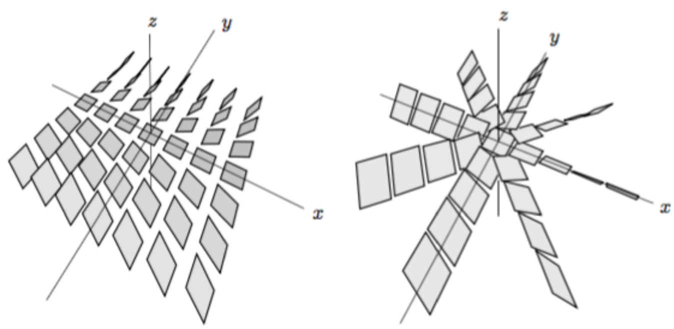

Figure 2.5: On the left, ξ_{std}, and on the right, ξ_{sym}. Figure courtesy of Schönenberger via Etnyre.

In 1971 Martinet proved that all closed, orientable 3-manifolds admit contact structures [69]; Lutz further showed that they admit a contact structure in each homotopy class of 2-plane fields [65]. For alternative proofs, see [87], [37], and [2]. Classifying all contact structures on a given 3-manifold has been a question of much interest. Due to work of many mathematicians (including Bennequin, Eliashberg, Etnyre, and Honda, among others), contact structures on some manifolds - for example, S^3 and lens spaces - have been classified (see [22], [10], [36], [42], [26]). However, this classification is unknown for most 3-manifolds.

Contact structures can give information about topological properties of the manifolds that admit them, and so the study of contact structures has many applications in low-dimensional topology. For example, contact geometry played a key role in Kronheimer and Mrowka's proof of the Property P conjecture in knot theory [57]. For more discussion of contact structures and their applications, see [34], [27], [28], and [29].

Transverse knots are knots that have the geometric property of being everywhere transverse to the standard contact structure $\xi_{sym} = \ker(dz + r^2 d\theta)$ on S^3. ("Transverse knots" could certainly be defined with respect to other contact structures and other manifolds, but already this case is not well understood.) The study of transverse knots is an important component of understanding contact structures on 3-manifolds: for example, studying transverse knots was part of the approach Bennequin used to distinguish non-standard contact structures on S^3 [10]. However, little is currently known about the classification of transverse knots. (While they will not be studied in this dissertation, Legendrian knots - knots that are everywhere tangent to the standard contact structure - are also of much mathematical interest.)

Transverse knots have a single classical invariant, the self-linking number [28]. We call a topological knot type transversely simple if its transverse representatives are classified by their self-linking number. A relatively small collection of knots are known to be transversely simple, including the unknot [23], torus knots [25], and the figure-eight knot [30]. In 2005-06 it was shown by Birman and Menasco and separately by Etnyre and Honda that there exist transversely non-simple knots ([17], [16], [31]), and there are now more families known (see [74, 88, 50, 51, 76, 21]). In general it is quite challenging to identify these non-simple families.

Transverse knots have a close mathematical relationship to braids. If we suppose that a given closed braid is wrapped around the z-axis (see Figure 2.5), we see that it is naturally transverse to the standard contact structure $\xi_{sym} = \ker(dz + r^2 d\theta)$ on S^3 and hence can be thought of as a transverse knot. (Notice that this implies that every knot type has a transverse representative.) Conversely, Bennequin proved that every transverse knot is transversely isotopic to a closed braid [10]. The self-linking number is straighforward to calculate from a closed braid representative β of a given transverse knot:

$$sl(\beta) = a(\beta) - n$$

where $a(\beta)$ is the sum of the exponents of β and n is the strand number.

In the early 2000s, Orevkov and Shevchishin (and independently Wrinkle) proved a "Markov theorem" for braids as transverse knots. Their result is that the closures of two braid words are isotopic as *transverse* knots in S^3 if they are related by the braid relations, conjugation, and only positive (de)stabilization ([75], [90]). This makes finding new potentially non-classical transverse invariants a more straightforward task: it is sufficient to find a mathematical quantity that one can calculate from a braid diagram and check if it is invariant under these three moves. This was the strategy used by Plamenevskaya in [80] (see Section 2.2.3).

2.2 Khovanov homology

2.2.1 Definition and applications

Khovanov homology, defined in the late 1990s by Khovanov [52], is a powerful knot invariant. In this exposition, we follow the notation introduced by Bar-Natan in [8]. A short description of the definition is given here; see [8] for details (additionally, the point of view in [9] is also quite useful). We work throughout with coefficients in $\mathbb{Z}/2\mathbb{Z}$.

Given a diagram for a link L, we pick an order for the n crossings of a projection of L, and associate to each vertex of the cube $\{0,1\}^n$ a resolution of L by resolving the crossings of the projection according to the rule in Figure 2.6. The resulting object is called the cube of resolutions of L.

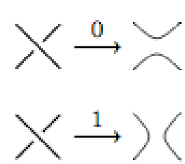

Figure 2.6: Resolutions of crossings

One now associates a chain complex to the cube of resolutions. To each circle in a resolution we assign a copy of V, a vector space generated by two basis elements v_+ and v_-. V is endowed with a grading referred to as the *quantum* grading (or q-grading) as follows: $gr_q(v_\pm) = \pm 1$. To each vertex of the cube of resolutions we associate the vector space $V^{\otimes c}\{i(\mathcal{I})\}$, where c is the number of circles in that resolution, and $\{i(\mathcal{I})\}$ denotes a shift in the q-grading by the *height* $i(\mathcal{I})$, that is, $i(\mathcal{I})$ is the number of 1's in the vertex $\mathcal{I}$.

We set the r'th chain group $\mathrm{CKh}(L)^r$ to be the direct sum of all of the vector spaces at height r. To each edge of the cube of resolutions, we associate a map between the vector spaces at either end; the map is determined by whether, topologically, one circle is splitting into two or two circles are merging into one. The maps are defined to be the identity on tensor factors corresponding to circles that are not involved, and on the involved tensor factors are:

$$\left.\begin{aligned} &v_+ \otimes v_+ \mapsto v_+ \\ &v_+ \otimes v_-, v_- \otimes v_+ \mapsto v_- \\ &v_- \otimes v_- \mapsto 0 \end{aligned}\right\} \qquad \text{Merge maps}$$

$$\left.\begin{aligned} &v_+ \mapsto v_+ \otimes v_- + v_- \otimes v_+ \\ &v_- \mapsto v_- \otimes v_- \end{aligned}\right\} \qquad \text{Split maps}$$

The differential ∂^r on the chain groups is defined to be the sum of all the maps from a vector space of height r to a vector space of height $r+1$. We now have a chain complex, $\mathrm{CKh}(L)$. Finally, after a shift of $-n_-$ in the homological grading and a shift of $n_+ - 2n_-$ in the q-grading (where n_+ is the number of positive crossings, and n_- is the number of negative crossings), we take the homology of this chain complex. The result is the Khovanov homology of L, a bigraded homology theory denoted $\mathrm{Kh}(L)$. It is invariant under the Reidemeister moves, and hence is an invariant of L up to smooth isotopy in S^3.

The Poincaré polynomial of Khovanov homology is:

$$\mathrm{Kh}(L)(q,t) = \sum_{i,j} t^i q^j \dim(\mathrm{Kh}(L)^{i,j})$$

where i denotes the homological grading and j denotes the q-grading. Plugging in -1 for t gives what is known as the *graded Euler characteristic* of Khovanov homology:

$$\chi_{\mathrm{Kh}}(L) = \sum_{i,j} (-1)^i q^j \dim(\mathrm{Kh}(L)^{i,j})$$

This is precisely equal, by construction, to the unnormalized Jones polynomial for L. This relationship between the Euler characteristic and the Jones polynomial is (partly) what is meant by the statement that Khovanov homology is the *categorification* of the Jones polynomial. For details, see [8].

Khovanov homology therefore captures all of the information of the Jones polynomial. Furthermore, Bar-Natan showed in [8] that it is strictly stronger than the Jones polynomial, as there are pairs of knots distinguished by Khovanov homology that are not distinguished by their Jones polynomials. While it is not known whether the Jones polynomial detects the unknot, Kronheimer and Mrowka showed in [58] that Khovanov homology does (see also [41]). Furthermore, it has several rich connections to low-dimensional topology and contact geometry. In [81] Rasmussen proved that Khovanov homology provides a lower bound for the smooth slice genus of a knot, and in [73] Ng showed that it gives a bound on the Thurston-Bennequin number of a Legendrian knot. Khovanov homology also has a rich connection to Heegaard Floer homology [77] (a powerful invariant of closed oriented 3-manifolds): there is a spectral sequence with E^2 page the (reduced, ref. Section 2.2.2) Khovanov homology of a knot K converging to the Heegaard Floer homology of the double branched cover of S^3 branched over K [79]. This spectral sequence has been the object of much study and interest.

2.2.2 Reduced Khovanov homology

Let p be a marked non-double point on a diagram $\mathcal{D}$ of a knot K. For convenience, we will assume that the last tensor factor of each generator of $\mathrm{CKh}(\mathcal{D})$ corresponds to the component containing p. There is a chain map $x_p\colon \mathrm{CKh}(\mathcal{D}) \to \mathrm{CKh}(\mathcal{D})$ defined on generators by

$$x_p(y \otimes v_+) = y \otimes v_-$$
$$x_p(y \otimes v_-) = 0$$

There are two flavors of *reduced Khovanov homology* [53]. The *reduced subcomplex* $\widetilde{\mathrm{CKh}}_p(\mathcal{D})$ is defined as $\ker(x_p)$. The *reduced quotient complex* $\underline{\mathrm{CKh}}_p(\mathcal{D})$ is defined as $\mathrm{coker}(x_p)$. The homologies of these complexes are both called reduced Khovanov homology. It is straightforward to check that the chain groups are isomorphic and the differential acts identically on the two chain groups, and hence this ambiguity is justified by the fact that their homologies are isomorphic as h- and q-graded complexes (with a constant shift in the q-gradings). The homology of the reduced quotient complex is the version of reduced Khovanov homology that is cited most commonly in the literature.

Reduced Khovanov homology is invariant under sliding the basepoint around the knot (or, if we are considering a link, sliding the basepoint around the component on which it is placed) [53]. The proof hinges on the observation that two marked diagrams $(\mathcal{D}, p)$ and $(\mathcal{D}', p')$ (where $\mathcal{D}$ and $\mathcal{D}'$ both represent K) can be related via Reidemeister moves that do not require sliding crossings over or under the marked point. Indeed, any arcs that need to be pulled over each other can instead be pulled around S^2 in the opposite direction.

2.2.3 Plamenevskaya's transverse invariant

In 2006, Plamenevskaya showed that the homology class of a distinguished element in the Khovanov chain complex is an invariant of transverse knots [80]. The construction is as follows. Suppose β is an n-strand braid whose closure $\bar{\beta}$ is a representative of a transverse link L. Let $\mathcal{D}$ be a diagram for $\bar{\beta}$. Consider the resolution of $\bar{\beta}$ which is given by taking the 0-resolution for each positive crossing and the 1-resolution for each negative crossing. This resolution looks like the trivial n-braid. The element $\psi(\mathcal{D})$ is defined as the generator in this resolution with only v_- labels. Suppose that β' is another braid whose closure is transversely isotopic to L. Then for any sequence of conjugations and positive (de)stabilizations that transforms β into β', Plamenevskaya showed that the naturally induced map $\mathrm{CKh}(\bar{\beta}) \to \mathrm{CKh}(\bar{\beta}')$ carries $\psi(\bar{\beta})$ to $\psi(\bar{\beta}')$. Thus $\psi(L) \in \mathrm{CKh}(L)$ is well-defined and the homology class of $\psi(L)$ is a transverse invariant.

This invariant is at least as strong as the classical self-linking number since an easy calculation shows that the q-grading of $\psi(L)$ is in fact the self-linking number of L. It is not known if it is effective; that is, it is not known if there exists a family of transversely non-simple knots that can be distinguished using Plamenevskaya's invariant.

2.3 Annular Khovanov homology

2.3.1 Definition and applications

Annular Khovanov homology (often referred to in the literature as *sutured Khovanov homology* or *sutured annular Khovanov homology*), first constructed by Asaeda, Przytycki, and Sikora in [5] and related to knot Floer homology by Roberts in [85], is an invariant for links equipped with an embedding into a thickened annulus $A \times I$. Specifically, $A \times I$ is embedded in $\mathbb{R}^2 \times \mathbb{R}$, and we project a link L into A embedded in $\mathbb{R}^2$. We call such links *annular*. As in the construction of Khovanov homology, we pick an order for the n crossings of a projection of L, and associate to each vertex of the cube $\{0,1\}^n$ a resolution of L by resolving the crossings of the projection according to the rule in Figure 2.6.

One now associates a chain complex to the cube of resolutions. A circle in a resolution is said to be trivial if it bounds a disk in A, and non-trivial if not. As in Khovanov homology, to each circle in a resolution we assign a copy of V, a vector space generated by the two basis elements v_+ and v_-. Here V is endowed with two gradings, the standard Khovanov q-grading and an extra k-grading. The k-grading is assigned as follows: $gr_k(v_\pm) = \pm 1$ when the corresponding V is assigned to a non-trivial circle, and $gr_k(v_\pm) = 0$ when the corresponding V is assigned to a trivial circle. As before, to each vertex of the cube we associate the vector space $V^{\otimes c}\{i(\mathcal{I})\}$, where c is the number of circles in that resolution, and $\{i(\mathcal{I})\}$ denotes a shift in the q-grading by the height $i(\mathcal{I})$, that is, $i(\mathcal{I})$ is the number of 1's in the vertex $\mathcal{I}$.

Roberts [85], following [5], shows that the Khovanov differential is non-increasing in k. Thus the subcomplexes $\mathcal{F}_i(\mathcal{D}) = \{x \in \mathrm{CKh}(\mathcal{D}) : k(x) \leq i\}$ form a bounded filtration of $\mathrm{CKh}(L)$. Moreover, the filtered chain homotopy type of $\mathrm{CKh}(\mathcal{D})$ is an invariant of L as an annular link. For a filtered complex $(X', d', \mathcal{F}'_i)$ the *associated graded object* is the direct sum of complexes $\bigoplus_i \mathcal{F}'_i/\mathcal{F}'_{i-1}$. The associated graded object of the Khovanov chain complex filtered by k is called *annular Khovanov homology* and is denoted by $\mathrm{AKh}(L)$. It is an invariant of the annular link: it is invariant under all smooth isotopies restricted to

$A \times I$.

Notice that braid closures may be naturally regarded as annular links. Indeed, AKh may be viewed as a conjugacy class invariant of braids, since conjugacy gives an annular isotopy of a closed braid. AKh has previously proven to be a powerful tool for studying braids: for instance, Grigsby and Ni proved in [38] that AKh can distinguish braids from other tangles, and Baldwin and Grigsby showed in [6] that AKh can distinguish the trivial braid among braid closures. (For more on braids and AKh, see, for example, [40].)

We note here that AKh is *not* a transverse invariant. Given a braid representative β of a transverse link L, the positive stabilization move on β cannot be performed in the complement of the braid axis, and indeed, this move does not generally preserve $\mathrm{AKh}(\beta)$.

Annular Khovanov homology has connections to several other constructions. The most straightforward is the following: there is always a spectral sequence from the associated graded object to the homology of the total complex (see [70]), and hence Roberts concludes in [85] that for any annular link L there is a spectral sequence from $\mathrm{AKh}(L)$ to $\mathrm{Kh}(L)$. In that same work, Roberts also proved that there is a spectral sequence relating (reduced) AKh and knot Floer homology via a double branched cover construction. Baldwin and Plamenevskaya built on Roberts' results in order to prove results about contact structures [7]. In [39], Grigsby and Wehrli extended the work of Roberts and proved that there is a spectral sequence relating AKh and sutured Floer homology via a (similar) double branched cover construction.

2.3.2 The Euler characteristic

The graded Euler characteristic of $\mathrm{AKh}(L)$ is:

$$\chi_{\mathrm{AKh}}(L) = \sum_{i,j,k} (-1)^i q^j t^k \dim(\mathrm{AKh}(L)^{i,j,k})$$

where $\mathrm{AKh}(L)^{i,j,k}$ is the homogeneous component of $\mathrm{AKh}(L)$ in homological grading i, q-grading j, and k-grading k.

An interpretation of the graded Euler characteristic of annular Khovanov homology in terms of the Reshetikhin-Turaev invariant [83] will be useful in Chapter 3. (Reshetikhin and Turaev defined in [83] a framework for associating a tangle invariant to a quantum group, and in [84] expanded this framework to 3-manifold invariants. Hence there are many invariants that could be called "Reshetikhin-Turaev invariants"; here we focus on the tangle invariant associated to $U_q(sl_2)$.) First we describe the construction of the Reshetikhin-Turaev invariant, which is a $U_q(sl_2)$-module map that is an invariant of tangles. A *tangle* is a proper embedding of some number of arcs and circles into $\mathbb{R}^2 \times [0,1]$. Notice that braids can naturally be viewed as tangles made up solely of arcs. Given a tangle $\mathbb{T}$, the Reshetikhin-Turaev invariant is calculated by constructing a matrix $J(\mathbb{T})$ and multiplying by final shifts of $(-1)^{n_-}(q)^{n_+-2n_-}$ to ensure invariance under the Reidemeister moves. We will refer to the matrix $J(\mathbb{T})$ as the Reshetikhin-Turaev matrix of $\mathbb{T}$. In what follows, we restrict our attention to braids and their closures and take advantage of some simplifications in notation that this yields. Refer to [40] for a more general description.

Grigsby and Wehrli showed in [40], building on work in [55], that the graded Euler characteristic of the annular Khovanov homology of a closed braid $\bar{\beta}$ can be calculated using the Reshetikhin-Turaev matrix $J(\beta)$ of any associated braid β whose closure is isotopic to $\bar{\beta}$ in $A \times I$ (n_+ denotes the number of positive crossings and n_- the number of negative crossings):

$$\chi_{\mathrm{AKh}}(\bar{\beta}) = (-1)^{n_-}(q)^{n_+-2n_-}\sum_k (qt)^k \operatorname{Tr}(J(\beta)|_{[k]})$$

For an n-braid β, the Reshetikhin-Turaev matrix $J(\beta)$ is a $U_q(sl_2)$-module map $V_1^{\otimes n} \to V_1^{\otimes n}$ intertwining the quantum group action. Here V_1 is the two-dimensional fundamental representation of $U_q(sl_2)$ with underlying vector space $V_1 = \mathbb{C}(q)v_+ \oplus \mathbb{C}(q)v_-$. The generators E, F, K of $U_q(sl_2)$ act by

$$Kv_+ = qv_+, \quad Kv_- = q^{-1}v_-, \quad Ev_+ = Fv_- = 0, \quad Ev_- = v_+, \quad Fv_+ = v_-.$$

In general, $J(\beta)$ is constructed diagrammatically by first calculating a matrix $J(\beta^I)$ for each choice of resolution I of β; the matrices associated to the resolutions are combined via

$$J(\beta) = \sum_I (-q)^{i(I)} J(\beta^I)$$

Each $J(\beta^I)$ is calculated as follows. We choose a basis for $V_1^{\otimes n}$ in one-to-one correspondence with n-tuples in $\{\uparrow, \downarrow\}^n$, namely by identifying $\uparrow$ with v_+ and identifying $\downarrow$ with v_-. For every $\mathbf{i}, \mathbf{j} \in \{\uparrow, \downarrow\}^n$, orient the top of β^I locally with $\mathbf{i}$ and the bottom of β^I locally with $\mathbf{j}$, reading left to right. The $(\mathbf{i}, \mathbf{j})$ entry of $J(\beta^I)$ is zero if any of the orientations in the resulting diagram are incompatible with each other.

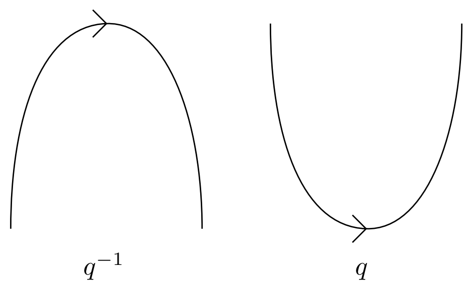

Figure 2.7: Assignments in the two special cases

If the orientations are compatible, we form the set $E_{\mathbf{i},\mathbf{j}}(\beta^I) = \{E(\beta^I) : t(\beta^I) = \mathbf{i}, b(\beta^I) = \mathbf{j}\}$ of all possible orientations of β^I satisfying that the top orientation of β^I is $\mathbf{i}$ and the bottom orientation of β^I is $\mathbf{j}$. (For example, if β^I contains one closed component, $E(\beta^I)$ contains two elements, one for each orientation of the closed component). Then the $\mathbf{i}, \mathbf{j}$ entry of $J(\beta^I)$ is a weighted sum over all elements of $E_{\mathbf{i},\mathbf{j}}(\beta^I)$:

$$J(\beta^I)_{\mathbf{i},\mathbf{j}} = \sum_{\mathbb{S} \in E_{\mathbf{i},\mathbf{j}}(\beta^I)} q^{j(\mathbb{S})}$$

where we describe $q^j(\mathbb{S})$, the appropriate power of q, here.

For each element in $E_{\mathbf{i},\mathbf{j}}(\beta^I)$, every arc is assigned a q^0 unless we have one of the cases shown in Figure 2.7, in which case the assignment is as shown. We multiply the assignments corresponding to each arc in the diagram to obtain a single power of q, written $q^{j(\mathbb{S})}$ for every element $\mathbb{S}$ in $E_{\mathbf{i},\mathbf{j}}(\beta^I)$.

The k in χ_{AKh} corresponds to $k = \#(\uparrow) - \#(\downarrow)$ in $\{\uparrow, \downarrow\}^n$. Notice that the $(\mathbf{i}, \mathbf{j})$ entry in $J(\beta)$ is non-zero if and only if $k(\mathbf{i}) = k(\mathbf{j})$. This implies that $J(\beta)$ is a block diagonal matrix, with a block for each k. We denote the block sub-matrix corresponding to a fixed k by $J(\beta)|_{[k]}$.

2.3.3 Reduced annular Khovanov homology

As in Khovanov homology (see Section 2.2.2), there are reduced versions of annular Khovanov homology. They are, however, much less versatile than the Khovanov construction, since the two versions are not isomorphic, and they are not invariant under moving the basepoint.

Here, we consider p to be a marked non-double point on a diagram $\mathcal{D}$ of an annular knot K. As before we will assume that the last tensor factor of each generator of $\mathrm{CKh}(\mathcal{D})$ corresponds to the component containing p. Here, we consider $\mathrm{CKh}(\mathcal{D})$ equipped with the annular differential - that is, we restrict to the Khovanov differential maps that preserve the k-grading. There is a chain map $x_p\colon\ \mathrm{CKh}(\mathcal{D}) \to \mathrm{CKh}(\mathcal{D})$ defined on generators by

$$x_p(y \otimes v_+) = y \otimes v_-$$
$$x_p(y \otimes v_-) = 0$$

Again, there are two flavors of *reduced annular Khovanov homology* [53]. The *reduced annular subcomplex* $\widetilde{\mathrm{CKh}}_p(\mathcal{D})$ is defined as $\ker(x_p)$; the homology of this subcomplex is one choice of what can be meant by "reduced annular Khovanov homology". This is the original definition given by Lawrence Roberts in [85]. We can also consider the *reduced annular quotient complex* $\underline{\mathrm{CKh}}_p(\mathcal{D})$, which is defined as $\mathrm{coker}(x_p)$, and could consider the homology of that complex as another choice for "reduced annular Khovanov homology". Here, however, the two versions are *not* isomorphic. This can easily be seen by calculating the two homologies for the closed 2-braid σ_1. While this example is extremely simple, this fact is, as far as the author knows, not mentioned in the literature.

In addition, neither of these versions is invariant under sliding the basepoint around the knot. (Indeed, it is easy to see that the proof outlined in Section 2.2.2 for Khovanov homology does not apply in the annular setting.) Here again this fact can be proved with a quick computation. For example, for both versions, the reduced annular Khovanov homology of the closed 3-braid $\sigma_1\sigma_2^{-1}$ changes depending on whether the basepoint is placed on the outermost strand of the braid versus the innermost strand of the braid. While this fact is also not to the author's knowledge mentioned explicitly in the literature, Roberts specifies in [85] that the basepoint should be placed on the innermost strand.

Chapter 3

Mutations on braids

In this chapter we prove the results introduced in Section 1.1.

3.1 On the Euler characteristic of AKh

We start with Theorem 2, which we restate here for reference:

Theorem 2. *Given a braid β,*

$$\chi_{\mathrm{AKh}}(\bar{\beta})|_{k=n-2} = (qt)^{n-2}(q)^{n_+-n_-}\,\mathrm{Tr}\left(\Phi(\beta, q^2)\right)$$

Recall that Φ is the classical Burau representation of braids, described in Section 2.1.3. In order to prove Theorem 2, we will first observe that the Reshetikhin-Turaev matrix takes a particularly nice form for braids when we restrict to $k = n-2$.

Lemma 6. *Consider the standard Artin generators $\sigma_1, \ldots, \sigma_{n-1}$ for the n-strand braid group. Then $J(\sigma_i)|_{[n-2]}$ is an n by n matrix that takes the following form: there is a 2 by 2 block*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - q\begin{pmatrix} q & 1 \\ 1 & q^{-1} \end{pmatrix} = q^{-1}\begin{pmatrix} q - q^3 & -q^2 \\ -q^2 & 0 \end{pmatrix}$$

with the 0 *in the* (i,i) *spot and* $q^{-1}(q)$ *along the diagonal everywhere else.*

$J(\sigma_i^{-1})|_{[n-2]}$ takes the following form: there is a 2 by 2 block

$$\begin{pmatrix} q & 1 \\ 1 & q^{-1} \end{pmatrix} - q \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -q^2 \begin{pmatrix} 0 & -q^{-2} \\ -q^{-2} & -q^{-3}+q^{-1} \end{pmatrix}$$

with the 0 in the (i,i) spot, and $-q^2(q^{-1})$ along the diagonal everywhere else.

For ease of notation in later calculations, we name the factored matrices L: e.g., $J(\sigma_i)|_{[n-2]} = q^{-1}L(\sigma_i))$ and $J(\sigma_i^{-1})|_{[n-2]} = -q^2 L(\sigma_i^{-1})$.

For example, for $\sigma_2 \in B_4$,

$$J(\sigma_2)|_{[2]} = q^{-1} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^3 & -q^2 & 0 \\ 0 & -q^2 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} = q^{-1}L(\sigma_2)$$

We note again here that the Reshetikhin-Turaev *matrix* is not an invariant of braids. For example, $J(\sigma_i)|_{[n-2]}J(\sigma_i^{-1})|_{[n-2]}$ is not the identity. However, the final grading shift $(-1)^{n_-}(q)^{n_+-2n_-}$ removes both the q^{-1} coefficient from each positive generator and the $-q^2$ coefficient from each negative generator, taking care of this problem:

$$(-1)^1(q)^{1-2}J(\sigma_i)|_{[n-2]}J(\sigma_i^{-1})|_{[n-2]} = (-1)^1(q)^{1-2}(-1)^1(q)^{2-1}L(\sigma_i)L(\sigma_i^{-1}) = I$$

Proof of Lemma 6. This is a calculation. We show it here for σ_1 in B_3; it will be clear how to extend the calculation to more strands and other σ_i. Restricting to $k=1$, we choose the basis ordering $\{\downarrow\uparrow\uparrow, \uparrow\downarrow\uparrow, \uparrow\uparrow\downarrow\}$. For example, for $\sigma_1 \in B_3$, the $(1,2)$ entry of $J(\sigma_1)|_{[1]}$ corresponds to orienting the top strands with $\downarrow\uparrow\uparrow$ and the bottom strands with $\uparrow\downarrow\uparrow$.

Calculating using the rules described in Section 2.3.2, the matrix associated to the 0-resolution of σ_1 is the identity matrix, and the matrix associated to the 1-resolution of σ_1 is

$$\begin{pmatrix} q & 1 & 0 \\ 1 & q^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

since the resolutions associated to σ_1 are as shown in Figure 3.1.

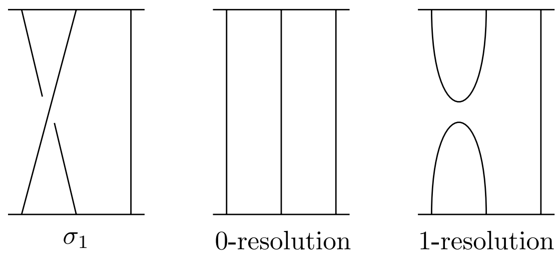

Figure 3.1: The resolutions of $\sigma_1 \in B_3$

So $J(\sigma_1)|_{[1]}$ is:

$$J(\sigma_1^0)_{[1]} - qJ(\sigma_1^1)_{[1]} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - q \begin{pmatrix} q & 1 & 0 \\ 1 & q^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} = q^{-1} \begin{pmatrix} q - q^3 & -q^2 & 0 \\ -q^2 & 0 & 0 \\ 0 & 0 & q \end{pmatrix}$$

For σ_1^{-1}, the 0 and 1 resolutions are switched.

□

We now show that given a braid β with braid word in the standard Artin generators, we can find $J(\beta)|_{[n-2]}$ by composing the matrices for each braid generator (as described in Lemma 6).

Lemma 7. *Given a braid* $\beta = \sigma_{l_1}\sigma_{l_2}\cdots\sigma_{l_m}$*,*

$$J(\beta)|_{[n-2]} = J(\sigma_{l_1})|_{[n-2]} J(\sigma_{l_2})|_{[n-2]} \cdots J(\sigma_{l_m})|_{[n-2]}$$

Proof. Recall that $J(\beta)$ is a block diagonal matrix with a block for each k; hence we can restrict to a specific k for matrix operations. In what follows we drop the $k = n-2$ notation for simplicity. We prove this by induction on the length of the braid word. The base case is trivial.

Any given resolution $\beta^{\mathcal{I}}$ of β restricts to a resolution of each individual braid generator; we write $\beta^{\mathcal{I}} = \sigma_{l_1}^{\mathcal{I}} \cdots \sigma_{l_m}^{\mathcal{I}}$ where by $\sigma_{l_j}^{\mathcal{I}}$ we mean σ_{l_j} restricted to the appropriate index within $\mathcal{I}$ (either a 0 or a 1). We first show the result on the resolution level: it suffices to show that

$$J(\beta^{\mathcal{I}}) = J(\sigma_{l_1}^{\mathcal{I}} \cdots \sigma_{l_{m-1}}^{\mathcal{I}}) J(\sigma_{l_m}^{\mathcal{I}})$$

We have

$$J(\beta^{\mathcal{I}})_{\mathbf{i},\mathbf{j}} = \sum_{\mathbb{S} \in E_{\mathbf{i},\mathbf{j}}(\beta^{\mathcal{I}})} q^{j(\mathbb{S})}$$

$$= \sum_{\mathbb{S}'' \in E_{\mathbf{k},\mathbf{j}}(\sigma_{l_m}^{\mathcal{I}})} \sum_{\mathbb{S}' \in E_{\mathbf{i},\mathbf{k}}(\sigma_{l_1}^{\mathcal{I}} \cdots \sigma_{l_{m-1}}^{\mathcal{I}})} q^{j(\mathbb{S}'\mathbb{S}'')}$$

where by $\mathbb{S}'\mathbb{S}''$ we mean the vertical stacking of these two enhanced resolutions. The expression becomes:

$$\sum_{\mathbb{S}'' \in E_{\mathbf{k},\mathbf{j}}(\sigma_{l_m}^{\mathcal{I}})} \sum_{\mathbb{S}' \in E_{\mathbf{i},\mathbf{k}}(\sigma_{l_1}^{\mathcal{I}} \cdots \sigma_{l_{m-1}}^{\mathcal{I}})} q^{j(\mathbb{S}')} q^{j(\mathbb{S}'')}$$

$$= \sum_{\mathbb{S}'' \in E_{\mathbf{k},\mathbf{j}}(\sigma_{l_m}^{\mathcal{I}})} q^{j(\mathbb{S}'')} \sum_{\mathbb{S}' \in E_{\mathbf{i},\mathbf{k}}(\sigma_{l_1}^{\mathcal{I}} \cdots \sigma_{l_{m-1}}^{\mathcal{I}})} q^{j(\mathbb{S}')}$$

$$= \sum_{k} J(\sigma_{l_m}^{\mathcal{I}})_{kj} J(\sigma_{l_1}^{\mathcal{I}} \cdots \sigma_{l_{m-1}}^{\mathcal{I}})_{ik} = \sum_{k} J(\sigma_{l_1}^{\mathcal{I}} \cdots \sigma_{l_{m-1}}^{\mathcal{I}})_{ik} J(\sigma_{l_m}^{\mathcal{I}})_{kj}$$

This shows

$$J(\beta^{\mathcal{I}}) = J(\sigma_{l_1}^{\mathcal{I}}) \cdots J(\sigma_{l_m}^{\mathcal{I}})$$

Now recall that

$$J(\beta) = \sum_{I} (-q)^{i(\mathcal{I})} J(\beta^{\mathcal{I}})$$

and

$$J(\sigma_{l_1}) \cdots J(\sigma_{l_m}) = (J(\sigma_{l_1}^0) - qJ(\sigma_{l_1}^1)) \cdots (J(\sigma_{l_m}^0) - qJ(\sigma_{l_m}^1))$$

Multiplying out the second expression gives the first.

□

Each of the factored matrices in Lemma 6 is (up to a constant) conjugate to the Burau representation:

Proof of Theorem 2. We prove here that $L(\sigma_1)$ is conjugate (up to a constant power of q)

to the Burau representation for $\sigma_1 \in B_3$, and then we will see that this is true for any $\sigma_i^{\pm 1} \in B_n$ for any n:

$$L(\sigma_1) = \begin{pmatrix} q - q^3 & -q^2 & 0 \\ -q^2 & 0 & 0 \\ 0 & 0 & q \end{pmatrix} = q \begin{pmatrix} 1 - q^2 & -q & 0 \\ -q & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Set

$$A = \begin{pmatrix} q^{-1} & 0 & 0 \\ 0 & -q^{-2} & 0 \\ 0 & 0 & q^{-3} \end{pmatrix}$$

Then

$$A \begin{pmatrix} 1 - q^2 & -q & 0 \\ -q & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} A^{-1} = \begin{pmatrix} 1 - q^2 & q^2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is the classical Burau matrix for σ_1 with $t = q^2$.

The process is similar for σ_1^{-1}:

$$L(\sigma_1^{-1}) = \begin{pmatrix} 0 & -q^{-2} & 0 \\ -q^{-2} & -q^{-3} + q^{-1} & 0 \\ 0 & 0 & q^{-1} \end{pmatrix} = q^{-1} \begin{pmatrix} 0 & -q^{-1} & 0 \\ -q^{-1} & -q^{-2} + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$A \begin{pmatrix} 0 & -q^{-1} & 0 \\ -q^{-1} & -q^{-2} + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ q^{-2} & -q^{-2} + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

One can check that the same conjugating matrix A works for each braid generator. We can expand A for an arbitrary number of strands by continuing alternating signs along the diagonal and ending at q^{-n}.

Now we have for $\beta = \sigma_{l_1} \cdots \sigma_{l_m}$, together with the fact that trace is invariant under conjugation,

$$\begin{aligned}
\chi_{\mathrm{AKh}}(\bar{\beta})|_{k=n-2} &= (-1)^{n_-}(q)^{n_+-2n_-}(qt)^{n-2}\operatorname{Tr}(J(\beta)|_{[n-2]}) \\
&= (-1)^{n_-}(q)^{n_+-2n_-}(qt)^{n-2}\operatorname{Tr}(J(\sigma_{l_1})|_{[n-2]}\cdots J(\sigma_{l_m})|_{[n-2]}) \\
&= (-1)^{n_-}(q)^{n_+-2n_-}(qt)^{n-2}(-1)^{n_-}(q)^{2n_--n_+}\operatorname{Tr}(L(\sigma_{l_1})\cdots L(\sigma_{l_m})) \\
&= (-1)^{n_-}(q)^{n_+-2n_-}(qt)^{n-2}(-1)^{n_-}(q)^{2n_--n_+}\operatorname{Tr}(AL(\sigma_{l_1})A^{-1}\cdots AL(\sigma_{l_m})A^{-1}) \\
&= (qt)^{n-2}q^{n_+-n_-}\operatorname{Tr}(\Phi(\beta,q^2))
\end{aligned}$$

proving Theorem 2.

□

3.2 Mutation and AKh

We are now ready to prove Theorem 1 using Theorem 2 (in particular using Corollary 3, since in each example the exponent sum of the braids are preserved under the mutation in question). We restate the theorem here for convenience; see Section 1.1 for the figures.

Theorem 1. *The annular Khovanov homology of a closed braid is not invariant under an axis-preserving mutation. Indeed, there exist infinite families of mutant 4-braid pairs and mutant 5-braid pairs, shown in Figures 1.3 and 1.4, whose annular Khovanov homologies differ.*

Proof. We have two families of examples.

Example 1: The following is an example of a family of 4-braid pairs related by a braid-axis preserving mutation whose annular Khovanov homologies differ. See Figure 1.3.

Given $A = \sigma_2^{-2}\sigma_3\sigma_2^{-1}\sigma_1^{-1}\sigma_2^3\sigma_3^{-1}\sigma_2\sigma_1$ and $B = \sigma_2^{-2}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2^3\sigma_3\sigma_2\sigma_1$, then the braids $(\sigma_1)^k A$ and $(\sigma_1)^k B$ are related by a mutation and their AKh differs for all $k \geq 0$.

For $k = 0$, a calculation shows that

$$\operatorname{Tr}(\Phi(A,t)) = -t^{-3} + 2t^{-2} - 4t^{-1} + 6 - 5t + 3t^2 - 2t^3 + t^4 \quad \text{and}$$

$$\operatorname{Tr}(\Phi(B,t)) = -2t^{-1} + 4 - 3t + t^2$$

and the result follows by Corollary 3.

For $k \geq 1$: first, we observe that the Burau matrix for σ_1^k is as follows:

$$\begin{pmatrix} \sum\limits_{m=0}^{k}(-t)^m & \sum\limits_{m=1}^{k}(-1)^{m+1}t^m & 0 & 0 \\ \sum\limits_{m=0}^{k-1}(-t)^m & \sum\limits_{m=1}^{k-1}(-1)^{m+1}t^m & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We prove this by induction on k. The base case $k = 1$ is trivially true.

Call the following matrix M:

$$\begin{pmatrix} \sum\limits_{m=0}^{k}(-t)^m & \sum\limits_{m=1}^{k}(-1)^{m+1}t^m & 0 & 0 \\ \sum\limits_{m=0}^{k-1}(-t)^m & \sum\limits_{m=1}^{k-1}(-1)^{m+1}t^m & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1-t & t & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We prove the result by examining the entries of M. We show one entry here; the rest are similar.

$$M_{(1,1)} = (1-t)\sum_{m=0}^{k}(-t)^m + \sum_{m=1}^{k}(-1)^{m+1}t^m =$$

$$\sum_{m=0}^{k}(-t)^m + \sum_{m=0}^{k}(-1)^{m+1}t^{m+1} + \sum_{m=1}^{k}(-1)^{m+1}t^m =$$

$$= \sum_{m=0}^{k}(-t)^m + \sum_{m=1}^{k+1}(-1)^{m}t^m + \sum_{m=1}^{k}(-1)^{m+1}t^m =$$

$$= \sum_{m=0}^{k}(-t)^m + (-1)^{k+1}t^{k+1} + \sum_{m=1}^{k} t^m((-1)^m + (-1)^{m+1})$$

Since $(-1)^m + (-1)^{m+1} = 0$, the desired result follows.

Now that we have established a formula for the Burau matrix of σ_1^k, we examine $\mathrm{Tr}(\Phi(\sigma_1^k A), t)$ and $\mathrm{Tr}(\Phi(\sigma_1^k B), t)$ (Mathematica gives us the entries in the matrices for A

and B):

$$\mathrm{Tr}(\Phi(\sigma_1^k A), t) = \left(\sum_{m=0}^{k} (-t)^m\right)(1 - 2t + 2t^2 - 2t^3 + t^4) +$$
$$\left(\sum_{m=1}^{k} (-1)^{m+1} t^m\right)(-t^{-2} + 3t^{-1} - 4 + 3t - t^2) +$$
$$\left(\sum_{m=1}^{k} (-1)^{m+1} t^m\right)(-t^{-2} + 2t^{-1} - 1) - t^{-3} + 3t^{-2} - 6t^{-1} + 6 - 3t + t^2$$

and

$$\mathrm{Tr}(\Phi(\sigma_1^k B), t) = \left(\sum_{m=0}^{k} (-t)^m\right)(1 - t) + 3 - 2t^{-1} - 2t + t^2$$

The largest power of t appearing in $\mathrm{Tr}(\Phi(\sigma_1^k A))$ is $k+4$, and the largest power appearing in $\mathrm{Tr}(\Phi(\sigma_1^k B))$ is 2 if $k = 0, 1$ and k if $k > 1$.

Example 2: See Figure 1.4.

The braids

$$(\sigma_1)^k \sigma_2^{-1} \sigma_3 \sigma_2 \sigma_3^{-1} \sigma_2 (\sigma_1^{-1})^k \sigma_4^{-1} \sigma_2^{-1} \sigma_3 \sigma_2^{-1} \sigma_3^{-1} \sigma_2 \sigma_4$$

and

$$(\sigma_1)^k \sigma_2^{-1} \sigma_3 \sigma_2 \sigma_3^{-1} \sigma_2 (\sigma_1^{-1})^k \sigma_4 \sigma_2^{-1} \sigma_3 \sigma_2^{-1} \sigma_3^{-1} \sigma_2 \sigma_4^{-1}$$

are related by a mutation and their AKh differ for all $k \geq 1$.

We prove the result by showing that the traces of the Burau matrices of these two braids are different when we set $t = -1$. First, it can be easily checked that the form for $(\sigma_1)^k$ when $t = -1$ is:

$$\begin{pmatrix} k+1 & -k & 0 & 0 & 0 \\ k & -k+1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and the form for $(\sigma_1^{-1})^k$ when $t = -1$ is:

$$\begin{pmatrix} 1-k & k & 0 & 0 & 0 \\ -k & k+1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

It can also be readily checked (for example, using Mathematica), that when $t = -1$, the Burau matrices for $X = \sigma_2^{-1}\sigma_3\sigma_2\sigma_3^{-1}\sigma_2$, $Y = \sigma_4^{-1}\sigma_2^{-1}\sigma_3\sigma_2^{-1}\sigma_3^{-1}\sigma_2\sigma_4$, and $Z = \sigma_4\sigma_2^{-1}\sigma_3\sigma_2^{-1}\sigma_3^{-1}\sigma_2\sigma_4^{-1}$ are as follows:

$$X = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & -2 & -2 & 0 \\ 0 & 6 & -2 & -3 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$Y = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -3 & 2 & 4 & -2 \\ 0 & -6 & 4 & 6 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & -1 & -2 & 2 \end{pmatrix}$$

$$Z = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -3 & 2 & 0 & 2 \\ 0 & -6 & 4 & 0 & 3 \\ 0 & -4 & 2 & 1 & 2 \\ 0 & -2 & 1 & 0 & 2 \end{pmatrix}$$

It can be calculated that

$$\mathrm{Tr}(\Phi(\sigma_1^k)X\Phi(\sigma_1^{-k})Y) = 6 + 8k + 16k^2$$

and

$$\mathrm{Tr}(\Phi(\sigma_1^k)X\Phi(\sigma_1^{-k})Z) = 6 - 8k + 16k^2$$

Hence for all $k \geq 1$, the traces of the Burau matrices are distinct.

□

The negative flype 4-braid examples mentioned in Section 1.1 are also a direct calculation, as their traces under the Burau representation differ as well. We show the calculations here. Suppose

$$\alpha = \sigma_3^2\sigma_2^2\sigma_3^{-1}\sigma_1^2\sigma_2\sigma_1^{-1}$$

$$\alpha' = \sigma_3^2\sigma_2^2\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_1^2$$

The braids α and α' are related by a negative flype [74], and hence they are smoothly isotopic in S^3. However,

$$\mathrm{Tr}(\Phi(\alpha, t)) = t - t^2 + t^4 - t^5$$

$$\mathrm{Tr}(\Phi(\alpha', t)) = -1 + 4t - 6t^2 + 5t^3 - 2t^4$$

and so their annular Khovanov homologies differ. In particular, this implies that they represent different isotopy classes of knots in $A \times I$.

Suppose

$$\beta = \sigma_3\sigma_2^{-2}\sigma_3^2\sigma_2\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_1^2$$

$$\beta' = \sigma_3\sigma_2^{-2}\sigma_3^2\sigma_2\sigma_3^{-1}\sigma_1^2\sigma_2\sigma_1^{-1}$$

The braids β and β' are also related by a negative flype [51] (see also [74]), and

$$\mathrm{Tr}(\Phi(\beta, t)) = -t^{-2} + 2t^{-1} - 5 + 9t - 9t^2 + 6t^3 - 2t^4$$

$$\mathrm{Tr}(\Phi(\beta', t)) = -t^{-1} + 1 + t - t^2 + t^4 - t^5$$

hence their annular Khovanov homologies differ as well.

Chapter 4

Annular refinements

In this chapter we discuss and prove the results outlined in Section 1.2.

4.1 Definition and invariance of κ

Let B_n be the braid group on n strands and let $\beta \in B_n$ be a braid with transverse element $\psi(\bar{\beta})$. The k-filtration on $\mathrm{CKh}(\bar{\beta})$ has the form

$$0 \subset \mathcal{F}_{-n} \subset \mathcal{F}_{2-n} \subset \cdots \subset \mathcal{F}_{n-2} \subset \mathcal{F}_n = \mathrm{CKh}(\bar{\beta})$$

where $\mathcal{F}_{-n}$ is generated by $\psi(\bar{\beta})$, so $\psi(\bar{\beta}) \in \mathcal{F}_i$ for $i \geq -n$. We restate Definition 1.2.1:

Definition. *Let $\beta \in B_n$ and suppose that $\psi(\bar{\beta})$ is a boundary in $\mathrm{CKh}(\bar{\beta})$. Define*

$$\kappa(\beta) = n + \min\{i : [\psi(\bar{\beta})] = 0 \in H(\mathcal{F}_i)\}.$$

If $\psi(\bar{\beta})$ is not a boundary, then define $\kappa(\beta) = \infty$.

We will say that $y \in \mathrm{CKh}(\bar{\beta})$ *realizes* $\kappa(\beta)$ if $dy = \psi(\bar{\beta})$ and $k(y) = \kappa(\beta) - n$. Note that κ is always even and that $2 \leq \kappa(\beta) \leq 2n$. The only element with k-grading n is the all v_+ labeling of the braidlike resolution, so in fact $\kappa(\beta) \leq 2(n-1)$. We now show that κ is a well-defined function on B_n. First, an algebraic lemma.

Lemma 8. *Let $(X, d, \mathcal{F})$ and $(X', d', \mathcal{F}')$ be complexes with bounded filtrations, and suppose that $f\colon X \to X'$ is a filtered chain map. For any non-zero cycle $x \in X$, define $\kappa(x) = \min\{i : [x] = 0 \in H_*(\mathcal{F}_i)\}$ or $\kappa(x) = \infty$ if x is not a boundary. Define κ' analogously on X'. Suppose that $f(x) = y \neq 0$. Then $\kappa(x) \geq \kappa'(y)$. If there is a filtered chain map $g : X' \to X$ with $g(y) = x$, then $\kappa(x) = \kappa'(y)$.*

Proof. Chain maps carry cycles to cycles, so if $\kappa(x)$ is defined then so is $\kappa'(y)$. There is nothing left to prove if $\kappa(x) = \infty$, so suppose that $\kappa(x)$ is finite. Then there is some $w \in \mathcal{F}_{\kappa(x)}$ so that $dw = x$, and $(f \circ d)(w) = y = (d \circ f)(w)$. As f is filtered, $f(w) \in \mathcal{F}'_{\kappa(x)}$, so $\kappa'(y) \leq \kappa(x)$. If there is a filtered chain map g with $g(y) = x$, then the opposite inequality shows that $\kappa(x) = \kappa'(y)$. □

The κ of Lemma 8 differs from that of Definition 1.2.1 in that the latter is normalized using the braid index, but the lemma clearly still applies to the Definition.

Proposition 9. *Suppose that β and β' are words in the Artin generators so that $\beta = \beta'$ in B_n. Then $\kappa(\beta) = \kappa(\beta')$.*

Proof. It will suffice to show that $\kappa(\beta)$ is invariant under Reidemeister 2 and Reidemeister 3 moves which do not cross the braid axis. These moves induce natural maps on the Khovanov chain complex which carry $\psi(\beta)$ to $\psi(\beta')$, see [80]. For a digestible summary of these maps, see [9]. If these maps are filtered, then Lemma 8 completes the proof.

The map induced by Reidemeister 2 (and its inverse) is a direct sum of identity maps and compositions of saddles with cups and caps. The saddles, cups, and caps do not cross the braid axis. Certainly the identity map is filtered. One may check directly that saddle maps are filtered; alternatively, observe that a saddle may be viewed as a component of the Khovanov differential of some annular link and so it must be filtered. Cups and caps that do not cross the braid axis cannot change the k-grading. Thus the Reidemeister 2 map is filtered. An identical analysis shows that the Reidemeister 3 maps are filtered. □

Considering braids instead of their closures, we obtain the following.

Proposition 10. *κ is an invariant of conjugacy classes in B_n.*

However, κ is certainly not a complete invariant of conjugacy classes and it is not known if it can be used to solve the conjugacy problem in the braid group. The conjugacy problem in the braid group was solved by Garside in [32] and has been extensively studied ever since (see [33]). It would be interesting to understand the meaning of κ in relation to this work.

A program to compute κ is available at `www2.bc.edu/adam-r-saltz/kappa.html`.

4.2 Examples and Properties of κ

4.2.1 Main example

An immediate first question is whether elements in k-grading $-n+2$ always suffice to kill $\psi(\bar{\beta})$ whenever $[\psi(\bar{\beta})] = 0$, that is, whether $\kappa = 2$ for all braids with $[\psi(\bar{\beta})] = 0$. Proposition 5, using examples from Theorem 1.1 in [51], shows that this is false. We restate it here for convenience:

Proposition. *For any $a, b \in \{0, 1, 2\}$, the pair of closed 4-braids*

$$A(a,b) = \sigma_3\sigma_2^{-2}\sigma_3^{2a+2}\sigma_2\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_1^{2b+2} \quad \textit{and}$$

$$B(a,b) = \sigma_3\sigma_2^{-2}\sigma_3^{2a+2}\sigma_2\sigma_3^{-1}\sigma_1^{2b+2}\sigma_2\sigma_1^{-1},$$

related by a negative flype, can be distinguished by κ: indeed, $\kappa(A(a,b)) = 4$ and $\kappa(B(a,b)) = 2$.

Proof. By computation. $\square$

We do not know if this relation holds for all $a, b \in \mathbb{Z}_{\geq 0}$. Recall that since the closed braids $A(a,b)$ and $B(a,b)$ are in the same isotopy class (as they are related by a flype), they have isomorphic Khovanov homologies. However, annular Khovanov homology can differentiate them (see Chapter 3) for $a, b \in \{0, 1, 2\}$.

4.2.2 Negative Stabilization

Proposition 11. *If a closed n-braid β is a negative stabilization of another braid, then $\kappa(\beta) = 2$.*

Proof. In Theorem 3 of [80], Plamenevskaya constructs an element $y \in CKh(\beta)$ such that $dy = \psi(\beta)$ as follows: consider the resolution formed from taking the 0-resolution of the negative crossing from the negative stabilization, the 1-resolution for all other negative crossings, and the 0-resolution for all positive crossings. The element y is obtained by assigning each circle in this resolution v_-. It is clear that y has k-grading $-n+2$. □

4.2.3 Positive Stabilization

Define an *arc* of a closed braid diagram to be a segment of the link that goes from one crossing to another crossing without traversing over or under any other crossings. An *innermost arc* is one for which we can draw a straight line from the braid axis to any point on the arc without crossing any other arcs. An *innermost point* is a point lying on an innermost arc.

Given an n-strand braid β, we define $S^p\beta$ to be β positively stabilized once at an innermost point p. That is: insert σ_n at the point p on the diagram.

Proposition 12. *$\kappa(\beta)$ is not a transverse invariant.*

Proof. This is due to the fact that the chain map corresponding to positive stabilization is not filtered (see Proposition 13). We have a concrete example: consider the braid $B(0,0)$ from Proposition 5. By computation, $\kappa(B(0,0)) = 2$ and $\kappa(S^pB(0,0)) = 4$ for all choices of innermost points p. □

We note here that we can define a transverse invariant using κ, though it is not clear how to compute it unless the transverse link is known to be represented by a braid with $\kappa = 2$.

Definition 4.2.1. For an n-braid β, define $\kappa_{min}(\beta)$ to be the minimum $\kappa(K)$ over all transverse representatives K of β. It is a transverse invariant.

We can give bounds on the behavior of κ under positive stabilization:

Proposition 13. *$\kappa(\beta) \leq \kappa(S^p\beta) \leq \kappa(\beta) + 2$.*

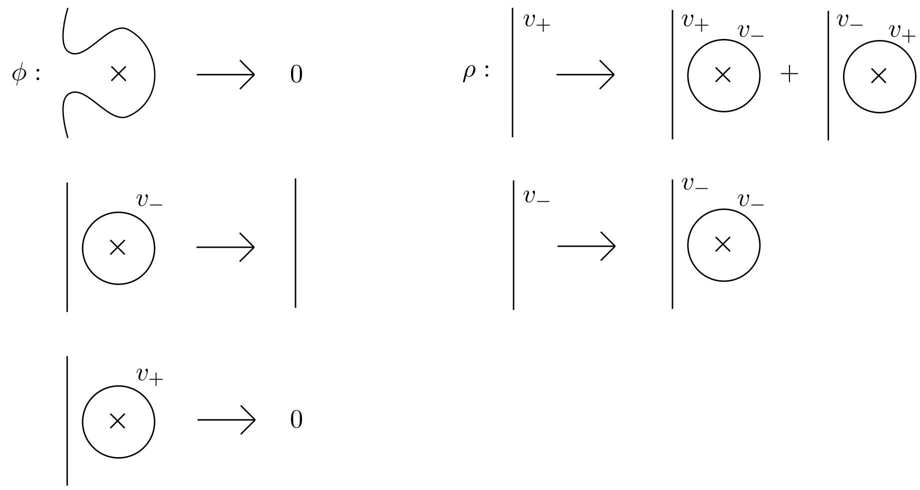

Figure 4.1: Chain maps for positive stabilization

Proof. $S^p\beta$ has a positive crossing at p, and for an n-strand braid β we refer to this crossing as $\sigma_{n,p}$. Suppose that $\sigma_{n,p}$ appears last in the crossing ordering. We show the first inequality. As described in [9], there is a chain map $\phi\colon\ CKh(S^p\beta) \to CKh(\beta)$ whose kernel contains all elements in resolutions of $S^p\beta$ where $\sigma_{n,p}$ is 1-resolved and satisfying

$$\phi(z \otimes v_-) = z$$
$$\phi(z \otimes v_+) = 0$$

for elements in resolutions where $\sigma_{n,p}$ is 0-resolved (see Figure 4.1). Consider an element $y \in CKh(S^p\beta)$ realizing $\kappa(S^p\beta)$. The element y takes the form $z_1 \otimes v_- + z_2 \otimes v_+ + z_3$. So

$$d(\phi(y)) = d(z_1) = \phi(dy) = \phi(\psi(S^p\beta)) = \psi(\beta)$$

Hence z_1 kills $\psi(\beta)$, and so we have

$$\begin{aligned}\kappa(S^p\beta) &= \max k(z_1 \otimes v_-, z_2 \otimes v_+, z_3) + n + 1 \\ &\geq k(z_1 \otimes v_-) + n + 1 = k(z_1) + n \\ &\geq \kappa(\beta)\end{aligned}$$

As described in [9] (see also [80]), there is a chain map $\rho\colon CKh(\beta) \to CKh(S^p\beta)$ satisfying $\rho(\psi(\beta)) = \psi(S^p\beta)$. It is given by

$$\begin{aligned}\rho(v_-) &= v_- \otimes v_- \\ \rho(v_+) &= v_+ \otimes v_- + v_- \otimes v_+\end{aligned}$$

Hence ρ can either decrease k-grading by one or increase it by one, depending on whether the circles in question are trivial or non-trivial. Now, suppose we have an element $y \in CKh(\beta)$ realizing $\kappa(\beta)$: then $\rho(y)$ kills $\psi(S^p\beta)$. The k-grading of $\rho(y)$ is at most $\kappa(\beta) - n + 1$. Stabilization increases strand number by one, so $\kappa(S^p\beta)$ could at most be

$$\kappa(\beta) - n + 1 + n + 1 = \kappa(\beta) + 2.$$

□

4.2.4 Other properties and consequences

Propositions 11 and 13 immediately give us bounds for κ of braids related by exchange moves and positive flypes:

Proposition 14. *If two braids σ and β are related by a single exchange move or a single positive flype, then $|\kappa(\sigma) - \kappa(\beta)| \leq 2$.*

Proof. Exchange moves and positive flypes can both be expressed as a composition of braid isotopies, one single positive stabilization, and one single positive destabilization (see for instance [18], [62]). □

Proposition 15. *Suppose a closed n-braid β can be represented by a braid word containing a factor of σ_i^{-1} but no σ_i's for some $i = 1, \ldots, n-1$. Then $\kappa(\beta) = 2$.*

Proof. The argument we give here is very similar to arguments found in [80]. Consider the resolution formed from taking the 0-resolution of one of the σ_i^{-1}'s, the 1-resolution for all other negative crossings, and the 0-resolution for all positive crossings. We claim that assigning each circle in this resolution v_- yields an element y with $dy = \psi$ and $k(y) = -n+2$. The differential d on y is the sum of all maps with y as their initial end. By our choice of resolution, any map corresponding to a merge map sends y to 0. Hence d is a sum of split maps. Topologically, the only split maps that can start from this resolution are in the i'th column; however, there are only negative crossings in this column, and at this resolution they are all 1-resolved except for the one that is 0-resolved. So the only contributor to dy is the map resolving that crossing, sending y to $\psi(\beta)$. □

The following two definitions (with more detail) can be found in [6]. Let D_n denote the standard unit disk with n marked points $p_1, \ldots, p_n$ positioned along the real axis.

Definition 4.2.2. An arc $\gamma\colon [0,1] \to D_n$ is admissible if it satisfies

(i) γ is a smooth imbedding transverse to ∂D_n

(ii) $\gamma(0) = -1 \in \mathbb{C}$ and $\gamma(1) \in \{p_1, \ldots, p_n\}$

(iii) $\gamma(t) \in D_n - (\partial D_n \cup \{p_1, \ldots, p_n\})$ for all $t \in (0,1)$ and

(iv) $\frac{d\gamma}{dt} \neq 0$ for all $t \in [0,1]$.

Definition 4.2.3. Let $\sigma \in B_n$. We say σ is right-veering if for all admissible arcs γ, $\sigma(\gamma)$ is right of γ when pulled tight.

Corollary 16. *If an n-braid σ is not right-veering, then $\kappa(\sigma) = 2$.*

Proof. By Proposition 3.1 of [6] and Proposition 6.2.7 of [20], σ is conjugate to a braid that can be represented by a word containing at least a factor of σ_i^{-1} but no σ_i's for some $i = 1, \ldots, n$. The result follows by Proposition 15. □

For a braid $\beta \in B_n$ we denote its mirror as $m(\beta) \in B_n$.

Corollary 17. *If $\kappa(\sigma) \neq 2$ and $\kappa(m(\sigma)) \neq 2$, then $\sigma = 1 \in B_n$.*

Proof. The proof is similar to that of Corollary 1 of [6]. By Corollary 16, σ and $m(\sigma)$ are right-veering and hence σ is also left-veering. By Lemma 3.1 of [6], σ is the identity braid. □

We note that this provides a solution to the word problem in the braid group. This solution is faster than that of [6] since it is only necessary to check if Plamenevskaya's invariant vanishes by the E^3 page of the spectral sequence from annular Khovanov homology to Khovanov homology. The word problem in the braid group has been solved in many ways; the first solution was presented by Artin in [4] (see also [3]). Garside presented a different solution in [32] which braid theorists like to use and has been implemented in readily available computer programs.

κ provides an obstruction to negative destabilization (Proposition 11). It can also provide an obstruction to positive destabilization for a braid in the case that $\kappa \neq 2$ for its mirror. Corollary 17 implies that it cannot provide an obstruction to destabilization in general. One might hope to show that $\kappa \neq 2$ for a braid and $\kappa \neq 2$ for its mirror, implying that the braid is neither negatively destabilizable nor positively destabilizable. However, Corollary 17 shows that if this is the case, the braid is trivial.

We end this section with a remark on spectral sequences. For any annular link $\mathbb{L}$, there is a spectral sequence whose E^0 page is the annular Khovanov complex of $\mathbb{L}$ and whose E^1 page is, as a group, the annular Khovanov homology of $\mathbb{L}$. Since there are no differentials that drop the k-grading by one, the E^2 page is identical to the E^1 page. Therefore the first page at which the spectral sequence could collapse is E^3. The following proposition provides a counterexample to Conjecture 4.2 from [44].

Proposition 18. *The spectral sequence from annular Khovanov homology to Khovanov homology does not always collapse at the E^3 page.*

Proof. We consider the braid $A(0,0)$ from Proposition 5. The distinguished element $\psi(A(0,0))$ lives in homological grading 4 (before any final shifts) and has k grading -4. Recall that $\psi(A(0,0))$ is unique in the lowest k-grading. By $E^r_{d,m}$ we mean the r'th page of the spectral sequence at homological grading d and k-grading m.

Following [45] (recall: the differentials on CKh increase homological grading),

$$\begin{aligned} E^3_{4,-4} &= \frac{\{x \in \mathcal{F}_{-4}CKh_4 | dx \in \mathcal{F}_{-7}CKH_5\}}{\mathcal{F}_{-5}CKh_4 + d(\mathcal{F}_{-2}CKh_3)} = \frac{\{x \in \mathcal{F}_{-4}CKh_4 | dx = 0\}}{d(\mathcal{F}_{-2}CKh_3)} \\ &= \frac{\mathrm{span}\{\psi(A(0,0))\}}{d(\mathcal{F}_{-2}CKh_3)} = [\psi(A(0,0))] \neq 0 \end{aligned}$$

since $\kappa(A(0,0)) \neq 2$. However, $[\psi(A(0,0))] = 0 \in Kh_4(A)$, and hence $Kh_4(A(0,0)) \neq \bigoplus_{k=-4}^{4} E^3_{4,k}$.

□

Precisely the same argument yields a more general statement:

Proposition 19. *Given a braid β, the length of the spectral sequence from $\mathrm{AKh}(\bar{\beta})$ to $\mathrm{Kh}(\bar{\beta})$ is bounded below by $\kappa(\beta)$.*

4.3 Invariants in reduced Khovanov homology

It is implicit in the proof of Proposition 13 that κ increases under positive stabilization at p precisely if every element which realizes κ has a canonical summand in which p lies on a trivial v_+-labeled circle. This situation cannot occur in (one version of) reduced Khovanov homology, and so one might hope that a "reduced κ" is an invariant of transverse links. That's not quite the case – the eager reader may skip to the examples at the end of this section – but the reduced invariants are interesting in their own right.

In this section let p be a non-double point on an n-strand annular braid diagram $\mathcal{D}$ of $\bar{\beta}$. For convenience, we will assume that the last tensor factor of each generator of $\mathrm{CKh}(\mathcal{D})$ corresponds to the component containing p. There is a chain map $x_p\colon\ \mathrm{CKh}(\mathcal{D}) \to \mathrm{CKh}(\mathcal{D})$ defined on generators by

$$\begin{aligned} x_p(y \otimes v_+) &= y \otimes v_- \\ x_p(y \otimes v_-) &= 0 \end{aligned}$$

Recall that there are two flavors of reduced Khovanov homology (see Section 2.2.2). The *reduced subcomplex* $\widetilde{\mathrm{CKh}}_p(\mathcal{D})$ is defined as $\ker(x_p)$. The *reduced quotient complex* $\underline{\mathrm{CKh}}_p(\mathcal{D})$

is defined as $\mathrm{coker}(x_p)$. It is clear that $\widetilde{\mathrm{CKh}}_p(\mathcal{D})$ has a basis of canonical generators. The projections of canonical generators of the form $y \otimes v_+$ form a basis of $\underline{\mathrm{CKh}}_p(\mathcal{D})$. Whenever we take a representative of an element in the quotient complex, we will assume it is a sum of these canonical generators.

The k-grading on $\mathrm{CKh}(\mathcal{D})$ induces a k-grading on each variant. On the subcomplex $\widetilde{\mathrm{CKh}}_p(\mathcal{D})$ this is simply the restriction. We define the k-grading on $\underline{\mathrm{CKh}}_p(\mathcal{D})$ via canonical representatives: if y is the canonical representative of $\underline{y} \in \underline{\mathrm{CKh}}_p(\mathcal{D})$, then $k(\underline{y}) = k(y)$. However, the isomorphism between the two variants is not in general k-filtered. Thus we will distinguish their homologies as the *reduced homology* $\widetilde{Kh}_p(\mathcal{D})$ and the *reduced quotient homology* $\underline{Kh}_p(\mathcal{D})$. We write $\widetilde{\mathcal{F}}_i$ and $\underline{\mathcal{F}}_i$ for the ith filtered levels of $\widetilde{\mathrm{CKh}}_p(\mathcal{D})$ and $\underline{\mathrm{CKh}}_p(\mathcal{D})$ respectively.

Each complex supports a variant of the transverse element $\psi(\mathcal{D})$. The cycle corresponding to $\psi(\mathcal{D})$ is also a cycle in the subcomplex $\widetilde{\mathrm{CKh}}_p(\mathcal{D})$ for any p. When we wish to emphasize that we are considering $\psi(\mathcal{D})$ as an element of the subcomplex, we will write it as $\widetilde{\psi}_{\mathrm{p}}(\mathcal{D})$. Plamenevskaya defines the reduced quotient invariant $\underline{\psi}_{\mathrm{p}}(\mathcal{D})$ to be the image of the chain $v_- \otimes \cdots \otimes v_- \otimes v_+$ in $\underline{\mathrm{CKh}}_p(\mathcal{D})$. Both $\widetilde{\psi}_{\mathrm{p}}$ and $\underline{\psi}_{\mathrm{p}}$ are invariant under braid conjugation and stabilization away from p in the same sense (and with the same proofs) as ψ. Both cycles have the lowest k-grading in their respective complexes, but $\underline{\psi}_{\mathrm{p}}$ does not necessarily generate that lowest level.

Note that these constructions depend on a choice of p on a particular diagram for a link, so we will not write "$\widetilde{\psi}_{\mathrm{p}}(\bar{\beta})$" or "$\underline{\psi}_{\mathrm{p}}(\bar{\beta})$".

Definition 4.3.1. Let $\beta \in B_n$, let $\mathcal{D}$ be an annular diagram for $\bar{\beta}$, and let $p \in \mathcal{D}$. If $\widetilde{\psi}_{\mathrm{p}}(\mathcal{D})$ is a boundary in $\widetilde{\mathrm{CKh}}_p(\mathcal{D})$, define

$$\tilde{\kappa}_{\mathrm{p}}(\mathcal{D}) = n + \min\{i : [\widetilde{\psi}_{\mathrm{p}}(\mathcal{D})] = 0 \in H_*(\widetilde{\mathcal{F}}_i(\mathcal{D}))\}.$$

If $\widetilde{\psi}_{\mathrm{p}}(\mathcal{D})$ is not a boundary, then define $\tilde{\kappa}_{\mathrm{p}}(\mathcal{D}) = \infty$. If $\psi_p(\mathcal{D})$ is a boundary in $\underline{\mathrm{CKh}}_p(\mathcal{D})$, define

$$\underline{\kappa}_{\mathrm{p}}(\mathcal{D}) = n + \min\{i : [\underline{\psi}_{\mathrm{p}}(\mathcal{D})] = 0 \in H_*(\underline{\mathcal{F}}_i(\mathcal{D}))\}.$$

If $\underline{\psi}_{\mathrm{p}}(\mathcal{D})$ is not a boundary, then define $\underline{\kappa}_{\mathrm{p}}(\mathcal{D}) = \infty$.

The arguments of Section 4.1 show that $\tilde{\kappa}_{\mathrm{p}}(\mathcal{D})$ and $\underline{\kappa}_{\mathrm{p}}(\mathcal{D})$ are invariant under positive stabilization away from p and conjugations that do not cross p.

Lemma 20. *For a fixed diagram $\mathcal{D}$, either $\kappa(\mathcal{D})$, $\tilde{\kappa}_{\mathrm{p}}(\mathcal{D})$, and $\underline{\kappa}_{\mathrm{p}}(\mathcal{D})$ are all finite or all infinite. In the finite case, $\kappa(\mathcal{D}) \leq \tilde{\kappa}_{\mathrm{p}}(\mathcal{D}) \leq \underline{\kappa}_{\mathrm{p}}(\mathcal{D}) \leq \tilde{\kappa}_{\mathrm{p}}(\mathcal{D}) + 2$.*

Proof. There is a short exact sequence of complexes

$$0 \to \widetilde{\mathrm{CKh}}_p(\mathcal{D}) \xrightarrow{i} \mathrm{CKh}(\mathcal{D}) \xrightarrow{\pi} \underline{\mathrm{CKh}}_p(\mathcal{D}) \to 0$$

where i is the inclusion and π is the projection to the quotient. The induced map on homology i_* carries $[\widetilde{\psi}_{\mathrm{p}}(\mathcal{D})]$ to $[\psi(\mathcal{D})]$, so if $[\psi(\mathcal{D})] \neq 0$ then $[\widetilde{\psi}_{\mathrm{p}}(\mathcal{D})] \neq 0$. If i_* is injective, then $[\widetilde{\psi}_{\mathrm{p}}(\mathcal{D})] \neq 0$ implies that $[\psi(\mathcal{D})] \neq 0$. To show that i_* is injective, we repeat Shumakovitch's argument [86] in our notation. Let $\nu\colon\ \mathrm{CKh}(\mathcal{D}) \to \mathrm{CKh}(\mathcal{D})$ be the chain map defined on V by the rule $\nu(v_+) = 0$ and $\nu(v_-) = v_+$ and extended to tensor powers by the Leibniz rule. Note that x_p defines a map $x'_p\colon\ \underline{\mathrm{CKh}}_p(\mathcal{D}) \to \mathrm{CKh}(\mathcal{D})$ by applying x_p to canonical representatives. Let $\underline{c} \in \underline{\mathrm{CKh}}_p(\mathcal{D})$ with canonical representative c. Then $(\nu \circ x'_p)(\underline{c}) = (\nu \circ x_p)(c)$, in which the only term with a v_+ label at p is exactly c. We conclude that $\pi \circ \nu \circ x'_p$ is the identity map, and therefore the short exact sequence splits. Thus i_* is injective.

The first piece of the inequality follows immediately from the fact that $\widetilde{\mathrm{CKh}}_p(\mathcal{D})$ is a subcomplex of $\mathrm{CKh}(\mathcal{D})$. For the next part, suppose that $\underline{z}$ realizes $\underline{\kappa}_{\mathrm{p}}(\mathcal{D})$. Then $d(x_p z) = \psi(\mathcal{D})$ and $k(x_p z) \leq k(\underline{z})$, so $\tilde{\kappa}_{\mathrm{p}}(\mathcal{D}) \leq \underline{\kappa}_{\mathrm{p}}(\mathcal{D})$. On the other hand, suppose that y realizes $\tilde{\kappa}_{\mathrm{p}}(\mathcal{D})$; every canonical summand of y has a v_- at p. Let y^+ be the element obtained from y by changing those v_-'s to v_+'s. Clearly $x_p d(\underline{y}^+) = \psi(\mathcal{D})$, so

$$dy^+ = \underline{\psi}_{\mathrm{p}}(\mathcal{D}) + \text{terms with } v_- \text{ at } p.$$

Therefore $d\underline{y}^+ = \underline{\psi}_{\mathrm{p}}(\mathcal{D})$ and $\underline{\kappa}_{\mathrm{p}}(\mathcal{D}) \leq k(y^+) + n \leq k(y) + 2 + n = \tilde{\kappa}_{\mathrm{p}}(\mathcal{D}) + 2$. (This also shows that $\tilde{\kappa}_{\mathrm{p}}(\mathcal{D})$ is finite if and only if $\underline{\kappa}_{\mathrm{p}}(\mathcal{D})$ is finite.) $\square$

The reduced invariants are stable under positive stabilization at p in the following sense: let p' be a point on the same arc as p. For each reduced complex, the positive stabilization map is filtered and preserves Plamanevskaya's invariant, so Lemma 8 implies that the appropriate version of κ does not change. But after this operation the image of p is not an innermost point. We instead study $\vec{S}^{p'}$, the operation of stabilizing at p' and then moving the basepoint to some point q on the new innermost strand.

Proposition 21. *Let $\mathcal{D}$ be a diagram of $\bar{\beta}$. Then $\tilde{\kappa}_q(\vec{S}^{p'}\mathcal{D}) \leq \tilde{\kappa}_{\mathrm{p}}(\mathcal{D})$ and $\underline{\kappa}_q(\vec{S}^{p'}\mathcal{D}) \leq \underline{\kappa}_{\mathrm{p}}(\mathcal{D}) + 2$.*

Proof. The first inequality follows from Lemma 8 once one makes the observation that the positive stabilization map carries $\widetilde{\mathrm{CKh}}_p(\mathcal{D})$ to a subcomplex of $\widetilde{\mathrm{CKh}}_q(\vec{S}^{p'}\mathcal{D})$ and carries $\widetilde{\psi}_{\mathrm{p}}(\mathcal{D})$ to $\widetilde{\psi}_q(\mathcal{D})$.

Suppose that $\underline{z}$ realizes $\underline{\kappa}_{\mathrm{p}}(\mathcal{D})$. Let q be a point on the innermost strand of $S^{p'}\mathcal{D}$. Recall that there is a map ρ on the Khovanov complex induced by positive stabilization. This map descends to a map $\rho\colon \underline{\mathrm{CKh}}_p(\mathcal{D}) \to \underline{\mathrm{CKh}}_p(S^{p'}\mathcal{D})$ which sends $\underline{z}$ to a sum of generators with v_- at q and v_+ at p. Let $\underline{z}' \in \underline{\mathrm{CKh}}_q(\vec{S}^{p'}\mathcal{D})$ be the element whose canonical representative z' is obtained from that of $\rho(\underline{z})$ by swapping these labels. Note that $dx_q z' = \rho(\psi(\mathcal{D})) = x_q dz'$, so $d\underline{z}' = \underline{\psi}_q(\mathcal{D})$. Clearly $k(\underline{z}') \leq k(\underline{z}) + 1$. The second inequality follows after taking into account that the operation $S^{p'}$ increases braid index by one. $\square$

Remark. *It is interesting to consider the sharpness of these inequalities using annular Khovanov homology. The map x_p is filtered and therefore induces a map on $\mathrm{AKh}(\mathcal{D}) = \bigoplus \mathcal{F}_i/\mathcal{F}_{i-1}$, the annular Khovanov homology of $\mathcal{D}$.*

Let p, p′, z, and z' be as in the previous proof. The point q lies on a non-trivial circle in every resolution of $S^{p'}\mathcal{D}$, so $k(z') > k(S^{p'}z) = k(z)$ precisely if p lies on a trivial circle in some canonical summand of z. Equivalently, $k(z') = k(z)$ precisely if p lies on a non-trivial circle in every canonical summand of z. Therefore $\underline{\kappa}_{\mathrm{p}}(\mathcal{D}) = \underline{\kappa}_q(\vec{S}^{p'}\mathcal{D})$ if and only if some z realizes $\underline{\kappa}_{\mathrm{p}}(\mathcal{D})$ and p lies on a non-trivial circle in every canonical summand of z. Write $\langle z\rangle$ for the image of z in $\mathrm{AKh}(\mathcal{D})$. Then $\underline{\kappa}_{\mathrm{p}}(\mathcal{D}) = \underline{\kappa}_q(\vec{S}^{p'}\mathcal{D})$ if and only if $\langle z\rangle \in \ker(x_p)$ for some z which realizes $\underline{\kappa}_{\mathrm{p}}(\mathcal{D})$.

While $\underline{\kappa}_{\mathrm{p}}$ is not preserved under stabilization, it is preserved under a certain sort of conjugation over p. Denote by C_p the operation of performing a braidlike Reidemeister 2 move over p. (In terms of braid words, this inserts $\sigma_{n-1}\sigma_{n-1}^{-1}$ or $\sigma_{n-1}^{-1}\sigma_{n-1}$.) Denote by $\vec{C}_p$ the operation C_p followed by moving the basepoint to the innermost strand at q. See Figure

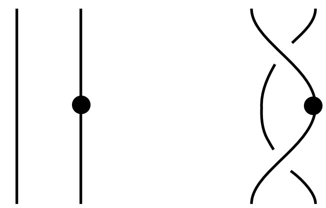

Figure 4.2: The result of the operation $\vec{C}_p$ on two strands

4.2. The Reidemeister 2 map induces a filtered map $\underline{\mathrm{CKh}}_p(\mathcal{D}) \to \underline{\mathrm{CKh}}_q(\vec{C}_p\mathcal{D})$ which carries ψ to ψ.

Proposition 22. $\underline{\kappa}_{\mathrm{p}}(\mathcal{D}) = \underline{\kappa}_q(\vec{C}_p\mathcal{D})$.

To dash any hope that $\underline{\kappa}_{\mathrm{p}}$ or $\tilde{\kappa}_{\mathrm{p}}$ might be transverse invariants, we note that both invariants depend on the placement of p. For $\underline{\kappa}_{\mathrm{p}}$ this is true even for negative stabilizations.

Example. *Let $\beta = \sigma_1\sigma_2^{-1} \in B_3$. Certainly ψ is null-homologous and $\kappa = \tilde{\kappa}_{\mathrm{p}} = 2$ for any p. Let p_1 and p_2 be points on the first and second strands of the braid. Then*

$$\underline{\kappa}_{p_1} = 2$$
$$\underline{\kappa}_{p_2} = 4$$

For a meatier example, we revisit the transversely non-simple family using the previously advertised computer program.

Example. *Recall that Ng and Khandhawit define two infinite families of braids $A(a,b)$ and $B(a,b)$ so that, for any $a,b \in \mathbb{Z}_{\geq 0}$, the closures of $A(a,b)$ and $B(a,b)$ have the same topological knot type and self-linking number but are not transversely isotopic. Write A_0 and B_0 for $A(0,0)$ and $B(0,0)$. We have already seen that $\kappa(A_0) = 4$ and $\kappa(B_0) = 2$. For any $p \in \bar{A}_0$ we have $\tilde{\kappa}_{\mathrm{p}}(A_0) = 4$ and $\underline{\kappa}_{\mathrm{p}}(B_0) = 4$. On the other hand, $\underline{\kappa}_{\mathrm{p}}(A_0)$ and $\tilde{\kappa}_{\mathrm{p}}(B_0)$ depend on p. See Figure 4.3.*

Remark. *It is straightforward to check that the two candidates for "reduced annular Khovanov homology" are not isomorphic: see Section 2.3.3. In addition, Shumakovitch's map ν (see Lemma 20) is not a chain map on the annular complex as it does not commute*

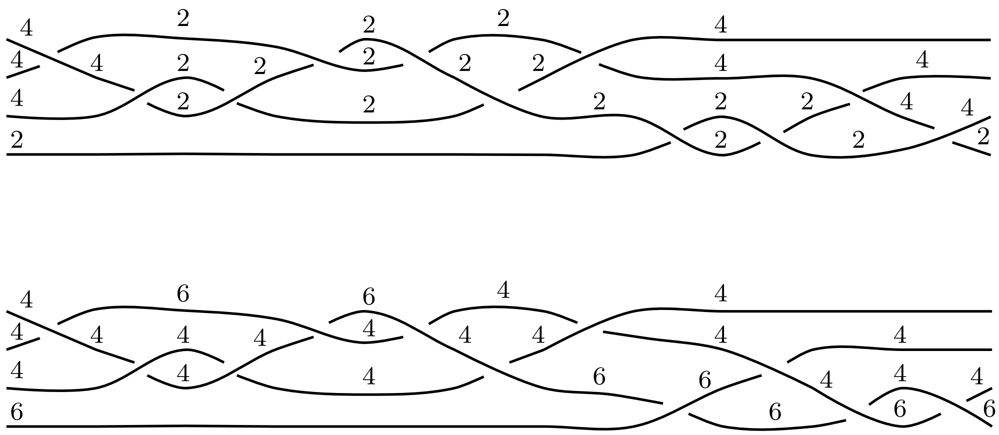

Figure 4.3: Values of $\underline{\kappa}_{\mathrm{p}}$ and $\tilde{\kappa}_{\mathrm{p}}$ may depend on p. The top braid is B_0 and the bottom braid is A_0. The number above each arc represents a value of $\underline{\kappa}_{\mathrm{p}}$ (for A_0) or $\tilde{\kappa}_{\mathrm{p}}$ (for B_0) when p is placed on that arc.

with the differential. These calculations show that the difference between the two versions is significant, and that the two reductions might provide different information.

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