# Designing fair tiebreak mechanisms for sequential team contests

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# Designing Fair Tiebreak Mechanisms for Sequential Team Contests*

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**Abstract**

Economists have long recognized that the effect of the order of actions in sequential contests on performance of the contestants is far from negligible. We model the tiebreak mechanisms, known as penalty shootouts, which have sequential move order and are used in several team-sports contests, as a dynamic mechanism-design problem with a fairness desideratum in mind. In most sports, such as soccer and hockey, the order in which teams take the penalties is fixed, known as ABAB. We first show that even with two balanced teams, this mechanism possesses many symmetric Markov-perfect equilibria giving various asymmetric winning probabilities to the first- and second-moving teams (but favoring the first mover under appropriate refinements) – an observation which also obtained some empirical support for soccer. Following that, we characterize all sequentially fair mechanisms in which two balanced teams have equal chances to win the shootout whenever the score is tied after equal numbers of attempts. Using additional desirable properties, we uniquely characterize practical mechanisms for easy and difficult shootouts, in which the success probability of a penalty kick is uniformly greater than 50% or less than 50%, as in soccer and hockey, respectively.

**Keywords:** Market design, mechanism design, penalty shootouts, tiebreak mechanisms, soccer, hockey, fairness in dynamic contests

**JEL Codes:** D78, D63, D47, C79

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# 1 Introduction

Economists have long recognized that the effect of the order of actions in sequential contests on performance of the contestants is far from negligible; examples in sequential individual and team contests are abound, e.g., R&D races (Fudenberg et al., 1983 and Harris and Vickers, 1985, 1987), job promotions (Rosen, 1986), political campaigns (Klumpp and Polborn, 2006), music competitions (Ginsburgh and van Ours, 2003) as well as penalty shootouts in soccer matches (Apesteguia and Palacios-Huerta, 2010) and tennis matches (Cohen-Zada, Krumer, and Shapir, 2018 and the references therein). Clearly, an order of actions that provides a systematic first- or second-mover advantage to one of the parties may decrease the probability of the 'better' contestant to win, causing efficiency and fairness issues. These unfair advantages may arise since the order of actions in contests may generate different strategic effects.

Understanding these strategic effects of move order on a well-defined sequential team contest, which has been empirically studied many times, is invaluable for empirical economists and mechanism designers alike. The history and experience of soccer and other sports' tiebreak mechanisms, known as penalty shootouts, present us a unique natural experiment to reckon and propose a plausible model and mechanism design.

Recently, in real-world contests that consist of multiple battles, where the final success or failure in a contest is determined by the outcome from multiple battles, the standard order of actions between two parties, A and B, in the form of ABABABAB... (ABAB hereafter; also known as *fixed order* as every party moves once in every round of the battle in a fixed order, A first, B second) has been modified to become ABBAABBA... (ABBA hereafter, also known as *alternating order*). For example, ABBA replaced ABAB in the U.S. presidential debates as well as in tiebreak shootout procedures at the U-17 Women's and Men's World Soccer Championships. The three 2016 U.S. presidential election debates between Donald Trump and Hillary Clinton, also followed the ABBA structure.[1] The International Football Association Board (IFAB) - the body that determines the rules of soccer - decided to implement the ABBA sequence in various trials before eventually replacing the current ABAB sequence.[2] In addition, the ABBA format for penalty shootouts is adopted in all English Football League (EFL) competitions in 2017-18, while the rest of the world still uses ABAB as of this writing.[3] Similarly, FIDE, the governing body of chess, has recently

[1]In each debate, each segment started with a two-minute speech by one candidate followed by a two-minute speech by the other. The first speaker in the first segment was determined according to a coin toss. But then in each segment the order of the first two speeches was reversed. E.g., on the final TV debate the order of the two-minute speeches went as follows: Clinton, Trump, Trump, Clinton, Clinton, Trump, and so on.

[2]See https://www.independent.co.uk/sport/football/international/uefa-penalty-shooutout-rules-system-new-trial-a7715026.html retrieved on Feb 13, 2018.

[3]Penalty shootouts currently constitute the only way to determine the winning team when the score is tied in major soccer elimination tournament matches after the regular 90-minute period and the 30-minute extra time, i.e., the overtime. It is customary to use tiebreak mechanisms in many other sports as well to determine the eventual winner when the regular match ends with a tie, e.g., tennis, ice hockey, field hockey, water polo, handball, cricket, and rugby. In a shootout, traditionally each team took five penalty kicks from the penalty mark in fixed order (ABAB),

changed the rules and regulations for the FIDE World Chess Championship and switched from ABAB to a generalized ABBA structure that reverses the sequence of players who play with white pieces at the half-way, i.e., ABABAB-BABABA. Traditionally ABBA has been used in tennis for serve order in tiebreak sets.

A particular observation shared by multiple empirical studies regarding soccer penalty shootouts is that the degree of how much kicking order in the traditional ABAB sequence matters may differ across different soccer competitions/traditions. For example, although kicking order does not matter for the German national cups, the Spanish national cup shootouts notoriously favor first-kicking teams significantly. On the other hand, in English national cups, the first-kicking team has only a slight advantage. These three countries represent different soccer styles and player characteristics, and their datasets consist of hundreds of shootouts.[4] Hence, in different tournament/country environments the current shootout mechanism leads to different focal outcomes in terms of first- and second-moving teams' winning chances. In addition, some researchers provide evidence that the first kicking team wins significantly more often overall with the current fixed-order shootout mechanism, while some others dispute that evidence.[5] No study, however, provides evidence that the second kicking team wins more often overall.

Thus, the rationale behind experimenting with ABBA in soccer is the belief that ABBA will eliminate or at least alleviate a first-mover advantage if it exists. Indeed, using simple calculations assuming exogenous first-kick and second-kick success probabilities, Brams and Ismail (2018) and Echenique (2017) show that ABBA is less biased than ABAB. For example, Echenique performs calculations to argue that if the first-mover advantage ranges between 60% and 70% in the ABAB system, it should move to the 52-55% range under the ABBA system.

In many multi-battle contests, players from rival teams, alliances or groups form pairwise matches and compete head-to-head, e.g., as in the soccer penalty shootout contests. Each team, alliance or group consists of independent players who share some common interests. Individual players in these teams take decentralized actions to obtain victories in their own battles, while their performance contributes to their team's or group's performance. In such dynamic team contests, a team's success or win benefits all its members, while likewise a team's failure or loss hurts all its members. In addition, each team member bears the cost of their own contributions and may

and the order of the kicks has always been decided by the referee's initial coin toss. Before 2003, the team that wins the coin toss gets to kick first in each round, including the sudden death rounds. Since 2003 the team that wins the toss decides which team kicks first. Although the winner of the coin toss has the option to go second, this has rarely been the case in actual shootouts. If the shootout score is tied after each team takes five penalty kicks, sudden-death rounds are reached, which go on until the tie is broken such that the kicking order is preserved or "fixed" in these extra rounds.

[4] See Table 5.1 in Palacios-Huerta (2014) and Table 1 in Kocher, Lenz, and Sutter (2012), which are summarized in Figure A.1 in Appendix A.

[5] See Apesteguia and Palacios-Huerta (2010) and Palacios-Huerta (2014) for evidence on the first kicking team winning significantly more often; Kocher, Lenz, and Sutter (2012), on the other hand, dispute this finding (see also Figure A.1).

also enjoy their individual success even if their team loses or suffer their individual failure even if their team wins. Overall, teams' expected winning odds and the expected total effort of the individual players can depend on the sequencing arrangements and the temporal structure of the contest. Thus, the probability of a player winning his battle can be determined by not only his and his matched opponent's abilities and strategies, but also depends on the state of the contest, such as how much one team is leading or lagging behind its rival at any given point.

This paper models shootouts as a mechanism design problem in sequential team contests with a fairness desideratum in mind. We can classify penalty shootouts as easy-task or difficult-task shootout/contest based on the goal scoring probability. A shootout is easy if the goal probability is greater than 50% for any kick in any round and it is difficult if the goal probability is less than 50%. A soccer shootout is an example of an easy task, while a hockey shootout is an example of a difficult task.[6]

Shootouts tend to be shorter and more structured than a regular match and can be modeled like dynamic versions of contests. We introduce such a model in which the kickers not only care about their team's winning the shootout but also about the outcome and even the individual performance they display during taking their penalty shot. Given that fairness, rather than revenue maximization and even efficiency, is the main desideratum of the design of tiebreaks, the first important question we address is what fairness means in this environment, where a coin toss determines the first-kicking team.

We tackle this question from two different angles: Whenever the score is tied after any round in a penalty shootout, having two teams (1) that are totally balanced in terms of their players' shootout abilities, a *sequentially fair* outcome should mean that each team is expected to win the shootout with 50% probability, and (2) where one team has higher-ability kickers than those of the other team, then the higher-ability team should have a higher probability of winning at state-symmetric perfect Bayesian equilibria. These two parts lead to nothing but the age-old Aristotelian Justice principle, which rests on a two-part criterion: equals should be treated equally and unequals unequally (see Aristotle, 1999).[7]

We first show that kicking order matters to each team's chance of winning the shootout in that the current fixed-order mechanism, ABAB, (1) is not sequentially fair and (2) can lead to many equilibria, with different winning probabilities for first- and second-moving teams. In Appendix C, we show that it is possible to devise a forward-looking equilibrium refinement that gets rid of the multiplicity of equilibria, in which the first-kicking team wins more often while any equilibrium candidate with the second-mover advantage does not survive it.

[6]Scoring a penalty shot is "easy" in soccer with a success probability greater than 59% in each round - see, for example see Apesteguia and Palacios-Huerta, 2010), while it is "difficult" in hockey with a probability less than 35% (see, for example, http://businessofhockey.wordpress.com/2015/01/04/a-deep-look-into-advanced-shootout-statistics/ retrieved on Feb 13, 2018).

[7]Observe that the use of a coin toss as a simple tiebreak mechanism, as FIFA did before 1970 (see Section 3), would be sequentially fair and satisfy the first part of the Aristotelian justice criterion, but it would fail its second part. Thus, this justice criterion rules this simple tiebreak mechanism out.

We then ask whether it is possible to devise a shootout mechanism that is sequentially fair. In our characterization of sequentially fair mechanisms in regular rounds, we find that there is only one *exogenous* mechanism, namely the unbiased random-order mechanism – in which the kicking order before any round is determined by an unbiased coin flip – that is sequentially fair. Non-random-order exogenous mechanisms, which, like the current fixed-order mechanism, ABAB, or the alternating order, ABBA sequence, have a predetermined kicking-order pattern by teams, are found not to be sequentially fair. Thus, not only the ABAB sequence but the recently adopted ABBA sequence is not sequentially fair, either.

Then the question becomes whether there are any other sequentially fair tiebreak mechanisms beside the random-order mechanism. As a result, we identify a continuum of sequentially fair mechanisms in which move order is *endogenous*. We show that sequential fairness fully coincides with "uneven score symmetry." That is, as long as the score is not tied at the end of a round, the probability of which team moves first in the next round is the same for both teams whenever they are in each other's shoes. E.g., when Team A is ahead $3-2$ in the beginning of Round 4, and when Team B is ahead in Round 4 with a score of $2-3$, in Round 4 Team A's probability of moving first in the first case is the same as Team B's probability of moving first in the second case. Thus, there is a continuum of sequentially fair mechanisms.

We also show that sequentially fair mechanisms satisfy the second part of the Aristotelian Justice principle: among two teams with unequal kicking skills, the better team will have a higher probability of winning the shootout at symmetric equilibria.

There are clearly many other factors that are known to affect the outcome of the current shootout mechanisms, maybe, unfairly. For example, performance under pressure is one that is investigated in both sports sciences (see Jordet et al., 2006 and Jordet, 2009) and in economics (see Apesteguia and Palacios-Huerta, 2010). Hence, the more we can equalize the stressful situations the teams face, the better it will be according to this criterion. For example, alternating the order of teams, like in the ABBA sequence, when score is tied is one way to facilitate this kind of fairness, or using coin flips throughout. One other factor that needs to be considered is practicality, i.e., easy to be implemented in the field. It turns out that if these concerns are answered appropriately, they would not be at odds with sequential fairness. We come up with simple fundamental properties to obtain a tight characterization among the continuum of the sequentially fair mechanisms in regular rounds we found.[8]

In Section 8, where we adopt the approach of market design, we discuss the relative merits of different sequentially fair mechanisms in terms of these additional criteria. We start with a property that can be used to give the opportunity of kicking to as many players as possible. For

[8]With this paper, we certainly do not claim to have addressed every concern in design of penalty shootouts. After reading our paper, some economists, especially those who are soccer fans, will no doubt come up with what we could have missed in our modeling or what else could be done additionally. This is the first paper using rational economic theory to address this design issue, and it addresses many empirical abnormalities surprisingly well through our approach of one parameter deviation from a model of players with only outcome-oriented preferences.

easy contests, we show that there is a class of sequentially fair mechanisms satisfying *maximization of the expected number of attempts*, which we term the *behind-first* mechanisms, such that the team that is behind in score after a round always kicks first in the next round, but if the score is tied after any round, then any random or fixed exogenous or endogenous order is admissible at the next round. In difficult contests, the same property is satisfied by *ahead-first* mechanisms.

Note that sequentially fair mechanisms, including behind-first and ahead-first mechanisms, leave unspecified how one should choose which team would kick first when the score is tied, which would be a major issue especially during the sudden-death rounds of a shootout. One thing we show is that the fixed-order scheme, ABAB would not be sequentially fair, and the order should somehow alter in sudden-death rounds. Hence, we embed the ABBA structure to our behind-first and ahead-first mechanisms in sudden-death rounds. Although there are many other ways to obtain sequential fairness by altering order in sudden-death rounds, this mechanism provides a *sudden-death equality of opportunity* to both teams in addition to its practicality. This also addresses almost all fairness concerns of previous investigators regarding "performance difficulties under stress" criteria: One team kicks at most one more or less time than the other team in stressful situations of kicking while behind (and equal times when the sudden-death rounds are even).

This change of order also naturally translates to the regular rounds with a formally defined simplicity property. Our *simplicity* property minimizes the patterns of how kicking order changes across rounds while keeping the probability of either team kicking first ex-ante positive. We uniquely characterize easy shootout mechanisms, such as the ones in soccer, satisfying sequential fairness and maximization of the expected number of attempts together with the other two properties, namely, simplicity and sudden-death equality of opportunity: The team that is behind in score after a round always kicks first in the next round, but if the score is tied after any round, then the team that kicked second at that round kicks first in the next round. We refer to this mechanism the *alternating-order behind-first* mechanism. Symmetrically, in difficult shootouts, such as in hockey, its symmetric counterpart, *alternating-order ahead first* uniquely characterizes this class.

## 2 Other Relevant Literature

Apart from the papers mentioned in the Introduction, our paper is also related to the following strands of research. In the first strand, Chiappori et al. (2002) studies soccer penalty kicks both theoretically and empirically to test mixed strategies, while Palacios-Huerta (2003) does so with a much more empirical focus. Both papers have considered regular penalty kicks during matches rather than penalty kicks in shootouts.

The second strand focuses on topics of economic design of sports contests, such as the optimal number of entrants/teams in a race/league, the optimal structure of prizes (revenue sharing) for a tournament (league), and so on (see Szymanski, 2003 for a review of this particular literature).

Before we finish this section, we need to make three remarks.

First, a precursor of our concept of sequential fairness can be found in Che and Hendershott (2008), who use a static one-shot version of it, ex-post fairness.[9]

Second, the similarity to the economic contest literature and our paper is not superfluous. Our model leads to a "difference in efforts"-based contest-success function and we can derive this endogenously from the specifics of our model (see Anbarcı et al., 2018). This is a rather non-standard contest-success function than the ones used in the contest literature in economics. Moreover, ties are allowed in our contest function within a round, while there is usually a winner or a loser in the standard contest functions such as Tullock's in every round of the contest.

Finally, in a general sense, our paper belongs to market design, a field seeking to provide practical solutions mainly to resource-allocation problems where the market mechanism may not be used either because monetary transfers are not allowed simply because such transfers would draw legal and ethical objections (e.g., public school slots and human kidneys are not allowed to be traded for money) or for other reasons such as unfairness or congestion. Fairness desideratum in the design of penalty shootouts also places these specific tiebreak mechanisms in this category of research. This field commonly studies problems that were viewed to be outside of the scope of traditional economics paradigm and proposes practical solutions using traditional tools of economic theory, such as problems of student admissions at schools (see Abdulkadiroğlu and Sönmez, 2003) or organ transplantation (see Roth, Sönmez, and Ünver, 2004).

# 3 Background on Soccer and Shootouts

Until 1970, elimination matches that were tied after extra time were either decided by a coin toss or replayed in two days if it was a finals match. Finally, the events in the 1968 European football championship led FIFA in 1970 to try penalty shootouts instead.[10] Given that the current shootout mechanism, ABAB, is no panacea, FIFA and England are now experimenting with alternating the orders of kicking teams, i.e., using ABBA scheme.

## 3.1 Unpredictability of Penalty Kicks

The soccer players who take the penalty kicks in shootouts are typically among the most skilled and elite professionals in the world, while the task they have to perform is a relatively easy but risky one. It involves hitting a spot with the ball from 12 yards (approximately 11 meters) at a sufficiently high speed to elude a high-caliber goalie who is scrambling to protect an 8-yard-wide

[9] In the National Football League (NFL), matches that end in a tie are determined by a sudden-death-like overtime. The initial coin toss yields a significant advantage to the team that wins it and the outcome fails to be ex-post fair. Che and Hendershott (2008, 2009) propose "auctioning off" or "dividing-and-choosing" the starting possession to restore ex-post fairness.

[10] A semifinal match was decided by a coin flip. The final match also ended in a tie after over time. But because of the growing public outrage of soccer fans and since there was no further match left in the tournament, authorities decided to replay this match in two days.

goal. Thus, each such kick involves an element of risk and can turn out to be costly for the kicker, especially if the miss is unambiguously his fault. The following quote by Italy's Roberto Baggio provides strong implications about plausible assumptions regarding players' preferences and the various basic physical aspects of a penalty kick (see the next subsection):[11]

> "As for the penalty, I don't want to brag but I've only ever missed a couple of penalties in my career. And they were because the goalkeeper saved them not because I shot wide. That's just so you understand that there is no easy explanation for what happened at Pasadena. When I went up to the spot I was pretty lucid, as much as one can be in that kind of situation. I knew [the Brazilian goalie] Taffarel always dived so I decided to shoot for the middle, about halfway up, so he couldn't get it with his feet. It was an intelligent decision because Taffarel did go to his left, and he would never have got to the shot I planned. Unfortunately, and I don't know how, the ball went up three meters and flew over the crossbar. ... I failed that time. Period. And it affected me for years. It is the worst moment of my career. I still dream about it. If I could erase a moment from my career, it would be that one."

## 3.2 Kickers' Preferences Over Outcomes and Physical Aspects of Penalty Kicks

The outcomes of players' kicks pertain to their teams as well as to themselves. From the team's perspective a goal is preferred to a non-goal, and clearly there is no difference at all between a saved kick and a kick that misses the goal. From Baggio's quote, we also infer that, from a player's perspective, while scoring a goal is the best outcome and the goalie's save has to some extent a face-saving value, missing the goal can be the worst of all outcomes for a kicker. Thus, a kick can be extremely costly for the kicker if his kick misses the goal, in which case the entire blame can be assigned to him unambiguously. One cannot, however, posit whether a player's individual utility from his kick or his collective utility from his team's winning the shootout should outweigh one another. For example, a player can be somewhat happy and heartbroken at the same time (1) if he scored his penalty kick while his team lost the shootout, or (2) if he missed his penalty kick outright while his team won the shootout.

We also infer from Baggio's quote that goalies typically feel the need to dive at the time the ball is kicked.[12] For that reason, for a significant portion of penalty kicks, the goalie and the ball end

[11] Baggio had a stellar career and his five goals in the tournament helped Italy to reach the final's match of the 1994 World Cup against Brazil. With the shootout score at 3-2, as the last kicker in regular rounds Baggio had to score to keep Italy's chances alive. He aimed for the middle but the ball sailed over the crossbar. The quote is from Baggio's (2001) autobiography.

[12] This is because, at the optimal speed-accuracy combinations of world-class kickers, the kicked ball typically takes around 0.3 seconds to reach the goal line (see, e.g., Harford, 2006, Chiappori et al., 2002, and Palacios-Huerta, 2003), which is less than the total of (1) roughly 0.2 seconds' reaction time of the goalie to clearly recognize the kick

up in opposite corners of the goal. Even when the goalie dives in the correct direction, he cannot save a goal with 100% chance, since he must also be able to reach the ball.

# 4 Model

## 4.1 The Setup

Two soccer teams, which we refer to as Teams 1 ($T1$ for short) and 2 ($T2$ for short), are facing off in a penalty shootout.[13] Each team shall take $n$ sequential rounds of penalty shots. Each round consists of one team kicking first, and, after observing the outcome of that shot, the second team taking the next shot. If one team scores more goals than the other at the end of $n$ rounds, then it wins the match. We refer to these $n$ rounds as the **regular rounds**. Throughout the paper we will assume that $n = 2$. This is sufficient to characterize sequential fairness and analyze the current scheme, ABAB, as well as other proposed mechanisms, such as the alternating-order mechanism, ABBA. Thus, with $n = 2$, the analysis is tractable and yet rich enough to capture the multi-round feature of penalty shootouts.[14]

If the shootout score is tied at the end of $n$ regular rounds, the format reverts to **sudden death**; that is, each team takes an additional round of shots, and then, if one team scores while the other one does not, the former team wins the match; otherwise a further round of sudden-death penalty shots is taken. We refer to the sudden-death rounds as $n + 1, n + 2, ....$

Since potentially the match can continue forever, we assume that each team consists of an infinite number of kickers and that each kicker takes at most one shot.[15]

A penalty kick consists of a probabilistic event with three outcomes: Either a goal is scored (G), the shot goes wide out (O), or the shot is saved by the goalie (S). The latter two outcomes lead to the same score for the team: a goal is not scored.

While each kicker is a strategic player, for tractability the goalie is modeled as a probabilistic machine (also see Fn 8).[16] The goalie waits in the middle of the goal line prior to the shot. He

direction of the ball first, plus (2) the time during his dive to reach the expected arrival spot of the ball before it reaches the goal plane. Hence, a goalie cannot afford to wait until he clearly observes the kick direction: to prevent a goal with non-trivial probability, he must commit to pick a side to dive - or alternatively to stay in the middle. As Baggio's quote also indicates, a shot aimed at the middle may be missed outright or may hit the feet or the legs of the diving goalie that cover part of the middle; thus, the shot can be saved even if the goalie dives.

[13] Instead of the conventional Team A vs Team B notation, we use $T1$ vs $T2$ because we reserve the letters A and B to the interim states of the dynamic contest when one team is *ahead* or *behind* the other.

[14] We have $n = 3$ results in an online appendix at http://www2.bc.edu/utku-unver/research/fair-tiebreak-asu-online-appendix.pdf, and no extra insight exists in this analysis. Similarly, we skip $n > 3$ as the analysis becomes extremely cumbersome and lengthy without providing any further insight.

[15] In reality, each soccer player can take at most one shot, unless all players in his team have already kicked penalty shots. As each team consists of 11 players, 11 shots need to be taken by each team before any player can kick a second shot. As $n = 5$, this happens very rarely.

[16] Alternatively, one can opt for a simple game in which the goalie is also strategic, as concluded by both Palacios-

jumps to one side or the other with probabilities $\frac{1}{2}:\frac{1}{2}$ prior to the penalty shot, as he needs to react early to have any realistic chance to save the kick (see our previous footnote, Fn 12). So with probability $\frac{1}{2}$ he reaches to the same side of the goal as the kick. Hence, we model the goal line as a one-dimensional line segment $[0,1]$, where $x=0$ refers to the center of the goal, and $x=1$ refers to the goal pole on the side of the kick (see Figure 1).

Each kicker, who is a single-round player in our game, has an action summarized as aiming at coordinate $x \in [0,1]$ of the goal line, which we refer to as the **intended spot**. When a kicker aims at $x$, the exact spot the ball reaches on the goal line is determined by a continuous probability density function $\sigma_x$ in a closed support $[\underline{\epsilon}_x, \overline{\epsilon}_x]$ for some $\overline{\epsilon}_x > x > \underline{\epsilon}_x$. The spot the ball reaches, $y$, is observable by all other players, but not the intended spot, $x$. Both $x$ and $y$ are observable by the kicker himself. Moreover, given that the shot is aimed at $x$, there is a $P_G(x)$ probability that a goal will be scored and a $P_O(x)$ probability that the shot will go out. Hence, the shot is saved by the goalie with probability $1 - P_G(x) - P_O(x)$.[17] We assume that $P_G$, $P_O$, and $\sigma_x$ for all $x \in [0,1]$ are all common knowledge.

We assume that $P_G$ is a twice continuously differentiable strictly concave function, which reaches its maximum at some interior $\overline{x} \in (0,1)$.[18]

Although so far we developed our theory taking soccer as our primary application, the insights we discover apply to other contests and sports. In particular, we can classify penalty shootouts as easy-task or difficult-task based on the goal scoring probability $P_G(x)$. A shootout is **easy** if $P_G(x) > \frac{1}{2}$ for all $x \in [0,\overline{x}]$. A shootout is **difficult** if $P_G(x) < \frac{1}{2}$ for all $x \in [0,\overline{x}]$. A soccer shootout is an example of an easy task, while a hockey shootout is an example of a difficult task.[19] This distinction will not matter in our results until we discuss practical market design in Section 8. We assume that the shootout is either easy or difficult, but not mixed, throughout the paper. Thus,

Huerta (2014) and Chiappori et al. (2002). In this simple 2x2 game, neutral-sided kickers kick and goalies dive. Then it would be clear that each kicker would kick the ball right or left with 1/2 probability each and the goalie would dive to each side with 1/2 probability. Thus, the equilibria of these games lead to our reduced-form goalie behavior.

[17] Actually $P_G$ and $P_O$ are summary functions obtained from the following process: As mentioned before, the spot the ball reaches, $y$, is observable by all other players, but not the intended spot, $x$. If $y > 1$, then the ball goes out. So $P_O(x) = \int_{y=1}^{\overline{\epsilon}_x} \sigma_x(y)dy$. On the other hand, the goalkeeper can save the ball that arrives at spot $y$ with probability $S(y)$, which is a continuous function. Hence, $P_G(x) = \int_{\underline{\epsilon}_x}^{1} [1 - S(y)]\sigma_x(y)dy$. Hence, we assume that the family of densities $\{\sigma_x\}_{x\in[0,1]}$ and save probability function $S$ have all the properties that need the below restrictions to hold for $P_G$ and $P_O$.

[18] In fact $P_G$ is not concave around 0, and it is decreasing, as the ball can go both sides of the middle, $x=0$, when it is aimed at $x=0$. Nevertheless, we assume the goal-maximizing point is farther to the right. Therefore, without loss of generality, we use a strictly concave $P_G$.

[19] As alluded to in the Introduction, scoring a penalty shot is "easy" in soccer with a success probability greater than 59% in each round (for example, see Apesteguia and Palacios-Huerta, 2010), while it is "difficult" in hockey with a probability less than 35% (for example, see http://businessofhockey.wordpress.com/2015/01/04/a-deep-look-into-advanced-shootout-statistics/ retrieved on Feb 13, 2018). As different from soccer, usually hockey goalies do not dive one side of the goal as the goal is substantially smaller, hence, the goal probabilities are uniformly less than 50%.

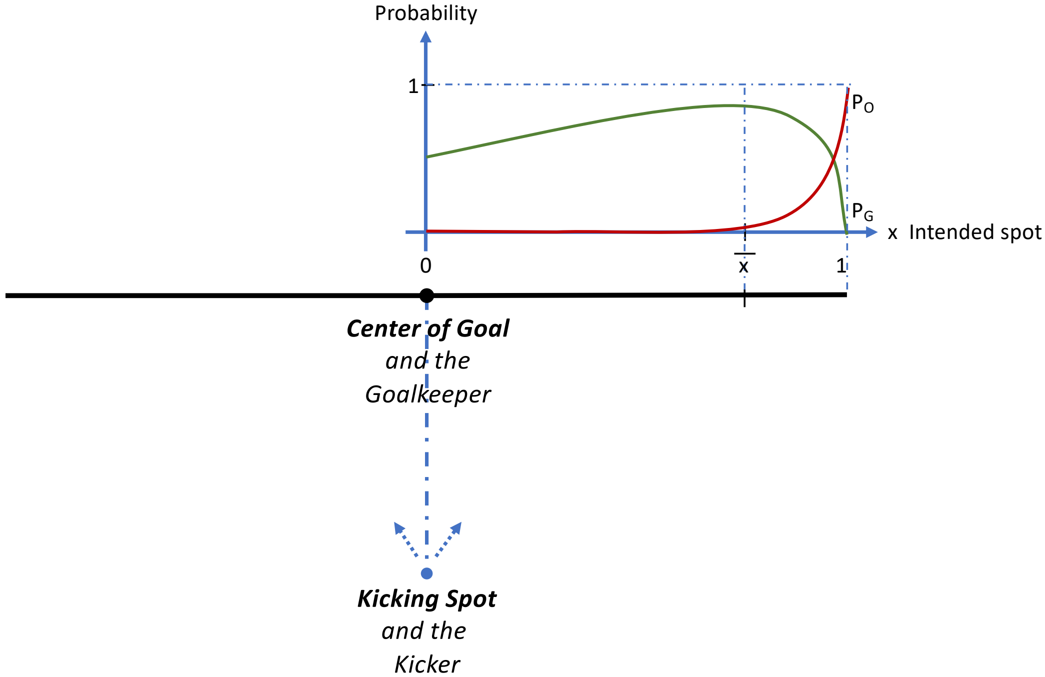


Figure 1: Depiction of the goal and kicking spot (see lower portion of the figure) and our modeling of the goal, the goal probability function for an *easy* shootout, $P_G$, and the probability function of kicking out, $P_O$, as functions of distance from the center of the goal, $x$, which we refer to as the intended spot (see higher portion of the figure). Observe that the left side of the goal would have a perfectly symmetric $P_G$ and $P_O$ curves around $x = 0$. Thus, modeling one side is sufficient for our purposes.

our analysis will focus on these two cases throughout.

Function $P_O$, on the other hand, is an increasing twice continuously differentiable convex function. Increasing $P_O$ is straightforward to motivate: the closer to the middle the ball is aimed, the lower is the chance that the ball will go out. Single-peakedness of $P_G$ is also easy to motivate: Whenever the ball is aimed at low $x$ values, it can be saved with a higher chance by the diving goalie (see the previous footnote). For higher $x$ values, although the goalie's chances of saving the ball decrease as he may no longer be able to reach it, the chances of the ball going out increase. Hence, it is easy to motivate the spot $\overline{x}$, which maximizes the goal probability. We will refer to it as the **goal-optimal** spot. Concavity of $P_G$ and convexity of $P_O$ are primarily assumed for the tractability of our analysis, and do not play any other major role for the interpretation of our results. See Figure 1 for a depiction of our model for an easy shootout.

We assume that each kicker on both teams is identical in ability and has the same goal-scoring and kicking-out probability unless otherwise noted (as we do in Subsection 7.1).

## 4.2 Shootout Mechanisms and the Shootout Game

A **shootout mechanism** is a function, $\phi$, that assigns a probability $\phi(h^{k-1}, g_{T1} : g_{T2})$ to $T1$ kicking first in Round $k$, given the sequence of first-kicking teams in the first $k-1$ rounds is $h^{k-1} = (h_r^{k-1})_{r=1}^{k-1}$ where $h_r^{k-1} \in \{T1, T2\}$ is the team that kicked first in Round $r$ and $g_{T1} : g_{T2}$ is the score (i.e., the goals scored by $T1$ and $T2$, respectively) at the beginning of Round $k$. Thus, the probability of $T2$ kicking first in Round $k$ is $1 - \phi(h^{k-1}; g_{T1} : g_{T2})$.

Each shootout mechanism $\phi$ induces a hidden-action extensive-form game, which we will simply refer to as **the game**, such that the exact spot that each kicker aims the ball on the goal line is unobservable by others. Given the current state $(h^{k-1}; g_{T1} : g_{T2})$, for Rounds $k = 1, 2, ...$, the order of first-kicking teams in the previous $k-1$ rounds $h^{k-1}$, and feasible scores $g_{T1} : g_{T2}$, the Nature determines with probability $\phi(h^{k-1}; g_{T1} : g_{T2})$ $T1$ kicking next first and probability $1-\phi(h^{k-1}; g_{T1} : g_{T2})$ $T2$ kicking next first. Then a kicker of the first-kicking team takes the penalty shot, observing the state and the history of the outcomes of all the shots up to that point as goal, out, or save. The kicker aims at his intended spot $x \in [0, 1]$ to maximize his expected individual payoff (which we explain in the next paragraph). Then the Nature determines with probability distribution $P_G(x), P_O(x), 1 - P_G(x) - P_O(x)$ whether the penalty kick results in a goal, goes out, or is saved, respectively. After the outcome of this shot is observed, the other team's kicker takes a penalty shot, observing the history of the outcomes of the shots up to that point. We continue until the end of regular rounds, Round $k = n$, similarly. If the score is tied, then we continue with the sudden-death rounds until the tie is broken at the end of a Sudden-death Round $k > n$.

Each kicker aims to maximize his expected individual payoff in the game. Each kicker's payoff function consists of two additive components. The first is the utility received when his team wins or loses the shootout: $V_W$ is the team win payoff and $V_L < V_W$ is the team loss payoff. This component of the payoff is common to each kicker on the team. The second component of the individual payoff consists of an individual outcome based valuation: If the kicker scores a goal he gets payoff $U_G > 0$, if he kicks the ball out he receives payoff $U_O < 0$, and if the goalie saves the kick he receives payoff $U_S = 0$. This is a normalization that guarantees that scoring a goal is the most desirable outcome, and kicking the ball out is less desirable than kicking the ball inside the goal frame and yet the goalie saves it. With this normalization, we can also drop a variable from our notation without affecting our analyses. Thus, the overall ex-post payoff of a kicker $i$ of Team $Tk$ is then

$$u_{i,Tk} = V_t + U_p \tag{1}$$

where $t \in \{W, L\}$ refers to the overall team outcome, win or loss; and $p \in \{G, O, S\}$ refers to the kicker's penalty outcome, goal, out, or save.

An **information set** is $H \in \mathcal{H}_{i,Tk}$, which is the collection of information sets that kicker $i \in \{1, 2, ...\}$ of Team $Tk \in \{T1, T2\}$ can move, consists of the exact spot the ball went to for each of the previous kicks, the team of the kick, and whether the kick was scored as a goal, went out, or was saved by the goalie. Thus, they are all observable by kicker $i$ of Team $Tk$ moving in information set $H$. What is not observable by him is the intended spots of previous kicks. Each information set also has an associated round (without loss of generality indexed with the kicker, i.e. $i$'th round), order of kicking in the round as $1^{st}$ or $2^{nd}$, and a current score difference between $T1$ and $T2$. We refer to all of this observable information as the **state** of the information set. Note that from the point of view of the kicker, who is a single-shot player in the game, all payoff-relevant information of an information set is given through its state.

A pure **strategy** $X_{i,Tk} : \mathcal{H}_{i,Tk} \to [0, 1]$ is a function from the collection of information sets that team $Tk$'s kicker $i$ can move to the intended spots that he can target while taking the penalty shot.

As alluded to before, this is a hidden-action sequential game, as each player observes only where the ball goes and whether the kick was a goal, out, or a save in previous kicks, but not the intended spot towards which the ball was kicked. Hence, as a kicker takes a penalty shot, he has a belief over intended spots of previous kicks. Formally, a **belief** $\mu(H)$ is a function that maps each information set $H \in \mathcal{H}_{i,Tk}$ that Team $Tk$'s $i$'th kicker moves with positive probability to a probability distribution over histories of actions taken that would lead to the same information set.

## 4.3 Markov Perfection and State-Symmetric Equilibria

Our solution concept is *state-symmetric perfect Bayesian equilibrium* (SPBE), in which strategies in regular rounds depend only on the state of the game, i.e., on the round number, kicking order, and score difference; strategies in sudden-death rounds depend only on the current kicking order and score difference. The strategies in SPBE are memoryless in that they depend only on the current state.

A **perfect Bayesian equilibrium** in the game of shootout mechanism $\phi$ is an assessment, i.e., a strategy profile and a belief profile pair $\left[X = (X_{i,Tk})_{i\in\{1,2,...\},Tk\in\{T1,T2\}}, \mu = (\mu(H))_{H\in\mathcal{H}_{i,Tk},i\in\{1,2,...\},Tk\in\{T1,T2\}}\right]$ such that for any $\{Tk, T\ell\} = \{T1, T2\}$, $i \in \{1, 2, ...\}$, and $H \in \mathcal{H}_{i,Tk}$,

- spot $X_{i,Tk}(H) \in [0, 1]$ maximizes the expected value over all possible ex-post payoffs $u_{i,Tk}$ at information set $H$, given $(X_{-i,Tk}), (X_{j,T\ell})$ among all spots in $[0, 1]$; and
- belief $\mu(H)$ is consistently derived by Bayes' rule from $\phi$, $X$, $P_G$, $P_O$, $\mu(H')$ for all $H' \neq H$.

Observe that each kicker is a one-shot player and maximizes his individual expected payoff over his ex-post payoffs $u_{i,Tk}$ defined in Equation 1. The exact formulation of this expected payoff will become clear later in our analysis.

In this game, once the equilibrium strategies are found, beliefs are straightforward to construct. At any information set $H$, the kicking player believes with probability one that other kickers before him used equilibrium strategies. This is because the payoffs explicitly depend on the actual outcome of each kick, which is observable as Goal (G) or No Goal (NG), not on the intended spots of kicks (which are not observable). Further, beliefs will not play a role in finding the optimal strategies in equilibria as the kicker decides on his best action by taking into consideration only future players' kicks, not those of the past ones. We will not explicitly calculate the beliefs from this point on, except when we refine the possible multiple equilibria of the fixed-order mechanism, ABAB, in Appendix C

Since we are making a fairness-centered analysis over different shootout mechanisms, we will focus on a Markovian symmetric equilibrium concept (i.e., unless we refine the possible multiple equilibria of the fixed-order mechanism):

A **state-symmetric assessment** $(X, \mu)$ is defined as

- **In regular rounds:** $X_{i,Tk}(H) = X_{i,T\ell}(H')$ and $\mu(H) = \mu(H')$ for teams $Tk, T\ell \in \{T1, T2\}$ where both information sets $H \in \mathcal{H}_{i,Tk}$ and $H' \in \mathcal{H}_{i,T\ell}$ pertain to the same Regular Round $i \le n$, and the same kicking order, $1^{st}$ or $2^{nd}$, in the round while the score difference between $T1$ and $T2$ in $H$, $s$, and in $H'$, $s'$, satisfy $s = -s'$ if $T\ell \ne Tk$ and $s = s'$ if $T\ell = Tk$.

- **In sudden-death rounds:** $X_{i,Tk}(H) = X_{j,T\ell}(H')$ and $\mu(H) = \mu(H')$ for any $Tk, T\ell \in \{T1, T2\}$ where information sets $H \in \mathcal{H}_{i,Tk}$ and $H' \in \mathcal{H}_{j,T\ell}$ involve (possibly different) Sudden-Death Rounds $i, j > n$ but they refer to the same kicking order, $1^{st}$ or $2^{nd}$, while the score difference between $T1$ and $T2$ in $H$, $s$, and in $H'$, $s'$, satisfy $s = -s'$ if $Tk \ne T\ell$ and $s = s'$ if $Tk = T\ell$.

A state-symmetric assessment in sudden-death rounds, for instance, dictates that two players on the same team or different teams will exactly aim at the same intended spot and have exactly the same beliefs if they were in each other's shoes. Note that before every sudden-death round the score is identical if the game reaches it, while before each regular round after Round 1 it could be different. Unlike the sudden-death rounds, the number of regular rounds is finite, and therefore the round number as well as the kicking order and score would matter in regular rounds. Therefore, even if two teams are tied in different regular rounds, the players who kick first need not use the same strategy in those two rounds.

A **state-symmetric equilibrium** of a shootout mechanism $\phi$ is defined as a state-symmetric Perfect Bayesian equilibrium of the game induced by $\phi$. This solution concept is identical to *symmetric Markov-perfect equilibrium* if one were to ignore the beliefs and focused only on strategies assuming that each state of the game spans a subgame of the game. As noted above, beliefs play no role other than equilibrium selection when there are multiple equilibria; this is without loss of generality.

## 4.4 Sequential Fairness and the Aristotelean Justice Criterion

Using the concept of state-symmetric equilibrium, we now define the key design concept in our analysis as follows: an assessment $(X, \mu)$ of the game induced by mechanism $\phi$ is **sequentially fair** if for all problems with balanced teams (i.e., for any underlying utility values $V_W, V_L, U_G, U_O$ and goal and out probability functions $P_G, P_O$), at any $(h^{k-1}; g_{T1} : g_{T2})$ with $g_{T1} = g_{T2}$ (i.e., when they are tied at the beginning of Round $k$ for any $k$), each team has exactly a 50% chance of winning.

We will seek shootout mechanisms whose *all* state-symmetric equilibria are sequentially fair. We will refer to such mechanisms, for short, as **sequentially fair mechanisms**. Note that it is not the shootout mechanism that is inherently fair, but its state-symmetric equilibria that need to be fair.

We will analyze sequential fairness in sudden-death rounds first. It will be useful to formally define this concept. A mechanism is **sequentially fair in sudden-death** rounds if, for all problems with balanced teams, for any Sudden-death Round $k > n$, at any $(h^{k-1}; g_{T1} : g_{T2})$ with $g_{T1} = g_{T2}$, (i.e., when they are tied at the beginning of Round $k$), each team has exactly a 50% chance of winning.

Our desiderata are determining whether the current mechanism's equilibria are sequentially fair, inspecting other plausible mechanisms, and characterizing the class of sequentially fair mechanisms.

Sequential fairness is the first part of the two-part Aristotelean justice criterion. We say that a mechanism satisfies the **Aristotelean justice criterion** if it is sequentially fair and, when there is team with higher-ability kickers than those of the other team, whenever scores are tied at the beginning of a round, the better team wins with a weakly higher probability (and with a strictly higher probability at least at one round) at all state-symmetric equilibria.[20]

# 5 Analysis: A Kicker's Optimization Problem

We first analyze each kicker's optimization problem for a given mechanism $\phi$ and other agents' strategies. The $i$'th kicker of Team $Tk$, denoted by $\ell \equiv Tk, i$'s best response determination problem boils down to

$$\max_{x_\ell \in [0,1]} U_\ell(x_\ell; W_{G,\ell}, W_{NG,\ell}) \equiv \Big( P_G(x_\ell) W_{G,\ell} + [1 - P_G(x_\ell)]\, W_{NG,\ell} \Big) + \Big( P_G(x_\ell) U_G + P_O(x_\ell) U_O \Big) \quad (2)$$

where $P_G(x_\ell)U_G + P_O(x_\ell)U_O$ is Kicker $\ell$'s expected individual kick payoff, and $P_G(x_\ell)W_{G,\ell} + [1 - P_G(x_\ell)]\, W_{NG,\ell}$ is Kicker $\ell$'s expected continuation team payoff given expected continuation values $W_{G,\ell}$ conditional on he scores and $W_{NG,\ell}$ conditional on he does not score. These values, $W_{G,\ell}$ and $W_{NG,\ell}$, are functions of the shootout mechanism, the score difference, round number ($i$ in this case), kicking order in that round, and the others' strategy profile. We drop them from our notation for simplicity.

Hence, the necessary first-order conditions for an interior maximum turn out to be

$$P'_G(x^*_\ell)(W_{G,\ell} - W_{NG,\ell} + U_G) + P'_O(x^*_\ell) U_O = 0. \quad (3)$$

The second-order conditions lead to the first-order conditions being sufficient, since we have

$$P''_G(x^*_\ell)(W_{G,\ell} - W_{NG,\ell} + U_G) + P''_O(x^*_\ell) U_O < 0, \quad (4)$$

[20] We introduce an instance of unbalanced teams in Subsection 7.1 such that we can rank the teams as better and worse teams.

which follows from the facts that $P''_G < 0$, $W_{G,\ell} - W_{NG,\ell} \geq 0$, $U_G > 0$, $P''_O \geq 0$ and $U_O < 0$. Hence, if an interior maximum exists it is unique given $W_{G,\ell} - W_{NG,\ell}$. We will refer to $W_{G,\ell} - W_{NG,\ell}$ as the **expected marginal contribution** of the kicker to his team. We turn our attention to analyze the properties of the optimum for a kicker.

**Proposition 1** *At any interior best response of Kicker $\ell$, $x^*_\ell < \overline{x}$ is the kicker-optimal spot, and the higher his expected marginal contribution, the higher is his goal-scoring probability.*

Also note that if kicking out and the goal being saved were valued equally, i.e., $U_O = U_S = 0$, then $x^*_\ell = \overline{x}$, i.e., $x^*_\ell$ would be also goal-optimal. But since $x^*_\ell < \overline{x}$ because $U_O < U_S = 0$, a kicker chooses to kick more conservatively. The relative magnitude $P'_G(x^*_\ell)/P'_O(x^*_\ell)$ as well as magnitudes of expected marginal contribution $W_{G,\ell} - W_{NG,\ell}$, $U_O$, and $U_{NG}$ determine how much he *shaves off* the goal-optimal spot to determine his kicker-optimal spot.

Observe that the choice of kicker-optimal spot is costly because of the probability of kicking the shot out, and therefore, it can be interpreted as a choice of an *effort level.* The goal-optimal spot is also the goal-optimal *effort.* The closer the spot gets (from the center) to the goal-optimal spot, it can be interpreted as exerting *higher effort.* From now on, when it is convenient, we will use this analogy more freely and refer to the choice of targeted spot as *exerting an effort.*

Next, we focus on fully solving $W_{G,\ell} - W_{NG,\ell}$ for the current scheme, the fixed-order mechanism, ABAB.

# 6 The Current Scheme: The Fixed-Order Mechanism, ABAB

The current shootout scheme is the fixed-order mechanism, ABAB, in which the first kicker is determined before Round 1 with an even lottery and then the procedure continues with the same kicking order throughout. Formally, the **fixed-order mechanism, ABAB,** $\phi$ is defined as follows:

$$\phi(\emptyset; 0:0) = 0.5 \quad \text{and} \quad \phi(h^{k-1}; g_{T1} : g_{T2}) = \begin{cases} 1 & \text{if } h_1^{k-1} = T1 \\ 0 & \text{if } h_1^{k-1} = T2 \end{cases}$$

for all Rounds $k \geq 2$, orders of first-kicking teams in the previous $k-1$ rounds $h^{k-1}$, and feasible scores $g_{T1} : g_{T2}$ at the beginning of Round $k$.

We will now characterize the state-symmetric equilibria of the fixed-order mechanism, ABAB, in the sudden-death rounds.

Without loss of generality assume that $T1$ wins the coin toss before Round 1 and kicks first throughout.

At state-symmetric equilibria, if they exist, each $T1$ kicker will use exactly the same action when he kicks in the sudden-death rounds, as $T1$ always goes first and the score is tied at the beginning of each sudden-death round.

Similarly, by symmetry, each $T2$ kicker will use exactly the same action when his team is behind (which can be by one goal at most); and he will use exactly the same action when the score is even (which can happen if the preceding $T1$ kicker kicks out or his kick is saved).

On the other hand, $T1$ and $T2$ kickers may potentially use different actions at state-symmetric equilibria, as they kick in different orders: in each round $T1$ goes first and $T2$ goes second.

Hence, if a state-symmetric equilibrium exists, for a given $k = 1, 2$, the probability of Team $Tk$ winning is the same at the beginning of each sudden-death round.

At a state-symmetric equilibrium, let us define $V_{T1}$ to be the *value function of* $T1$, that is the expected utility it contributes by winning or losing to its all kickers, in the first sudden-death round. Denote by $x$ the kicking strategy for $T1$'s kickers. Define $V_{T2}^B$ to be the value function of $T2$ in the first sudden-death round when $T2$ is currently behind by one goal, $V_{T2}^E$ and to be the value function of $T2$ in the first sudden-death round when the score is currently even. $T2$'s kickers' optimal kicking strategy in each scenario is $y_B$ and $y_E$ respectively.

We can write the following *Bellman* equation for $V_{T1}$:

$$V_{T1} = \; P_G(x) W_{G,T1} + [1 - P_G(x)]\, W_{NG,T1} \tag{5}$$

where $W_{G,T1}$ is the expected future value conditional on the kicker scoring and $W_{NG,T1}$ is the expected future value conditional on the kicker not scoring. We have

$$W_{G,T1} = P_G(y_B) V_{T1} + [1 - P_G(y_B)]\, V_W \tag{6}$$

$$W_{NG,T1} = P_G(y_E) V_L + [1 - P_G(y_E)]\, V_{T1} \tag{7}$$

For $T2$, we have

$$V_{T2}^B = P_G(y_B) \underbrace{V_{T2}}_{=W_{G,T2}^B} + [1 - P_G(y_B)] \underbrace{V_L}_{=W_{NG,T2}^B} \tag{8}$$

$$V_{T2}^E = P_G(y_E) \underbrace{V_W}_{=W_{G,T2}^E} + [1 - P_G(y_E)] \underbrace{V_{T2}}_{=W_{NG,T2}^E} \tag{9}$$

where

$$V_{T2} = V_W + V_L - V_{T1} \tag{10}$$

is the continuation payoff attributed to $T2$ in our win-or-lose game.

Next, we solve the decision problem faced by each kicker given other players' actions and beliefs using the first-order necessary and sufficient conditions given in Equation 3. Recall that for a Kicker $\ell$

$$P_G'(x_\ell)(W_{G,\ell} - W_{NG,\ell} + U_G) + P_O'(x_\ell) U_O = 0 \tag{11}$$

where $x_\ell$ is the optimal spot for Kicker $\ell$.

We can then solve $(x, y_B, y_E)$ by plugging Equations 5 – 10 into Equation 11. To do that we need to resolve the continuation values $V_{T1}$ and $V_{T2}$ for each team.

From Equation 5,

$$V_{T1}=\frac{P_G(x)[1-P_G(y_B)]V_W+[1-P_G(x)]P_G(y_E)V_L}{P_G(x)[1-P_G(y_B)]+[1-P_G(x)]P_G(y_E)}=\alpha V_W+(1-\alpha)V_L \tag{12}$$

where the winning probability of $T1$, $\alpha$, is given by

$$\alpha=\frac{P_G(x)\left[1-P_G(y_B)\right]}{P_G(x)\left[1-P_G(y_B)\right]+\left[1-P_G(x)\right]P_G(y_E)}. \tag{13}$$

A value for $\alpha > 0.5$ at a state-symmetric equilibrium will signal that the fixed-order mechanism, ABAB, is biased in favor of the first-kicking team in the sudden-death rounds (and $\alpha < 0.5$ is vice versa for the second-kicking team). On the other hand, the fixed-order mechanism is a sequentially fair mechanism if and only if $\alpha = 0.5$ at every state-symmetric equilibrium. For $T2$, then, we get by Equation 10,

$$V_{T2}=(1-\alpha)V_W+\alpha V_L. \tag{14}$$

Hence, Equations 5 – 10 through Equation 11 become self-contained to solve for $x, y_B$ and $y_E$. The following theorem characterizes the state-symmetric equilibrium strategy candidates solving these equations:[21]

**Theorem 1 (The fixed-order mechanism, sudden-death rounds)** *(i) In sudden-death rounds, a state-symmetric equilibrium of the fixed-order mechanism, ABAB, exists if and only if* $P'_G(0)[\frac{V_W-V_L}{2}+U_G]+P'_O(0)U_O\geq 0$.

*(ii) When it exists, there may be multiple state-symmetric equilibria with strategy profiles* $(x, y_B, y_E)$*, all of which are to the left of the goal-optimal spot, satisfying*

- $x = y_E$*, i.e., the* $T1$ *kicker and* $T2$ *kicker, when the score is even, kick at the same spot; and*
- *for every equilibrium with* $(y_E, y_B)$*, there exists another equilibrium with* $(\hat{y}_E, \hat{y}_B)$ *such that* $\hat{y}_E = y_B$ *and* $\hat{y}_B = y_E$.

It will be useful to quantify "may be" in the above theorem. The below proposition answers this question:

**Proposition 2** *Suppose that in the sudden-death rounds of the fixed-order mechanism, ABAB, a state-symmetric equilibrium exists. Then, multiple state-symmetric equilibria exist if and only if there are multiple solutions* $\beta$ *to the equation*

$$\beta=\frac{1-P_G\left(y(1-\beta)\right)}{2-P_G\left(y(\beta)\right)-P_G\left(y(1-\beta)\right)}, \tag{15}$$

[21] To reiterate, in our analysis, we do not have to model the beliefs of agents explicitly. We use the summary functions $P_G$ and $P_O$, and the agents have to best respond to what the other players are doing at equilibrium. As mentioned before, the beliefs will be crucial in using the equilibrium selection criterion, though, later in Appendix C.

*where* $y(\beta) = f^{-1}\left(\frac{-U_O}{(V_W - V_L)\beta + U_G}\right)$ *for* $f(x) = P'_G(x)/P'_O(x)$ *for all* $x \in [0,1]$.

*Moreover, there is an odd number of solutions with* $\beta = \frac{1}{2}$ *always being a solution and others being located symmetrically around it. We also have* $y_B = y(\beta)$ *and* $x = y_E = y(1-\beta)$ *for any solution* $\beta$.[22]

Thus, generically, the fixed-order mechanism, ABAB, is *not sequentially fair* as the winning probability of $T1$ $\alpha \neq \frac{1}{2}$ in equilibrium, whenever $y_B \neq y_E$.

**Example 1** ***(Sequentially unfair equilibria)*** *Suppose the game has the following structure:*

$$V_W - V_L = 7;\ U_G = \frac{556.08}{879};\ U_O = -15s$$
$$P_G(x) = 0.82 - 1.2(0.5-x)^2;\ P_O(x) = \frac{270.232704}{879s}x$$

*It can be readily verified that for every* $s \geq 1$, $(x, y_E, y_B) = (0.04, 0.04, 0.03)$ *(and hence* $(x, y_E, y_B) = (0.03, 0.03, 0.04)$*) constitutes an equilibrium.*

**Theorem 2** *The fixed-order mechanism, ABAB, is not sequentially fair in general.*

Its proof is immediately implied by Theorem 1. We will provide some intuition for this result when we discuss sequentially fair mechanisms in the sudden-death rounds in Section 7.2. As alluded to in a footnote in the Introduction briefly, we also show in Appendix C that it is possible to devise a forward-looking equilibrium refinement – similar in vein to the Intuitive Criterion of Cho and Kreps (1987) – to get rid of the multiplicity of equilibria, so that the first-kicking team wins more often. Any potential equilibrium candidate with the second-mover advantage does not survive this refinement.

# 7 Mechanism Design: Sequentially Fair Mechanisms

In the previous section, we have concluded that the currently used fixed-order mechanism, ABAB, is not sequentially fair. It turns out that even if we introduced a sequentially fair extension to it in sudden-death rounds, such as an unbiased coin flip to determine the winner, it would still be sequentially unfair.

In fact, a large class of intuitive mechanisms turns out to be sequentially unfair. A fitting example of such mechanisms is the alternating-order mechanism, ABBA, in which the kicking order reverses in every round. It turns out that even this mechanism is sequentially unfair. In addition, a large class of mechanisms, which we refer to as *exogenous mechanisms*, turns out to be sequentially unfair. A mechanism $\phi$ is **exogenous** if, for all rounds $k$, and kicking orders $h^{k-1}$ regarding the

[22] We thank Randy Silvers for his help in generating examples.

beginning of round $k$, $\phi(h^{k-1} : g_{T1} : g_{T2}) = \rho(k)$ for some function $\rho$, i.e., who goes first in each round is determined independent of the current score but as a function of the current round. Hence, both fixed order, ABAB, and alternating order, ABBA, are exogenous, and even the version of the alternating-order mechanism in which the 5th round's kicking order is randomly determined is exogenous.

Another interesting exogenous mechanism is the **random-order** mechanism $\phi$, which determines who goes first in every round using an unbiased lottery at the beginning of that round, that is $\phi(h^{k-1}; g_{T1}, g_{T2}) = \frac{1}{2}$ for all $k$. Despite its impracticality, one may expect this exogenous mechanism to be sequentially fair. Indeed, this turns out to be the case.[23] However, the class of sequentially fair mechanisms is far richer than the random-order mechanism. There are some practical mechanisms in this class.

We will next characterize all sequentially fair mechanisms in the regular rounds. We will assume that a mechanism that gives sequential fairness in the sudden-death rounds exists (and we then show that there are uncountably many such mechanisms).

We introduce a class of mechanisms that will be crucial in our analysis of sequentially fair mechanisms. A mechanism $\phi$ is **uneven score symmetric** if for all $(h^{k-1}; g_{T1} : g_{T2})$ and $(h'^{\,k-1}; g_{T2} : g_{T1})$ such that $g_{T1} \neq g_{T2}$ and $k \leq n$, we have $\phi(h^{k-1}; g_{T1} : g_{T2}) = 1 - \phi(h'^{\,k-1}, g_{T2} : g_{T1})$. That is, as long as the score is not tied at the end of a round, the probability of who kicks first in the next round is the same for $T1$ and $T2$ whenever they are in each other's shoes. E.g., when $T1$ is ahead $3-2$ in (the beginning of) Round 4, and when $T2$ is ahead in Round 4 with a score of $2-3$, in Round 4 $T1$'s probability of kicking first in the first case is the same as $T2$'s probability of kicking first in the second case.

It turns out that such mechanisms fully characterize the sequentially fair mechanisms in the regular rounds.

**Theorem 3 (Sequentially fair mechanisms)** *Suppose a mechanism $\phi$ is sequentially fair in sudden-death rounds. Then $\phi$ is sequentially fair if and only if it is uneven score symmetric in regular rounds.*

The intuition behind this result can be given as follows: When a round starts even, the first team's kicker and the second team's kicker both exert the same effort and kick to the same intended spot. This is almost like asserting that when the score is even, kicking order is of minimum importance. The importance of kicking order, on the other hand, stems from the fact that when the score is uneven at the beginning of a round, kickers assert different levels of effort in kicking penalties depending on when they kick. Under an uneven-score-symmetric mechanism, each team's kickers foresee that their team will be treated symmetrically as the other team, in case either team falls behind or jumps ahead in score. Therefore, this assurance takes the reason behind the importance of kicking order out of the equation.

[23] However, a biased random-order mechanism where the probability of which team kicks first does not depend on the current score is sequentially unfair.

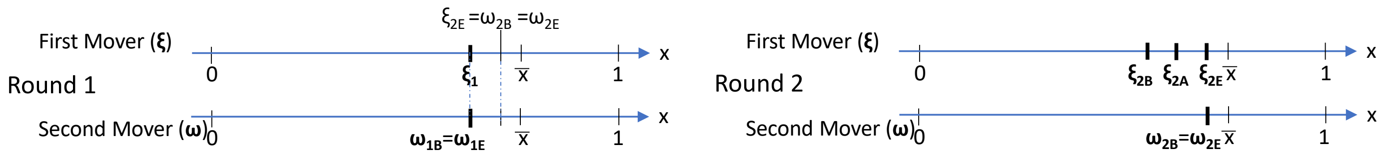


Figure 2: The effort levels of teams under a sequentially fair mechanism is state-symmetric equilibria in an easy shootout.

A corollary to the proof of this theorem is of independent interest. It tells how effort levels can be compared between the teams in different states and rounds (see also Figure 2).

**Corollary 1** *Let $\phi$ be a sequentially fair mechanism. Suppose effort levels, i.e., the state-symmetric equilibrium intended spots, are denoted as follows:*

- *In Round 1: (1) for first kicking team as $\xi_1$; and (2) for second kicking team as $\omega_{1E}$ when score is even and as $\omega_{1B}$ when it is behind.*
- *In Round 2: (1) for first kicking team as $\xi_{2A}$ when it is ahead, as $\xi_{2E}$ when the score is even, and as $\xi_{2B}$ when it is behind; and (2) for second kicking team as $\omega_{2E}$ when score is even and as $\omega_{2B}$ when it is behind.*

*Then the following relations hold for easy shootouts:*

*Round 1.* $\xi_1 = \omega_{1E} = \omega_{1B} < \omega_{2B}$.

*Round 2.* $\xi_{2B} < \xi_{2A} < \xi_{2E} = \omega_{2E} = \omega_{2B}$.

*For difficult shootouts, only the first inequality reverses in Round 2 relations above.*

The theorem makes another interesting point: There is only one sequentially fair exogenous mechanism; the random-order mechanism that determines which team will kick first with an unbiased coin toss in each round. We formalize it below, and it follows directly from Theorem 3.

**Proposition 3** *Random order is the only exogenous mechanism that is sequentially fair.*

Note that one does not need to treat both teams symmetrically all the time to obtain sequential fairness. In fact, when the score is tied, it does not matter which team kicks first. However, when the score is not tied, teams need to be treated symmetrically. This feature opens the door for some interesting practical mechanisms to be sequentially fair. Two examples of such mechanisms are the **behind-first** and **ahead-first** mechanisms. In behind first (ahead first), the team who is behind (ahead) in score after a round kicks first in the next round, and otherwise the order of the teams is determined in some other manner. There are also many other uneven-score-symmetric mechanisms in which lotteries play a significant role. For example, a lottery mechanism that forces the behind team to go first in 75% of the time and $T1$ to go first 60% of the time when the score is tied is also sequentially fair.

## 7.1 Better Teams under Sequentially Fair Mechanisms

Uneven-score-symmetric mechanisms have another nice feature. Theorem 3 states that when two teams have the same kicking ability, they have equal winning probability. What if one team is better than the other? Suppose there is one player who has a better kicking ability than the rest of the players, i.e., the player has a higher $P_G(x)$ and a lower $P_O(x)$ for every $x \in [0, 1]$. We formally define a **better player** as follows: Let $\{P_G, P_O\}$ represent all players' kicking ability except the better player, and $\{\widetilde{P}_G, \widetilde{P}_O\}$ represent the better player's kicking ability. We assume (a) $P_G(x) < \widetilde{P}_G(x)$ and $P_O(x) > \widetilde{P}_O(x)$, and (b) $\frac{P'_G(x)}{\widetilde{P}'_G(x)} = \frac{P'_O(x)}{\widetilde{P}'_O(x)}$ for all $x \in [0, 1]$. We show that the team with this better player – now named the **better team** – has a higher winning probability under uneven-score-symmetric mechanisms.

**Theorem 4** *Suppose a mechanism that is sequentially fair in sudden-death rounds and uneven score symmetric in regular rounds is used in the shootout. Then a better team has a higher ex-ante chance of winning at the unique state-symmetric equilibrium of the shootout induced by this mechanism, if the better player is used strategically in the best kicking order possible by the better team.*

Therefore, sequentially fair mechanisms satisfy the Aristotelean Justice criterion according to the definition of better/worse teams above.

## 7.2 Sequential Fairness in Sudden-Death Rounds

The class of sequentially fair mechanisms is larger when sudden-death rounds are also considered. The analysis of sudden-death rounds differs from that of regular rounds, as they make the game potentially an infinite game.

First we introduce a practical sequentially fair mechanism for the sudden-death rounds.

As we concluded in the previous section, the fixed-order mechanism, ABAB, clearly fails sequential fairness in the sudden-death rounds. So is there a simple and deterministic mechanism that is sequentially fair in the sudden-death rounds? The answer is affirmative, and the alternating-order mechanism, ABBA, *is* sequentially fair in sudden-death rounds, although it is not in regular rounds. The intuition is straightforward: Under the alternating-order mechanism, ABBA, one can have uneven scores, such as $T1$ being ahead, in an intermediate regular round. Hence, it cannot satisfy uneven score symmetry as required in a sequentially fair mechanism. On the other hand, in the sudden-death rounds, the score is never uneven at the beginning of a round. Hence, the exogeneity of the alternating order does not prevent sequential fairness.

**Theorem 5** *The alternating-order mechanism, ABBA, is sequentially fair in sudden-death rounds.*

The intuition behind this result and its relationship to Theorem 1 about the multiplicity of equilibria in the fixed-order can be given as follows: All mechanisms span an infinite game in the sudden-death rounds. Typically this gives rise to multiplicity of equilibria. However, we are interested in state-symmetric equilibria for sequential fairness. In the fixed-order mechanism, ABAB, not all histories and information sets are reached in the path of the shootout, since fixed-order always dictates the same team to kick first. That is, two teams are "never in each other's shoes" during sudden-death rounds. Hence, the game has total freedom to choose among many different equilibria, i.e., the rounds that dictate that $T2$ kicks first are never reached and have no restrictions on the equilibrium behavior. On the other hand, the alternating-order mechanism, ABBA, is just the opposite in that sense: both teams are "in each other's shoes" in every other round. This puts more restrictions on the state-symmetric equilibria, and only the 50%-50% winning equilibria survive state-symmetry.

Actually, for such a restriction to hold, we do not even need the teams to be "in each other's shoes" as frequently as in the alternating-order mechanism. In fact, there are uncountably many other mechanisms that are sequentially fair in sudden-death rounds:

**Theorem 6** *Take any mechanism $\phi$ that is uneven score symmetric in regular rounds and any sequentially fair mechanism $\varphi$ in sudden-death rounds. Construct a mechanism $\psi$ as follows: Fix a Sudden-death Round $k$.*

- *For all $\ell$ such that $n < \ell < k$, feasible scores $g_{T1} : g_{T2}$, and beginning of Round $\ell$ kicking orders $h^{\ell-1}$, let $\psi(h^{\ell-1}; g_{T1} : g_{T2}) = \phi(h^{\ell-1}; g_{T1} : g_{T2})$, and*
- *for all $\ell \geq k$ and $\ell \leq n$, feasible scores $g_{T1} : g_{T2}$, and beginning of Round $\ell$ kicking orders $h^{\ell-1}$, let $\psi(h^{\ell-1}; g_{T1} : g_{T2}) = \varphi(h^{\ell-1}; g_{T1} : g_{T2})$.*

*Then $\psi$ is sequentially fair.*

That is, we can replace the continuation of any uneven-score-symmetric mechanism after some sudden-death round with a sequentially fair mechanism in sudden-death rounds (i.e., such as with the alternating-order mechanism, ABBA), and regardless of initial part of the mechanism, the newly constructed mechanism becomes sequentially fair. The intuition of this result is as follows: Take the last round before sequential fairness kicks in, say Round $k$. By backward induction, as teams are tied at the beginning of Round $k$ and in Round $k+1$ they have a $50\% - 50\%$ chance of winning, in all situations the two kickers of Round $k$ exert the same effort regardless of kicking order (as we explained in the intuition behind Theorem 3). Therefore, at the beginning of Round $k$, both teams have an equal chance of winning as well. An example of such a mechanism is a behind-first mechanism such that in the first $n + 10$ rounds $T1$ kicks first whenever the game is tied, and then we alternate the order. Note that in the first 10 sudden-death rounds $T1$ kicks first, and yet, the mechanism is sequentially fair as it is appended by a sequentially fair mechanism in sudden-death rounds, namely the alternating-order mechanism, ABBA.

# 8 Market Design and Practical Criteria

Sequential fairness is capable of ruling out many mechanisms in regular rounds, including the current fixed-order mechanism, ABAB. Interestingly, it also rules out a seemingly fair exogenous mechanism, namely the alternating-order mechanism, ABBA. In terms of endogenous mechanisms, however, sequential fairness does not pose as much of a restriction. One needs further desirable properties to help refine the set of sequentially fair mechanisms. We will next define additional criteria to provide concrete practical advice in that regard.

## 8.1 Goal Efforts and Maximizing the Expected Number of Attempts

We can talk about efficiency or inefficiency of an attempt, to the extent it maximizes the effort levels of kickers. In other economic contests, efficiency could be even more important. Among sequentially fair mechanisms, it turns out that behind first and ahead first are the two extremes in terms of inducing kickers' efforts in Round 1:

**Proposition 4** *Behind-first (ahead-first) mechanisms maximize (minimize) Round 1 efforts of all kickers among all sequentially fair mechanisms at the state-symmetric equilibrium.*

The intuition behind this result can be summarized as follows for behind first (ahead first is symmetric). First we summarize the incentives facing Round 2 kickers. In Round 2, kicking first is not good at all for higher goal efforts: the first-kicking team's player (if his team is either behind or ahead) will always exert less effort than he would in the case when he kicks second in Round 2. This is true because his marginal contribution will be less in the first case, as the other team's kicker – who will go second – can always miss or offset the first kicker's failure. So he has higher incentives to shirk when he kicks first.

Now, we turn our attention to Round 1 kickers' marginal contributions under both mechanisms. First, observe that both teams' kickers under any uneven-score-symmetric mechanism exert the same effort in Round 1, by Corollary 1. Therefore, understanding the first-kicking team player's incentives is sufficient to draw the difference between the two mechanisms regardless of the kicking order or score during Round 1. A Round 1 kicker, if he does not exert high effort under behind first, may cause his team to fall behind with higher probability. This causes his teammate to shirk more, when he goes first, and the other team's second player to exert higher effort, when he goes second in Round 2. On the other hand, under ahead first, the Round 1 kicker's incentives are exactly the opposite! If he does not exert high effort in Round 1, his team may fall behind with higher probability, but his teammate will exert relatively higher effort under ahead first by going second in Round 2 (with respect to behind first) and the other team's second kicker will exert less effort in Round 2 (with respect to behind first). Hence, Round 1 kicker's possible failure can still be salvaged with higher probability under ahead first. So he shirks under ahead first vis-a-vis behind first. Therefore, behind first dominates any random (i.e., convex combination of ahead first and behind first) and ahead-first mechanisms among all uneven-score-symmetric mechanisms.

On the other hand, observe that ahead first and behind first cannot be compared with each other in Round 2 whenever the score is not tied: in ahead first when $T1$ is ahead, $T1$ kicks first while in behind first, it kicks second under the same scenario. So there are no two comparable information sets that are reached with positive probability under both mechanisms in Round 2. When the score is tied however, all uneven-score-symmetric mechanisms lead to the same goal efforts and are equivalent in Round 2.

Thus, round by round we are not able to establish an effort ranking among different sequentially fair shootout mechanisms. Nevertheless, we can still obtain a partial ranking based on the expected number of attempts.

Maximizing the expected number of attempts can be desirable from a social perspective, as well as a team perspective, if the attempt cost is negligible or negative. Spectators may wish to see more scoring attempts. Recall that we had defined two particular cases of shootouts earlier. In easy shootouts, the goal probability of each kick is greater than 50% for each intended spot, as in soccer, while in difficult shootouts, the goal probability of each kick is less than 50% for each intended spot, as in hockey.

Then we have the following result as a proposition:

**Proposition 5** *A sequentially fair mechanism maximizes the expected number of attempts taken by both teams if and only if*

- *it is behind first in an easy shootout, and*
- *it is ahead first in a difficult shootout.*

The intuition for this result is as follows for easy shootouts: In Round 1, all teams take penalties. In Round 2 however, both teams take penalties for sure if and only if the score is tied after Round 1. It turns out that this occurs with the most probability in a behind-first shootout (see Proposition 4). Moreover, when the score is not tied after Round 1, the second-moving team in Round 2 does not take a kick if the ahead team moves first and scores or the behind team moves first and misses. Given that in an easy shootout the probability of scoring is higher than the probability of missing, this probability is minimized under behind first. Hence, overall, behind first maximizes the expected number of attempts among all sequentially fair mechanisms. The intuition is reversed for difficult shootouts.

## 8.2 Alternating-Order Mechanisms and Sequential Fairness

Although behind-first mechanisms have nice features when the score is uneven, as mentioned before they are silent on how to define the kicking order when the score is tied. Sequential fairness in regular rounds, by our characterization in Theorem 3, is also mute on this issue, but reversing the kicking order is a sure way of establishing sequential fairness in sudden-death rounds (Theorem 5).

The alternating-order mechanism, ABBA, which is not sequentially fair in regular rounds since it does not satisfy uneven score symmetry, does possess a nice property: When the score is tied in most crucial rounds, i.e., in sudden-death rounds, it gives both teams an equality of opportunity of kicking first. Clearly, such an equality-of-opportunity property is nowhere more important than in sudden-death rounds in which the score must be tied before every round.

We would like to preserve the equality-of-opportunity feature of this mechanism, especially in sudden-death rounds. The behind-first (ahead-first) mechanism defined below has this feature.

**The alternating-order behind-first (ahead-first) mechanism**: The team that is behind (ahead) in score after any Round $r$ kicks first in Round $r+1$. If the score is tied after Round $r$, then the team that kicked second in Round $r$ kicks first in Round $r+1$.[24]

Besides its simplicity, this mechanism possesses several nice features. We will start with sudden-death equality of opportunity. This property would emerge naturally since a simple but strong case could be easily made against the same team kicking multiple times in a row in those rounds in a lop-sided fashion:

**Sudden-death equality of opportunity**: Whenever the shootout ends after the Sudden-death Round $n+r$ with $r$ even, each team will have kicked first exactly $r/2$ times in the sudden-death rounds.

Then we have the following corollary:

**Corollary 2** *The alternating-order behind-first (ahead-first) mechanism satisfies sudden-death equality of opportunity.*

Another justification of alternating-order behind first and ahead first is as follows: Eclectic mechanisms could be confusing for players, coaches, referees, and fans. One can combine a sequentially fair mechanism in regular rounds with another sequentially fair mechanism in sudden-death rounds in an eclectic fashion to come up with an overall sequentially fair mechanism. For example, consider the following mechanism in regular rounds coupled with the alternating-order mechanism in sudden-death rounds: $T1$ kicks first in Round $r$ as long as the score is tied or $T1$ is behind in Round $r-1$ ; once $T2$ falls behind after some Round $r' > r$, $T2$ kicks first until $T1$ falls behind in score after some Round $r'' > r'$, after which $T1$ kicks first. One can improve on such a patchy mechanism by requiring that such an eclecticism should be eliminated. We will introduce two properties such that the latter uses the former in its definition to formalize this intuition of simplicity. Before introducing the first property, we formally introduce how an order pattern can be recognized in a mechanism:

A **finite machine representation** of a mechanism is a triple $(Q, A, t)$ such that

- $Q$ is a finite set of **(machine) states** such that state $q = (Tk)_w \in Q$ denotes that team $Tk$ taking the first penalty shot in the round associated with this state and $w$ is just an

[24] We are agnostic about how Round 1 order is determined in the definition of the mechanism. It can be determined in any manner. However, in practice we suggest it be determined by a fair coin toss as in the current fixed-order mechanism.

index number. Thus, $Q$ can be partitioned into two as $Q_{T1} = \{(T1)_1, ..., (T1)_{w_1}\}$ and $Q_{T2} = \{(T2_1, ..., (T2)_{w_2}\}$ for some $w_1$ and $w_2$ as the sets of states in which team $T1$ and $T2$ kick first, respectively.

- $A = \{(g_1 : g_2)\}$ is the set of **possible scores**.

- $t : Q \cup \{\emptyset\} \times A \times Q \to [0, 1]$ is a **state transition probability function** such that $\sum_{q' \in Q} t(q, (g_1 : g_2), q') = 1$ for all $q \in Q \cup \{\emptyset\}$ and $(g_1 : g_2) \in A$. Here, $t(q, (g_1 : g_2), q')$ is the probability of moving from state $q$ to state $q'$ when after round associated with $q$ is played and the score is $g_1 : g_2$ just before $q'$ and after $q$.

We refer to null state $\emptyset$, as the **start of the shootout**. In this representation, we envision that each machine state is associated with a round of penalty kicks taken by each team consecutively. However, as round numbers proceed, the game will have to come back to some previous machine state, as the set of states is finite whereas a game can last arbitrarily long in theory.

A mechanism $\phi$ is said to **have finite machine representation** $(Q, A, t)$, if (1) $t(\emptyset, (0 : 0), (T1)_1) = \phi(\emptyset, 0 : 0)$ and $t(\emptyset, (0 : 0), (T2)_1) = 1 - \phi(\emptyset, 0 : 0)$; and (2) recursively, for any kicking-order history $h^{r-1}$ at the beginning of Round $r$, and feasible score $g_{T1} : g_{T2}$ at the beginning of Round $r$, if the associated machine state with round $r - 1$ was $q \in Q$, then we have $t(q, (g_{T1} : g_{T2}), (T1)_w) = \phi(h^{r-1}, g_{T1} : g_{T2})$ for some state $(T1)_w \in Q_{T1}$ and $t(q, (g_{T1} : g_{T2}), (T2)_w) = 1 - \phi(h^{r-1}, g_{T1} : g_{T2})$ for some state $(T2)_w \in Q_{T2}$; and once a transition occurs to a state $q'$ from $q$, ex post we refer to $q'$ as the machine state associated with round $r$.

Note that a machine representation does not specify when the shootout game ends, as no round information is kept in the machine representation. It only keeps track of how transitions are made between different kicking orders in a well-defined pattern. We are now ready to introduce our next property:

A mechanism is **stationary** if it has a finite machine representation $(Q, A, t)$ such that for all states $q_i \in Q \cup \{\emptyset\}$ and $q_j \in Q$, $t(q_i, (g_{T1} : g_{T2}), q_j) = t(q_i, (g'_{T1} : g'_{T2}), q_j)$ for all scores such that $g_{T1} - g_{T2} = g'_{T1} - g'_{T2}$.

Thus, stationarity implies that state transitions are made in the same manner whenever score differences are the same.

For example, the alternating-order behind-first mechanism has this type of a representation as shown in Figure 3.

We state the following proposition whose proof is given in the figure for behind first:

**Proposition 6** *The alternating-order behind-first (ahead-first) mechanism is stationary.*

Machine representations can be used to measure the complexity of an algorithm.[25] However,

[25]For example, in game theory, they are used to represent the recall requirement needed for implementing a repeated game strategy (see Rubinstein, 1998, for an excellent survey).

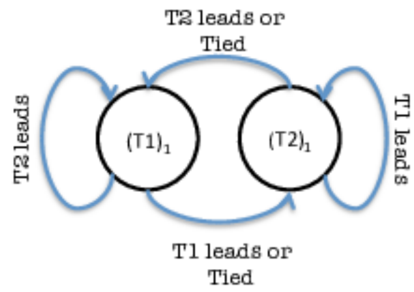


Figure 3: The state transition representation for the alternating-order behind-first mechanism. Transitions from the start of the shootout are omitted for simplicity. In general one of the two states in the figure will be chosen randomly with an unbiased lottery. Alternating-order ahead first's representation is symmetrically defined.

very complicated mechanisms can also be stationary.[26] On the other hand, if we would like to have a chance of both teams kicking first in at least one round, we need at least two states, one $T1$-kicking-first state and one $T2$-kicking-first state. Thus, $|Q| = 2$ is the minimum we can hope for in a reasonable mechanism.[27] Our alternating-order behind-first mechanism also has this property (see Figure 3). We formalize this property as follows:

A mechanism is **simple** if it has a stationary machine representation with only two states such that in one state $T1$ kicks first and in the other $T2$ kicks first.

An important motivation for simplicity stems from the FIFA soccer rules. These rules state that a rule violation by the referees during a game necessitates replay of the game. Shootout mechanisms that satisfy the simplicity property will make the process easier to administer for the referees and will make the process less prone to rule violations. We see simplicity as a vital requirement of a real-life shootout mechanism. The current mechanism satisfies simplicity but none of the other properties we have introduced in this paper. We formalize the simplicity of the alternating-order behind first (ahead first) with the following proposition. We gave its proof earlier through Figure 3:

[26]Consider a modified Prouhet-Thue-Morse behind-first mechanism. First we define the fractal *Prouhet-Thue-Morse* mechanism (see Palacios-Huerta, 2014): The kicking order proceeds in an exogenous manner as follows: ABBABAAB..., i.e., the order sequence since the beginning of the shootout reverses after $2^k$ rounds for each $k = 1, 2, \ldots$. We define the following modified Prouhet-Thue-Morse behind-first mechanism: If one team is behind, it kicks first; otherwise, at even scores the first-kicking team follows the sequence ABBABAAB; then this sequence reverses starting with $T2$ and keeps reversing until the shootout ends. Any behind-first mechanism compatible with a Prouhet-Thue-Morse order is stationary, and the simplest stationary machine representation of such a mechanism cannot have fewer than $|Q| = 16$ states. On the other hand, the truly fractal Prouhet-Thue-Morse sequence is not stationary.

[27]Indeed, the current fixed-order mechanism, ABAB, has $|Q| = 2$, as according to the initial coin toss, either team can go first. However, it is not sequentially fair. The random-order mechanism has also $|Q| = 2$ and is sequentially fair, however, does not maximize the expected number of attempts.

**Proposition 7** *The alternating-order behind-first (ahead-first) mechanism is simple.*

We state the main result of this section as follows:

**Theorem 7** *In an easy shootout, alternating order behind first is the unique sequentially fair mechanism that maximizes the expected number of attempts and satisfies simplicity and sudden-death equality of opportunity.*

*On the other hand, in a difficult shootout, alternating order ahead first is the unique sequentially fair mechanism that maximizes the expected number of attempts and satisfies simplicity and sudden-death equality of opportunity.*

We next demonstrate the independence of properties in Theorem 7: A sequentially fair mechanism that satisfies all properties but violates the maximization of expected number of attempts is the alternating-order ahead-first (behind-first) mechanism for easy (difficult) shootouts. A sequentially fair mechanism that satisfies all properties but the sudden-death equality of opportunity is a behind-first (an ahead-first) mechanism for easy (difficult) shootouts, which randomly determines with an even lottery who goes first when the score is tied. A sequentially fair mechanism that satisfies all properties but is not simple is a Prouhet-Thue-Morse behind-first (ahead-first) mechanism for easy (difficult) shootouts.

# 9 Discussion and Concluding Remarks

Like the current fixed-order shootout mechanism, ABAB, in soccer, some sequential tournaments may be conducive to a first- or second-mover advantage, which may impede the efficiency and/or fairness of these tournaments. We propose fairness as the first-order design desideratum instead of efficiency or revenue maximization unlike previous economic models, while efficiency is a second-order criterion in our setting. Our fairness approach can further be utilized in other economic dynamic contest design problems, such as fair design of dynamic labor tournaments (Lazear and Rosen, 1981; see Connelly et al., 2014 for a survey of the tournament theory literature).

Our approach adds only one parameter to the standard dynamic tug-of-war framework. A kicker has different utility levels for saved or missed kicks, even though they are outcome equivalent. This small friction in the model generates rich results and explains certain stylized empirical facts of the current shootout mechanism. Moreover, it helps us to characterize the optimal shootout mechanism in this setting.

Moreover, our behind-first and ahead-first mechanisms and the additional properties we have considered here can be of help in sports competitions other than soccer and hockey. For example, field hockey, as well as water polo, handball, cricket, and rugby also have tiebreak or penalty shootout mechanisms the same as or similar to that of soccer, and thus can benefit from our analysis.

A relevant question would be whether any of these properties are already being used in real life and how successful they are in their domains. Perhaps the behind-first property is nowhere more blatant and effectively at work than in the rules of "petanque" (a.k.a. boules or bocce), which was invented in ancient times by the Greeks, later modified by Romans, and is now popular in various parts of the world including France and Italy - and currently expanding.[28] In this game, the goal is to throw metal or wooden balls as close as possible to a small special wooden target, while standing inside a small starting circle. The rules are as follows: A player from the team that threw (and established) the target also throws the first ball. Then a player from the other team throws the second ball. The team with the ball that is closest to the target is said to "have the point" or "be winning" and other team is "losing." Then the "losing" team gets to throw the next ball.[29] Thus, in essence, just like our behind-first mechanisms, petanque too intends to give the "losing" team a chance to recover. Further, if the two balls closest to the target are from opposing teams and equidistant, teams play alternately until one team becomes the "winning" team and the other one the "losing" team.[30]

Our related current and future work involves testing the predictions of our model in the field and other controlled media such as virtual reality labs. We hope this approach will help us direct comparison of different shootout formats such as ABAB, ABBA, and our behind first and ahead first.

We also hope that our approach will be a first in rigorously designing (and testing) fair rules and institutions for sports contests and economic competitions such as labor or promotion tournaments, akin to auction design for radio spectrum and electricity sales and matching mechanism design for school choice and organ allocation.

[28]We thank William Thomson for bringing the sports of petanque to our attention.

[29]See Article 16 of the world governing body of petanque FIPJP's official rulebook at http://fipjp.org/index.php/en/2015-05-10-11-11-42/petanque-rules retrieved on Feb 13, 2018.

[30]See Article 29 of the FIPJP's official rulebook.

# Appendix A Shootout Winning Percentages in Major Tournaments

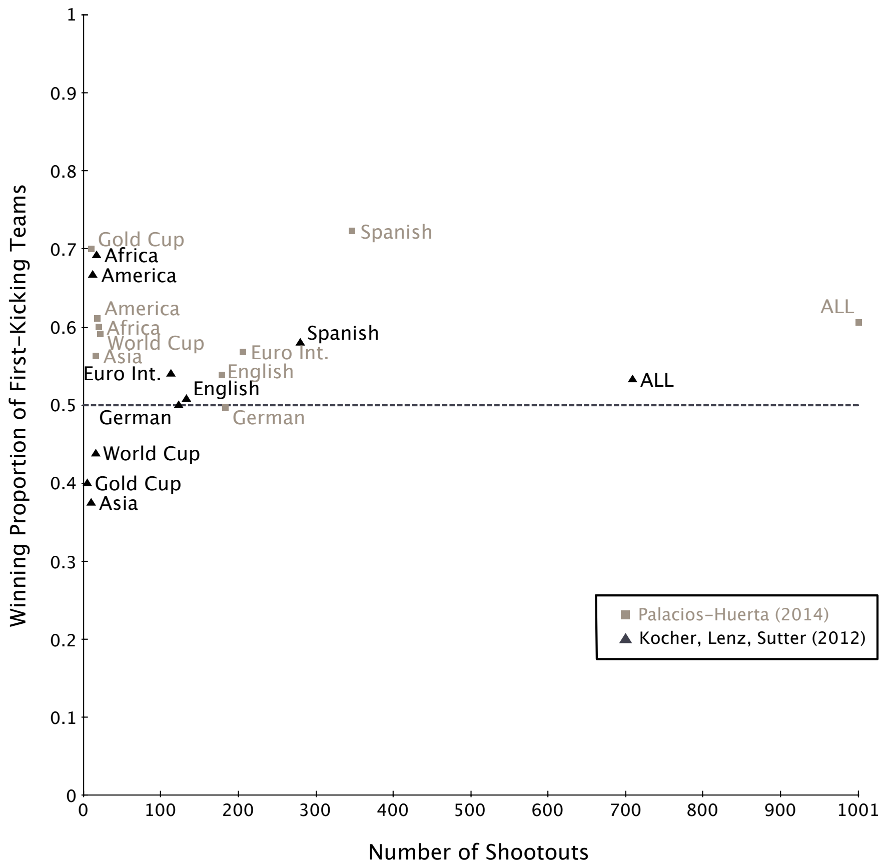


Figure A.1: Empirical Evidence from Table 5.1 in Palacios-Huerta (2014) and Table 1 in Kocher, Lenz, and Sutter (2012): The winning proportions of first-kicking teams are given on the vertical axis while the numbers of shootouts in the considered championships are given on the horizontal axis. Euro int refers to combined proportion for all European international championships such as European Championship, Champions League, Cup Winners Cup, and UEFA Cup. Observe that as sample size increases (i.e., data points 50 or more) second-mover advantage disappears in major soccer data tournaments. While there is undisputed first-mover advantage in Spanish Cup, Euro int and English Cup display somewhat first-mover advantage, and German Cup displays neither first- nor second-mover advantage.[31]

# Appendix B Proofs of the Results in the Main Text

**Proof of Proposition 1.** First observe that $\overline{x}$ solves Equation 4 when $U_O = 0$. As the partial derivative w.r.t. $U_O$ on the (left-hand side of) first-order condition is $P_O'(x_\ell^*) > 0$, the implicit

[31]We also thank Martin Kocher, Marc Lenz, and Matthias Sutter for providing us their data set.

function theorem implies that $x_\ell^* < \overline{x}$. Moreover, as the partial derivative w.r.t. $W_{G,\ell} - W_{NG,\ell}$ on the first-order condition is $P_G'(x_\ell^*) > 0$ (as $x_\ell^* < \overline{x}$), the implicit function theorem again implies that the higher the expected marginal contribution, $W_{G,\ell} - W_{NG,\ell}$, the higher is $x_\ell^*$; and the higher is $P_G(x_\ell^*)$. ■

**Proof of Theorem 1.** We write the three first-order conditions using Equation 11 (or 3) as:

$$P_G'(x)[P_G(y_B)V_{T1} + (1 - P_G(y_B))V_W - P_G(y_E)V_L - (1 - P_G(y_E))V_{T1} + U_G] + P_O'(x)U_O = 0$$

$$P_G'(y_B)[V_{T2} - V_L + U_G] + P_O'(y_B)U_O = 0$$

$$P_G'(y_E)[V_W - V_{T2} + U_G] + P_O'(y_E)U_O = 0$$

We first prove that $x = y_E$ in any state-symmetric equilibrium.

*Claim* 1. $x = y_E$.

*Proof of Claim* 1. Define

$$\Delta = P_G(y_B)V_{T1} + (1 - P_G(y_B))V_W - P_G(y_E)V_L - [1 - P_G(y_E)]V_{T1} - V_W + V_{T2}.$$

From the first-order conditions of $x$ and $y_E$, $x \geq y_E$ if and only if $\Delta \geq 0$. Recall that the winning probability of $T1$ in equilibrium, $\alpha$, is given in Equation 13. Hence,

$$\begin{aligned}\Delta &= P_G(y_B)(V_{T1} - V_W) + P_G(y_E)(V_{T1} - V_L) + V_{T2} - V_{T1} \\ &= P_G(y_B)(1 - \alpha)(V_L - V_W) + P_G(y_E)\alpha(V_W - V_L) + (1 - 2\alpha)(V_W - V_L) \\ &= [-P_G(y_B)(1 - \alpha) + P_G(y_E)\alpha + 1 - 2\alpha](V_W - V_L) \\ &= [1 - P_G(y_B) + (P_G(y_E) + P_G(y_B) - 2)\alpha](V_W - V_L)\end{aligned}$$

We substitute $\alpha$ from Equation 13 as follows:

$$\begin{aligned}\Delta &= [1 - P_G(y_B) + (P_G(y_E) + P_G(y_B) - 2)\frac{P_G(x)(1 - P_G(y_B))}{P_G(x)(1 - P_G(y_B)) + (1 - P_G(x))P_G(y_E)}](V_W - V_L) \\ &= (1 - P_G(y_B))[1 + \frac{(P_G(y_E) + P_G(y_B) - 2)P_G(x)}{P_G(x)(1 - P_G(y_B)) + (1 - P_G(x))P_G(y_E)}](V_W - V_L) \\ &= \left[\frac{(1 - P_G(y_B))(V_W - V_L)}{P_G(x)(1 - P_G(y_B)) + (1 - P_G(x))P_G(y_E)}\right] \\ &\quad \times [P_G(x)(1 - P_G(y_B)) + (1 - P_G(x))P_G(y_E) + (P_G(y_E) + P_G(y_B) - 2)P_G(x)] \\ &= \frac{(1 - P_G(y_B))(V_W - V_L)}{P_G(x)(1 - P_G(y_B)) + (1 - P_G(x))P_G(y_E)}[P_G(y_E) - P_G(x)]\end{aligned}$$

Suppose $x > y_E$, then as both $x, y_E < \overline{x}$ and $P_G$ is increasing on the left of $\overline{x}$, we have $P_G(x) > P_G(y_E)$. But then $\Delta < 0$, contradicting that $x > y_E$. Supposition $x < y_E$ leads to a similar contradiction. Therefore, we must have $x = y_E$. ⋄

Given $x = y_E$, $\alpha$ can be simplified as

$$\alpha = \frac{P_G(x)(1 - P_G(y_B))}{P_G(x)(1 - P_G(y_B)) + (1 - P_G(x))P_G(y_E)} = \frac{1 - P_G(y_B)}{2 - P_G(y_B) - P_G(y_E)},$$

and $\alpha = \frac{1}{2}$ iff $x = y_B$. Then the first-order condition w.r.t. $y_B$ can be simplified as:

$$\begin{aligned} P_G'(y_B)[V_{T2} - V_L + U_G] + P_O'(y_B)U_O = 0 \\ \implies P_G'(y_B)[(1-\alpha)(V_W - V_L) + U_G] + P_O'(y_B)U_O = 0 \\ \implies P_G'(y_B)[(V_W - V_L)\frac{1 - P_G(y_E)}{2 - P_G(y_B) - P_G(y_E)} + U_G] + P_O'(y_B)U_O = 0 \end{aligned} \tag{16}$$

Similarly, the first-order condition w.r.t. $y_E$ can be simplified as:

$$\begin{aligned} P_G'(y_E)[V_W - V_{T2} + U_G] + P_O'(y_E)U_O = 0 \\ \implies P_G'(y_E)[\alpha(V_W - V_L) + U_G] + P_O'(y_E)U_O = 0 \\ \implies P_G'(y_E)[(V_W - V_L)\frac{1 - P_G(y_B)}{2 - P_G(y_B) - P_G(y_E)} + U_G] + P_O'(y_E)U_O = 0 \end{aligned} \tag{17}$$

Now we are ready to prove part (i).

First we show that $P_G'(0)[\frac{V_W - V_L}{2} + U_G] + P_O'(0)U_O \geq 0$ implies the existence of equilibrium. Define $H(z) \equiv P_G'(z)[\frac{V_W - V_L}{2} + U_G] + P_O'(z)U_O$. $H(z)$ is continuous with $H'(z) < 0$ as $P_G''(z) < 0$ and $P_O''(z) \geq 0$. Then $H(0) = P_G'(0)[\frac{V_W - V_L}{2} + U_G] + P_O'(0)U_O \geq 0$ and $H(\overline{x}) = P_O'(\overline{x})U_O < 0$ implies that there exists some $a \in [0, \overline{x})$ such that $H(a) = 0$. It can readily be seen that $(x, y_B, y_E) = (a, a, a)$ solves the two first-order conditions and hence constitutes an equilibrium.

On the other hand, assume now $P_G'(0)[\frac{V_W - V_L}{2} + U_G] + P_O'(0)U_O = H(0) < 0$. As $H'(z) < 0$, $H(z) < 0$ for every $z \in [0, 1]$. Suppose to the contrary that there exists an equilibrium $(x, y_B, y_E)$. Clearly $y_B \neq y_E$, for otherwise $\frac{1 - P_G(y_E)}{2 - P_G(y_B) - P_G(y_E)} = \frac{1}{2}$ and the first-order condition of $y_B$ becomes $H(y_B) < 0$. Suppose $y_B > y_E$. Then the first-order condition w.r.t. $y_E$ in Equation 17 becomes:

$$\begin{aligned} &P_G'(y_E)[(V_W - V_L)\frac{1 - P_G(y_B)}{2 - P_G(y_B) - P_G(y_E)} + U_G] + P_O'(y_E)U_O \\ &< P_G'(y_E)[\frac{V_W - V_L}{2} + U_G] + P_O'(y_E)U_O = H(y_E) < 0, \end{aligned}$$

a contradiction. Then $y_B < y_E$; and similarly, the first-order condition for $y_B$ is negative, leading to a contradiction. Therefore, an equilibrium exists if and only if $P_G'[\frac{V_W - V_L}{2} + U_G] + P_O'(0)U_O = H(0) \geq 0$.

There may be multiple solutions $(y_E, y_B)$, and whenever one exists, then $(\hat{y_E}, \hat{y_B})$ satisfying $\hat{y_E} = y_B$ and $\hat{y_B} = y_E$ also leads to a state-symmetric equilibrium. ■

**Proof of Proposition 2.** The first-order conditions are given by Equations 16 and 17 for $y_B$ and $y_E$ in the proof of Theorem 1, respectively. We get $y_B = y(\beta)$ and $y_E = y(1 - \beta)$, since $f = P_G'/P_O'$ is an invertible differentiable decreasing function in the region $[0, \bar{x}]$ by assumption that $P_O$ is convex and increasing and $P_G$ is strictly concave and increasing in the interval $[0, \bar{x}]$. Thus, circularly, plugging in $y_B$ and $y_E$, we get Equation 15. Optimal spots $y_B$ and $x = y_E$ are multiple valued if and only if $\beta$ is multiple valued. $\beta = \frac{1}{2}$ always solves Equation 15, and if $\beta = \alpha$ is a solution then $\beta = 1 - \alpha$ is also a solution. Thus, there is an odd number of solutions. ■

**Proof of Theorem 3.** We will prove the theorem for easy shootouts. The proof is symmetric for difficult shootouts. We solve it by backward induction. As both teams have an equal chance of winning in sudden-death rounds, the value function is $\frac{V_W+V_L}{2}$ for each team at the end of the regular rounds.

**Round 2, Second Kick**:

Whether the last-kicking team is currently even or behind, it can readily be verified that the optimal kicking strategy is always $\xi < \overline{x}$, where $\xi$ is determined by the following first-order condition:

$$P'_G(\xi)[\frac{V_W - V_L}{2} + U_G] + P'_O(\xi)U_O = 0 \tag{18}$$

**Round 2, First Kick**:

Next we look at the optimal kicking strategy for the first-kicking team in Round 2. Consider two cases:

Case 1: When $T2$ kicks first in Round 2

There are three possible states: when the score is currently even, when $T2$ is currently behind (by one goal), and when $T2$ is currently ahead (by one goal).

• *When the score is currently even*: Let $y_{2E}$ denote the optimal kicking strategy for $T2's$ kicker in Round 2 when the score is even. The value function for $T2$ is

$$\begin{aligned} V_{T2,P2,E} = {} & (P_G(y_{2E})P_G(\xi) + (1 - P_G(y_{2E}))(1 - P_G(\xi)))\frac{V_W + V_L}{2} \\ & + P_G(y_{2E})(1 - P_G(\xi))V_W + (1 - P_G(y_{2E}))P_G(\xi)V_L \end{aligned}$$

By Equation 11, $y_{2E}$ solves the following first-order condition:

$$\begin{aligned} P'_G(y_{2E})[(P_G(\xi) - (1 - P_G(\xi)))\frac{V_W + V_L}{2} + (1 - P_G(\xi))V_W - P_G(\xi)V_L + U_G] + P'_O(y_{2E})U_O = 0 \\ \implies P'_G(y_{2E})[\frac{V_W - V_L}{2} + U_G] + P'_O(y_{2E})U_O = 0 \end{aligned}$$

Therefore,

$$y_{2E} = \xi \tag{19}$$

and $V_{T2,P2,E} = \frac{V_W+V_L}{2}$.

• *When $T2$ is currently behind*: Let $y_{2B}$ denote the optimal kicking strategy for $T2's$ kicker in Round 2 when the score is currently behind. The value function for $T2$ is

$$V_{T2,P2,B} = P_G(y_{2B})P_G(\xi)V_L + P_G(y_{2B})(1 - P_G(\xi))\frac{V_W + V_L}{2} + (1 - P_G(y_{2B}))V_L$$

$y_{2B}$ satisfies the following first-order condition:

$$\begin{aligned} P'_G(y_{2B})[P_G(\xi)V_L + (1 - P_G(\xi))\frac{V_W + V_L}{2} - V_L + U_G] + P'_O(y_{2B})U_O = 0 \\ \implies P'_G(y_{2B})[(1 - P_G(\xi))\frac{V_W - V_L}{2} + U_G] + P'_O(y_{2B})U_O = 0 \end{aligned}$$

• *When $T2$ is currently ahead*: Let $y_{2A}$ denote the optimal kicking strategy for $T2's$ kicker in Round 2 when the score is currently ahead. The value function for $T2$ is

$$V_{T2,P2,A} = P_G(y_{2A})V_W + (1 - P_G(y_{2A}))[(1 - P_G(\xi))V_W + P_G(\xi)\frac{V_W + V_L}{2}]$$

The optimal kicking strategy, $y_{2A}$, satisfies the following first-order condition:

$$P'_G(y_{2A})[V_W - (1-P_G(\xi))V_W - P_G(\xi)\frac{V_W+V_L}{2} + U_G] + P'_O(y_{2A})U_O = 0$$
$$\Longrightarrow P'_G(y_{2A})[P_G(\xi)\frac{V_W-V_L}{2} + U_G] + P'_O(y_{2A})U_O = 0 \tag{20}$$

As

$$P_G(\xi) > \frac{1}{2} \implies y_{2A} > y_{2B}. \tag{21}$$

Moreover, since $P_G(\xi) < 1$, Equations 18 and 20 imply

$$y_{2A} < \xi. \tag{22}$$

Case 2: When $T1$ kicks first in Round 2

Let $x_{2E}$, $x_{2B}$, and $x_{2A}$ denote the optimal kicking strategy for $T1's$ kicker in Round 2 when the score is even, when $T1$ is behind, and when $T1$ is ahead respectively. By symmetry, we have the following results:

- *When the score is currently even*: The optimal kicking strategy is

$$x_{2E} = y_{2E} = \xi, \tag{23}$$

and the value function for $T1$ is $V_{T1,P2,E} = \frac{V_W+V_L}{2}$.

- *When $T1$ is currently behind*: The optimal kicking strategy is

$$x_{2B} = y_{2B} < \xi, \tag{24}$$

and the value function for $T1$ is

$$V_{T1,P2,B} = P_G(x_{2B})P_G(\xi)V_L + P_G(x_{2B})(1-P_G(\xi))\frac{V_W+V_L}{2} + (1-P_G(x_{2B}))V_L$$

- *When $T1$ is currently ahead*: The optimal kicking strategy is

$$x_{2B} < x_{2A} = y_{2A} < \xi, \tag{25}$$

and the value function for $T1$ is

$$V_{T1,P2,A} = P_G(x_{2A})V_W + (1-P_G(x_{2A}))[(1-P_G(\xi))V_W + P_G(\xi)\frac{V_W+V_L}{2}]$$

**Round 1, Second Kick**:

Next we study the second team's optimal kicking strategy in Round 1. There are two possible states:

- *When $T1$ does not score in Round 1*:

The value function for $T2$ in this case is

$$V_{T2,P1,E} = P_G(y_{1E})[\phi(T1;0:1)(V_W+V_L-V_{T1,P2,B})+(1-\phi(T1;0:1))V_{T2,P2,A}]+(1-P_G(y_{1E}))\frac{V_W+V_L}{2},$$

where

$$
\begin{aligned}
V_{T1,P2,B} &= P_G(x_{2B})P_G(\xi)V_L + P_G(x_{2B})(1-P_G(\xi))\frac{V_W+V_L}{2} + (1-P_G(x_{2B}))V_L \\
&= \frac{V_W+V_L}{2} - [1-P_G(x_{2B})(1-P_G(\xi))]\frac{V_W-V_L}{2} \\
V_{T2,P2,A} &= P_G(y_{2A})V_W + (1-P_G(y_{2A}))[(1-P_G(\xi))V_W + P_G(\xi)\frac{V_W+V_L}{2}] \\
&= \frac{V_W+V_L}{2} + [1-(1-P_G(y_{2A}))P_G(\xi)]\frac{V_W-V_L}{2}
\end{aligned}
$$

We substitute the equations of $V_{T1,P2,B}$ and $V_{T2,P2,A}$ into $V_{T2,P1,E}$ as follows:

$$
\begin{aligned}
V_{T2,P1,E} = &\frac{V_W+V_L}{2} + P_G(y_{1E})\{\phi(T1;0:1)[1-P_G(x_{2B})(1-P_G(\xi))] \\
&+(1-\phi(T1;0:1))[1-(1-P_G(y_{2A}))P_G(\xi)]\}\frac{V_W-V_L}{2}
\end{aligned}
$$

The optimal kicking strategy, $y_{1E}$, satisfies the following first-order condition:

$$
P'_G(y_{1E})\{\alpha_1\frac{V_W-V_L}{2} + U_G\} + P'_O(y_{1E})U_O = 0, \text{ where}
$$

$$
\alpha_1 = \phi(T1;0:1)[1-P_G(x_{2B})(1-P_G(\xi))] + (1-\phi(T1;0:1))[1-(1-P_G(y_{2A}))P_G(\xi)].
$$

- *When $T1$ scores in Round 1:*

  The value function for $T2$ is

$$
V_{T2,P1,B} = P_G(y_{1B})\frac{V_W+V_L}{2} + (1-P_G(y_{1B}))[(1-\phi(T1;1:0))V_{T2,P2,B} + \phi(T1;1:0)(V_W+V_L-V_{T1,P2,A})],
$$

where

$$
\begin{aligned}
V_{T2,P2,B} &= P_G(y_{2B})P_G(\xi)V_L + P_G(y_{2B})(1-P_G(\xi))\frac{V_W+V_L}{2} + (1-P_G(y_{2B}))V_L \\
&= \frac{V_W+V_L}{2} - [1-P_G(y_{2B})(1-P_G(\xi))]\frac{V_W-V_L}{2} \\
V_{T1,P2,A} &= P_G(x_{2A})V_W + (1-P_G(x_{2A}))[(1-P_G(\xi))V_W + P_G(\xi)\frac{V_W+V_L}{2}] \\
&= \frac{V_W+V_L}{2} + [1-(1-P_G(x_{2A}))P_G(\xi)]\frac{V_W-V_L}{2}
\end{aligned}
$$

We substitute the equations of $V_{T2,P2,B}$ and $V_{T1,P2,A}$ into $V_{T2,P1,B}$ as follows:

$$
\begin{aligned}
V_{T2,P1,B} = &\frac{V_W+V_L}{2} - (1-P_G(y_{1B}))[(1-\phi(T1;1:0))[1-P_G(y_{2B})(1-P_G(\xi))] \\
&+\phi(T1;1:0)[1-(1-P_G(x_{2A}))P_G(\xi)]]\frac{V_W-V_L}{2}
\end{aligned}
$$

The optimal kicking strategy, $y_{1B}$, satisfies the following first-order condition:

$$
P'_G(y_{1B})\{\alpha_2\frac{V_W-V_L}{2} + U_G\} + P'_O(y_{1B})U_O = 0, \tag{26}
$$

where

$$\alpha_2 = (1-\phi(T1;1:0))[1-P_G(y_{2B})(1-P_G(\xi))] + \phi(T1;1:0)[1-(1-P_G(x_{2A}))P_G(\xi)]$$

Then $y_{1B} = y_{1E}$ iff $\alpha_1 = \alpha_2$ iff

$$\begin{aligned}
&\phi(T1;0:1)[1-P_G(x_{2B})(1-P_G(\xi))] + (1-\phi(T1;0:1))[1-(1-P_G(y_{2A}))P_G(\xi)]] \\
&= (1-\phi(T1;1:0))[1-P_G(y_{2B})(1-P_G(\xi))] + \phi(T1;1:0)[1-(1-P_G(x_{2A}))P_G(\xi)] \\
&\iff (1-\phi(T1;0:1)-\phi(T1;1:0))[1-(1-P_G(y_{2A}))P_G(\xi)] \\
&= (1-\phi(T1;0:1)-\phi(T1;1:0))[1-P_G(x_{2B})(1-P_G(\xi))] \\
&\iff (1-\phi(T1;0:1)-\phi(T1;1:0))[(1-P_G(y_{2A}))P_G(\xi) - P_G(x_{2B})(1-P_G(\xi))] = 0
\end{aligned}$$

However, $(1-P_G(y_{2A}))P_G(\xi) - P_G(x_{2B})(1-P_G(\xi)) > 0$ as $\overline{x} > \xi > x_{2B}$ and $y_{2A} < \xi$. Accordingly,

$$y_{1B} = y_{1E} \iff \phi(T1;0:1) + \phi(T1;1:0) = 1. \tag{27}$$

**Round 1, First Kick**:

Finally, we solve for $T1$'s optimal kicking strategy in Round 1. The value function for $T1$ is

$$\begin{aligned}
V_{T1} &= P_G(x_1)[V_W + V_L - V_{T2,P1,B}] + (1-P_G(x_1))[V_W + V_L - V_{T2,P1,E}] \\
&= V_W + V_L - P_G(x_1)V_{T2,P1,B} - (1-P_G(x_1))V_{T2,P1,E}
\end{aligned}$$

We substitute the equations of $V_{T2,P1,B}$ and $V_{T2,P1,E}$ into $V_{T1}$ as follows:

$$V_{T1} = \frac{V_W + V_L}{2} + [P_G(x_1)(1-P_G(y_{1B}))\alpha_2 - (1-P_G(x_1))P_G(y_{1E})\alpha_1]\frac{V_W - V_L}{2}$$

The optimal kicking strategy, $x_1$, satisfies the following first-order condition:

$$P_G'(x_1)\{[(1-P_G(y_{1B}))\alpha_2 + P_G(y_{1E})\alpha_1]\frac{V_W - V_L}{2} + U_G\} + P_O'(x_1)U_O = 0$$

Therefore

$$x_1 \gtreqless y_{1E} \iff (1-P_G(y_{1B}))\alpha_2 \gtreqless (1-P_G(y_{1E}))\alpha_1 \tag{28}$$

On the other hand, we have

$$V_{T1} = \frac{V_W + V_L}{2} \iff P_G(x_1)(1-P_G(y_{1B}))\alpha_2 = (1-P_G(x_1))P_G(y_{1E})\alpha_1$$

Given that both teams have an equal chance of winning in sudden-death rounds and $V_{T2,P2,E} = V_{T1,P2,E} = \frac{V_W+V_L}{2}$, $\phi$ is sequentially fair if and only if $V_{T1} = \frac{V_W+V_L}{2}$. We first make the following claim:

*Claim* 1. $P_G(x_1)(1-P_G(y_{1B}))\alpha_2 = (1-P_G(x_1))P_G(y_{1E})\alpha_1$ if and only if $(1-P_G(y_{1B}))\alpha_2 = (1-P_G(y_{1E}))\alpha_1$.

*Proof of Claim* 1.

($\Rightarrow$) Suppose to the contrary that $(1 - P_G(y_{1B}))\alpha_2 \neq (1 - P_G(y_{1E}))\alpha_1$ but $P_G(x_1)(1 - P_G(y_{1B}))\alpha_2 = (1-P_G(x_1))P_G(y_{1E})\alpha_1$. If $(1-P_G(y_{1B}))\alpha_2 > (1-P_G(y_{1E}))\alpha_1$, then from the first-order condition of $x_1$ we have $\overline{x} > x_1 > y_{1E}$. Then $P_G(x_1)(1 - P_G(y_{1B}))\alpha_2 > P_G(y_{1E})(1 - P_G(y_{1B}))\alpha_2 > P_G(y_{1E})(1-P_G(y_{1E}))\alpha_1 > (1-P_G(x_1))P_G(y_{1E})\alpha_1$, a contradiction. The other case can be analyzed in a similar fashion.

($\Leftarrow$) If $(1 - P_G(y_{1B}))\alpha_2 = (1 - P_G(y_{1E}))\alpha_1$, then from the first-order condition of $x_1$ we have $x_1 = y_{1E}$, which in turn implies

$$P_G(x_1)(1-P_G(y_{1B}))\alpha_2 = P_G(y_{1E})(1-P_G(y_{1B}))\alpha_2 = P_G(y_{1E})(1-P_G(y_{1E}))\alpha_1 = (1-P_G(x_1))P_G(y_{1E})\alpha_1$$

Hence the Claim is established. $\diamond$

Accordingly, $\phi$ is sequentially fair if and only if

$$(1 - P_G(y_{1B}))\alpha_2 = (1 - P_G(y_{1E}))\alpha_1. \tag{29}$$

This equality holds for an arbitrary pair of (feasible) probabilities, $\{P_G, P_O\}$, if and only if $\alpha_1 = \alpha_2$, which holds if and only if $\phi(T1; 0 : 1) + \phi(T1; 1 : 0) = 1$, i.e., $\phi$ is uneven score symmetric. $\blacksquare$

**Proof of Corollary 1.** It can readily be seen from the proof of Theorem 3 that the optimal kicking strategy at that state is solely determined by "the state" (the score difference and the kicking order in Round 2), and hence is independent of which sequentially fair mechanism leads to that state. W.l.o.g., suppose $T1$ moves first in Round 1 and $T2$ moves second. Consider Round 1 first. Equation 27 in the proof of Theorem 3 implies that $\omega_{1E} = \omega_{1B}$. Equations 28 and 29 imply that $\xi_1 = \omega_{1E}$. Next consider Round 2. Equation 18 implies that $\omega_{2B} = \omega_{2E}$. Equations 18 and 19 when $T1$ moves second in Round 2 and Equation 23 when $T1$ moves first in Round 1 imply that $\xi_{2E} = \omega_{2E}$. For an easy shootout, Equations 21 and 22 when $T1$ moves second in Round 2 and Equations 24 and 25 when $T1$ moves first in Round 2 imply that $\xi_{2E} > \xi_{2A} > \xi_{2B}$. (where easiness of the shootout is only used in Equation 21, for a difficult shootout, we would have $\xi_{2E} > \xi_{2B} > \xi_{2A}$). Finally, Equations 18 and 26 and the fact that $\alpha_2 < 1$ imply $\omega_{1B} < \omega_{2B}$.

$\blacksquare$

**Proof of Theorem 4.** We show that by having the better player kick in Round 1, the better team has a higher chance of winning under an uneven-score-symmetric mechanism. Consider two subcases:

**(i) When the better player is in T2.**

Since the better player is placed in Round 1, the second-round maximization problems remain unchanged. Following the proof of Theorem 3, we have $x_{2A} = y_{2A} > x_{2B} = y_{2B}$, and the last kicker's optimal kicking strategy is $\xi$. Next we study the second team's optimal kicking strategy in Round 1. When $T1$ does not score in Round 1, the value function for $T2$ is[32]

$$V_{T2,P1,E} = \widetilde{P}_G(y_{1E})[\phi(T1; 0 : 1)(V_W + V_L - V_{T1,P2,B}) + (1-\phi(T1; 0 : 1))V_{T2,P2,A}] + (1-\widetilde{P}_G(y_{1E}))\frac{V_W + V_L}{2},$$

[32]Recall that kicking order $T1$ in expression $\phi(T1; g_{T1} : g_{T2})$ refers to the beginning of Round 2 when $T1$ kicked first in Round 1.

where

$$\begin{aligned}V_{T1,P2,B} &= P_G(x_{2B})P_G(\xi)V_L + P_G(x_{2B})(1-P_G(\xi))\frac{V_W+V_L}{2} + (1-P_G(x_{2B}))V_L \\ &= \frac{V_W+V_L}{2} - [1-P_G(x_{2B})(1-P_G(\xi))]\frac{V_W-V_L}{2}\end{aligned}$$

$$\begin{aligned}V_{T2,P2,A} &= P_G(y_{2A})V_W + (1-P_G(y_{2A}))[(1-P_G(\xi))V_W + P_G(\xi)\frac{V_W+V_L}{2}] \\ &= \frac{V_W+V_L}{2} + [1-(1-P_G(y_{2A}))P_G(\xi)]\frac{V_W-V_L}{2}\end{aligned}$$

Therefore

$$V_{T2,P1,E} = \frac{V_W+V_L}{2} + \widetilde{P}_G(y_{1E})\{\phi(T1;0:1)[1-P_G(x_{2B})(1-P_G(\xi))]$$
$$+(1-\phi(T1;0:1))[1-(1-P_G(y_{2A}))P_G(\xi)]\}\frac{V_W-V_L}{2}$$

The optimal kicking strategy, $y_{1E}$, satisfies the following first-order condition:

$$\widetilde{P}'_G(y_{1E})\left[\alpha_1\frac{V_W-V_L}{2} + U_G\right] + \widetilde{P}'_O(y_{1E})U_O = 0, \text{ where}$$

$$\alpha_1 = \phi(T1;0:1)[1-P_G(x_{2B})(1-P_G(\xi))] + (1-\phi(T1;0:1))[1-(1-P_G(y_{2A}))P_G(\xi)].$$

When $T1$ scores in Round 1, the value function for $T2$ is

$$V_{T2,P1,B} = \widetilde{P}_G(y_{1B})\frac{V_W+V_L}{2} + (1-\widetilde{P}_G(y_{1B}))[(1-\phi(T1;1:0))V_{T2,P2,B} + \phi(T1;1:0)(V_W+V_L-V_{T1,P2,A})],$$

where

$$\begin{aligned}V_{T2,P2,B} &= P_G(y_{2B})P_G(\xi)V_L + P_G(y_{2B})(1-P_G(\xi))\frac{V_W+V_L}{2} + (1-P_G(y_{2B}))V_L \\ &= \frac{V_W+V_L}{2} - [1-P_G(y_{2B})(1-P_G(\xi))]\,\frac{V_W-V_L}{2}\end{aligned}$$

$$\begin{aligned}V_{T1,P2,A} &= P_G(x_{2A})V_W + (1-P_G(x_{2A}))\left[(1-P_G(\xi))V_W + P_G(\xi)\frac{V_W+V_L}{2}\right] \\ &= \frac{V_W+V_L}{2} + [1-(1-P_G(x_{2A}))P_G(\xi)]\,\frac{V_W-V_L}{2}\end{aligned}$$

We substitute the equations of $V_{T2,P2,B}$ and $V_{T1,P2,A}$ into $V_{T2,P1,B}$ as follows:

$$V_{T2,P1,B} = \frac{V_W+V_L}{2} - (1-\widetilde{P}_G(y_{1B}))\Big[(1-\phi(T1;1:0))\,[1-P_G(y_{2B})(1-P_G(\xi))]$$
$$+\phi(T1;1:0)\,[1-(1-P_G(x_{2A}))P_G(\xi)]\,\Big]\,\frac{V_W-V_L}{2}$$

The optimal kicking strategy, $y_{1B}$, satisfies the following first-order condition:

$$\widetilde{P}'_G(y_{1B})\Bigg[\Big[[(1-\phi(T1;1:0))[1-P_G(y_{2B})(1-P_G(\xi))] \\ +\phi(T1;1:0)[1-(1-P_G(x_{2A}))P_G(\xi)]\Big]\frac{V_W-V_L}{2}+U_G\Bigg]+\widetilde{P}'_O(y_{1B})U_O=0$$

Given that $y_{2B}=x_{2B}$ and $x_{2A}=y_{2A}$, the first-order condition can be rewritten as

$$P'_G(y_{1B})\{\alpha_2\frac{V_W-V_L}{2}+U_G\}+P'_O(y_{1B})U_O=0, \text{ where}$$

$$\alpha_2=(1-\phi(T1;1:0))[1-P_G(x_{2B})(1-P_G(\xi))]+\phi(T1;1:0)[1-(1-P_G(y_{2A}))P_G(\xi)].$$

Under a sequentially fair mechanism, $\phi(T1;0:1)+\phi(T1;1:0)=1$, and we have $\alpha_1=\alpha_2$. Accordingly, $y_{1E}=y_{1B}$. Finally, we solve for $T1$'s optimal kicking strategy in Round 1. The value function for $T1$ is

$$\begin{aligned}V_{T1}&=P_G(x_1)[V_W+V_L-V_{T2,P1,B}]+(1-P_G(x_1))[V_W+V_L-V_{T2,P1,E}]\\&=V_W+V_L-P_G(x_1)V_{T2,P1,B}-(1-P_G(x_1))V_{T2,P1,E}\\&=\frac{V_W+V_L}{2}+[P_G(x_1)(1-\widetilde{P}_G(y_{1B}))\alpha_2-(1-P_G(x_1))\widetilde{P}_G(y_{1E})\alpha_1]\frac{V_W-V_L}{2}\end{aligned}$$

The optimal kicking strategy, $x_1$, satisfies the following first-order condition:

$$\begin{aligned}P'_G(x_1)\{[(1-\widetilde{P}_G(y_{1B}))\alpha_2+\widetilde{P}_G(y_{1E})\alpha_1]\frac{V_W-V_L}{2}+U_G\}+P'_O(x_1)U_O=0\\ \Longrightarrow P'_G(x_1)\{\alpha_1\frac{V_W-V_L}{2}+U_G\}+P'_O(x_1)U_O=0\end{aligned}$$

Therefore $x_1=y_{1E}=y_{1B}$, and

$$V_{T1}=\frac{V_W+V_L}{2}+[P_G(x_1)(1-\widetilde{P}_G(x_1))-(1-P_G(x_1))\widetilde{P}_G(x_1)]\alpha_1\frac{V_W-V_L}{2}<\frac{V_W+V_L}{2}.$$

Hence $T2$ has a higher chance of winning.

**(ii) When the better player is in T1.**

Following the same procedure in (i), we conclude $x_1=y_{1E}=y_{1B}$. But now $V_{T1}$ becomes

$$\begin{aligned}V_{T1}&=\widetilde{P}_G(x_1)[V_W+V_L-V_{T2,P1,B}]+(1-\widetilde{P}_G(x_1))[V_W+V_L-V_{T2,P1,E}]\\&=V_W+V_L-\widetilde{P}_G(x_1)V_{T2,P1,B}-(1-\widetilde{P}_G(x_1))V_{T2,P1,E}\\&=\frac{V_W+V_L}{2}+[\widetilde{P}_G(x_1)(1-P_G(x_1))-(1-\widetilde{P}_G(x_1))P_G(x_1)]\alpha_1\frac{V_W-V_L}{2}>\frac{V_W+V_L}{2}.\end{aligned}$$

Again, the team with a better player has a higher chance of winning. ■

**Proof of Theorem 5.** Without loss of generality, assume $T1$ kicks first in the first sudden-death round (i.e., in Round $n+1$). In a state-symmetric equilibrium, denote by $x_I$ the optimal kicking strategy for the first kicker in each sudden-death round, and $x_B$ ($x_E$) the optimal kicking strategy for the second kicker in each sudden-death round when the score is behind (tied). Let $V_{T1}$ ($V_{T2}$) denote $T1$'s ($T2$'s) value function at the beginning of the first sudden-death round (Round $n+1$). Then

$$\begin{aligned}
V_{T1} &= [P_G(x_I)P_G(x_B) + (1-P_G(x_I))(1-P_G(x_E))]\, V_{T2} \\
&\quad + P_G(x_I)(1-P_G(x_B))V_W + (1-P_G(x_I))P_G(x_E)V_L \\
V_{T2} &= [P_G(x_I)P_G(x_B) + (1-P_G(x_I))(1-P_G(x_E))]\, V_{T1} \\
&\quad + P_G(x_I)(1-P_G(x_B))V_L + (1-P_G(x_I))P_G(x_E)V_W
\end{aligned}$$

We substitute $V_{T2}$ into the equation of $V_{T1}$ as follows:

$$\begin{aligned}
V_{T1} &= [P_G(x_I)P_G(x_B) + (1-P_G(x_I))(1-P_G(x_E))]^2 V_{T1} \\
&\quad + \{[P_G(x_I)P_G(x_B) + (1-P_G(x_I))(1-P_G(x_E))]P_G(x_I)(1-P_G(x_B)) + (1-P_G(x_I))P_G(x_E)\}V_L \\
&\quad + \{[P_G(x_I)P_G(x_B) + (1-P_G(x_I))(1-P_G(x_E))](1-P_G(x_I))P_G(x_E) + P_G(x_I)(1-P_G(x_B))\}V_W
\end{aligned}$$

Then $V_{T1}$ can be solved as:

$$V_{T1} = \gamma V_W + (1-\gamma)V_L, \text{ where}$$
$$\gamma = \frac{1-(1-P_G(x_I))P_G(x_E)}{2-(1-P_G(x_I))P_G(x_E) - P_G(x_I)(1-P_G(x_B))}.$$

As this is a zero-sum game, we have $V_{T2} = (1-\gamma)V_W + \gamma V_L$.

The optimal kicking strategy, $x_I$, satisfies the following first-order condition:

$$P_G'(x_I)\{[P_G(x_B) - (1-P_G(x_E))]V_{T2} + (1-P_G(x_B))V_W - P_G(x_E)V_L + U_G\} + P_O'(x_I)U_O = 0.$$

Similarly, the optimal kicking strategies $x_B$ and $x_E$ are determined by the following conditions:

$$\begin{aligned}
P_G'(x_B)\{V_{T1} - V_L + U_G\} + P_O'(x_B)U_O = 0 \\
P_G'(x_E)\{V_W - V_{T1} + U_G\} + P_O'(x_E)U_O = 0
\end{aligned}$$

We are going to claim that all three kicking strategies are equivalent, i.e., $x_I = x_B = x_E$, which in turn implies that $V_{T1} = V_{T2} = \frac{V_W+V_L}{2}$ as $\gamma = \frac{1}{2}$, and sequential fairness is established.

First we compare $x_I$ and $x_E$. Define

$$\Delta_{IE} = [P_G(x_B) - (1-P_G(x_E))]V_{T2} + (1-P_G(x_B))V_W - P_G(x_E)V_L - (V_W - V_{T1})$$

By comparing the first-order conditions of $x_I$ and $x_E$, we observe that

$$\Delta_{IE} \gtreqless 0 \text{ if and only if } x_I \gtreqless x_E.$$

Substituting the equations of $V_{T2}$ and $V_{T1}$ into $\Delta_{IE}$ gives us

$$\begin{aligned}
\Delta_{IE} &= V_{T1} - V_{T2} - P_G(x_B)(V_W - V_{T2}) + P_G(x_E)(V_{T2} - V_L) \\
&= [2\gamma - 1 - P_G(x_B)\gamma + P_G(x_E)(1-\gamma)](V_W - V_L) \\
&= [(2 - P_G(x_B) - P_G(x_E))\gamma - 1 + P_G(x_E)](V_W - V_L).
\end{aligned}$$

Plugging in the expression of $\gamma$ and doing some simplifications, we have

$$\Delta_{IE} = \frac{P_G(x_I)(1 - P_G(x_B)) - P_G(x_B)(1 - P_G(x_E))}{2 - (1 - P_G(x_I))P_G(x_E) - P_G(x_I)(1 - P_G(x_B))}(V_W - V_L)$$

We can then conclude that $x_I \gtreqless x_E$ if and only if $x_I \gtreqless x_B$. Next we compare $x_I$ and $x_B$. Define

$$\Delta_{IB} = [P_G(x_B) - (1 - P_G(x_E))]V_{T2} + (1 - P_G(x_B))V_W - P_G(x_E)V_L - (V_{T1} - V_L)$$

By comparing the first-order conditions of $x_I$ and $x_B$, we observe that

$$\Delta_{IB} \gtreqless 0 \text{ if and only if } x_I \gtreqless x_B.$$

By the same token, we can simplify $\Delta_{IB}$ as

$$\Delta_{IB} = \frac{P_G(x_E)(1 - P_G(x_I)) - P_G(x_B)(1 - P_G(x_E))}{2 - (1 - P_G(x_I))P_G(x_E) - P_G(x_I)(1 - P_G(x_B))}(V_W - V_L)$$

Therefore

$$x_I \gtreqless x_B \text{ if and only if } x_E \gtreqless x_B.$$

Finally we compare $x_E$ and $x_B$. Define

$$\Delta_{EB} = V_W - V_{T1} - (V_{T1} - V_L)$$

$\Delta_{EB}$ can be simplified as

$$\Delta_{EB} = \frac{P_G(x_E)(1 - P_G(x_I)) - P_G(x_I)(1 - P_G(x_B))}{2 - (1 - P_G(x_I))P_G(x_E) - P_G(x_I)(1 - P_G(x_B))}(V_W - V_L)$$

Accordingly,

$$x_E \gtreqless x_B \text{ if and only if } x_E \gtreqless x_I.$$

Combining all three observations (inequalities) above, we conclude that in a state-symmetric equilibrium we must have $x_I = x_E = x_B$. ■

**Proof of Theorem 6.** Take any mechanism $\phi$ and any sequentially fair mechanism $\varphi$. Construct a mechanism $\psi$ such that for a given Sudden-death Round $k$, for all $n < \ell < k$, kicking-order histories $h^{\ell-1}$, and feasible scores $g_{T1} : g_{T2}$, $\psi(h^{\ell-1}; g_{T1} : g_{T2}) = \phi(h^{\ell-1}; g_{T1} : g_{T2})$ and for all $\ell \geq k$ and $\ell \leq n$, kicking-order histories $h^{\ell-1}$, and feasible scores $g_{T1} : g_{T2}$, $\psi(h^{\ell-1}; g1 : g_{T2}) = \varphi(h^{\ell-1}; g_{T1} : g_{T2})$.

Now in the Sudden-death Round $k$ and after, whenever the game reaches this round, the probability of winning is given as $\frac{1}{2}$ for each team. By backward induction, consider Round $k-1$. Consider the team that kicks second. Without loss of generality suppose it is $T2$, and $T1$ goes first in Round $k-1$. We can reuse the same first-order conditions for both teams that we used in the proof of Theorem 1, setting

$$V_{T1} = V_{T2} = \frac{V_W + V_L}{2}$$

as the continuation value under the sequentially fair mechanism in Round $k$. Suppose $x$ is $T1$'s kicker's optimal spot, $y_E$ is $T2$'s kicker's optimal spot when they are still tied, and $y_B$ is $T1$'s kicker's optimal spot when $T1$ is ahead (by one goal). Recall the first-order conditions through Equation 11 (or 3):

$$P'_G(x)[P_G(y_B)V_{T1} + (1 - P_G(y_B))V_W - P_G(y_E)V_L - (1 - P_G(y_E))V_{T1} + U_G] + P'_O(x)U_O = 0$$

$$P'_G(y_B)[V_{T2} - V_L + U_G] + P'_O(y_B)U_O = 0$$

$$P'_G(y_E)[V_W - V_{T2} + U_G] + P'_O(y_E)U_O = 0$$

We rewrite $T2$'s kicker's first-order conditions plugging in $V_{T1} = V_{T2}$:

$$P'_G(y_B)[\frac{V_W - V_L}{2} + U_G] + P'_O(y_B)U_O = 0$$

$$P'_G(y_E)[\frac{V_W - V_L}{2} + U_G] + P'_O(y_E)U_O = 0$$

The last two equations yield $y_B = y_E$ (each has a unique solution by assumptions). Given that $T1$'s equation yields:

$$P'_G(x)[\frac{V_W - V_L}{2} + U_G] + P'_O(x)U_O = 0$$

As $T1$ has the same first-order conditions as $T2$, we get $x = y_B = y_E$. So each team's winning probability is the same, $\frac{1}{2}$ in Round $k$, as well. The mechanism $\psi$ is sequentially fair starting from Round $k$. We repeat this argument for each Sudden-death Round $\ell = k-2, k-3, ..., n+1$ and obtain the desired result. ■

**Proof of Proposition 4.** From the proof of Theorem 3, we observe that any sequentially fair mechanism $\phi$ must satisfy the condition $\phi(T1; 0:1) + \phi(T1; 1:0) = 1$.[33] Moreover, under this condition, the three optimal kicking strategies in Round 1 are the same: $x_1 = y_{1E} = y_{1B}$ (see Corollary 1) and they are determined by the following first-order condition:

$$P'_G(x_1)\left[\alpha_1 \frac{V_W - V_L}{2} + U_G\right] + P'_O(x_1)U_O = 0, \text{ where}$$

$$\alpha_1 = \phi(T1; 1:0)[1 - (1 - P_G(y_{2A}))P_G(\xi)] + (1 - \phi(T1; 1:0))[1 - P_G(x_{2B})(1 - P_G(\xi))].$$

Hence the higher the value of $\alpha_1$, the higher $x_1$. As $x_{2B} < \xi$, which is Round 2 second kicking team's intended spot, and $y_{2A} < \xi$, we obtain $1 - P_G(x_{2B})(1 - P_G(\xi)) > 1 - (1 - P_G(y_{2A}))P_G(\xi)$. Therefore maximum $x_1$ is achieved in a sequentially fair mechanism when $\phi(T1; 1:0) = 0$, i.e., when $\phi$ is a behind-first mechanism. On the other hand, minimum $x_1$ is achieved when $\phi(T1; 1:0) = 1$, i.e., when $\phi$ is an ahead-first mechanism. ■

**Proof of Proposition 5.** When a tie occurs at the end of Round 1, both teams take a shot with probability 1 in Round 2. So an attempt is not taken only if the game is not tied at the end of

[33]Recall that kicking order $T1$ in expression $\phi(T1; g_{T2} : g_{T2})$ refers to the beginning of Round 2 when $T1$ kicked first in Round 1.

Round 1. As both team players always exert the same effort $\xi_1$ in a sequentially fair mechanism at all state-symmetric equilibria in Round 1 (see the Corollary 1), the probability of one team being ahead at the beginning of Round 2 is $2P_G(\xi_1)(1 - P_G(\xi_1))$.

Consider any uneven-score-symmetric mechanism in which the ahead team moves first with probability $\alpha \in [0, 1]$ in Round 2. An attempt from the second-moving team in Round 2 is not made if and only if the ahead team moves first and scores or the behind team moves first and misses. This happens with a probability $2P_G(\xi_1)(1 - P_G(\xi_1))[\alpha P_G(\xi_{2A}) + (1 - \alpha)(1 - P_G(\xi_{2B}))]$, where $\xi_1$ is the Round 1 effort level of this shootout and $\xi_{2B}$, $\xi_{2A}$ are the Round 2 effort levels at state-symmetric equilibrium for the first-moving team.

For an easy shootout, we have $P_G(x) > 1/2$ for all $x \in [0, \bar{x}]$. Given that the optimal first-period effort $\xi_1 < \bar{x}$, $2P_G(\xi_1)(1 - P_G(\xi_1))$ is decreasing in $\xi_1$. Since behind first maximizes $\xi_1$ among all sequentially fair shootouts by Proposition 4, $2P_G(\xi_1)(1 - P_G(\xi_1))$ is minimized under behind first. Moreover, in an easy shootout $1 - P_G(\xi_{2B}) < 1/2 < P_G(\xi_{2A})$. Hence, the probability of not making an attempt is minimized for behind first among all sequentially fair easy shootouts.

For a difficult shootout, we have $P_G(x) < 1/2$ for all $x \in [0, \bar{x}]$. Given that the optimal first-period effort $\xi_1 < \bar{x}$, $2P_G(\xi_1)(1 - P_G(\xi_1))$ is increasing in $\xi_1$. Since ahead first minimizes $\xi_1$ among all sequentially fair shootouts by Proposition 4, $2P_G(\xi_1)(1 - P_G(\xi_1))$ is minimized under ahead first. Moreover, in a difficult shootout $1 - P_G(\xi_{2B}) > 1/2 > P_G(\xi_{2A})$. Hence, the probability of not making an attempt is minimized for ahead first among all sequentially fair easy shootouts. ∎

**Proof of Theorem 7.** Observe that the mechanisms that satisfy the properties should be behind-first, since behind-first mechanisms are the only ones that satisfy sequential fairness and maximizing expected number of attempts (by Theorem 5). The mechanisms that satisfy the sudden-death equality of opportunity (SDEO from now on) have to have each team kicking first in every two sudden-death rounds exactly once. Hence, the only kicking order that is simple and SDEO in the sudden-death rounds is alternating-order. Stationarity (as implied by simplicity) implies that the order of kicking switches when the score stays even between two rounds - i.e., if the state was reached after a tie in score, the order switches after this state if the tied score continues. But this does not imply how the kicking order changes if we transition to a tied score from an uneven score. Simplicity implies that we have two states as $Q = \{(T1)_1, (T2)_1\}$. Thus, we need to use the same states of sudden-death rounds also in the regular rounds. Hence, as kicking order switches when the score is tied, i.e. we transition from $(T1)_1$ to $(T2)_1$ or the other way around in the sudden-death rounds, we should do the same in the regular rounds as well. Thus, whenever a round ends with a tied score, we should reverse the kicking order. We end up with the unique machine representation in Figure 3, i.e. with the alternating-order behind-first. ∎

# Appendix C First-Mover Advantage: A Refinement

Here, we address the question as to which state-symmetric equilibrium is more likely to be observed when there are multiple state-symmetric equilibria in the fixed-order mechanism. To that end, we use a selection criterion similar to Cho and Kreps's (1987) "Intuitive Criterion."

Suppose there are multiple state-symmetric equilibria. Let the state-symmetric equilibrium with $(x^*, y_E^*, y_B^*)$ be the one with highest $x$, i.e., the intended spot by $T1$'s kickers is the closest to the goal-optimal spot among all state-symmetric equilibria. We will refer to this equilibrium as *the most aggressive equilibrium for* $T1$ for the following reason: As $x^* = y_E^* > y_B^*$, we have the winning probability of $T1$, $\alpha = \frac{1-P_G(y_B^*)}{2-P_G(y_E^*)-P_G(y_B^*)} > \frac{1}{2}$ by Equation 13; and moreover, such a winning probability for $T1$ is the highest among all state-symmetric equilibria.

As a result, $T1$'s kickers can collectively enforce the most aggressive kicking equilibrium for their team and win more often, in which the first kicker can set the tone of aggressiveness for his team. Being the first mover, if $T1$ can credibly "signal" $T2$ that they are indeed playing this most aggressive equilibrium, this would be the most beneficial for $T1$. In this case, we can use such a signaling through beliefs in the state-symmetric equilibrium to obtain a refinement. For example, if $\sigma_{x^*}$, the probability density function of the ball reaching a particular spot on the goal line when it is aimed at $x^*$ has the support set $[x^* - \underline{\epsilon_{x^*}}, x^* + \overline{\epsilon_{x^*}}]$. Suppose that this support is disjoint from such support sets of other equilibria. Then, whenever $T2$ kickers observe a kick spot in $\sigma_{x^*}$'s support, they can credibly deduce that indeed $T1$ is playing this aggressive equilibrium. Hence, the beliefs of $T2$'s kickers in information sets that are never reached in a state-symmetric equilibrium can be fine-tuned so that less aggressive equilibria can be eliminated.

**Definition 1 (Refinement Criterion)** *If the most aggressive state-symmetric equilibrium for $T1$ involves aiming at $x^*$ for each kicker, and the possible spots that the ball can go under $x^*$ (as determined by the support of $\sigma_{x^*}$, $[x^* - \underline{\epsilon}_{x^*}, x^* + \overline{\epsilon}_{x^*}]$) are different from any of the spots that the ball can go under all other state-symmetric equilibria, then $T1$ can credibly enforce the most aggressive state-symmetric equilibrium.*

Hence, we get the following corollary:

**Corollary 3 (Team 1 wins more often)** *If the state-symmetric equilibria can be refined, then $T1$, the team that kicks first, wins with a higher probability than $T2$ in the sudden-death rounds of the fixed-order mechanism, ABAB.*

Hence, in our analysis with equally-skilled players and goalies, the fixed-order mechanism, ABAB, is biased toward the first-moving team and further multiple equilibria certainly exist. Indeed, empirically as well, these multiple equilibria and the overall first-mover advantage are evident. The relative frequency figures regarding the winning probability of the teams that kick first vary significantly across tournaments throughout the world (see Figure A.1).

# References


[1] Abdulkadiroğlu, A. and T. Sönmez. 2003. "School Choice: A Mechanism Design Approach." *American Economic Review*, 93, 729-747.

[2] Anbarcı, N, M. Büyükboyacı, C.J. Sun, and M.U. Ünver. 2018. "Efficiency and Fairness in Dynamic Team Contests: Theory and Evidence." In progress.

[3] Apesteguia, J., and I. Palacios-Huerta. 2010. "Psychological Pressure in Competitive Environments: Evidence from a Randomized Natural Experiment." *American Economic Review*, 100, 2548-2564.

[4] Aristotle. 1999. *Nicomachean Ethics* (translated by W.D. Ross). Batoche Books.

[5] Baggio, R. 2001. *Una Porta Nel Cielo: Un'autobiografia*. Limina (for quote in the paper see: http://www.theguardian.com/sport/2002/may/19/worldcupfootball2002.football).

[6] Brams, S. and M. Ismail. 2018. "Making the Rules of Sports Fairer." *SIAM Review*.

[7] Che, Y-K. and T. Hendershott. 2008. "How to Divide the Possession of a Football?" *Economics Letters*, 99, 561-565.

[8] Che, Y-K. and T. Hendershott. 2009. "The NFL Should Auction Possession in Overtime Games." *Economists' Voice* 6.9, Article 5.

[9] Cho, I-K. and D.M. Kreps. 1987. "Signaling Games and Stable Equilibria." *Quarterly Journal of Economics*, 102, 179-221.

[10] Chiappori, P.-A., S. Levitt, and T. Groseclose. 2002. "Testing Mixed-Strategy Equilibria When Players Are Heterogeneous: The Case of Penalty Kicks in Soccer." *American Economic Review*, 92, 1138-1151.

[11] Cohen-Zada, D., A. Krumer, and O.M. Shapir. 2018. "Testing the Effect of Serve Order in Tennis Tiebreak." *Journal of Economic Behavior & Organization,* 146, 106-115.

[12] Connelly, B. L, L. Tihanyi, T. R. Crook, and K. A. Gangloff. 2014. "Tournament Theory: Thirty Years of Contests and Competitions." *Journal of Management*, 40, 16-47.

[13] Echenique, F. 2017. "ABAB or ABBA? The Arithmetics of Penalty Shootouts in Soccer." CalTech, mimeo.

[14] Fudenberg, D., G. Richard, J. Stiglitz, and J. Tirole. 1983. "Preemption, Leapfrogging, and Competition in Patent Races." *European Economic Review*, 22, 3-31.

[15] Ginsburgh, V.A. and J.C. Van Ours. 2003. "Expert Opinion and Compensation: Evidence from a Musical Competition." *American Economic Review,* 93, 289-296.

[16] Harford, T. 2006. "World Cup Game Theory: What Economics Tells Us about Penalty Kicks." *The Undercover Economist,* http://www.slate.com/articles/arts/the_undercover_economist/2006/06/world_cup_game_theory.html.

[17] Harris, C. and J. Vickers. 1985. "Perfect Equilibrium in a Model of a Race." *Review of Economic Studies*, 52, 193-209.

[18] Harris, C. and J. Vickers. 1987. "Racing with Uncertainty." *Review of Economic Studies*, 54, 1-21.

[19] Jordet, G. 2009. "Why Do English Players Fail in Soccer Penalty Shootouts? A Study of Team Status, Self-Regulation, and Choking under Pressure." *Journal of Sports Sciences*, 27, 97-106.

[20] Jordet, G., E. Hartman, C. Visscher, and K. A. P. M. Lemmink. 2006. "Kicks from the Penalty Mark in Soccer: The Roles of Stress, Skill, and Fatigue for Kick Outcomes." *Journal of Sports Sciences*, 25, 121-129.

[21] Klumpp, T. and M.K. Polborn. 2006. "Primaries and the New Hampshire Effect." *Journal of Public Economics*, 90, 1073-1114.

[22] Kocher, M., M. Lenz, and M. Sutter. 2012. "Psychological Pressure in Competitive Environments: New Evidence from Randomized Natural Experiments." *Management Science*, 58, 1585-1591.

[23] Lazear, E. P. and Rosen, S. 1981. "Rank-Order Tournaments as Optimum Labor Contracts." *Journal of Political Economy*, 89, 841-864.

[24] Palacios-Huerta, I. 2003. "Professionals Play Minimax." *Review of Economic Studies*, 70, 395-415.

[25] Palacios-Huerta, I. 2014. *Beautiful Game Theory: How Soccer Can Help Economics.* Princeton University Press.

[26] Rosen, S., 1986. "Prizes and Incentives in Elimination Tournaments." *American Economic Review*, 76, 701-715.

[27] Roth, A. E., Sönmez, T. and M. U. Ünver. 2004. "Kidney Exchange." *Quarterly Journal of Economics*, 119, 457-488.

[28] Rubinstein, A. 1998. *Modeling Bounded Rationality.* MIT Press.

[29] Szymanski, S. 2003. "The Economic Design of Sporting Contests." *Journal of Economic Literature*, 41, 1137-1187.