

# Essays in Market Design

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Boston College Electronic Thesis or Dissertation, 2015

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Boston College

The Graduate School of Arts and Sciences

Department of Economics

“ESSAYS IN MARKET DESIGN”

a dissertation

by

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submitted in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

May 2015



# ESSAYS IN MARKET DESIGN

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## Abstract

This dissertation consists of two chapters. The first chapter: Dynamic reserves in matching markets with contracts. In this paper we study a matching problem where agents care not only about the institution they are assigned to but also about the contractual terms of their assignment so that they have preferences over institution-contractual term pairs. Each institution has a target distribution of its slots reserved for different contractual terms. If there is less demand for some groups of slots, then the institution is given opportunity to redistribute unassigned slots over other groups. The choice function we construct takes the capacity of each group of seats to be a function of number of vacant seats of groups considered earlier. We advocate the use of a cumulative offer mechanism (COM) with overall choice functions designed for institutions that allow capacity transfer across different groups of seats as an allocation rule. In applications such as engineering school admissions in India, cadet-branch matching problems at the USMA and ROTC where students are ranked according to test scores (and for each group of seats, corresponding choice functions are induced by them), we show that the COM with a monotonic capacity transfer scheme produces stable outcomes, is strategy proof, and respect improvements in test scores. Allowing capacity redistribution increases efficiency. The outcome of the COM with monotone capacity transfer scheme Pareto dominates the outcome of the COM with no capacity transfer.

The second chapter: On relationships between substitutes conditions. In the matching with contracts literature, three well-known conditions on choice functions (from stronger to weaker)- substitutability,

unilateral substitutability (US) and bilateral substitutability (BS) have proven to be critical. This paper aims to deepen our understanding of them by separately axiomatizing the gap between the BS and the other two. We first introduce a new “doctor separability” (DS) condition and show that BS, DS and irrelevance of rejected contracts (IRC) are equivalent to IRC and US. Due to Hatfield and Kojima (2010) and Aygün and Sönmez (2012), it is known that US, “Pareto separability” (PS), and IRC are equivalent to substitutability and IRC. This, along with our result, implies that BS, DS, PS, and IRC are equivalent to substitutability and IRC. All of these results are given without IRC whenever hospital choices are induced from preferences.

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# Chapter 1: Dynamic Reserves in Matching Markets

## With Contracts: Theory and Applications

May 4, 2015

### 1 Introduction

Engineering school admissions in India functions through a centralized matching market in which students with different privileged backgrounds such as caste and tribes are treated with different admission criteria. Students have different preferences over how they are treated in admission to the same engineering program. Therefore, students may prefer not to reveal their caste and tribe information in the application process. Besides this strategic calculation burden on students, the current system suffers from a crucial market failure: The centralized assignment mechanism fails to transfer some unfilled seats reserved for under-privileged castes and tribes to the use of remaining students. Hence, it is vastly wasteful.

In this paper, we propose a remedy to this problem in a new matching model with contracts and ability to utilize unfilled seats of certain types for other students. Moreover, our remedy removes strategic manipulation burden from students' shoulders about for which seat types they should apply at an engineering program. We propose a strategy proof and stable mechanism in this framework.

There are other direct applications of our model such as USMA and ROTC cadet-branch matching and assignment procedures in hierarchical firms. We discuss all three motivating examples in Section 2.

More specifically our model addresses the real-life applications as follows: There

are institutions and agents to be matched. Each institution initially reserves a certain number of its slots for different privilege groups (agent types). For a single agent, there might be more than one possible contract to obtain a slot at a given institution. Each institution has a pre-specified order (*precedence order*) in which these different privilege groups are to be considered. Different institutions might have different orders. Each agent is a member of at least one privilege group. Since an agent might have more than one privilege type, the set of agents cannot be partitioned according to privilege groups. Each agent has a preference over institution-privilege type pairs. Agents care not only about which institution they are matched to but also about the contractual terms or privilege type under which they are admitted. Each institution has a target distribution of its slots over privilege types, but we do not consider these target distribution as hard bounds.<sup>1</sup> If there is less demand from at least one privilege type institutions are given opportunity to utilize these vacant seats by transferring them over other privilege groups. Institutions might have preferences over how to redistribute these unassigned slots. Each institution has a complete plan where they state how they want to redistribute these slots, so we take capacity transfers to be exogenous. The only condition imposed on the capacity transfer scheme is monotonicity<sup>2</sup> which requires that if more slots are left unassigned from one or more privilege types, the capacity of other privilege types is required to be weakly higher.

The novel design part of this paper is the construction of a choice function of institutions that allows them to transfer capacities from low-demand privilege types to high-demand privilege types. Each institution respects an exogenously given precedence order between different privilege types of student groups when it fills its slots. For each of the privilege types there is an associated choice function, we call *a sub-choice function*. Given the target distribution of the institution and the set of contracts, the first privilege type (according to its precedence order) of the institution fills its slots according to its sub-choice function. Then, it moves

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<sup>1</sup>Hard bounds and soft bounds are analyzed in detail in Ehlers et al. (2014).

<sup>2</sup>This condition is first introduced by Westkamp (2013) in the context of German university admissions.

to the second privilege type. Sub-choice functions are linked to each other by two components. Firstly, since we take a pre-specified precedence order, the choice in each privilege group depends on what has been chosen by the privileges groups that were considered earlier according to the precedence order. Given the chosen contracts from the first privilege type, the remaining set of contracts for the second privilege type can be found as follows: if an agent has one of her contracts chosen by the first privilege type, then all of her contracts are removed (rejected) for the rest of the choice process. The second component that links sub-choice functions of different privilege types is that the capacity of a privilege type changes dynamically according to the number of unassigned slots in the privilege types considered earlier in the choice procedure, i.e., the possible transfer of unassigned slots from privilege groups to other privilege groups. The idea here is that the capacity of the privilege type following the first privilege type according to precedence order is a function of the number of unassigned seats in the first privilege type. The capacity of the third privilege group is a function of unassigned seats in the first and second privilege types, and so on. In short, each sub-choice function has two inputs: the set of remaining contracts to consider, which depends on the choices of the privilege types considered before it, and its capacity which changes dynamically according to the number of unassigned slots of the privilege types considered earlier. In this modeling choice, both of these two inputs of a given sub-choice function depends on the choices of the sub-choice function of the privilege types preceding it. The overall choice of an institution is the union of choices by its different privilege types.

In applications, which we describe in part 2, there is a strict ranking of individuals according to test scores. In the cadet-branch matching problem, cadets are ranked according to test scores, i.e., the order of merit list. In the school-choice application from India, students are ranked according to test scores, as well. Then, for each privilege type, students with that privilege type are ordered, according to the test score ranking. Choice functions for each privilege type, then, are induced from these strict rankings. These types of choice functions are common in practice and are called *q-responsive*.

We present the cumulative offer algorithm as an allocation rule with overall functions of institutions described above. As in Kominers-Sönmez (2013), our overall choice functions fail to satisfy unilateral substitutability and the law of aggregate demand.<sup>3</sup> Even with this complication, we are able to show that the cumulative offer mechanism yields stable outcomes, is strategy proof, and respects improvements in test scores. However, there might not be an agent-optimal stable allocation in our framework. Moreover, even when an agent-optimal stable allocation does exist, the cumulative offer mechanism might not find it. The main purpose of introducing dynamic reserves, i.e., capacity transfers, is to increase efficiency. We show that the outcome of the cumulative offer process under any monotonic capacity transfer scheme Pareto dominates the outcome of the cumulative offer mechanism outcome without a capacity transfer.

## 2 Motivating Applications

The theoretical framework we develop in this paper has a wide applications in matching problems with distributional concerns. In this section we give three main applications of our analysis : caste-based affirmative action in engineering school admissions in India, the cadet-branch matching problem with multiple branch-of-choice contract possibilities at the USMA and ROTC programs, and firm-worker matching in the context of hierarchical organization structures.

### 2.1 Engineering School Admissions in India

Countries in which minority groups have suffered from historic discrimination are commonly characterized by considerable schooling inequalities between these groups and the majority of the population. Particularly when the inequality is great, governments have adopted strong *affirmative action* policies in higher education to remedy it, eschewing a voluntary preferential system in favor of a “reservation system” that *reserves* a fixed percentage of seats in higher education institutions for the relevant

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<sup>3</sup>These two conditions on choice functions are sufficient for the cumulative offer mechanism to be strategy proof. See Hatfield and Kojima (2010) and Aygün and Sönmez (2012).

groups. The fundamental assumption underlying the imposition of a reservation system is that minority students gain admission into selective programs they would otherwise not have access to, and such gains generates social return in the near future.<sup>4</sup>

India is one of the few countries that practices affirmative action on a large scale. *Reservation in India* is the process of setting aside a certain percentage of seats in government institutions for members of under represented communities defined primarily by castes and tribes. Reservation is a form of quota-based affirmative action that is governed by constitutional laws, statutory laws, and local rules and regulations. *Scheduled castes* (SC), *Scheduled Tribes* (ST), and *Other Backward Classes* (OBC) are the primary beneficiaries of the reservation policies under the constitution with the objective of ensuring to level the playing field.<sup>5</sup>

Among all higher education institutions in India, engineering schools are the most prestigious. The admission procedure in engineering schools is organized and regulated by the Indian Institute of Technology (IIT). The IIT practices affirmative action and offers reservation to minority sectors of the society. The following table shows the reservation structure of engineering schools in the State of Maharashtra.<sup>6</sup>

<i>Category of Reservation</i>	<i>Reservation</i>
Scheduled Castes (SC)	13%
Scheduled Tribes (ST)	7%
Other Backward Classes (OBC)	30%
General Category (including SC, ST and OBC)	50%

As shown in the above table, the reservation system sets aside a proportion of all possible positions for members of a specific group. Those not belonging to the designated communities can compete only for general category positions, while mem-

<sup>4</sup>See Bertrand et al. (2010). They argue that affirmative action successfully targets the financially disadvantaged in India. The authors find that despite poor entrance exam scores, lower-caste entrants obtain a positive return for admission.

<sup>5</sup>For a brief history of affirmative action policies in India, see Bertrand et al. (2010) and Weiskopf (2004).

<sup>6</sup>See “Rules for Admissions to First year of Degree Courses in Engineering/Technology in Government, Govt. Aided and Unaided Engineering institutes in Maharashtra State-Academic year 2014-2015”.

bers of the designated communities can compete for both reserved seats and open seats. Students who are not coming from designated communities are considered only for general category seats. However, a student who belongs to one of the designated groups is given opportunity to use his or her caste (or tribe) background as a privilege. If students from designated communities do not use their caste or tribe privileges they are considered only for general category seats. Claiming a reserved seat for students from designated communities is optional. If they state their privilege and get accepted to a program with a reserved seat in that category, they have to prove their membership in the group by providing a legal document.

### **2.1.1 Engineering School Admission Procedure and the DTE Mechanism**

In the Maharashtra Engineering school admission procedure, students are ranked based on their total scores in the “Maharashtra Common Entrance Test (MT CET).” This ranking is used to assign students to general category seats. Rankings for privilege types SC, ST, and OBC are derived as follows: For each category, the relative rankings of the same-category students are preserved and the students from other categories are removed. For students with the same score, students are ranked first by their math scores, then by chemistry scores, and finally by physics scores. In the circumstance that students have the same three scores in each field, age determines the priority, i.e., the older student is given priority. As such, each student has a unique ranking. Each student submits his or her preferences over engineering programs. They can rank at most 100 programs. Together with their program rankings, they can also submit their privilege type if they are coming from SC, ST, or OBC communities and want to use this privilege.

The Directorate of Technical Education (DTE), the institution in charge of the admissions to engineering schools in Maharashtra, uses the following mechanism to allocate seats to students in the centralized admission process (CAP):

**Step 1 :** Each student applies to his or her top choice. Each school considers the applications for the general seat category first, following the ranking  $\succ$ . Students are assigned general category seats one by one following  $\succ$  up to the capacity of

the general category seats. If there are more students than allowed by the capacity of the general seat category, the remaining students are considered for the reserved categories depending on their submitted privilege type. For each reserved category *SC*, *ST*, and *OBC*, students are assigned seats one by one following the priority order of privilege type up to the capacity of that category. The remaining students are rejected.

In general, at step  $n$ :

**Step  $n$**  : Each student who was rejected in the previous step applies to his or her next-choice school. Each school fills its general seats first following  $\succ$  from the tentatively held students and new applicants. Students are assigned general category seats one at a time following  $\succ$  up to the capacity of the general category seats. If there are more students than allowed by the capacity of the general seat category, the remaining students are considered for the reserved categories depending on their stated privilege type. For each reserved category *SC*, *ST*, and *OBC*, students are assigned seats one at a time following the priority ranking in each privilege type up to the capacity of that category. The remaining students are rejected.

This algorithm ends in finitely many steps. When outcomes are announced, all students learn their program assignments together with the privilege type under which they were accepted. DTE announces privilege types together with the program assignment for each student to show the public that reservations are actually respected.

After the above centralized admissions process is done, if there are empty seats in OBC category these seats are converted into general seats and filled by general category applicants according to test score rankings. This process is called “counseling” process.

### **2.1.2 The Shortcomings of the DTE mechanism**

The mechanism used by the DTE has many shortcomings. Two main problems with their admission procedure are listed below. The Indian authorities either are not aware of the first problem or they find it insignificant; however, they realize that the

second problem is important and are trying to solve it.

(i) Students are asked to state their preferences over the set of programs even though their assignments specify a program name together with a seat type. The preference domain is narrower than the allocation domain in that students' preferences over seat types are not investigated but are assumed in a specific way. For example, suppose a student, say from OBC background, submits two schools in his preference list, school A and school B, such that he prefers school A to school B. However, when his assignment is announced, it is going to be in the following form: "general category seat from school B" or "OBC category seat from school A." The DTE assignment procedure simply *assumes* that students only care about which program they are admitted to. They assume that for each program a student ranks in his or her preference list, he or she prefers the open category seat type of that program over the reserved type seat if the student submitted any privilege along with his application. However, for several reasons, which we will discuss below, students may actually care about what type of seats they receive together with their program assignments. Their true preferences might be over program name-seat type pairs, not just program names. Similar to the problem of narrower preference domain explained in the cadet-branch matching papers of Switzer and Sönmez (2013) and Sönmez (2013), the DTE assumes each student prefers the general seats over the reserved seats given a program. Hence, given a preference relation over schools, the DTE generates a new preference profile such that the relative ranking of schools is the same, and in each school the general seat is preferred over the reserved category seat for every student. However, across different programs with different types of seats, students might have more complicated preferences.

- Some students might not want to reveal their caste and tribe information and hence would prefer general category seats over type-specific seats. One of the main reasons for this is the fact that students who obtain a seat from a reserved category are discriminated against in some universities. Opponents of the reservation policies in India argue that the policy is anti-meritocracy and

decreases the average quality of Indian engineering schools. As a result, many students who obtain reserved seats feel discriminated against, as the following item illustrates:

*“A survey among first year students (2013-14 batch) belonging to various SC, ST and OBC categories, has revealed that an alarming 56% of them feel discriminated against in the institution, albeit in a discreet manner. Nearly 60% of those in the reserved category also said they experienced more academic pressure than those in the general category.”*<sup>7</sup>

Because of this pressure, some SC, ST and OBC students prefer general category seats rather than reserved category seat for personal reasons such as pride and dignity.<sup>8</sup> However, current mechanism in use does not let students express these concerns in their preferences.

- If a student from a designated community uses her privilege and is assigned to a reserved seat, then she is exempt from school fees (or pay very low fee), will receive book grants, and will be able to live for free in college housing. Because of financial reasons a high-score, poor student from a designated community would prefer a reserved seat over a general seat. This point is illustrated in the following quote from an online education forum:

*“It’s estimated that 70% of Below Poverty Line in India comprises of Scheduled caste people. It’s very difficult for an SC/ST/OBC student to crack JEE advanced and once they crack this exam, they have to face even a bigger problem. How will they afford at least 1.20 lakh Rupees per year for this technical education? I mean come on, this comprises only of tuition and hostel fee. What about other expenses? I think at least 40,000 rs would be enough in minimal living condition. So a total of 1.6 lakh rs per year. Oh, did we include the cost of a laptop, a bicycle and food? No. So what we conclude from all this is that it’s not an easy task for reserved category students to get education in IITs.*

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<sup>7</sup><http://www.dnaindia.com/mumbai/report-caste-discrimination-in-india-s-elite-institutions-students-2016745>

<sup>8</sup><http://www.quora.com/How-does-it-feel-to-be-an-SC-ST-category-student-in-IIT>

*I do agree that there are some reserved category students who take advantage of all this. I guess at least 30% of reserved category students are economically well and they can afford all this on their own. This is a flaw in the system and we have to accept it.*<sup>9</sup>

- General category seats are regarded as more prestigious. Students from designated communities who care about obtaining prestigious seats have more complicated preferences than simple preferences only over programs. Also, some give political reasons for arguing against the reservation policy. Many students from designated communities are against caste-based reservation policy and do not claim caste or tribe privileges. In that case, they are considered for only general seats.

**Example 1.** Suppose that student  $i$  who has privilege ST submits the following preference over schools:  $s_1 P_i s_2 P_i s_3$ . The DTE generates the following preference relation from the stated preference:  $s_1^{Gen} P_i s_1^{ST} P_i s_2^{Gen} P_i s_2^{ST} P_i s_3^{Gen} P_i s_3^{ST}$ . However, student  $i$ 's true preference might be as follows:  $s_1^{Gen} P_i s_2^{Gen} P_i s_3^{Gen} P_i s_1^{ST} P_i s_2^{ST} P_i s_3^{ST}$ .

This student can manipulate the DTE mechanism by misrepresenting her preferences. Also, the mechanism may create an adverse incentive to have lower test scores if a student from a designated category wants to gain admission only through reserved category seats; i.e., in the above example, a student from ST community might have the following true preference:  $s_1^{ST} P_i s_2^{ST} P_i s_3^{ST} P_i s_1^{Gen} P_i s_2^{Gen} P_i s_3^{Gen}$ .

As such, it is obvious that the DTE mechanism is not fair, does not respect improvements, and is manipulable. Furthermore, it is actually very easy to manipulate the DTE-mechanism. In our model, we expand the preference domain to program-seat type pairs to fully alleviate this problem. Every preference profile over only schools can be represented when preferences are defined over program type-seat type pairs.

The second problem regarding the DTE mechanism is that every year, many reserved seats remain vacant and the public (especially general category applicants) react negatively to this fact.

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<sup>9</sup><http://www.quora.com/Reservation-in-India/As-per-the-policy-of-my-institute-IIT-Roorkee-SC-ST-students-are-provided-with-concession-and-waiver-in-fee-irrespective-of-their-family-income>

(ii) The capacities of reserved seats in the SC and ST categories are taken to be hard bounds. In other words, if there are not enough applications for one of the privilege types SC or ST, some of the seats will remain empty. In Maharashtra, the data show that most years applications from ST students have been low. Hence, some reserved seats for the ST students have remained vacant.<sup>10</sup> However, if there is any vacant seat from the OBC category, the DTE converts that seat into a general category seat.<sup>11</sup> Also, the number of applications from designated communities is volatile over time. Due to insufficient demand from some of these communities, every year many seats that are reserved for SC and ST students remains vacant :

*“As admissions to engineering colleges across the state closed, seats in some of the finest institutes that charge almost nothing have gone abegging. Not only are seats open in some of the most prestigious colleges of the state, slots are vacant in some of the top streams too: 69 in electronics, 38 in mechanical engineering, 27 in civil engineering, 23 in computer science and 10 in electrical engineering. The Directorate of Technical Education (DTE) on Thursday kickstarted the special admission round to fill vacant seats in government institutes; 269 seats are yet to be filled.”<sup>12</sup>*

In our model, we introduce dynamic reserves such that capacity can be transferred from one group of seats to another. Allowing capacity transfer increases efficiency by utilizing slots that would otherwise remain vacant.

## 2.2 Cadet-Branch Matching Problem

Motivated by the low retention rates of the US Military Academy (USMA) and Reserved Officer Training Corps (ROTC) graduates, the army introduced an incentive program in which cadets could bid three years of additional service obligation to obtain higher priority for their desired branches. The full potential of this incen-

<sup>10</sup>See Weisskopf (2004). See also Bertrand et al. (2010).

<sup>11</sup>For the details of the admission procedure for engineering schools, see Weisskopf (2004), Kochar (2009), and Bertrand et al. (2010).

<sup>12</sup><http://timesofindia.indiatimes.com/city/mumbai/Prestigious-government-engineering-colleges-still-have-vacant-seats/articleshow/39833944.cms>

tive program is not utilized, however, because of the USMA's and ROTC's deficient matching mechanism. Switzer and Sönmez (2013) propose a design that eliminates the mechanism's shortcomings and mitigates several policy problems the Army has identified.

*The Officer Career Satisfaction Program (OCSF)* was designed by a group of economists and officers at West Point's Office of Economic and Manpower Analysis to boost career satisfaction and retention. According to this program, cadets are given the opportunity to obtain higher priority for branches that they will sign branch-of-choice contracts with in exchange for serving an additional three years of active duty, which is normally five years if a cadet does not sign a branch-of-choice contract. In this setup, two different contractual terms are possible to obtain a seat from a given branch: the cheap option (five years of active service duty) and the expensive option (eight years of active service duty). The army reserves certain slots for cadets who sign branch-of-choice contracts. Cadets at the USMA have a strict priority ranking known as an *order-of-merit-list (OML)* that is based on a weighted average of academic performance, physical fitness test scores, and military performance. Prior to the implementation of the OCSF, the army had been using the *serial dictatorship* induced by the OML to assign slots at 16 branches to cadets. With the introduction of the OCSF, the army decided to change branch priorities as follows: For the first 75% of slots at any branch, OML is used, and for the remaining 25% of slots cadets who sign branch-of-choice contracts receive higher priority, while OML is used to rank them. If the last 25% of the slots cannot be filled, then OML is used to rank cadets who do not sign a branch-of-choice contract.

One of the problems about the army's design was that the mechanism they propose is not a direct mechanism. They ask cadets to choose (i) a ranking of branches alone, and (ii) a number of branches (possibly none) for which the cadet is asked to sign a branch-of-choice contract.<sup>13</sup> Sönmez and Switzer (2013) carefully redesign this incentive program for the army as follows: Each cadet is asked to

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<sup>13</sup>Similar to the problem of the engineering school admissions in India, the preference domain is not large enough to contain all possible preferences even though the outcomes are announced as branch-contractual terms for each cadet.

state his or her preferences over branch-contractual term pairs. They propose to use a cadet optimal stable mechanism with a specific choice function they designed. According to their choice function, for the first 75% of slots the contracts with highest-priority cadets will be chosen one at a time according to OML. If contracts remain to be considered, for the last 25% of slots first consider contracts with the expensive option following OLM among them. If there are not enough contracts with the expensive option to fill the last 25% of slots, fill the remaining slots with contracts with the cheap option following the OML.

In their choice function design, one should notice that the army reserves 25% of slots for contracts with the expensive option, but if there is not enough demand for these slots, they are transferred to the group of slots for contracts with the cheap option. Consider the following design problem: Suppose that the army offers more than one *branch-of-choice* contract possibility in the sense that, say, there are different types of slots reserved for cadets who want to serve six additional years of active duty and for cadets who want to serve three years of additional active service duty (both in exchange of higher priorities at those slots). In our framework, these different branch-of-choice options correspond to different privilege types and a cadet might belong to all of them. If the army has a target distribution over these types of cadets and initially reserves them certain available slots while there is not enough demand for one of the privilege types, then the army can express its preference over how to redistribute unassigned slots from the low-demand privilege type over the others.

### **2.3 Assignment Procedures in Hierarchical Institutions**

Alva (2014) offers an explanation of why hierarchies are a common organizational structure in institutions from a matching-theoretic perspective, which emphasizes the significance of stable outcomes for the persistence of organizational structures. He studies the matching of individual talents via contracts with institutions that are composed of different divisions enjoined by an institutional governance structure. The term *precedence order* in Kominers and Sönmez's (2013) framework cor-

responds to *institutional governance structure* in Alva’s model. Talents (students in the school-choice models) have preferences not only over institutions but over institution-division pairs. Conflicts over contracts between the divisions of an institution are resolved by a hierarchical (linear) governance structure, whereas conflicts between divisions across institutions are resolved by the preferences of agents. The author shows that stable market outcomes exist whenever each institution has a linear order of its divisions in where, given a set a contracts, divisions choose contracts according to this specified order and the choice of each division is *bilaterally substitutable* and satisfies the *irrelevance of rejected contracts* condition.

In Alva’s model each division within an institution has a pre-specified capacity. The author takes capacities of divisions to be hard bounds. If there is not enough demand for a certain division, then some of the available slots in that division will remain empty. Our main deviation from Alva (2014) is that if there is not enough demand for some of the divisions we utilize these remaining slots by transferring them to other divisions. This transfer scheme simply increases efficiency. To be able to accomplish capacity transfers we introduce choice functions for each division, where the capacity of a division becomes a function of number of the unassigned slots of the divisions that fill their slots earlier.

### 3 The Model

#### 3.1 Agents, Institutions, Contracts, and Privileges

In a matching problem with dynamic reserves, there is a set of agents  $I = \{i_1, \dots, i_n\}$ , a set of institutions  $S = \{s_1, \dots, s_m\}$ , a set of privileges  $\Theta = \{t_1, \dots, t_k\}$ , and a (finite) set of contracts  $X = I \times S \times \Theta$ . Each agent  $i \in I$  has a set of privileges  $\tau(i) \subseteq \Theta$  he or she can claim, where  $\tau : I \Rightarrow \Theta$  is a privilege correspondence. Each contract  $x \in X$  is between an agent  $i(x) \in I$  and an institution  $s(x) \in S$ , and states the privilege  $t(x) \in \tau(i(x))$ . We extend the notations  $i(\cdot)$ ,  $s(\cdot)$ , and  $t(\cdot)$  to sets of contracts by setting  $i(Y) \equiv \cup_{y \in Y} \{i(y)\}$ , and  $s(Y) \equiv \cup_{y \in Y} \{s(y)\}$ . For  $Y \subseteq X$ , we denote  $Y_i \equiv \{y \in Y : i(y) = i\}$ ; analogously, we denote  $Y_s \equiv \{y \in Y : s(y) = s\}$  and

$Y_t \equiv \{y \in Y : t(y) = t\}$ .

Each agent  $i \in I$  has a (linear) preference order  $P^i$  (with weak order  $R^i$ ) over contracts in  $X_i = \{x \in X : i(x) = i\}$ . For ease of notation, we assume that each  $i \in I$  also ranks a “null contract”  $\emptyset_i$ , which represents remaining unmatched (and hence is always available), so that we may assume that  $s$  ranks all the contracts in  $X_i$ .<sup>14</sup> We say that the contracts  $x \in X_i$  for which  $\emptyset_i P^i x$  are *unacceptable to  $i$* . Let  $\mathcal{P}$  denote the set of all preferences over  $S \times \Theta$ . A preference profile of agents is denoted by  $P = (P^{i_1}, \dots, P^{i_n}) \in \mathcal{P}^n$ . A preference profile of all agents except agent  $i_l$  is denoted by  $P_{-i_l} = (P^{i_1}, \dots, P^{i_{l-1}}, P^{i_{l+1}}, \dots, P^{i_n}) \in \mathcal{P}^{n-1}$ .

An *allocation*  $X' \subset X$  is a set of contracts such that each agent appears in at most one contract and no institution appears in more contracts than its capacity allows. Let  $\mathcal{X}$  denote the set of all allocations. Given an agent  $i \in I$  and an allocation  $X'$  with  $(i, s, t) \in X'$ , we refer to the pair  $(s, t)$  as the *assignment* of agent  $i$  under allocation  $X'$ . Agent preferences over allocations are induced by their assignments under these allocations.

**Definition 1.** (*Pareto dominance*) Outcome  $Y \subseteq X$  Pareto dominates outcome  $Z \subseteq X$  if  $Y_i R^i Z_i$  for all  $i \in I$  and  $Y_i P^i Z_i$  for at least one  $i \in I$ .

A *mechanism* is a *strategy space*  $S_i$  for each agent  $i$  along with an *outcome function*  $\varphi : (S_{i_1}, \dots, S_{i_n}) \rightarrow \mathcal{X}$  that selects an allocation for each strategy vector  $(s_{i_1}, s_{i_2}, \dots, s_{i_n}) \in S_{i_1} \times S_{i_2} \times \dots \times S_{i_n}$ . Given an agent  $i$  and a strategy profile  $s \in S$ , let  $s_{-i}$  denote the strategy of all agents except agent  $i$ .

A *direct mechanism* is a mechanism where the strategy space is the set of preferences  $\mathcal{P}$  for each agent  $i$ . Hence a direct mechanism is simply a function  $\psi : \mathcal{P}^n \rightarrow \mathcal{X}$  that selects an allocation for each preference profile.

### 3.2 Choice Procedure of Schools

Each institution  $s \in S$  reserves certain parts of its capacity for special agent groups in order to make some reserved seats available to other privilege types to accommodate

<sup>14</sup>We use the convention that  $\emptyset_i P^i x$  if  $x \in X \setminus X_i$ .

to the characteristics of applicants. These kinds of constraints are encoded in the choice procedure of institution  $s$ . First of all, each institution pre-specifies a linear order in which privilege types are considered. We assume that for each  $s \in S$ , the privileges are ordered to a (linear) *order of precedence*  $\triangleright^s$ . The interpretation of  $\triangleright^s$  is that if  $t \triangleright^s t'$  then, whenever possible, the slots reserved for agents with privilege  $t$  are filled before the slots reserved for agents with privilege  $t'$ . Note that an agent might have multiple privileges, so that set of agents  $I$  may not be partitioned into disjoint sets of agents with different privileges. In particular, a given agent may be considered multiple times by a choice procedure.

Institution  $s$  initially has a target distribution of its seats over different groups of agents with different privileges. Let  $\bar{q}_s$  denote the total capacity of institution  $s$ . The number of reserved for agents with privilege  $t_j$  is denoted by  $\bar{q}_{t_j}^s$ . Then, we have  $\bar{q}_s = \sum_{j=1}^k \bar{q}_{t_j}^s$ . Institution  $s$  has a strict preference for filling these slots according to its target distribution. If the target distribution cannot be achieved because too few agents from one or more of the  $k$  privilege groups apply, then institution  $s$  can express its preferences over possible alternative distributions of privilege types by specifying how its capacity is to be redistributed.

For a given institution  $s \in S$ ,  $C^s(\cdot) : 2^X \rightarrow 2^X$  denotes the *overall choice function* of institution  $s$ . Without loss of generality, assume that the precedence order is  $t_1 \triangleright^s t_2 \triangleright^s \dots \triangleright^s t_k$ . Given a set of contracts  $Y \subseteq X$ ,  $C^s(Y)$  is determined as follows:

- Given  $\bar{q}_{t_1}^s$  and  $Y = Y^0 \subseteq X$ , let  $Y_1 \equiv C_{t_1}^s(Y^0, \bar{q}_{t_1}^s)$  be the set of chosen contracts with privilege  $t_1$ . Then, let  $r_1 = \bar{q}_{t_1}^s - |Y_1|$  be the number of unused seats that were initially reserved for agents with privilege  $t_1$ . Define  $\tilde{Y}_1 \equiv \{y \in Y : i(y) \in i(Y_1)\}$ . This is the set of all contracts of agents whose contract is chosen by the sub-choice function  $C_{t_1}^s(\cdot, \cdot)$ . If a contract of an agent with privilege  $t_1$  is chosen, then all of the contracts naming that agent shall be removed from the set of available contracts for the rest of the procedure. The set of remaining contracts is then  $Y^1 = Y^0 \setminus \tilde{Y}_1$ .

- Given the set of remaining contracts  $Y^1$  and the capacity  $q_{t_2}^s = q_{t_2}^s(r_1) \geq \tilde{q}_{t_2}^s$ , let  $Y_2 = C_{t_2}^s(Y^1, q_{t_2}^s)$  be the set of chosen contracts with privilege  $t_2$ , where the capacity of the group of seats for agents with privilege  $t_2$  is the function of the number of unused seats from the first group. Let  $r_2 = q_{t_2}^s - |Y_2|$  be the number of unused seats that were reserved for agents with privilege  $t_2$ . Define  $\tilde{Y}_2 \equiv \{y \in Y^1 : i(y) \in i(Y^1)\}$ . If a contract of an agent with privilege  $t_2$  is chosen by the sub-choice function  $C_{t_2}^s(\cdot, \cdot)$ , then all of the contracts belonging to that agent will be removed from the set of available contracts. Then, the remaining set of contracts is  $Y^2 = Y^1 \setminus \tilde{Y}_1$ .
- In general, let  $Y_j = C_{t_j}^s(Y^{j-1}, q_{t_j}^s)$  be the set of chosen contracts with privilege  $t_j$  from the set of available contracts  $Y^{j-1}$ , where  $q_{t_j}^s = q_{t_j}^s(r_1, \dots, r_{j-1}) \geq \tilde{q}_{t_j}^s$  is the capacity of the group of seats for agents with privilege  $t_j$  as a function of the vector of the number of unused seats  $(r_1, \dots, r_{j-1})$  that are initially reserved for agents with privileges  $t_1, \dots, t_{j-1}$ , respectively. Let  $r_j = q_{t_j}^s - |Y_j|$  be the number of unused seats that were reserved for agents with privilege  $t_j$ . Define  $\tilde{Y}_j = \{y \in Y^{j-1} : i(y) \in i(Y_j)\}$ . The set of remaining contracts is then  $Y^j = Y^{j-1} \setminus \tilde{Y}_j$ .
- Given the set of contracts  $Y = Y^0$  and the capacity  $q_{t_1}^s$  of the group of seats reserved for agents with privilege  $t_1$ , which comes first in the precedence order, we define the *overall choice function* of institution  $s$  as  $C^s(Y) = \bigcup_{j=1}^k C_{t_j}^s(Y^{j-1}, q_{t_j}^s(r_1, \dots, r_{j-1}))$ .

### 3.3 Stability

An *outcome* is a set of contracts  $Y \subseteq X$ . We follow the Gale and Shapley (1962) tradition in focusing on match outcomes that are *stable* in the sense that

- neither agents nor institutions wish to unilaterally walk away from their assignments, and
- agents and institutions cannot benefit by recontracting outside of the match.

**Definition 2.** We say that an outcome  $Y$  is **stable** if it is

(i) **individually rational**-  $C^i(Y) = Y_i$  for all  $i \in I$  and  $C^s(Y) = Y_s$  for all  $s \in S$ , and

(ii) **unblocked**- there does not exist an institution  $s \in S$  and **blocking set**  $Z \neq C^s(Y)$  such that  $Z = C^s(Y \cup Z)$  and  $Z_i = C^i(Y \cup Z)$  for all  $i \in i(Z)$ .

Note that if the first condition fails, then there is either an agent or an institution who prefers rejecting a contract that involves him/it. If the second condition fails, then there exists an unselected contract  $x$  where not only agent  $i(x)$  prefers  $(s(x), t(x))$  over his assignment but also contract  $x$  can be selected by institution  $s(x)$  given its composition.

**Definition 3.** A stable outcome  $Y \subseteq X$  that *Pareto dominates* all other stable outcomes is called *an agent-optimal stable outcome*.

### 3.4 Monotone Capacity Transfers

The idea behind the class of problems we study is that each institution is required to reserve certain parts of its capacity for different privilege types and may prefer or be required to make some of these reserved seats available to other privilege types if its capacity cannot be filled by the first privilege types. Each institution has a pre-specified order in which different privileges are considered while filling its slots and also has a target capacity distribution over these privilege groups. If its target distribution cannot be achieved because too few agents from one or more privilege types apply, the institution would like to have an alternative distribution over privilege types. To guarantee the existence of stable matchings along with many other possibility results under capacity transfers, in our framework we require the capacity transfer scheme to be monotonic.

**Definition 4.** A capacity transfer scheme is **monotonic**, if for all  $j \in \{2, \dots, k\}$  and all pairs of sequences  $(r_s \tilde{r}_s)_{s=1}^{j-1}$  such that  $\tilde{r}_s \geq r_s$  for all  $s \leq j-1$ ,  $q_{t_j}^s(\tilde{r}_1, \dots, \tilde{r}_{j-1}) \geq q_{t_j}^s(r_1, \dots, r_{j-1})$ .

Monotonicity of capacity transfer scheme requires that whenever weakly more seats are left unassigned in *every* privilege type from  $t_1$  to  $t_{j-1}$ , weakly more seats should be available for privilege type  $t_j$ . Notice that *no capacity transfer* trivially satisfies this definition, so it is considered a monotonic capacity transfer. If the reserve structure is defined as hard bounds, then there is no capacity transfer. In this paper, we propose the control constraints to be interpreted as *soft bounds*-flexible capacities rather than hard bounds. For example, transferring all of the unassigned seats from privilege types (other than general category) that have empty slots to only general category satisfies the monotonic capacity transfer definition and can be considered a flexible capacity scheme. Even though transferring all of unassigned seats from other privilege types to the general category might seem more likely to occur in real-life merit-based assignment procedures to promote competition among agents (students), in our framework, the capacity transfer schemes that institutions can implement are very flexible because different institutions might have different distributional concerns. As long as the capacity transfer scheme is monotonic, each institution can express its preferences over different capacity transfers where it prefers to fill its slots according to its initial target distribution.

### 3.5 Conditions on Preferences and Choice Functions

Let  $X$  be the set of contracts.  $\mathcal{P}(X) = 2^X$  is the power set of  $X$ . A choice function is  $C : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$  such that for every  $Y \subseteq X$ ,  $C(Y) \subseteq Y$ . We now discuss the extent to which institutions' choice functions and sub-choice functions satisfy the conditions that have been key to previous analyses of matching with contracts models.

**Definition 5.** A choice function  $C^s$  satisfies **substitutability** if for all  $z, z' \in X$  and  $Y \subseteq X$ ,  $z \notin C^s(Y \cup \{z\}) \implies z \notin C^s(Y \cup \{z, z'\})$ .

Hatfield and Milgrom (2005) introduced this substitutability condition, which generalizes the earlier *gross substitutes* condition of Kelso and Crawford (1982). Hatfield and Milgrom (2005) also showed that substitutability is sufficient to guar-

antee the existence of stable outcomes. However, their analysis implicitly assumes the **irrelevance of rejected contracts (IRC)**<sup>15</sup> condition defined below:

**Definition 6.** Given a set of contracts  $X$ , a choice function  $C^s : 2^X \rightarrow 2^X$  satisfies IRC if  $\forall Y \subset X, \forall z \in X \setminus Y, z \notin C^s(Y \cup \{z\}) \implies C^s(Y) = C^s(Y \cup \{z\})$ .

Aygün and Sönmez (2013) show that the substitutability condition together with the IRC condition assures the existence of a stable allocation.

Choice function substitutability is necessary in the maximal domain sense for guaranteed existence of stable outcomes in a variety of settings. However, substitutability is *not* necessary for the guaranteed existence of stable outcomes in settings where agents have unit demand (Hatfield and Kojima (2008, 2010)). Indeed, as Hatfield and Kojima (2010) showed, the following condition which is weaker than substitutability, not only suffices for the existence of stable outcomes but also guarantees that there is no conflict of interest among agents. As in the work of Hatfield and Milgrom (2005), an irrelevance of rejected contracts condition is implicitly assumed throughout the work of Hatfield and Kojima (2010).<sup>16</sup> In a matching with contracts framework, the IRC condition is crucial. In a recent study, Afacan (2014) gives an example in which without assuming the IRC condition, the cumulative offer algorithm (which will be defined in Section 4 and is what we propose as an allocation mechanism) does not even produce an allocation.

**Definition 7.** A choice function  $C^s$  satisfies **unilateral substitutability (US)** if  $z \notin C^s(Y \cup \{z\}) \implies z \notin C^s(Y \cup \{z, z'\})$  for all  $z, z' \in X$  and  $Y \subseteq X$  for which  $i(z) \notin i(Y)$  (i.e., no contracts in  $Y$  is associated to agent  $i(z)$ ).

Unilateral substitutability has been proven to be crucial in market design applications. The choice functions of branches in the cadet-branch problem (Switzer and Sönmez (2013) and Sönmez (2013)) do not satisfy substitutability. However, they do satisfy unilateral substitutability. Unilateral substitutability, together with the law of aggregate demand, guarantees the existence of an agent-optimal stable

<sup>15</sup>Alkan (2002) refers to it as “consistency.”

<sup>16</sup>See Aygün and Sönmez (2012) for details.

allocation, and under them the agent-proposing deferred acceptance mechanism is strategy proof. Also, in a recent study, Afacan (2014) shows that the cumulative offer mechanism (which is the main mechanism to be used in matching with contracts framework and will be defined in Section 4) is both resource and population monotonic whenever the choice functions of institutions satisfy unilateral substitutability and irrelevance of rejected contracts.

**Definition 8.** A choice function  $C^s$  satisfies **bilateral substitutability** (BS) if  $z \notin C^s(Y \cup \{z\}) \implies z \notin C^s(Y \cup \{z, z'\})$  for all  $z, z' \in X$  and  $Y \subseteq X$  for which  $i(z), i(z') \notin i(Y)$ .

Bilateral substitutability of a choice function is implied by unilateral substitutability, so it is a weaker condition than US. The BS together with the IRC of overall choice functions guarantees the existence of a stable allocation in a matching with contracts framework under no capacity transfer. However, BS and IRC together are weak conditions (even under no capacity transfer) in the sense that many well-known properties of stable allocations in the standard matching problem do not carry over to the matching with contracts setting. For instance, the agent-optimal stable allocation fails to exist. Strengthening BS to US restores most of these well-known properties.<sup>17</sup>

The choice functions  $C^s$  do satisfy substitutability whenever each agent offers at most one contract to school  $s$ .

**Definition 9.** A choice function  $C^s(\cdot)$  satisfies **weak substitutability** (WS) if  $z \notin C^s(Y \cup \{z\}) \implies z \notin C^s(Y \cup \{z, z'\})$  for all  $z, z' \in X$  and  $Y \subseteq X$  for which  $|Y \cup \{z, z'\}| = |i(Y \cup \{z, z'\})|$ .

This WS condition, first introduced by Hatfield and Kojima (2008), is in general necessary (in the maximal domain sense) for the guaranteed existence of stable outcomes (Proposition 1 of Hatfield and Kojima (2008)). Notice that if every agent has only one privilege type WS corresponds to substitutability.

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<sup>17</sup>See Afacan and Turhan (2014) for the axiomatization of the gap between US and BS.

**Definition 10.** A choice function  $C^s(\cdot)$  satisfies **the law of aggregate demand** (LAD) if  $Y \subseteq Y' \implies |C^s(Y)| \leq |C^s(Y')|$ .

That is, the size of the chosen set never shrinks as the set of contracts grows under the law of aggregate demand.<sup>18</sup> Hatfield and Milgrom (2005) introduce the LAD condition in a matching with contracts framework, and it has proven to be critical. Hatfield and Kojima (2010) show that if choice functions of institutions all satisfy US and LAD, every agent and institution signs the same number of contracts at every stable allocation (i.e., the rural hospital theorem holds). Moreover, the cumulative offer mechanism becomes strategy proof and weakly Pareto efficient for agents. If institutions do not have preferences that generate their choices, then all of these results are obtained under the additional IRC condition of Aygün and Sönmez (2012).

**Definition 11.** A choice function  $C^s(\cdot)$  satisfies **quota monotonicity** (QM) if for any  $q, q' \in \mathbb{Z}_+$  such that  $q < q'$ , for all  $Y \subseteq X$

$$C^s(Y, q) \subseteq C^s(Y, q')$$

$$|C^s(Y, q')| - |C^s(Y, q)| \leq q' - q$$

Quota monotonicity requires choice functions to satisfy two conditions. First, given any set of contracts, if there is an increase in the capacity we require the choice function to select every contract it was choosing before increasing its capacity. It might choose some additional contracts. Second, if, say, the capacity of a privilege type is increased by 2, then the difference between the numbers of contracts chosen after and before the capacity increase cannot exceed 2. Since we allow capacities of privilege types to change dynamically during the choice procedure by exogenously given monotonic capacity transfer schemes, quota monotonicity will be a crucial regulative condition on privileges' sub-choice functions to obtain positive results. However, it will be trivially satisfied if the sub-choice functions are derived from

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<sup>18</sup>In a different setting, Alkan (2002) refers to the LAD as “cardinal monotonicity.”

strict priority rankings induced by test scores in merit-based allocation problems.

### 3.6 Conditions on Sub-choice Functions for Applications

In the Indian engineering school admission problem and cadet-branch matching problem in USMA and ROTC, each sub-choice function for a privilege type is induced from a strict ranking of agents according to test scores. Since each agent (cadet) from a particular privilege type is acceptable for the privilege types she announces at every institution (branch), the sub-choice functions of every privilege type is acceptant.

**Definition 12.** A sub-choice function  $C_{t_j}^s(\cdot, q)$  is  $q$ -acceptant if  $|C(Y)| = \min\{q, |Y|\}$  for every  $Y \subseteq X$ . A sub-choice function is *acceptant* if it is  $q$ -acceptant for some  $q$ .

This definition basically says that if the number of applicants is less than the capacity of the privilege type, every contract (each is associated with a different student/cadet) must be chosen, and if the number of applicants is more than the capacity of the privilege type then the capacity must be filled.

The following is the standard responsiveness definition presented in the literature.

**Definition 13.** (*Responsive priorities (Roth, 1985)*) The preferences of school  $s$  are responsive with capacity  $q$  if (i) for any  $i, j \in I$ , if  $\{i\} \succ_s \{j\}$ , then for any  $I' \subseteq I \setminus \{i, j\}$ ,  $I' \cup \{i\} \succ_s I' \cup \{j\}$ , (ii) for any  $i \in I$ , if  $\{i\} \succ_s \emptyset$ , then for any  $I' \subseteq I$  such that  $|I'| < q$ ,  $I' \cup \{i\} \succ_s I'$ , (iii)  $\emptyset \succ_s I'$  for any  $I' \subseteq I$  with  $|I'| > q$ .

In our framework, we can state both acceptance and responsiveness in a single condition following Chambers and Yenmez (2014). Note that each agent (cadet) has only one contract with a given privilege type in our framework.<sup>19</sup> Let  $\succ$  be the strict ranking of agents according to test scores. For privilege type  $t_j$ , the priority ranking associated with it,  $\succ_{t_j}$ , is obtained from  $\succ$  as follows: for every  $i, j \in I$  such that  $t_j \in \tau(i) = \tau(j)$ ,  $i \succ_{t_j} j$  if and only if  $i \succ j$ , and for every  $k \in I$  such that  $\tau(k) \neq t$ ,  $\emptyset \succ_{t_j} k$ .

<sup>19</sup>This is not necessarily the case in Kominers and Sönmez (2013). In their slot-specific priorities setting, an agent may have multiple contracts with a privilege type for a given institution.

**Definition 14.** A sub-choice function  $C_{t_j}^s(\cdot, q)$  of institution  $s$  for privilege type  $t_j$  is  $q$ -responsive if there exists a strict priority ordering  $\succ_{t_j}$  on the set of contracts naming privilege type  $t_j$ ,  $X_{t_j}$ , and a positive integer  $q$  such that for any  $Y \subseteq (X_s \cap X_{t_j})$

$$C_{t_j}^s(Y, q) = \bigcup_{i=1}^q \{y_i^*\}$$

where  $y_i^*$  is defined as  $y_1^* = \max_Y \succ_{t_j}$  and, for  $2 \leq i \leq q$ ,  $y_i^* = \max_{Y \setminus \{y_1^*, \dots, y_{i-1}^*\}} \succ_{t_j}$ .

Responsiveness and acceptance are both crucial for matching applications where admissions are merit-based. A sub-choice function  $C_{t_j}^s(\cdot, q)$  is  $q$ -responsive if there is a strict priority ordering over the agents for which the sub-choice function always selects the highest-ranked available agents. If a school's sub-choice functions are  $q$ -responsive, then for each privilege type the school acts as if it has preferences over contracts with a capacity constraint, and the school takes the highest-ranking students available to that privilege type up to its capacity.

### 3.7 Respect for Unambiguous Improvements

One of the most important parameters of the Indian engineering school admission problem and cadet-branch matching problem is the strict ranking of agents according to test scores. Let  $\succ$  be the strict ranking of students. For each school  $s \in S$  the strict ranking of contracts in privilege type  $t_j$  is obtained from  $\succ$  as follows:  $x \succ_{t_j}^s y$  if and only if  $i(x) \succ i(y)$  and  $t(x) = t(y) = t_j$ . If  $t_j \notin \tau(i)$ , then  $\emptyset_s \succ_{t_j}^s x$  for all  $x$  such that  $i(x) = i$ . The choice function for each privilege type is obtained from these strict rankings, i.e.,  $C_{t_j}^s(Y, q_{t_j}^s) = C_{t_j}^s(Y, q_{t_j}^s | \succ_{t_j}^s)$ , which is  $q$ -responsive.

Clearly, a reasonable mechanism would never penalize a student as a result of an improvement in his standing in the strict ordering according to test scores. Given two strict rankings of students according to test scores  $\succ$  and  $\succ'$ , we say that  $\succ'$  is an *unambiguous improvement* for student  $i$  over  $\succ$  if

1. the relative ranking between all students except student  $i$  remains exactly the same between  $\succ$  and  $\succ'$ , although

2. the standing of student  $i$  is strictly better under  $\succ'$  than under  $\succ$ .

**Definition 15.** A mechanism *respects improvements* if a student never receives a strictly worse assignment as a result of an unambiguous improvement of his priority ranking.<sup>20</sup>

Violation of this condition may create adverse incentives for some agents to lower their test scores to obtain a better outcome according to their true preferences, as in the current application procedure of engineering school admissions in India.<sup>21</sup>

## 4 The Cumulative Offer Process

The cumulative offer algorithm, which is the generalization of the agent-proposing deferred acceptance algorithm of Gale and Shapley, is the central allocation mechanism used in matching with contracts framework. We now introduce the cumulative offer process for matching with contracts (see Hatfield and Kojima (2010); Hatfield and Milgrom (2005); Kelso and Crawford (1982)).

Here, we provide an intuitive description of this algorithm; we give a more technical statement in Appendix A.

**Definition 16.** In the **cumulative offer process**, students propose contracts to schools in a sequence of steps  $l = 1, 2, \dots$  :

*Step 1* : Some student  $i^1 \in I$  proposes his most-preferred contract,  $x^1 \in X_{i^1}$ . School  $s(x^1)$  holds  $x^1$  if  $x^1 \in C^{s(x^1)}(\{x^1\})$ , and rejects  $x^1$  otherwise. Set  $A_{s(x^1)}^2 = \{x^1\}$ , and set  $A_{s'}^2 = \emptyset$  for each  $s' \neq s(x^1)$ ; these are the sets of contracts available to schools at the beginning of Step 2.

*Step 2* : Some student  $i^2 \in I$  for whom no contract is currently held by any school proposes his most-preferred contract that has not yet been rejected,  $x^2 \in X_{i^2}$ . School  $s(x^2)$  holds the contract in  $C^{s(x^2)}(A_{s(x^2)}^2 \cup \{x^2\})$  and rejects all other contracts in  $A_{s(x^2)}^2 \cup \{x^2\}$ ; schools  $s' \neq s(x^2)$  continue to hold all contracts they held at the end of Step 1. Set  $A_{s(x^2)}^3 = A_{s(x^2)}^2 \cup \{x^2\}$ , and set  $A_{s'}^3 = A_{s'}^2$  for each  $s' \neq s(x^2)$ .

<sup>20</sup>This property was first formulated by Balinski and Sönmez (1999).

<sup>21</sup>See Sönmez (2013), where the author discusses how cadets intentionally lower their OML to obtain better outcomes.

*Step  $l$*  : Some student  $i^l \in I$  for whom no contract is currently held by any school proposes his most-preferred contract that has not yet been rejected,  $x^l \in X_{i^l}$ . School  $s(x^l)$  holds the contract in  $C^{s(x^l)}(A_{s(x^l)}^l \cup \{x^l\})$  and rejects all other contracts in  $A_{s(x^l)}^l \cup \{x^l\}$ ; schools  $s' \neq s(x^l)$  continue to hold all contracts they held at the end of Step  $l - 1$ . Set  $A_{s(x^l)}^{l+1} = A_{s(x^l)}^l \cup \{x^l\}$ , and set  $A_{s'}^{l+1} = A_{s'}^l$  for each  $s' \neq s(x^l)$ .

If at any time no student is able to propose a new contract, that is, if all students for whom no contracts are on hold have proposed all contracts they find acceptable, then the algorithm terminates. The *outcome of the cumulative offer process* is the set of contracts held by schools at the end of the last step before termination.

In the cumulative offer process, agents propose contracts sequentially. Schools accumulate offers, choosing at each step (according to  $C^s$ ) a set of contracts to hold from the set of all previous offers. The process terminates when no agent wishes to propose a contract.

*Remark 1.* Note that we do not explicitly specify the order in which students make proposals. Hirata and Kasuya (2014) show that in the matching with contracts model, the outcome of the cumulative offer process is *order - independent* if the overall choice function of every institution satisfies the bilateral substitutability and the irrelevance of rejected contracts condition. In our setup, the overall choice function of every institution satisfies BS and IRC, and hence, the order - independence result holds for our choice functions.

## 5 Main Results

We now develop our general theoretical results. Overall choice functions of institutions were defined in Section 4.2 as the union of choices by sub-choice functions. Sub-choices are linked by both their choices and the monotonic capacity transfer scheme. Each sub-choice function has two inputs: the set of remaining (rejected) contracts by the sub-choice functions that precede it and the capacity of the privilege type as a function of number of unassigned seats from all of the privilege types considered before it. For overall choice function, to guarantee the existence of stable

allocation under monotonic capacity transfer schemes, we impose certain conditions on sub-choice functions. As shown by Aygün and Sönmez (2012) and Aygün and Sönmez (2013), the IRC condition is needed for the overall choice functions of institutions to guarantee the existence of stable allocation. To achieve this we require that every sub-choice function satisfies IRC. Alva (2014) shows that if sub-choice functions satisfy the BS together with the IRC, then the overall choice function of institutions satisfies BS and IRC if there are no capacity transfers across different privilege types (“divisions” in his terminology). These two conditions are not enough to obtain an overall choice function that satisfies BS and IRC if we allow capacity transfers across privilege types. Since sub-choice functions are linked by their two inputs in our framework, we need to impose further axioms, namely, the law of aggregate demand and quota monotonicity under monotonic capacity transfer schemes.

### 5.1 The Existence of Stable Allocation under Monotonic Capacity Transfers

To ensure that overall choice functions satisfy IRC, it suffices to impose IRC on sub-choice functions for any capacity transfer scheme (not necessarily monotonic).

**Proposition 1.** *Suppose that all sub-choice functions satisfy IRC. Then, the overall choice function satisfies IRC.*

*Proof.* See Appendix B. □

*Remark 2.* For the rest of the paper we always assume that sub-choice functions satisfy IRC so that the overall choice functions of institutions satisfy it as well.

When each agent has only one contract associated with an institution, then substitutability becomes identical to weak substitutability (WS). To obtain an overall choice function that satisfies WS, it suffices for sub-choice functions to satisfy WS, LAD, and QM.

**Proposition 2.** *Suppose that all sub-choice functions satisfy WS, LAD, and QM.*

*If the capacity transfer scheme is monotonic, then the overall choice function also satisfies WS and IRC.*

*Proof.* See Appendix B. □

The following proposition is key to guaranteeing the existence of a stable allocation. The BS condition on overall choice functions, together with IRC, is sufficient to guarantee the existence of stable outcomes.

**Proposition 3.** *Suppose that sub-choice functions satisfy BS, LAD, and QM. If the capacity transfer scheme is monotonic, then the overall choice function satisfies BS and IRC.*

*Proof.* See Appendix B. □

If overall choice functions of institutions satisfy BS and IRC, then by Hatfield and Kojima (2010) and Aygün and Sönmez (2012), a stable allocation exists.

**Theorem 1.** *Suppose that all sub-choice functions satisfy BS, LAD, and QM. If the capacity transfer scheme is monotonic then there exists a stable allocation.*

*Proof.* By Proposition 1 and Proposition 3 we know that the overall choice function of each school satisfies BS and IRC. Then, by the Theorem 1 of Hatfield and Kojima (2010), together with Theorem 1 of Aygün and Sönmez (2012), the set of stable outcomes is non-empty. □

In the Indian engineering school admission problem and the cadet-branch matching problem, sub-choice functions are derived from strict priority rankings according to exam scores. These type of sub-choice functions trivially satisfy BS, IRC, LAD, and QM. By Theorem 1, we have existence of stable allocation under these type of sub-choice functions. We state it as a corollary below:

**Corollary 1.** *Suppose that all-sub-choice functions are q-responsive. Then, under a monotonic capacity transfer scheme, there exists a stable allocation.*

## 5.2 Incentive Issues

Hatfield and Kojima (2010) and Aygün and Sönmez (2012) show that if the overall choice functions of institutions satisfy US and LAD, then the cumulative offer mechanism is (group) strategy proof. Even though US and LAD are sufficient for strategy-proofness, they are not necessary in some frameworks. Kominers and Sönmez (2013) provide a choice function that violates both US and LAD, but even with this complication they show that the cumulative offer mechanism is strategy-proof in their slot-specific priorities setup. In our problem, if we set the capacity of each privilege type equal to 1 and do not allow capacity transfer our problem collapses to a *specific version* of the slot-specific priorities model of Kominers and Sönmez (2013). Notice that in our setting, each agent has only one contract associated with a certain privilege type of a given institution, whereas in their setting this might not be the case. Also, in our Indian school choice application and in the cadet-branch matching framework, each sub-choice function is induced from a strict priority ranking of contracts (also agents since each agent can have at most one contract for a certain privilege type) that is obtained from a strict ranking of agents according to test scores. However, some negative results from Kominers and Sönmez (2013) hold in our model as well. Precisely, examples in the proofs of Proposition 4 and Proposition 5 are both from Kominers and Sönmez (2013).

As in Kominers and Sönmez (2013), overall choice functions fail to satisfy US in our setup as well:

**Proposition 4.** *Suppose that sub-choice functions are  $q$ -responsive and capacity transfer scheme is monotonic. The overall choice functions of schools may fail to satisfy unilateral substitutability.*

*Proof.* Consider  $X = \{x_1, x_2, y\}$  with  $S = \{s\}$ ,  $I = \{i, j\}$  where  $i(x_1) = i(x_2) = i$  and  $i(y) = j$ . Student  $i$  has a higher test score than student  $j$ . Also,  $s(x_1) = s(x_2) = s(y) = s$ . The school has two slots and the  $s_1 \triangleright^s s_2$  with the following priorities:

$$\Pi^{s_1} : x_1 \succ \emptyset_{s_1} \text{ and } \Pi^{s_2} : x_2 \succ y \succ \emptyset_{s_2}.$$

Suppose that the school set the following capacity transfer scheme:  $\bar{q}_{s_1} = 1$  is

given.  $q_{s_2}(r_1) = 1$  for both  $r_1 = 0$  and  $r_1 = 1$ , i.e., even if the first slot remains empty there will be no transfer of this empty seat. Note that the monotonicity of capacity transfer scheme is satisfied when there is no capacity transfer.

Then,  $C^s$  fails to satisfy unilateral substitutability. To see why consider  $C^s(\{x_2, y\}) = \{x_2\}$  and  $C^s(\{x_1, x_2, y\}) = \{x_1, y\}$ . Note that  $y \notin C^s(\{x_2, y\})$  but  $y \in C^s(\{x_1, x_2, y\})$ .

□

Furthermore, overall choice functions in our setting need not satisfy LAD.

**Proposition 5.** *Suppose that sub-choice functions are  $q$ -responsive and capacity transfer scheme is monotonic. The overall choice functions of schools may fail to satisfy the law of aggregate demand.*

*Proof.* Consider  $X = \{x_1, x_2, y\}$  with  $S = \{s\}$ ,  $I = \{i, j\}$  where  $i(x_1) = i(x_2) = i$  and  $i(y) = j$ . Also,  $s(x_1) = s(x_2) = s(y) = s$ . The school has two slots and the  $s_1 \triangleright^s s_2$  with the following priorities:

$$\Pi^{s_1} : x_1 \succ y \succ \emptyset_{s_1} \text{ and } \Pi^{s_2} : x_2 \succ \emptyset_{s_2}$$

Suppose that the school sets the following capacity transfer scheme:  $\bar{q}_{s_1} = 1$  is given.  $q_{s_2}(r_1) = 1$  for both  $r_1 = 0$  and  $r_1 = 1$ , i.e., even if the first slot remains empty there will be no transfer of this empty seat. Then,  $C^s$  fails to satisfy the law of aggregate demand. Consider  $C^s(\{x_2, y\}) = \{x_2, y\}$  and  $C^s(\{x_1, x_2, y\}) = \{x_1\}$ . □

Even though overall choice functions fail to satisfy US and LAD, in the cumulative offer algorithm if a contract is rejected at any step of the algorithm, then that contract cannot be held at any further step. In other words, there is no renegotiation of a rejected contract.

**Proposition 6.** *Suppose that sub-choice functions are  $q$ -responsive. If a contract  $z$  is rejected by school  $s$  at any step of the cumulative offer algorithm, then it cannot be held by school  $s$  in any subsequent step.*

*Proof.* See Appendix B. □

When no renegotiation occurs in the cumulative offer process, the algorithm coincides with the standard agent-proposing deferred acceptance algorithm.<sup>22</sup>

**Proposition 7.** *Suppose that all sub-choice functions are  $q$ -responsive. Then, the cumulative offer algorithm outcome under any monotone capacity transfer scheme is stable.*

The standard definition of dominant strategy incentive compatibility, i.e., strategy-proofness, is as follows:

**Definition 17.** A direct mechanism  $\varphi$  is *strategy-proof* if  $\forall i \in I, P_{-i} \in \mathcal{P}_{-i}, P^i, \tilde{P}^i \in \mathcal{P}$  such that  $\varphi(\tilde{P}^i, P_{-i}) P^i \varphi(P)$ .

That is, no matter which agent we consider, no matter what her true preferences  $P^i$  are, no matter what other preferences  $P_{-i}$  other cadets report (true or not), and no matter which potential “misrepresentation”  $\tilde{P}^i$  agent  $i$  considers, truthful preference revelation is in her best interests. Hence, agents can never benefit from “gaming” the mechanism  $\varphi$ .

**Theorem 2.** *Suppose that all sub-choice functions are  $q$ -responsive and the capacity transfer scheme is monotonic. Then, the cumulative offer mechanism  $\Phi$  as a direct mechanism is strategy-proof.*

*Proof.* See Appendix B. □

In a setting with no capacity transfer, Alva (2104) shows that if sub-choice functions satisfy US and LAD, even though overall choice functions do not satisfy US and LAD, the cumulative offer mechanism is strategy proof.

### 5.3 Agent-Optimal Stable Outcomes

In our framework, an agent-optimal stable outcome need not exist.

**Proposition 8.** *An agent-optimal stable outcome might not exist.*

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<sup>22</sup>See Hatfield and Kojima (2010).

*Proof.* Consider  $X = \{x_1, x_2, y\}$  with  $S = \{s\}$ ,  $I = \{i, j\}$  where  $i(x_1) = i(x_2) = i$  and  $i(y) = j$ . Also,  $s(x_1) = s(x_2) = s(y) = s$ . The school has two slots, each with a different privilege type  $t_1$  and  $t_2$ . The precedence order is  $t_1 \succ^s t_2$ . The priorities of each privilege type is as follows:  $\Pi^{t_1} : x_1 \succ \emptyset_{t_1}$  and  $\Pi^{t_2} : x_2 \succ y \succ \emptyset_{t_2}$ . Without capacity transfer, the cumulative offer algorithm outcome is  $\{x_2\}$ . However, the outcome  $\{x_1, y\}$  is also stable. Since there is no Pareto-domination relationship between the two outcomes  $\{x_2\}$  and  $\{x_1, y\}$  and they are the only stable outcomes, there is no agent-optimal stable outcome in this example.  $\square$

In the above example, suppose school  $s$  uses the following capacity transfer scheme : if  $r_1 = 1$ , then  $q_{t_2} = 2$ . The cumulative offer algorithm outcome is now  $\{x_2, y\}$ , which is the only stable outcome under the given monotonic capacity transfer.

Even when agent-optimal stable outcomes *do* exist, the cumulative offer process might not select them.

**Proposition 9.** *The cumulative offer algorithm outcome might be Pareto dominated by the agent-optimal stable outcome.*

*Proof.* Consider the following example:  $I = \{i, j, k\}$  where  $i$  has the highest test score and  $k$  has the lowest one. There is only one school, i.e.,  $S = \{s\}$ . There are five different privilege types, i.e.,  $\Theta = \{t_1, t_2, t_3, t_4, t_5\}$ . The set of contracts is  $X = \{x_2, x_4, x_5, y_1, y_3, y_4, z_1, z_3\}$  where  $i(x_2) = i(x_4) = i(x_5) = i$ ,  $i(y_1) = i(y_3) = i(y_4) = j$  and  $i(z_1) = i(z_3) = k$ . The privilege type specific to each contract is as follows:  $t(y_1) = t(z_1) = t_1$ ,  $t(x_2) = t_2$ ,  $t(y_3) = t(z_3) = t_3$ ,  $t(x_4) = t(y_4) = t_4$ , and  $t(x_5) = t_5$ . The agent preferences are

$$P_i : x_2 \succ_i x_4 \succ_i x_5 \succ_i \emptyset_i$$

$$P_j : y_3 \succ_j y_4 \succ_j y_1 \succ_j \emptyset_j$$

$$P_k : z_1 \succ_k z_3 \succ_k \emptyset_k$$

Suppose we have the following monotonic capacity transfer scheme:

1. The first seat,  $s_1$ , is for privilege type  $t_1$  and  $\bar{q}_{s_1} = 1$ .
2. The second seat,  $s_2$ , is for privilege type  $t_2$  and initially  $\bar{q}_{s_2} = 0$ . However, if  $r_1 = 1$ , then  $q_{s_2} = 1$ .
3. The third seat,  $s_3$ , is for privilege type  $t_3$  and initially  $\bar{q}_{s_3} = 0$ . If  $r_1 = r_2 = 1$ , then  $q_{s_3} = 1$ . Otherwise  $q_{s_3} = 0$ .
4. The fourth seat,  $s_4$ , is for privilege type  $t_4$  and initially  $\bar{q}_{s_4} = 1$ . For any  $r_1, r_2$  and  $r_3$ ,  $q_{s_4} = 1$ . (no capacity transfer)
5. The fifth seat,  $s_5$ , is for privilege type  $t_2$  and initially  $\bar{q}_{s_5} = 0$ . However, if  $r_4 = 1$ , then  $q_{s_5} = 1$ . Otherwise, it is 0.
6. The sixth seat,  $s_6$ , is for privilege type  $t_5$  and initially  $\bar{q}_{s_6} = 0$ . However, if  $r_4 = r_5 = 1$ , then  $q_{s_6} = 1$ . Otherwise, it is 0.
7. The seventh seat,  $s_7$ , is for privilege type  $t_3$  and initially  $\bar{q}_{s_7} = 0$ . However, if  $r_4 = r_5 = r_6 = 1$ , then  $q_{s_7} = 1$ . Otherwise, it is 0.
8. The last seat,  $s_8$ , is for privilege type  $t_1$  and initially  $\bar{q}_{s_8} = 0$ . However, if  $r_4 = r_5 = r_6 = r_7 = 1$ , then  $q_{s_8} = 1$ . Otherwise, it is 0.

For this example, the cumulative offer process is run with the precedence order  $s_1 \triangleright^s s_2 \triangleright^s s_3 \triangleright^s s_4 \triangleright^s s_5 \triangleright^2 s_6 \triangleright^s s_7 \triangleright^s s_8$  as follows:

$Y$	$C^s(Y)$
$\{x_2\}$	$\{x_2\}$
$\{x_2, y_3\}$	$\{x_2, y_3\}$
$\{x_2, y_3, z_3\}$	$\{x_2, y_3\}$
$\{x_2, y_3, z_3, z_1\}$	$\{z_1, x_2\}$
$\{x_2, y_3, z_3, z_1, y_4\}$	$\{z_1, y_4\}$
$\{x_2, y_3, z_3, z_1, y_4, x_4\}$	$\{z_1, x_4\}$
$\{x_2, y_3, z_3, z_1, y_4, x_4, y_1\}$	$\{y_1, x_4\}$

As shown above, the cumulative offer algorithm outcome in this example is  $\{y_1, x_4\}$ . However, it is Pareto dominated by the outcome  $\{y_1, x_2\}$ , which is stable. Moreover,  $\{y_1, x_2\}$  is the agent-optimal stable allocation in this example.  $\square$

## 5.4 Respect for Unambiguous Improvements

The failure of respecting improvement property hurts the mechanism not only from a normative perspective but also via the adverse incentives it creates if student effort plays any role in determining the strict ranking of agents according to test scores. As in most merit-based resource allocation problems, this is the case for both engineering school admissions in India and also the cadet-branch matching problem in USMA and ROTC.

**Theorem 3.** *The cumulative offer mechanism  $\Phi$  respects unambiguous improvements under any monotonic capacity transfer scheme.*

*Proof.* See Appendix B. □

## 5.5 Increasing Efficiency through Monotonic Capacity Transfer Schemes

The following example illustrates the idea that the outcome of the cumulative offer algorithm under monotonic capacity transfers Pareto dominates the outcome of the cumulative offer algorithm under no capacity transfers.

**Example 2.** Consider  $X = \{x_1, x_2, y_1, y_3, z_1, z_2, w_2, w_3\}$  with  $S = \{s\}$ ,  $I = \{i, j, k, l\}$  where  $i(x_1) = i(x_2) = i$ ,  $i(y_1) = i(y_3) = j$ ,  $i(z_1) = i(z_2) = k$  and  $i(w_2) = i(w_3) = l$ . All the contracts are with school  $s$ .  $\Theta = \{t_1, t_2, t_3\}$  where  $t(x_1) = t(y_1) = t(z_1) = t_1$ ,  $t(x_2) = t(z_2) = t(w_2) = t_2$  and  $t(y_3) = t(w_3) = t_3$ . School  $s$  has three seats, one for each type of student, i.e.,  $\bar{q}_{t_1} = \bar{q}_{t_2} = \bar{q}_{t_3} = 1$  is the target distribution of the school. Students are ranked according to test scores from highest to lowest as :  $i - j - k - l$ . Hence, the following priorities for each type are derived:

$$\Pi^{t_1} : x_1 \succ y_1 \succ z_1 \succ \emptyset_{t_1}$$

$$\Pi^{t_2} : x_2 \succ z_2 \succ w_2 \succ \emptyset_{t_2}$$

$$\Pi^{t_3} : y_3 \succ w_3 \succ \emptyset_{t_3}$$

The student preferences over contracts naming them are as follows:

$$P_i : x_2 P_i x_1 P_i \emptyset_i$$

$$P_j : y_3 P_j y_1 P_j \emptyset_j$$

$$P_k : z_2 P_k z_1 P_k \emptyset_k$$

$$P_l : w_2 P_l w_3 P_l \emptyset_l$$

If there is no capacity transfer then the cumulative offer algorithm outcome is  $\{x_2, y_3, z_1\}$ . Now, suppose that the school has the following monotonic capacity transfer scheme:  $\bar{q}_{t_1} = 1$ . If  $r_1 = 0$ , then  $q_{t_2} = 1$ . If  $r_1 = 1$ , then  $q_{t_2} = 2$ . If  $r_1 = 0$  and  $r_2 = 0$ , then  $q_{t_3} = 1$ . If  $r_1 = 1$  and  $r_2 = 0$ , then  $q_{t_3} = 1$ . If  $r_1 = 0$  and  $r_2 = 1$ , then  $q_{t_3} = 2$ . If  $r_1 = 1$  and  $r_2 = 1$ , then  $q_{t_3} = 2$ . Under this capacity transfer scheme the outcome of the cumulative offer process is  $\{x_2, y_3, z_2\}$ . The important observation here is that the outcome of the cumulative offer algorithm under monotonic capacity transfer scheme Pareto dominates the outcome of the cumulative offer algorithm under no capacity transfers. Even though agents  $i$  and  $j$  obtain the same assignment, agent  $k$  obtains a strictly better assignment under the monotonic capacity transfer described above.

Now, we generalize the observation obtained from the example above:

**Theorem 4.** *If the sub-choice functions are derived from an underlying strict ranking of students  $\succ$  according to test scores, then the outcome of the cumulative offer algorithm under any monotonic capacity transfer,  $\Phi_\succ(P, q)$ , Pareto dominates the outcome of the cumulative offer algorithm under no capacity transfer,  $\Phi_\succ(P, \bar{q})$ .*

*Proof.* See Appendix B. □

Hence, introducing monotonic capacity transfer increases efficiency by utilizing seats that would remain unassigned without capacity transfer.

## 6 Related Matching Problems with Distributional Concerns

We now discuss some other matching models and approaches to matching problems with complex distributional concerns to outline the most important differences between our work and others. For the discussion, we fix a strict ranking of all agents  $\succ$  according to test scores that is respected at every institution when the strict priority rankings of different privilege groups are constructed. Also, for each institution  $s \in S$ , we fix an initial target distribution of the seats over privilege types  $\bar{q}_s = (\bar{q}_{s,t_j})_{j=1}^k$  and a monotonic capacity transfer scheme  $q_s = (q_{s,t_j})_{j=1}^k$ .

Suppose that for all institutions and for every privilege types, the set of acceptable students of two different institutions for the same privilege type is the same. Even for this case, a matching problem with dynamic reserves cannot be reduced to a college admission problem with responsive preferences because (i) a stable outcome cannot in general be achieved by splitting each institution into  $k$  (number of different privilege groups) smaller institutions, one for each privilege type and then running separate assignment procedures for each privilege type, since their capacities change dynamically in our model, and also an agent may have more than one privilege type; and (ii) it is not possible in general to eliminate the possibility of capacity transfers by modifying institutions' priorities/sub-choice functions.

*Alva (2014)* studies a matching problem similar to ours. However, in his model, there is no possibility for capacity transfers. The author interprets capacities of privilege types as hard bounds. In his model, agents have preferences over institution-privilege type pairs. In our application, our sub-choice functions are induced from a strict priority ranking that is obtained from a common test score ranking. *Alva (2014)* is working on more general choice functions than the one we use in our merit-based admission applications. Some results we obtain are similar to his in the sense that both *Alva (2014)* and our work analyze the relationship between the conditions satisfied by the overall choice functions of institutions and conditions imposed on the sub-choice functions of privilege types. However, our analysis is

neither a generalization nor a special case of his.

In the context of German university admissions, Westkamp (2013) introduces a general class of *matching problems with complex constraints*. In such a problem, schools may decide or be required to reserve a certain part of its capacity for special student groups (e.g., siblings, minorities, and so on) and may want to make some of these reserved seats available to other student groups to accommodate to the characteristics of applicants. Such constraints are encoded in the overall choice function of the institution in his model: First, the institution specifies an order in which special student groups are considered. A student may belong to multiple special student groups. For each special student group there is a strict ranking of the students. The institution fill its groups of slots by following the priority order of each group. The idea in this paper is that how much capacity is reserved for each group is a function of the number of seats left vacant by groups considered earlier, starting from some fixed value for the first group to be considered. Each institution has a *target distribution* of its slots. An institution initially intends to allocate a fixed number of slots and has a strict preference for filling these slots according to its target distribution. If its target distribution cannot be achieved because too few students from one or more of the groups apply, an institution can express its preference over possible alternative distributions of student groups by specifying how its capacity is to be redistributed through its choice of the capacity functions. It is very important note here that the formulation of Westkamp (2013) implicitly assumes that there are no specific advantages or disadvantages associated with being admitted because one belongs to a particular group, so that students do not care about the type of the slot they receive, but care only about their assigned institutions. Even though students only care about the school they are assigned to, Westkamp (2013) resolves the indifferences over different type of seats in a given school by using the order of precedence in that school. By contrast, the applications we use as our motivation in this paper, some agents/students do care about what type seat they receives and their true preferences among the different types of seats of a given institution might be different from the given precedence order of the institution. To accommodate such

preferences over school-seat type pairs we use the matching with contracts framework rather than the conventional matching model used by Westkamp (2013). When students have preferences over institutions-privilege type pairs the problem becomes outside of the scope of the problems he considers. Notice that every preference profile in his setting (after breaking the ties) can be represented in our setting as well but not vice versa.

*Kominers and Sönmez (2013)* introduced a two-sided, many-to-one matching with contracts model in which agents with unit demand match to branches, which may have multiple slots available to accept contracts. Agents care only about their institutional assignments so that they are indifferent between different contracts that name the same institution. Each slot of every institution has its own linear priority order over contracts, and a branch chooses contracts by filling its slots sequentially. They demonstrate that in these *matching markets with slot-specific priorities*, branches' choice functions may not satisfy the substitutability condition typically crucial for matching with contracts. Despite this complication, they are able to show that stable outcomes exist in their framework and can be found by a cumulative offer mechanism that is strategy proof and respects unambiguous improvements in priority. There are significant differences between our framework and theirs. First of all, they do not allow for capacities to be transferred from one slot to another. In our setting, each privilege type has initially set target capacities that may be greater than one. If we consider each slot as a different privilege type and set the capacity to 1 without the possibility of capacity transfer, we obtain a "specific version" of their slot-specific priority models because in our applications each sub-choice function is derived from a strict priority ordering that is induced by a common ordering of agents according to test scores. For example, if we have two students, say student  $i$  and  $j$ , such that student  $i$ 's test score is higher than student  $j$ 's test score, then for each privilege type that both  $i$  and  $j$  have the rankings of agents will be the same in our setup:  $i$  has higher priority than  $j$ . In their setting, this does not need to be the case. Hence, their analysis is neither a generalization nor a special case of the dynamic capacity approach that we use. Most of our results are similar to theirs in

the sense that we also show that the cumulative offer mechanism is stable, strategy proof, and respects improvements in test scores, even though the overall choice functions of institutions in our framework do not satisfy the unilateral substitutability and the law of aggregate demand conditions.

*Ehlers et al. (2014)* take two different approaches to analyze controlled school choice problems. In the first approach, controlled choice constraints define feasibility of assignments, i.e., they are hard bounds. In this case, they show that it may be impossible to eliminate justified envy across types. However, justified envy can be eliminated among students of the same type by their student exchange algorithm. In the second approach, they provide a new interpretation of controlled choice constraints as soft bounds. Our dynamic reserve interpretation can be thought as a soft bound for controlled choice constraints. With the soft bound view, they describe school preferences through choice rules that satisfy substitutability and the law of aggregate demand. It must be noted here that in their model students preferences are only over schools and each student has one type only, whereas in our setup students have preferences over school-type pairs and each student may have more than one type.

*Hafalir et al. (2011)* analyzes a model of school choice with minority and majority students where certain slots at each school are reserved for minority students but convert into regular slots if not claimed by minority students. They show that there exists a stable mechanism with minority reserves that is group-strategy-proof for students. This result is a special case of Theorem 2 of Westkamp (2013).

## 7 Conclusion

In this paper, we have studied a matching problem with distributional concerns where agents care not only about the institution they are matched with but also about the contractual terms of the contract with the institution. In other words, we expand the preference domain of agents from institutions only to institutions-contractual terms pairs. Each institution can be thought as a union of different divisions, where

each division is associated with exactly one contractual term. Institutions have target distributions over their divisions in the form of reserves. If these reserves are considered as hard bounds, then in the case of demand for a particular division that is less than its target capacity, some slots will remain empty. To overcome this problem and to increase efficiency we introduce capacity transfers across divisions when one or more of the divisions is not able to fill its target capacity. The capacity transfer scheme is embedded into divisions' choice functions, i.e., sub-choice functions. The overall choice function of an institution can be thought of as the union of choices with these sub-choice functions.

We offer the cumulative offer mechanism under monotonic capacity transfers as an allocation rule in merit-based object allocation problems where agents are ranked strictly according to certain test scores. When each privilege has a q-responsive choice function obtained from a strict priority ranking, the cumulative offer mechanism is stable and strategy proof. Moreover, the cumulative offer mechanism respects improvement in test scores, i.e., improvement in the ranking of an agent. By introducing monotonic capacity transfers in the matching with contracts framework, we obtain a gain in efficiency in the sense outcome of the the cumulative offer algorithm under monotonic capacity transfer Pareto dominates the outcome of the cumulative offer algorithm without capacity transfer.

## 8 Appendix

### A. Formal Description of the Cumulative Offer Process

The **cumulative offer process associated to proposal order**  $\Gamma$  is the following algorithm

1. Let  $l = 0$ . For each  $s \in S$ , let  $D_s^0 \equiv \emptyset$ , and let  $A_s^1 \equiv \emptyset$ .
2. For each  $l = 1, 2, \dots$

Let  $i$  be the  $\Gamma_l$  – maximal agent  $i \in I$  such that  $i \notin i(\bigcup_{s \in S} D_s^{l-1})$  and  $\max_{P^i}(X \setminus (\bigcup_{s \in S} A_s^l))_i \neq \emptyset$  – that is, the agent highest in the proposal order who wants

to propose a new contract- if such agent exist. (If no such agent exist, then proceed to Step 3, below.)

(a) Let  $x = \underset{P^i}{\max}(X \setminus (\bigcup_{s \in S} A_s^l))_i$  be  $i$ 's most preferred contract that has not been proposed.

(b) Let  $s = s(x)$ . Set  $D_s^l = C^s(A_s^l \cup \{x\})$  and set  $A_s^{l+1} = A_s^l \cup \{x\}$ . For each  $s' \neq s$ , set  $D_{s'}^l = D_{s'}^{l-1}$  and set  $A_{s'}^{l+1} = A_{s'}^l$ .

3. Return the outcome

$$Y \equiv (\bigcup_{s \in S} D_s^{l-1}) = (\bigcup_{s \in S} C^s(A_s^l))$$

consisting of contracts held by institutions at the point when no agents want to propose additional contract.

Here, the sets  $D_s^{l-1}$  and  $A_s^l$  denote the set of contracts **held by** and **available to** institution  $s$  at the beginning of the cumulative offer process step  $l$ . We say that a contract  $z$  is **rejected** during the cumulative offer process if  $z \in A_{s(z)}^l$  but  $z \notin D_{s(z)}^{l-1}$  for some  $l$ .

## B. Proofs Omitted from the Main Text

- *Proof of Proposition 1:*

*Proof.* Take a set of contracts  $Y \subseteq X$  and a contract  $z \in X \setminus Y$  such that  $z \notin C^s(Y \cup \{z\})$ . We need to prove that  $C^s(Y) = C^s(Y \cup \{z\})$ . Suppose that  $t(z) = t_j$ . Then the contract  $z$  is not chosen by the sub-choice function of the privilege types  $t_l$ ,  $l = 1, \dots, j - 1$ . Note that if any other contract of the agent  $i(z)$  is chosen by the sub-choice functions of privileges  $t_1, \dots, t_{j-1}$  the proof is done because when another contract of agent  $i(z)$  is chosen at any step, the contract  $z$  is removed from the process for the remaining steps. So, we will consider the non-trivial case where none of the contracts of agent  $i(z)$  is chosen up to the privilege type  $t_j$ . Since all the sub-choice functions satisfy IRC, up to privilege type  $t_j$ , the same contracts will be chosen from the sets  $Y$  and  $Y \cup \{z\}$  by the sub-choice functions  $C_{t_1}^s(\cdot, \cdot), \dots, C_{t_{j-1}}^s(\cdot, \cdot)$ ,

respectively. Let us denote the number of unused seats for privilege type  $t_l$  from the initial contracts sets  $Y$  and  $Y \cup \{z\}$  as  $r_l$  and  $\tilde{r}_l$ , respectively. Since  $t(z) = t_j$  we have  $r_l = \tilde{r}_l$  for  $l = 1, \dots, j-1$ . It implies that  $q_{t_j}^s(r_1, \dots, r_{j-1}) = q_{t_j}^s(\tilde{r}_1, \dots, \tilde{r}_{j-1})$ . Let us denote the remaining set of contracts after the choice by the choice function of privilege type  $t_l$  from the initial contract sets  $Y$  and  $Y \cup \{z\}$  as  $Y^l$  and  $\tilde{Y}^l$ , respectively.

By our assumption we know that  $z \notin C_{t_j}^s(\tilde{Y}^{j-1}, q_{t_j}^s(\tilde{r}_1, \dots, \tilde{r}_{j-1}))$  and  $\tilde{Y}^{j-1} = Y^{j-1} \cup \{z\}$ . By the IRC of the sub-choice function  $C_{t_j}^s(\cdot, \cdot)$ , we obtain  $C(\tilde{Y}^{j-1}, q_{t_j}^s(\tilde{r}_1, \dots, \tilde{r}_{j-1})) = C_{t_j}^s(Y^{j-1}, q_{t_j}^s(r_1, \dots, r_{j-1}))$ . Also,  $r_j = \tilde{r}_j$ . If  $i(z) \in i[C_{t_j}^s(\tilde{Y}^{j-1}, q_{t_j}^s(\tilde{r}_1, \dots, \tilde{r}_{j-1}))]$ , then the contract  $z$  is removed from the process. Otherwise, we have  $\tilde{Y}^j = Y^j \cup \{z\}$ . Since  $q_{t_{j+1}}^s(r_1, \dots, r_j) = q_{t_{j+1}}^s(\tilde{r}_1, \dots, \tilde{r}_j)$  the same argument holds for the privilege type  $t_{j+1}$ . By proceeding in the same fashion we obtain  $C(\tilde{Y}^l, q_{t_j}^s(\tilde{r}_1, \dots, \tilde{r}_l)) = C_{t_j}^s(Y^l, q_{t_j}^s(r_1, \dots, r_l))$  for all  $l = 1, \dots, k$ . Hence, we have  $C^s(Y) = C^s(Y \cup \{z\})$ .  $\square$

- *Proof of Proposition 2:*

*Proof.* Since all sub-choice functions satisfy IRC, by Proposition 1, the overall choice function satisfies IRC as well. In order to prove that the overall choice function satisfies WS we take a set of contracts  $Y \subseteq X$  and two contracts  $x, z \in X \setminus Y$  such that  $|Y \cup \{x, z\}| = |i(Y \cup \{x, z\})|$ . Suppose that  $z \notin C^s(Y \cup \{z\})$ . We need to show that  $z \notin C^s(Y \cup \{x, z\})$ . We consider two cases:

**Case 1:**  $x \notin C^s(Y \cup \{x, z\})$ . Since the overall choice function satisfies IRC, we then have  $C^s(Y \cup \{x, z\}) = C^s(Y \cup \{z\})$ . Hence, by our assumption, we have  $z \notin C^s(Y \cup \{x, z\})$ .

**Case 2:**  $x \in C^s(Y \cup \{x, z\})$ . Let the privilege type of agent  $i(x)$  be  $t(x) = t_j$  where  $j \in \{1, \dots, k\}$ . Then for each  $l \notin \{1, \dots, j-1\}$ , neither  $x$  nor  $z$  are chosen by sub-choice functions. By IRC of sub-choice functions since  $x$  is not chosen by the sub-choice functions of privileges  $t_1, \dots, t_{j-1}$ , sub-choices from the sets  $(Y \cup \{z\})$  and  $(Y \cup \{x, z\})$  for privilege types  $t_1, \dots, t_{j-1}$  are identical. Hence, the number of unused seats of privilege types  $t_1, \dots, t_{j-1}$  from the sets  $(Y \cup \{z\})$  and  $(Y \cup \{x, z\})$  are the same, i.e.,  $r_l = \tilde{r}_l$  for every  $l \in \{1, \dots, j-1\}$ . It implies that capacity of privilege type  $t_j$   $q_{t_j}^s(r_1, \dots, r_{j-1})$  is equal to  $q_{t_j}^s(\tilde{r}_1, \dots, \tilde{r}_{j-1})$ . Let  $Y^l$  be the set of remaining

contracts after sub-choice for privilege type  $t_j$  from the set  $(Y \cup \{z\})$  and  $\tilde{Y}^l$  be the set of remaining contracts after sub-choice for privilege  $t_j$  from the set  $(Y \cup \{x, z\})$ . Note that  $\tilde{Y}^l = Y^l \cup \{x\}$  for all  $l \in \{1, \dots, j-1\}$ .

Let  $Y_j$  and  $\tilde{Y}_j$  be the set of chosen contracts by sub-choice functions for privilege  $t_j$  from the sets  $(Y \cup \{z\})$  and  $(Y \cup \{x, z\})$ , respectively. By the weak substitutability of sub-choice function for privilege  $t_j$  we have  $z \notin \tilde{Y}_j$ . It is easy to see that  $Y^j \subseteq \tilde{Y}^j$  because otherwise there exists a contract  $y \in Y^j$  (means  $y \notin Y_j$ ) but  $y \notin \tilde{Y}^j$  (means  $y \in \tilde{Y}_j$ ). Since each agent has only one contract we have contradiction with the fact that sub-choice functions satisfy weak substitutability (WS). By the law of aggregate demand (LAD) of the sub-choice functions we have  $|Y_j| \leq |\tilde{Y}_j|$ . Hence, we have  $q_{t_{j+1}}^s = q_{t_{j+1}}^s(r_1, \dots, r_j) \geq \tilde{q}_{t_{j+1}}^s = q_{t_{j+1}}^s(\tilde{r}_1, \dots, \tilde{r}_j)$  by monotonicity of the capacity transfer scheme as  $r_j \geq \tilde{r}_j$  and  $r_l = \tilde{r}_l$  for every  $l \in \{1, \dots, j-1\}$ .

By our assumption, we know that  $z \notin C_{t_{j+1}}^s(Y^j, q_{t_{j+1}}^s)$ . By quota monotonicity (QM) of sub-choice functions we have  $z \notin C_{t_{j+1}}^s(Y^j, \tilde{q}_{t_{j+1}}^s)$ . Then, WS and IRC of sub-choice functions imply that  $z \notin \tilde{Y}_{j+1} = C_{t_{j+1}}^s(\tilde{Y}^j, \tilde{q}_{t_{j+1}}^s)$ . By the LAD of sub-choice functions we have  $|Y_{j+1}| \leq |\tilde{Y}_{j+1}|$ . It implies that  $r_{j+1} \geq \tilde{r}_{j+1}$ , and, hence,  $q_{t_{j+2}}^s(r_1, \dots, r_{j+1}) \geq q_{t_{j+2}}^s(\tilde{r}_1, \dots, \tilde{r}_{j+1})$  by the monotonicity of the capacity transfer scheme. Also, it is easy to see that  $Y^{j+1} \subseteq \tilde{Y}^{j+1}$ . Repeating the same arguments for the rest of the privileges gives us  $z \notin C^s(Y \cup \{x, z\})$  and completes the proof.  $\square$

**Lemma 1.** *Take  $Y \subseteq X$  and  $x, z \in X \setminus Y$  such that  $i(x) \neq i(y)$  and  $i(x), i(z) \notin i(Y)$ . Suppose that  $z \notin C^s(Y \cup \{z\})$ . Set  $Y^0 = Y \cup \{z\}$  and  $\tilde{Y}^0 = Y^0 \cup \{x\}$ . Suppose also that  $x \in \tilde{Y}_j = C_{t_j}^s(\tilde{Y}^{j-1}, q_{t_j}^s(\tilde{r}_1, \dots, \tilde{r}_{j-1}))$ . Let  $Y^j = Y^{j-1} \setminus \{x \in Y^{j-1} : i(x) \notin i(Y_j)\}$  and  $\tilde{Y}^j = \tilde{Y}^{j-1} \setminus \{x \in \tilde{Y}^{j-1} : i(x) \notin i(\tilde{Y}_j)\}$ . Then,  $Y^j \subseteq \tilde{Y}^j$ .*

*Proof.* Assume not. Then there exists a contract  $y \in Y^j$  such that  $y \in \tilde{Y}_j$  (hence,  $y \notin \tilde{Y}^j$ ) and  $i(y) \notin i(Y_j)$ . Since none of the contracts of agent  $i(y)$  is chosen from  $Y^{j-1}$  removing them from  $Y^{j-1}$  does not change the set of chosen contracts by IRC of the sub-choice function, i.e., construct the set  $A = Y^{j-1} \setminus \{y' \in Y^{j-1} : i(y') = i(y)\}$  and we have  $C_{t_j}^s(A, q) = C_{t_j}^s(Y^{j-1}, q)$ . Now consider the choice from the sets  $A \cup \{y\}$  and  $\tilde{Y}^{j-1}$ . We have  $y \notin C_{t_j}^s(A \cup \{y\}, q)$ . Notice that  $y$  is the only contract of agent

$i(y)$  in  $A \cup \{y\}$ . Now consider the set  $A \cup \{x, y\}$ . Since  $y \in \tilde{Y}_j$ , by the IRC of the sub-choice function we have  $y \in C_{t_j}^s(A \cup \{x, y\}, q)$ . This contradicts with the BS of the sub-choice function because  $y \notin C_{t_j}^s(A \cup \{y\}, q)$  and yet  $y \in C_{t_j}^s(A \cup \{x, y\}, q)$ . This completes the proof.  $\square$

• *Proof of Proposition 3:*

*Proof.* Since all sub-choice functions satisfy IRC, by Proposition 1, the overall choice function satisfies IRC as well. To prove that the overall choice function also satisfies bilateral substitutability consider a set of contracts  $Y \subseteq X$  and contracts  $x, z \in X \setminus Y$  such that  $i(x), i(z) \notin i(Y)$ . Suppose that  $z \notin C^s(Y \cup \{z\})$ . We need to show that  $z \notin C^s(Y \cup \{x, z\})$ . There are two cases to consider:

**Case 1 :**  $x \notin C^s(Y \cup \{x, z\})$

Since the overall choice function satisfies IRC, we then have  $C^s(Y \cup \{x, z\}) = C^s(Y \cup \{z\})$ . Hence, by our assumption, we have  $z \notin C^s(Y \cup \{x, z\})$ .

**Case 2:**  $x \in C^s(Y \cup \{x, z\})$

There exist  $j \in \{1, \dots, k\}$  such that  $x \in \tilde{Y}_j = C_{t_j}^s(\tilde{Y}^{j-1}, q_{t_j}^s(\tilde{r}_1, \dots, \tilde{r}_{j-1}))$ . For all  $i \in \{1, \dots, j-1\}$ , we know that  $x \notin \tilde{Y}_i$  and  $z \notin Y_i$  by our assumptions. Then, by the BS of sub-choice functions of the privileges  $t_1, \dots, t_{j-1}$ , we have  $z \notin \tilde{Y}_i$ . Also note that  $\tilde{Y}^i = Y^i \cup \{x\}$  and  $z \in Y^i$  for all  $i \in \{0, 1, \dots, j-1\}$ . By Lemma 1, we know that  $Y^j \subseteq \tilde{Y}^j$ . Also, since  $r_1 = \tilde{r}_1, \dots, r_{j-1} = \tilde{r}_{j-1}$  we have  $q_{t_j}^s(r_1, \dots, r_{j-1}) = q_{t_j}^s(\tilde{r}_1, \dots, \tilde{r}_{j-1})$ . By the LAD, we know that  $|Y_j| \leq |\tilde{Y}_j|$ . Hence we have  $q_{t_{j+1}}^s(r_1, \dots, r_j) \geq q_{t_{j+1}}^s(\tilde{r}_1, \dots, \tilde{r}_j)$  by the monotonicity of the capacity transfer scheme.

We need to prove that  $z \notin C_{t_{j+1}}^s(\tilde{Y}^j, q_{t_{j+1}}^s(\tilde{r}_1, \dots, \tilde{r}_j))$ . We know, by our assumption, that  $z \notin C_{t_{j+1}}^s(Y^j, q_{t_{j+1}}^s(r_1, \dots, r_j))$  where  $Y^j \subseteq \tilde{Y}^j$  and  $q_{t_{j+1}}^s(r_1, \dots, r_j) \geq q_{t_{j+1}}^s(\tilde{r}_1, \dots, \tilde{r}_j)$ . Also, notice that  $i(\tilde{Y}^j \setminus Y^j) \cap i(Y^j) = \emptyset$ . By quota monotonicity (QM) of the sub-choice functions  $z \notin C_{t_{j+1}}^s(Y^j, q_{t_{j+1}}^s(r_1, \dots, r_j))$  implies  $z \notin C_{t_{j+1}}^s(Y^j, q_{t_{j+1}}^s(\tilde{r}_1, \dots, \tilde{r}_j))$ . If  $i(\tilde{Y}^j \setminus Y^j) \notin i(\tilde{Y}_{j+1})$ , then by the IRC of the sub-choice function we have  $z \notin \tilde{Y}_{j+1}$ . Otherwise, there must exist  $y' \in \tilde{Y}^j \setminus Y^j$  such that  $y' \in \tilde{Y}_{j+1} = C_{t_{j+1}}^s(\tilde{Y}^j, q_{t_{j+1}}^s(\tilde{r}_1, \dots, \tilde{r}_j))$ . Note that  $i(y') \notin i(Y^j)$ . Let  $\{y', \dots, w'\}$  be the set of contracts in  $\tilde{Y}^j \setminus Y^j$  such that each of them is chosen

by  $\tilde{Y}_{j+1}$ . By the IRC of the sub-choice function, removing the other contracts of the doctors  $i(\{y', \dots, w'\})$  from the set  $\tilde{Y}^j$  does not change the chosen set. Therefore,  $C_{t_{j+1}}^s(\tilde{Y}^j, q_{t_{j+1}}^s(\tilde{r}_1, \dots, \tilde{r}_j)) = C_{t_{j+1}}^s(Y^j \cup \{y', \dots, w'\}, q_{t_{j+1}}^s(\tilde{r}_1, \dots, \tilde{r}_j))$ . The BS of the sub-choice function implies  $z \notin C_{t_{j+1}}^s(Y^j \cup \{y', \dots, w'\}, q_{t_{j+1}}^s(\tilde{r}_1, \dots, \tilde{r}_j))$ . Hence,  $z \notin \tilde{Y}_{j+1}$ .

We now need to prove  $Y^{j+1} \subseteq \tilde{Y}^{j+1}$ . Take  $y \in Y^{j+1}$ . We know that  $y \notin Y_{j+1}$ . Then, by QM, it implies that  $y \notin C_{t_{j+1}}^s(Y^j, q_{t_{j+1}}^s(\tilde{r}_1, \dots, \tilde{r}_j))$ . Finally, BS and IRC implies that  $y \notin \tilde{Y}_{j+1} = C_{t_{j+1}}^s(\tilde{Y}^j, q_{t_{j+1}}^s(\tilde{r}_1, \dots, \tilde{r}_j))$ , i.e.,  $y \in \tilde{Y}^{j+1}$ .

To finish the proof we need to show that  $\tilde{r}_{j+1} \leq r_{j+1}$ , i.e.,  $q_{j+1}(\tilde{r}) - |\tilde{Y}_{j+1}| \leq q_{j+1}(r) - |Y_{j+1}|$ . By the monotonicity of the capacity transfer scheme we have  $\tilde{q}_{j+1} \leq q_{j+1}$ . By the LAD, it implies  $|C_{t_{j+1}}^s(Y^{j+1}, q_{j+1}^s(r))| - |C_{t_{j+1}}^s(Y^{j+1}, q_{j+1}^s(\tilde{r}))| \leq q_{j+1}^s(r) - q_{j+1}^s(\tilde{r})$ . Again by the LADS we obtain  $|C_{t_{j+1}}^s(\tilde{Y}^{j+1}, q_{j+1}^s(\tilde{r}))| \geq |C_{t_{j+1}}^s(Y^{j+1}, q_{j+1}^s(\tilde{r}))|$ . The last two inequalities together implies that  $|Y_{j+1}| - |\tilde{Y}_{j+1}| = |C_{t_{j+1}}^s(Y^{j+1}, q_{j+1}^s(r))| - |C_{t_{j+1}}^s(\tilde{Y}^{j+1}, q_{j+1}^s(\tilde{r}))| \leq q_{j+1}^s(r) - q_{j+1}^s(\tilde{r})$ .

Since the same observations applies to all of the remaining privileges after  $t_{j+1}$ , this observation ends the proof.  $\square$

- *Proof of Proposition 6:*

*Proof.* Towards a contradiction let  $t'$  be the first step a school holds a contract  $z$  it previously rejected at Step  $t < t'$ . Since  $z$  is rejected by school  $s$  at Step  $t$  there are two cases to consider:

- (i)  $z$  was on hold at Step  $(t-1)$ , i.e.,  $z \in C^s(A_s(t-1))$ , or
- (ii)  $z$  was offered to school  $s$  at Step  $t$ , i.e.,  $z = A_s(t) \setminus A_s(t-1)$ .

In either case no other contract of student  $i(z)$  could be on hold by school  $s$  at Step  $(t-1)$ . But then, since  $z$  is the first contract to be held after an earlier rejection, school  $s$  cannot have held another contract by student  $i(z)$  at Step  $t$ . That is,

$$i(z) \notin i[C^s(A_s(t))]$$

Then by IRC  $z \in A_s(t) \setminus C^s(A_s(t))$  implies that

$$z \notin C^s(C^s(A_s(t)) \cup \{z\})$$

and yet

$$z \in C^s(A_s(t'))$$

Consider every step  $t''$  in the cumulative offer algorithm where  $t < t'' \leq t'$ . In each stage one of the following cases occurs:

(i) a new contract,  $x$ , from another student with the same privilege type as  $t(z)$  is offered, i.e.,  $i(x) \neq i(z)$  but  $t(x) = t(z) = t_j$ ,

(ii) a new contract,  $x$ , from another student with a different privilege type than  $t(z)$  is offered, i.e.,  $i(x) \neq i(z)$  and  $t(x) \neq t(z) = t_j$ ,

(iii) a new contract from student  $i(z)$ ,  $z'$ , with a different privilege type than  $t(z)$  is offered, i.e.,  $i(z') = i(z)$  but  $t(z') \neq t(z) = t_j$ .

In each case and for each step of the cumulative offer algorithm between steps  $t$  and  $t'$  we will show that  $z$  is not going to be recalled.

(i) In this case note that both  $r_l$  and  $Y_l$  for  $l = 1, \dots, j-1$  remain unchanged. Hence the capacity of the privilege type  $t_j$  will be as same as the capacity before receiving the offer  $x$ . Since  $\succ^{t_j}$  is responsive with capacity  $q_{t_j}^s$ ,  $z$  will be rejected as it was before the arrival of the contract  $x$  since now competition for slots is higher.

(ii) There are several sub-cases to consider in this case. If the contract  $x$  is chosen by a sub-choice function of a privilege  $t_l$  where  $l > j$  then the contract  $z$  will be rejected again since the capacity of the privilege type  $t_j$  and all the chosen contracts  $Y_k$  where  $k < j$  will be the same. If the contract  $x$  is chosen by any privilege type  $t_l$  where  $l < j$ , the number of unused seats for all the privileges after the privilege  $t_l$  will be weakly smaller. By the monotonicity of capacity transfer scheme, the capacity of the privilege type  $t_j$  will be weakly smaller. Note that the contract  $x$  cannot be the contract of any student whose contract is on hold at the privilege type  $t_j$  by the dynamics of the cumulative offer algorithm. Finally, if the

contract  $x$  is not chosen by any of privileges, then by the IRC of the overall choice function  $z$  will be rejected.

(iii) For this case there are several cases to consider as well. If  $z'$  is not chosen by any privileges by the IRC of the overall choice function  $z$  will be rejected. If  $z'$  is chosen by the privilege  $t(z') = t_l$  where  $l < j$ , then the contract  $z$  will be removed from the process by the definition of our choice function and, hence,  $z$  will be rejected again. If  $z'$  is chosen by a privilege  $t(z') = t_l$  where  $l > j$ , then neither the number of unused seats  $r_k$  where  $k < j$  nor the set of chosen contracts  $Y_k$  where  $k < j$  changes. Privilege type  $t_j$  will have the same capacity as it had before the arrival of  $z'$ . Therefore,  $z$  will be rejected.

Hence it contradicts with  $z \in C^s(A_s(t'))$ . □

- *Proof of Proposition 7:*

*Proof.* Let  $Y$  be the outcome of the cumulative offer algorithm. Since agents/students only offer their acceptable contracts during the cumulative offer process we have  $C^i(Y) = Y_i$  for all  $i \in I$ . Towards a desired contradiction suppose that  $Y$  is not stable. Then, there must exist a school  $s \in S$  and a set of blocking contracts  $Z \neq C^s(Y)$  such that  $Z = C^s(Y \cup Z)$  and  $Z_i = C^i(Y \cup Z)$  for all  $i \in i(Z)$ . Consider an agent/student  $j \in i(Z)$  where  $Z_j P^j Y_j$ . By the definition of the cumulative offer algorithm agent  $j$  must have offered contract  $Z_j$  before offering the contract  $Y_j$ . Since  $Z_j \notin Y$  then  $Z_j$  must have been rejected at some step of the cumulative offer process. It holds for every agent whose more preferred contract in  $Y$  compared to their contract in  $Z$ . So, there is a step  $t$  of the cumulative offer process in which  $(Y \cup Z) \subseteq A_s(t)$ . By Proposition 6, a rejected contract during the cumulative offer algorithm can not be on hold at a further step under monotone capacity transfer scheme, i.e., there is no renegotiation. It contradicts with our assumption that  $Z = C^s(Y \cup Z)$ . □

- *Proof of Theorem 2:*

*Proof.* Let  $\succ$  be the strict priority ranking of agents according to test scores. The

priority ranking of each privilege type  $t_j$  ( $j = 1, \dots, k$ ) at each institution  $s \in S$ ,  $\succ_{t_j}^s$ , is derived from  $\succ$ . Each sub-choice function of privilege type  $t_j$  at institution  $s \in S$  is induced by the priority order  $\succ_{t_j}^s$ . In other words,  $C_{t_j}^s(Y^{j-1}, q_{t_j}^s(r_1, \dots, r_{j-1})) = C_{t_j}^s(Y^{j-1}, q_{t_j}^s(r_1, \dots, r_{j-1}) \mid \succ_{t_j}^s)$ . Note that each sub-choice function is q-responsive.

Fix an agent  $i \in I$ . Consider the following proposal order  $\Gamma$  such that for all  $l = 1, 2, \dots$

$$j\Gamma_l k \iff j \succ k \text{ for all } j, k \neq i$$

$$j\Gamma_l i \text{ for all } j \neq i$$

that is, the order obtained from  $\succ$  by moving agent  $i$  to the bottom of each linear order  $\Gamma_l$ . By the order-independence of the cumulative offer process result of Hirata and Kasuya (2014) no matter which proposal order we choose, outcome of the cumulative offer process will be the same since overall choice functions satisfy the BS and IRC conditions in our setting.

*Claim 1* : Suppose that agent  $i$  obtains contract  $x$  from the cumulative offer algorithm when he submits preference  $P^i : z_1 P^i z_2 P^i \dots z_n P^i x P^i \dots$ . If agent  $i$  submits the preference  $\tilde{P}^i : x \tilde{P}^i \emptyset_i$ , then she obtains contract  $x$  in the cumulative offer process.

*Proof of Claim 1* : Consider two different problems that are the same except agent  $i$ 's preference and also consider two different cumulative offer processes associated with these two different problems: In the first problem agent  $i$  submits preference  $P^i$  and in the other one she submits  $\tilde{P}^i$ . Let  $X$  be the outcome of the cumulative offer algorithm if we exclude agent  $i$  from both of the problems, i.e.,  $X$  is the set of contracts that are on hold by institutions before agent  $i$  proposes a contract in both problems. Note that agent  $i$  is the last agent to propose a contract according to proposal order  $\Gamma$ . In the cumulative offer process where she submits the preference  $P^i$  since she obtains the contract  $x$  all the contracts he prefers to  $x$  according to  $P^i$  are rejected, i.e.,  $z_1, z_2, \dots, z_n$  are all rejected. However, note that when she offers these contracts rejection cycles that return back to agent  $i$  occur.

Note that there are two possible ways for a contract that are currently on hold in privilege type  $t_j$  at institution  $s$  to be rejected as a result of a rejection chain: (i) Current capacity of a privilege type is full, the rejected contract belongs to the lowest scored agent among the agents whose contracts are currently on hold at that privilege type and some other agent who has a higher score offers a contract to that privilege type, and (ii) Contracts of some other agents are chosen to be on hold in the privilege types that are considered before the privilege type  $t_j$  so that the number of unassigned slots in privilege types  $t_1, \dots, t_{j-1}$  decreases, and as a result the capacity of privilege type  $t_j$  decreases by the monotonicity of the capacity transfer scheme.

Suppose agent  $i$  offers  $z_1$  to institution  $s(z_1)$  in privilege type  $t(z_1)$ . Note that for  $z_1$  to get rejected capacity of privilege type  $t(z_1)$  must be exhausted when agent  $i$  offers  $z_1$  because otherwise  $z_1$  would be accepted and the cumulative offer algorithm would terminate as agent  $i$  is the last agent to propose according to  $\Gamma$ . There are two possible ways for  $z_1$  to be rejected:

(1) agent  $i$  has a lower score than the lowest scored agent whose contract is currently on hold in privilege type  $t(z_1)$  at  $s(z_1)$ . If this is the case  $z_1$  is automatically rejected.

(2) agent  $i$  has higher score than some agents whose contracts are currently on in privilege type  $t(z_1)$  at  $s(z_1)$ . Upon arrival of  $z_1$  the contract of lowest scored agent, call it  $y$ , will be rejected. Agent  $i$  has higher score than agent  $i(y)$ . Then, agent  $i(y)$  offers her next best contract and starts a rejection chain. In the last stage of the rejection chain a contract from some agent must be accepted in institution  $s(z_1)$  in one of privilege types which is considered earlier than  $t(z_1)$ , so that by monotone capacity transfer scheme capacity of privilege type  $t(z_1)$  is decreased and  $z_1$  gets rejected. During the rejection cycle, until it reaches to institution  $s(z_1)$ , capacities in every privilege type in every institution other than  $s(z_1)$  remain unchanged because otherwise the rejection chain would not reach to institution  $s(z_1)$ . So, if  $s(z_1) \neq s(x)$ , then capacities of privilege types in  $s(x)$  remains unchanged. Also, note that if  $s(z_1) = s(x)$ , then the capacity of  $t(x)$  either remains the same or decreases. In this

scenario one should note that if a contract replace another one during the choice procedure it means the newly offered contract must be associated with agent whose score is higher than the agent which has the replaced contract.

Since  $z_1, z_2, \dots, z_n$  are all rejected, one of the two ways should occur for each contract  $z_l$ ,  $l = 1, \dots, n$ .

Let the privilege type associated with contract  $x$  is  $t_j = t(x)$ . In the first problem, due to rejection cycles some other agents might propose some additional contracts to privilege types  $t_1, t_2, \dots$ , and  $t_{j-1}$  to institution  $s(x)$ . Since sub-choice functions of privilege types are all q-responsive following test scores and the capacity transfer scheme is monotonic, capacity of privilege type  $t_j$  in institution  $s(x)$  weakly decreases as agent  $i$  continues to propose her contracts  $z_1, \dots, z_n$ . Also, the score of the agent who has minimum scored contract among the ones that are currently on hold in privilege type  $t(x)$  weakly increases. However, we know that contract  $x$  is chosen in this scenario.

Now, consider the cumulative offer process where agent  $i$  finds contract  $x$  acceptable only. Compared to the first scenario privilege type  $t_j$  has weakly more slots and score of the agent who is associated with the minimum scored contract weakly lower. Therefore, if  $x$  is chosen by  $s(x)$  for privilege type  $t_j$  in the first scenario, then it must be chosen by  $s(x)$  for privilege type  $t_j$  in the second scenario. This completes the proof of the claim.

The above claim basically says that if any agent wants contract  $x$  to be chosen and if it is possible by submitting some preference, then she can do it by truncating her preferences such that she finds only contract  $x$  acceptable. By Proposition 6 we know that if each sub-choice function is q-responsive then the cumulative offer process collapses to agent-proposing deferred acceptance algorithm, i.e., a rejected contract at some step of the cumulative offer algorithm can not be on hold at a further step of the algorithm. Since the agent-proposing deferred acceptance algorithm is immune to truncation strategies, this observation completes the proof.  $\square$

- *Proof of Theorem 3:*

*Proof.* Fix a student  $i$  and let  $\succ'$  be an unambiguous improvement for student  $i$  over  $\succ$ .

We will first consider the outcome of the cumulative offer mechanism under a monotone capacity transfer when the sub-choice functions for each school are induced from strict priority rankings  $\succ'_{t_1}, \succ'_{t_2}, \dots, \succ'_{t_k}$ , respectively. Recall that by Remark 1, the order of students making offers has no impact on the outcome of the cumulative offer algorithm. Therefore, we can obtain the outcome of the cumulative offer algorithm when the strict ranking of students according to test scores is  $\succ'$ : First, entirely ignore student  $i$  and run the cumulative offer algorithm until it stops. Let  $X'$  be the resulting set of contracts. At this point, student  $i$  makes an offer for her first-choice contract  $x^1$ . His offer may cause a chain of rejections, which may eventually cause contract  $x^1$  to be rejected as well. If that happens, student  $i$  makes an offer for his second choice  $x^2$ , which may cause another chain of rejections, and so on. Let this process terminate after student  $i$  makes an offer for his  $l$ th choice contract  $x^l$ . There may still be a chain of rejections after this offer, but it does not reach student  $i$  again. Hence, student  $i$  receives his  $l$ th choice under  $\Phi^{COM}(\succ')$ .

Next consider the outcome of the cumulative offer mechanism under the same monotone capacity transfer when the sub-choice functions for each school are induced from strict priority rankings  $\succ_{t_1}, \succ_{t_2}, \dots, \succ_{t_k}$ , respectively. Initially entirely ignore student  $i$  and run the cumulative offer algorithm until it stops. Since the only difference between the two scenarios is the standing of student  $i$  in the priority list,  $X'$  will again be the resulting set of contracts. Next, student  $i$  makes an offer for her first-choice contract  $x^1$ . Since  $\succ'$  is an unambiguous improvement for student  $i$  over  $\succ$ , precisely the same sequence of rejections will take place until he makes an offer for her  $l$ th choice contract  $x^l$ . Therefore, student  $i$  cannot receive a better contract than his  $l$ th choice under  $\Phi^{COM}(\succ)$  even though she can receive a worse contract than her  $l$ th choice if the rejection chain returns back to her.  $\square$

- *Proof of Theorem 4:*

*Proof.* Consider two problems  $(I, S, P^{|I|}, \succ, (\bar{q}_{t_j}^s)_{s \in S})$  and  $(I, S, P^{|I|}, \succ, (q_{t_j}^s(r_1, \dots, r_{j-1}))_{s \in S})$

in which there is no capacity transfer in the first one while the second one allows monotone capacity transfer across different privilege types, everything else is the same in both problems. Note that for every institution  $s \in S$  and all privilege types  $t_j, j = 1, \dots, k$ , we have  $q_{t_j}^s(r_1, \dots, r_{j-1}) \geq \bar{q}_{t_j}^s$ .

We need to show that each agent  $i \in I$  obtains weakly better outcome in the cumulative offer algorithm with monotone capacity transfer than she obtains in the cumulative offer algorithm without capacity transfer. Consider the following proposal order  $\succ$ - the strict ranking of agents according to test scores. Let  $i_1 - i_2 - \dots - i_n$  be the enumeration of agents according to  $\succ$  where  $i_1$  has the highest test score,  $i_2$  is the second highest test score, and so on. Let  $I'_l \equiv \{i_j \in I: j < l\}$  be the set of agents who have higher test scores than agent  $i_l$ . We are going to prove the theorem by induction on students following the proposal order  $\succ$ .

First ranked student according to  $\succ$  obtains the same outcome under both monotone capacity transfer scheme and no capacity transfer. Hence, he weakly prefer the assignment from the second problem over the assignment from the first problem. Suppose that  $x'_l$  is the contract agent  $i_l$  obtains in the cumulative offer algorithm with monotone capacity transfer and  $x_l$  is the contract she obtains from the cumulative offer algorithm with no capacity transfer. Assume that for all  $l \leq L$ ,  $x'_l R^{i_l} x_l$ . We need to show that it also hold for agent  $i_{L+1}$ , i.e.,  $x'_{L+1} R^{i_{L+1}} x_{L+1}$ . Assume not. Suppose that agent  $i_{L+1}$  obtains a contract  $y$  in the cumulative offer algorithm with monotone capacity transfer such that  $x_{L+1} P^{i_{L+1}} y$  where  $x_{L+1}$  is the contract she obtain in the cumulative offer algorithm without capacity transfer. We know that  $q_{t(x_{L+1})}^{s(x_{L+1})}(r) \geq \bar{q}_{t(x_{L+1})}^{s(x_{L+1})}$  by the monotone capacity transfer. Also, by our inductive hypothesis, the set of agents in  $(I'_{i_{L+1}} \cap X_{s(x_{L+1})} \cap X_{t(x_{L+1})})$  whose contract are not on hold in the cumulative offer algorithm with monotone capacity transfer at the step where agent  $i_{L+1}$  offer her contract  $x_{L+1}$  is contained by the set of agents in  $(I'_{i_{L+1}} \cap X_{s(x_{L+1})} \cap X_{t(x_{L+1})})$  whose contracts are not on hold in the cumulative offer algorithm without capacity transfer at the step where agent  $i_{L+1}$  offer her contract  $x_{L+1}$ . Then, it means when there are weakly more seats available and there are less agents whose score are higher than agent  $i_{L+1}$  in the privilege type  $t(x_{L+1})$  at

institution  $s(x_{L+1})$  her contract  $x_{L+1}$  is rejected while it is accepted when there are weakly more students whose score higher than  $i_{L+1}$  vying for a seat in the same institution and for the same privilege type and there are weakly less seats available. This contradicts with the construction of our sub-choice functions which are q-responsive. Hence,  $x'_{L+1} R^{i_{L+1}} x_{L+1}$  completes the proof.

□

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# On Relationships Between Substitutes Conditions

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## Abstract

In the matching with contracts literature, three well-known conditions (from stronger to weaker)– substitutes, unilateral substitutes ( $US$ ), and bilateral substitutes ( $BS$ )– have proven to be critical. This paper aims to deepen our understanding of them by separately axiomatizing the gap between  $BS$  and the other two. We first introduce a new “doctor separability” condition ( $DS$ ) and show that  $BS$ ,  $DS$ , and irrelevance of rejected contracts ( $IRC$ ) are equivalent to  $US$  and  $IRC$ . Due to Hatfield and Kojima (2010) and Aygün and Sönmez (2012), we know that  $US$ , “Pareto separability” ( $PS$ ), and  $IRC$  are the same as substitutes and  $IRC$ . This, along with our result, implies that  $BS$ ,  $DS$ ,  $PS$ , and  $IRC$  are equivalent to substitutes and  $IRC$ . All of these results are given without  $IRC$  whenever hospitals have preferences.

**JEL classification:** C78, D44, D47

**Keywords:** Bilateral substitutes, Unilateral substitutes, Substitutes, Doctor separability, Pareto separability, Irrelevance of rejected contracts.

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‡The authors thank Utku Ünver, Fuhito Kojima, İsa Hafalir, Vikram Manjunath, Jens Gudmundsson, and especially Orhan Aygün, Samson Alva, and the anonymous referee. Afacan gratefully acknowledges the Marie Curie International Reintegration Grant (no: 618263) within the European Community Framework Programme and TÜBİTAK (The Scientific and Technological Research Council of Turkey) Grant (no: 113K763) within the National Career Development Program.

# 1 Introduction

In the matching with contracts framework of Hatfield and Milgrom (2005), Hatfield and Kojima (2010) obtain the existence of a stable allocation under a bilateral substitutes ( $BS$ ) condition. Aygün and Sönmez (2012) then show that if hospital choices are not necessarily induced by preferences, an irrelevance of rejected contracts ( $IRC$ )<sup>1</sup> assumption is also needed. Nevertheless,  $BS$  and  $IRC$  are still weak in the sense that many well-known results in the standard matching problem do not carry over to the matching with contracts setting under them. For instance, the doctor-optimal stable allocation fails to exist. Hatfield and Kojima (2010) then introduce a stronger unilateral substitutes condition ( $US$ ), and the existence of the doctor-optimal stable allocation is obtained under both  $US$  and  $IRC$ . With an additional law of aggregate demand condition ( $LAD$ ),<sup>2</sup> Hatfield and Kojima (2010) recover both the strategy-proofness of the doctor-optimal stable rule and a version of the so-called “rural hospitals theorem.”

Given that many well-known properties are restored by strengthening  $BS$  to  $US$  or substitutes, it is important to understand the relations between them. While the extant literature clarifies the difference between the  $US$  and substitutes conditions through axiomatizing the gap between them, such an analysis is yet to be done for the difference between them and  $BS$ . In this study, we pursue this analysis and separately axiomatize the gap between  $BS$  and the other two. To this end, we introduce a *doctor separability* ( $DS$ ) condition, which says that if no contract of a doctor is chosen from a set of contracts, then that doctor continues not to be chosen unless a contract of a new doctor (we refer to a doctor as new doctor if he does not have any contract in the initially given set of contracts) becomes available. We then show that  $US$  and  $IRC$  are equivalent to  $DS$ ,  $BS$ , and  $IRC$ .<sup>3</sup>

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<sup>1</sup>Alkan (2002) refers to it as “consistency.”

<sup>2</sup>In a different setting, Alkan (2002) refers to  $LAD$  as “cardinal monotonicity.”

<sup>3</sup>Alva (2014) gives some necessary (but not sufficient) conditions for  $US$  and  $BS$  to hold. We will come back to those conditions in Remark 3.

Hatfield and Kojima (2010) show that *US* and “Pareto Separability” (*PS*) are equivalent to substitutability. By additionally imposing *IRC*, Aygün and Sönmez (2012) extend it to the case where hospital choices are primitives. This result, along with our axiomatization, yields that *BS*, *DS*, *PS*, and *IRC* are equivalent to substitutes and *IRC*.<sup>4</sup> As *IRC* is automatically satisfied whenever hospitals have preferences, all of our results hold without *IRC* when hospitals are assumed to have preferences.

As summarized above, strengthening *BS* to either *US* or substitutes recovers important properties. Indeed, it is not only restricted to the ones mentioned above. In a recent study, Afacan (2014) shows that the Cumulative Offer Process (Hatfield and Milgrom (2005)) is both population and resource monotonic under *US* and *IRC*, and it respects doctors’ improvements with the additional *LAD*. The theoretical appeal of understanding the difference between *US* and *BS* is therefore clear. In addition, our paper has practical appeal as recent works in the literature (notably, Sönmez and Switzer (2013), Sönmez (2013), and Kominers and Sönmez (2013)) illustrate that the *US* and *BS* conditions are critical for practical market design.

## 2 Model and Results

There are finite sets  $D$  and  $H$  of doctors and hospitals, and a finite set of contracts  $X$ . Each contract  $x \in X$  is associated with one doctor  $x_D \in D$  and one hospital  $x_H \in H$ . Given a set of contracts  $X' \subseteq X$ , let  $X'_D = \{d \in D : \exists x \in X' \text{ with } x_D = d\}$ . Each hospital  $h$  has a choice function  $C_h : 2^X \rightarrow 2^X$  defined as follows: for any  $X' \subseteq X$ :

$$C_h(X') \in \{X'' \subseteq X' : (x \in X'' \Rightarrow x_H = h) \text{ and } (x, x' \in X'', x \neq x' \Rightarrow x_D \neq x'_D)\}.$$

**Definition 1.** *Contracts satisfy irrelevance of rejected contracts (IRC) for hospital  $h$  if, for any  $X' \subset X$  and  $z \in X \setminus X'$ , if  $z \notin C_h(X' \cup \{z\})$  then  $C_h(X') = C_h(X' \cup \{z\})$ .*

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<sup>4</sup>Alva (2014) provides another characterization of substitutability by using different properties, which are not directly related to the currently used ones.

**Definition 2.** *Contracts are bilateral substitutes (BS) for hospital  $h$  if there do not exist contracts  $x, z \in X$  and a set of contracts  $Y \subseteq X$  such that  $x_D, z_D \notin Y_D$ ,  $z \notin C_h(Y \cup \{z\})$ , and  $z \in C_h(Y \cup \{x, z\})$ .*

**Definition 3.** *Contracts are unilateral substitutes (US) for hospital  $h$  if there do not exist contracts  $x, z \in X$  and a set of contracts  $Y \subseteq X$  such that  $z_D \notin Y_D$ ,  $z \notin C_h(Y \cup \{z\})$ , and  $z \in C_h(Y \cup \{x, z\})$ .*

Below we introduce our new condition.

**Definition 4.** *Contracts are doctor separable (DS) for hospital  $h$  if, for any  $Y \subset X$  and  $x, z, z' \in X \setminus Y$  with  $x_D \neq z_D = z'_D$ , if  $x_D \notin [C_h(Y \cup \{x, z\})]_D$ , then  $x_D \notin [C_h(Y \cup \{x, z, z'\})]_D$ .*

Less formally, *DS* says that if a doctor is not chosen from a set of contracts in the sense that no contract of him is selected, then that doctor should still not be chosen unless a contract of a new doctor (that is, doctor having no contract in the given set of contracts) becomes available. For practical purposes, we can consider *DS* as capturing contracts where certain groups of doctors are substitutes.<sup>5</sup>

**Theorem 1.** *Contracts are US and IRC if and only if they are BS, DS, and IRC.*

*Proof.* **“If” Part.** Let  $Y \subset X$  and  $x \in X$  such that  $x_D \notin Y_D$  and  $x \notin C_h(Y \cup \{x\})$ . We now claim that  $x \notin C_h(Y \cup \{x, z\})$  for any  $z \in X$  as well. If  $z_D \notin Y_D$ , then by *BS*, the result follows. Let us now assume that  $z_D \in Y_D$ . Then, we can write  $Y = Y' \cup \{z'\}$  for some  $z'$  where  $z'_D = z_D$ . This means that  $x \notin C_h(Y' \cup \{x, z'\})$ , and since  $x_D \notin Y_D$ , it in particular implies that  $x_D \notin [C_h(Y' \cup \{x, z'\})]_D$ . By *DS*, then, we have  $x_D \notin [C_h(Y' \cup \{x, z', z\})]_D$ ; in other words,  $x_D \notin [C_h(Y \cup \{x, z\})]_D$ . Hence, in particular,  $x \notin C_h(Y \cup \{x, z\})$ .

**“Only If” Part.** Let contracts be *US* satisfying *IRC*. By definition, they are *BS* as well. Let  $x_D \notin [C_h(Y \cup \{x, z\})]_D$  and  $Y' = Y \setminus \{x' \in Y : x_D = x'_D \text{ and } x \neq x'\}$ . By

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<sup>5</sup>If  $x_D \notin [C_h(Y \cup \{x, z\})]_D$ , then doctor  $x_D$  is not chosen. And under *DS*, he continues not to be chosen unless a new doctor comes. Hence, we can interpret it as the doctors in the given set of contracts are substitutes.

*IRC*,  $C_h(Y \cup \{x, z\}) = C_h(Y' \cup \{x, z\})$ . Let us now add a new contract  $z'$  where  $z_D = z'_D$ . By *US*,  $x \notin C_h(Y' \cup \{x, z, z'\})$ . If  $x \in C_h(Y \cup \{x, z, z'\})$ , then by *IRC*, it has to be that  $C_h(Y \cup \{x, z, z'\}) = C_h(Y' \cup \{x, z, z'\})$ . This, however, contradicts  $x \notin C_h(Y' \cup \{x, z, z'\})$ . Hence,  $x \notin C_h(Y \cup \{x, z, z'\})$ . For any other contract  $x' \in Y$  of doctor  $x_D$ , we can define  $Y' = [Y \setminus \{x'\}] \cup \{x\}$ . Then, by above,  $x_D \notin [C_h(Y' \cup \{x', z\})]_D$  (note that  $Y' \cup \{x', z\} = Y \cup \{x, z\}$ ). By easily following the same steps above, we can conclude that  $x' \notin C_h(Y \cup \{x, z, z'\})$  as well. Hence,  $x_D \notin [C_h(Y \cup \{x, z, z'\})]_D$ , showing that contracts are *DS*.  $\square$

**Definition 5.** *Contracts are substitutes for hospital  $h$  if there do not exist contracts  $x, z \in X$  and a set of contracts  $Y \subseteq X$  such that  $z \notin C_h(Y \cup \{z\})$  and  $z \in C_h(Y \cup \{x, z\})$ .*

**Definition 6.** *Contracts are Pareto separable (PS) for hospital  $h$  if, for any two distinct contracts  $x, x'$  with  $x_D = x'_D$  and  $x_H = x'_H = h$ , if  $x \in C_h(Y \cup \{x, x'\})$  for some  $Y \subseteq X$ , then  $x' \notin C_h(Y' \cup \{x, x'\})$  for any  $Y' \subseteq X$ .*

**Fact 1** (Hatfield and Kojima (2010) and Aygün and Sönmez (2012)). *Hospital choices are US and PS, satisfying IRC, if and only if they are substitutes satisfying IRC.*

As a corollary of Theorem 1 and Fact 1 above, we obtain the following characterization.

**Corollary 1.** *Contracts are substitutes satisfying IRC if and only if they are BS, DS, PS, satisfying IRC.*

**Remark 1.** As *IRC* is automatically satisfied whenever hospital choices are generated by certain preferences, all of the above results work without *IRC* in that case.

**Remark 2.** It is easy to verify that *DS* is independent of both *BS* and *PS*.

**Remark 3.** By following our notation, Alva (2014) says that contracts (of hospital  $h$ ) satisfy “recall rejected talents” (*RRT*) if there are  $Y \subset X$  and  $x, z \in X$  such that  $x \in Y$ ,  $x_D \notin [C_h(Y)]_D$ , and  $x \in C_h(Y \cup \{z\})$ . Moreover, he says that both *RRT* and another condition “New Offer From New Talent” (*NOFNT*) are satisfied if the  $z$  contract in the *RRT* definition is such that  $z_D \notin Y_D$ . He shows that (i) *US* fails if *RRT* is satisfied and

(ii) *BS* fails if both *RRT* and *NOFNT* are satisfied. It is easy to verify that the absence of *RRT* implies *DS* (the converse is not true even under *IRC*); however, the absence (or the presence) of both *RRT* and *NOFNT* does not imply *DS* even under *IRC*.

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