Essays on Matching Theory and Networks

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Boston College Electronic Thesis or Dissertation, 2013

Boston College

The Graduate School of Arts and Sciences

Department of Economics

ESSAYS ON MATCHING THEORY AND NETWORKS

a dissertation

by

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submitted in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

August 2013

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This dissertation consists of three essays in microeconomic theory. The first and second essays are in the theory of matching, with hierarchical organizations and complementarities being their respective topic. The third essay is in on electoral competition and political polarization as a result of manipulation of public opinion through social influence networks.

Hierarchies are a common organizational structure in institutions. In the first essay, I offer an explanation of this fact from a matching-theoretic perspective, which emphasizes the importance of stable outcomes for the persistence of organizational structures. I study the matching of individuals (talents) via contracts with institutions, which are aggregate market actors, each composed of decision makers (divisions) enjoined by an institutional governance structure. Conflicts over contracts between divisions of an institution are resolved by the institutional governance structure, whereas conflicts between divisions across institutions are resolved by talents' preferences. Stable market outcomes exist whenever institutional governance is hierarchical and divisions consider contracts to be bilaterally substitutable. In contrast, when governance in institutions is non-hierarchical, stable outcomes may not exist. Since market stability does not provide an impetus for reorganization, the persistence of markets with hierarchical institutions can thus be rationalized. Hierarchies in institutions also have the attractive incentive property that in a take-it-or-leave-it bargaining game with talents making offers to institutions, the choice problem for divisions is straightforward and realized market outcomes are pairwise stable, and stable when divisions have substitutable

preferences.

Complementarity has proved to be a challenge for many-to-one matching theory, because the core and group stable matchings may fail to exist. Less well understood is the more basic notion of pairwise stability. In a second essay, I define a class of complementarity, *asymmetric complements*, and show that pairwise stable matchings are guaranteed to exist in matching markets where no firm considers workers to be asymmetric complements. The lattice structure of the pairwise stable matchings, familiar from the matching theory with substitutes, does not survive in this more general domain. The simultaneous-offer and sequential-offer versions of the worker-proposing deferred acceptance algorithm can produce different matchings when workers are not necessarily substitutable. If no firm considers workers to be *imperfect complements*, then the simultaneous-offer version produces a pairwise stable matching, but this is not necessarily true otherwise. If no firm considers workers to be asymmetric complements, a weaker restriction than no imperfect complements, then the sequential-offer version produces a pairwise stable matching, though the matching produced is order-dependent.

In a third essay, I examine electoral competition in which two candidates compete through policy and persuasion, and using a tractable two-dimensional framework with social learning provide an explanation for increasing political polarization. Voters and candidates have policy preferences that depend upon the state of the world, which is known to candidates but not known to voters, and are connected through a social influence network that determines through a learning process the final opinion of voters, where the voters' initial opinions and the persuasion efforts of the candidates affect final opinions, and so voting behavior. Equilibrium level of polarization in policy and opinion (of both party and population) increases when persuasion costs decrease. An increase in homophily increases the equilibrium level of policy polarization and population opinion polarization. These comparative static results help explain the increased polarization in both the policy and opinion dimensions in the United States.

To my mother, and to my father, who knew an absent-minded professor when he saw one

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Acknowledgments

I thank Utku Ünver, Hideo Konishi and Tayfun Sönmez for their constant support and counsel, and for their unwavering belief in me even when I didn't believe in myself. Each in their own way has left an indelible mark on my life as a scholar, and I am forever grateful.

I thank the Department of Economics at Boston College for the many opportunities afforded me and its exceptional support over my graduate career. I appreciate the helpful advice given so freely by everyone at the Theory Workshop and the Dissertation Workshop, especially Inácio Guerberoff, Ko Chiu Yu, and Orhan Aygün. My first two years in the graduate program would have been dull without the energetic Murat Mungan. I have many friends in the department with whom I have shared bread and wine and words and laughs. I thank them all, especially Federico Mantovanelli. I appreciate also the hospitality of Starbucks, Athan's Bakery, Crema Café, and the many other cafés dotting the Boston area, where much of this work was born and nurtured on a diet of coffee and pastries.

In the course of my twelve year adventure in the United States I have made many good friends and received their encouragement from near and far, and I thank each and every one of them for their faith and friendship. I thank Ben Dunn, Chris Licciardi, and Nam Bui for teaching me about freedom, being-in-the-world, and mischief. I thank the Dunn family, Susan especially, for being my safe harbor on these once foreign shores. HMS Putnam, the Empire, and the Phoenix provided sustenance and warmth, especially Iyar Mazar and Bryan Plummer. And most importantly I thank Aaron Fix, a true and noble friend. He never doubted the outcome. I will always look up to him as a role-model of disciplined and ethical behavior, though I will inevitably fail to emulate it. I also owe a tremendous debt of gratitude to Rossella Calvi for many illuminating conversations. I would have no words without her.

To my loving and supportive family I offer my eternal gratitude. I thank in particular my brother Solomon, who has made many sacrifices on our family's behalf. Finally, I thank my mother, the rock of our family, the embodiment of faith and love. She believed in my choices without having to understand them.

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Chapter 1

Stability and Matching with Aggregate Actors

1.1 Introduction

Hierarchies of decision-makers are the dominant form of organizational design in a wide variety of institutions, from social institutions such as families and communities, to political institutions such as the executive branch of government, to economic institutions such as large corporations or small firms. This robust empirical fact of real-world organizations has prompted many theories to explain their existence and their functioning. Given the key role firms play in the operation of the economy, the hierarchical firm is of particular interest to economists and organizational theorists. Managerial hierarchies determine the allocation of resources within the firm, particularly through their role in conflict resolution, and also enable coordination of activities in the firm. A potential alternative to hierarchies for internal allocation is a market-like exchange mechanism, where claims on resources are more widely distributed within the organization, in the manner of cooperatives. However, while firms may have lateral equity, they usually still possess a clear vertical structure[1].

[1] For evidence on hierarchies and decentralization in firms, their impact on productivity, see Bloom et al. (2010).

Many theories have been proposed to explain the existence of hierarchies in real-world organization of production, a structure at odds with the decentralized market mechanism coordinating economic activity. The transactions costs and incomplete contracts theories and the procedural rationality theory are some responses to this limitation of the basic theory of the firm. One goal of these theories has been to explain why firms exist or why they may be hierarchical, usually taking the market as exogenous and unaffected by the organizational design of the firm. I wish, instead, to turn the question on its head and ask how the organizational design of institutions can impact the performance of the market as a whole, where the market constitutes the free environment with institutions and individuals.

In this paper I argue that the organizational structure within each institution, what I identify as its *governance structure*, can indeed have important implications for market-level outcomes and market performance. Specifically I study how *complex* institutions, each composed of multiple actors called *divisions* with varying interests mediated by an institutional governance structure, come to make market-level choices. The governance structure is a defining feature of the institution, a product of its internal rules of coordinated resource allocation, conflict resolution, and culture. A production team in a firm, for example, could demand the same skilled worker as another team, creating a conflict for the human resource. The skilled worker may have a preference for one team over another, but this preference may not be sufficient to effect a favorable institutional decision, due to a governance structure that in this case strongly empowers the less-preferred production team. Thus, unlike the market governance structure, where parties can freely negotiate and associate, an institutional governance structure can restricts how parties inside the institution can do so.

The main result of this paper is that whenever institutions have governance structures that are inclusive hierarchies then stable market outcomes will exist. This existence result for the aggregate actors matching model relies upon the existence result of Hatfield and Kojima (2010), who generalize the many-to-one matching with contracts model of Hatfield and Milgrom (2005). The emergent choice behavior of institutions that have inclusive hierarchies is bilaterally substitutable whenever the divisions have bilaterally substitutable choice functions. In essence, inclusive hierarchies preserve the property of bilaterally substitutability of choice, leading to the existence result. Also preserved by this aggregation procedure is the Irrelevance of Rejected Contracts condition introduced by Aygün and Sönmez (2012b), which is a maintained assumption throughout this paper. As shown by those authors in Aygün and Sönmez (2012a), this condition is required when working with choice functions rather than with preferences as primitive. Other choice properties that are preserved include the weak substitutes condition of Hatfield and Kojima (2008) and the Strong Axiom of Revealed Preference.

Many transactions in the real world have the feature that one side is an individual such as a supplier of labor or intermediate inputs and the other side is an institution such as a large buyer firm, where the individual seeks just one relationship but the institution usually seeks many with different individuals. The standard model of matching where institutional welfare matters assumes that the institution is a single-minded actor with preferences, just like the individuals on the other side, but this black-box approach does not allow for an analysis of institutional level details. In practice, institutional choice behavior is determined by multiple institutional actors within a governance structure, which is the set of rules and norms regulating the internal functioning of the institution. As institutions seek to allocate resources amongst competing internal objectives, perhaps embodied in the divisions of the institutions, they often do so often without resorting to a price mechanism, but to a hierarchical mechanism instead. A central contribution of my work is to explain this fact by analyzing the interplay between institutional governance and market governance of transactions, which in spite of being an empirical feature of many real-world markets has been relatively unstudied from the matching perspective.

I use the matching model with aggregate actors to provide a theory for the widespread presence in firms of hierarchies with partial decentralization in decision-making in the context of factor markets. I show that hierarchical firms transacting with heterogeneous individuals in a market leads to outcomes that are in the core of the economy and are stable in a matching-theoretic sense. I support this observation by showing via examples how even in a simple setting with basic contracts (where a contract only specifies the two parties involved) and with unit-demand for factors by every division within the firms, an internal governance structure that distributes power more broadly amongst divisions and allows for trading by divisions of claims to contracts can create market-level instabilities that result in non-existence of stable or core outcomes. While this example does not rule out the possibility of market stability with such internal governance structures, it does demonstrate the difficulty of constructing a general theory in this regard while maintaining the importance of stability of market outcomes.

The importance of institutional-level analysis of choice has been amply demonstrated in the recent market design work of Sönmez and Switzer (2012), Sönmez (2011) and Kominers and Sönmez (2012). These authors study market design where the objectives of institutions can be multiple and complex, and the manner in which these objectives are introduced into the design has a material effect on design desiderata such as stability and strategyproofness. My work is similar to these authors' works in the feature that choice is realized by an institutional procedure, though in the case of market design the only agents for the purposes of welfare are the individuals. My work is also similar to Westkamp (2012), who studies a problem of matching with complex constraints using a sequential choice procedure.

This paper, and the previously mentioned work in market design, rests upon the theory of stable matchings, initiated by Gale and Shapley (1962), which has been one of the great successes of economic theory, providing an analytical framework for the study of both non-monetary transactions and transactions with non-negligible indivisibilities.[2] This theory underpins the work in market design, where solutions to real-world allocation problems cannot feature monetary transfers and centralized mechanisms can overcome limitations of a decentralized market. Matching theory is also illuminating in the study of heterogeneous labor markets and supply chain networks, where transactions between agents are conducted in a decentralized setting. The approach of studying a heterogeneous labor market using a matching-theoretic framework was pioneered by Crawford and Knoer (1981) and Kelso and Crawford (1982), and further explored by Roth (1984b) and Roth (1985). Hatfield and Milgrom (2005) provide the modern matching with contracts framework on which much new work in matching theory is built, this paper included. Ostrovsky (2008) studies supply networks using the matching with contracts approach, work that has been followed by Westkamp (2010), Hatfield and Kominers (2012b), and Hatfield et al. (2012).

The real-world relevance of stability has been part of the extensive evidence collected by Alvin Roth for the usefulness of the matching framework for understanding *inter alia* professional labor markets. In Roth (1984a), the author describes and analyzes the history of the market for medical residents in the United States, and makes the case that stability of outcomes affected the evolution of the organizational form of the market, and that the success and persistence of the National Residency Matching Program should be attributed to the stability of the outcomes it produces under straightforward behavior. Further support for the relevance of stability comes from the evidence provided in Roth (1991), where the author documents a natural experiment in the use of a variety of market institutions in a number of regional British markets for physicians and surgeons. In regions with matching procedures that under straightforward behavior produce stable outcomes, the procedures were successful in making the market operate smoothly and persisted. In some regions where the procedures in use did not necessarily produce stable outcomes, the market eventually failed to work well and these procedures were abandoned.[3] While this evidence might be construed as support for centralization of matching, the market forces are unrelated to the centralization or decentralization of the market, most clear in the fact that some of the centralized regional procedures in Britain failed to survive. Instead, the evidence points to

[2]The theoretical argument that final market outcomes will be stable can be traced back to the Edgeworth's approach to realized transactions as "finalized settlements", which are "contract[s] which cannot be varied with the consent of all parties to it [and] …which cannot be varied by recontract within the field of competition" (see pg. 19 of Edgeworth (1881)). The core of a game is a generalization of Edgeworth's recontracting notion, and the stability concept of Gale and Shapley the analogue of the core for the class of two-sided matching problems, when considered in the cooperative game framework.

the importance of the final outcome being a stable one.

In order to provide a non-cooperative game-theoretic understanding of my model, I study a two-stage game where talents make offers to institutions in the first stage, and then divisions within institutions choose from the available set of offers by using the internal mechanism of the institution. Focusing on subgame perfect Nash equilibria, I show that with hierarchical structures these equilibria yield pairwise stable outcomes. This supports the argument for inclusive hierarchical governance structures, in this case relying upon the notion that as internal mechanisms they have good local incentive properties for a given choice situation, in addition to their market-stability properties.

The positive and normative properties of hierarchies as allocative mechanisms when modeled as dictatorial structures has been explored in the indivisible goods setting (see Sönmez and Ünver (2011) for a survey) and in the continuous setting; for a hierarchical counterpart to the classic exchange economy model, see for example Piccione and Rubinstein (2007).[4]

The closest line of inquiry, in terms of both question and method, is Demange (2004). Her work focuses on explaining hierarchies as an organizational form for a group given a variety of coordination problems facing this group, using a cooperative game approach with a characteristic function to represent the value of various coalitions. With superadditivity, she finds that hierarchies distribute blocking power in such a way that the core exists. An important difference in this paper is the presence of multiple organizations in a bigger market. My analysis complements her study in showing that hierarchies are important not only because they produce stability in her sense, but also because they behave well in competition in a bigger market.

A well-established theory of hierarchies in organizations is the transaction costs theory,

[3]The British study is all the more intriguing because of the survival of a particular class of unstable procedures. Roth (1991) suggests that the smallness of these particular markets (numbering two) might be playing a role by removing the "impersonal" aspect of the other larger markets.

[4]There are a host of papers studying non-price mechanisms, some of which can serve as models of hierarchies. Some important works include Satterthwaite, Sonnenschein (1981), Svensson (1999), Pápai (2000), Piccione and Razin (2009), and Jordan (2006).

introduced by Ronald Coase in 1937 and then thoroughly pursued by Oliver Williamson (see Williamson (2002) for a more recent summary). In the transactions costs theory, not all market transactions can be secured solely through contracts, because the governance rules of the market do not allow for it. For example, the buyer of a specific input could contract with one of a number of potential suppliers, but the relationship is plagued by the problem of hold up, since the outside value of the input is low. This example of a transaction cost, it is argued, is avoided by a vertical integration of production into the buying firm.[5]

Yet another perspective on hierarchies is the procedural rationality approach of Herbert Simon, perhaps best captured by the following quotation from a lecture in his book *The New Science of Management Decision*:

> An organization will tend to assume hierarchical form whenever the task environment is complex relative to the problem-solving and communicating powers of the organization members and their tools. Hierarchy is the adaptive form for finite intelligence to assume in the face of complexity.

Simon explained how the complexity of decision problems facing large firms cannot be solved by the individual entrepreneur, as is the characteristic assumption of the neoclassical theory of the firm. Instead, the organizational response to these problem-solving difficulties is to divide decision-making tasks within the organization and use procedures to coordinate and communicate smaller decisions in the pursuit of large goals. This information processing approach has been studied by a host of researchers, especially early on by Jacob Marschak and Roy Radner.[6]

In this paper, I abstract from informational concerns with decision-making, concentrating instead on the relationship between the capabilities of coalitions and outcomes to

[5]Hierarchies also arise in the literature on property rights and incomplete contracts, where a fundamental inability to write comprehensive contracts makes arms-length transactions less attractive in comparison to direct control. See the seminal works of Grossman and Hart (1986) and Hart and Moore (1990), and Gibbons (2005) for a survey on theories of the firm.

[6]See Radner (1992) for a survey on hierarchies with a focus on the information processing approach. Other important works in a similar vein include the communication network of Bolton and Dewatripont (1994) and the knowledge-based hierarchy of Garicano (2000).

understand what relational structures are compatible with the preferences of actors (operationalized through the notion of stability). The origins of the decision hierarchies might be multiple, but their persistence too deserves explanation.

The remainder of the paper is organized as follows. In section 2, I describe and explain the formal framework, which I then use towards a theory of hierarchical institutions in section 3, where I also foray into an larger class of institutional structures to demonstrate that hierarchies are distinguished. In section 4, I take a non-cooperative approach and study a take-it-or-leave-it bargaining game. I conclude in section 5. Some proofs are to be found in the appendix, which also contains a section on useful comparative statics of combinatorial choice in matching and a section on the relationship between stability and the weaker notion of pairwise stability.

1.2 A Model of Matching with Institutions

1.2.1 The Elements

Let I be the set of **talents** and $(D(k))_{k \in K}$ be the collection of the set of **divisions** indexed by the set K, which is the set of (local) **institutions**, where all these sets are disjoint from each other. Associated with an institution k is a **governance structure** ψ^k, which are institutional-level rules and culture that determine how transactions involving institutional members can be secured. In the background is the **market governance structure**, which is the ambient framework within which talents and institutions conduct **market transactions**. The market governance structure determines the security of transactions between talents and institutions, but is superceded by the institutional governance structure for intra-institutional transactional details.

Transactions are modeled as bilateral **contracts** that describe the parties to the transaction as well as other details of the transaction. A contract x will name one talent $I(x)$ and one institution $K(x)$. A contract might also name one or more divisions from $D(K(x))$, but this, as with other intra-institutional details, are left free. Contracts are *comprehensive* in the sense that they describe completely all talent-institution transactional matters.[7] For a given talent i and institution k, let $X(i,k)$ be the set of all possible contracts between them. Then, $X(i) \equiv \bigcup_{k\in K} X(i,k)$ is the collection of all possible contracts involving talent i, $X(k) \equiv \bigcup_{i\in I} X(i,k)$ is the collection of all possible contracts involving institution k, and $X \equiv \bigcup_{i\in I} \bigcup_{k\in K} X(i,k)$ is the contract set for this economy, which is taken to be exogenous.

An institution transacts with potentially multiple talents in pursuit of its goals, but a talent transacts with at most one institution. Let $\mathcal{X}(i)$ be the collection of subsets of $X(i)$ that are **feasible** for i, where the empty set $\emptyset$, representing the outside option (being unmatched) for i, is always assumed to be feasible. In keeping with the assumption that a talent can have at most one contract with *any* institution, it must be that for any $Y \in \mathcal{X}(i)$, $|X(i) \cap Y| \leq 1$. We will identify these singleton sets with the element they contain for notational convenience.[8] If $|X(i) \cap X(k)| = 1$ for all $i \in I$ and $k \in K$, then the contract set is **classical**.

Each talent i has strict preferences[9] $\mathcal{P}^i$ over the set of feasible contracts $\mathcal{X}(i)$ naming him. Let $\mathcal{R}^i$ be the associated weak preference relation, where $Y\mathcal{R}^iY'$ if $Y\mathcal{P}^iY'$ or $Y = Y'$. The choice behavior in a given **choice situation** $Y \subseteq X(i)$ for a talent i is determined by the preferences of this talent.[10] Let C^i denote the choice function of talent i, defined for

[7]To the extent that a contract encodes all the details of a relationship that matter to either party, and that the set of contracts allows for every combination that could matter, this assumption is innocuous.

[8] A brief description of notation is in order. An arbitrary map f from domain E to codomain F associates each element $e \in E$ with a subset $f(e) \subseteq F$ of the codomain i.e. it is a correspondence. If for all $e \in E$, $|f(e)| = 1$, then f is a function. I will use maps from a set to some other set (where typically one of these two sets is a subset of X) to work with the relational information encoded in contracts, using the symbol for the target set as the symbol for the mapping as well.

So, for any $x \in X$, $I(x)$ is the subset of talents associated with contract x, and $K(x)$ the subset of institutions. With this notation, the set of all contracts in an arbitrary subset $Y \subseteq X$ associated with some talent $i \in I$ by $Y(i)$ (the map is $Y : I \rightrightarrows Y$) is defined by $Y(i) \equiv \{y \in Y \subseteq X : i \in I(y)\}$. Another typical practice in this paper will be the identification of singleton sets with the element it contains, as above. For any map f from domain E to codomain F, the following extension of this map over the domain 2^E will also be denoted by f: $f(E') \equiv \bigcup_{e\in E'} f(e)$ for every $E' \subseteq E$ (note that $f(\emptyset) \equiv \emptyset$). Given a subset of contracts $Y \subseteq X$, $I(Y)$ is the subset of talents associated with at least one contract in Y. Consider the following more complex example: suppose we have two subsets of contracts Y and Z, and we want to work with the set of all contracts in Z that name some talent that is named by some contract in Y. This is exactly $Z(I(Y))$, since $I(Y)$ is the set of talent that have a contract in Y, and $Z(I')$ is the set of all contracts that name a talent in the set I'.

[9]A strict preference relation on a set is complete, asymmetric, transitive binary relation on that set. A weak preference relation is a complete, reflexive, transitive binary relation.

every possible choice situation $Y \subseteq X(i)$, such that $C^i(Y) \subseteq Y$ is feasible. The assumption of preference maximization is that $C^i(Y)\mathcal{R}^i Z$ for all feasible $Z \subseteq Y$. Strict preferences implies that the maximizer is unique and thus that choice *functions* are appropriate.

In keeping with the purpose of building a model of market behavior of the institution, we will focus on the choice behavior of the institution with respect to contracts with talents. A **choice situation** for k is a subset of contracts $Y \subseteq X(k)$, a set of potential transactions that is available to the institution. Because institutions are *complex* entities, composed of many divisions with various interests, the choice behavior of an institution is an *emergent* phenomenon, shaped by the institutional governance structure ψ^k that mediate the interests of these divisions. The ideal choice of the institution in a given choice situation Y is a feasible subset $C \subseteq Y$. But whence choice?

Towards the goal of understanding institutional choice within the cooperative framework, I recognize the workings of the institution depend on the particulars of its governance structure and the interests of its actors, with the choice behavior of the institution in the market being thereby determined. I model this theory of the institution as follows: for every division $d \in D(k)$, there is an associated **domain of interest** $X(d) \subseteq X(k)$ (domains of interest of different divisions may overlap). A division d has strict preferences $\mathcal{P}^d$ over subsets of contracts in its domain of interest $X(d)$. Fixing the collection of domains of interest $\mathcal{D}(k) \equiv \{X(d)\}_{d \in D(k)}$ and the preferences of the divisions $\mathcal{P}(k) \equiv \{\mathcal{P}^d\}_{d \in D(k)}$, the institutional governance structure ψ^k determines for every choice situation $Y \subseteq X(k)$ the choice of the institution. Let C^k be the institution's **derived choice**, where the dependence on ψ^k, $\mathcal{D}(k)$, and $\mathcal{P}(k)$ has been suppressed. Choice behavior of an institution does not necessarily arise from the preference maximization of a single preference relation, unlike a talent. To the extent that a profit function can be modeled as the preference relation of a firm, the neoclassical model of the firm as a profit-maximizer, while compatible with the framework here, is not assumed.

[10] The assumption that the only determinant of choice behavior is the preference relation is in keeping with the spirit of cooperative game theory and matching theory.

1.2.2 Internal Assignments, Governance and Stability

Fix an institution k and take as given $X(k)$ and $\{X(d)\}_{d \in D(k)}$. Let $Y \subseteq X(k)$ be a choice situation for the institution k. The governance structure ψ^k determines the institution's choice from Y, $C^k(Y)$, via an **internal assignment** f_Y, which is a correspondence from $D(k)$ to Y such that the feasibility condition of one contract per talent is satisfied: $|\bigcup_{d \in D(k)} f_Y(d) \cap X(i)| \leq 1$. Any contract $y \in Y$ such that $f_Y^{-1}(y) = \emptyset$ is considered to be *unassigned* at Y. A contract $y \in Y$ may contain terms that disallow certain divisions from accessing this contract. For example, divisions may be geographical offices of a firm and the contract may specify geographical restrictions. Any such restrictions are respected by ψ^k and are formally captured by excluding the contract from the domain of interest of the disallowed divisions. Thus, any internal assignment f_Y will respect these contract restrictions. Let F_Y be the set of all internal assignments given $Y \subseteq X(k)$ and let $F \equiv \bigcup_{Y \subseteq X(k)} F_Y$ be the set of all internal assignments. The institutional choice from Y given some internal assignment f_Y is defined as $C^k(Y; f_Y) \equiv \bigcup_{d \in D(k)} f_Y(d)$. Note that given Y, all unassigned contracts are *rejected* from Y.

Given a choice situation Y and the list of preferences of divisions $\mathcal{P}(k)$, the governance structure ψ^k determines an **internally stable** assignment $\psi^k(Y, \mathcal{P}^d) \in F_Y$.[11] For this paper I focus on governance structures that satisfy **institutional efficiency** i.e. for any Y, if f_Y is internally stable, then there does not exist $f'_Y \in F_Y$ such that $f'_Y \mathcal{R}^d f_Y$ for all $d \in D(k)$ and $f'_Y \mathcal{P}^d f_Y$ for some d. Let Ψ^k be the family of institutionally efficient governance structures for k.

1.2.3 Market Outcomes, Governance and Stability

For the sake of notational convenience, I extend the definition of choice functions for talents and institutions to choice situations where contracts not naming them are present: for any $Y \subseteq X$ and for any $j \in I \cup K$, $C^j(Y) \equiv C^j(Y(j))$. So, for a choice situation the

[11] One could allow for multiple internally stable assignments but I focus in this paper on single-valuedness.

only contracts that matter for j are those contracts that name it.

A market **outcome** (or **allocation**) is a feasible collection of contracts $A \subseteq X$, i.e. for all $i \in I$, $Y(i) \in \mathcal{X}(i)$. Let $\mathcal{A}$ be the set of all feasible outcomes. I extend preferences of talent from $\mathcal{X}(i)$ to $\mathcal{A}$ (keeping the same notation for the relations) as follows: for any $i \in I$ and $A, A' \in \mathcal{A}$, $A\, P(i)\, A'$ if $A(i)\, P(i)\, A'(i)$ and $A\, R(i)\, A'$ if $A(i)\, R(i)\, A'(i)$. So, talents are indifferent about the presence or absence of contracts in an outcome that do not name them.

The market governance structure within which talents and institutions transact determines what each of these market participants are capable of securing. That a talent is free to contract with any institution, or not at all, is an outcome of the market governance structure enabling this. Similarly, that an institution may cancel a contract with a talent also reflects the rules of the marketplace. In matching theory, and cooperative game theory more generally, this is modeled by describing the way in which a market outcome can be *blocked* or *dominated*. Thus, any market outcome that is not blocked is considered to be consonant with the rules of market governance, and is considered *stable*. An important question is whether a given market governance structure, together with the interests and behavior of the market participants, allows for stable market outcomes.

An outcome A is **individually rational** for talent i if $A(i)\, R(i)\, \emptyset$. This captures the notion that i is not compelled to participate in the market by holding a contract that he prefers less than his outside option. An outcome A is **institutionally blocked** by institution k if $C^k(A(k)) \neq A(k)$. This captures the notion that k can unilaterally sever relationships with some talent without disturbing relationships with other talents and that the outcome has to be consistent with internally stable assignments. An outcome A is **institutionally stable** if it is not institutionally blocked by any institution. An outcome A is **individually stable** if it is individually rational for all talent and institutionally stable at every institution.

An outcome A is **pairwise blocked** if there exists a contract $x \in X \backslash A$ such that the talent $I(x)$ strictly prefers outcome $A \cup \{x\}$ to A and the institution $K(x)$ will choose this contract from $A \cup \{x\}$, that is $x \in C^{I(x)}(A \cup \{x\})$ and $x \in C^{K(x)}(A \cup \{x\})$. This captures the notion that the possibility of a new mutually chosen relationship will upset an outcome, and so the initial outcome is not secure. An outcome A is **pairwise stable** if it is individually stable and it is not pairwise blocked.

An outcome A is **setwise blocked** if there exists a *blocking set* of contracts $Z \subseteq X \backslash A$ such that every talent $i \in I(Z)$ strictly prefers $A \cup Z$ to A and every institution $k \in K(Z)$ will choose all its contracts in Z from choice situation $A \cup Z$ i.e. for all $i \in I(Z)$, $Z(i) \in C^i(A \cup Z)$ and for all $k \in K(Z)$, $Z(k) \subseteq C^k(A \cup Z)$. This captures the notion that the possibility of a collection of new relationships that would be chosen if available together with existing relationships will upset an allocation. An outcome A is **stable** if it is individually stable and it is not setwise blocked.

An outcome A is **dominated by A' via J**, where A' is an alternate outcome and $J \subseteq I \cup K$ is a deviating coalition, if

1. the deviating coalition's contracts in the alternate outcome is different from that in the original allocation: $A'(J) \neq A(J)$.
2. every deviating actor $j \in J$ holds contracts with other deviating actors only: for all $i \in J \cap I$, $K(A'(i)) \in J$, and for all $k \in J \cap K$, $I(A'(k)) \subseteq J$.
3. every deviating actor $j \in J$ would choose its contracts in the alternate outcome A' over those in the original outcome: for all $i \in J \cap I$, $C^i(A \cup A') = A'$ and for all $k \in J \cap K$, $C^k(A \cup A') = A'$.

An outcome A is in the **core** (is **core stable**) if there does not exist another outcome that dominates it via some coalition.

The concept of pairwise stability was first introduced by Gale and Shapley (1962), in a setting where pairwise stability and (setwise) stability coincide. Like the cooperative game concept of the core, the solution concept of stability appeals to outcomes of the economy to generate predictions, without considering strategic aspects that require the level of detail common in non-cooperative game theory. The stability concepts are closer in spirit to the concept of competitive equilibrium; in the stability concept the choice situation is taken as given just as in the competitive equilibrium concept the prices are taken as given (see Ostrovsky (2008) for an elaboration of this argument in the context of supply chain markets).

In the present setting of many-to-one matching, the set of core outcome and the set of stable outcome coincides. This is the content of the following lemma, analogues[12] of which have been proved in many-to-one matching settings where choice is generated by preferences for all market participants.

Lemma 1.1. *An outcome is in the core if and only if it is stable.*

Proof. First, we will show that every stable outcome is in the core, by proving the contrapositive. Suppose A is dominated by A' via coalition J. Suppose J contains no institution. Then, every deviating talent receives his outside option, and by domination requirement 1 at least one of these deviators held a different contract in A than the null contract $\emptyset$ in A'. Pick one such talent $i \in J$. Then A is not individually rational for i and so A is not stable. Instead, suppose J contains at least one institution k. If every institution holds exactly the same set of contracts in A' and A, then we are back to the case where at least one worker holds a different contract in A and A'. Moreover, it must be the case, given all $k \in J \cap K$ hold the same contracts in A and A', that this one worker holds the null contract in A', and so again we have that A is not individually rational for this worker and hence not stable. So, in the final case, we have at least one institution $k \in J$ and moreover this institution holds different contracts in A and A'. Then the set of contracts $Z \equiv A'(k)$ constitutes a block of A, since domination condition 3 implies $C^k(A \cup Z) = Z$ and $C^i(A \cup Z) = Z$ for any $i \in I(Z)$, proving A is not stable.

Second, we will show that every core outcome is stable, by proving the contrapositive. Suppose A is setwise blocked by $Z \subseteq X \backslash A$. Define $J \equiv I(Z) \cup K(Z)$ and for each $j \in J$, define $B_j \equiv C^j(A \cup Z)$. Define $A' \equiv \left(A \backslash \bigcup_{j \in J} A(j)\right) \cup \left(\bigcup_{j \in J} B_j\right)$. Note that

[12] See Echenique and Oviedo (2004) for a proof of this in the classic many-to-one matching model, and see Hatfield and Milgrom (2005) for a similar statement.

A' is an outcome by construction. Now, define $J' \equiv I\left(\bigcup_{k \in K \cap J}(B_j \backslash Z)\right)$; these are the talents not in the blocking coalition J whose contracts with blocking institutions are chosen after the block. There is no analogous set of institutions, since the unit-demand condition of talents' preferences implies that blocking talents do not hold any contracts with non-blocking institutions after the block. It follows from the construction of A' that A is dominated by A' via coalition $J \cup J'$. $\square$

This coincidence of the more widely-known concept of the core with the matching solution concept of stability supports the argument that stability is an important condition for market outcomes to satisfy. In the Walrasian model of markets, similar results relating the core to the competitive equilibrium lend support to the latter as a market outcome. While in that setting equivalence of the two does not hold generally, the core convergence result of Debreu and Scarf (1963) shows that in sufficiently large markets every core outcome can be supported as a competitive equilibrium outcome and vice versa, and provides a proof of the Edgeworth conjecture. Similar large market results have been obtained in matching models.[13]

1.2.4 Conditions on Preferences and Choice

Certain conditions on choice are needed to ensure existence of stable outcomes in many-to-one matching models.[14] Perhaps the most important of these conditions is substitutability.

Definition 1.1 (Substitutability)**.** *A choice function C^k on domain $X(k)$ satisfies **substitutability** if for any $z, x \in X(k)$ and $Y \subseteq X(k)$, $z \notin C^k(Y \cup \{z\})$ implies $z \notin C^k(Y \cup \{z, x\})$.*

Substitutability, introduced in its earliest form by Kelso and Crawford (1982), is sufficient for the existence of stable outcomes in many-to-one matching models when choice is determined by preferences, both in the classical models without contracts and in the more

[13]See Kojima and Pathak (2009) and Azevedo and Leshno (2012).

[14]For the sake of collecting definitions in one subsection, I define and discuss the important conditions on choice that will be used in this paper. The reader may wish to skip these and proceed to the next section on hierarchical institutions, using this subsection as a useful reference.

general framework with contracts, this last result due to Hatfield and Milgrom (2005). In addition, the set of stable matchings has a lattice structure, with two extremal stable matchings, each distinguished by simultaneously being the most preferred stable matching of one side and the least preferred stable matching of the other side.

Substitutability has also proved useful as a sufficient condition for existence of weakly setwise stable outcomes in the many-to-many matching with contracts model, a concept introduced and studied in Klaus and Walzl (2009). These authors follow the early literature in assuming that contracts are comprehensive, so that any pair has at most one contract with each other in an outcome. Hatfield and Kominers (2012a) instead assume that a pair may have multiple contracts with each other in an outcome and show that substitutability is sufficient under their definition of stability.[15] Substitutability is not sufficient for existence of outcomes that satisfy a solution concept stronger than weak setwise stability, though Echenique and Oviedo (2006) show that strengthening the condition for one side to strong substitutes restores existence for this stability notion in the classical setting.

While providing the maximal Cartesian domain for existence of stable outcomes in the classical many-to-one matching model (the college admissions model), substitutability is not the weakest condition ensuring existence of stable outcomes in many-to-one matching with contracts. Hatfield and Kojima (2010) provide a weaker substitutability condition that ensures existence of stable outcomes in models with preferences as primitives.

Definition 1.2 (Bilateral Substitutability). *A choice function C^k on domain $X(k)$ satisfies **bilateral substitutability** if for any $z, x \in X(k)$ and $Y \subseteq X(k)$ with $I(z) \notin I(Y)$ and $I(x) \notin I(Y)$, $z \notin C^k(Y \cup \{z\})$ implies $z \notin C^k(Y \cup \{z, x\})$.*

Bilateral substitutability guarantees existence in the many-to-one setting, but the structure of the stable set is no longer a lattice, and extremal outcomes need not exist. Hatfield

[15]The stability definition of Hatfield and Kominers (2012a) coincides with the weak setwise stability of Klaus and Walzl (2009) under the assumption of comprehensive contracts (which Kominers (2012) call *unitarity*), but is stronger under the assumption of non-comprehensive contracts. They also prove that substitutability provides a maximal Cartesian domain for existence of stable outcomes, with the caveat that contracts are not comprehensive.

and Kojima (2010) provide an intermediate condition, unilateral substitutability, that restores the existence of one of the extremal stable outcome, the doctor-optimal stable outcome, which is simultaneously the hospital-pessimal stable outcome.[16]

Definition 1.3 (Unilateral Substitutability). *A choice function C^k on domain $X(k)$ satisfies **unilateral substitutability** if for any $z, x \in X(k)$ and $Y \subseteq X(k)$ with $I(z) \notin I(Y)$, $z \notin C^k(Y \cup \{z\})$ implies $z \notin C^k(Y \cup \{z, x\})$.*

Bilateral substitutability does not provide a maximal Cartesian domain for sufficiency of existence, unlike substitutability in the college admissions model. Hatfield and Kojima (2008) introduced the weak substitutes condition, which mimics substitutability for a *unitary set* of contracts, defined to be a set in which no talent has more than one contract present. The authors show that any Cartesian domain of preferences that guarantees existence of stable outcomes must satisfy weak substitutability.

Definition 1.4 (Weak Substitutability). *A choice function C^k on domain $X(k)$ satisfies **weak substitutability** if for any $z, x \in X(k)$ and $Y \subseteq X(k)$ with $I(z) \notin I(Y)$, $I(x) \notin I(Y)$ and $|I(Y)| = |Y|$, $z \notin C^k(Y \cup \{z\})$ implies $z \notin C^k(Y \cup \{z, x\})$.*

The common assumption about choice behavior in the matching literature has been that agents choose by maximizing a preference relation or objects are allocated while respecting a priority relation. With the definition of stability introduced in Hatfield and Milgrom (2005), however, one that makes reference only to choice functions, it is no longer necessary to make reference to underlying preferences for the model to be studied, since substitutability is a condition on choice functions as well. For this more abstract setting however, substitutability is no longer a sufficient condition for existence, as shown by Aygün and Sönmez (2012b). These authors introduce the Irrelevance of Rejected Contracts condition on choice that restores the familiar results of matching models under substitutable preferences, such as the lattice structure and the opposition of interests at extremal matchings.

[16] In their setting, doctors are the talents who can hold only one contract in an outcome and hospitals are the institutions which can hold many contracts in an outcome. Moreover, hospitals have preferences as primitives that define choice behavior.

Definition 1.5 (Irrelevance of Rejected Contracts)**.** *A choice function C^k on domain $X(k)$ satisfies the* ***Irrelevance of Rejected Contracts (IRC)*** *condition if for any $Y \subseteq X(k)$ and $z \in X(k)\backslash Y$, $z \notin C^k(Y \cup \{z\})$ implies $C^k(Y \cup \{z\}) = C^k(Y)$.*

Choice derived from preferences must satisfy the Strong Axiom of Revealed Preference (SARP)[17], and it is the combination of this choice assumption and substitutability that yields the results of Hatfield and Milgrom (2005). However, under the substitutes condition, IRC is no weaker than SARP. However, the IRC condition is also sufficient to restore all the results of Hatfield and Kojima (2010) under the weaker substitutes conditions introduced therein, and Aygün and Sönmez (2012a) also show that in this setting IRC is strictly weaker than SARP.

While substitutability and unilateral substitutability are strong enough conditions to provide useful structure on the stable set, particularly in ensuring the existence of a talent-optimal stable outcome, they are not strong enough to yield the result that a strategyproof mechanism exists for this domain, a result that is familiar from the college admissions model with responsive preferences. Hatfield and Milgrom (2005) show that under a condition on choice they call Law of Aggregate Demand, a generalized version of the Gale-Shapley Deferred Acceptance algorithm serves as a strategyproof mechanism for talent.

Definition 1.6 (Law of Aggregate Demand)**.** *A choice function C^k on domain $X(k)$ satisfies the* ***law of aggregate demand (LAD)*** *if for any $Y, Y' \subseteq X(k)$, $Y \subseteq Y'$ implies $|C^k(Y)| \leq |C^k(Y')|$.*

Alkan (2002) introduced the analog of this condition, *cardinal monotonicity*, for the classical matching model to prove a version of the *rural hospital theorem*[18]. He demonstrates that with cardinal monotonicity, in every stable matching every agent is matched to the same number

[17] In a matching setting, where choice is combinatorial, a choice function C with domain X satisfies the **Strong Axiom of Revealed Preference (SARP)** if there does not exist a sequence of distinct $X_1, \ldots, X_n, X_{n+1} = X_1$, $X_m \subseteq X$, with $Y_m \equiv C(X_m)$ and $Y_m \subseteq X_m \cap X_{m+1}$ for all $m \in 1, \ldots, n$.

[18] Roth (1986) showed that in the college admissions model with responsive preferences, any college that does not fill its capacity in some stable matching then in every stable matching it is matched to exactly the same set of students.

of partners. The analog for the contracts setting is that under the Law of Aggregate Demand, every institution holds the same number of contracts in every stable outcome.

One last condition that will prove useful in the later section on a decentralized bargaining game is the condition of Pareto Separable choice.

Definition 1.7 (Pareto Separable). *A choice function C of an institution k (or division d) is Pareto Separable if, for any $i \in I$ and distinct $x, x' \in X(i, k)$, $x \in C(Y \cup \{x, x'\})$ for some $Y \subseteq X(k)$ implies that $x' \notin C(Y' \cup \{x, x'\})$ for any $Y' \subseteq X(k)$.*

Hatfield and Kojima (2010) prove that substitutability is equivalent to unilateral substitutability and the Pareto Separable condition. A partial analog to this result is that weak substitutability and the Pareto Separable condition is equivalent to bilateral substitutability, though the converse is not true.

Proposition 1.1. *Suppose institution k has a choice function C satisfying IRC, weak substitutes and the Pareto Separable condition. Then C satisfies bilateral substitutes.*

The Pareto Separable condition states that if in a choice situation some contract with a talent is not chosen but an alternative contract with this talent is, then in any other choice situation where the alternative is present the first cannot be chosen. So, in particular, suppose a new contract with a new talent becomes available and is chosen. With the Pareto Separable assumption, we can conclude that there cannot be any *renegotiation with held talents*, since such a renegotiation would involve a violation of this assumption. Therefore, given the assumption of IRC, we can remove these unchosen alternatives with talents held in the original choice situation without altering choice behavior. Moreover, IRC allows us to remove any contracts with talents who are not chosen in either the original situation or in the new situation with the arrival of a previously unseen talent. Thus, we can reduce the set of available contracts in the original situation to contain no more than one contract per talent. Thus, if any previously rejected talent (or contract) is recalled with the arrival of a new talent (violating bilateral substitutes), then this behavior would prevail in the pruned choice situation, resulting in a violation of weak substitutes. This argument is formalized in the following proof.

Proof. Let $Y \subseteq X(k)$ and $z, x \in X(k)\backslash Y$ such that $z \neq x$ and $I(z) \neq I(x)$. Moreover, suppose $I(z), I(x) \notin I(Y)$. Suppose $z \notin C(Y \cup \{z\})$. Now, suppose $z \in C(Y \cup \{z, x\})$, which constitutes a violation of bilateral substitutability. First, suppose there exist $w \in Y$ such that $w \notin C(Y \cup \{z\})$ and $w \notin C(Y \cup \{z, x\})$. Then by IRC we can remove w from Y without affecting choice i.e. $C(Y' \cup \{z\}) = C(Y \cup \{z\})$ and $C(Y' \cup \{z, x\}) = C(Y' \cup \{z, x\})$. Repeatedly delete such contracts, and let Y' denote the set remaining after all such deletions from Y.

If there exist $y, y' \in Y'$ with $I(y) = I(y')$ such that $y \in C(Y' \cup \{z\})$ and $y' \in C(Y' \cup \{z, x\})$, then C would violate the Pareto Separable condition, given that no more than one contract with $I(y)$ can be chosen. Thus, if $y \in C(Y' \cup \{z\})$ then for any $y' \in Y'$ with $I(y') = I(y)$, $y' \notin C(Y' \cup \{z, x\})$. So, by IRC, $C(Y'' \cup \{z\}) = C(Y' \cup \{z\})$ and $C(Y'' \cup \{z, x\}) = C(Y' \cup \{z, x\})$, where $Y'' = Y'\backslash\{y'\}$. We can repeat this deletion procedure and let Y'' denote the set remaining after all such deletions from Y.

It should be clear that $|Y''| = |I(Y'')|$. Moreover, we have that $z \notin C(Y'' \cup \{z\})$ but $z \in C(Y'' \cup \{z, x\})$, constituting a violation of weak substitutes, and concluding our proof. $\square$

1.3 The Theory of Hierarchical Institutions

In this section, I define and examine a particular institutional governance structure, the *inclusive hierarchical* governance structure. Unlike the market governance structure, which is a rather permissive type of governance structure that allows talents and institutions to freely re-contract, inclusive hierarchical governance structures greatly enhance the bargaining power of divisions versus talents. The view taken in this section is that talents are human resources to be allocated within the institution, and the institutional governance structures considered reflects this aim. The inclusive hierarchical governance structure provides talents with *weak veto power* since they can leave any contract with the institution for another institution, is *institutionally efficient* since there does not exist any internal assignment of contracts to divisions that is weakly improving for every division and strictly improving for some, and is *situationally strategyproof* since for a fixed take-it-or-leave-it choice situation every division has a dominant strategy reveal its preferences when the governance structure ψ is viewed as a mechanism. Proofs of results can be found in the appendix.

1.3.1 The Inclusive Hierarchical Governance Structure

A governance structure $\psi \in \Psi^k$ has a **hierarchy** if it is parametrized by a linear order $\rhd^k$ on $D(k)$. **Inclusive Hierarchical (IH)** governance structures constitute a class of governance structures where the hierarchy $\rhd^k$ determines how conflicts between divisions over contracts are resolved, and where divisions have the power to choose contracts without approval of other divisions, except in the case of conflicts for talents already mentioned. For example, given a choice situation Y, if there is a contract $y \in Y$ such that distinct divisions $d, d' \in D(k)$ both have y as part of their most preferred bundle of contracts in Y, then the governance structure resolves this conflict in favor of the division with higher rank, where $d \rhd^k d'$ means that division d has a higher rank than d'. However, if given any two divisions their most preferred bundles in Y are such that there is no conflict over a contracts or talents, then the divisions have the autonomy to choose these bundles on behalf of the institution. The order $\rhd^k$ defines a *ranking* of divisions, where division d is said to be *higher-ranked* than division d' if $d \rhd^k d'$, where $d, d' \in D(k)$ for some institution k. Since it should not cause any confusion, let $\rhd^k : D(k) \to \{1, \ldots, |D(k)|\}$ be the rank function, where $\rhd^k(d) < \rhd^k(d')$ if and only if $d \rhd^k d'$. Also, for any $n \in \{1, \ldots, |D(k)|\}$, let d_n^k denote the n-th ranked division i.e. $\rhd^k(d_n^k) = n$.[19]

The inclusive hierarchical governance structure ψ^k parametrized by $\rhd^k$ can be modeled

[19] A division d higher-ranked than another division d' if and only if its rank number $\rhd^k(d)$ is *smaller* $\rhd^k(d')$.

using the following choice aggregation procedure, the **inclusionary hierarchical procedure**. This procedure determines the internal assignment of contracts for a given choice situation $Y \subseteq X(k)$, and thence the derived institutional choice $C^k(Y)$. The procedure is analogous to a serial dictatorship in the resource allocation literature, with the hierarchy $\rhd^k$ serving as the serial ordering. The highest ranked division d_1^k is assigned its most preferred set of contracts from Y. The next highest ranked division d_2^k is assigned its most preferred set of contracts from the remain set of contracts, and so on. Importantly, after a division's assignment is determined, any unassigned contracts that name a talent assigned at this step are removed (though still unassigned), and the remaining contracts constitute the availability set for the next step. At every step, the assignment must be feasible, so that no division d is assigned a contract outside of its domain of interest $X(d)$.

The formal description of the procedure requires some notation. Let $Y \subseteq X(k)$ be a subset of contracts naming the institution k. There are $N^k = |D(k)|$ steps in the procedure. For the sake of notational convenience and readability, I will suppress dependence on the institution k, which will be fixed. For any $n \in \{1, \ldots, N\}$, let λ_n^Y be the set of contracts *available at step* n, let α_n^Y be the set of contracts *available and allowed at step* n, β_n^Y be the set of contracts *available and not allowed at step* n, γ_n^Y be the set of contracts *assigned at step* n, δ_n^Y be the set of contracts *eliminated at step* n, and ρ_n^Y be the set of contracts *rejected at step* n.

Step 1 Define $\lambda_1^Y \equiv Y$. Define $\alpha_1^Y \equiv \lambda_1^Y \cap X(d_1)$, $\beta_1^Y \equiv \lambda_1^Y \backslash \alpha_1^Y$, $\gamma_1^Y \equiv C^{d_1}(\alpha_1^Y)$, $\delta_1^Y \equiv \left(\lambda_1^Y \cap X(I(\gamma_1^Y))\right) \backslash \gamma_1^Y$, and $\rho_1^Y \equiv \alpha_1^Y \backslash (\gamma_1^Y \cup \delta_1^Y)$.

$\vdots$

Step n Define $\lambda_n^Y \equiv (\beta_{n-1}^Y \backslash \delta_{n-1}^Y) \cup \rho_{n-1}^Y$. Define $\alpha_n^Y \equiv \lambda_n^Y \cap X(d_n)$, $\beta_n^Y \equiv \lambda_n^Y \backslash \alpha_n^Y$, $\gamma_n^Y \equiv C^{d_n}(\alpha_n^Y)$, $\delta_n^Y \equiv \left(\lambda_n^Y \cap X(I(\gamma_n^Y))\right) \backslash \gamma_n^Y$, and $\rho_n^Y \equiv \alpha_n^Y \backslash (\gamma_n^Y \cup \delta_n^Y)$.

The *internal assignment* $f_Y(d)$ of division $d \in D(k)$ given a choice situation Y is $f_Y(d) = \gamma_{\rhd^k(d)}^Y$. The derived *institutional choice* $C^k(Y)$ from set Y is defined by $C^k(Y) \equiv \bigcup_{n=1}^{N^k} \gamma_n^Y$. Note that both f_Y and $C^k(Y)$ depend upon the hierarchy $\rhd^k$.

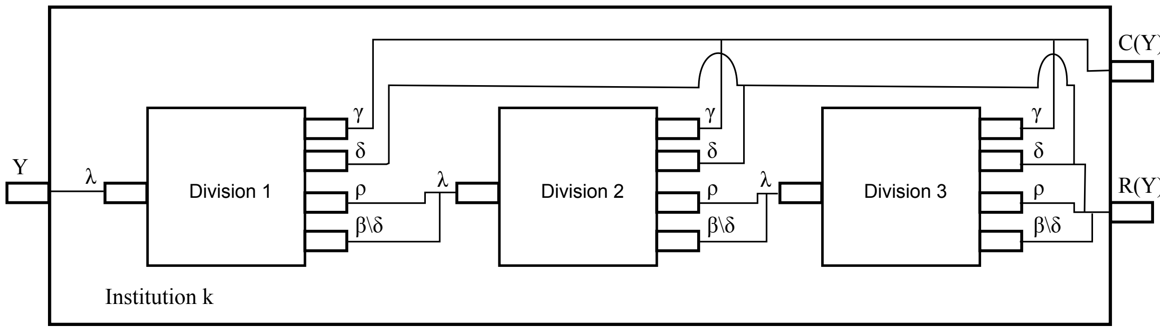

Figure 1.1: Graphical Depiction of a Hierarchical Institution with three Divisions, with the various contract-pathways of the Inclusionary Hierarchical Procedure displayed.

Figure 1.1 illustrates the inclusionary hierarchical procedure for an institution with three divisions. In this case, the choice procedure has three steps, one for each division. One can imagine that the set of contracts available to the institution "flow" through the institution along the "paths" illustrated, where divisions "split" the flow into various components that then travel along different paths. Some of these paths meet at a "union junction" (every junction in this figure is a union junction); some paths lead to a division of the institution. The paths form an "acyclic network" beginning at the "entry port" of the institution and ending at either the "acceptance port" or "rejection port", and so every contract that enters the institution will exit after encountering a finite number of nodes. While this description choice is not meant to be taken literally, it is a useful mnemonic for understanding the forthcoming results.

In summary, for any choice situation $Y \subseteq X(k)$, the internal assignment f that is internally stable given an inclusive hierarchical governance structure ψ^k with hierarchy $\triangleright^k$ coincides with the assignment $\left(\gamma_n^Y\right)_{n=1}^{N^k}$ produced by the corresponding inclusionary hierarchical procedure.

1.3.2 Properties of Inclusive Hierarchical Governance

I now turn to answering the main question posed by this paper: why hierarchies? In this subsection I will demonstrate that inclusive hierarchical governance structures have the positive property that the institutional choice function derived from the internally stable assignment satisfies two key choice properties, the Irrelevance of Rejected Contracts and bilateral substitutability, under the assumption that divisions have bilaterally substitutable preferences. This important result will then straightforwardly lead to the theorem that markets featuring institutions with inclusive hierarchical governance are guaranteed to have stable outcomes. Other interesting results about this governance structure will also be discussed.

Fix an institution k with divisions $D(k)$, where $(\mathcal{P}^d)_{d \in D(k)}$ are the preferences of each division, which respect the domain of interest restrictions $\mathcal{D}(k)$.[20] Let ψ^k be the inclusive hierarchical governance structure of k, parameterized by $\rhd^k$. In order to ease exposition and readability, I will suppress notation indicating the institution. Thus, for the purposes of this subsection, we will denote $X(k)$, the set of all contracts naming institution k, simply by X, and $D(k)$, the set of all divisions in k, simply by D.

The first property of inclusive hierarchical choice aggregation is that the IRC property of division choice will be preserved at the institutional level. As discussed previously, this condition states that the presence of "dominated" contracts in particular choice situation has no bearing on the choice, and so their removal from the available set does not alter the chosen set.

Theorem 1.1. *The institutional choice function C derived from the inclusive hierarchical governance structure parametrized by $\rhd^k$ satisfies the IRC condition if for every division $d \in D$, C^d satisfies the IRC condition.*

The following theorem is the key choice property with inclusive hierarchical governance. The property of bilateral substitutes is preserved by aggregation, given that divisional choice satisfies it and IRC.

Theorem 1.2. *The institutional choice function C derived from the inclusive hierarchical governance structure parametrized by $\rhd^k$ satisfies bilateral substitutes if for every division $d \in D$, C^d satisfies bilateral*

[20]The results of this subsection also hold if division choice is taken to be primitive with the additional assumption of IRC.

substitutes and the IRC condition.

The important observation in the proof is that an expansion of the choice situation through the introduction of a contract with a new or unchosen talent improves the array of contract options for every division in the institution, and not just for the highest-ranked division, given the assumptions of bilateral substitutability and IRC of division choice.

It is also the case that choice aggregation with inclusive hierarchies preserves the property of weak substitutes.

Proposition 1.2. *The institutional choice function C derived from the inclusive hierarchical governance structure parametrized by $\rhd^k$ satisfies weak substitutes if for every division $d \in D$, C^d satisfies weak substitutes and the IRC condition.*

The proof follows a similar strategy to that of Theorem 1.2, showing a monotonic relationship between certain choice situations of the institution and the resultant choice situations of each division.

Intriguingly, this preservation by inclusive hierarchical aggregation does not hold when divisions have substitutable choice, as shown by Kominers and Sönmez (2012) in the slot-specific priorities model, where slots are analogous to unit-demand divisions and the order of precedence is analogous to the institutional hierarchy. They provide an example where institutional choice violates substitutes and unilateral substitutes with two divisions of unit-demand. These authors also obtain results that correspond to Theorems 1.1 and 1.2 and Proposition 1.2. It is also the case that the unilateral substitutes property cannot be preserved through this aggregation. Thus, bilateral substitutes is the strongest substitutability property that is preserved through inclusive hierarchical governance.

That the property of weak substitutes is preserved through aggregation leads naturally to the following result for the classical matching setting, since weak substitutes is a property that places conditions on choice in situations where no talent has more than one contract available.

Proposition 1.3. *If $X(k)$ is a classical contract set and if C^d satisfies Subs and IRC for all $d \in D(k)$, then C^k satisfies Subs and IRC.*

Another novel result of the inclusive hierarchical procedure is that SARP is preserved. Thus, in the baseline case where divisions are assumed to have preferences, the institutional choice can in fact be rationalized by some preference relation. Nevertheless, as shown in Aygün and Sönmez (2012a), there exist unilaterally substitutable choice functions that satisfy IRC and the law of aggregate demand that violate the SARP, and so if divisional choice was not generated by preferences it could well be that the institutional choice cannot be rationalized either.

Theorem 1.3. *The institutional choice function C derived from the inclusive hierarchical governance structure parametrized by $\rhd^k$ satisfies SARP if for every division $d \in D$, C^d satisfies SARP.*

The following is an example of a bilaterally substitutable and IRC choice function that cannot be decomposed into a sequential dictatorship of unit-demand divisions with strict preference relations. In fact, it cannot be non-trivially generated by an institution with at least two divisions with bilaterally substitutable choice functions.

Example 1.1. Suppose we have a choice function C defined as follows:

$$\begin{array}{llll}
 & C(\emptyset) = \emptyset & & \\
C(x) = x & C(x') = x' & C(z) = z & C(z') = z' \\
C(\{x, x'\}) = x & C(\{x, z\}) = x & C(\{x, z'\}) = x & \\
C(\{x', z\}) = z & C(\{x', z'\}) = \{x', z'\} & C(\{z, z'\}) = z & \\
C(\{x, x', z\}) = x & C(\{x, x', z'\}) = \{x', z'\} & C(\{x, z, z'\}) = x & C(\{x', z, z'\}) = \{x', z'\} \\
 & C(\{x, x', z, z'\}) = \{x', z'\} & &
\end{array}$$

Contracts x and x' are with talent t_x and contracts z and z' are with talent t_z.

Since x' and z' are selected from the largest offer set, one of these two contracts must be the highest priority (amongst contracts with these two talents) for the division with the highest rank that ever holds a contract with any one of these two talents. Without loss of generality, suppose it is x'. Then, since x' will always be picked by this division over any contract with talents t_x, t_z, if available, it must be that contract x' is never rejected. But this is not the case for choice function C, proving that this choice function cannot be generated by a sequential dictatorship of unit-demand divisions.

The key feature of this example is that $\{x', z'\}$ are complementary. This is illustrated by supposing there are two divisions d and d', where $d \rhd d'$, with preferences $\{x', z'\} \succ_d \emptyset$ and $x \succ_{d'} z' \succ_{d'} z \succ_{d'} x' \succ_{d'} \emptyset$; the institutional choice function is identical to C. However, in this case, the choice function of the first division does not satisfy bilateral substitutes (in fact, violates weak substitutes). Furthermore, there does not exist any *non-trivial* institution with at least two divisions that generates this choice function. Thus, we have shown that there exist bilaterally substitutable choice functions that cannot be generated from a non-trivial inclusive hierarchy with bilaterally substitutable divisions.

Proposition 1.4. *In the setting with classical contracts, if C^d satisfies substitutability and the LAD for every d $\in D$ and the set of acceptable talents $X(d)$ is the same for every division, then with inclusive hierarchical governance the derived choice function C satisfies substitutability and LAD.*

Proof. Let $Y \subseteq X$ and $z \in X \backslash Y$. Define $Z \equiv Y \cup \{z\}$. The first thing to note is that C_d satisfies IRC since it satisfies Subs and LAD. Thus, from Proposition 1.1, C satisfies IRC. Thus, if $x \notin C(Z)$, then $C(Z) = C(Y)$ and so the condition for LAD is satisfied. So, suppose $z \in C(Z)$. Now, consider the first division according to $\rhd$. If z is rejected, then the division chooses exactly the same contracts it would choose with z present, and so the cardinality of the set of contracts rejected by the division increases by exactly one, and the cardinality of the chosen set stays the same. If z is accepted, then by the Subs condition, every previously rejected contract remains rejected and by LAD the cardinality of the chosen set does not decrease. Thus, these restrictions imply that at most one previously chosen contract is now rejected due to the acceptance of z, and so the set of contracts rejected by the first division increases by at most one contract. Next, suppose that the set of contracts rejected increases by at most one for every division up to and including k. Then, the previous argument can be repeated to show that the set of rejected contracts increases by at most one, thereby demonstrating that that set of contracts which are unchosen using C increases in cardinality by at most one, and so we have that C satisfies LAD. $\square$

Corollary 1.1. *In the setting with classical contracts, if every division $d \in D$ has unit-demand with strict preferences and the set of acceptable talents is the same for every division, then C satisfies Subs and LAD.*

Proof. This follows from the observation that the condition of unit-demand with strict preferences induces a substitutable choice function for the division satisfying LAD, combined with the previous proposition. $\square$

1.3.3 On Markets and Hierarchies

With the results of the previous subsection, we know that an institution with an inclusive hierarchy will have a derived choice function that satisfies the properties of IRC and bilateral substitutability, amongst other properties. Consider now an economy with some set of institutions K, each of which is organized by an inclusive hierarchy of divisions, and some set of talents I and some set of contracts X. The key existence result for this economy is that the set of stable market outcomes, and so the core, is nonempty.

Theorem 1.4. *If for every institution $k \in K$ the choice functions C^d of every division $d \in D(k)$ satisfies IRC and bilateral substitutability, then the set of stable market outcomes is nonempty.*

Proof. By Theorem 1.2, we know that choice function of every institution satisfies IRC and bilateral substitutability. Then by Theorem 1 of Hatfield and Kojima (2010), the conditions of which are satisfied by the talent-institution matching economy, the set of stable outcomes is nonempty. $\square$

The existence of a market stable outcome means that there does not exist any group of talents and divisions that can find an arrangement each of them prefers that is institutionally stable. It may be the case that some talent and division wish to hold a contract with each other, but this does not block the market outcome because the institution to which the division belongs prevents such a block from being secure. As we shall see in the next subsection, it is a property of inclusive hierarchical governance that a market stable outcome exists, and not merely that there is an institutional governance structure, even though the presence of a governance structure can limit the types of blocks to market outcomes that might be possible.

1.3.4 Non-Hierarchical Conflict Resolution

With inclusive hierarchical governance in institution k, conflicts between divisions over contracts are resolved through hierarchical ranking $\rhd^k$, with division d obtaining a favorable resolution in any dispute with division d' if and only if $d \rhd^k d'$. In this subsection, I will consider a more flexible conflict resolution system, where conflicts over a particular contract are resolved in a manner that is dependent on the contract in question.

Fix an institution k and now suppose that there exists a collection $\left(\rhd^k_x\right)_{x \in X(k)}$ of linear order on $D(k)$. The role of any order $\rhd^k_x$ in the institutional governance is to determine which division can claim contract x in a conflict between two or more divisions. Given some choice situation $Y \subseteq X(k)$ and contract $x \in Y$, if for some distinct $d, d' \in D(k)$ with $d \rhd^k_x d'$, $x \in X(d) \cap X(d')$, and if $x \in C^d(Y) \cap C^{d'}(Y)$, then the divisions are in conflict over x. This conflict is resolved in favor of the division with the higher rank according to $\rhd^k_x$, which in this case is d, which means that an internal assignment f where x is assigned to d', $x \in f(d')$, and d would choose x given its assignment i.e. $x \in C^d(f(d) \cup \{x\})$ is a disputed assignment and so not internally stable.

Let ψ^k be an internally efficient governance structure parametrized by a flexible conflict resolution system $\left(\rhd^k_x\right)_{x \in X(k)}$. The requirement of internal efficiency, which is the condition that in any choice situation $Y \subseteq X(k)$ there is no feasible internal allocation g such that $g(d)\mathcal{R}^d f(d)$ for all divisions $d \in D(k)$ and $g(d)\mathcal{P}^d f(d)$ for some division $d \in D(k)$, where

$f \equiv \psi^k(Y, (\mathcal{P}^d)_{d \in D(k)})$.

Theorem 1.5. *Suppose the contracts is classical. If all divisions are unit-demand and the institutional governance structure ψ^k is internally efficient and has a flexible conflict resolution system, then the institutional choice satisfies IRC but can violate substitutability.*

Proof. For the institution k in question, let $Y \subseteq X(k)$ be the set of contracts available to it, and let $z \in X(k) \backslash Y$. Define $\hat{Y} \equiv Y \cup \{z\}$.

Given the hierarchical priority structure at situation Y, $\mathcal{H}(Y)$, we can use the hierarchical exchange mechanism ϕ with $\mathcal{H}(Y)$ to get an assignment of contracts to divisions μ by using the preferences of the divisions as an input to ϕ.

Some notation: I assume there is some fixed exogenous tie-breaking rule that determines the order in which cycles are removed in the situation where there are multiples cycles, so that only one cycle is removed per step, where such a rule always removes older cycles before younger ones. In particular, I use an exogenous ordering of the divisions to determine the ordering of cycles to be removed when there are multiple cycles at a step, where the cycles at a step are ordered for removal as follows. There is a queue for cycle removal. In every step, have all divisions point to their favorite available contract. Order all cycles that newly appear in this step by cycle-removal order and place it into the removal queue, where a new cycle enters the queue before another new cycle if it has a division in the cycle that is cycle-removal-smaller than every agent in the other cycle. Then, remove in this round the the cycle at the front of the queue. Update the control rights of any contracts whose previously controlling division has been assigned and removed. Go to the next step.

Note that in every step, if the queue as any cycles remaining, one cycle is removed, though it is not the case that in every step new cycles are created. However, in any step where the queue is empty at the beginning of the step, a new cycle must be created if there are any divisions remaining. Let $T(Y)$ be number of steps for all divisions to be assigned or removed.

Let $(\gamma_t(Y))_{t \in T(Y)}$ be the sequence of trading cycles realized by the mechanism when the set of available contracts is Y. Then, $C(Y) \equiv \bigcup_{t \in T(Y)} X(\gamma_t(Y))$. Also, $(\gamma_t(Y))_{t \in T(Y)}$ determines the internal allocation μ_Y.

Now, let us study what occurs when a new contract z is introduced. Since the hierarchical priority structure is contract-consistent, every contract $y \in Y$ has the same division controlling it in $\mathcal{H}(Y)$ and $\mathcal{H}(\hat{Y})$. Let d be the division that controls z at $\hat{Y}$.

To demonstrate that C satisfies IRC, we will assume that $z \notin C(\hat{Y})$ and prove that $C(\hat{Y}) = C(Y)$. Given that $z \notin C(\hat{Y})$, $z \notin \gamma_t(\hat{Y})$ for any $t \in T(\hat{Y})$. Since the only way that z is removed from the assignment procedure is by removal via a trading cycle and since a division that does not have z in its domain of interest is not allowed to point to it, we know that no division could have pointed to z at any step. Thus, in every step, contracts pointed to remains the same as it did in situation Y, and so $T(\hat{Y}) = T(Y)$ and $\gamma_t(\hat{Y}) = \gamma_t(Y)$. Thus, $C(\hat{Y}) = C(Y)$, proving IRC.

To show that substitutability can be violated, consider the following example. Suppose three divisions 1, 2, and 3 with preferences: $w\mathcal{P}^1 y\mathcal{P}^1\emptyset$, $x\mathcal{P}^2 z\mathcal{P}^2\emptyset$, and $x\mathcal{P}^3 w\mathcal{P}^3\emptyset$. Suppose that the priority structure is $1 \rhd_x 3 \rhd_x 2$, $2 \rhd_y 3 \rhd_y 1$, $2 \rhd_z 3 \rhd_z 1$, and $1 \rhd_w 2 \rhd_w 3$. For this problem, with $Y \equiv \{x, y, z\}$, we have that $C(Y) = \{x, y\}$, but with $\hat{Y} \equiv Y \cup \{w\}$, we have $C(\hat{Y}) = \{w, x, z\}$. The problem here is that the introduction of a new contract can make some division worse off, because the new contract can result in the loss of access to a contract that that division used to get through trading, as a consequence of the partner to that trade leaving earlier, and the inheritor of the desired contract not being interested in trading with the division in question. $\square$

As demonstrated in the counterexample, the problem with more flexible conflict-resolution together with the goal of efficiency is that the resolution process might not be consistent in the way it treats a division in terms of its welfare. Even a three-way trading cycle can lead to this non-harmonious welfare impact of an extra contract opportunity, and possibly lead to complementarity of choice at the institutional level.

1.4 Take-it-or-leave-it Bargaining

Towards an understanding of the impact of strategic behavior by talents and by institutional actors, consider a multi-stage game form $\mathbb{G}$, where each talent makes a take-it-or-leave-it offer of a set of contracts to an institution in the first stage, and institutions choose contracts which to accept in the second stage, with the final outcome being determined by these institutional choices. I will focus on Subgame Perfect Nash Equilibria (SPNE).

While it is certainly the case that the take-it-or-leave-it assumption places a great deal of the bargaining power in the hands of the talents, it is also worth recognizing that this bargaining power is mitigated by the presence of talent competition in the first stage, enhanced by the possibility of making offers that have multiple acceptable contracts, and so effective bargaining power of any particular talent is endogenous. We shall see that the set of outcomes realizable in SPNE are pairwise stable when institutions have an inclusive hierarchical governance structure.

It is possible that SPNE outcomes are unstable, though pairwise stable. The equilibria of such outcomes feature a coordination failure on the part of talents and an institution, due to the complementarities that are present even in bilaterally substitutable preferences of a division. With a strengthening of conditions on institutional choice to include the Pareto Separable condition, introduced by Hatfield and Kojima (2010), I obtain the stronger result of stability of SPNE outcomes. More generally, restrictions on division preferences that ensure equivalence between pairwise stability and stability ensure that SPNE outcomes are stable. This is the case when all divisions have substitutable preferences, even though the derived institutional choice fails substitutability.

There exists a literature on non-revelation mechanisms and hiring games like the take-it-or-leave-it game studied here. Alcalde (1996) studied the marriage problem using such a game form, and showed that the set of (pairwise) stable outcomes can be implemented in undominated Nash Equilibria. Alcalde et al. (1998) study a hiring game in the Kelso-Crawford setting with firms and workers where firms propose salaries for each worker in the first stage, and workers choose which firm to work given the proposed salaries. In this firm-offering take-it-or-leave-it game, they obtain implementability of the stable set in Subgame Perfect Nash Equilibria. Under the assumption of additive preferences, they show that in the worker-offering version of the hiring game, the worker optimal stable outcome is implementable in SPNE. Alcalde and Romero-Medina (2000) show SPNE implementability of the set of stable outcomes for the college admission model using the two-stage game form with students proposing in the first stage. In Sotomayor (2003) and Sotomayor (2004), the author provides SPNE implementation results for the pairwise stable set of the marriage model and the many-to-many matching (without contracts) model. Finally, Haeringer and Wooders (2011) study a sequential game form, where firms (which have capacity one) are proposers and workers can accept or reject offers, with acceptance being final, and show that in all SPNE the outcome is the worker optimal stable outcome.[21]

The side that moves first in the two-stage game has a material impact on the stability of the outcome of the game. Stability is a group rationality concept, and tests for the presence of groups of agents that can be made better off by a coordinated alternative action. When talents propose, a deviation by a worker cannot be coordinated in the SPNE solution concept, and so at most the talent and an institution (via a division) is involved in altering the outcome. In games where colleges or firms propose (see Alcalde and Romero-Medina (2000) and Alcalde et al. (1998), respectively), a deviation by a college or firm can involve a group of workers, since many "offers" can be change in a deviation. Thus, it is not surprising that SPNE outcomes of a college- or firm-proposing bargaining game are stable without any assumptions on preferences, but outcomes of a student- or worker-proposing game are only pairwise stable for this domain. Obtaining stability in this latter version requires a strengthening of assumptions to identify stability with pairwise stability.

The distinction between the college admissions model and the Kelso-Crawford model is also important to understand the implementation results in the literature. In the latter

[21] They also show that if workers make decisions simultaneously, then the set of SPNE outcomes expands to include all stable outcomes and possibly some unstable ones as well.

model, the presence of a salary component, or more abstractly of multiple potential contracts between a firm-worker pair, means that implementability should not be expected, given that as first movers the workers/talents can take advantage of their proposing power to "select out" less preferred stable outcomes. In my setting, given the weak assumptions on preferences, stability under SPNE cannot be assured, though pairwise stability can. However, for the stronger condition of Pareto Separable preferences, together with the Weak Substitutes and IRC conditions, stability of SPNE outcomes is assured, a novel result considering the weakened domain.

Throughout this section, assume that we have a hierarchical matching problem $\mathcal{E} \in \mathbb{E}^{\mathcal{H}}$, where divisions have preferences instead of merely choice functions. Also, assume that all divisions have bilaterally substitutable preferences. Suppose the game is one of complete information, so that the preferences of talents, contract sets, preferences of divisions, and the institutional hierarchies are common knowledge amongst the talents and divisions. The formal description of the game $\mathbb{G}(\mathcal{E})$ is as follows. There are two stages, the Offer stage (Stage 1) and the Internal Choice stage (Stage 2). The players are the set of talents I and the set of divisions $D \equiv \bigcup_{k \in K} D(k)$. In Stage 1, the Offer stage, every talent simultaneously makes one offer to one institution i.e. the action ω_i taken by a talent i is an element of $\Omega_i \equiv X(i)$. Let h_0 be the history of the game *at the end* of the Offer stage. Then, if $\omega \equiv (\omega_i)_{i \in I}$ is the action profile at the Offer stage, $h_0 \equiv (\omega)$.

In Stage 2, divisions choose amongst the contract offers to their institutions. Define $\omega_k \equiv X(k) \cap \bigcup_{i \in I} \omega_i$ to be the set of offers made to institution k. For each $k \in K$, label divisions in $D(k)$ according to the linear order $\rhd^k$, so that $d^k_m \rhd^k d^k_n$ if and only $m < n$, where $m, n \in \{1, \ldots, |D(k)|\}$ and $d^k_m, d^k_n \in D(k)$. Define $\mathcal{G}^k(\omega)$ to be the *internal choice game* amongst divisions $D(k)$ of institution k given offers $\omega \in \Omega \equiv \prod_{i \in I} \Omega_i$. This internal choice game is a sequential game with $|D(k)|$ rounds from 1 to $|D(k)|$, where the player at round n is $d^k_n \in D(k)$ and takes action λ^k_n. Let $h^k_1 \equiv h_0$ be the history at the start of the internal choice game and let h^k_n be the history of play at the start of round n, where

$h_m^k \equiv \left(h_{m-1}^k, \lambda_{m-1}^k\right)$. The action that a division takes is to choose a subset of contracts from the available set of contracts at round n. Define $\Lambda_1^k(h_1^k) \equiv \omega_k$ and

$$\Lambda_{n+1}^k(h_{n+1}^k) = \Lambda_{n+1}^k((h_n^k, \lambda_n^k)) \equiv \Lambda_n^k(h_n^k) \backslash \left(\bigcup_{i' \in I(\lambda_n^k)} X(i') \right),$$

where $\Lambda_1^k(h_1^k)$ is the set of offers available to division d_1^k in round 1 and $\Lambda_n^k(h_n^k)$ is the set of offers available to division d_n^k in round n given the history of play h_n^k. Thus, the action λ_n^k is an element of $2^{\Lambda_n^k(h_n^k)}$, the action space for d_n^k. Finally, for any two distinct institutions k and k', I shall treat the internal choice games $\mathcal{G}(k)$ and $\mathcal{G}(k')$ as independent of each other.[22]

Given the list of actions a, where

$$a \equiv \left((\omega_i)_{i \in I}, \left(\left(\lambda_n^k\right)_{n=1}^{n=|D(k)|} \right)_{k \in K} \right),$$

the outcome of the game $\mathbb{G}(\mathcal{E})$ is a set of contracts $A(a) \equiv \bigcup_{k \in K} \bigcup_{n=1}^{n=|D(k)|} \lambda_n^k$. A strategy for a division $d_n^k \in D(k)$, denoted σ_n^k, is a map from the set of all possible histories at round n in the second stage, $H_n^k \equiv \{h_n^k\}$, to the *feasible* set of actions $\Lambda_n^k(h_n^k) \subseteq X(k)$. Let Σ_n^k be the set of all strategies for division d_n^k. A strategy for a talent i, denoted σ_i, is a map from $\prod_i \Omega_i$ to Ω_i. Let Σ_i be the set of all strategies for talent i. Define the strategy space Σ by

$$\Sigma \equiv (\Sigma_i)_{i \in I} \times \left(\left(\Sigma_n^k\right)_{n=1}^{n=|D(k)|} \right)_{k \in K}.$$

Every strategy profile $\sigma \in \Sigma$ induces a *path of play* $a(\sigma)$, which is a list of actions of each talent and division, and an *outcome* $A(\sigma) \equiv A(a(\sigma))$.

[22]To be completely strict, an extensive game formalization of the second stage would require some specification of how rounds of an institution's internal choice game relates to the rounds of another's, and might therefore allow for the strategy of a division in one institution to depend on the choice of a division in another institution. The assumption of these internal choice games as being independent of each other is tantamount to analyzing a strict formalization with one division per round with a restriction of the class of strategies allowed. However, given the focus on subgame perfection, this restriction will not have a material impact on the equilibrium outcomes. An alternative formalization would be to model all institutional choice games occurring simultaneously, but with each choice game being sequential.

A strategy profile $\sigma \in \Sigma$ is a Subgame Perfect Nash Equilibrium (SPNE) if

- for every division d_n^k and every $\tilde{\sigma} \in \Sigma_n^k \times \{\sigma_{-d_n^k}\}$, it is the case that $A(\sigma) R^d A(\tilde{\sigma})$ at every history $h_n^k \in H_n^k$.
- for every talent i and every $\tilde{\sigma} \in \Sigma_i \times \{\sigma_{-i}\}$, it is the case that $A(\sigma) R^i A(\tilde{\sigma})$.

Since every list of talent offers induces a subgame for the divisions in each institution, we will first study the internal choice game induced by a particular list of offers $\omega \in \Omega$. The internal choice game induced by a hierarchical governance structure gives each division a unique weakly dominant strategy to choose at each realization of history its preference maximizing set of offers, taking ω as a parameter. Once ω is endogenized by embedding the internal choice game into the two-stage bargaining game, the unique weak dominance of this strategy remains. Denote this dominant strategy by $\hat{\sigma}_n^k$, where for any history $h_n^k \in H_n^k$,

$$\hat{\sigma}_n^k(h_n^k) = \max_{\mathcal{P}^{d_n^k}} \Lambda_n^k(h_n^k).$$

Moreover, requiring subgame perfection eliminates the use of any other strategy in equilibrium. Therefore, the divisions actions and the final outcome of the internal choice game $\mathcal{G}^k$ corresponds with the internal allocation and institutional choice produced by the inclusionary hierarchical procedure.

Lemma 1.2. *In any SPNE of $\mathbb{G}$, the strategy of any division d_n^k is $\hat{\sigma}_n^k$. For any SPNE σ^*, $\mathcal{G}^k(\omega)$ yields the outcome $C^k(\omega_k)$, where $\omega \equiv \prod_{i \in I} \sigma_i^*$.*

Proof. At any history $h \in H_n^k$, division d_n^k can determine its contracts in the outcome of the game by its choice from the available offers $\Lambda_n^k(h)$, no matter what subsequent actions are taken by other players. Therefore, the unique best response of d_n^k at history h is to choose the action of that corresponds to picking its preference-maximizing bundle from $\Lambda_n^k(h)$, which is exactly the prescribed action according to strategy $\hat{\sigma}_n^k$.

Since in SPNE every division takes the action of choosing its most preferred bundle of contracts, the outcome at this equilibrium coincides with the revelation mechanism induced by the institutional governance structure *qua* mechanism ψ^k given ω, which is strategyproof, and immediately yields the conclusion that the internal choice game $\mathcal{G}^k$ at ω reproduces the derived institutional choice function $C^k(\omega_k, \psi^k(\omega^k, (\mathcal{P}^d)_{d\in D(k)}))$, denoted $C^k(\omega_k)$ for simplicity, where $(\mathcal{P}^d)_{d\in D(k)})$ are the true preferences of divisions in $D(k)$. $\square$

The previous lemma justifies the reduction of the second stage in the subsequent propositions to a list of choice functions C^k. The interpretation is that with the inclusionary hierarchical governance, the internal game amongst divisions can be separated from the game between talent and institutions as a whole, given the focus on SPNE.

The first result will be to demonstrate pairwise stability of the outcome in SPNE. Note that the proof, and hence the result, does not require any assumption on preferences of divisions (and would only require the assumption of IRC on institutional choice if this choice is taken to be the primitive).

Proposition 1.5. *Let $\sigma^* \in \Sigma$ be an SPNE of the bargaining game $\mathbb{G}$ and let $a(\sigma^*)$ be the associated equilibrium actions and $A(\sigma^*)$ be associated equilibrium outcome. Then $A(\sigma^*)$ is pairwise stable.*

Proof. We know from lemma 1.2 that in SPNE, the subgame at any talent strategy profile ω, $\mathcal{G}^k(\omega)$ yields as the outcome the institutional choice function C^k derived from the inclusionary hierarchical procedure. That is, for any $(\sigma_i)_{i\in I} \in \prod_i \Sigma_i$, the outcome of the subgame at history $h_0 = (\omega)$ is exactly $C^K(h_0) \equiv \bigcup_{k\in K} C^k(\omega_k)$. The game $\mathbb{G}$ is thereby reduced to a simultaneous game amongst the talent.

Now, suppose that the SPNE outcome $A(\sigma^*)$ is not pairwise stable. Then there exists $i \in I, k \in K$ and $z \in X(i,k)\backslash A(\sigma^*)$ such that $z \in C^k(A(\sigma^*)\cup\{z\})$ and $z \in C^i(A(\sigma^*)\cup\{z\})$. Suppose talent i were to deviate from offering σ_i^* to offering z. Then, since C^k satisfies IRC, $z \in C^k(A(\sigma^*) \cup \{z\})$ and $\sigma_i^* \notin C^k(A(\sigma^*) \cup \{z\})$ implies $z \in C^k\left((A(\sigma^*) \cup \{z\})\backslash\sigma_i^*\right)$, and so $z \in A((\tilde{\sigma}_i, \sigma_{-i}^*))$, where $\tilde{\sigma}_i = z$. But then i strictly prefers the outcome from playing

$\tilde{\sigma}_i$ to playing σ_i^*, contradicting our assumption that σ^* is SPNE. Thus, $A(\sigma^*)$ is pairwise stable. $\square$

Subgame perfection is not strong enough to ensure stability of outcomes because talents can fail to "coordinate" with their proposed contracts, as described in the following example.

Example 1.2. Suppose there are two talents Ian i and John j and an institution Konsulting Group k. Let x and x' be two potential contracts between Ian and Konsulting, and let y and y' be two potential contracts between John and Konsulting. Imagine, perhaps, that contracts x and y stipulate working on the East Coast and contracts x' and y' stipulate working on the West Coast. Suppose Ian prefers the West Coast contract to the East Coast contract, as does John i.e. $x'\mathcal{P}^i x \mathcal{P}^i \emptyset$ and $y'\mathcal{P}^j y \mathcal{P}^j \emptyset$. Also, suppose that Konsulting Group is composed of just one division d, which would like to hire at least one of Ian or John in either geographical region, but does not want to hire both in *different* regions: $\{x', y'\}\mathcal{P}^d\{x, y\}\mathcal{P}^d x \mathcal{P}^d y \mathcal{P}^d x' \mathcal{P}^d y' \mathcal{P}^d \emptyset$. While other talents and institutions may be present, they are not required to demonstrate the "coordination failure" amongst talents; assume that no other talents are acceptable to Konsulting Group and that Ian and John are unacceptable to every other institution $k' \neq k$. Suppose in the non-cooperative bargaining game described above Ian offers only contract x and John offers only contract y, and suppose the one division in Konsulting Group chooses according to its preference, which it has a weakly dominant strategy to do. Then both x and y are chosen, and moreover are SPNE strategies for each talent, since Ian cannot improve by offering x' instead of (or as well as) x, given that John is offering only y, and vice versa. Notice also that the division's preferences satisfy bilateral substitutes, and that $\{x, y\}$ is pairwise stable but not stable. The only stable outcome is $\{x', y'\}$, which constitutes another SPNE outcome, supported for example by Ian offering x and John offering y. Both Ian and John prefer the equilibrium outcome $\{x', y'\}$ to $\{x, y\}$, but cannot unilaterally prevent the less-preferred outcome. In fact, even the division prefers $\{x', y'\}$ to $\{x, y\}$, and so SPNE outcomes can be inefficient.

When viewing institutional choice as primitive, stability of SPNE outcomes can be recovered by strengthening the assumptions on these choice functions. Suppose that every institution has a choice function satisfying IRC, bilateral substitutes and the Pareto Separable condition. Now, SPNE outcomes are stable and not just pairwise stable.

The power of the Pareto Separable condition comes from the property that the set of contracts between an institution and a talent now has a structure that is independent of the set of contracts with other talents available to the institution. A pair of contracts on which the institution and the talent have opposing choice behavior in some choice situation will never be harmonized in some other choice situation. This property is satisfied by substitutable choice, but is not a characteristic of it, since bilaterally substitutable choice functions that are not substitutable can still be Pareto Separable.

Proposition 1.6. *Suppose institutional choice functions are Pareto Separable and satisfy IRC and weak substitutes. Then every SPNE outcome is stable.*

The proof of the proposition lies in the recognition that under the assumption of bilateral substitutes and Pareto Separability, every group block can be reduced to an appropriate pairwise block, and thus every pairwise stable outcome is also stable. In fact, we can weaken the assumption from bilateral substitutability to weak substitutability, because these two substitutes conditions are equivalent given the Pareto Separable condition, stated in Proposition 1.1.

The equivalence of stability concepts under the Pareto Separable condition is the key lemma to the proof of stability of SPNE outcomes, and can be understood by recognizing that a block of an outcome that involves a contract between an institution and talent who have a contract with each other in the blocked allocation, a *renegotiation*, can be reduced to a block by just this contract. Similarly, any group block that does not have a renegotiation cannot involve more than one contract, if bilateral substitutability is to remain inviolate. But then any block can be reduced to a singleton block, and so stability is equivalent to pairwise stability.

Lemma 1.3. *Suppose institutional choice functions are Pareto Separable and satisfy IRC and weak substitutes. Then the set of stable outcome coincides with the set of pairwise stable outcomes.*

Proof. It is clear that every stable outcome is pairwise stable, by definition. To prove the converse, suppose A is pairwise stable. Assume that A is not stable. Then there exists an institution k and $Z \subseteq X(k)\backslash A(k)$ such that $Z \subseteq C^k(A \cup Z)$ and $Z(i)\mathcal{P}^i A(i)$ for every $i \in I(Z)$, and such that no $Z' \subsetneq Z$ has this same blocking property as Z. We say that such a Z is a *minimal* blocking group. We will show that $|Z| = 1$, contradicting the assumption that A is not pairwise blocked.

First, suppose that there exists $z \in Z$ such that the talent $I(z)$ has a contract with k in A i.e. $I(z) \in I(A(k))$. Let $y \in A(k)$ be the contract between $I(z)$ and k in A that is renegotiated via the block Z. Since $z \in C^k(A(k) \cup Z)$ and $y \in A(k)$, from the Pareto Separable condition we have that $y \notin C^k(A(k) \cup \{z\})$. Now, suppose $z \notin C^k(A(k) \cup \{z\})$. Then, by IRC we know that $C^k(A(k) \cup \{z\}) = C^k(A(k)) \ni y$, a contradiction. Thus, $z \in C^k(A(k) \cup \{z\})$, which implies that $\{z\}$ blocks A. Given that Z is a minimal blocking set, this implies $Z = \{z\}$ and so A is not pairwise stable, a contradiction.

Second, suppose that for every $z \in Z$, talent $I(z)$ does not have a contract with k in A i.e. $I(z) \notin I(A(k))$. Suppose that there exist $z, x \in Z$ where $z \neq x$. Clearly, $I(z) \neq I(x)$ given IRC and the assumption that a talent-institution pair can sign at most one contract in an allocation. Define $Y = A(k) \cup (Z\backslash\{z, x\})$. Since Z is a minimal block, $z \notin C^k(Y \cup \{z\}) = C^k(A(k))$ where the equality follows from IRC. However, $z \in C^k(Y \cup \{z, x\}) = C^k(A(k) \cup Z)$ by definition of a block. However, given that $I(z), I(x) \notin I(A(k))$ and since $|A(k)| = |I(A(k)|$, this block would violate assumption that C^k satisfies weak substitutes. Thus, Z must contain no more than one contract and so A is not pairwise stable, a contradiction.

Thus, we have proved that every pairwise stable outcome is stable. $\square$

Hence our proof of Proposition 1.6 is an immediate application of our previous results.

Proof. From Proposition 1.5 we have that every SPNE outcome is pairwise stable. From Lemma 1.3 we have that every pairwise stable outcome is stable. $\square$

Another result is that the SPNE outcomes of the bargaining game are stable under the assumption that all divisions have substitutable preferences. Given the discussion of the previous section that substitutability of preferences of divisions does not ensure substitutability or even unilateral substitutability of institutional choice, this result proves stability of the non-cooperative bargaining game outcomes for this class of bilaterally substitutable institutional choice functions. Note that the following proposition does not following from Proposition 1.6, because the property of Pareto Separability need not be preserved by inclusionary hierarchical procedures.

Proposition 1.7. *Suppose that every division has substitutable preferences. Then every SPNE outcome of the game $\mathbb{G}$ is stable.*

The proof of the proposition follows immediately given the following lemma.

Lemma 1.4. *Suppose every division has substitutable preferences. Then every pairwise stable outcome is stable.*

Proof. Let $A \subseteq X$ be a pairwise stable outcome. Suppose that there exists a blocking set $Z \subseteq X \backslash A$ involving institution k, so that $Z \subseteq C^k(A(k) \cup Z)$ and $z\mathcal{P}^{I(z)}A(I(z))$ for every $z \in Z$. Under the inclusionary hierarchical procedure, every contract in Z is allocated divisions in $D(k)$. Denote by f the internally stable allocation given choice situation $A(k)$ and by g the internally stable allocation given the choice situation $A(k) \cup Z$ i.e. $f \equiv \psi^k \left(A(k), (\mathcal{P}^d)_{d \in D(k)}\right)$ and $g \equiv \psi^k \left(A(k) \cup Z, (\mathcal{P}^d)_{d \in D(k)}\right)$. Let $\hat{d}$ be highest-ranked division to obtain one or more contracts from Z, define as follows: $Z \cap g(\hat{d}) \neq \emptyset$ and for every $d \rhd^k \hat{d}$, $Z \cap g(d) = \emptyset$. We will show that there exists some contract $\hat{z} \in Z$ such that $\hat{z}$ constitutes a pairwise block of A, contradicting the opening assumption.

Let $\hat{z} \in Z' \equiv Z \cap g(\hat{d}) \neq \emptyset$. By definition no division $d \rhd^k \hat{d}$ is allocated a contract in Z in choice situation $A(k) \cup Z$. Also, none of the talents with contracts in Z have alternative contracts in A that are allocated under g to any division higher-ranked than $\hat{d}$, since feasibility of the internal allocation would then prevent any such talent's contract in Z being chosen by the institution. We know that for every division $d \triangleright^k \hat{d}$ $g(d) = f(d)$ by IRC of division choice, trivially satisfied since divisions have preferences. In fact, IRC yields another conclusion, that $g'(d) = f(d)$ for every $d \triangleright^k \hat{d}$, where $g \equiv \psi^k \left(A(k) \cup \{\hat{z}\}, (\mathcal{P}^d)_{d \in D(k)} \right)$. Consider also that when the inclusionary hierarchical procedure determines the allocation from $A(k) \cup Z$ for $\hat{d}$, every contract that is available at this stage when the choice situation for the institution is $A(k)$, call it A', is still available for $\hat{d}$ in the expanded choice situation $A(k) \cup Z$. By IRC of division's choice, we know that removing contracts in Z that are not in Z' has no effect on choice of $\hat{d}$. By substitutability of division's choice, we know that $\hat{z} \in C^{\hat{d}}(A' \cup Z')$ implies $\hat{z} \in C^{\hat{d}}(A' \cup \{\hat{z}\})$. But then $\hat{z} \in C^k(A(k) \cup \{\hat{z}\})$, and so $\hat{z}$ blocks A, which contradicts the assumption of pairwise stability of A, and concludes the proof. $\square$

An implementation result analogous to some in the literature, however, is not forthcoming, as the following example shows. The difficulty with achieving implementation in SPNE in a setting with multiple potential contracts between the two sides and with talents offering first is that there is very little competition over institutions, since talents do not make offers to more than one institution. This gives a lot of bargaining power to the talents, and makes it so that any bilateral "surplus" consistent with stability goes to the first mover, the talents.

Example 1.3. Suppose there is one institution k trivially consisting of one division d and three talents i_x, i_y, i_z, where the choice function of the division is given as follows:

$$
\begin{array}{ccc}
Y \to C(Y) & Y \to C(Y) & Y \to C(Y) \\
\{x\} \to \{x\} & \{x, y\} \to \{x, y\} & \{x, y'\} \to \{x, y'\} \\
\{y\} \to \{y\} & \{x, z\} \to \{x, z\} & \{y, y'\} \to \{y'\} \\
\{z\} \to \{z\} & \{y, z\} \to \{y, z\} & \{y', z\} \to \{y', z\} \\
\{y'\} \to \{y'\} & & \\
\{x, y, z\} \to \{x, y, z\} & \{x, y', z\} \to \{x, y'\} & \{x, y, y', z\} \to \{x, y, z\}
\end{array}
$$

with contract x belonging to i_x, contracts y and y' to i_y, and contract z to i_z.

Suppose preferences of the three agents are: $xP^{i_x}\emptyset$, $yP^{i_y}y'P^{i_y}\emptyset$ and $zP^{i_z}\emptyset$. The choice function satisfies BLS and IRC, and is (for example) consistent with the following preferences:

$$\{x,y,z\}\mathcal{P}^d\{x,y'\}\mathcal{P}^d\{y',z\}\mathcal{P}^d\{x,y\}\mathcal{P}^d\{y,z\}\mathcal{P}^d\{x,z\}\mathcal{P}^d\{y'\}\mathcal{P}^d\{y\}\mathcal{P}^d\{x\}\mathcal{P}^d\{z\}\mathcal{P}^d\emptyset$$

for the division.

There is only one stable allocation $A_1 \equiv \{x,y,z\}$, which is also pairwise stable. However, $A_2 \equiv \{x,y'\}$ is also pairwise stable, though unstable.

Note that A_2 cannot be supported as a SPNE of the game $\mathbb{G}$, because t_y could strictly improve by offering y instead of y', keeping fixed the offers of other talents, which must be x by t_x and could be either z or $\emptyset$. If t_z is offering z, then if t_y offers y the division picks $\{x,y,z\}$. If t_z is offering $\emptyset$, then the division picks $\{x,y\}$. Thus, A_2 cannot be an SPNE outcome.

1.5 Conclusion

Stability has proven to be an important requirement that market outcomes should satisfy if the market is to function well. Using a matching-theoretic model, in this paper I show how hierarchies as a governance mode in institutions might persist in the market as a result of choice behavior that ensures stable market outcomes, a property that is not shared by some other organizational modes within institutions.

The novel approach complements existing theories for the presence of hierarchies in institutions in a market setting. Hierarchies induce institutionally efficient and strategyproof internal assignment rules while also producing market-level choice behavior that ensures stability. An important departure taken in this paper from the standard matching with contracts framework is that institutions are groups of decision-makers enjoined by a governance structure, which is modeled as an internal assignment rule. The decentralized market, studied as a noncooperative take-it-or-leave-it bargaining game, supports the conclusion that market outcomes will be pairwise stable generally, and stable under the assumption of substitutable preferences for divisions.

While the focus of this paper is on hierarchical governance within institutions, other governance structures could be considered, especially ones that allow for multiple internally stable assignments. Broadly speaking, the institutions could be thought of as competing allocation systems, with talents selecting into a particular institution. The framework allows for the study of buyer-seller relationships with institutional rules that may vary by jurisdiction. An axiomatic approach to this problem is a topic of ongoing research.

1.6 Proofs

Definition 1.8. *Given a combinatorial choice function C with domain X, define the Blair relation $\succsim^R$ as follows: for any $A \subseteq X$, $B \subseteq X$, $A \succsim^R B$ if $A = C(A \cup B)$. Let $\succ^R$ be the asymmetric component of $\succsim^R$.*

The proofs of the main results (Theorems 1.1, 1.2, 1.3 and Proposition 1.2) are obtained by a simple induction argument, given the results below.

For the following proofs, let C_1 and C_2 be choice functions defined on some domain X, where $I(x)$ is the talent associated with contract $x \in X$. Let $C_1 \rightarrowtail C_2$ denote the choice function derived from the inclusionary hierarchical procedure, where division 1 ranks higher than division 2.

Proposition 1.8. *Suppose C_1 and C_2 satisfy IRC. Then $C \equiv C_1 \rightarrowtail C_2$ satisfies IRC.*

Proof. Let $Y \subseteq X$ and $x \notin Y$ such that $x \notin C(\hat{Y})$, where $\hat{Y} \equiv Y \cup \{x\}$. Then $x \in C_1(\hat{Y})$ and so $C_1(\hat{Y}) = C_1(Y)$, since C_1 satisfies IRC. If $I(x) \in I(C_1(Y))$, then $x \notin \tilde{R}_1(\hat{Y})$ implying $\tilde{R}_1(\hat{Y}) = \tilde{R}_1(Y)$ and so $C_2(\tilde{R}_1(\hat{Y})) = C_2(\tilde{R}_1(Y))$. Thus, $C(\hat{Y}) = C_1(\hat{Y}) \cup C_2(\tilde{R}_1(\hat{Y})) = C_1(Y) \cup C_2(\tilde{R}_1(Y)) = C(Y)$, so IRC is satisfied in this case.

Instead, if $I(x) \notin I(C_1(Y))$, then $x \in \tilde{R}_1(\hat{Y})$. Now, since $x \notin C(\hat{Y})$, it must be that $x \in C_2(\tilde{R}_1(\hat{Y}))$ and since $\tilde{R}_1(\hat{Y}) = \tilde{R}_1(Y) \cup \{x\}$, IRC of C_2 implies $C_2(\tilde{R}_1(\hat{Y})) = C_2(\tilde{R}_1(Y) \cup \{x\}) = C_2(\tilde{R}_1(Y))$, implying $C(\hat{Y}) = C(Y)$ and establishing that C satisfies IRC.

□

Proposition 1.9. *Suppose C_1 and C_2 satisfy SARP. Then $C \equiv C_1 \rightarrowtail C_2$ satisfies SARP.*

Proof. Assume that C violates SARP, in order to obtain a contradiction. Given that SARP implies IRC, we know that C_1 and C_2 satisfy IRC. Then from Proposition 1.8 we know that C satisfies IRC. Finally, from Alva (2012) we know that if C satisfies IRC it satisfies WARP. So, if C violates SARP but not WARP, there exists a sequence $X_1, \ldots, X_n, X_{n+1} = X_1$, with $n \geq 3$, such that $Y_{m+1} \succsim^R Y_m$ for all $m \in \{1, \ldots, n\}$ and $Y_{l+1} \succ^R Y_l$ for at least one l, where $Y_m \equiv C(X_m)$ and $\succsim^R$ is the previously defined Blair relation associated with C. To see the connection between the condition in the definition of SARP and the Blair relation, notice that the cycle condition for SARP requires $Y_m \subseteq X_{m+1}$. Now, by IRC we get $Y_{m+1} = C(X_{m+1}) = C(Y_{m+1} \cup Y_m)$, which means that $Y_{m+1} \succsim^R Y_m$.

Next, define $a_m \equiv C_1(X_m) = C_1(Y_m)$, where the latter equality follows from IRC, define $b_m \equiv C_2(\tilde{R}_1(X_m))$, where $\tilde{R}_1(X_m) \equiv \{x \in X_m : I(x) \notin I(C_1(X_m))\}$. Notice that $b_m = Y_m \backslash a_m$ and that $a_m \cap b_m = \emptyset$. Also, for any $Z \subseteq X$, $a_m \succsim_1^R Z$, where $\succsim_1^R$ is the Blair relation generated by C_1. Since $a_m \subseteq X_m$ and $a_m \subseteq X_{m+1}$, and $a_{m+1} \subseteq X_{m+1}$, we have that $a_{m+1} \succ_1^R a_m$ or $a_{m+1} = a_m$. However, given that C_1 satisfies SARP, we cannot have $a_{m+1} \succsim_1^R a_m$ for all m and $a_{l+1} \succ_1^R a_l$ for some l. Thus, for any m, $a_m = a_{m+1}$.

Now, define $Z_m \equiv \tilde{R}_1(X_m)$. Notice that $b_m \subseteq Z_m$. Moreover, since $a_m = a_{m+1}$ and $b_m \cap a_m = \emptyset$, we have that $b_m \cap a_{m+1} = \emptyset$ and so $b_m \subseteq Z_{m+1}$. However, this means $b_{m+1} \succsim_2^R b_m$, where $\succsim_2^R$ is the Blair relation generated by C_2. Given that C_2 satisfies SARP, an analogous argument to the one in the previous paragraph, given for C_1, applies here and allows us to conclude that $b_m = b_{m+1}$ for any m. But then $Y_m = Y_{m+1}$ for all m, contradicting our assumption of a choice cycle. Thus, $C \equiv C_1 \rightarrowtail C_2$ satisfies SARP.

□

Proposition 1.10. *Suppose C_1 and C_2 satisfy IRC and WeakSubs. Then $C \equiv C_1 \rightarrowtail C_2$ satisfies IRC and WeakSubs.*

Proof. We have already proved that C satisfies IRC under the given assumptions.

Let $Y \subseteq X$ such that $|I(Y)| = |Y|$. Let $x \in X \backslash Y$ and $I(x) \notin I(Y)$ and $z \in X \backslash Y$, $z \neq x$, $I(z) \notin I(Y \cup \{x\})$. Suppose $z \notin C(Y \cup \{z\})$. If $x \notin C(\hat{Y})$, where $\hat{Y} \equiv Y \cup \{z, x\}$, then by IRC of C_1 and C_2, and hence of C, we have that $C(\hat{Y}) = C(Y \cup \{z\})$ implying $z \notin C(\hat{Y})$. Instead, suppose $x \in C(\hat{Y})$. Now, $z \notin C(Y \cup \{z\})$ implies $z \notin C_1(Y \cup \{z\})$. By IRC of C_1, $x \notin C_1(\hat{Y})$ implies $z \notin C_1(\hat{Y})$, so, given $I(z) \notin I(Y \cup \{z\})$, $z \in \tilde{R}_1(\hat{Y})$. If $x \in C_1(\hat{Y})$, then $x \notin \tilde{R}_1(\hat{Y})$. Moreover, by WeakSubs of C_1, for any $y \notin C_1(Y \cup \{z\})$, $y \notin C_1(\hat{Y})$. Thus, given that there is no more than one contract per talent in the available sets, if $y \in \tilde{R}_1(Y \cup \{z\})$, then $y \in \tilde{R}_1(\hat{Y})$. Thus, by WeakSubs and IRC of C_2, given that $z \notin C_2(\tilde{R}_1(Y \cup \{z\}))$, it must be that $z \notin C_2(\tilde{R}_1(\hat{Y}))$. Finally, if $x \notin C_1(\hat{Y})$, then $\tilde{R}_1(\hat{Y}) = \tilde{R}_1(Y \cup \{z\}) \cup \{x\}$ and so again IRC and WeakSubs of C_2 implies $z \notin C_2(\tilde{R}_1(\hat{Y}))$. Thus, C satisfies WeakSubs.

□

Proposition 1.11. *Suppose C_1 and C_2 satisfy IRC and BLS. Then $C \equiv C_1 \rightarrowtail C_2$ satisfies IRC and BLS.*

Proof. We have already proved that C satisfies IRC under the given assumptions.

Let $Y \subseteq X$, $x, z \in X \backslash Y$, $I(x) \neq I(z)$, $I(x), I(z) \notin I(Y)$. Suppose $z \notin C(Y \cup \{z\})$. Define $\hat{Y} \equiv Y \cup \{z, x\}$.

In the first case, suppose $x \notin C(\hat{Y})$. Then $x \notin C_1(\hat{Y})$. Since $I(x) \notin I(Y \cup \{x\})$, $x \in \tilde{R}_1(\hat{Y})$. By IRC of C_1, $z \notin C_1(\hat{Y})$ and $I(z) \notin I(Y \cup \{z\})$ implies $z \in \tilde{R}_1(\hat{Y})$. Thus, $\tilde{R}_1(\hat{Y}) = \tilde{R}_z(Y \cup \{z\}) \cup \{x\} = \tilde{R}_1(Y) \cup \{z, x\}$. Now, we know that $z \notin C_2(\tilde{R}_1(Y \cup \{z\}))$ and so by BLS of C_2, $z \notin C_2(\tilde{R}_1(\hat{Y}))$. Thus, $z \notin C_1(\hat{Y}) \cup C_2(\tilde{R}_1(\hat{Y})) = C(\hat{Y})$, proving that C satisfies the BLS condition for this case.

In the second case, suppose $x \in C(\hat{Y})$. In the first subcase, suppose $x \in C_1(\hat{Y})$. By BLS of C_1, $z \notin C_1(\hat{Y})$. Since $I(z) \notin I(Y \cup \{x\})$, $z \in \tilde{R}_1(\hat{Y})$. Moreover, by BLS of C_1, if $y \in \tilde{R}_1(Y \cup \{z\})$ and $I(y) \notin C_1(Y \cup \{z\})$ then $y \in \tilde{R}_1(\hat{Y})$, keeping in mind that $I(y) \neq I(x)$. Thus, $\tilde{R}_1(\hat{Y}) \supseteq \tilde{R}_1(Y \cup \{z\})$ and $I(z)$ has only one contract in $\tilde{R}_1(\hat{Y})$. Now if for all $y \in \tilde{R}_1(\hat{Y}) \backslash \tilde{R}_1(Y \cup \{z\})$, we have that $y \notin C_2(\tilde{R}_1(\hat{Y}))$, then IRC implies $z \notin C_2(\tilde{R}_1(\hat{Y}))$. Instead, if $y \in C_2(\tilde{R}_1(\hat{Y}))$ then by IRC we have $y \in C_2(\tilde{Y} \cup \{y\})$, where $\tilde{Y} \equiv \tilde{R}_1(\hat{Y}) \backslash \{w \in \tilde{R}_1(\hat{Y}) : I(w) = I(y)\}$. But now, since $I(y) \neq I(\tilde{Y})$ and since $|\tilde{Y}(I(z))| = 1$, BLS of C_2 implies that $z \notin C_2(\tilde{Y} \cup \{y\})$ and so by IRC $z \notin C_2(\tilde{R}_1(\hat{Y}))$. Thus, $z \notin C(\hat{Y})$.

In the second subcase of the second case, suppose $x \notin C_1(\hat{Y})$. Since $x \in C(\hat{Y})$, it must be that $x \in C_2(\tilde{R}_1(\hat{Y}))$. By IRC of C_1, we have that $\tilde{R}_1(\hat{Y}) = \tilde{R}_1(Y \cup \{z\}) \cup \{x\} = \tilde{R}_1(Y) \cup \{z, x\}$. By BLS of C_2, we have $z \notin C_2(\tilde{R}_1(Y) \cup \{z\})$, implying $z \notin C_2(\tilde{R}(Y) \cup \{z, x\}) = C_2(\tilde{R}_1(\hat{Y}))$ and so $z \notin C(\hat{Y})$.

Having established that $z \notin C(\hat{Y})$ in every case, we have that C satisfies BLS.

$\square$

1.7 Appendix: The Comparative Statics of Combinatorial Choice

Fix a choice function. For any set of contracts Y, let $R(Y)$ be the set of contracts rejected from Y and $C(Y)$ the set of contracts chosen from Y, and let $I(Y)$ be the set of talents with contracts in Y. Let A be the current set of contracts available, and let a be a contract not in A. Define $\hat{A} \equiv A \cup \{a\}$.

- The condition *NewOfferChosen* (NOC) is satisfied if and only if the following is true: $a \in C(\hat{A})$.

- The condition *NewOfferFromNewTalent* (NOFNT) is satisfied if and only if the following is true: $I(a) \notin I(A)$.

- The condition *NewOfferFromHeldTalent* (NOFHT) is satisfied if and only if the following is true: $I(a) \in I(C(A))$.

- The condition *NewOfferFromRejectedTalent* (NOFRT) is satisfied if and only if the following is true: $I(a) \notin I(C(A))$.

- The condition *RenegotiateWithHeldTalent* (RWHT) is satisfied if and only if the following is true: $\Big(\exists x \in R(A), x \in C(\hat{A}) \wedge I(x) \in I(C(A))\Big)$.

- The set $\mathcal{RRT}$ is the set of talents rejected at A but recalled at $\hat{A}$, excepting the talent making the new offer i.e. $\mathcal{RRT} \equiv (I(A) \backslash I(C(A))) \cap I(C(\hat{A}))$.

- The condition *RecallRejectedTalent* (RRT) is satisfied if and only if the following is true: $\Big(\exists x \in R(A), x \in C(\hat{A}) \wedge I(x) \notin I(C(A))\Big)$. Equivalently, RRT is satisfied if and only if $\mathcal{RRT} \neq \emptyset$.

- The set $\mathcal{RHT}$ is the set of talents held at A but rejected at $\hat{A}$, excepting the talent making the new offer i.e. $\mathcal{RHT} \equiv I(C(A)) \cap \Big(I(A) \backslash I(C(\hat{A}))\Big)$.

- The condition *RejectHeldTalent* (RHT) is satisfied if and only if the following is true: $\Big(\exists i \in I(C(A)), i \notin I(C(\hat{A}))\Big)$. Equivalently, RHT is satisfied if and only if $\mathcal{RHT} \neq \emptyset$.

- The condition *UnitarySet* (UnitS) is satisfied if and only the following is true: $|I(A)| = |A|$.

Let A be a subset of contracts and $a \notin A$, with $\hat{A} \equiv A \cup \{a\}$.

1. A choice function fails IRC if $\neg$*NewOfferChosen* and (*RejectHeldTalent* or *RecallRejectedTalent* or *RenegotiateWithHeldTalent*).

2. A choice function fails ParSep if *RenegotiateWithHeldTalent*.

3. A choice function fails ULS if *RecallRejectedTalent*.

4. A choice function fails BLS if *NewOfferFromNewTalent* and *RecallRejectedTalent*.

5. A choice function satisfies Subs if and only if it is never the case that *RenegotiateWith-HeldTalent* or *RecallRejectedTalent* is true.

6. A choice function fails WS if (IRC or *UnitarySet*) and *NewOfferFromNewTalent* and *NewOfferChosen* and ¬*RenegotiateWithHeldTalent* and *RecallRejectedTalent*.

For a summary of these comparative statics results, see Table 1.1.

New Offer Chosen: $a \in C(A \cup \{a\})$

		Recall Rejected Talent	***¬Recall Rejected Talent***
New Offer From New Talent: $I(a) \notin I(A)$	***Renegotiate With Held Talent***	Fails ParSep Fails ULS Fails BLS	Fails ParSep
	¬Renegotiate With Held Talent	Fails ULS Fails BLS IRC or UnitS $\Longrightarrow$ Fails WS	
New Offer From Held Talent: $I(a) \in I(C(A))$	***Renegotiate With Held Talent***	Fails ParSep Fails ULS	Fails ParSep
	¬Renegotiate With Held Talent	Fails ULS	
New Offer From Rejected Talent: $I(a) \in I(A) \backslash I(C(A))$	***Renegotiate With Held Talent***	Fails ParSep Fails ULS IRC $\Longrightarrow$ Fails BLS	Fails ParSep
	¬Renegotiate With Held Talent	Fails ULS IRC $\Longrightarrow$ Fails BLS IRC $\Longrightarrow$ Fails WS	

Table 1.1: Categorizing Choice Behavior where A is initially available and $a \notin A$ is a new contract offer

1.8 Appendix: Concepts of Stability

An allocation $A \in \mathcal{A}$ is **pairwise stable** (or **contractwise stable**) if it is individually stable and there does not exist a contract $x \in X \backslash A$ such that $x \in C^{K(x)}(A \cup \{x\})$ and

$x \in C^{I(x)}(A \cup \{x\})$.

An allocation $A \in \mathcal{A}$ is **renegotiation-proof** if it is individually stable and there does not exist $k \in K$ and $Y \subseteq X(I(A(k)), k) \backslash A$ such that $Y \subseteq C^k(A \cup Y)$ and $Y(j) \in C^j(A \cup Y)$ for every $j \in I(Y)$. This notion of stability rules out allocations where an institution and some subset of agents with which it holds contracts have alternate contracts amongst themselves that they would all choose over their current contracts if available. Thus, renegotiation-proof allocations are intra-coalitionally efficient.

An allocation $A \in \mathcal{A}$ is **strongly pairwise stable** if it is individually stable, renegotiation-proof, and there does not exist an agent-institution pair $(i, k) \in I \times K$ that have no contract with each other in A i.e. $A \cap X(i, k) = \emptyset$, a contract $x \in X(i, k$, and a collection of contracts $Y \subseteq X(I(A(k)), k) \backslash A(k)$ such that $Y \cup \{x\} \subseteq C^{K(x)}(A \cup Y \cup \{x\})$ and $x \in C^{I(x)}(A \cup \{x\})$ and $Y(j) \in C^j(A \cup Y)$ for every $j \in I(Y)$. This notion of stability rules out blocks coming from an institution and agent without an existing relationship where the institution can renegotiate with some agents with which it has an existing relationship. It is an enjoining of the renegotiation-proof concept and of the pairwise stable concept.

Note that the strongly pairwise stable outcomes need not be stable, because a blocking set of contracts in the latter concept can include more than one agent that does not have a held contract with the blocking institution (where w.l.o.g. there is one blocking institution). However, if all divisions have choice functions that satisfy BLS and IRC, then every strongly pairwise stable outcome is also stable.

Proposition 1.12. *If choice functions satisfy BLS and IRC, then the strongly pairwise stable set is equivalent to the stable set.*

Proof. Every stable outcome is strongly pairwise stable, so we shall prove the converse, and do so by contradiction. Suppose A is strongly pairwise stable but not stable. Since it is not stable, there exists an institution k, a subset of talents $J \subseteq I$, and a collection of contracts $Z \subseteq X \backslash A$ where every contract in Z involves k and some talent in J and no two distinct contracts in Z name the same talent, such that for every $j \in J$, $Z(j) \mathcal{P}^j A(j)$ and $Z \subseteq C^k(A \cup Z)$. This set of contracts Z blocks A. Without loss of generality, let us suppose that Z is a *minimal* blocking set i.e. there does not exist $Z' \subseteq Z$ such that $Z' \subseteq C^k(A \cup Z')$. Given that A is strongly pairwise stable, we also know that there exists at least two talents $i_1, i_2 \in J$ who do not have contracts in A with institution k. Let $z_1 \equiv Z(i_1)$ and $z_2 \equiv Z(i_2)$, and define $Y \equiv Z \backslash \{z_1, z_2\}$. Since Z is a minimal blocking set, we know that $Y \cap C^k(A \cup Y) = \emptyset$ and $(Y \cup \{z_1\}) \cap C^k(A \cup Y \cup \{z_1\}) = \emptyset$, so $z_1 \notin C^k(A \cup Y \cup z_1)$. But since $Z \subseteq C^k(A \cup Z)$, it must be that $z_1 \in C^k(A \cup Z)$. However, implies that C^k violates bilateral substitutes, since z_1 and z_2 are contracts with distinct talents who do not have any contracts with k in $A \cup Y$, which is a contradiction. $\square$

This result is the counterpart to the well-known result on pairwise stability and stability under the assumption of substitutability, stated here for completeness.

Result 1.1. *In the classical matching model, the set of pairwise and strongly pairwise stable allocations is identical. Moreover, if choice functions satisfy substitutability and IRC, then the set of stable matchings and the set of pairwise stable matchings coincide, and these sets coincide with the strongly pairwise stable set and the renegotiation-proof set.*

The following propositions document that the strong pairwise stability concept in the domain of BLS and IRC divisional choice functions is distinct from the weaker concepts of pairwise stability and renegotiation-proofness.

Proposition 1.13. *If choice functions satisfy BLS and IRC, then the pairwise stable set is distinct from the renegotiation-proof set, which is distinct from the strongly pairwise stable set.*

Proof. Consider the following example with one institution and three agents, where the choice function of the institution is given as follows:

$$\begin{array}{ccc}
Y \to C(Y) & Y \to C(Y) & Y \to C(Y) \\
x \to x & xy \to xy & xy' \to xy' \\
y \to y & xz \to xz & yy' \to y' \\
z \to z & yz \to yz & y'z \to y'z \\
y' \to y' & & \\
xyz \to xyz & xy'z \to xy' & xyy'z \to xyz
\end{array}$$

Suppose preferences of the three agents are: $xP_x\emptyset, yP_yy'P_y\emptyset$ and $zP_z\emptyset$. The choice function satisfies BLS and IRC, and is (for example) consistent with the following preferences:

$$xyz \succ xy' \succ y'z \succ xy \succ yz \succ xz \succ y' \succ y \succ x \succ z \succ \emptyset$$

for the institution. The set of stable allocations is

$$\{\{x, y, z\}\},$$

the set of strongly pairwise stable allocations is

$$\{\{x, y, z\}\},$$

the set of renegotiation-proof allocations is the set of all individually stable allocations, and the set of pairwise stable allocations is

$$\{\{x, y, z\}, \{x, y'\}\}.$$

$\square$

Finally, I show by example that under a notion of substitutability weaker than BLS, the notion of Weak Substitutes introduced in Hatfield and Kojima (2008), the equivalence between strong pairwise stability and stability is broken.

Proposition 1.14. *If choice functions satisfy WeakSubs and IRC, then the strongly pairwise stable set is distinct from the stable set.*

Proof. Consider the following example with one institution and three agents, where the choice function of the institution is given as follows:

$$\begin{array}{ccc}
Y \to C(Y) & Y \to C(Y) & Y \to C(Y) \\
x \to x & xy \to xy & xy' \to y' \\
y \to y & xz \to xz & yy' \to y' \\
z \to z & yz \to yz & y'z \to y' \\
y' \to y' & & \\
xyz \to xyz & xy'z \to y' & xyy'z \to xyz
\end{array}$$

Suppose preferences of the three agents are: $xP_x\emptyset, yP_yy'P_y\emptyset$ and $zP_z\emptyset$. The choice function satisfies Weak Subs and IRC, though it fails BLS, and is (for example) consistent with the following preferences:

$$xyz \succ y' \succ xy \succ yz \succ xz \succ y \succ x \succ z \succ \emptyset$$

for the institution. The set of stable allocations is

$$\{\{x, y, z\}\},$$

the set of strongly pairwise stable allocations is

$$\{\{x, y, z\}, \{y'\}\},$$

the set of renegotiation-proof allocations is the set of all individually stable allocations, and the set of pairwise stable allocations is

$$\{\{x, y, z\}, \{y'\}\}.$$

$\square$

Chapter 2

Pairwise Stability and Asymmetric Complementarity

2.1 Introduction

Matching theory has had great success in both the study of decentralized labor markets and in the design of real-world mechanisms for both centralized labor markets and indivisible goods allocation. Case studies of actual centralized mechanisms, evidence from laboratory experiments, and theoretical work has lent support for stability-type concepts, particularly pairwise stability, as the appropriate solution for matching models. The stronger notion of group-stability has also been studied, and has the appeal of being equivalent to the core concept in many-to-one matching models.

However, group stable outcomes are not guaranteed to exist in matching problems where at least one side can have multiple partners. In the setting of classical many-to-one matching, where the terms of the match between a pair of agents is of a fixed type, substitutability is the weakest preference requirement that ensures the existence of group stable matchings over a Cartesian domain of preferences. With this domain restriction, not only is existence assured, but also the set of group stable outcomes has a lattice structure, and in particular, for each side there is an outcome that is best amongst all group-stable outcomes. This restriction on the domain of preferences has another important consequence – the group-stable outcomes are the same as the pairwise-stable outcomes, and so the concept of group-stability has no refining power on the more basic concept of pairwise-stability. Therefore, pairwise stability, introduced together with the original matching framework by the ground-breaking paper Gale and Shapley (1962), is the key stability notion that unifies a large variety of matching models.[1]

The study of the stable set in markets with complementarities requires either restrictions on the preferences or the family of allowable relationships, both of which have been explored in recent work. I take a different approach by using the observation that every existence result for the stable set is for domains where the stable set coincides with the pairwise stable set. I argue that pairwise stability is a well-motivated solution concept, and I study the existence of pairwise stable matchings in problems where group stable matchings are not guaranteed to exist, focusing on particular classes of complementarity. An important finding is that the problem of existence of pairwise stable (and hence stable) matchings is connected to a type of complementarity that I term *subjectively asymmetric complements* (SAC). The presence of such complementarity in the choice of some firm implies that there can be otherwise regular markets where no pairwise stable matching exists. In a positive result, I show that if no firm exhibits SAC, then a pairwise stable matching is guaranteed to exist. Existence is demonstrated by use of a sequential-offer worker-proposing deferred acceptance algorithm.

Kelso and Crawford (1982) significantly generalized the early matching models by considering a many-to-one model with salaries (contracts) and with a weaker assumption on preferences, the gross substitutes condition, and using a stronger notion of stability, group stability, which they argue is the preferred notion because it is equivalent here to the core. However, it is well known that in a general many-to-one matching model (where preferences

[1] In their marriage model, these authors show that pairwise stable allocations exist for any profile of strict preferences of both sides of the market. This is also true for their college admissions model, a model of many-to-one matching, where preferences are assumed to be responsive with quotas.

are unrestricted), a pairwise stable allocation may not exist (and the core is therefore empty).

Echenique and Oviedo (2004) studies the Core of general many-to-one matching markets, and characterizes the Core as the set of fixed points of an operator. See also Roth and Sotomayor (1990) for an argument why pairwise stable matchings is the relevant solution concept in certain settings, particularly decentralized ones.

2.2 Model

2.2.1 The Elements

Let F be the set of firms and W be the set of workers, these sets being mutually exclusive. For any agent $i \in F \cup W$, the set P_i is the set of possible partners of agent i; the two-sidedness of this market is captured by the requirement that $P_f \subseteq W$ for any $f \in F$ and $P_w \subseteq F$ for any $w \in W$. Workers can only work for at most one firm, but firms can hire teams of workers, or none at all. A *matching* is a correspondence μ from $F \cup W$ to $F \cup W$ satisfying:

1. For all $f \in F$, $\mu(f) \subseteq P_f$,

2. For all $w \in W$, $\mu(w) \subseteq P_w$ and $|\mu(w)| \leq 1$.

Each worker w has strict preferences over his partners and over being unmatched, embodied by a complete, transitive, asymmetric binary relation $\succ_w$ on the set $\{A \subseteq 2^{P_f} : |A| \leq 1\}$, and is assumed to choose from any set of available firms by selecting the maximum from the set according to this preference relation; this choice function is denoted C_w. This asymmetric relation has an antisymmetric counterpart $\succsim_w$, where, for any $A, B \subseteq P_w$, $A \succsim_w B$ if and only $A \succ_w B$ or $A = B$.

Each firm f is assumed to have a choice process, the outcome of which is captured by a choice function $C_f : 2^{P_f} \to 2^{P_f}$, which necessarily satisfies, for any $A \in 2^{P_f}$, $C(A) \subseteq A$. For example, the firm f might have strict preferences over the collection of subsets of its partner set P_f, the maximization of which yields a chosen set from a given choice set.

2.2.2 Stability Concepts

A matching describes the active partnerships in the economy. However, some matchings have agents who have an incentive to dissolve one or more partnerships. For a worker, for example, being matched to a firm that is less preferred than having no partner violates individual rationality, because the worker could be better off by himself. Formally, a matching μ is *Individually Rational (IR)* if for all workers $w \in W$ if $\mu(w) \succsim_w \emptyset$, i.e. there is no worker w that *individually blocks* the matching μ. A firm, too, could have the incentive to dissolve a partnership. Given the underlying assumption that workers are independent actors, a firm can dissolve a relationship with one matched worker while maintaining its relationship with some other matched worker. This would be firm f's course of action at some matching μ if $C_f(\mu(f)) \neq \mu(f)$, which captures the idea that given the choice, the firm would choose a strict subset of its current set of partners; this is an *individual block* by the firm. A matching is *Individually Stable (IS)* if for every firm f, $C(\mu(f)) = \mu(f)$ and if it is IR i.e. there is no individual block by any agent.

Individual Stability captures the actions individual agents could unilaterally take to better themselves. However, a firm and a worker could also desire to create a partnership between each other, thereby upsetting a matching. Formally, a firm-worker pair $(f, w) \in F \times W$ is creates a *block* if $\{f\} \succ_w \mu(w)$ and $w \in C_f(\mu(f) \cup \{w\})$ i.e. firm f and worker w are a block of matching μ if worker w prefers f to its current match and is included in firm f's choice from the set consisting of its current partners together with w. A matching is *Stable* if it is IS and there is no block.

Another blocking concept that is relevant for our matching model is the group blocking notion used in connection with the *core* solution concept. Formally, a group of agents $J \subseteq F \cup W$ forms a *corewise block* of a matching μ via a matching $\mu' \neq \mu$ if 1) for every $i \in J$, $\mu'(i) \subseteq J$ and 2) for every $i \in J$, $\mu'(i) = C_i(\mu(i) \cup \mu'(i))$. A matching is a *Core* matching if it cannot be corewise blocked.[2]

[2] Note that the definition of a corewise block allows for some blocking agents to have the same partners

2.2.3 Conditions on Choice

We will assume throughout the analysis that every choice function satisfies the following consistency condition.

Definition 2.1 (Irrelevance of Rejected Partners). *A choice function $C_i : 2^{P_i} \to 2^{P_i}$ of an agent i with partner set P_i satisfies the* ***Irrelevance of Rejected Partners (IRP)*** *condition if for any $A, A' \subseteq P_i$, $C_i(A) \subseteq A' \subseteq A$ implies $C_i(A') = C_i(A)$.*

This condition has appeared in various guises in both the classical matching literature and in the literature on matching with contracts[3]. It is an essential requirement for stable matchings to exist, as discussed in Aygün and Sönmez (2012a).

As discussed in the introduction, without some restriction on choice, neither Stable nor Core matchings are guaranteed to exist. The following condition, Substitutability, has played a critical role both in the theory of matching and in the application of matching to market design problems.

Definition 2.2 (Substitutability). *A choice function $C_i : 2^{P_i} \to 2^{P_i}$ of an agent i with partner set P_i satisfies the* ***Substitutability (Subs)*** *condition if for all $j, k \in P_i$ and $A \subseteq P_i \backslash \{j, k\}$, $j \notin C_i(A \cup \{j\})$ implies $j \notin C_i(A \cup \{j, k\})$.*

Essentially, a choice function satisfies Subs if it is never the case that the addition of some new partner to the choice situation results in a rejected available partner now being chosen. Equivalently, if a chosen partner from a group of chosen partners becomes unavailable, it should not be the case that one (or more) of the still available and chosen partners is rejected.

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before and after the block, and so the core defined here is the strict Core. There is also the notion of a *groupwise block* of a matching μ by a set of agents J via μ', where for every $i \in J$, $\mu'(i)\backslash J \subseteq \mu(i)$ and $\mu'(i) \subseteq C_i(\mu(i) \cup \mu'(i))$. A matching is *Groupwise Stable (GWS)* if there exists no groupwise block. Given our many-to-one setting with no peer effects, it is straightforward to show that the set of GWS matchings is identical to the set of Core matchings. See Echenique and Oviedo (2004) for a proof.

[3]See Blair (1988), Alkan (2001), Alkan (2002), Alkan and Gale (2003), Fleiner (2003), Echenique (2007), and Aygün and Sönmez (2012a). See also Alva (2012), where this author shows that the IRP condition is equivalent to the Weak Axiom of Revealed Preference. Moreover, in the present context, the assumption of IRP for a worker's choice, given that no more than one partner is chosen, is equivalent to the assumption of a strict preference ordering.

Definition 2.3. *Workers w_1 and w_2 are* ***subjective complements*** *for firm f if there exists a choice situation $A \subseteq P_f$ where both workers are available ($w_1, w_2 \in A$) and worker w_1 is chosen from A ($w_1 \in C_f(A)$) but not chosen when the other worker w_2 is removed from the choice situation ($w_1 \notin C_f(A \backslash \{w_2\})$).*

Note that workers are subjective complements for a firm if and only if the firm's choice function does not satisfy the Subs condition. Thus, a choice function that has no subjective complements satisfies Subs. The key result of this paper is that not all complementarities are bad for existence. Next, I describe two particular classes of complementarity, with the goal of demonstrating that these are the problematic forms of complementarity.

Subjective Imperfect and Subjective Asymmetric Complements

I define two novel types of complementarity, Subjective Imperfect Complements and Subjective Asymmetric Complements.

Definition 2.4 (Subjective Imperfect Complements)**.** *Workers w_1 and w_2 are* ***Subjective Imperfect Complements (SIC)*** *for firm f if:*

1. *w_1 and w_2 are subjective complements for firm f and*
2. *there exists $Y' \ni w_1$ where $w_2 \in C_f(Y') \cap C_f(Y' \backslash \{w_1\})$.*

A choice function profile $(C_f)_{f \in F}$ satisfies ***No Subjective Imperfect Complements (NSIC)*** *if there does not exist any $f \in F$, $w_1, w_2 \in W$ such that w_1 and w_2 are SIC for f.*

If two workers w_1 and w_2 are Subjective Imperfect Complements for firm f, it means that there are some situations where gaining access to w_2 leads the firm to hire w_1 but that there are other situations where w_2 is hired despite w_1 being unavailable. The contrapositive can be stated as follows:

Definition 2.5. *Workers w_1 and w_2 are* ***Subjective Perfect Complements*** *for firm f if for any Y, $w_1 \in C_f(Y)$ if and only if $w_2 \in C_f(Y)$.*

This is a very particular type of complementarity, where some workers are considered to be valuable for a firm if and only if they are available as a team, and is a minimal weakening of the substitutes condition. In the next section, we shall see that it is relatively straightforward to show the existence of the Stable set if firm choice never contains Subjective Imperfect Complements.

The second type of complementarity is defined as follows.

Definition 2.6 (Subjective Asymmetric Complements)**.** *Workers w_1 and w_2 are* ***Subjective Asymmetric Complements (SAC)*** *for firm f if there exists $Y \subseteq P_f$ such that*

1. $w_1, w_2 \in C_f(Y)$,
2. $w_1 \notin C_f(Y \backslash \{w_2\})$ *and*
3. $w_2 \in C_f(Y \backslash \{w_1\})$.

A choice function profile $(C_f)_{f \in F}$ satisfies ***No Subjective Asymmetric Complements (NSAC)*** *if there does not exist any $f \in F$, $w_1, w_2 \in W$ such that w_1 and w_2 are SAC for f.*

If two workers w_1 and w_2 are Subjective Asymmetric Complements for firm f, it means that there are some situations where losing access to w_2 leads the firm to reject w_1 but losing access to w_1 (in the same situation) does not induce the firm to reject w_2, thereby treating the two workers differently. The contrapositive of the previous definition can be stated as follows:

Definition 2.7. *Workers w_1 and w_2 are* ***Subjective Symmetric Complements*** *for firm f if for any Y where $\{w_1, w_2\} \subseteq C_f(Y)$, $w_1 \notin C_f(Y \backslash \{w_2\})$ implies $w_2 \notin C_f(Y \backslash \{w_1\})$.*

Clearly, if w_1 and w_2 are Subjective Perfect Complements for f, they are Subjective Symmetric Complements, but the converse is not true (see Example 2.7 below).

2.3 The Theory of Stability with Complementarities

2.3.1 The Trouble with Complementarity

For any matching problem, there necessarily exists a matching that is IS. In fact, the empty matching, where no partnerships exist, is IS. If both firms and workers can have at most one partner then there exists a Stable[4] matching. In this class of problems, the Core coincides with the Stable set, and so the Core is nonempty. However, in the many-to-one class of problems, the Core can be empty (and distinct from the Stable set), as demonstrated in Examples 2.1 and 2.2. This also means that the set of Groupwise Stable matchings is empty (see footnote 2 for the definition of this stability concept).

Example 2.1 (Empty Core)**.** Suppose there are two firms f and f' and two workers w and w'. The preferences of workers are:

w: $f' \succ_w f \succ_w \emptyset$

w': $f \succ_{w'} f' \succ_{w'} \emptyset$

and the preferences of firms are:

f: $\{w, w'\} \succ_f \emptyset$

f': $w' \succ_{f'} w \succ_{f'} \emptyset$.

For notational convenience, I identify a singleton set with the element it contains. Teams of partners that are less preferred than being unmatched are omitted, for they have no impact on the set of Stable or Core matchings. It is straightforward to demonstrate that any choice function generated by the maximization of a strict preference relation over the available set will satisfy IRP.

There is no Core matching for this problem. If f were to be matched to $\{w, w'\}$, then $J = \{f', w\}$ forms a corewise block via any matching where they partner with each other

[4]Recall that throughout this paper, for the sake of brevity, stable refers to what is often known as pairwise stable in the literature.

exclusively. If f were to be matched with only one of w or w', then this matching would not be IS, because it would be individually blocked by f. Thus, f must be unmatched in any Core matching. Assume this. If f' were to be unmatched, then $J = \{f', w\}$ forms a corewise block via any matching where they partner with each other exclusively. If f' were to be matched with w, then $J = \{f', w'\}$ forms a corewise block via any matching where they partner with each other exclusively (we use the assumption that f is unmatched here). Finally, if f' were to be matched with w', then $J = \{f, w, w'\}$ forms a corewise block. Thus, all matching possibilities are exhausted, and the Core is empty.

Nevertheless, the Stable set is nonempty in Example 2.1, for the matching where f has no workers and f' has worker w' is Stable, because the only corewise block is $\{f, w, w'\}$, which does not form a (pairwise) block. Since every Core matching is necessarily Stable, this example demonstrates that the Core is generally a strict subset of the Stable set. Unfortunately, there exist matching problems where the Stable set is also empty.

Example 2.2 (Empty Stable set)**.** Consider the same problem in Example 2.1 but with firm f having the following preferences instead:

f: $\{w, w'\} \succ_f w \succ_f \emptyset$

All the blocks involving two agents described in Example 2.1 continue to be blocks here. Additionally, if f' were to be matched with w' and f were to be unmatched, then f and w form a block. If f were to be matched to w, then f and w' form a block via the matching that gives f both w and w'. Thus, no matching is immune to individual and blocks, and the Stable set is empty.

These negative results demonstrate that some restriction on the domain of the problem is required to ensure existence of these two solutions. The most general condition known to guarantee existence of the Core in the class of many-to-one matching problems is a condition that rules out complementarities between potential partners, the Substitutes condition (see Definition 2.2). The problem complementarity poses for existence is clear in Example

2.1, where firm f would like to have both workers or neither of them. In fact, as shown in Example 2.2, it does not help that firm f might be fine with one particular worker w if both are not available, since the other worker w' is not acceptable to the firm by himself, and thus w' is complemented by w.[5]. In a setting where firms are assumed to have preferences as primitives (yielding derived choice functions), the existence result states that the Core is nonempty if firms' derived choice functions satisfy Subs (workers' choice functions trivially satisfy this condition). The importance of the Subs condition for existence of the Core has been explored in many other models of matching, both many-to-one and many-to-many, where 1) agents match to each other directly (classical variety); 2) agents negotiate a salary in addition to being matched (salary variety); or 3) agents match to each other via arbitrary contractual relations (contracts variety). For almost all these settings, the many-to-one matching with contracts model being an exception, the Subs condition (with preferences as primitives) is necessary for guaranteed existence of the Core i.e. there is no Cartesian product domain of preferences that strictly contains this domain and guarantees existence of the Core.

As important as the Core concept may be, the assumption of Substitutability is clearly a strong one, limiting the scope of the existing theory to problems where no complementarities of any sort exist. Focusing on the weaker solution concept of Stability will not eliminate the difficulty of existence in the presence of some complementarities, as demonstrated in Example 2.2, but the question of whether there exists a Cartesian product domain of choice functions more general than Substitutable domain that guarantees existence of a nonempty Stable set has not been previously asked or answered. It is clear from Example 2.1 that complementarity does not automatically rule out the existence of a Stable matching, but this is also true about existence of a Core matching, as shown in Example 2.3.

[5]This problem posed by complementarities was recognized by Kelso and Crawford (1982), who proposed a condition on demand functions they called *Gross Substitutability* for a variant of the many-to-one matching model considered here that allows for salaries to be negotiated. Theirs is a quasilinear environment, which is unnecessary for their existence result (a fact recognized by Kelso and Crawford). In the many-to-many matching problem without salaries, the equivalent condition imposed on choice functions is Substitutability, introduced in Roth (1984b)

Example 2.3 (Nonempty Core does not imply Coincidence with the Stable Set)**.** Consider the same problem in Example 2.1 but with worker w having the following preferences instead:

w: $f \succ_w f' \succ_w \emptyset$

The matching μ where f is matched to w and w' and f' is unmatched is a Core matching, since all matched agents receive their most-preferred partner set. This shows that the complementarity of firm f's preferences does not automatically rule out existence of the Core. Note, however, that the coincidence of the Core with the Stable set is no longer true when complementarities exist; the matching where firm f' is matched to w' and the other agents are unmatched is Stable (though not in the Core), since it is Individually Stable and no block exists.

Even if there is a coincidence of the Core with the Stable set, the lattice structure that is present when all choice functions satisfy Subs is lost.

Example 2.4 (Core = Stable $\neq \emptyset$ does not imply Canonical Lattice Structure)**.** Suppose there are two firms f and f' and three workers x, y, and z. The preferences of firms are:

f: $\{x, y\} \succ_f z \succ_f \emptyset$

f': $z \succ_{f'} y \succ_{f'} x \succ_{f'} \emptyset$.

and the preferences of workers are:

x: $f \succ_x f' \succ_x \emptyset$

y: $f' \succ_y f \succ_y \emptyset$

z: $f \succ_z f' \succ_z \emptyset$

The Core for this problem contains two matchings, μ and μ':

$$\mu(f) = \{x, y\}, \quad \mu(f') = z$$

and

$$\mu'(f) = z, \quad \mu'(f') = y.$$

The Stable set coincides with the Core. The matching μ is the firm-optimal stable matching and μ' the firm-pessimal stable matching. However, there is no worker-optimal or -pessimal matching. Workers y and z prefer μ' to μ but worker x prefers μ to μ'. The firm-proposing simultaneous deferred acceptance algorithm produces matching μ and the worker-proposing simultaneous deferred acceptance algorithm produces matching μ' (see the sequel for the definition of these algorithms).

2.3.2 Deferred Acceptance Algorithms

I describe two variants of Deferred Acceptance algorithms, the worker-proposing simultaneous deferred acceptance algorithm (W-SimDAA algorithm) and the worker-proposing sequential deferred acceptance algorithm (W-SeqDAA algorithm). The first variant, W-SimDAA, is just a convenient renaming of the Gale-Shapley's student-proposing deferred acceptance algorithm (Gale and Shapley (1962)). The second variant, W-SeqDAA, is a formalization of the algorithm used by Dubins and Freedman (1981).

Simultaneous DAA

The *worker-proposing simultaneous deferred acceptance algorithm* (W-SimDAA) consists of a sequence of rounds, with each round consisting of two stages. In the first stage of every round, every worker $w \in W$ that is not currently *held* (was *rejected* in the previous round) by some firm proposes to his most preferred partner in its partner set P_w *to which he has not previously proposed*. In the second stage of every round, every firm $f \in F$ chooses (*holds*) its most preferred set of workers from amongst those it is holding from the previous round and those who have newly proposed in this round, *rejecting* the unchosen workers. Since in the first round no worker was previously rejected, every worker proposes to some acceptable firm (unless there are no firms that the worker finds acceptable). The algorithm terminates at the beginning of a round where all workers are held or have no more acceptable firms left to which to propose.

The W-SimDAA ends in a finite number of steps. This is because in any round where at least one worker is rejected, that worker will move down his list of acceptable firms and will never make a second offer to a firm that has rejected him. Since the number of firms is finite, the number of rounds has to be finite.

Next, consider the outcome of the W-SimDAA. It is feasible by construction, since no worker can be matched to more than one firm. Is it individually stable? Yes, since in every round, firms choose from amongst the union of the workers held from the previous round and the new offers, the choice from which, given IRP of choice functions, is an individually stable set. The outcome is individually rational for workers because no worker makes an offer to an unacceptable firm.

For arbitrary choice functions for firms (that satisfy IRP), there is no guarantee that the outcome of the W-SimDAA is Stable. Nevertheless, it is well-known (Kelso and Crawford (1982), Roth (1984b)) that under the assumption of Subs, the outcome is not only Stable, but also in the Core.

Sequential DAA

The *worker-proposing sequential deferred acceptance algorithm* (W-SimDAA), parametrized by an linear ordering $\rhd$ over the set of workers W, consists of a sequence of rounds, each round consisting of two stages. In the first stage of a round, the highest-ranked worker, according to $\rhd$, that is not *held* by a firm and has not yet proposed to every acceptable firm proposes to his most-preferred acceptable firm that has previously rejected him. In the second stage of a round, the firm which received a proposal in this round chooses its most preferred set of workers from amongst those it is holding from the previous round and the newly arrived worker, rejecting the unchosen ones. The algorithm terminates at the beginning of a round where all workers are held or have no more acceptable firms left at which to propose.

Fix an order $\triangleright$. The W-SeqDAA ends in a finite number of steps and always produces an Individually Stable matching, for the same reason that these properties are true of the W-SimDAA. If every firm's choice function satisfies Subs, then the outcome of the W-SeqDAA and of the W-SimDAA are equivalent (regardless of the ordering $\triangleright$), and is thus Stable and in the Core. Without Subs, however, Stability is not assured.

An Example Illustrating the Difference in Algorithms

Example 2.5. [6] Suppose there are three firms f, g, and h and three workers w_1, w_2, and w_3. The preferences of firms are:

f: $\{w_1, w_2\} \succ_f \emptyset$

g: $\{w_2, w_3\} \succ_g \emptyset$

h: $\{w_3, w_1\} \succ_h \emptyset$

and the preferences of workers are:

w_1: $g \succ_{w_1} f \succ_{w_1} h \succ_{w_1} \emptyset$

w_2: $g \succ_{w_2} f \succ_{w_2} h \succ_{w_2} \emptyset$

w_3: $f \succ_{w_3} h \succ_{w_3} g \succ_{w_3} \emptyset$

There are four pairwise stable matchings:

$$\mu_1(f) = \{w_1, w_2\}, \quad \mu_1(g) = \mu_1(h) = \emptyset$$

$$\mu_2(g) = \{w_2, w_3\}, \quad \mu_2(h) = \mu_2(f) = \emptyset$$

$$\mu_3(h) = \{w_3, w_1\}, \quad \mu_3(f) = \mu_3(g) = \emptyset$$

$$\mu_4(f) = \mu_4(g) = \mu_4(h) = \emptyset$$

[6] The following example was suggested by Hideo Konishi.

The Core is empty. The W-SimDAA produces μ_1. In the first round, workers w_1, w_2, w_3 propose to firms g, g, and f, respectively. Firm g rejects the proposals of both w_1 and w_2 and firm f rejects w_3. In the second round, workers w_1, w_2, w_3 propose to firms f, f, and h, respectively. Firm f accepts the proposal of workers w_1 and w_2, while firm h rejects the proposal of workers w_3. In the third round, worker w_3 proposes to firm g, which rejects him. At the beginning of the fourth round there are no workers who have any firms left to propose to so the algorithm terminates, with the matching μ_1 being realized.

On the other hand, the W-SeqDAA produces μ_4, regardless of the order $\rhd$ over workers. Notice that μ_4 is weakly Pareto dominated by each of μ_1, μ_2, and μ_3.

2.3.3 The Basic Theory

Illustrating NSIC and NSAC

To understand better the novel choice conditions of NSIC and NSAC, consider the following three examples.

Example 2.6. Suppose a firm f has the following preferences over the set of partners $P_f \equiv \{w_1, w_2, w_3\}$:

$$f\colon\ \{w_1, w_2, w_3\} \succ_f \{w_1, w_2\} \succ_f \{w_3\} \succ_f \emptyset.$$

Workers w_1 and w_2 are (the only) subjective complements. Since w_1 is chosen if and only if w_2 is chosen, the choice function of firm f satisfies NSIC (and so NSAC).

Example 2.7. Suppose a firm f has the following preferences over the set of partners $P_f \equiv \{w_1, w_2, w_3\}$:

$$f\colon\ \{w_1, w_2\} \succ_f \{w_1, w_3\} \succ_f \{w_2, w_3\} \succ_f \emptyset.$$

Any pair of workers are subjective complements for firm f. However, no pair of workers are Subjective Perfect Complements. For example, w_1 can be teamed with w_2 or w_3 (but not both), so firm f does not consider w_1 and w_2 to be acceptable only if hired together. Thus, C_f derived from $\succ_f$ failed the NSIC condition.

However, the NSAC condition is satisfied. For any team $\{w_i, w_j\}$, $i, j \in \{1, 2, 3\}$ and $i \neq j$, the removal of one member results in the rejection of the other. Finally, if all three workers are available, note that removal of w_1 does *not* result in the rejection of w_2, and vice versa, and removal of w_3 leaves the chosen team intact, illustrating that C_f satisfies the IRP condition.

Example 2.8. Suppose a firm f has the following preferences over the set of partners $P_f \equiv \{w_1, w_2, w_3, w_4\}$:

f: $\{w_1, w_2\} \succ_f \{w_1, w_3\} \succ_f \{w_2, w_3\} \succ_f w_4 \succ_f \emptyset$.

The restriction of the choice function C_f to the subset $\{w_1, w_2, w_3\}$ is equivalent to the choice function in Example 2.7, and so does not contain any violation of NSAC. Moreover, in situations where w_4 is available, he is not chosen unless one or fewer of the workers w_1, w_2, and w_3 is available, again implying that the complementary pairs are chosen in a symmetric fashion, even though no group of workers are subjective perfect complements.

Now, suppose instead that firm f had the following, slightly modified, preference ranking:

f: $\{w_1, w_2\} \succ_f \{w_1, w_3\} \succ_f w_4 \succ_f \{w_2, w_3\} \succ_f \emptyset$.

Consider choice situation $Y = P_f$. Both w_1 and w_2 are chosen from Y. However, while removal of w_1 from the choice set leads to w_2 being let go, the converse is not true, since $\{w_1, w_3\}$ is preferred to w_4. Thus, the choice condition NSAC is violated.

Results

The main result of this paper is the following existence theorem for Stable matchings.

Theorem 2.1. *If the choice function of every firm satisfies the NSAC condition, then the Stable set is nonempty.*

The proof of existence is shown by demonstrating that, for any given order $\triangleright$, the W-SeqDAA produces a matching that is Stable. However, as the following example shows, even if NSAC is satisfied, the Stable matching produced by W-SeqDAA is dependent upon the order $\triangleright$ over W.

Example 2.9 (Order Dependence of W-SeqDAA)**.** Even if every firms' choice function satisfies NSAC, the Stable matching produced by W-SeqDAA depends upon the order $\triangleright$. Suppose the preferences of the only firm f over partners $P_f \equiv \{w_1, w_2, w_3, w_4\}$ are:

$$f\colon\ \{w_1, w_2\} \succ_f \{w_3, w_4\} \succ_f w_1 \succ_f w_2 \succ_f w_3 \succ_f w_4 \succ_f \emptyset,$$

and all workers prefer f to being unmatched. There are two Stable matchings, $\mu_1(f) = \{w_1, w_2\}$ and $\mu_2(f) = \{w_3, w_4\}$. Suppose the order $\triangleright^1 : w_1, w_2, w_3, w_4$ is used with W-SeqDAA. Then matching μ_1 is produced. If instead the order $\triangleright^2 : w_4, w_3, w_2, w_1$ is used, the matching μ_2 is produced.

The next result relates the NSIC condition (which is stronger than NSAC) and the better-known version of the deferred acceptance algorithm, the W-SimDAA.

Theorem 2.2. *If the choice function profile of firms satisfies the NSIC condition, then the W-SimDAA yields a Stable matching.*

The following example demonstrates that the W-SimDAA produces an unstable matching even when a Stable (and a Core) matching exists, if the NSIC condition is violated.

Example 2.10. Suppose there are two firms f and g and three workers w_1, w_2, and w_3. The preferences of firms are:

$$f:\ \{w_1, w_2, w_3\} \succ_f \{w_1, w_2\} \succ_f \{w_2, w_3\} \succ_f \{w_3, w_1\} \succ_f \emptyset$$

$$g:\ w_1 \succ_g \emptyset$$

and the preferences of workers are :

w_1: $f \succ_{w_1} g \succ_{w_1} \emptyset$

w_2: $g \succ_{w_2} f \succ_{w_2} \emptyset$

w_3: $g \succ_{w_3} f \succ_{w_3} \emptyset$

The only Core matching is $\hat{\mu}$: $\hat{\mu}(f) = \{w_1, w_2, w_3\}$ and $\hat{\mu}(g) = \emptyset$. The matching $\tilde{\mu}(f) = \emptyset$ and $\tilde{\mu}(g) = w_1$ is the only other Stable matching. However, the W-SimDAA produces the matching μ': $\mu'(f) = \{w_2, w_3\}$ and $\mu'(g) = w_1$, which is blocked by firm f and worker w_1. Note that firm f's choice function violates the NSIC condition because in situation $Y = \{w_2, w_3\}$, $\{w_2, w_3\} \subseteq C(Y)$, but in situation $Y' = \{w_1, w_2\}$, $\{w_2, w_3\} \nsubseteq C(Y')$ and $\{w_2, w_3\} \cap C(Y') \neq \emptyset$.

With the W-SeqDAA (with any order), the Stable matching $\tilde{\mu}$ will be realized.

2.4 Conclusion

Complementarities are an important feature of many real-world markets, including market-design applications such as the National Residency Matching Program for doctors and hospitals, and supply-chain networks. Nevertheless, complementarities has always been theoretically difficult to deal with, and this is no exception in matching theory. In this paper, we have a positive theory of pairwise stable matchings for a new domain of preferences that allows for a class of complementarities called subjective symmetric complementarities. However, other than existence, few of the positive results from the theory under substitutable choice survive in the new domain.

Chapter 3

Electoral Competition and Social Influence Networks

3.1 Introduction

> *The greatest deception men suffer is from their own opinions.*
>
> Leonardo da Vinci, *The Notebooks of Leonardo da Vinci*

Consider the US presidential race in 2008 between the Republican candidate Senator John McCain and Democratic candidate Senator Barack Obama. The two main dimensions of the debate about the candidates' qualifications and abilities to be the president were national security and economic policy. McCain had strong national security credentials, both by virtue of individual achievements and by association with the Republican Party. Obama, on the other hand, was a more technocratic candidate particularly on economic matters, in no small part because of his affiliation with the Democratic Party. Given this framing of the debate between McCain and Obama as one about the relative importance of terrorism versus the economy, both McCain and Obama spent a significant amount of time and resources making and publicizing statements about what they believed to be the more important class of issues facing the country: McCain played up the importance of national security and the threat of terrorism, while Obama instead called attention to the importance of dealing with a failing economy.[1]

The candidates disagreed on what the true state of the world was, even though they both had access to a great deal of information, most of which came from the same sources. While some differences in their information could have lead to their different opinions, it is unlikely that these differences could have resulted is such vast differences in their opinions about the importance of the issue of terrorism relative to that of the economy. I argue that this difference-of-stated-opinion is the equilibrium outcome of a game between the candidates, where each candidate publicly states his opinion about the state of the world, commonly known to the candidates, and the population, each member of which has some private opinion, learns from others in the population and from the candidates' stated opinions.

I investigate the role of persuasion within social influence networks in generating policy polarization in electoral competition. Voters have single-peaked policy preferences and candidates are ideological, with single-peaked policy preferences as well. The impact of policy, however, is state-dependent, where the state of the world can be understood as the relationship between an underlying set of outcomes that voters actually care about and the manner in which particular policies translate into outcomes. Due to their uncertainty about the true state, voters update their opinions about the true state, learning only from their social neighbors, described by a social influence network. The candidates have targets within this network, which captures the media penetration of candidates, and thus influence voters directly through the media and indirectly through other voters.

Focusing on a symmetric setting, I find that a unique pure strategy equilibrium exists under some reasonable parameter restrictions, where in the first stage of the game the two competing parties choose policies and public opinions, and then in the second stage the population engages in social learning to update their opinions followed by a voting stage.

[1] McCain famously stated that "[T]he fundamentals of our economy are strong" in an attempt to downplay the importance of economic issues.

I show that there is a substitution effect at work between the persuasion strategy through public opinions and the policy position. The persuasion dimension is like a tug-of-war, because a party "pulls" the distribution of voters's ideal policies towards its ideological position, while the policy dimension has the usual moderation towards the median. Opening up this new dimension of competition (by lowering the cost of the persuasion channel) can result in a substitution away from moderation along the policy dimension, leading to greater policy polarization and persistent population differences in opinion about the state of the world. Homophily exacerbates these polarization effects by reducing competition for centrist votes.

A large fraction of the population are non-partisan in their preferences over potential candidates – members of the population have state-dependent preferences over the two candidates, where the state of the world includes such aspects as the condition of the economy and of foreign relations. I focus on this segment of the population, assuming the remaining members are ideological in their voting behavior, and thus unimportant in the candidates' strategic consideration of policy platform and public opinion. In particularly, there are some states of the world where a non-partisan voter would prefer the left-party policy but other states of the world where he would prefer the right-party policy, for any given pair of policies from the left and right party. Voters state-dependent preferences over policy are comonotonic, which means that for any pair of policies when the state increases no voter who preferred the more rightist policy would now prefer the more leftist policy. In the example of the competition between McCain/Republicans and Obama/Democrats, the commonly-held dimension of preferences is that as the economy becomes a bigger issue relative to terrorism, all voters would agree that Obama's leftist policy is a (weakly-) better that McCain's rightist policy.

Political parties expend a great deal of effort persuading voters to adopt particular beliefs about the true state of the world. A party will point to evidence supporting their assessment about the state of the world, and frequently the other party will proffer a different assessment about the state of the world, generally by pointing to other pieces of evidence (often through filtering out information that is not favorable).

When a voter becomes aware of the candidates' public opinion, the voter must account for the difference by considering the above three explanations. A rational Bayesian voter would recognize that political candidates have a strategic reason to present particular pieces of evidence or to espouse a particular belief about the unknown state. The extent to which a voter could extract information from the espoused belief of a political candidate depends, amongst other things, upon that voter's prior about the candidate's type. The information extraction problem would also require that the voter have a prior about the types of all the other voters, and furthermore require that the voter understand the formation of the equilibrium and hence the equilibrium bias in the candidates' espoused beliefs.

Alternatively, the voter could use some heuristic to determine how much he discounts the beliefs of the candidates, and in that manner attempt to correct for candidate bias in stated beliefs. This behavioral assumption weakens the conditions on both the amount of information about the structure of the world (preferences and types of other voters and of candidates) and the level of rationality required by a voter, a bounded rationality approach that I pursue in this paper.

The boundedly rational learning model is quite flexible and intuitive, because with multi-dimensional opinions, different dimensions can have different weights. Consider a particular topic of interest, for example the health effects of genetically-modified foods. Some people have an *a priori* interest in the topic while others less so. Moreover, given that topic, a person has an associated influence network, which indicates how much the person trusts the opinion of another. These influence networks reflect the history of previous interactions and the local knowledge the person has about the expertise of her network neighbors. The opinion of a friend with knowledge about or training in genetics is given more credence that the opinion of a friend with no background in the subject, but this latter friend's opinion may well be given more credence than the opinion of a completely unknown geneticist. I do not build a general theory of such influence networks, but rather assume the existence of such networks in order to study the evolution and long-run state of opinions about a given topic, particularly when some agents are strategic in their choice of opinions. To be more clear, I assume that some agents can freely "choose" their opinions, while others use the local averaging heuristic to form their opinions. The agents that choose their opinions are deemed strategic, while those using the heuristic capturing the idea of persuadability are deemed boundedly rational or behavioral. For the purposes of this paper, the strategic agents are two political parties, and the behavioral agents are a voting populace.

Robinson (1976) examines the two-step flow hypothesis in a national election campaign. The two-step flow hypothesis posits that ideas flow from mass media to opinion leaders, and then from opinion leaders to less interested sections of the population, and is the paradigm that is modeled in this paper, using social influence networks. Pattie and Johnston (1999) furnish evidence from the 1992 British Election Study demonstrating political discussion with partisans influences the vote of undecided voters. The importance of social networks for decision-making in a variety of settings has been well-documented and studied. For example, the diffusion of microcredit in India has been shown to depend crucially on the structure of the social network (Banerjee, Chandrasekhar, Duflo, Jackson (2012)).

Banks (1990) analyzes electoral competition when voters are uncertain about the policy the winning candidate will enact i.e. the policy position announcement is not assumed to be carried out. Essentially, his paper constructs a signaling model that yields a weakening of the convergence result of the standard median voter model. Along different lines, Roemer (1997) demonstrated that policy convergence need not occur when there is uncertainty about the median voter's policy ideal and when candidates are motivated by policy rather than the spoils of office.

Glaeser et al. (2005) explain that extremism in the messages of candidates arise when affiliates of a candidate's party have a higher probability of learning about the platform than other citizens, and as a result deviation from the median towards the position of the party affiliates energizes the party base, therefore increasing turnout of loyalists more than the opposing party's base. That the deviation is less likely to be detected by non-affiliates implies that the opposing party base is relatively less energized.

Galeotti and Mattozzi (2011) is the only paper I am aware of that examines the effect of homophily in social networks on electoral competition.[2] They consider a citizen-candidate model with two parties, where voters are uncertain about the type of the candidates representing each party. Parties can choose to advertise truthfully their candidate's type, but pay a cost convex in the share of the population reached. Voters then randomly sample a finite subset of the population for any information on candidate types, update their beliefs in a Bayesian manner and then vote. In mixed-strategy equilibria, they find that the probability that a more extreme candidate wins can decrease when the cost of advertising decreases.[3]

This paper differs in a number of crucial ways. Most importantly, I study the importance of the structure of the social network by explicitly modeling a social learning model, albeit one where individuals are boundedly rational. The power of this framework is that many questions related to communication structure and media can be handled in a tractable and realistic manner. Secondly, the nature of informational problem is different in the two models. I focus on the role of a second dimension of competition, the role of opinion and "spin", in shaping the perceptions of policy efficacy, and thus altering voting behavior. This allows me to simultaneously answer questions relating to polarization in party platforms and polarization in population beliefs about the state of the world. For example, there is a lot of disagreement about the impact of government debt levels on economic growth, and thus the size of fiscal stimulus through government spending. An individual's policy preference depends not just on the policy devoid of context, but on the theory of the world that connects policies to outcomes. I maintain that the more relevant uncertainty for voters is not the policy platform of a party, but the relationship between policy and outcomes that is captured by the state of the world.

[2] Lever (2010) studies Colonel Blotto games of influence on a social network, focusing on how the network structure affects where two parties will allocate a fixed amount of resources.

[3] They do not make clear in the paper when this is a monotonic relationship. The equilibrium condition is an implicit function of the strategy so the analysis is not straightforward.

There is a vast literature on learning, and a growing one on learning in networks. In this paper, voters learn about the underlying state of the world by truthfully sharing[4] their best guess about the state of the world with each person in their immediate social network. Golub and Jackson (2010) dub this "naïve" learning, because the agents fail to account for repetition of information from common but distant-in-the-network people. DeMarzo et al. (2003) introduced this boundedly-rational Bayesian learner model to the economics, but studied the implications for a more limited set of networks than Golub and Jackson (2010). However, DeMarzo et al. (2003) furnish a defense for this boundedly-rational behavior, documenting evidence of what they term persuasion bias, which is defined to be the overweighting of repeated but uninformative signals in updating beliefs.

3.2 A Framework for Elections with Influenceable Voters

3.2.1 The Elements

There is a population of voters modeled as a measure space $\tilde{\mathcal{V}}$ with measure λ where $\lambda(\tilde{\mathcal{V}}) = 1$. The policy space X is one-dimensional and taken to be equivalent to $\mathbb{R}$. The effect of a policy $x \in X$ on outcomes and hence a voter's utility depends upon the state of the world $\theta \in \Theta \equiv \mathbb{R}$. Voters have single-peaked preferences over X, where voter i's ideal policy $\hat{x}_i$ is a function of the state of the world θ, given by $\hat{x}_i(\theta) = \hat{x}_i + b\theta$, and measurable over $\tilde{\mathcal{V}}$. For example, voters might have utility $u_i(x, \theta) = -\frac{a}{2}(x - \hat{x}_i - b\theta)^2$. When convenient, I will assume this utility function, but most results do not depend upon the exact form of the single-peaked preferences.

The population $\tilde{\mathcal{V}}$ is partitioned into the measurable subsets $\mathcal{L}, \mathcal{R}, \{\mathcal{N}_n\}_{n \in N}$ for some

[4]Other papers about learning in networks include Bala and Goyal (1998) and Acemoglu et al. (2008), but here the agents learn not through truthful communication but through observation of actions and outcomes in the case of Bala and Goyal (1998) and just of actions in the case of Acemoglu et al. (2008). Since my interest here is to understand how candidates might use the fact that people are susceptible to persuasion bias, I have chosen to work with the simpler and tractable naïve learning model.

$N \equiv \{1, \ldots, N \in \mathbb{N}\}$, where N is used to indicate both the set and its cardinality. Define $\mathcal{V} \equiv \bigcup_{n \in N} \mathcal{N}_n$. Let $\lambda_n \equiv \lambda(\mathcal{N}_n)$ for all $n \in N$. There are two political parties labeled L and R, the strategic agents in the model. The two sets $\mathcal{L}$ and $\mathcal{R}$ are *party members* associated with party L and party R, respectively, and the other sets consist of N groups (types) of independent voters (the set $\mathcal{V}$). Party L represents the interests of the voter group $\mathcal{L}$ and party R represents the interests of voter group $\mathcal{R}$. The policy preferences of party $p \in \{L, R\}$ are single-peaked over X, with ideal policies $\hat{x}_p(\theta) = \hat{x}_p + b\theta$, where $\hat{x}_p$ is the average of the ideal policies of the corresponding party members p.[5]

The political parties know the true state of the world θ^*, which I normalize to zero, but the voters do not. Each voter i has an initial opinion about the true state of the world θ_i^0, measurable over $\mathcal{V}$. The average of every voter's opinion $\bar{\theta}^0 \equiv \int_{\mathcal{V}} \theta_i^0 \lambda(di)$ is θ^*. Parties also have public opinions about the state of the world, denoted θ_L and θ_R. These publicly-held opinions are strategic opinions and need not be equal to the true state. The opinion of a voter can be influenced by the opinions of other voters and of the candidates. The process by which social influence affects the opinion of a voter will be described later. The final opinion of a voter will depend upon the public opinions θ_L, θ_R. Let $\mathcal{M}$ be the space of all measurable functions from $\mathcal{V}$ to Θ. Then the process by which voter opinions change from θ^0 to their final opinion θ^∞ is modeled by an operator M on $\mathcal{M}$, the influence operator, that depends upon the opinions θ_L and θ_R of the parties.

Let Ψ denote the share of votes for party R. The utility function for party R is

$$U_R \equiv \Psi - \frac{\alpha_x}{2}(x_R - \hat{x}_R)^2 - \frac{\alpha_\theta}{2}(\theta_R - \theta^*)^2$$

[5]This aggregation can be justified rigorously by assuming all voters i in party p have quadratic utility $u_i(x, \theta) = -\frac{a}{2}(x - \hat{x}_i(\theta))$ with $\hat{x}_i$ distributed according to some density function $f_p(\hat{x}_i)$. Then if the party utility is just the average utility of its members, the party utility has the quadratic utility representation $u_p(x, \theta) = \mathrm{E}[u_i(x, \theta)] = -\frac{a}{2}(x - \hat{x}_p)^2 - \frac{a}{2}\operatorname{var}[\hat{x}_i]$. Since the variance term is independent of x and θ, we can ignore it.

and the utility function for party L is

$$U_L \equiv 1 - \Psi - \frac{\alpha_x}{2}(x_L - \hat{x}_L)^2 - \frac{\alpha_\theta}{2}(\theta_L - \theta^*)^2.$$

Parties derive utility from vote share rather than just winning an election. Parties competing for seats in a proportional representation legislative body would have marginal incentives for votes that are roughly constant. For presidential elections the marginal incentives are nonlinear, potentially even discontinuous. However, if the population votes in a probabilistic fashion, then the marginal incentives are approximately linear in symmetric equilibria, and so the vote share maximization model is a reasonable model. When a symmetric equilibrium exists in a model with a commonly known state (so that persuasion plays no role) and no policy preferences ($\alpha_x = 0$), the equilibrium policy choice is the same with either vote-share maximization or winning probability maximization.

Parties derive utility from the policy they propose. Alternatively, parties bear an ideological cost from proposing policies different from their ideal. This could reflect a reputation cost to the party, left as a primitive, of compromising on policy.

Finally, parties face a cost[6] of holding public opinions that differ from the true state of the world, which reflects the cost of maintaining party line and being more aggressive in efforts to distort the truth. Depending upon the interpretation of the model, this could also be thought of as the cost of influencing voters such as through advertising.

Party members $\mathcal{L}$ and $\mathcal{R}$ always cast votes for their respective parties, unless otherwise specified. Non-partisan voters cast their vote according to the utility they would derive from having the proposed policy in place, given their opinion of the state of the world. If both proposed policies yield the same utility, the voter randomizes with equal probability. Given the opinions of the population $\theta_\mathcal{V}$, let $F(\theta_\mathcal{V}) : X \to [0, 1]$ be the conditional distribution of the ideal policies of $\mathcal{V}$, and let $F_{\mathcal{N}_n}(\theta_\mathcal{V}) : X \to [0, 1]$ be the conditional distribution of the

[6]The assumption of quadratic ideological or opinion costs is only made for expositional convenience. Cost functions that are strictly convex, symmetric about the policy ideal/true state of the world and twice continuously differentiable would suffice.

ideal policies of $\mathcal{N}_n$. Let $f(\theta_{\mathcal{V}}) : X \to \mathbb{R}$ and $f_{\mathcal{N}_n}(\theta_{\mathcal{V}}) : X \to \mathbb{R}$ be the corresponding conditional density functions[7] of the ideal policies over the policy space X. These distributions and densities will be parameterized and studied in subsequent sections.

3.2.2 Political Equilibrium

The electoral competition game takes place in three stages. In the first stage, the *announcement stage*, parties choose their policy x_L, x_R and their public opinion θ_L, θ_R. In the second stage, the *opinion updating stage*, voters update their opinions. In the final stage, the *election stage*, voters cast their votes, without abstention, and the outcome of the election and payoffs are realized.

A strategy for a party $k \in \{L, R\}$ is a policy and opinion pair $(x_k, \theta_k) \in X \times \Theta$.

Definition 3.1 (Political Equilibrium)**.** *A political equilibrium*[8] *is a strategy profile* $((x_L^*, \theta_L^*), (x_R^*, \theta_R^*))$ *such that:*

- *The final opinion θ^∞ of voters is given by $\theta^\infty = \mathcal{M}(\theta_L^*, \theta_R^*)\theta^0$.*
- *Partisans vote for their party and others vote for the party with the most preferred equilibrium policy, breaking ties independently and uniformly at random:*
  - $i \in \mathcal{V}$ *votes for R if* $u_i(x_R^*, \theta_i^\infty) > u_i(x_L^*, \theta_i^\infty)$
  - $i \in \mathcal{V}$ *votes for L if* $u_i(x_R^*, \theta_i^\infty) < u_i(x_L^*, \theta_i^\infty)$
  - $i \in \mathcal{V}$ *votes for R with probability $\frac{1}{2}$ (independently of others) if* $u_i(x_R^*, \theta_i^\infty) = u_i(x_L^*, \theta_i^\infty)$
- *For $k, k' \in \{L, R\}$ with $k \neq k'$,* $U_k(x_k^*, \theta_k^*, x_{k'}^*, \theta_{k'}^*) \geq U_k(\tilde{x}, \tilde{\theta}, x_{k'}^*, \theta_{k'}^*)$ *for all $\tilde{x} \in X$ and $\tilde{\theta} \in \Theta$.*

Through the opinion update operator $\mathcal{M}(\theta_L, \theta_R)$, the distribution and density functions of voter ideal policy are parametrized by θ_L and θ_R; we continue to denote this functions by $F_{\mathcal{N}_n}$ and $f_{\mathcal{N}_n}$, indicating the party opinions when useful. Define $\mu_x = \frac{x_R + x_L}{2}$, $\delta_x = x_R - x_L$,

[7] We will make assumptions later that guarantee the existence of these density functions.

[8] We will focus only on pure-strategy equilibria in this paper.

$\mu_\theta = \frac{\theta_R+\theta_L}{2}$ and $\delta_\theta = \theta_R - \theta_L$. Then, δ_x is a measure of the level of policy polarization and δ_θ a measure of the level of opinion polarization. We will study how these two dimensions of polarization are affected in equilibrium by changes in the fundamentals of the economy, such as the network structure, the preferences of voters, or ideological and opinion distortion costs of parties.

3.3 Elections with Naïve Social Learning: Two Types

> No man is an island entire of itself…
>
> John Donne, Devotions upon Emergent Occasions, Meditation XVII

There are two groups of voters, in addition to the partisan groups $\mathcal{L}$ and $\mathcal{R}$, i.e. $N = 2$, with $\lambda(\mathcal{N}_1) = \lambda(\mathcal{N}_2) = \lambda_1$. Voters i from group $\mathcal{N}_n$ have ideal policies given by $\hat{x}_i(\theta) = \hat{x}_n + \Delta\hat{x}_i + b\theta_i$, where $\Delta\hat{x}_i$ is a mean zero independent random variable with a symmetric single-peaked distribution h_n with full support on X and where $\hat{x}_n$ is a group-specific parameter (the average ideal policy for the group when $\theta_i = 0$ for all $i \in \mathcal{N}_n$). Note that all random variables Δx_i with $i \in \mathcal{V}$ are independent of each other.

We will assume that $\hat{x}_1 = -\hat{x}_2$ and $h_1 = h_2 = h$, so that, given a fixed opinion $\theta_\mathcal{V}$ common to all voters, the distribution of ideal policies is symmetric about zero, the center of the policy space (according to our normalization). Assume that voters have distance preferences symmetric about the ideal policy[9]

Thus, $f_{\mathcal{N}_n}$ the density of ideal policies for voters in $\mathcal{N}_n$ can be derived from $h, \hat{x}_n$ and $\theta_{\mathcal{N}_n}$. Moreover, the vote share function Ψ can be defined as follows:

$$\Psi \equiv \underbrace{\left(\frac{1}{2} - \lambda_1\right)}_{\text{partisan vote}} + \lambda_1 \int_{\mu_x}^{\infty} \left(f_{\mathcal{N}_1}(x) + f_{\mathcal{N}_2}(x)\right) dx$$

where $\mu_x \equiv \frac{x_R+x_L}{2}$ is the midpoint between the policies proposed by the two parties.

[9] Utility of a policy is a function of the Euclidean distance between the policy and the ideal policy (such that single-peakedness is preserved).

3.3.1 The Social Influence Network: Two Types

Consider a voter i from some group $n \in N$. Voter i has an initial opinion about the true state of the world, which denoted θ_i^0. In the course of the electoral campaign, the voter engages in discussions with other voters (of his and of other groups) about the state of the world and in the process changes his opinion. The voter also pays attention to the opinions of the two candidates L and R about the state of the world, affecting his opinion. The voter updates his opinion by taking a weighted average of the opinions of all voters and candidates opinions, where weights could be zero to reflect no (valuable) interaction.

Let γ_i be the fraction of attention that voter i devotes to the candidates/partisans, and $1-\gamma_i$ be the fraction devoted to the non-partisan citizenry. Of the fraction $1-\gamma_i$, let η_i be towards members of the same group and $1-\eta_i$ be towards members of other groups. Of the fraction γ_i, group $\mathcal{N}_1$ voters divide their attention in ν_i to party/group L and $1-\nu_i$ to party/group R, while group $\mathcal{N}_2$ voters divide their attention in $1-\nu_i$ to L and ν_i to R. Voter $i \in \mathcal{N}_1$ updates his opinion according to the rule

$$\theta_i^{t+1} = \frac{\int_{\mathcal{N}_1}(1-\gamma_j)\eta_j\theta_j^t\lambda_1(dj)}{\int_{\mathcal{N}_1}(1-\gamma_j)\eta_j\lambda_1(dj)} + \frac{\int_{\mathcal{N}_2}(1-\gamma_j)(1-\eta_j)\theta_j^t\lambda_2(dj)}{\int_{\mathcal{N}_2}(1-\gamma_j)(1-\eta_j)\lambda_2(dj)} + \gamma_i\nu_i\theta_L^t + \gamma_i(1-\nu_i)\theta_R^t \tag{3.1a}$$

and voter $i \in \mathcal{N}_2$ updates opinions according to the analogous rule

$$\theta_i^{t+1} = \frac{\int_{\mathcal{N}_1}(1-\gamma_j)(1-\eta_j)\theta_j^t\lambda_1(dj)}{\int_{\mathcal{N}_1}(1-\gamma_j)(1-\eta_j)\lambda_1(dj)} + \frac{\int_{\mathcal{N}_2}(1-\gamma_j)\eta_j\theta_j^t\lambda_2(dj)}{\int_{\mathcal{N}_2}(1-\gamma_j)\eta_j\lambda_2(dj)} + \gamma_i(1-\nu_i)\theta_L^t + \gamma_i\nu_i\theta_R^t. \tag{3.1b}$$

To avoid serious technical difficulties with measurability of the opinions of voters $\mathcal{V}$, we will assume that all voters in the same group have the same values of η, γ and ν.[10] With this assumption, it is clear that $\int_{\mathcal{N}_1}(1-\gamma_j)\eta_j\theta_j^t\lambda_1(dj) = (1-\gamma_1)\eta_1\int_{\mathcal{N}_1}\theta_j^t\lambda_1(dj)$. Similarly,

[10] In fact, we can easily relax the assumption of identical ν across voters of the same group, but none of the results would be altered by this additional complexity.

the network parameters can be extracted from the other integrals in these updating rules. However, for the integrals to be well defined, we need θ^t to be a measurable function.

Lemma 3.1. *Suppose θ^0, the initial opinions of the voters, is a measurable function with respect to the measure space $\mathcal{V}$ of voters. Then, for any $n \in \{1, 2\}$ and $i \in \mathcal{N}_n$, the update rules in equations 3.1a and 3.1b are well defined.*

Proof. Proceed by induction. First, suppose θ^t is a measurable function for a given t. Since each of the sets $\mathcal{N}_1$ and $\mathcal{N}_2$ are measurable subsets of $\mathcal{V}$, then θ_i^{t+1} is well defined for all $i \in \mathcal{V}$. This follows because γ_i and η_i are measurable functions on $\mathcal{V}$ by virtue of being constants over $\mathcal{N}_1$ and over $\mathcal{N}_2$, and so the integrals in equations 3.1a and 3.1b are well defined.

Next, to establish the induction step, we need to show that θ^{t+1} is a measurable function on $\mathcal{V}$. Define $\bar{\theta}^t_{\mathcal{N}_n} \equiv \frac{\int_{\mathcal{N}_n} \theta_i^t \lambda_n(di)}{\int_{\mathcal{N}_n} \lambda_n(di)}$ for $n \in \{1, 2\}$, the average opinion of each of the two groups, which is well defined as just established. But then equation 3.1a at time $t + 1$ can be written as

$$\theta_i^{t+1} = (1 - \gamma_1)\eta_1 \bar{\theta}^t_{\mathcal{N}_1} + (1 - \gamma_1)(1 - \eta_1)\bar{\theta}^t_{\mathcal{N}_2} + \gamma_1 \nu_1 \theta^t_L + \gamma_1(1 - \nu_1)\theta^t_R$$

which is independent of i for all $i \in \mathcal{N}_1$. Then, $\theta_i^{t+1} = \theta_{i'}^{t+1}$ for all $i, i' \in \mathcal{N}_1$. The analogous argument applied to equation 3.1b yields the conclusion that $\theta_i^{t+1} = \theta_{i'}^{t+1}$ for all $i, i' \in \mathcal{N}_2$. Thus, θ^{t+1} is a measurable function on $\mathcal{V}$. Since we have assumed that θ^0 is a measurable function, we have the base case for induction. $\square$

So, we can rewrite equations 3.1a and 3.1b as

$$\theta^{t+1}_{i \in \mathcal{N}_1} = (1 - \gamma_1)\eta_1 \bar{\theta}^t_{\mathcal{N}_1} + (1 - \gamma_1)(1 - \eta_1)\bar{\theta}^t_{\mathcal{N}_2} + \gamma_1 \nu_1 \theta^t_L + \gamma_1(1 - \nu_1)\theta^t_R \tag{3.2a}$$

$$\theta^{t+1}_{i \in \mathcal{N}_2} = (1 - \gamma_2)(1 - \eta_2)\bar{\theta}^t_{\mathcal{N}_1} + (1 - \gamma_2)\eta_2 \bar{\theta}^t_{\mathcal{N}_2} + \gamma_2(1 - \nu_2)\theta^t_L + \gamma_2 \nu_2 \theta^t_R \tag{3.2b}$$

In order to study the evolution of opinions in the population, let us study first the evolution of the group averages. Integrating the individual opinion updating equations we get

$$\bar{\theta}_{\mathcal{N}_1}^{t+1} = (1-\gamma_1)\eta_1\bar{\theta}_{\mathcal{N}_1}^{t} + (1-\gamma_1)(1-\eta_1)\bar{\theta}_{\mathcal{N}_2}^{t} + \gamma_1\nu_1\theta_L^t + \gamma_1(1-\nu_1)\theta_R^t$$
$$\bar{\theta}_{\mathcal{N}_2}^{t+1} = (1-\gamma_2)(1-\eta_2)\bar{\theta}_{\mathcal{N}_1}^{t} + (1-\gamma_2)\eta_2\bar{\theta}_{\mathcal{N}_2}^{t} + \gamma_2(1-\nu_2)\theta_L^t + \gamma_2\nu_2\theta_R^t$$

which written in matrix form is

$$\underbrace{\begin{bmatrix}\bar{\theta}_{\mathcal{N}_1}^{t+1}\\ \bar{\theta}_{\mathcal{N}_2}^{t+1}\end{bmatrix}}_{\bar{\theta}^{t+1}} = \underbrace{\begin{bmatrix}(1-\gamma_1)\eta_1 & (1-\gamma_1)(1-\eta_1)\\ (1-\gamma_2)(1-\eta_2) & (1-\gamma_2)\eta_2\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}\bar{\theta}_{\mathcal{N}_1}^{t}\\ \bar{\theta}_{\mathcal{N}_2}^{t}\end{bmatrix}}_{\bar{\theta}^{t}} + \underbrace{\begin{bmatrix}\gamma_1\nu_1 & \gamma_1(1-\nu_1)\\ \gamma_2(1-\nu_2) & \gamma_2\nu_2\end{bmatrix}}_{B}\underbrace{\begin{bmatrix}\theta_L^t\\ \theta_R^t\end{bmatrix}}_{\theta_K^t}$$

or more concisely

$$\bar{\theta}^{t+1} = A\bar{\theta}^t + B\theta_K^t. \tag{3.3}$$

The opinion update weights A and B are assumed to be a constant during the updating process. The weights in A could be interpreted as the product of the probability that voters i and j discuss opinions and the persuasiveness of j from the perspective of i. Similarly, B can be interpreted as the product of the probability that voter i pays attention to candidate k through various media or to members of this candidates party and the persuasiveness of k from the perspective of i.

As discussed previously, candidates choose their opinions strategically at the beginning of the opinion updating process (i.e. at time 0) and do not subsequently change these opinions, so we know that $\theta_k^t = \theta_k^0 = \theta_k$ for all $t \in \mathbb{N}$. Moreover, with our focus on symmetric environments, we have that $\eta_1 = \eta_2 = \eta$, $\gamma_1 = \gamma_2 = \gamma$ and $\nu_1 = \nu_2 = \nu$. To avoid the trivial situation where voters cannot be influenced, I assume that $\gamma > 0$.

I assume that the election stage takes place after a sufficiently large number of rounds of updating, so that the opinions of the voters are arbitrarily well-approximated by the limit opinion $\lim_{t\to\infty}\bar{\theta}^t$, which I call the *equilibrium voter opinions*. Proposition 3.1 describes the equilibrium weights voters place on the opinions of the candidates. The weights are a function of the network parameters γ, η and ν. Figure 3.1 shows a schematic diagram of the social influence network. The parameter γ captures the overall penetration of non-partisan

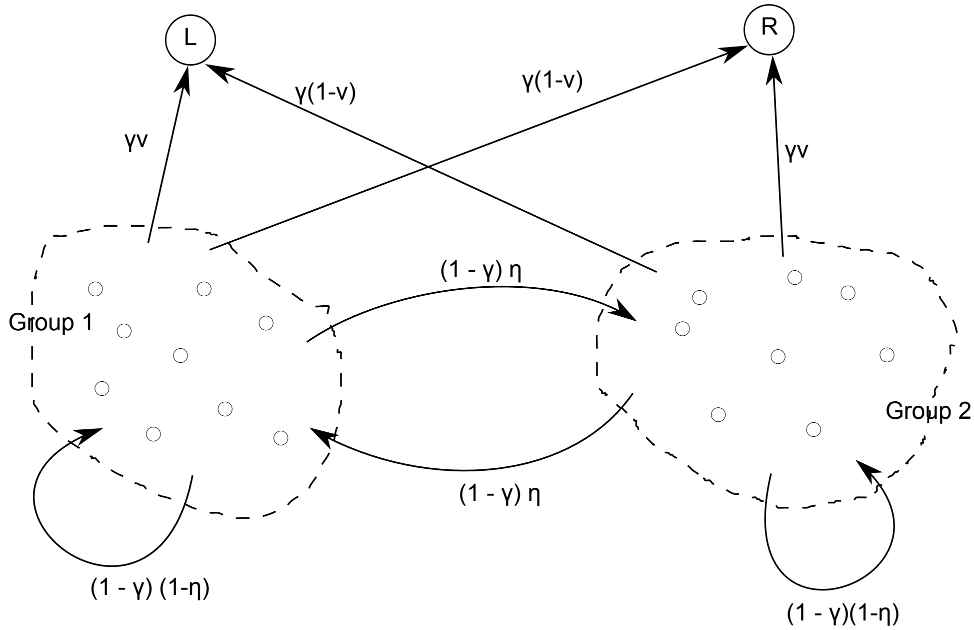

Figure 3.1: Schematic of a symmetric network with two groups of voters $\mathcal{N}_1$ and $\mathcal{N}_2$

citizen networks by party candidates and members, with relevant values ranging from γ close to zero when parties have very little influence to γ close to one when parties directly determine opinions of the voters. The parameter ν describes the direct listening bias of a group of non-partisan voters to their corresponding party, with $\nu = \frac{1}{2}$ indicating no bias and $\nu = 0$ or $\nu = 1$ indicating maximal bias (for one of the two parties). Finally η is a measure of homophily with $\eta = \frac{1}{2}$ indicating no homophily (in our symmetric case with $\lambda(\mathcal{N}_1) = \lambda(\mathcal{N}_2)$) and $\eta = 1$ indicating maximal homophily, where voters are only influenced by members of their own group. Assume that $\nu \geq \frac{1}{2}$. This is without loss of generality since we can relabel the groups to achieve this condition.

Proposition 3.1. *Suppose $\gamma > 0$. Then, $\lim_{t\to\infty} \bar{\theta}^t$ exists. Moreover, the equilibrium opinion of voter $i \in \mathcal{N}_1$ is given by $\theta_i^\infty = (\Delta_w + \frac{1}{2})\theta_L + (\frac{1}{2} - \Delta_w)\theta_R$ and that of voter $i \in \mathcal{N}_2$ is given by $\theta_i^\infty = (\frac{1}{2} - \Delta_w)\theta_L + (\frac{1}{2} + \Delta_w)\theta_R$, where $\Delta_w = \frac{\gamma(\mu - \frac{1}{2})}{1-2(1-\gamma)(\eta-\frac{1}{2})}$ is the equilibrium influence bias.*

Proof. Equation 3.3 is a first-order difference equation that can be solved given initial conditions θ^0 and θ_K, where we use the assumption that $\theta_K^t = \theta_K$ for all t:

$$\bar{\theta}^{t+1} = A^{t+1}\bar{\theta}^0 + \left(\sum_{\tau=0}^{t} A^\tau\right) B\theta_K.$$

With $\gamma > 0$, we know that each row of the non-negative matrix A has a sum that is strictly less than one, which implies that the spectral radius of A is strictly less than one. But then $\lim_{t\to\infty} A^t$ is the zero matrix. Additionally, since $(I + A + A^2 + \dots)$ is a Neumann series with spectral radius strictly less than one, $\sum_{\tau=0}^{t} A^\tau = (I - A)^{-1}$. Thus, $\bar{\theta}^\infty \equiv \lim_{t\to\infty} \bar{\theta}^t = \lim_{t\to\infty} A^{t+1}\bar{\theta}^0 + \lim_{t\to\infty} \left(\sum_{\tau=0}^{t} A^\tau\right) B\theta_K = (I - A)^{-1}B\theta_K$.

Define $\kappa \equiv (1-\gamma)(1-\eta)$. Then, $(I - A) = \begin{bmatrix} \kappa+\gamma & -\kappa \\ -\kappa & \kappa+\gamma \end{bmatrix}$, with $\det(I - A) = (\kappa+\gamma) - \kappa^2 = \gamma(2\kappa+\gamma)$. Then,

$$\begin{aligned}(I-A)^{-1}B &= \frac{1}{\gamma(2\kappa+\gamma)} \begin{bmatrix} \kappa+\gamma & \kappa \\ \kappa & \kappa+\gamma \end{bmatrix} \begin{bmatrix} \gamma\nu & \gamma(1-\nu) \\ \gamma(1-\nu) & \gamma\nu \end{bmatrix} \\ &= \frac{1}{2\kappa+\gamma} \begin{bmatrix} \kappa+\gamma\nu & \kappa+\gamma(1-\nu) \\ \kappa+\gamma(1-\nu) & \kappa+\gamma\nu \end{bmatrix}.\end{aligned}$$

Define

$$w \equiv \frac{\kappa+\gamma\nu}{2\kappa+\gamma} = \frac{(1-\gamma)(1-\eta)+\gamma\nu}{2(1-\gamma)(1-\eta)+\gamma} = \frac{(1-\gamma)(1-\eta)+\gamma\nu}{1-2(1-\gamma)(\eta-\frac{1}{2})}.$$

Then, $(I-A)^{-1}B = \left[\begin{smallmatrix} w & 1-w \\ 1-w & w \end{smallmatrix}\right]$, so $\bar{\theta}^\infty_{\mathcal{N}_1} = w\theta_L + (1-w)\theta_R$ and $\bar{\theta}^\infty_{\mathcal{N}_2} = (1-w)\theta_L + w\theta_R$.

Finally, using equation 3.2a, we have $\theta^\infty_{i\in\mathcal{N}_1} = (1-\gamma)\eta\bar{\theta}^\infty_{\mathcal{N}_1} + (1-\gamma)(1-\eta)\bar{\theta}^\infty_{\mathcal{N}_2} + \gamma\nu\theta_L + \gamma(1-\nu)\theta_R = w\theta_L + (1-w)\theta_R$ after simplification. Similarly, using equation 3.2b we obtain that $\theta^\infty_{i\in\mathcal{N}_2} = (1-\gamma)(1-\eta)\bar{\theta}^\infty_{\mathcal{N}_1} + (1-\gamma)\eta\bar{\theta}^\infty_{\mathcal{N}_2} + \gamma(1-\nu)\theta_L + \gamma\nu\theta_R = w\theta_L + (1-w)\theta_R$ after simplification, completing the proof. □

The weight w is the equilibrium influence of candidate R on group $\mathcal{N}_2$ (symmetrically, the equilibrium influence of candidate L on group $\mathcal{N}_1$), defined as

$$w \equiv \Delta_w + \frac{1}{2} = \frac{(1-\gamma)(1-\eta) + \gamma\nu}{1 - 2(1-\gamma)(\eta - \frac{1}{2})}.$$

The structure of the influence network affects the equilibrium influence bias Δ_w (and weight w). The following proposition describes the impact of changes in structural parameters on this bias Δ_w.

Proposition 3.2. *Any of the following results in an increase in the equilibrium influence bias Δ_w:*

1. *An increase in the penetration parameter γ.*
2. *An increase in the level of homophily η.*
3. *An increase in the direct listening bias ν.*

In other words, an increase in γ, η, or ν will increase the magnitude of the equilibrium influence bias.

Proof. The derivative of Δ_w with respect to γ is

$$\frac{\partial \Delta_w}{\partial \gamma} = \frac{(\nu - \frac{1}{2}) - 2(\eta - \frac{1}{2})\Delta_w}{1 - 2(1-\gamma)(\eta - \frac{1}{2})} = \frac{\Delta_w}{\gamma} \cdot \frac{2(1-\eta)}{1 - 2(1-\gamma)(\eta - \frac{1}{2})} \tag{3.4}$$

so $\frac{\partial \Delta_w}{\partial \gamma} \geq 0$ if $\Delta_w \geq 0$, which is the case since $\nu \geq \frac{1}{2}$.

The derivative of Δ_w with respect to η is

$$\frac{\partial \Delta_w}{\partial \eta} = \frac{2(1-\gamma)}{1 - 2(1-\gamma)(\eta - \frac{1}{2})}\Delta_w \tag{3.5}$$

so $\frac{\partial \Delta_w}{\partial \eta} \geq 0$ if $\Delta_w \geq 0$, which is the case since $\nu \geq \frac{1}{2}$.

The derivative of Δ_w with respect to ν is

$$\frac{\partial \Delta_w}{\partial \nu} = \frac{\gamma}{1 - 2(1-\gamma)(\eta - \frac{1}{2})} \tag{3.6}$$

so $\frac{\partial \Delta_w}{\partial \nu} \geq 0$. $\square$

3.3.2 Equilibrium Characterization

We will now examine the equilibrium of the two-type model with naïve social learning. We will establish equilibrium existence in a more general model in a later section, but for now the following theorem will suffice.

Theorem 3.1 (Existence with Two Types)**.** *Let $\alpha_x^* = \frac{\xi}{4}$ and $\alpha_\theta^* = \frac{b^2\xi}{2}$, where $\xi \equiv \max|h'|$ is the maximum value of the slope of the density function h. Then, for any $\alpha_x \geq \alpha_x^*$ and $\alpha_\theta \geq \alpha_\theta^*$ there is a unique symmetric political equilibrium $x_L^* = -x_R^*$ and $\theta_L^* = -\theta_R^*$.*

Essentially, if the marginal costs of ideological deviation or opinion distortion are sufficiently steep, then equilibrium exists.[11] Moreover, in our symmetric setting there is a unique equilibrium that is symmetric. This existence theorem can be weakened substantially by parametrizing the cutoff values of α_x^* and α_θ^* on the network and policy ideal parameters as well.

The following theorem characterizes the unique symmetric political equilibrium.

Theorem 3.2 (Equilibrium Characterization)**.** *When the unique symmetric political equilibrium exists,*

- *the equilibrium influence bias* $\Delta_w^* = \frac{\gamma(\nu-\frac{1}{2})}{(1-2(1-\gamma)(\eta-\frac{1}{2})}$,
- $\theta_R^* = \frac{b\lambda_2}{\alpha_\theta} f_{\mathcal{N}_2}(\mu_x^* = 0; \theta_L^*, \theta_R^*)$,
- $\theta_L^* = -\theta_R^*$,
- $x_R^* = \max\{0, \hat{x}_R - \frac{\alpha_\theta \theta_R^*}{\alpha_x b}\}$,
- $x_L^* = -x_R^*$.

The following is an immediate corollary that describes the equilibrium in terms of policy and opinion polarization.

Corollary 3.1. *When the unique symmetric political equilibrium exists,*

- $\mu_x^* = 0$, $\mu_\theta^* = 0$, *and party R's vote share* $\Psi^* = \frac{1}{2}$,

[11] The condition we define ensures that the objective functions of the parties are quasiconcave.

- $\alpha_\theta \delta_\theta^* = 2b\lambda_2 f_{\mathcal{N}_2}(0; \theta_L^*, \theta_R^*)$,
- $\Delta_\theta^* \equiv \bar{\theta}_{\mathcal{N}_2}^* - \bar{\theta}_{\mathcal{N}_1}^* = 2\Delta_w^* \delta_\theta^*$,
- $\delta_x^* = \max\{0, \hat{\delta}_x - \frac{\alpha_\theta \delta_\theta^*}{\alpha_x b}\}$,

where $\hat{\delta}_x \equiv \hat{x}_R - \hat{x}_L$, the difference between the policy ideals of the parties.

Given the symmetry of equilibrium, it is no surprise that policy proposals of the parties are symmetric about the median ideal policy of the population, nor that an equally weighted average of the opinions of the two parties provides an accurate opinion about the true state. Nevertheless, unless the the equilibrium influence bias Δ_w^* is zero, there will be opinion polarization among the voters ($\Delta_\theta^* > 0$).

The key relationship is the one between the marginal value and the marginal cost of increasing opinion distortion, as embodied in the implicit function for θ_R^*

$$\theta_R^* = \frac{b\lambda_2}{\alpha_\theta} f_{\mathcal{N}_2}(0; \theta_L^*, \theta_R^*)$$

In the symmetric equilibrium, the indifferent voter is the median voter, where the distribution of voters has been altered by the persuasion efforts from $f(\cdot; 0, 0)$ to $f(\cdot; \theta_L^*, \theta_R^*)$. The within-group distribution of voter ideal policies h gives shape to the marginal value of increasing θ_R (which is the increase in the vote share accruing to party R). So, in our simple two-type model, the equilibrium shape of f mirrors that of h, since the within-group ideal policy distributions are symmetric about zero. For example, if h is a normal density function, then f is also a normal density function.

In the canonical case, where $\hat{x}_2 \geq 0$, so that there is a positive correlation between a voter's ideological leaning and their listening bias (rightward ideological bias and rightward listening bias are positively correlated), the marginal vote share with respect to increasing θ_R is negative i.e. $f_{\mathcal{N}_2}$ is downward sloping with respect to θ_R in equilibrium, whereas marginal cost of increasing θ_R is strictly increasing. This equilibrium condition is illustrated in Figure 3.2.

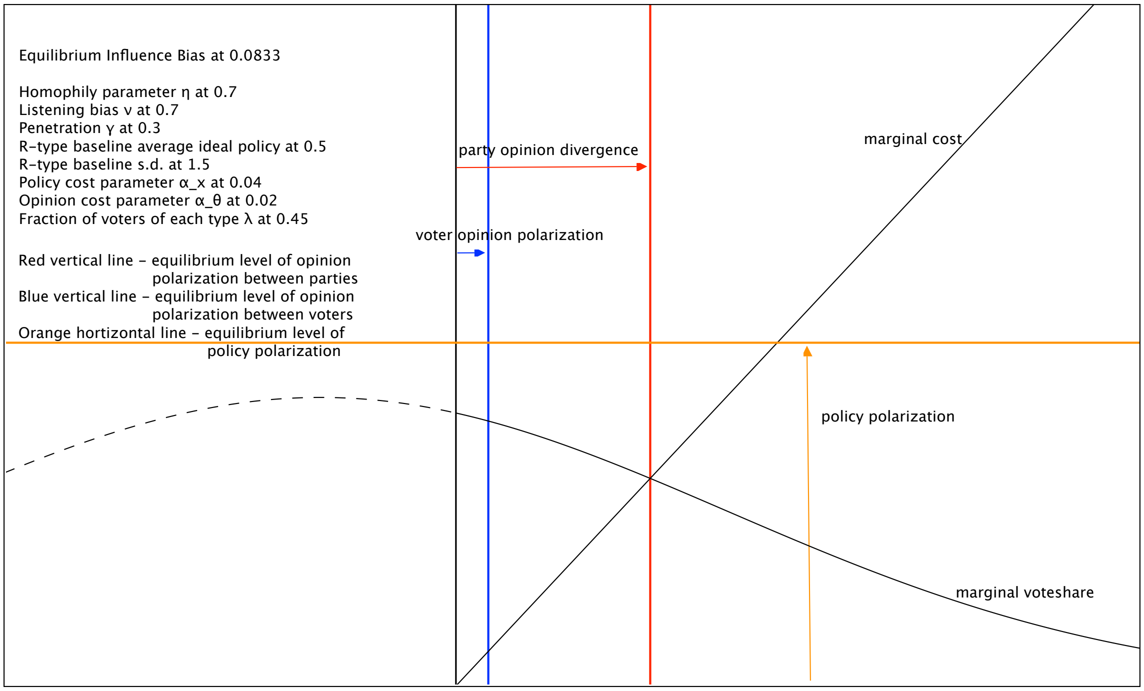

Figure 3.2: Diagram of marginal vote share and marginal opinion distortion cost in symmetric equilibrium with voter-type normally distributed: θ_R^* on the horizontal axis, with marginal cost intersecting axis at $\theta_R^* = 0$.

Equilibrium effects on party opinion divergence due to parameter changes can qualitatively be studied within this diagrammatic framework, as demonstrated in the following subsections.

3.3.3 Comparative Statics of the Network Structure

There are three network parameters that determine the social influence network and related metrics. Perhaps the property of greatest interest is homophily, which is the propensity of individuals to be socially connected to other people of a similar type. In the context of the two-type social network, there are group $\mathcal{N}_1$ and group $\mathcal{N}_2$ individuals, distinguished by their expected ideological position ($\hat{x}$) and their expected listening bias (ν). If a voter was "blind" to the type of others, so that his immediate social neighbors were a representative sample of the population, then the proportion of $\mathcal{N}_1$ individuals in his social network would be $\frac{\lambda_1}{\lambda_1+\lambda_2}$ i.e. the population proportion of $\mathcal{N}_1$ individuals. In our symmetric setting, this would be $\frac{1}{2}$. Thus, when $\eta = \frac{1}{2}$, there is no homophily, since any individual's social neighbors is a representative sample. As η increases from $\frac{1}{2}$ to 1, the level of homophily increases to its maximum level, where voters only communicate with members within their group.

The listening bias ν is the second important parameter that describes the structure of social influence. Unlike homophily, the listening bias parameterizes the relationship between a voter and the parties through media. When $\nu = \frac{1}{2}$, the voter has no listening bias, because he equally weights the partisan outlets of the left and the right. When $\nu > \frac{1}{2}$, voters from group $\mathcal{N}_1$ pay more attention to left outlets than right outlets and voters from group $\mathcal{N}_2$ do the opposite. Thus, ν is a measure of the direct influence bias in the population. However, the equilibrium influence bias is generally larger than the direct bias as a result of amplification due to homophily.

The final parameter of study is the penetration (or media influence) parameter γ, which captures the importance of direct media and party communication relative to the social communication of voters. If $\gamma = 0$, then opinion formation of the voters is completely independent of party influence. In this case, the model reduces to the standard one-dimensional policy competition, with voter opinions converging to a consensus opinion (which in our symmetric setting would be the true state of the world).

Higher levels of homophily has two counteracting effects – voters' bias in media consumption is amplified by an "echo chamber" effect, simultaneously making it easier to keep voters who already have a favorable listening bias but harder to steal voters who have a listening bias in favor of the opposing party. This amplification of the initial bias through homophily implies that policy competition at the median is reduced by the skimming out voters at the center, who are pulled away from the median. The lowered level of policy competition leads to less moderation in the policy dimension, increasing policy polarization. The effect of higher homophily is embodied by the equilibrium influence bias, which increases with level of homophily. This increase in the equilibrium influence bias reduces the divergence in public opinions of the parties, by reducing the marginal value of opinion distortion. However, the divergence of opinion within the population actually increases because this reduction in party-level opinion divergence is counteracted by the increase in the equilibrium influence bias. Thus, variance of opinions in the population of voters increases.

Lemma 3.2. *Suppose $\alpha_x \geq \alpha_x^*$ and $\alpha_\theta \geq \alpha_\theta^*$. The level of policy polarization δ_x and of opinion polarization Δ_θ are positively related to the level of equilibrium influence bias Δ_w^*.*

Combining this above lemma with our earlier lemma on the impact of the three network parameters on the level of equilibrium influence bias, we obtain the important theorem.

Theorem 3.3. *Suppose $\alpha_x \geq \alpha_x^*$ and $\alpha_\theta \geq \alpha_\theta^*$. When the degree of homophily η in the network increases, the level of policy polarization δ_x^* and the level of voter opinion polarization Δ_θ^* increases, but the level of party opinion polarization δ_θ^* decreases.*

3.4 Conclusion

In this paper I provide a tractable framework for analyzing electoral competition and political persuasion that reconciles greater polarization in policy and public opinion with the changing landscape of communication networks. Sorting into like-minded groups through innovations in communication technology, thereby creating social networks displaying a greater degree of homophily, can increase the degree of policy and voter opinion polarization.

Bibliography

Acemoglu, Daron, Munther A. Dahleh, Ilan Lobel, and Asuman Ozdaglar, "Bayesian Learning in Social Networks," *Working Paper, MIT*, 2008.

Alcalde, José, "Implementation of Stable Solutions to Marriage Problems," *Journal of Economic Theory*, May 1996.

_ **and Antonio Romero-Medina**, "Simple Mechanisms to Implement the Core of College Admissions Problems," *Games and Economic Behavior*, May 2000, *31* (2), 294–302.

_ **, David Pérez-Castrillo, and Antonio Romero-Medina**, "Hiring Procedures to Implement Stable Allocations," *Journal of Economic Theory*, Sep 1998.

Alkan, Ahmet, "On preferences over subsets and the lattice structure of stable matchings," *Review of Economic Design*, 2001.

_ , "A class of multipartner matching markets with a strong lattice structure," *Economic Theory*, 2002.

_ **and David Gale**, "Stable Schedule Matching Under Revealed Preference," *Journal of Economic Theory*, 2003.

Alva, Samson, "A Note on the Weak Axiom of Revealed Preference in Matching with Contracts," *Working Paper*, 2012.

Aygün, Orhan and Tayfun Sönmez, "The Importance of Irrelevance of Rejected Contracts in Matching under Weakened Substitutes Conditions," *Working Paper*, 2012.

_ **and** _ , "Matching with Contracts: The Critical Role of Irrelevance of Rejected Contracts," *Working Paper*, 2012.

Azevedo, Eduardo and Jacob Leshno, "A Supply and Demand Framework for Two-Sided Matching Markets," *Working Paper*, 2012.

Bala, Venkatesh and Sanjeev Goyal, "Learning from Neighbors," *Review of Economic Studies*, 1998, *65*, 595–621.

Banks, Jeffrey, "A model of electoral competition with incomplete information," *Journal of Economic Theory*, Jan 1990.

Blair, Charles, "The lattice structure of the set of stable matchings with multiple partners," *Mathematics of Operations Research*, Jan 1988.

Bloom, Nicholas, Raffaella Sadun, and John Van Reenen, "Recent Advances in the Empirics of Organizational Economics," *Annu. Rev. Econ.*, Sep 2010, *2* (1), 105–137.

Bolton, Patrick and Mathias Dewatripont, "The Firm as a Communication Network," *Quarterly Journal of Economics*, 1994.

Coase, Ronald, "The nature of the firm," *Economica*, Jan 1937.

Crawford, Vincent and Elsie Marie Knoer, "Job matching with heterogeneous firms and workers," *Econometrica*, Jan 1981.

Debreu, Gerard and Herbert Scarf, "A Limit Theorem on the Core of an Economy," *International Economic Review*, Jan 1963.

Demange, Gabrielle, "On Group Stability in Hierarchies and Networks," *Journal of Political Economy*, Jul 2004.

DeMarzo, Peter, Dimitri Vayanos, and Jeffrey Zwiebel, "Persuasion Bias, Social Influence, and Unidimensional Opinions," *Quarterly Journal of Economics*, Jan 2003.

Dubins, Lester and David Freedman, "Machiavelli and the Gale-Shapley algorithm," *American Mathematical Monthly*, Jan 1981.

Echenique, Federico, "Counting combinatorial choice rules," *Games and Economic Behavior*, Jan 2007.

_ **and Jorge Oviedo**, "Core many-to-one matchings by fixed-point methods," *Journal of Economic Theory*, Apr 2004, *115* (2), 358–376.

_ **and** _ , "A theory of stability in many-to-many matching markets," *Theoretical Economics*, Jan 2006.

Edgeworth, Francis Ysidro, *Mathematical Psychics: An Essay on the Application of Mathematics to the Moral Sciences*, C. Kegan Paul & Co., London, 1881.

Fleiner, Tamás, "A fixed-point approach to stable matchings and some applications," *Mathematics of Operations Research*, Jan 2003.

Gale, David and Lloyd Shapley, "College admissions and the stability of marriage," *American Mathematical Monthly*, Jan 1962.

Galeotti, Andrea and Andrea Mattozzi, ""Personal Influence": Social Context and Political Competition," *American Economic Journal: Microeconomics*, Feb 2011, *3* (1), 307–327.

Garicano, Luis, "Hierarchies and the Organization of Knowledge in Production," *Journal of Political Economy*, 2000.

Gibbons, Robert, "Four formal(izable) theories of the firm?," *Journal of Economic Behavior & Organization*, 2005.

Glaeser, Edward, Giacomo Ponzetto, and Jesse Shapiro, "Strategic Extremism: Why Republicans and Democrats Divide on Religious Values," *Quarterly Journal of Economics*, Jan 2005.

Golub, Benjamin and Matthew O Jackson, "Naive Learning in Social Networks and the Wisdom of Crowds," *American Economic Journal: Microeconomics*, Jan 2010.

Grossman, Sanford and Oliver Hart, "The costs and benefits of ownership: A theory of vertical and lateral integration," *Journal of Political Economy*, 1986.

Haeringer, Guillaume and Myrna Wooders, "Decentralized job matching," *International Journal of Game Theory*, Feb 2011, *40* (1), 1–28.

Hart, Oliver and John Moore, "Property Rights and the Nature of the Firm," *Journal of Political Economy*, Jan 1990.

Hatfield, John William and Fuhito Kojima, "Matching with contracts: Comment," *American Economic Review*, Jan 2008.

_ **and** _ , "Substitutes and stability for matching with contracts," *Journal of Economic Theory*, Jan 2010.

_ **and Paul Milgrom**, "Matching with contracts," *American Economic Review*, Jan 2005.

_ **and Scott Duke Kominers**, "Contract Design and Stability in Many-to-Many Matching," *Working Paper*, 2012.

_ **and** _ , "Matching in Networks with Bilateral Contracts," *American Economic Journal: Microeconomics*, 2012.

_ **,** _ **, Alexandru Nichifor, Michael Ostrovsky, and Alexander Westkamp**, "Stability and Competitive Equilibrium in Trading Networks," *Working Paper*, 2012.

Jordan, James S, "Pillage and property," *Journal of Economic Theory*, 2006.

Kelso, Alexander and Vincent Crawford, "Job matching, coalition formation, and gross substitutes," *Econometrica*, Jan 1982.

Klaus, Bettina and Markus Walzl, "Stable many-to-many matchings with contracts," *Journal of Mathematical Economics*, Jan 2009.

Kojima, Fuhito and Parag A Pathak, "Incentives and Stability in Large Two-Sided Matching Markets," *American Economic Review*, 2009.

Kominers, Scott Duke, "On The Correspondence of Contracts to Salaries in (Many-to-Many) Matching," *Games and Economic Behavior*, Jan 2012.

_ **and Tayfun Sönmez**, "Designing for Diversity: Matching with Slot-Specific Priorities," *Working Paper*, 2012.

Lever, Carlos, "Strategic Spending in Voting Competitions with Social Networks," *Working Paper*, 2010.

Ostrovsky, Michael, "Stability in Supply Chain Networks," *American Economic Review*, 2008.

Pápai, Szilvia, "Strategyproof assignment by hierarchical exchange," *Econometrica*, Jan 2000.

Pattie, Charles and Ron Johnston, "Context, conversation and conviction: Social networks and voting at the 1992 British General Election," *Political Studies*, Jan 1999.

Piccione, Michele and Ariel Rubinstein, "Equilibrium in the Jungle," *Economic Journal*, Jan 2007.

_ **and Ronny Razin**, "Coalition formation under power relations," *Theoretical Economics*, Jan 2009.

Radner, Roy, "Hierarchy: The Economics of Managing," *Journal of Economic Literature*, 1992.

Robinson, John, "Interpersonal influence in election campaigns," *Public Opinion Quarterly*, Jan 1976.

Roemer, John, "Political–economic equilibrium when parties represent constituents: The unidimensional case," *Social Choice and Welfare*, 1997.

Roth, Alvin, "The Evolution of the Labor Market for Medical Interns and Residents: A Case Study in Game Theory," *Journal of Political Economy*, 1984.

_ , "Stability and polarization of interests in job matching," *Econometrica*, Jan 1984.

_ , "Conflict and coincidence of interest in job matching: some new results and open questions," *Mathematics of Operations Research*, Jan 1985.

_ , "On the Allocation of Residents to Rural Hospitals: A General Property of Two-Sided Matching Markets," *Econometrica*, 1986.

_ , "A Natural Experiment in the Organization of Entry-Level Labor Markets: Regional Markets for New Physicians and Surgeons in the United Kingdom," *American Economic Review*, 1991.

Roth, Alvin E. and Marilda A. Oliveira Sotomayor, *Two-Sided Matching*, Cambridge University Press, 1990.

Sönmez, Tayfun, "Bidding for Army Career Specialties: Improving the ROTC Branching Mechanism," *Working Paper*, 2011.

_ **and Tobias Switzer**, "Matching with (Branch-of-Choice) Contracts at United States Military Academy," *Working Paper*, 2012.

_ **and Utku Ünver**, "Matching, allocation, and exchange of discrete resources," *Handbook of Social Economics*, Jan 2011.

Sotomayor, Marilda, "Reaching the core of the marriage market through a non-revelation matching mechanism," *International Journal of Game Theory*, Dec 2003, *32* (2), 241–251.

_ , "Implementation in the many-to-many matching market," *Games and Economic Behavior*, Jan 2004.

Svensson, Lars-Gunnar, "Strategy-proof allocation of indivisible goods," *Socal Choice and Welfare*, Jan 1999.

Westkamp, Alexander, "Market structure and matching with contracts," *Journal of Economic Theory*, Jan 2010.

_ , "An analysis of the German university admissions system," *Economic Theory*, Apr 2012, pp. 1–29.

Williamson, Oliver, "The theory of the firm as governance structure: from choice to contract," *Journal of Economic Perspectives*, Jan 2002.