

Dehn paternity bounds and hyperbolicity tests

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DEHN PATERNITY BOUNDS AND HYPERBOLICITY TESTS

a dissertation

by

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submitted in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

May 2015

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2015

Dehn paternity bounds and hyperbolicity tests

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Abstract

Recent advances in normal surface algorithms enable the determination by computer of the hyperbolicity of compact orientable 3-manifolds with zero Euler characteristic and nonempty boundary. Recent advances in hyperbolic geometry enable the determination by computer of the Dehn paternity relation between two orientable compact hyperbolic 3-manifolds. Presented here is an exposition of these developments, along with prototype implementations of one of these determinations in software. These have applications to two questions about Mom technology.

Contents

1	Introduction	1
2	Triangulations in Regina	5
3	Normal Surface Theory in Brief	12
4	Determining hyperbolicity	20
4.1	Rewriting Thurston's Haken theorem	21
4.2	On Faults	31
5	Practical Bounds on Dehn Surgery Space	33
5.1	Rewriting the Hodgson-Kerckhoff Bounds	35
5.1.1	Monotonicities	36
5.1.2	Complicated upper bound on ℓ	41
5.1.3	Complicated bounds on $\hat{L}(c)$	42
5.1.4	Nice bounds	43
5.1.5	Numerical approximations	52
5.2	Estimates in Maxima for Lemma 5.1.11	53
6	Prospects	54

Acknowledgments

Thanks first to God, Father, Son, and Holy Spirit. Here are the first fruits of my labors.

Thanks to Rob, my advisor! I have learned much about topology and hyperbolic geometry, but more about myself as a mathematician. Without your guidance this thesis would be uninteresting and nonexistent, and I would not be so gainfully employed. You know how much I am in your debt.

Thanks to Dave Gabai! This thesis is, in part, an attempt to thank him for his guidance and help in getting into grad school and getting employed. Words cannot properly express my thanks to him.

Thanks to Craig Hodgson! This thesis is, in part, an extension of his work with Steven Kerckhoff, and I appreciate their careful attention to detail and good taste.

Thanks to Sol, Marilyn, and Jo Ann! You have kept this department running; thank you so much. May the coffee ever flow like wine, and the theorems and Sloan fellowships too.

Thanks to Martin Bridgeman, Dave Futer, Tao Li, and Andrew Yarmola, for helpful conversations about this work. Naturally, since they have vetted this, any remaining errors herein are *their* problem, not mine! (Just kidding.)

Thanks to my home away from home at Holy Trinity Orthodox Cathedral! Without their support, I would be depressed and this thesis would likely be a year late.

Thanks to Alireza Golsefidy! I think I am now coming to understand how to learn and do mathematics; I began to learn these things under your tutelage. Thank you.

Finally, thanks to my immediate family: to wit, Dad, Mom, T, Meema, and Lin-Lin. Your forbearance with me and your prayers, your love and your support made this possible.

Chapter 1

Introduction

Fundamental to our intuitive model of the universe is local spatiality, that our motion has three degrees of freedom. In other words, we model the universe as a 3-manifold. The class of 3-manifolds constitutes the gamut of possible inhabitable universes. As such, 3-manifolds are interesting objects of study in their own right.

The classification of n -manifolds for $n < 3$ is at least a century old now, its proof streamlined. The classification of n -manifolds for $n > 3$ has been proven impossible. The classification of 3-manifolds is not yet finished, but the story so far is one of the triumphs of recent mathematics.

Modern surface topology and modern 3-manifold topology have in common a striking characteristic: the pervasive influence of hyperbolic geometry. For instance, one may put the classification of closed orientable surfaces thus:

Except for the sphere and the torus, a closed orientable surface admits a complete hyperbolic metric of finite area, and is characterized uniquely by this area.

In particular, all but finitely many surfaces admit hyperbolic metrics, and area is a topological invariant. On the other hand, the theorem of Lickorish and Wallace, Thurston's Haken hyperbolization theorem, and Thurston's Dehn surgery theorem

imply that, in some sense, most closed 3-manifolds are also hyperbolic. Furthermore, by Mostow rigidity, all geometric invariants are topological invariants; in particular, volume is a topological invariant. Finally, Thurston also showed that there are only finitely many hyperbolic 3-manifolds of given volume, so volume virtually characterizes hyperbolic 3-manifolds, so to speak.

Continuing this line of thought about volume, we can strengthen our claims easily when speaking of hyperbolic surfaces:

The more area a hyperbolic surface has, the more complex its topological structure.

This is made precise by the Gauss-Bonnet theorem: the area of a hyperbolic surface S is $-2\pi \cdot \chi(S)$, where χ is Euler characteristic, a measure of topological complexity.

One would like to make a similar claim about 3-manifolds, but the Gauss-Bonnet theorem does not work in odd dimensions. Indeed, for all closed 3-manifolds and for all hyperbolic 3-manifolds of finite volume, the Euler characteristic is just 0. Much work in 3-manifold topology can be viewed as an attempt to define a decent notion of topological complexity to replace Euler characteristic.

The most obvious geometric such notion is the minimal number $\Delta(M)$ of tetrahedra necessary to triangulate the manifold M in question. In the case of surfaces, this turns out to be equivalent to χ , up to a multiplicative constant. Unfortunately for 3-manifolds, this measure does not behave well with respect to volume; there are, for instance, infinitely many hyperbolic 3-manifolds whose volume is less than 2.03, but only finitely many 3-manifolds with bounded Δ .

By the work of Thurston, such infinities of hyperbolic 3-manifolds with close volumes must come from Dehn fillings on a finite set of “parent” hyperbolic 3-manifolds. So one might ask instead for a measure of complexity that does not increase under Dehn filling. Such measures include the Mom number of Gabai, Meyerhoff, and Milley, and the treewidth of Burton and Downey. The present work is motivated by *Mom technology*, the body of work surrounding Mom number.

The significance of Mom number, briefly, is that Gabai, Meyerhoff, and Milley proved in [2] and [3] that

- a minimum-volume closed orientable hyperbolic 3-manifold is a Dehn filling of an orientable hyperbolic 3-manifold with one cusp¹ and with volume less than 2.848; and
- a hyperbolic 3-manifold with one cusp and with volume less than 2.848 has Mom number less than 4.

The bound on Mom number enabled Milley in [10] to construct a finite list of parents for the set of all hyperbolic one-cusped 3-manifolds of volume less than 2.848. Indeed, this is the list of topological Mom-2 and Mom-3 manifolds; a 3-manifold has Mom-number n when it is a Dehn filling of a topological Mom- n . Milley then determined which among the hundred-or-so parents were hyperbolic and which were not by hand; then determined which fillings might be hyperbolic with small volume via a theorem of Futer, Kalfagianni, and Purcell; and then did the same all over again for the resulting list of small one-cusped manifolds.

It would have been useful to have a computer program that could test whether or not a compact 3-manifold were hyperbolic. Presented here is a partial result along these lines—to wit, an algorithm and implementation thereof in Regina to determine, given a compact orientable 3-manifold with nonempty boundary, whether or not it admits a complete hyperbolic metric of finite volume. The algorithm depends essentially upon Thurston’s Haken hyperbolization theorem, which links hyperbolicity to the nonexistence of certain surfaces of small complexity embedded in the given manifold; and upon the algorithms already marvellously implemented in Regina for finding such surfaces.

This implementation answers a question posed by Gabai, Meyerhoff, and Milley about manifolds with Mom-number 4. Namely, we confirm their list of parent manifolds by disproving approximately 700 manifolds to be hyperbolic.

¹That is, one torus boundary component.

Finally, there remains the question of minimal Mom number. A 3-manifold can be both Mom-2 and Mom-4; indeed, the Whitehead link complement is both a topological Mom-2 and Mom-4, so any Dehn filling thereof has Mom numbers both 2 and 4. One may ask for the minimal Mom number of a 3-manifold. In particular, Dave Gabai has asked [1] for the minimal Mom number of all SnapPea census manifolds with maximal cusp area less than 5.1.

To prove that a 3-manifold has minimal Mom number 2 is simple; just express it as a Dehn filling of a topological Mom-2. But to prove a 3-manifold has minimal Mom number 3, one must show that it is *not* a Dehn filling of any Mom-2. To this end, presented here are bounds based on the work of Hodgson and Kerckhoff which lay the foundations for a future algorithm to tell whether or not a given compact orientable hyperbolic 3-manifold of finite volume is a Dehn filling of another such manifold.

Chapter 2

Triangulations in Regina

Since `Regina` is such a nice general program for studying both hyperbolic and non-hyperbolic 3-manifolds, we first will describe how to represent a topological $Mom-n$ manifold therein.

A topological $Mom-n$ manifold is defined as an ideal triangulation, so first, we make the following definition.

Definition 2.0.1. An oriented *ideal 3-triangulation* is a space resulting from a orientable face-pairing of oriented solid tetrahedra, such that no face-pairing identifies an oriented edge to itself backwards.

An orientable ideal triangulation of an orientable 3-manifold $(M, \partial M)$ is a homeomorphism $\phi : T \setminus T_0 \rightarrow M \setminus \partial M$, where T is an oriented ideal 3-triangulation, and T_0 is its vertices.

We note that the link of a vertex of an ideal triangulation corresponds through ϕ to a connected component of ∂M . In particular, closed 3-manifolds have no ideal triangulations as defined above. One may, however, express a closed 3-manifold as a Dehn filling of a “parent” manifold, and then ideally triangulate the parent. This is how `SnapPy` represents closed hyperbolic 3-manifolds.

Onward to the translation program. The relevant quotation from Milley’s docu-

mentation of his code for representing Mom- n manifolds as SnapPy triangulations is in the file `README.txt` in the outermost folder of the `enum.Moms` data from [3].

As an introduction to coding in Regina, we begin with a simple program, which constructs an n -dipyramid:

```
<construct dipyr>≡
def make_dipyr(n):
    """Returns an n-dipyr."""
    newt = regina.NTriangulation()
    for i in range(0,n):
        newt.newTetrahedron()
    for i in range(0,n):
        me = newt.getTetrahedron(i)
        you = newt.getTetrahedron((i+1)%n)
        me.joinTo(2,you,NPerm4(2,3))
    return newt
```

This code chunk illustrates the Python and Regina formalisms that will be used throughout this work. Those unfamiliar with these may visit their respective websites [13] and [14] to learn more. For now, here is what the above code chunk means, line by line.

`def make_dipyr(n):` means “The following code is a definition for a procedure called `make_dipyr` which takes one argument, which we will call `n`.”

““Returns an n-dipyr.”” is a Python-docstring, where “doc” is short for “documentation”. It explains briefly what the procedure does. I say “procedure,” because it is not a function. Every time it is called with argument n , it will make a new n -dipyr.

`newt = regina.NTriangulation()` basically means “Let `newt` start off as an empty triangulation.”

The next two lines basically mean “Let `newt` be the disjoint union of n tetrahedra.”

The next four lines require more explanation.

Let T and B be the top and bottom polar vertices of an n -dipyramid, and let $v_i, i \in \mathbb{Z}/n\mathbb{Z}$ be its equatorial vertices in order. By “in order”, I mean that v_i and v_{i+1} are connected by an edge. Then we may triangulate the n -dipyramid by a cyclic list of n tetrahedra, the i^{th} element of which is the tetrahedron T_i with vertices $T, B, v_i, v_{(i+1)}$, for $i \in \mathbb{Z}/n\mathbb{Z}$. Equivalently, the i^{th} element of this list is the tetrahedron T_i with vertices $T, B, v_i, v_{(i+1)\%n}$, where $0 \leq i < n$ and $k\%n$ is the least natural number a such that $k \equiv a \pmod{n}$.

Conversely, we may construct an n -dipyramid from these n tetrahedra by gluing them up appropriately. The appropriate gluings glue the face TBv_{i+1} in T_i to TBv_{i+1} in T_{i+1} , preserving incidence. The code implements this in `Regina`. Note that in `Regina`, the vertices of a tetrahedron are labelled `0, 1, 2, 3`, and the faces are labelled by the vertices they omit.

We know we want to glue T_i to $T_{(i+1)\%n}$ for each $0 \leq i < n$. The last four lines do this. The first line means “For all $0 \leq i < n$, run the following indented code block:”. The next two lines mean “Let `me` be T_i and `you` be $T_{(i+1)\%n}$.”

The last line of the indented code block under the `for` statement means “Glue the face of `me` opposite the vertex `2` to the face of `you` using the gluing map that permutes the vertices by the transposition $(2\ 3)$.”

The explanation for this is as follows. Vertex `2` of `me` should be $v_{(i+1)\%n}$ in the n -dipyramid, and vertex `3` of `you` should also be v_{i+1} . Furthermore, since both `2` and `3` will become equatorial vertices in the n -dipyramid, `0, 1` will become polar vertices. Then the gluing map should send the face `013` (the face opposite `2`) of T_i to the face `012` (the face opposite `3`) of $T_{(i+1)\%n}$. So on vertices, this gluing map acts as the transposition $(2\ 3)$. This concludes the explanation of the last line of the `for` loop block, and the explanation of the penultimate four lines.

The last line means “The final result of this procedure is the value of `newt`,” the value of `newt` being, of course, the newly-constructed n -dipyramid, represented as a

`Regina NTriangulation`. That concludes the explanation for this code chunk.

This code will not suffice, for Mom-4 manifolds may be glued from multiple dipyrs. So we should implement a procedure to construct a disjoint union of dipyrs according to a Mom prefix. That is, we want a procedure that constructs a `Regina NTriangulation` with an n -dipyr for every n in `prefix`, with multiplicity, where `prefix` is a finite tuple of positive integers. (We break from Milley's convention here, since in the future one may wish to have n -dipyrs with n having two or more digits.)

(make dipyrs by prefix) \equiv

```
def make_dipyr(prefix):
    """Return dipyrs specified by prefix."""
    newt = regina.NTriangulation()
    offset = 0
    for i in prefix:
        for j in range(0,i):
            newt.newTetrahedron()
        for j in range(0,i):
            me = newt.getTetrahedron(offset+j)
            jj = (j+1) % i
            you = newt.getTetrahedron(offset+jj)
            me.joinTo(2,you,regina.NPerm(2,3))
        offset += i
    return newt
```

Now we need to write code to glue up the remaining faces of the dipyrs according to the rest of Milley's Mom-strings. The rest of the Mom-string is a permutation in Cayley notation on $2n$ elements, where n is the sum of the prefix. Eventually, we would like to represent this not as a string, but as a $2n$ -tuple. Suppose then that `perm` is such a $2n$ -tuple. Then the face Milley calls `i` should be glued to the face Milley calls `perm[i]`.

The question, then, is which face of `make_dipyrs(prefix)` is the face Milley calls `i`? It must be a face opposite either vertex 0 or vertex 1 of some tetrahedron. If $0 \leq i < n$, then it is a face opposite vertex 0; otherwise, it is a face opposite vertex 1. The question remains, on which tetrahedron does Milley's face i lie? Note first that for $0 \leq j < n$, the faces j and $j + n$ lie on the same tetrahedron. In fact they both lie on tetrahedron j . In other words, the face of `make_dipyrs(prefix)` which Milley calls `i` is the face opposite vertex `depth` of tetrahedron `tet_idx`, where `(tet_idx,depth)` is `face(prefix,i)`, and where the latter is given by the following.

(tuples to triangulations) \equiv

```
def face(prefix,i):
    n = sum(prefix)
    if i < n:
        depth = 0
    else:
        depth = 1
    return (i%n, depth)
```

Therefore, a Mom manifold specified by one of Milley's Mom-strings is given by the following procedure.

(Mom manifold) \equiv

```
def make((prefix, perm)):
    newt = make_dipyrs(prefix)
    for i in perm:
        if i < perm[i]:
            <glue up the pair of faces>
    label = str((prefix,perm))
    newt.setPacketLabel(label)
    return newt
```

```

<glue up the pair of faces>≡
    (tet_i,depth_i) = face(prefix, i)
    (tet_j,depth_j) = face(prefix, perm[i])
    me = newt.getTetrahedron(tet_i)
    you = newt.getTetrahedron(tet_j)
    if depth_i == depth_j:
        p = regina.NPerm(2,3)
    else:
        p = regina.NPerm(0,1)
    me.joinTo(depth_i,you,p)

```

This bears some explanation. Northern faces have depth 0, and southern faces depth 1. A gluing map (of the sort considered above) between faces of the same depth will send $0\ 1$ to $0\ 1$. Since the map must be orientation reversing, its action on the vertices can't be the identity. So it must be given by $(2\ 3)$. Similarly, a gluing of faces with different depth must send $0\ 1$ to $1\ 0$. It can't also switch $2\ 3$, for then it would preserve orientation. So it must be given by $(0\ 1)$.

Having written a program to make a Mom manifold from two tuples representing, respectively, the prefix and permutation of a Mom string, let us now write a function to transform a Mom string into such a pair of tuples. We'll do it in `Regina-Python` since `Python` has good string-manipulation libraries.

(Milley to Regina-Python)≡

```
import shlex
def parse(mill):
    tokenized = shlex.split(mill)
    perm_str = tokenized[1:]
    perm = tuple(map(int, perm_str))
    pref_tok = tokenized[0]
    a = pref_tok.find("(") + 1
    b = pref_tok.find(")")
    pref_str = pref_tok[a:b]
    prefix = tuple(map(int, pref_str))
    reginafied = (prefix, perm)
    return reginafied
```

`map` is a higher-order function that applies its initial argument to all the elements of the following argument (assumed to be a `list` or `tuple` or some other `Iterable`) and returns the resulting values in a `list`.

In this case, its first argument is the function `int`, which attempts to interpret a string as a number in the usual way—e.g. `int("394")` returns the integer value three hundred ninety four, but `int("Fangorn")` fails.

`shlex.split` regards its argument as a series of *tokens*—space-free strings—separated by spaces. It returns the `list` of these tokens in order.

The first part of this method will need to be changed in the event that the prefix notation starts using spaces to distinguish numbers. Namely, it will need to tokenize `mill_prefix` using `shlex.split`, as for `perm`.

That concludes this chapter on turning descriptions of *Mom- n* manifolds as prefix-permutation pairs into *Regina* triangulations. We now move on to determining whether or not such triangulations represent hyperbolic 3-manifolds.

Chapter 3

Normal Surface Theory in Brief

Throughout, all surfaces, all 3-manifolds, and all functions between them shall be in some tame category like \mathcal{PL} or \mathcal{C}^∞ . All maps and manifolds shall be orientable (except for face-pairing maps).

The classification theorem for compact 2-manifolds can be interpreted as saying that essential curves in surfaces determine these surfaces. In dimension 2, determining these curves is essentially a homological problem. When extrapolating this to 3-manifolds, one can either guess that essential curves determine a 3-manifold or that essential codimension one objects determine a 3-manifold. The former claim isn't exactly true, and homology is no longer sufficient. The latter claim is nearer to the truth, since if there is an essential surface in a 3-manifold, then the homeomorphism problem is solved.

As vague as the above introduction is, it should at least make clear the fact that embeddings of surfaces into 3-manifolds are very important for understanding 3-manifold topology.

Now, one of the most natural representations of a 3-manifold is as a triangulation. We have already introduced the notions of ideal 3-triangulation and ideal triangulation of a 3-manifold. A more familiar notion is what we will call a *finite triangulation* of a 3-manifold:

Definition 3.0.2. An orientable finite triangulation of an orientable 3-manifold $(M, \partial M)$ is a homeomorphism $\phi : (T, \partial T) \rightarrow (M, \partial M)$ where T is an oriented ideal 3-triangulation.

We will have occasion to use both ideal and finite triangulations of 3-manifolds. We will refer to both of them as triangulations.

Since embedded surfaces are so important, we should ask how to represent them with respect to a triangulation T . The first thing to notice is that any surface embedding $\phi : (\Sigma, \partial\Sigma) \rightarrow (M, \partial M)$ may be isotoped to be transverse to (or in general position with respect to) the 1-skeleton $T^{(1)}$ of T . In fact, we can do better:

Lemma 3.0.3. *Let T triangulate a 3-manifold M . Let $\phi : (\Sigma, \partial\Sigma) \rightarrow (M, \partial M)$ be a surface embedding.*

ϕ may be isotoped so that for all 3-simplices $\Delta \in T$, for all components C of $\phi \cap \partial\Delta$, for all edges $e \in \partial\Delta$, $|C \cap e| \leq 1$.

We may isotope ϕ to be transverse to or in general position with respect to $T^{(1)}$. The weight function $w(\psi) = |\psi \cap T^{(1)}|$ is well-defined on such embeddings. This function is a variant for a while-loop whose invariant predicate is the existence of an edge with more than two points in common with ψ . This is how the proof goes. For more details, see [8].

There are, up to isotopy, only three sorts of curves on $\partial\Delta$ with this property: circles in the interior of a face, triangular boundaries of regular neighborhoods of vertices, and quadrilateral boundaries of regular neighborhoods of edges.

The latter two curves are boundaries of triangle discs and quad discs, respectively. Such discs are called *normal discs* in Δ .

Definition 3.0.4. A surface embedding ϕ is *normal* with respect to a triangulation T when for all 3-simplices Δ of T , every component F of $\phi \cap \Delta$ is a normal disc.

This is quite a restrictive condition. One can show, moreover, that any such

surface is determined, up to isotopy preserving the triangulation's incidence structure (so-called *normal isotopy*), by its normal discs.

More precisely, let Δ be the set of normal isotopy classes of triangle discs in 3-simplices of T , and likewise let \square be the set of normal isotopy classes of quad discs in 3-simplices of T . For any normal surface ϕ , and for any $t \in \Delta$, letting Δ be the 3-simplex supporting t , define $t.\phi$ to be the number of components of $\phi \cap \Delta$ in t ; likewise for $q \in \square$. We define the normal coordinates $c.\phi$ of ϕ as the element of $\mathbf{N}^\Delta \times \mathbf{N}^\square$ such that $(\pi_0(c.\phi))(t) = t.\phi$ and likewise $(\pi_1(c.\phi))(q) = q.\phi$.

The vague statement above is made more precise by the following (see, e.g. [9]):

Lemma 3.0.5. *The normal coordinates of a normal surface determine that surface up to normal isotopy.*

The first surprise of normal surface theory is that (in spite of normality's restrictiveness) if there is an interesting surface, then there must be an interesting *normal* surface:

Lemma 3.0.6. *Let M be a closed 3-manifold, and let T triangulate M .*

- *If M contains an essential sphere (a sphere not bounding a 3-ball), then it contains an essential sphere normal to T .*
- *If M is irreducible and it contains an incompressible surface, then M contains an incompressible surface normal to T .*

Proof. See lemmas 2.11 and 2.12 of [8]. □

This still does not provide a way to detect whether or not there are such surfaces in a 3-manifold with a given triangulation.

Notice, however, that if two 3-simplices Δ and Δ' are glued together along faces f, f' as depicted, then the arcs of $\phi \cap f$ must be identified with arcs of $\phi \cap f'$.

More precisely, suppose a is an arc of f , and suppose a is identified to the arc a' of f' . Then the number of components of $\phi \cap f$ normally isotopic to a must equal the number of components of $\phi \cap f'$ normally isotopic to a' .

But the number of components of $\phi \cap f$ normally isotopic to a is equal to $t_a \cdot \phi + q_a \cdot \phi$, where t_a is the normal isotopy class of a normal triangle disc with a on its boundary, and likewise q_a is the normal isotopy class of a normal quad disc with a on its boundary.

Therefore, every face-pairing in T yields three homogeneous linear equations on $\mathbf{N}^\Delta \times \mathbf{N}^\square$ of the form

$$t_a + q_a = t_{a'} + q_{a'}.$$

These are called the (Haken) *matching equations*.

Next, notice that if q, q' are distinct quads supported in the same 3-simplex, then there is no normal surface ϕ such that $q \cdot \phi > 0$ and $q' \cdot \phi > 0$. This is called the *quad condition*, or the *admissibility criterion*.

From the above, it is obvious that the normal coordinates of every normal surface constitute an admissible solution to the matching equations. Conversely (see [9]),

Lemma 3.0.7. *Any admissible solution to the matching equations is a set of normal coordinates for a normal surface.*

The set of solutions to the matching equations is closed under addition. A solution s is called *fundamental* when for all solutions t, u , $s = t + u$ is equivalent to $\{t, u\} = \{0, s\}$. One may prove (see [9], p. 114)

Theorem 3.0.8. *The fundamental solutions to a system of homogeneous linear equations over \mathbf{N} is finite and computable, and every solution is a finite linear combination of fundamental solutions.*

The second surprise of normal surface theory is that if there is an interesting surface of low genus, then there is an interesting *fundamental* surface of low genus.

Theorem 3.0.9. *Let M be a compact 3-manifold. Let T triangulate M .*

- *If M has an essential sphere or RP^2 , then T admits a fundamental normal sphere or RP^2 .*
- *Suppose M is irreducible.*
 - *If M has a compressing disc, then T admits a fundamental compressing disc.*
 - *Suppose M has incompressible boundary.*
 - * *If M has an incompressible torus, then T admits an incompressible fundamental torus or an embedded fundamental Klein bottle.*
 - * *If M has an essential (i.e. incompressible and ∂ -incompressible) annulus, then T admits either an essential fundamental annulus or an embedded fundamental Möbius band.*

One can find the parts of this theorem scattered in various places in [9]. Their proofs all involve showing that a least-weight normal essential surface must be fundamental.

The easiest fundamental surfaces to calculate are the *vertex surfaces*, so-called for the following reason. Notice that since the matching equations are homogeneous, they descend to linear equations on some projective space, if we start taking rational coordinates. The admissible rational solutions project to a convex polytope in this projective space. A vertex solution is a fundamental solution that projects to a vertex of this convex polytope.

The third surprise of normal surface theory is that, for *finite* triangulations, if there is an interesting surface, then there is an interesting *vertex* surface (see e.g. [7]):

Theorem 3.0.10. *Let M be a compact 3-manifold. Let T finitely triangulate M .*

- *If M has an essential sphere, then T admits a vertex essential sphere.*

- *Suppose M is irreducible.*
 - *If M has a compressing disc, then T admits a vertex compressing disc.*
 - *Suppose M is ∂ -irreducible.*
 - * *If M has an essential two-sided annulus or torus, then T admits a vertex essential annulus or torus.*

We will regard boundary-parallel surfaces as uninteresting. The fourth surprise of normal surface theory is that one may develop matching equations and admissibility criteria on just \mathbf{N}^\square and get analogous results to the above. That is, there is a set of linear homogeneous equations on \mathbf{N}^\square called the Q -matching equations, and linear inequalities on \mathbf{N}^\square called the Q -admissibility criteria, such that the following theorem is true.

Theorem 3.0.11 (Thm. 1 of [11]). *Let M be a compact, irreducible, ∂ -irreducible 3-manifold finitely triangulated by T .*

If M admits an incompressible, ∂ -incompressible surface, then T admits a Q -vertex such surface.

Moreover, every normal surface yields an admissible solution to the Q -matching equations, and conversely, every admissible solution to these equations is a set of quad coordinates for a unique normal surface with no boundary-parallel components.

We will find the following lemmas useful in the next chapter. First a definition.

Definition 3.0.12. *A medium Seifert fibering is a Seifert fibering over base orbifold a sphere with b punctures and c cone points such that $b + c \leq 3$.*

Lemma 3.0.13. *Let M be a medium Seifert fibering with nonempty boundary.*

Every finite triangulation of M admits a vertex Q -normal annulus fault.

Lemma 3.0.14. *Let M be a medium Seifert fibering with at least two boundary tori.*

Every finite triangulation of M admits a non-separating vertex Q -normal annulus fault.

Proof of Lemma 3.0.13. We briefly recall some terminology. The *carrier* \mathcal{C} of a point on a convex polytope is the smallest face of the polytope containing the point.

Suppose M is a medium Seifert fibering with nonempty boundary, and suppose T finitely triangulates M .

M 's base orbifold has at least one boundary component S . Suppose there were no essential simple arc from S to itself. Then M has a disc with no cone points for its base orbifold, and M is a solid torus. But solid tori are ∂ -compressible, contrary to our assumptions on M . Hence there is an essential simple arc from S to itself.

The vertical fiber a over this arc is an annulus. a is essential, so it isotopes to a normal annulus. Let A be such an annulus such that the number of intersections of A with the 1-skeleton of T is minimal among normal surfaces isotopic to a . That is, let A have least weight in its isotopy class.

Now, every vertex surface in $\mathcal{C}(A)$ is an essential annulus or an essential torus (Cor. 6.8, [7]). There are no essential tori, by assumption. Consequently, each vertex surface in $\mathcal{C}(A)$ is an essential annulus. The proof of Theorem 2 in [11] shows that every two-sided vertex surface in $\mathcal{C}(A)$ is isotopic to a Q-vertex surface. Thus T admits some essential Q-vertex annulus. \square

Proof of Lemma 3.0.14. Suppose M is a medium Seifert fibering with at least two boundary tori, and suppose T is a finite triangulation of M .

M 's base orbifold now, by assumption, has at least two boundary components. Up to isotopy, there is a unique essential simple arc running between them. Let a be a vertical fiber over such an arc, a non-separating annulus. Since a is non-separating, it is essential. So it isotopes to a normal annulus. Let A be a least weight such annulus. Again, every vertex surface in $\mathcal{C}(A)$ is an essential annulus (or an essential torus, of which we've assumed there are none). Furthermore, every such essential annulus is isotopic to a Q-vertex surface, as above. So we just need a *non-separating* vertex annulus in $\mathcal{C}(A)$.

If there is a horizontal vertex annulus in $\mathcal{C}(A)$, then that is a non-separating vertex

annulus in $\mathcal{C}(A)$.

Otherwise, A is a sum of some vertical vertex annuli in $\mathcal{C}(A)$.

The geometric sum (see [9], pp. 136–7 or [7], Fig. 2.1, p. 363) on the boundary just resolves intersections $\times \rightarrow \asymp$, which preserves homology mod 2. Therefore, for all normal surfaces S, S' , $\partial(S + S') \equiv \partial(S) + \partial(S')$ in $H_1(\partial M; \mathbb{Z}/2\mathbb{Z})$.

The boundaries of separating annuli in M are 0 in $H_1(\partial M; \mathbb{Z}/2\mathbb{Z})$, but ∂A is not 0 in this homology. So A is not a sum of separating annuli. Consequently one of the summands must be non-separating, and be a non-separating vertex annulus in $\mathcal{C}(A)$.

□

Chapter 4

Determining hyperbolicity

The work of Jørgensen, Thurston, and Gromov in the late '70s showed that the set of volumes of orientable hyperbolic 3-manifolds has order type ω^ω . Cao and Meyerhoff in 2001 showed that the first limit point is the volume of the figure eight knot complement. Agol in 2010 showed that the first limit point of limit points is the volume of the Whitehead link complement. Most significantly for this paper, Gabai, Meyerhoff, and Milley (in the series of papers [2], [3], and [10]) showed that the smallest, closed, orientable hyperbolic 3-manifold is the Weeks-Matveev-Fomenko manifold.

The proof of the last result required distinguishing hyperbolic 3-manifolds from non-hyperbolic 3-manifolds in a large list of 3-manifolds; this was carried out in [10]. The method of proof was to see whether SnapPea's *canonicalize* procedure succeeded or not; identify the successes as census manifolds; and then examine the fundamental groups of the 66 remaining manifolds by hand. This method made the analysis of non-hyperbolic Mom-4 manifolds, of which there are 762 combinatorial types, prohibitively time-consuming.

The algorithm presented here determines whether or not a compact 3-manifold admits a complete finite-volume hyperbolic metric, i.e. is *hyperbolic*, assuming the manifold in question has nonempty boundary consisting of tori.

4.1 Rewriting Thurston's Haken theorem

Because it is so fundamental to modern 3-manifold topology, Thurston's hyperbolicity theorem for Haken manifolds merits a succinct formulation. Shoving some complications from the original theorem into definitions and restricting attention to manifolds with nonempty torus boundary yields

Theorem 4.1.1. *Let M be a compact orientable 3-manifold with nonempty boundary consisting of tori.*

M is hyperbolic with finite volume if and only if M has no faults.

The above uses the following definitions.

Definition 4.1.2. A manifold is *hyperbolic* when its interior admits a complete hyperbolic metric—a complete Riemannian metric of constant negative curvature.

Definition 4.1.3. Let s be an embedding of a manifold into a connected manifold M . By abuse of notation, also let s denote the image of s in M . Suppose s has codimension 1. Pick a metric on M compatible with its p.l. structure, and let M' be the path-metric completion of $M \setminus s$.

When M' is disconnected, s *separates* M .

When M' has two connected components N, N' , s *cuts off* N from M , or, if M is understood from context, s *cuts off* N .

If N is homeomorphic to some common 3-manifold X , s *cuts off an* X ; if, in addition, N' is not homeomorphic to X , s *cuts off one* X .

Definition 4.1.4. A properly embedded surface s in an orientable 3-manifold M is a *fault* when $\chi(s) \geq 0$ and it satisfies one of the following:

- s is nonorientable.
- s is a sphere which does not cut off a 3-ball.
- s is a disc which does not off one 3-ball.

- s is a torus which does not cut off a $T^2 \times I$, and does not cut off a ∂ -compressible manifold.
- s is an annulus which does not cut off a 3-ball, and does not cut off one solid torus.

Sketch of theorem. 4.1.1's proof. This is a corollary of common knowledge surrounding Thurston's hyperbolization theorem for Haken manifolds. Specifically, it's commonly known that an irreducible, ∂ -incompressible, geometrically atoroidal 3-manifold with nonempty boundary consisting of tori is either hyperbolic or Seifert-fibered, where "hyperbolic" means "admits a complete hyperbolic metric." All Seifert-fibered spaces with at least two boundary components admit essential tori, which are faults. A Seifert-fibered space with one boundary component admits no essential tori. But it still admits an annulus fault, namely a vertical fiber over an arc separating the cone points of the base orbifold, which is a disc with two cone points. Hence all Seifert-fibered spaces with nonempty boundary admit faults.

Consequently, a compact orientable 3-manifold with nonempty boundary consisting of tori which admits no faults is irreducible, ∂ -incompressible, Haken, and geometrically atoroidal, and it admits no annulus faults. So it must be hyperbolic.

In fact, Thurston proved something more, namely that unless this manifold is $T^2 \times I$, then its metric has finite volume. Now, $T^2 \times I$ admits faults—non-separating annuli, in fact. Since we assumed the manifold had no faults, its metric must have finite volume.

Conversely, hyperbolic 3-manifolds of finite volume admit no orientable faults—they have no essential spheres, no compressing discs, no incompressible tori which aren't ∂ -parallel, and no annuli which are both incompressible and ∂ -incompressible. Finally, orientable hyperbolic 3-manifolds of finite volume don't admit any faults at all, since they admit no properly embedded nonorientable surfaces of nonnegative Euler characteristic. \square

Having finished this first reformulation, we note the following theorems from normal surface theory.

Theorem 4.1.5. *Let T ideally triangulate a compact orientable 3-manifold M . Then M has a closed fault precisely when T has a fundamental normal fault.*

Theorem 4.1.6. *Let T finitely triangulate an irreducible, ∂ -incompressible, geometrically atoroidal 3-manifold M with nonempty boundary consisting of tori.*

1. *M has a fault if and only if T has a vertex Q -normal annulus fault.*
2. *If M has at least two boundary components, then M has a fault if and only if T has a non-separating vertex Q -normal annulus fault.*

The last section of this chapter contains proofs of these statements.

Theorems 4.1.1, 4.1.5, and 4.1.6 together yield the following useful results amenable to computer implementation.

Corollary 4.1.7. *Let M be a compact orientable 3-manifold with nonempty boundary consisting of tori.*

Let T, T' triangulate M ideally and finitely, respectively.

M is hyperbolic precisely when T has no fundamental normal closed fault, T' has no disc fault, and T' has no vertex Q -normal annulus fault.

Corollary 4.1.8. *The last condition in Corollary 4.1.7 can be relaxed to having no non-separating vertex Q -normal annulus fault in case $|\partial M| \geq 2$.*

Therefore, assuming T is an ideal triangulation of a compact orientable 3-manifold M with nonempty boundary consisting of tori,

```

l := list of fundamental normal surfaces in T
for surf in l:
  if surf is fault:
    return False

```

```

T' := truncation of T to finite triangulation
if T' has a compressing disc:
    return False
l' := list of vertex Q-normal surfaces in T'
for annulus in l':
    if M has at least two boundary tori:
        if annulus is non-separating:
            return False
    else:
        if annulus is fault:
            return False
else:
    return True

```

describes an algorithm determining whether or not M is hyperbolic.

Of course, this algorithm depends upon

- enumerating fundamental normal surfaces of ideal triangulations;
- truncating ideal triangulations into finite triangulations;
- the predicate “has a compressing disc”;
- enumerating vertex Q -normal surfaces of finite triangulations;
- the predicate “is non-separating annulus”; and
- the predicate “is fault.”

All but the last two are already described in the existing literature and implemented conveniently in Regina.

The relevant tests and algorithms for detecting non-separating annuli are in Regina already—calculating Euler characteristic, cutting along a surface, and determining whether or not a manifold is connected.

The relevant tests for faultiness (all but the last of which are in Regina) are

- “is a 3-ball”
- “is ∂ -compressible”
- “is a solid torus”, and
- “is $T^2 \times I$ ”.

We can notice first that admitting a non-separating annulus is a necessary condition for being $T^2 \times I$. We note that a further necessary condition for being $T^2 \times I$ is that splitting along any such annulus is a solid torus. Finally, $T^2 \times I$ has exactly two boundary components.

Now, if a 3-manifold M split along a non-separating annulus is a solid torus, then M is a Seifert fibering with base orbifold an annulus or a Möbius band with at most a single cone point, i.e. $M = M(+0, 2; r)$ or $M = M(-0, 1; r)$ for some $r \in \mathbb{Q}$. Of course, if M is to be homeomorphic to $T^2 \times I$, it must have two boundary components, so $M = M(+0, 2; r)$, not $M(-0, 1; r)$.

Recall the following results about Seifert fiberings:

Proposition 4.1.9 ([4], 2.1). *Every orientable Seifert fibering is isomorphic to one of the models $M(\pm g, b; s_1, \dots, s_k)$. Any two Seifert fiberings with the same $\pm g$ and b are isomorphic when their multisets of slopes are equal modulo 1 after removing integers, assuming $b > 0$.*

Theorem 4.1.10 ([4], 2.3). *Orientable manifolds admitting Seifert fiberings have unique such fiberings up to isomorphism, except for $M(0, 1; s)$ for all $s \in \mathbb{Q}$ (the solid torus), $M(0, 1; 1/2, 1/2) = M(-1, 1;)$ (not the solid torus), and three others without boundary.*

Proposition 4.1.11. *Among manifolds of the form $M(\pm 0, 2; r)$, only $T^2 \times I$ has all Dehn fillings being solid tori.*

Proof. Plainly $T^2 \times I$ has this property.

Suppose $M(0, 2; r)$ is not $T^2 \times I$. Then by Proposition 4.1.9 and Theorem 4.1.10, $r \notin \mathbb{Z}$. We wish to show that $M(0, 2; r)$ admits some Dehn filling which is not a solid torus. Let s, s' be two slopes differing mod 1. Then $M(0, 1; r, s)$ and $M(0, 1; r, s')$ are Dehn fillings of $M(0, 2; r)$. They are not homeomorphic, by Theorem 4.1.10 and the fact that $r \notin \mathbb{Z}$. So one of them is not a solid torus. \square

It is quite easy to compute slopes differing mod 1 after simplifying the cusps' induced triangulations.

Proposition 4.1.12. *In a triangulation of the torus T^2 by one vertex, three edges, and two faces, for any nontrivial element g of $H_1(T^2)$, the edges represent homology classes not all equivalent mod g .*

Proof. Suppose $v, w, x \in H_1(T^2)$ and $v + w = x$. Let $=_g$ denote equivalence in $H_1(T^2)$ mod g . Then

$$\begin{aligned} & \Rightarrow \begin{aligned} v &= _g x \\ v + w &= _g x + w \end{aligned} \\ & \equiv \{ v + w = _g x \} \\ & \equiv \begin{aligned} x &= _g x + w \\ 0 &= _g w; \end{aligned} \end{aligned}$$

assuming $v =_g w$ then implies v and x also are 0 mod g . Therefore they are all multiples of g . But $H_1(T^2)$ is not cyclic. So v, w, x cannot generate $H_1(T^2)$.

Now, one may pick homology classes v, w, x representing the three edges such that $v + w = x$. These generate $H_1(T^2)$. Therefore, by the above argument, they cannot satisfy $v =_g w =_g x$ for any element g . \square

Corollary 4.1.13. *The following pseudocode describes an algorithm determining whether or not a compact, orientable, 3-manifold M with nonempty boundary consisting of tori is $T^2 \times I$:*

```

if M splits along no annulus into a solid torus:
    return False
let D be M's triangulation
let T be a boundary component of M
let tr(T,D) be the triangulation on T induced from D
change D so tr(T,D) has 2 faces, 3 edges, and 1 vertex
if M filled along one of the 3 edges' slopes is not a solid torus:
    return False
else:
    return True

```

Proof. Suppose M is $T^2 \times I$. Then the first if-statement doesn't activate, for M splits along an annulus into a solid torus. Also, M filled along any edge's slope whatever is a solid torus, so the second if-statement doesn't activate. So the algorithm returns **True**.

Suppose instead that M is not $T^2 \times I$. If M splits along no non-separating annulus into a solid torus, then the algorithm correctly returns **False**. Otherwise, M does so split, and therefore $M = M(0, 2; r)$ for some $r \in \mathbb{Q} \setminus \mathbb{Z}$. The algorithm then establishes that M 's triangulation induces a minimal triangulation on the boundary component T . By Proposition 4.1.12, the edges represent at least two different slopes modulo 1. Therefore, by Proposition 4.1.9 and Theorem 4.1.10, the Dehn fillings of M along these slopes are not all homeomorphic. In particular, one of them is not a solid torus. Therefore, the if-statement activates, and the algorithm correctly returns **False**. \square

It remains to describe

- splitting along a non-separating annulus into a solid torus,
- changing a triangulation to induce a minimal triangulation on a cusp, and
- filling along a slope in a simplified cusp.

Proposition 4.1.14. *The following pseudocode describes an algorithm implementing the first item:*

```

for every vertex Q-normal surface s in M:
  if s is a non-separating annulus:
    if M splits along s into a solid torus:
      return True
return False

```

Proof. Suppose M doesn't split along a non-separating annulus into a solid torus. Then not both of the if-statements can activate, so the for loop ends without returning, and so the algorithm correctly returns **False**.

On the other hand, if M does split along a non-separating annulus into a solid torus, then M is of the form $M(0, 2; r)$. By Lemma 3.0.14, every finite triangulation of such a manifold admits a non-separating Q-vertex annulus. Hence the if-statements eventually activate, and the algorithm correctly returns **True**. \square

Now for the next item, simplifying cusps. One may find a nice algorithm in SnapPea for doing this, a special, simpler case of which is presented here. We use the following terminology.

Definition 4.1.15. First, suppose M is finitely triangulated. Let T, T' be boundary triangles adjacent along an edge e . Orient e so that T lies to its left and T' to its right. Let Δ be a fresh tetrahedron, and let τ, τ' be boundary triangles of Δ adjacent along an edge η . Orient η so that τ lies to its left and τ' to its right. Without changing M 's topology we may glue Δ to T by gluing η to e , τ to T' and τ' to T . This is called a *two-two* move.

In the above definition, the edge η' opposite η in Δ becomes a boundary edge of the new finite triangulation.

Definition 4.1.16. We say e is *embedded* when its vertices are distinct. We say e is *coembedded* when η' as defined above is embedded. Equivalently, e is coembedded when the vertices in T, T' opposite e are distinct.

Given a boundary edge e between two boundary triangles T and T' , one may glue T to T' and e to itself via a valid, orientation-reversing map from T to T' . This identification we call “folding along e ”. (Weeks, in the SnapPea source code, calls this a “close-the-book” move.) This gluing will change the topology of M when the vertices opposite e in T and T' are the same vertex. Conversely, when these vertices are distinct, the folding preserves the topology. In other words, folding along e preserves topology if and only if e is coembedded.

Notice that folding along a coembedded edge decreases the number of boundary triangles, and performing a two-two move on an embedded edge produces a coembedded edge and preserves the number of boundary triangles. Therefore, the following while-loops terminate, using number of boundary triangles as a variant function:

```
while there's an embedded boundary edge e:
  do a two-two move on e
  while there's a coembedded boundary edge f:
    fold along f
```

The obvious postcondition of the outer while loop is that there is no embedded boundary edge. Since the boundary is still triangulated, this is equivalent to each boundary component having only one vertex on it. Since each boundary component is a torus, $V - E + F = 0$. Now, $V = 1$, and since the cellulation is a triangulation,

$$3 * F = 2 * E.$$

$$1 - E + F = 0$$

$$2 - 2 * E + 2 * F = 0$$

$$2 - 3 * F + 2 * F = 0$$

$$2 - F = 0$$

$$2 = F,$$

and there are only two triangles and three edges.

The routine in SnapPea is more complicated because, rather than filling in a cusp any old way, SnapPea wants to make sure the filling compresses some given slope in the cusp.

In conclusion,

Proposition 4.1.17. *The following pseudocode changes a finite triangulation D with boundary consisting of tori so that D induces a minimal triangulation on every boundary component:*

while D has an embedded boundary edge e :

do a two-two move on e

while D has a coembedded boundary edge f :

fold along f

Proof. See above discussion. □

Finally,

Proposition 4.1.18. *Assuming a triangulation D has a torus boundary component T and induces a minimal triangulation thereon, the following pseudocode determines whether folding along one of the edges in T yields a solid torus:*

```

for each edge e in T:
    let N be D folded along e
    if N is a solid torus:
        return True
return False

```

Proof. See above. □

This concludes the present sketch of an algorithm to determine hyperbolicity of a compact, orientable 3-manifold with nonempty boundary consisting of tori. Both literate and raw implementations of this algorithm as a Regina-Python module `unhyp` reside at [15]. Also available at [15] is a Regina Python module `mom` for interpreting Milley’s data as manifolds in Regina.

4.2 On Faults

Proof of Thm. 4.1.5. This is just breaking down the definition of closed fault and using the theorems of normal surface theory cited in Chapter 3.

In particular, the first item of Theorem 3.0.9 can be recast as saying M has a closed fault of $\chi > 0$ precisely when T has a fundamental closed such fault.

If this is not the case, then M is irreducible, so the third item of Theorem 3.0.9 applies. Hence, M admits an essential torus—that is, an incompressible torus which is not ∂ -parallel (see [9], p. 245), i.e. a torus fault—or injective Klein bottle precisely when T admits a fundamental essential torus or a fundamental injective Klein bottle, both of which are closed faults. □

Proof of Theorem 4.1.6. For the first part, note that since M is irreducible, it has no sphere fault or P^2 fault. Since M is ∂ -incompressible, it has no disc fault. Since M is geometrically atoroidal, it has neither torus fault nor Klein bottle fault. Hence any fault must be an annulus fault or a Möbius band fault.

Suppose M has such a fault Σ . If Σ is a Möbius band, then the boundary of a regular neighborhood is an essential annulus A ; otherwise, Σ itself is an essential annulus A . Isotope A to have least possible weight. By Corollary 6.8 of [7], every vertex surface in the carrier $\mathcal{C}(A)$ is either an essential annulus or torus. By assumption, there are no essential tori; hence, every such vertex surface is an essential annulus. Let F be one of these surfaces. The proof of Theorem 2 of [11] shows that F is isotopic to a Q-vertex surface. So there is a Q-vertex essential annulus. That is, T admits a Q-vertex annulus fault.

For the second part, suppose M has at least two boundary components. M is compact, orientable, irreducible, ∂ -irreducible, and geometrically atoroidal. By Thurston's Haken theorem, this implies that M is either hyperbolic or Seifert-fibered. If M is hyperbolic, it has no faults. If instead M is Seifert-fibered, then because M is geometrically atoroidal its base orbifold is a sphere with b holes and c cone points such that $b + c \leq 3$ —i.e., M is a medium Seifert fibering. Lemma 3.0.14 implies that a finite triangulation T of M admits a non-separating Q-vertex annulus fault. \square

Chapter 5

Practical Bounds on Dehn Surgery Space

The presumptive title of this chapter is only possible because of the groundbreaking work of Craig Hodgson and Steve Kerckhoff on making effective Thurston's original landmark Dehn surgery theorem. The following work is the third step in the general pattern seen in algorithmic 3-manifold topology: first come existence results (e.g. Kneser-Milnor prime decomposition theorem), then come algorithms (e.g. Jaco-Oertel-Tollefson algorithms), then come refinements suitable for computer implementation (e.g. Jaco-Rubinstein-Burton crushing), then come censuses and running time estimates (e.g. Burton's nine-tetrahedron census), then come more questions (e.g. Luo's alternative).

Let us get right to it. The theorem inspiring the following work is

Theorem 5.0.1 (Thm. 5.11, Cor. 5.13 [5]). *Let X be an orientable 3-manifold with nonempty boundary and a complete finite-volume hyperbolic metric on its interior.*

Let \hat{L} be a normalized length function on Dehn filling coefficients (see pp. 1068 and 1076 of [5]), and suppose $c \in H_1(\partial X; \mathbb{R})$ is a Dehn filling coefficient such that $\hat{L}(c) > 7.5832$. Then

- $X(c)$ admits a complete finite-volume hyperbolic metric on its interior;
- $\text{vol}(X) - \text{vol}(X(c)) < 0.198$; and
- the geodesic core of the filling has length at most 0.156012.

After suitably rephrasing this, it seems to give a practical method for solving our problem:

Corollary 5.0.2. *Let M, N be orientable 3-manifolds admitting complete hyperbolic metrics of finite volume on their interior. N is a Dehn filling of M if and only if⁴ either*

- N is a Dehn filling of M along a slope of normalized length less than or equal to 7.5832, or
- M is isometric to $N \setminus \gamma$ for a simple closed geodesic of length less than 0.156012.

The collection of slopes of ∂M with normalized length less than 7.5832 is computable, and likewise the length spectrum of N is computable, and SnapPy can drill out curves and determine isometries, so that is that. Right?

Unfortunately not. The problem is in drilling out curves. SnapPy can only drill out simple closed curves in the dual 1-complex of an ideal triangulation. As explained in [6] on page 264, these may or may not be isotopic (or even homotopic) to a given geodesic which one wishes to drill out.

Fortunately, Theorem 5.0.1 is a corollary of a much more powerful theorem, Theorem 5.1.1, about volume change under drilling. Theorem 5.0.1 follows from the upper bounds in Theorem 5.0.1, but Theorem 5.0.1 contains lower bounds as well. We use both bounds in what follows to enable a solution to the problem in terms of procedures either already made rigorous or with a reasonable hope of being made rigorous soon, viz. length spectra, cusp area, slope length, and (to a lesser extent) isometry testing.

⁴The only-if part is the content of Theorem 5.0.1.

5.1 Rewriting the Hodgson-Kerckhoff Bounds

The stronger theorem alluded to above is

Theorem 5.1.1 (theorem. 5.12, [5]). *Let X, \hat{L} , and c be as in Theorem 5.0.1. Let $\Delta V = \text{vol}(X) - \text{vol}(X(c))$. Let ℓ be the length of the geodesic core of the filling. Then*

$$\frac{1}{4} \cdot \int_{\tilde{z}}^1 \frac{H'(z)}{H(z) \cdot (H(z) - \tilde{G}(z))} dz \leq \Delta V, \quad (5.1.1)$$

$$\Delta V \leq \frac{1}{4} \cdot \int_{\tilde{z}}^1 \frac{H'(z)}{H(z) \cdot (H(z) + G(z))} dz, \quad (5.1.2)$$

and

$$1/H(\tilde{z}) \leq 2\pi \cdot \ell \leq 1/H(\hat{z}), \quad (5.1.3)$$

where $H, G, \tilde{G}, \tilde{z}$, and \hat{z} have the following definitions.

Definition 5.1.2.

$$\begin{aligned} K &= 3.3957, & h(z) &= \frac{1+z^2}{z \cdot (1-z^2)}, \\ \tilde{g}(z) &= \frac{(1+z^2)^2}{2 \cdot z^3 \cdot (3-z^2)}, & g(z) &= \frac{1+z^2}{2 \cdot z^3}, \\ H &= h/K, & G &= g/K, & \tilde{G} &= \tilde{g}/K, \\ F(z) &= \frac{H'(z)}{H(z) + G(z)} - \frac{1}{1-z} = \frac{h'(z)}{h(z) + g(z)} - \frac{1}{1-z}, \\ \tilde{F}(z) &= \frac{H'(z)}{H(z) - \tilde{G}(z)} - \frac{1}{1-z} = \frac{h'(z)}{h(z) - \tilde{g}(z)} - \frac{1}{1-z}, \\ f(z) &= K \cdot (1-z) \cdot e^{-\Phi(z)}, & \Phi(z) &= \int_1^z F(w) dw, \\ \tilde{f}(z) &= K \cdot (1-z) \cdot e^{-\tilde{\Phi}(z)}, & \tilde{\Phi}(z) &= \int_1^z \tilde{F}(w) dw, \\ f(\hat{z}) &= \frac{(2\pi)^2}{\hat{L}(c)^2}, & \tilde{f}(\tilde{z}) &= \frac{(2\pi)^2}{\hat{L}(c)^2} \end{aligned}$$

These definitions are from pp. 1079, 1080, and 1088 of [5]. The reader should note

that the above theorem has $2\pi \cdot \ell$ in place of \mathcal{A} . This is valid—see, e.g., Corollary 5.13 of [5].

This gives complicated bounds on ΔV and ℓ in terms of \hat{z} and \tilde{z} . We require simple but not necessarily tight upper and lower bounds on ℓ and $\hat{L}(c)$ in terms of ΔV . The bounds on ℓ will be used most often; the upper bounds on $\hat{L}(c)$ will be used when the volumes of the putative parent and child P and C are close, *and* C has a very short geodesic. (For instance, P might be the Whitehead link complement, and C might be a high-order Dehn filling on the Whitehead link complement sibling.)

To get these bounds, we will approximate solutions to inequalities (5.1.1) and (5.1.2) in \tilde{z} and \hat{z} , respectively, for given ΔV .

5.1.1 Monotonocities

Let

$$LB(z) = \frac{1}{4} \cdot \int_z^1 \frac{H'(w)}{H(w) \cdot (H(w) - \tilde{G}(w))} dw \quad (5.1.4)$$

and

$$UB(z) = \frac{1}{4} \cdot \int_z^1 \frac{H'(w)}{H(w) \cdot (H(w) + G(w))} dw. \quad (5.1.5)$$

We intend to solve the inequalities by inverting LB and UB . This will work if we know the monotonicity of LB and UB . We will require the monotonicity of several other functions as well, and the (very calculational) proofs are in proof-hint notation. It behooves us then to introduce “ \sim .”

Definition 5.1.3. For all real x and y , $x \sim y$ when $sgn(x) = sgn(y)$, where $sgn(x)$ is the signum function $sgn(0) = 0$, else $sgn(x) = |x|/x$.

Lemma 5.1.4. LB is decreasing on $(\sqrt{\sqrt{5}-2}, 1)$.

Lemma 5.1.5. UB is decreasing on $(\sqrt{\sqrt{5}-2}, \infty)$.

Proof of Lemma 5.1.4.

$$\begin{aligned}
& LB'(z) \\
&= \{ \text{by definition of } LB \} \\
&\quad \frac{1}{4} \cdot \frac{H'(z)}{H(z) \cdot (H(z) - \tilde{G}(z))} \\
&= \{ \text{algebra} \} \\
&\quad -K/4 \cdot \frac{h'(z)}{h(z) \cdot (h(z) - \tilde{g}(z))} \\
&\sim \{ K > 0 \} \\
&\quad \frac{h'(z)}{h(z) \cdot (h(z) - \tilde{g}(z))} \\
&= \{ \text{algebra} \} \\
&\quad \frac{z^2 \cdot (z^2 - 3) \cdot (z^4 + 4 \cdot z^2 - 1)}{(z^2 + 1)^2 \cdot (z^2 - 2 \cdot z - 1) \cdot (z^2 + 2 \cdot z - 1)} \\
&\sim \{ z > \sqrt{\sqrt{5} - 2} \Rightarrow z^2 - 3 < 0 \} \\
&\quad \frac{z^2 \cdot (z^4 + 4 \cdot z^2 - 1)}{(z^2 + 1)^2 \cdot (z^2 - 2 \cdot z - 1) \cdot (z^2 + 2 \cdot z - 1)} \\
&\sim \{ z > \sqrt{\sqrt{5} - 2} \Rightarrow z^2 / (z^2 + 1)^2 > 0 \} \\
&\quad \frac{z^4 + 4 \cdot z^2 - 1}{(z^2 - 2 \cdot z - 1) \cdot (z^2 + 2 \cdot z - 1)} \\
&\sim \{ z > \sqrt{\sqrt{5} - 2} \Rightarrow z^4 + 4 \cdot z^2 - 1 > 0 \} \\
&\quad \frac{1}{(z^2 - 2 \cdot z - 1) \cdot (z^2 + 2 \cdot z - 1)} \\
&\sim \{ z > \sqrt{\sqrt{5} - 2} \Rightarrow z > \sqrt{2} - 1, \\
&\quad z > \sqrt{2} - 1 \Rightarrow z^2 + 2 \cdot z - 1 > 0 \}
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{z^2 - 2 \cdot z - 1} \\
\sim & \left\{ \begin{aligned} z > \sqrt{\sqrt{5} - 2} &\Rightarrow z > 1 - \sqrt{2}, \\ z < 1 &\Rightarrow z < 1 + \sqrt{2}, \\ 1 - \sqrt{2} < z < 1 + \sqrt{2} &\Rightarrow z^2 - 2 \cdot z - 1 < 0 \end{aligned} \right\} \\
& -1.
\end{aligned}$$

By calculus, therefore, LB is decreasing on $(\sqrt{\sqrt{5} - 2}, 1)$. \square

Proof of Lemma 5.1.5.

$$\begin{aligned}
& UB'(z) \\
= & \quad \{\text{by definition of } UB\} \\
& -\frac{1}{4} \cdot \frac{H'(z)}{H(z) \cdot (H(z) + G(z))} \\
= & \quad \{\text{algebra}\} \\
& -\frac{K}{4} \cdot \frac{h'(z)}{h(z) \cdot (h(z) + g(z))} \\
= & \quad \{\text{more algebra}\} \\
& -\frac{K}{2} \cdot \frac{z^2 \cdot (z^4 + 4 \cdot z^2 - 1)}{(z^2 + 1)^3} \\
\sim & \quad \{K > 0; \quad z \neq 0\} \\
& - (z^4 + 4 \cdot z^2 - 1) \\
\sim & \quad \left\{ z > \sqrt{\sqrt{5} - 2} \Rightarrow z^4 + 4 \cdot z^2 - 1 > 0 \right\} \\
& -1.
\end{aligned}$$

Again, by calculus, UB is decreasing, on $(\sqrt{\sqrt{5} - 2}, \infty)$. \square

Therefore, the first two inequalities of Theorem 5.1.1 are equivalent, respectively, to $\tilde{z} \geq LB^{-1}(\Delta V)$ and $UB^{-1}(\Delta V) \geq \hat{z}$.

Next, we should do the same to the inequalities (5.1.3), getting bounds for \tilde{z} and \hat{z} in terms of ℓ . To do that we need H 's monotonicity. We can then play the various inequalities off one another to get our desired result. Also, we should determine the monotonicities of f and \tilde{f} ; they will prove useful later.

Lemma 5.1.6. H is increasing on $\left(\sqrt{\sqrt{5}-2}, \infty\right)$.

Lemma 5.1.7. f is decreasing on $\left(\sqrt{\sqrt{5}-2}, \infty\right)$.

Lemma 5.1.8. \tilde{f} is decreasing on $\left(\sqrt{\sqrt{5}-2}, \sqrt{3}\right)$.

Proof of Lemma 5.1.6.

$$\begin{aligned}
& H'(z) \\
&= \quad \{\text{by definition}\} \\
& \quad h'(z)/K \\
&\sim \quad \{K > 0\} \\
& \quad h'(z) \\
&= \quad \{\text{calculus}\} \\
& \quad \frac{z^4 + 4 \cdot z - 1}{(z-1)^2 \cdot z^2 \cdot (z+1)^2} \\
&\sim \\
& \quad z^4 + 4 \cdot z - 1 \\
&\sim \quad \left\{ z > \sqrt{\sqrt{5}-2} \Rightarrow z^4 + 4 \cdot z - 1 > 0 \right\} \\
& \quad 1.
\end{aligned}$$

By calculus, H is increasing if $z > \sqrt{\sqrt{5}-2}$ —in particular, if $z \in \left(\sqrt{\sqrt{5}-2}, 1\right)$. \square

Proof of Lemma 5.1.7.

$$\begin{aligned}
& f'(z) \\
&= \quad \{\text{calculus, algebra}\} \\
&\quad K \cdot e^{-\Phi(z)} \cdot (-1 + (1 - z) \cdot (-\Phi'(z))) \\
&\sim \quad \{K > 0; \quad e^{-\Phi(z)} > 0\} \\
&\quad (z - 1) \cdot \Phi'(z) - 1 \\
&= \quad \{\text{fund. thm. of calculus}\} \\
&\quad (z - 1) \cdot F(z) - 1 \\
&= \quad \{\text{algebra}\} \\
&\quad - \frac{2 \cdot z \cdot (z^4 + 4 \cdot z^2 - 1)}{(z + 1) \cdot (z^2 + 1)^2} \\
&\sim \\
&\quad - (z^4 + 4 \cdot z^2 - 1) \\
&\sim \quad \left\{ z > \sqrt{\sqrt{5} - 2} \Rightarrow z^4 + 4 \cdot z - 1 > 0 \right\} \\
&\quad - 1.
\end{aligned}$$

By calculus, f is decreasing if $z > \sqrt{\sqrt{5} - 2}$ —in particular, if $z \in \left(\sqrt{\sqrt{5} - 2}, 1\right)$. \square

Proof of Lemma 5.1.8.

$$\begin{aligned}
& \tilde{f}'(z) \\
&= \quad \{\text{calculus, algebra}\} \\
& \quad K \cdot e^{-\tilde{\Phi}(z)} \cdot \left(-1 + (1-z) \cdot (-\tilde{F}(z))\right) \\
&\sim \quad \left\{ K > 0; \quad e^{-\tilde{\Phi}(z)} > 0 \right\} \\
& \quad (z-1) \cdot \tilde{F}(z) - 1 \\
&= \quad \{\text{algebra}\} \\
& \quad \frac{-2 \cdot z \cdot (z^2 - 3) \cdot (z^4 + 4 \cdot z - 1)}{(z+1) \cdot (z^2 + 1) \cdot (z^2 - 2 \cdot z - 1) \cdot (z^2 + 2 \cdot z - 1)} \\
&\sim \quad \left\{ z \in \left(\sqrt{\sqrt{5} - 2}, \sqrt{3} \right) \Rightarrow z^2 - 3 < 0 \right\} \\
& \quad \frac{2 \cdot z \cdot (z^4 + 4 \cdot z - 1)}{(z+1) \cdot (z^2 + 1) \cdot (z^2 - 2 \cdot z - 1) \cdot (z^2 + 2 \cdot z - 1)} \\
&\sim \quad \left\{ z > \sqrt{\sqrt{5} - 2} \Rightarrow z > 0 \right\} \\
& \quad \frac{z^4 + 4 \cdot z - 1}{(z^2 - 2 \cdot z - 1) \cdot (z^2 + 2 \cdot z - 1)} \\
&\sim \quad \{\text{latter half of Lemma 5.1.4}\} \\
& \quad -1.
\end{aligned}$$

□

5.1.2 Complicated upper bound on ℓ

Plainly we already have an upper bound on ℓ , viz. $\ell \leq 1/(2\pi \cdot H(\hat{z}))$. We just need to put the right-hand side in terms of ΔV .

In fact, since H is increasing, $1/(2\pi \cdot H)$ is decreasing. Therefore we just need a lower bound on \hat{z} ; applying $1/(2\pi \cdot H)$ to this lower bound will give us a bound on ℓ .

At this point, one could use the standing assumption in [5] after p. 1079 that all

variables named z represent $\tanh \rho$ for some $\rho > \operatorname{artanh}(1/\sqrt{3})$. Therefore, $\hat{z} > \sqrt{1/3}$. As a matter of fact, this is where the bounds in Theorem 5.0.1 come from. But we would like a better bound for small ΔV .

Now, $UB(\hat{z}) \geq \Delta V$. Unfortunately UB is decreasing, so this doesn't give a lower bound on \hat{z} . Also, \hat{z} is defined by $f(\hat{z}) = (2\pi)^2/(\hat{L}(c)^2)$, but all we know about $\hat{L}(c)$ is $\hat{L}(c) > 7.5832$. In fact, this bound is taken from the standing assumption on z .

However, we also know $f(\hat{z}) = \tilde{f}(\tilde{z})$ and f and \tilde{f} both are decreasing. Therefore, if we can get a lower bound on \tilde{z} , we get a lower bound on \hat{z} , via upper bounds on $f(\hat{z}) = \tilde{f}(\tilde{z})$.

Finally, (5.1.1) from Theorem 5.1.1 says $LB(\tilde{z}) \leq \Delta V$, and LB is decreasing on $(\sqrt{\sqrt{5}-2}, 1)$. $\sqrt{1/3} > \sqrt{\sqrt{5}-2}$, so this yields a lower bound on \tilde{z} , and hence an upper bound on ℓ , in terms of ΔV ; to wit,

$$\ell \leq \frac{1}{2\pi \cdot (H \circ s \circ \tilde{f} \circ BL)(\Delta V)}, \quad (5.1.6)$$

where $s(f(\hat{z})) = \hat{z}$ and $BL(LB(\tilde{z})) = \tilde{z}$ for $\tilde{z}, \hat{z} \in (\sqrt{1/3}, 1)$, and $s : (0, f(\sqrt{1/3})) \rightarrow (\sqrt{1/3}, 1)$, $BL : (0, LB(\sqrt{1/3})) \rightarrow (\sqrt{1/3}, 1)$.

This bound is valid only when ΔV is in the domain of BL . If this is not the case, then the right-hand side should be replaced by Hodgson and Kerckhoff's original bound 0.156012.

5.1.3 Complicated bounds on $\hat{L}(c)$

We know $\frac{(2\pi)^2}{\hat{L}(c)^2} = f(\hat{z}) = \tilde{f}(\tilde{z})$. We just got upper bounds on this, yielding a lower bound for $\hat{L}(c)$. More explicitly,

$$\hat{L}(c)^2 \geq \frac{(2\pi)^2}{\tilde{f}(BL(\Delta V))}. \quad (5.1.7)$$

To get an upper bound on $\hat{L}(c)$, we can get a lower bound on $f(\hat{z})$, which would result from an upper bound on \hat{z} (since f is decreasing), which would result from a lower bound on $UB(\hat{z})$ (since UB is decreasing). But $\Delta V \leq UB(\hat{z})$ by assumption. So

$$\hat{L}(c)^2 \leq \frac{(2\pi)^2}{f(BU(\Delta V))}, \quad (5.1.8)$$

where $BU : (0, UB(\sqrt{1/3})) \rightarrow (\sqrt{1/3}, 1)$ satisfies $BU(UB(\hat{z})) = \hat{z}$ for $\hat{z} \in (\sqrt{1/3}, 1)$.

5.1.4 Nice bounds

Since these bounds depend upon inverting functions defined by integrals, one cannot expect a computer to calculate the bounds very quickly. But if we approximate the functions and relax the bounds, we can get decent running times.

The conditions which the approximations should satisfy (in order to accord with (5.1.6), (5.1.7), and (5.1.8)) are not difficult to derive. For instance, an approximation η to $1/(2\pi \cdot H)$ should be decreasing, since $1/(2\pi \cdot H)$ is itself decreasing and we want a reasonable approximation; and η should be greater than $1/(2\pi \cdot H)$ so that we can deduce

$$\ell \leq (\eta \circ s \circ \tilde{f} \circ BL)(\Delta V)$$

from (5.1.6). In fact, $\eta(z) = K \cdot (1 - z)/(2\pi)$ suffices. Useful approximations for all the necessary functions are as follows:

Lemma 5.1.9.

$$1/h(z) \leq 1 - z, \quad (5.1.9)$$

$$f(z) \geq A \cdot (1 - z), \quad (5.1.10)$$

$$\tilde{f}(z) \leq B \cdot (1 - z), \quad (5.1.11)$$

$$LB(z) \geq C \cdot (1 - z), \quad (5.1.12)$$

$$UB(z) \leq D \cdot (1 - z). \quad (5.1.13)$$

where

$$\begin{aligned}
A &= K \cdot e^{-\Phi(\sqrt{1/3})}; \\
\tilde{F}(\beta) &= 0, \quad \beta \in (\sqrt{1/3}, 1); \\
B &= K \cdot e^{-\tilde{\Phi}(\beta)}; \\
t &= \frac{h'}{h \cdot (h - \tilde{g})}; \\
C &= K \cdot t(\sqrt{1/3})/4; \\
D &= K/4;
\end{aligned}$$

Proof of (5.1.9).

$$1 - z - \frac{1}{h(z)} = \frac{(1 - z)^2}{1 + z^2} \geq 0.$$

□

Proof of (5.1.10). Assume $z \in (\sqrt{1/3}, 1)$. Now, by definition,

$$F(z) = -\frac{z^4 + 6 \cdot z^2 + 4 \cdot z + 1}{(z + 1) \cdot (z^2 + 1)^2}.$$

But

$$\begin{aligned}
F(z) &= -\frac{z^4 + 6 \cdot z^2 + 4 \cdot z + 1}{(z + 1) \cdot (z^2 + 1)^2} \\
\Rightarrow \quad &\{ \text{algebra} \} \\
&F < 0 \text{ on } (\sqrt{1/3}, 1) \\
\Rightarrow \quad &\{ \text{calculus; } z \in (\sqrt{1/3}, 1) \} \\
&\int_z^1 F(w) dw \geq \int_{\sqrt{1/3}}^1 F(w) dw \\
\equiv \quad &\{ \text{calculus, algebra} \}
\end{aligned}$$

$$\begin{aligned}
& \int_1^z F(w) dw \leq \int_1^{\sqrt{1/3}} F(w) dw \\
\equiv & \quad \{ \text{definition of } \Phi \} \\
& \Phi(z) \leq \Phi(\sqrt{1/3}) \\
\equiv & \quad \{ x \mapsto e^{-x} \text{ is decreasing} \} \\
& e^{-\Phi(z)} \geq e^{-\Phi(\sqrt{1/3})} \\
\equiv & \quad \{ z \in (\sqrt{1/3}, 1) \Rightarrow 1 - z > 0; K > 0 \} \\
& K \cdot e^{-\Phi(z)} \cdot (1 - z) \geq K \cdot e^{-\Phi(\sqrt{1/3})} \cdot (1 - z) \\
\equiv & \quad \{ \text{definition of } f \} \\
& f(z) \geq K \cdot e^{-\Phi(\sqrt{1/3})} \cdot (1 - z) \\
\equiv & \quad \{ \text{definition of } A \} \\
& f(z) \geq A \cdot (1 - z)
\end{aligned}$$

□

Proof of (5.1.11). $\tilde{F}(1) = 1$, $\tilde{F}(\sqrt{1/3}) < 0$, and \tilde{F} has exactly one root β in $(\sqrt{1/3}, 1)$.

Thus if $z \in (\sqrt{1/3}, 1)$, then

$$\begin{aligned}
& \int_z^1 \tilde{F}(w) dw \leq \int_\beta^1 \tilde{F}(w) dw \\
\equiv & \quad \{ \text{calculus} \} \\
& \int_1^z \tilde{F}(w) dw \geq \int_1^\beta \tilde{F}(w) dw \\
\equiv & \quad \{ \text{definition of } \tilde{\Phi} \} \\
& \tilde{\Phi}(z) \geq \tilde{\Phi}(\beta) \\
\equiv & \quad \{ \text{algebra} \} \\
& -\tilde{\Phi}(z) \leq -\tilde{\Phi}(\beta) \\
\equiv & \quad \{ x \mapsto e^x \text{ is increasing} \}
\end{aligned}$$

$$\begin{aligned}
& e^{-\tilde{\Phi}(z)} \leq e^{-\tilde{\Phi}(\beta)} \\
& \equiv \quad \{ \text{algebra} \} \\
& \quad K \cdot (1 - z) \cdot e^{-\tilde{\Phi}(z)} \leq K \cdot (1 - z) \cdot e^{-\tilde{\Phi}(\beta)} \\
& \equiv \quad \{ \text{definition of } \tilde{f} \} \\
& \quad \tilde{f}(z) \leq K \cdot (1 - z) \cdot e^{-\tilde{\Phi}(\beta)} \\
& \equiv \quad \{ \text{definition of } B \} \\
& \quad \tilde{f}(z) \leq B \cdot (1 - z).
\end{aligned}$$

But the initial statement is just equation (5.1.1). □

Proof of (5.1.12). For variety, we do this proof backwards. We seek a C such that for all $z \in (\sqrt{1/3}, 1)$, $LB(z) \geq C \cdot (1 - z)$:

$$\begin{aligned}
& \langle \forall z : LB(z) \geq C \cdot (1 - z) \rangle \\
& \equiv \quad \{ \text{let } lb(z) = \int_z^1 h'/(h \cdot (h - \tilde{g})) \} \\
& \quad \langle \forall z : K \cdot lb(z)/4 \geq C \cdot (1 - z) \rangle \\
& \equiv \quad \{ \text{algebra} \} \\
& \quad \langle \forall z : lb(z) \geq 4 \cdot C \cdot (1 - z)/K \rangle \\
& \Leftarrow \quad \{ \text{calculus} \} \\
& \quad h'/(h \cdot (h - \tilde{g})) \geq 4 \cdot C/K \text{ on } (\sqrt{1/3}, 1).
\end{aligned}$$

In other words, we just need a lower bound on $t = h'/(h \cdot (h - \tilde{g}))$ over $(\sqrt{1/3}, 1)$.

Now,

$$t'(z) = \frac{4 \cdot (1 - z) \cdot (z + 1) \cdot p(z)}{(z^2 + 1)^3 \cdot (z^2 - 2 \cdot z - 1)^2 \cdot (z^2 + 2 \cdot z - 1)^2},$$

where

$$p(z) = 5 \cdot z^8 - 6 \cdot z^6 + 88 \cdot z^4 - 26 \cdot z^2 + 3.$$

It is clear that on $(\sqrt{1/3}, 1)$, $t' \sim p$. Now,

$$p(z)$$

$$\begin{aligned}
&= 5 \cdot z^8 - 6 \cdot z^6 + 2 \cdot z^4 + 86 \cdot z^4 - 26 \cdot z^2 + 3 \\
&= z^4 \cdot (5 \cdot (z^2)^2 - 6 \cdot (z^2) + 2) \\
&+ 86 \cdot (z^2)^2 - 26 \cdot z^2 + 3.
\end{aligned}$$

$(-6)^2 - 4 \cdot 5 \cdot 2 < 0$ and $(-26)^2 - 4 \cdot 86 \cdot 3 < 0$. Therefore, $5 \cdot z^2 - 6 \cdot z + 2$ has constant sign, and $86 \cdot z^2 - 26 \cdot z + 3$ does too. By evaluation at 0, this sign is positive on both. Therefore p is positive. That is, $t' > 0$ on $(\sqrt{1/3}, 1)$. Consequently, t achieves its smallest value at $\sqrt{1/3}$. That is, $t \geq t(\sqrt{1/3})$. So we have, finally,

$$\begin{aligned}
&\langle \forall z : LB(z) \geq C \cdot (1 - z) \rangle \\
&\Leftrightarrow \{ \text{see above} \} \\
&C = K \cdot t(\sqrt{1/3})/4.
\end{aligned}$$

□

Proof of (5.1.13). Likewise, we do this proof backwards. We seek a D such that for all $z \in (\sqrt{1/3}, 1)$, $UB(z) \leq D \cdot (1 - z)$:

$$\begin{aligned}
&\langle \forall z : UB(z) \leq D \cdot (1 - z) \rangle \\
&\equiv \{ \text{let } ub(z) = \int_z^1 h'/(h \cdot (h + g)) \} \\
&\langle \forall z : K \cdot ub(z)/4 \leq D \cdot (1 - z) \rangle \\
&\equiv \{ \text{algebra} \} \\
&\langle \forall z : ub(z) \leq 4 \cdot D \cdot (1 - z)/K \rangle \\
&\Leftrightarrow \{ \text{calculus} \} \\
&h'/(h \cdot (h + g)) \leq 4 \cdot D/K \text{ on } (\sqrt{1/3}, 1).
\end{aligned}$$

In other words, we just need an upper bound on $T = h'/(h \cdot (h + g))$ over $(\sqrt{1/3}, 1)$.

Now,

$$T'(z) = -\frac{4 \cdot z \cdot (z^4 - 10 \cdot z^2 + 1)}{(z^2 + 1)^4}.$$

Plainly, $T'(z) \sim -z^4 + 10 \cdot z^2 - 1$ on $(\sqrt{1/3}, 1)$. This has four real roots, $\pm\sqrt{5 \pm 2 \cdot \sqrt{6}}$, none of which lies in $(\sqrt{1/3}, 1)$. $-1 + 10 - 1 > 0$, so T' is positive on $(\sqrt{1/3}, 1)$. That is, T is increasing on $(\sqrt{1/3}, 1)$. So it takes its maximum at 1, where its value is just 1! In conclusion, then,

$$\begin{aligned} & \langle \forall z : UB(z) \leq D \cdot (1 - z) \rangle \\ \Leftrightarrow & \quad \{ \text{see above} \} \\ & D = K/4. \end{aligned}$$

□

Lemma 5.1.10.

$$\frac{1}{2\pi \cdot (H \circ s \circ \tilde{f} \circ BL)(\Delta V)} \leq \alpha \cdot \Delta V, \quad (5.1.14)$$

$$\frac{(2\pi)^2}{\tilde{f}(BL(\Delta V))} \geq \delta \cdot \frac{1}{\Delta V}, \quad (5.1.15)$$

and

$$\frac{(2\pi)^2}{f(BU(\Delta V))} \leq \gamma \cdot \frac{1}{\Delta V}, \quad (5.1.16)$$

where

$$\begin{aligned} \alpha &= \frac{2 \cdot e^{\Phi(\sqrt{1/3}) - \tilde{\Phi}(\beta)}}{\pi \cdot t(\sqrt{1/3})}, \\ \delta &= \frac{(2\pi)^2 \cdot e^{\tilde{\Phi}(\beta)} \cdot t(\sqrt{1/3})}{4}, \end{aligned}$$

and

$$\gamma = \frac{(2\pi)^2 \cdot e^{\Phi(\sqrt{1/3})}}{4}.$$

Proof of (5.1.14).

$$\begin{aligned} & 1/(h \circ s \circ \tilde{f} \circ BL)(\Delta V) \\ = & \quad \{ \text{algebra} \} \end{aligned}$$

$$\begin{aligned}
& \left(\frac{1}{h} \circ s \circ \tilde{f} \circ BL \right) (\Delta V) \\
\leq & \quad \{ (5.1.9) \} \\
& 1 - (s \circ \tilde{f} \circ BL)(\Delta V) \\
\leq & \quad \{ s, f \text{ inverse; (5.1.10)} \} \\
& 1 - (1 - \tilde{f}(BL(\Delta V))/A) \\
= & \quad \{ \text{algebra} \} \\
& \tilde{f}(BL(\Delta V))/A \\
\leq & \quad \{ (5.1.11) \} \\
& \frac{B}{A} \cdot (1 - BL(\Delta V)) \\
\leq & \quad \{ BL, LB \text{ inverse; (5.1.12)} \} \\
& \frac{B}{A} \cdot (1 - (1 - \Delta V/C)) \\
= & \quad \{ \text{algebra} \} \\
& \frac{B}{A \cdot C} \cdot \Delta V.
\end{aligned}$$

Consequently,

$$\begin{aligned}
& 1/(2\pi \cdot (H \circ s \circ \tilde{f} \circ BL)(\Delta V)) \\
= & \quad \{ \text{definition of } H \} \\
& K/(2\pi \cdot (h \circ s \circ \tilde{f} \circ BL)(\Delta V)) \\
= & \quad \{ \text{algebra} \} \\
& \frac{K}{2\pi} \cdot 1/(h \circ s \circ \tilde{f} \circ BL)(\Delta V) \\
\leq & \quad \{ \text{see above} \} \\
& \frac{K}{2\pi} \cdot \frac{B}{A \cdot C} \cdot \Delta V.
\end{aligned}$$

Finally,

$$\begin{aligned}
& \frac{K}{2\pi} \cdot \frac{B}{A \cdot C} \\
= & \{ \text{definitions of } A, B, C \} \\
& \frac{K \cdot K \cdot e^{-\tilde{\Phi}(\beta)} \cdot 4}{2\pi \cdot K \cdot e^{-\Phi(\sqrt{1/3})} \cdot K \cdot t(\sqrt{1/3})} \\
= & \{ \text{algebra} \} \\
& 2 \cdot e^{\Phi(\sqrt{1/3}) - \tilde{\Phi}(\beta)} / (\pi \cdot t(\sqrt{1/3})) \\
= & \{ \text{definition of } \alpha \} \\
& \alpha.
\end{aligned}$$

□

Proof of (5.1.15).

$$\begin{aligned}
& 1/(\tilde{f}(BL(\Delta V))) \\
\geq & \{ (5.1.11); \text{algebra} \} \\
& 1/(B \cdot (1 - BL(\Delta V))) \\
\geq & \{ (5.1.12); \text{algebra} \} \\
& 1/(B \cdot (1 - (1 - \Delta V/C))) \\
= & \{ \text{algebra} \} \\
& (C/B) \cdot (1/\Delta V)
\end{aligned}$$

Consequently,

$$\frac{(2\pi)^2}{\tilde{f}(BL(\Delta V))} \leq \frac{(2\pi)^2 \cdot C}{B} \cdot \frac{1}{\Delta V}.$$

Finally,

$$\begin{aligned}
& (2\pi)^2 \cdot C/B \\
= & \{ \text{definitions of } B, C \}
\end{aligned}$$

$$\begin{aligned}
& \frac{(2\pi)^2 \cdot K \cdot t(\sqrt{1/3})}{K \cdot e^{-\tilde{\Phi}(\beta)} \cdot 4} \\
&= \{ \text{algebra} \} \\
& (2\pi)^2 \cdot t(\sqrt{1/3}) \cdot e^{\tilde{\Phi}(\beta)}/4. \\
&= \{ \text{definition of } \delta \} \\
& \delta.
\end{aligned}$$

□

Proof of (5.1.16).

$$\begin{aligned}
& 1/(f(BU(\Delta V))) \\
& \leq \{ (5.1.10); \text{algebra} \} \\
& 1/(A \cdot (1 - BU(\Delta V))) \\
& \leq \{ (5.1.13) \} \\
& 1/(A \cdot (1 - (1 - \Delta V/D))) \\
& = \{ \text{algebra} \} \\
& (D/A) \cdot (1/\Delta V).
\end{aligned}$$

Consequently,

$$\frac{(2\pi)^2}{f(BU(\Delta V))} \leq \frac{(2\pi)^2 \cdot D}{A} \cdot \frac{1}{\Delta V}.$$

Finally,

$$\begin{aligned}
& (2\pi)^2 \cdot D/A \\
&= \{ \text{definitions of } A, D \} \\
& \frac{(2\pi)^2 \cdot K}{K \cdot e^{-\Phi(\sqrt{1/3})} \cdot 4} \\
&= \{ \text{algebra} \}
\end{aligned}$$

$$\begin{aligned}
& (2\pi)^2 \cdot e^{\Phi(\sqrt{1/3})}/4 \\
& = \{ \text{definition of } \gamma \} \\
& \gamma.
\end{aligned}$$

□

5.1.5 Numerical approximations

To make the bounds from Lemma 5.1.10 implementable in software, we just need some simple estimates on α, δ, γ . Using a computer algebra system one may show

Lemma 5.1.11. $\alpha \leq 2.879$, $\delta \geq 4.563$, and $\gamma \leq 20.633$.

In other words,

Corollary 5.1.12. *Let M, N be orientable 3-manifolds admitting complete hyperbolic metrics of finite volume on their interiors. Let $\Delta V = \text{Vol}(M) - \text{Vol}(N)$.*

N is a Dehn filling of M if and only if either

- *N is a Dehn filling of M along a slope of normalized length less than or equal to 7.5832, or*
 - *N has a closed simple geodesic of length less than $2.879 \cdot \Delta V$, and*
 - *N is a Dehn filling of M along a slope of normalized length \hat{L} such that*

$$\frac{4.563}{\Delta V} \leq \hat{L}^2 \leq \frac{20.633}{\Delta V}. \quad (5.1.17)$$

5.2 Estimates in Maxima for Lemma 5.1.11

```

⟨estimates for Lemma 5.1.11⟩≡
K : 3.3957;
h : (1+z^2)/(z*(1-z^2)); H : h/K;
g : (1+z^2)/(2*z^3); G:g/K;
gt : (1+z^2)^2/(2*z^3*(3-z^2)); Gt : gt/K;
hh : factor(ratsimp(derivative(h,z)));
F : partfrac(ratsimp(hh/(h+g) -1/(1-z)),z);
Ft : partfrac(ratsimp(hh/(h-gt)-1/(1-z)),z);
assume(z>sqrt(1/3.0)); assume(z<1.0);
Phi : integrate(ev(F ,z=w),w,1,z);
Phit : integrate(ev(Ft,z=w),w,1,z);
f : K*(1-z)*exp(-Phi);
ft : K*(1-z)*exp(-Phit);
lbintegrand : partfrac(ratsimp(hh/(h*(h-gt))));
t : lbintegrand;
beta : rhs(realroots(Ft,1/1000000000000000000)[4]);
alpha : bfloat( 2 * exp( ev(Phi,z=sqrt(1/3.0))
                        -ev(Phit,z=beta)
                      )
            / (%pi * ev(t,z=sqrt(1/3.0)) ) );
delta : bfloat( (2 * %pi)^2 * exp(ev(Phit,z=beta))
                * ev(t,z=sqrt(1/3.0)) / 4);
gamma : bfloat( (2 * %pi)^2
                * exp(ev(Phi,z=sqrt(1/3.0))) / 4);

```

⟨dad.mac⟩≡

⟨estimates for Lemma 5.1.11⟩

The reader running this code is reminded that Maxima displays big-floats in scientific notation with, e.g, 1.0b1 denoting 10, instead of 1.0e1.

Chapter 6

Prospects

To develop a rigorous Dehn parteral test, it remains to develop programs to rigorously estimate volume, cusp area, parabolic translation length along maximal cusp tori, and length spectra. The rigorous estimates of hyperbolic structures given by HIKMOT should be useful in this regard for cusped manifolds. For estimating the length spectra of closed manifolds, M. Trnkova has developed some `Mathematica` code [12].

The unhyperbolicity algorithm detailed above does not work for closed 3-manifolds. However, because of 3-manifold geometrization, it is now known that a closed irreducible geometrically atoroidal 3-manifold must either be hyperbolic or *small* Seifert-fibered, which is to say a Seifert fibering over a sphere with at most three cone points. Algorithms for detecting whether or not a 3-manifold is such a fibering have been developed, but not implemented in code. Refining these algorithms into something suitable to code is the last step in getting the homeomorphism problem for compact 3-manifolds solved completely constructively and effectively. We should do that soon; we are so very close!

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