Identities on hyperbolic manifolds and quasiconformal homogeneity of hyperbolic surfaces

Author: Nicholas George Vlamis

Boston College Electronic Thesis or Dissertation, 2015

Boston College

The Graduate School of Arts and Sciences

Department of Mathematics

Identities on hyperbolic manifolds and quasiconformal homogeneity of hyperbolic surfaces

a dissertation

by

Nicholas G. Vlamis

submitted in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

May 2015

Identities on hyperbolic manifolds and quasiconformal homogeneity of hyperbolic surfaces

Nicholas G. Vlamis

Advisors: Martin Bridgeman and Ian Biringer

Abstract

The first part of this dissertation is on the quasiconformal homogeneity of surfaces. In the vein of Bonfert-Taylor, Bridgeman, Canary, and Taylor we introduce the notion of quasiconformal homogeneity for closed oriented hyperbolic surfaces restricted to subgroups of the mapping class group. We find uniform lower bounds for the associated quasiconformal homogeneity constants across all closed hyperbolic surfaces in several cases, including the Torelli group, congruence subgroups, and pure cyclic subgroups. Further, we introduce a counting argument providing a possible path to exploring a uniform lower bound for the nonrestricted quasiconformal homogeneity constant across all closed hyperbolic surfaces.

We then move on to identities on hyperbolic manifolds. We study the statistics of the unit geodesic flow normal to the boundary of a hyperbolic manifold with non-empty totally geodesic boundary. Viewing the time it takes this flow to hit the boundary as a random variable, we derive a formula for its moments in terms of the orthospectrum. The first moment gives the average time for the normal flow acting on the boundary to again reach the boundary, which we connect to Bridgeman's identity (in the surface case), and the zeroth moment recovers Basmajian's identity. Furthermore, we are able to give explicit formulae for the first moment in the surface case as well as for manifolds of odd dimension. In dimension two, the summation terms are dilogarithms. In dimension three, we are able to find the moment generating function for this length function.

Contents

Acknowledgments

I would first like to thank my advisors Martin Bridgeman and Ian Biringer for all their guidance, time, and patience. I would also like to thank my fellow graduate students for their support, especially my mathematical brother Andrew Yarmola for many helpful conversations and for collaborative work that will appear in the near future.

Lastly, I would like to thank my family for their love, support, and encouragement, and especially my wife, Diana Hubbard, for her never ending willingness to edit everything I write.

Chapter 1

Introduction

1.1 Quasiconformal Homogeneity and Subgroups of the Mapping Class Group

Let M be a hyperbolic manifold and $\mathrm{QC}(M)$ be the associated group of quasiconformal homeomorphisms from M to itself. Given any subgroup $\Gamma \leq \mathrm{QC}(M)$, we say that M is Γ-*homogeneous* if the action of Γ on M is transitive. Furthermore, we say M is Γ_K*-homogeneous* for $K \in [1, \infty)$ if the restriction of the action of Γ on M to the subset

$$\Gamma_K = \{f \in \Gamma \colon K_f \leq K\}$$

on M is transitive, where $K_f = \inf\{K \colon f \text{ is } K\text{-quasiconformal}\}$ is the *dilatation* of f.

If $\Gamma = \mathrm{QC}(M)$ and there exists a K such that M is Γ_K-homogeneous, then this manifold is said to be K*-quasiconformally homogeneous*, or K*-qch*. In [BTCMT05] it is shown that for each $n \geq 3$ there exists a constant $K_n > 1$ such that if $M \neq \mathbb{H}^n$ is an n-dimensional K-quasiconformally homogeneous hyperbolic manifold, then $K \geq K_n$. This result relies on rigidity in higher dimensions, which does not occur in dimension two. The natural question motivating this paper is as follows:

Question 1.1.1. *Does there exist a constant $K_2 > 1$ such that every K-qch surface $X \neq \mathbb{H}^2$ satisfies $K \geq K_2$?*

Given a closed hyperbolic surface and $f \in \mathrm{QC}(X)$, let $[f] \in \mathrm{Mod}(X)$ denote its homotopy class, which gives a surjection $\pi : \mathrm{QC}(X) \to \mathrm{Mod}(X)$, where $f \mapsto [f]$. If $H \leq \mathrm{Mod}(X)$, we say that X is *H-homogeneous* if X is $\pi^{-1}(H)$-homogeneous. Similarly, we say X is *H_K-homogeneous* if it is $\pi^{-1}(H)_K$-homogeneous.

The focus of this chapter will be to restrict ourselves to homogeneity with respect to subgroups of the mapping class group of closed hyperbolic surfaces and find lower bounds for the associated homogeneity constants. We will go about this by leveraging lower bounds on the quasiconformal dilatations for maps in a given homotopy class.

Torelli and Congruence Subgroups. Let S be a closed orientable surface, then $\mathrm{Mod}(S)$ acts on the first homology $H_1(S, \mathbb{Z})$ by isomorphisms and the kernel of this action is called the *Torelli group*, denoted $\mathcal{I}(S)$. Similarly, the kernel of the action of $\mathrm{Mod}(S)$ on $H_1(S, \mathbb{Z}/r\mathbb{Z})$ is called the *level r congruence subgroup* and is denoted by $\mathrm{Mod}(S)[r]$. The first theorem gives a universal bound on the quasiconformal homogeneity constant with respect to these subgroups for closed hyperbolic surfaces.

Theorem 1.1.2. *There exists a constant $K_T > 1$ such that if X is a closed hyperbolic surface that is Γ_K-homogeneous for $\Gamma = \mathcal{I}(X)$ or $\Gamma = \mathrm{Mod}(X)[r]$ with $r \geq 3$, then $K \geq K_T$.*

The case of $\Gamma = \mathcal{I}(X)$ was independently discovered by Greenfield [Gre13].

Since $H_1(S, \mathbb{Z}/r\mathbb{Z})$ is a finite group, so is its automorphism group; hence, $\mathrm{Mod}(S)[r]$ is finite index in Mod(S). Theorem 1.1.2 provides an optimistic outlook for answering Question 1.1.1 in the positive for the case of closed surfaces.

Homogeneity and Teichmüller Space. The rest of the paper is flavored by a technique, introduced in Section 3.2, which translates questions about homogeneity constants to questions about orbit points under the action of the mapping class group on Teichmüller space. Given a closed hyperbolic surface S, we define its associated *Teichmüller space* $\mathrm{Teich}(S)$ to be the space of equivalence classes of pairs (X, φ), where X is a hyperbolic surface and $\varphi : S \to X$ is a homeomorphism called the *marking*. Two such pairs (X, φ) and (Y, ψ) are equivalent if $\psi \circ \varphi^{-1} : X \to Y$ is homotopic to an isometry (see [Hub06]). The mapping class group $\mathrm{Mod}(S)$ acts on $\mathrm{Teich}(S)$ by changing the marking:

$$[f] \cdot [(X, \varphi)] = [(X, \varphi \circ f^{-1})].$$

Furthermore, this action is by isometries with respect to the Teichmüller metric on $\mathrm{Teich}(S)$, which is defined by

$$d_T([(X,\varphi)], [(Y,\psi)]) = \frac{1}{2}\log(\min K(h)),$$

where the minimum of the quasiconformal dilatation is over all quasiconformal maps $h : X \to Y$ homotopic to $\psi \circ \varphi^{-1}$. The fact that this minimum exists is a well-known theorem of Teichmüller (a proof can be found in [Hub06]).

Our next theorem is a direct result of the technique mentioned above and gives a possible path to finding a lower bound for the quasiconformal homogeneity constant for closed hyperbolic surfaces. It is shown in [BTCMT05] (see Theorem 2.5.1 below) that surfaces with short curves have large homogeneity constants. We let

$$\mathrm{Teich}_{(\epsilon,\infty)}(S) = \{[(X,\varphi)] \in \mathrm{Teich}(S) \colon \ell(X) > \epsilon\},$$

where $\ell(X)$ is the length of the systole. Also, given a point $\mathcal{X} \in \mathrm{Teich}(S)$, let $B_R(\mathcal{X})$ be the ball of radius R about $\mathcal{X}$ in $(\mathrm{Teich}(S), d_T)$. We let S_g be an oriented closed genus g surface.

Theorem 1.1.3. *Suppose there exist constants $\epsilon, R, C > 0$ such that for any $\mathcal{X} \in \mathrm{Teich}_{(\epsilon,\infty)}(S_g)$ with $g > 1$*

$$|\{f \in \mathrm{Mod}(S_g) \colon f \cdot \mathcal{X} \in B_R(\mathcal{X})\}| \leq Cg.$$

Then, there exists a constant $K_2 > 1$ such that any closed K-qch surface must have $K \geq K_2$.

Question 1.1.4. *Does there exist such an ϵ, R, C?*

Note that ϵ and C can be chosen to be arbitrarily large and R can be chosen to be arbitrarily small.

Finite, Cyclic, and Torsion-Free Subgroups.

Returning to more restrictive forms of homogeneity, we use this counting method to consider finite and cyclic subgroups of the mapping class group:

Theorem 1.1.5. *There exists a constant $K_F > 1$ such that if a closed hyperbolic surface X is Γ_K-homogeneous, where $\Gamma < \mathrm{Mod}(X)$ has finite order, then $K \geq K_F$. Furthermore, we have*

$$K_F \geq \sqrt{\psi\left(2\,\mathrm{arccosh}\left(\frac{1}{42}+1\right)\right)} = 1.11469\ldots,$$

where ψ is defined in equation (3.1).

Theorem 1.1.6. *There exists a constant $K_C > 1$ such that if a closed hyperbolic surface X is Γ_K-homogeneous, where $\Gamma = \langle [f] \rangle$ with $[f] \in \mathrm{Mod}(X)$ a pure mapping class, then $K \geq K_C$. Furthermore, we have $K_C \geq 1.09297$.*

It is particularly difficult to understand the orbit of points in $\mathrm{Teich}(S)$ under periodic mapping classes; hence, our last theorem deals with torsion-free subgroups of $\mathrm{Mod}(S)$.

Theorem 1.1.7. *Let X be a closed hyperbolic surface and suppose $\Gamma < \mathrm{Mod}(X)$ is torsion-free. If X is Γ_K-homogeneous, then*

$$\log K \geq \frac{1}{7000g^2},$$

where g is the genus of X.

Question 1.1.8. *Can one find a constant C such that every closed K-qch surface satisfies $K \geq Cg^{-2}$?*

The rest of the paper discusses how to define continuous functions on Teichmüller space and Moduli space using subgroups of the mapping class group and the associated homogeneity constants for surfaces.

Related Results in the Literature.

In recent years there have been several papers published that make progress towards understanding quasiconformal homogeneity of surfaces. In [BTBCT07] the authors bound the quasiconformal constant of hyperbolic surfaces having automorphisms with many fixed points away from 1, in particular, all hyperelliptic surfaces. In the same paper, they also consider homogeneity with respect to $\Gamma = \{e\} < \text{Mod}(X)$ and $\text{Aut}(X)$. They prove that a surface is $\{e\}_K$-homogeneous for some K if and only if it is closed; furthermore, there exists a constant $K_e > 1$ such that $K \geq K_e$. In a similar fashion, the authors find that a hyperbolic surface X is $\text{Aut}(X)_K$-homogeneous for some K if and only if it is a regular cover of a hyperbolic orbifold; furthermore, there exists a constant $K_{aut} > 1$ such that $K \geq K_{aut}$. A sharp bound is found for the constant K_{aut} in [BTMRT11]. The authors in [KM11] show the existence of a lower bound $K_0 > 1$ for the quasiconformal homogeneity constant of genus zero surfaces, which answers a question about quasiconformal homogeneity of planar domains posed by Gehring and Palka in [GP76].

1.2 A Generalization of Basmajian's Identity

Let M be a compact hyperbolic manifold with non-empty totally geodesic boundary. An *orthogeodesic* for M is an oriented geodesic arc with endpoints normal to ∂M (see [Bas93]). We will denote the collection of orthogeodesics by $O_M = \{\alpha_i\}$. Let ℓ_i denote the length of α_i, then the collection $|O_M| = \{\ell_i\}$ (with multiplicities) is known as the *orthospectrum*. As we will be summing over the orthospectrum, it is important to note that O_M is a countable collection: this can be seen by doubling the manifold and observing that the orthogeodesics correspond to a subset of the closed geodesics in the double.

Given $x \in \partial M$, let α_x be the geodesic emanating from x normal to ∂M. Then, as the limit set is measure zero, for almost every $x \in \partial M$ we have that α_x terminates in ∂M; hence, the length of α_x is finite. This allows us to define the measurable function $L\colon \partial M \to \mathbb{R}$ given by $L(x) = \mathit{length}(\alpha_x)$. Let dV denote the hyperbolic volume measure on ∂M, then $V(\partial M)$ is finite allowing us to define the probability measure $dm = dV/V(\partial M)$ on ∂M, so that $(\partial M, dm)$ is a probability space. This lets us view $L\colon \partial M \to \mathbb{R}$ as a random variable. Given a random variable X on a probability space with measure p, the k^{th}-*moment of* X is defined to be $E[X^k] = \int X^k dp$, where $E[X]$ denotes the expected value. Let $A_k(M)$ be the k^{th} moment of L. In particular, $A_1(M)$ is the expected value of L. In this paper we will show that the positive moments of L are finite and encoded in the orthospectrum:

Theorem 1.2.1. *Let $M = M^n$ be an n-dimensional compact hyperbolic manifold with nonempty totally geodesic boundary, then $A_k(M)$ is finite for all $k \in \mathbb{Z}^{\geq 0}$.*

Theorem 1.2.2. *Let $M = M^n$ be an n-dimensional compact hyperbolic manifold with nonempty totally geodesic boundary, then for all $k \in \mathbb{Z}^{\geq 0}$*

$$A_k(M) = \frac{1}{V(\partial M)} \sum_{\ell \in |O_M|} F_{n,k}(\ell),$$

where

$$F_{n,k}(x) = \Omega_{n-2} \int_0^{\log\coth(x/2)} \left[\log\left(\frac{\coth x + \cosh r}{\coth x - \cosh r}\right)\right]^k \sinh^{n-2}(r)\ dr$$

and Ω_n is the volume of the standard n-sphere. Furthermore, the identity for $A_0(M)$ is Basmajian's identity.

Basmajian's identity gives the volume of the boundary in terms of the orthospectrum:

Theorem 1.2.3 (Basmajian's Identity, [Bas93])**.** *If M is a compact hyperbolic n-manifold with totally geodesic boundary, then*

$$V(\partial M) = \sum_{\ell_i \in |O_M|} V_{n-1}\left(\log\coth\frac{\ell_i}{2}\right),$$

where $V_n(r)$ is the volume of the hyperbolic n-ball of radius r.

Note that by combining Theorem 1.2.2 and Basmajian's identity we see that $A_k(M)$ depends solely on the orthospectrum.

As corollaries to Theorem 1.2.2 we can write the function $F_{n,1}(x)$ in dimension 2 and all odd dimensions without integrals. In the following corollary $\mathrm{Li}_2(x)$ is the standard dilogarithm (see [Lew91]). We will also write $\ell(\partial S)$ for sum of the lengths of each boundary component of a surface S.

Corollary 1.2.4. *Let S be a compact hyperbolic surface with nonempty totally geodesic* *boundary. Then*

$$A_1(S) = \frac{2}{\ell(\partial S)} \sum_{\ell \in |O_S|} \left[\mathrm{Li}_2\left(-\tanh^2 \frac{\ell}{2}\right) - \mathrm{Li}_2\left(\tanh^2 \frac{\ell}{2}\right) + \frac{\pi^2}{4} \right]$$

Corollary 1.2.5. *Let M be an n-dimensional compact hyperbolic manifold with nonempty totally geodesic boundary where n is odd. Then*

$$A_1(M) = \frac{2\Omega_{n-2}}{Vol(\partial M)} \sum_{\ell \in |O_M|} \sum_{j=0}^{\frac{n-3}{2}} \frac{(-1)^{\frac{n-3}{2}-j}\binom{\frac{n-3}{2}}{j}}{2j+1} \coth^{2j+1}(\ell) \left[\log(2\cosh \ell) - \ell_i \tanh^{2j+1}(\ell) + \sum_{k=1}^{j} \frac{1-\tanh^{2k}(\ell)}{2k} \right].$$

The rest of the paper is dedicated to understanding the asymptotics of the $F_{n,k}$'s and finding the moment generating function in dimension 3. The motivation of this paper comes from recent work of Bridgeman and Tan in [BT13], where the authors study the moments of the hitting function associated to the unit tangent bundle of a manifold (i.e. the time it takes the geodesic flow of a vector to reach the boundary). In the paper they are able to show the moments are finite and give an explicit formula for the expected value in the surface case as well as relate the orthospectrum identities of Basmajian and Bridgeman (see [Bri11], [BK10], and §4.4 below) as different moments of the hitting function. In §4.4 we give a relationship between Bridgeman's identity and $A_1(S)$ in dimension 2.

Chapter 2

Background

2.1 Kleinian Groups

For $n \geq 2$, let $\mathrm{Isom}^+(\mathbb{H}^n)$ be the space of orientation preserving isometries of hyperbolic n-space. With the topology of uniform convergence on the space of isometries, we define a *Kleinian group* to be a discrete torsion-free subgroup of $\mathrm{Isom}^+(\mathbb{H}^n)$. If $\Gamma < \mathrm{Isom}^+(\mathbb{H}^n)$ is a Kleinian group, then $\mathbb{H}^n/\Gamma$ is a hyperbolic manifold, i.e. a Riemannian manifold of constant curvature -1.

In the Poincaré model of hyperbolic space we can identify the boundary of $\mathbb{H}^n$ with the $(n-1)$-sphere called the *sphere at infinity* and denoted S^{n-1}_∞. Pick $x \in \mathbb{H}^n$ and define the *limit set of* Γ to be the set $\Lambda_\Gamma = \overline{\Gamma x} \cap S^{n-1}_\infty$. Note that this definition is independent of the choice of x. Define the *convex hull* $\mathrm{CH}(\Lambda_\Gamma)$ of the limit set Λ_Γ to be the smallest convex subset of $\mathbb{H}^n$ containing all the geodesics in $\mathbb{H}^n$ with endpoints in Λ_Γ. As Λ_Γ is Γ-invariant, so is $\mathrm{CH}(\Lambda_\Gamma)$ and so we can take the quotient of $\mathrm{CH}(\Lambda_\Gamma)$ by Γ, which we call the *convex core* and denote $\mathrm{C}(\Gamma)$. A Kleinian group is *convex cocompact* if its associated convex core is compact (see [Thu79]).

2.2 Teichmüller Space

For the entirety of this dissertation, S_g denotes a connected oriented closed surface of genus g. Assuming $g \geq 2$, let $S = S_g$. A hyperbolic surface X and a diffeomorphism $f\colon S \to X$

determines a hyperbolic structure on S by pulling back the structure on X to S. We will denote this hyperbolic structure by the pair (X, f), which is commonly referred to as a *marked hyperbolic surface* with f being the *marking*. Two such pairs (X, f) and (Y, g) are said to be equivalent if there exists an isometry $I : X \to Y$ isotopic to $g \circ f^{-1}$, that is, the following diagram commutes up to isotopy:

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X \arrow[rr, "I"] & & Y \end{tikzcd}$$

Let $[(X, f)]$ denote the equivalence class associated to the marked hyperbolic surface (X, f). Then, the *Teichmüller space* associated to S is defined as

$$\mathrm{Teich}(S) = \{[(X, f)] \colon X \text{ a hyperbolic surface, } f : S \to X \text{ a diffeomorphism}\}.$$

Let Γ be a *Fuchsian* (i.e. discrete) subgroup of $\mathrm{Isom}^+(\mathbb{H}^2) \cong \mathrm{PSL}(2, \mathbb{R})$. Then a marked hyperbolic surface (X, f) determines a discrete and faithful representation

$$f_* \colon \ \pi_1(S) \to \mathrm{PSL}(2, \mathbb{R})$$

with $f_*(\pi_1(S)) = \Gamma$. In fact the converse relationship holds as well: let

$$\mathrm{DF(S)} \subset \mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2, \mathbb{R}))$$

be the set of discrete and faithful homomorphisms from $\pi_1(S)$ to $\mathrm{PSL}(2, \mathbb{R})$, then we have

$$\mathrm{Teich}(S) = \mathrm{DF(S)} / \mathrm{PSL}(2, \mathbb{R}),$$

where the action of $\mathrm{PSL}(2, \mathbb{R})$ is by conjugation. (The proof of this fact can be found in [FM11, Proposition 10.2].) This definition allows one to put a topology on $\mathrm{Teich}(S_g)$ by realizing $\mathrm{Hom}(\pi_1(S_g), \mathrm{PSL}(2, \mathbb{R}))$ as a subspace of $\mathrm{PSL}(2, \mathbb{R})^{2g}$ (note that $2g$ is the minimum number of generators of $\pi_1(S_g)$). Though the above definitions are for closed surfaces, minor modifications generalize the definitions for punctured and bordered surfaces.

For a surface S as above, the uniformization theorem (see [Hub06, Theorem 1.1.1]) implies that there is a bijection between the isometry classes of hyperbolic surfaces diffeomorphic to S and the isomorphism classes of Riemann surfaces diffeomorphic to S. This implies that the discussion above defining Teichmüller space could have been done using marked Riemann surfaces instead of hyperbolic surfaces. This viewpoint will allow us to define a metric on Teichmüller space; but, we first must introduce the notion of a quasiconformal map.

2.3 Quasiconformal Maps

A quadrilateral in a domain $U \subset \mathbb{C}$ is a Jordan region Q, whose closure is contained in U, together with a pair of disjoint arcs on the boundary. The Riemann mapping theorem tells us there exists $a, b \in \mathbb{R}^+$ such that Q can be conformally mapped onto the rectangle $[0, a] \times [0, ib] \subset \mathbb{C}$ such that the distinguished arcs in Q map to $\{0\} \times [0, ib]$ and $\{a\} \times [0, ib]$. The module of Q is then $m(Q) = a/b$. For U, V open sets in $\mathbb{C}$, a homeomorphism $f : U \to V$ is said to be K-*quasiconformal* if for every quadrilateral $Q \subset U$

$$K^{-1} \cdot m(Q) \leq m(f(Q)) \leq K \cdot m(Q).$$

We define the *dilatation* of f to be

$$K_f = K(f) = \inf\{K\colon f \text{ is } K\text{-quasiconformal}\}.$$

This is in fact a location condition: [Ahl66, Theorem 1] tells us that if f is K-quasiconformal in a neighborhood of every point, then it is K-quasiconformal.

There are two properties of quasiconformal maps that will play a key role in what follows. The first property shows us that quasiconformal maps retain some of the nicety of conformal maps. Let $\mathbb{D}$ denote the unit disk in $\mathbb{C}$. The following theorem and proof can be found in [Hub06].

Theorem 2.3.1. *Denote by $\mathcal{F}_K(\mathbb{D})$ the set of K-quasiconformal homeomorphisms $f : \mathbb{D} \to \mathbb{D}$ with $f(0) = 0$. Then $\mathcal{F}_K(\mathbb{D})$ is a normal family.*

We will rely heavily on this theorem for the convergence of sequences of quasiconformal maps.

The next property relates the quasiconformal condition of a homeomorphism $f : \mathbb{D} \to \mathbb{D}$ to the geometry of the hyperbolic plane. We say that $f : \mathbb{D} \to \mathbb{D}$ is an (A, B)-quasi-isometry if there are constants $A, B > 0$ such that

$$\frac{d(z, w)}{A} - B \leq d(f(z), f(w)) \leq Ad(z, w) + B,$$

for all $z, w \in \mathbb{D}$ and where d is the hyperbolic metric on $\mathbb{D}$. The following theorem can be found in [Vuo88].

Theorem 2.3.2. *Let $f : \mathbb{D} \to \mathbb{D}$ be K-quasiconformal, then f is a $(K, K \log 4)$-quasi-isometry with respect to the hyperbolic metric.*

In particular, the image of a geodesic $\gamma \in \mathbb{D}$ under a K-quasiconformal map $f : \mathbb{D} \to \mathbb{D}$ is a $(K, K \log 4)$-quasi-geodesic. It is well known (see [Kap01]) that a quasi-geodesic stays within a bounded distance of a geodesic. In our case, we know there exists some $C(K)$ and some geodesic $\tilde{\gamma}$ such that $f(\gamma) \subset N_{C(K)}(\tilde{\gamma})$, where $N_{C(K)}$ is the $C(K)$-neighborhood.

Now, as quasiconformality is a local condition, the above definition easily extends to maps between Riemann surfaces (and hence hyperbolic surfaces) as the transition maps are biholomorphic. Given a Riemann surface X, let $\mathrm{QC}(X)$ be the set of quasiconformal homeomorphisms $X \to X$.

Proposition 2.3.3. *([FM11, Proposition 11.3]) Let X be a Riemann surface and let f and g be quasiconformal homeomorphisms of X with dilatations K_f and K_g. Then:*

1) The composition $f \circ g$ is quasiconformal and $K_{f\circ g} \leq K_f K_g$.

2) The inverse f^{-1} is quasiconformal and $K_{f^{-1}} = K_f$.

3) If g is conformal, then $K_{f\circ g} = K_f = K_{g\circ f}$.

In particular, $\mathrm{QC}(X)$ *is a group.*

With the notion of quasiconformal maps between Riemann surfaces, we can define a metric on Teichmüller space as follows: Let $S = S_g$ with $g \geq 2$. Given $[(X, f)], [(Y, g)] \in \text{Teich}(S)$, we define the distance between them to be

$$d_T([(X,f)],[(Y,g)]) = \frac{1}{2}\log\left(\inf_{h \simeq g \circ f^{-1}} K_h\right),$$

where $\simeq$ denotes isotopy. This is the *Teichmüller metric.* It is a theorem of Teichmüller [Tei44] that the infimum in the definition of the metric is in fact a minimum realized by a quasiconformal map referred to as the *Teichmüller mapping.*

2.4 Mapping Class Groups

Let $S = S_g$ with $g \geq 2$. Let $\text{Diff}^+(S)$ be the group of orientation-preserving self-diffeomorphisms of S and $\text{Diff}_0(S) < \text{Diff}^+(S)$ be the group of diffeomorphisms isotopic to the identity. The *mapping class group* $\text{Mod}(S)$ is then defined to be

$$\text{Mod}(S) = \text{Diff}^+(S)/\text{Diff}_0(S).$$

An element of $\text{Mod}(S)$ is referred to as a mapping class. By an abuse of notation, throughout this dissertation we will treat a mapping class as a diffeomorphism.

The first property we note is that $\text{Mod}(S)$ acts by isometries on $\text{Teich}(S)$ with respect to the Teichmüller metric via

$$\varphi \cdot [(X, f)] = [(X, f \circ \varphi^{-1})].$$

In fact, for $g \geq 2$ [Roy71]

$$\text{Isom}((\text{Teich}(S_g), d_T)) = \text{Mod}(S_g).$$

Hence, the mapping class group is often referred to as the Teichmüller modular group. It is also important to note that this action of $\text{Mod}(S)$ on $\text{Teich}(S)$ is properly discontinuous (see [FM11, Theorem 12.2]). The quotient space $\text{Teich}(S)/\text{Mod}(S)$ is the moduli space of

Riemann surfaces $\mathcal{M}(S)$.

Let's introduce three types of mapping classes: Let $f \in \mathrm{Mod}(S)$, then

- f is *periodic* if it has finite order.
- f is *reducible* if it fixes the homotopy class of a multicurve (a collection of disjoint simple closed curves).
- f is *pseudo-Anosov* if there exists a transverse pair of measured foliations $(\mathcal{F}^u, \mu_u)$ and $(\mathcal{F}^s, \mu_s)$ on S, a number $\lambda > 1$ (called the *dilatation of* f), and a representative homeomorphism φ so that

$$\varphi \cdot (\mathcal{F}^u, \mu_u) = (\mathcal{F}^u, \lambda\mu_u) \text{ and } \varphi \cdot (\mathcal{F}^s, \mu_s) = (\mathcal{F}^s, \lambda^{-1}\mu_s).$$

Briefly, a *measured foliation* is a (singular) foliation on S equipped with a measure that assigns arcs transverse to the leaves of the foliation a length. If $[f] \in \mathrm{Mod}(S)$ and $(\mathcal{F}, \mu)$ is a measured foliation on S, then we define $f \cdot (\mathcal{F}, \mu) = (f(\mathcal{F}), f_*\mu)$.

The Nielsen-Thurston classification of surface diffeomorphisms states that every element of $\mathrm{Mod}(S)$ falls into one of the above three categories and was proved in [Thu88]. It is convenient for our purposes to put this classification in terms of the action of $\mathrm{Mod}(S)$ on $\mathrm{Teich}(S)$. This was the method Bers [Ber78] used to prove the Nielsen-Thurston classification. A full proof of this theorem can be found in [FM11].

Let X be a metric space and $\varphi \in \mathrm{Isom}(X)$, then we can define the *translation length* $\tau(\varphi)$ by

$$\tau(\varphi) = \inf_{x \in X} \{d(x, \varphi(x))\}.$$

Every isometry falls into one of three categories: (1) $\tau(\varphi) = 0$ and is realized, (2) $\tau(\varphi)$ is not realized, or (3) $\tau(\varphi) > 0$ and is realized.

Theorem 2.4.1. *(Nielsen-Thurston classification of surface diffeomorphisms, [FM11, Theorem 13.2]) Let $S = S_g$ with $g \geq 2$ and $f \in \mathrm{Mod}(S)$. Let $\tau(f)$ denote the translation length of f acting on $(\mathrm{Teich}(S), d_T)$, then there are three distinct possibilities:*

1) $\tau(f) = 0$ and realized if and only if f is periodic.

2) $\tau(f)$ is not realized if and only if f is reducible and has infinite order.

3) $\tau(f) > 0$ and is realized if and only if f is pseudo-Anosov. Moreover, $\tau(f) = \log \lambda(f)$.

2.5 Quasiconformal Homogeneity

The first project covered in this dissertation is in the field of quasiconformal homogeneity of hyperbolic manifolds. We give here a quick introduction to the landscape and provide the motivating question for this project.

Above we describe quasiconformal homeomorphisms of surfaces, but a similar concept holds for hyperbolic manifolds in higher dimensions, which we will not give the details of here. Let M be an oriented hyperbolic manifold and let $\mathrm{QC}(M)$ be the group of quasiconformal self-mappings of M. We say that M is *quasiconformally homogeneous*, or *qch*, if the action of $\mathrm{QC}(M)$ on M is transitive. Let $\mathrm{QC}_K(M) = \{f \in \mathrm{QC}(M)\colon K_f \leq K\}$, then we say that M is *uniformly quasiconformally homogenous* if there exists a K such that the action of $\mathrm{QC}_K(M)$ on X is transitive, that is, given any two points $x, y \in M$ there exists $f \in \mathrm{QC}_K(M)$ such that $f(x) = y$. The work of Bonfert-Taylor, Canary, Martin, and Taylor in [BTCMT05] shows that being uniformly quasiconformally homogeneous puts strong geometric restrictions on a manifold. Let $d(M)$ be the supremum of the diameters of embedded hyperbolic balls in M and let $\ell(M)$ be the infimum of the lengths of homotopically non-trivial curves in M.

Theorem 2.5.1. *([BTCMT05, Theorem 1.1]) For each dimension $n \geq 2$ and each $K \geq 1$, there is a positive constant $m(n, K)$ with the following property. Let $M = \mathbb{H}^n/\Gamma$ be a K-quasiconformally homogenous hyperbolic n-manifold, which is not $\mathbb{H}^n$. Then*

1) $d(M) \leq K\ell(M) + 2K \log 4$.

2) $\ell(M) \geq m(n, K)$.

3) Every nontrivial element of Γ is hyperbolic and the limit set $\Lambda(\Gamma)$ of Γ is $\partial\mathbb{H}^n$.

In addition, every closed manifold is K-quasiconformally homogeneous for some K (also in [BTCMT05]). These facts tell us that a geometrically-finite hyperbolic surface is K-quasiconformally homogeneous for some K if and only if it is closed. Observe that if $G < G' < \mathrm{Mod}(X)$ for some hyperbolic surface X, then if X is G_K-homogeneous we have that X is also G'_K-homogeneous. In particular, a geometrically-finite hyperbolic surface X is

G_K-homogeneous for $G < \mathrm{Mod}(X)$ if and only if X is closed. This fact will be our motivation for stating our theorems in terms of closed surfaces as opposed to the geometrically-finite terminology.

The other key tool we will need comes from understanding the quasiconformal homogeneity constant under geometric convergence and the fact that the only hyperbolic n-manifold that is 1-quasiconformally homogeneous is $\mathbb{H}^n$.

Proposition 2.5.2 (Proposition 3.2 in [BTBCT07]). *Let $\{M_i\}$ be a sequence of hyperbolic manifolds with M_i being K_i-quasiconformally homogeneous. If $\lim_{i\to\infty} K_i = 1$, then $\lim_{i\to\infty} \ell(M_i) = \infty$.*

2.6 Identities on Hyperbolic Manifolds

The second project covered in this dissertation is in the subject of spectral identities on hyperbolic manifolds. This project provides a generalization of Basmajian's identity, which is described here along with several other famous identities.

The study of identities on hyperbolic manifolds was initiated in the thesis of G. McShane [McS91], where he discovered the following identity:

Theorem 2.6.1 (McShane's Identity). *Let M be a once-punctured torus with a complete finite-area hyperbolic structure, then*

$$\sum_{\gamma} \frac{1}{1+e^{\ell(\gamma)}} = \frac{1}{2},$$

where the sum is over all simple closed geodesics γ on M and $\ell(\gamma)$ denotes the length of γ in M.

The above identity was later extended by McShane to include all finite-area hyperbolic punctured surfaces in [McS98]. In what follows we will give the form for four different identities and a framework in which they all sit.

Let M be a compact hyperbolic manifold with non-empty totally geodesic boundary. An *orthogeodesic* for M is an oriented geodesic arc with endpoints normal to ∂M (see [Bas93]).

We will denote the collection of orthogeodesics by $O_M = \{\alpha_i\}$. Let ℓ_i denote the length of α_i, then the collection $|O_M| = \{\ell_i\}$ (with multiplicities) is known as the *orthospectrum*. As we will be summing over the orthospectrum, it is important to note that O_M is a countable collection: this can be seen by doubling the manifold and observing that the orthogeodesics correspond to a subset of the closed geodesics in the double.

Given $x \in \partial M$, let α_x be the geodesic emanating from x normal to ∂M. Then, as the limit set is measure zero, for almost every $x \in \partial M$ we have that α_x terminates in ∂M; hence, the length of α_x is finite. For each $\alpha_i \in O_M$ define

$$U_i = \{x \in \partial M \colon \alpha_x \text{ is properly homotopic to } \alpha_i\}.$$

The U_i are pairwise disjoint and give a full measure decomposition of the boundary and investigating the volume of the U_i yields the following:

Theorem 2.6.2 (Basmajian's Identity, [Bas93])**.** *If M is a compact hyperbolic n-manifold with totally geodesic boundary, then*

$$\text{Volume}(\partial M) = \sum_{\ell_i \in |O_M|} V_{n-1}\left(\log\coth\frac{\ell_i}{2}\right),$$

where $V_n(r)$ is the volume of the hyperbolic n-ball of radius r.

We can view the above as partitioning the unit normal bundle associated to ∂M by proper homotopy classes associated to the orthogeodesics. We now want to do the same with the unit tangent bundle. Let $v \in T_1 M$, then associated to v is a geodesic arc α_v obtained by flowing both forwards and backwards in time along v. The set of vectors such that $\ell(\alpha_v)$ is finite is full measure in $T_1 M$. For each $\alpha_i \in O_M$, define

$$V_i = \{v \in T_1(M) \colon \alpha_v \text{ is properly homotopic to } \alpha_i\},$$

then the V_i are pairwise disjoint and are full measure in $T_1 M$. Observing that the volume of U_i only depends on ℓ_i yields the following:

Theorem 2.6.3 (Bridgeman-Kahn Identity, [BK10])**.** *Given $n \geq 2$ there exists a continuous* *monotonically decreasing function $F_n : \mathbb{R}_+ \to \mathbb{R}_+$ such that if M is a compact hyperbolic n-manifold with non-empty totally geodesic boundary, then*

$$\text{Volume}(M) = \sum_i F_n(\ell_i).$$

In the surface case $M = S$, $F_2(x) = 8\,\mathcal{L}\left(\operatorname{sech}^2 \frac{x}{2}\right)$, where $\mathcal{L}$ is the Rogers dilogarithm. This yields Bridgeman's identity [Bri11]:

$$\sum_i \mathcal{L}\left(\operatorname{sech}^2 \frac{\ell_i}{2}\right) = \frac{\pi^2}{2}|\chi(S)|,$$

where $\chi(S)$ denotes the Euler characteristic of S.

We now restrict our focus to surfaces. Let X be a compact hyperbolic surface with non-empty totally geodesic boundary. Let β be a component of ∂X. For any $x \in \beta$ there is a corresponding geometric pair of pants P_x embedded in X, which we construct as follows: as above, we can associate to x a geodesic arc α_x, which we obtain by flowing along the unit normal at x. If α_x is simple with its other endpoint in β' (possibly $\beta = \beta'$), then P_x is the unique geodesic pair of pants homotopic to a regular neighborhood of $\beta \cup \beta' \cup \alpha_x$. If α_x is not simple, then let $t \in \mathbb{R}_+$ such that $\alpha_x([0,t))$ is embedded, but $\alpha_x([0,t])$ is not. We then define P_x to be the unique geodesic pair of pants homotopic to a regular neighborhood of $\beta \cup \alpha_x([0,t])$. Let α_1, α_2 be two disjoint simple closed geodesics in X bounding a pair of pants with β, then define P_{α_1,α_2} to be the unique geodesic pair of pants embedded in X with boundary components $\beta, \alpha_1, \alpha_2$. Given such a pair α_1, α_2, define

$$U_{\alpha_1,\alpha_2} = \{x \in \beta \colon P_x = P_{\alpha_1,\alpha_2}\}.$$

The collection of sets U_{α_1,α_2} are pairwise disjoint and full measure in β, so calculating the measures of each we obtain:

Theorem 2.6.4 (McShane-Mirzakhani Identity, [Mir07])**.** *There exist functions $\mathcal{D}, \mathcal{R} : \mathbb{R}_+^3 \to \mathbb{R}_+$ such that for any hyperbolic surface X with n geodesic boundary components*

$\beta_1, \ldots, \beta_n$ of lengths $L_1, \ldots, L_n$, we have

$$\sum_{\{\alpha_1,\alpha_2\}} \mathcal{D}(L_1, \ell_X(\alpha_1), \ell_X(\alpha_2)) + \sum_{i=2}^{n} \sum_{\gamma} \mathcal{R}(L_1, L_i, \ell_X(\gamma)) = L_1.$$

Here the first sum is over all unordered pairs of simple closed geodesics $\{\alpha_1, \alpha_2\}$ bounding a pair of pants with β_1, and the second sum is over simple closed geodesics γ bounding a pair of pants with β_1 and β_i.

We note that by allowing the length of β_1 to approach zero one recovers McShane's original (extended) identity.

For our final identity, let X be a closed hyperbolic surface. Let $v \in T_1X$ and let α_v be the complete geodesic associated to the flow of v. The set of v in T_1X such that α_v is simple is measure 0. To a generic $v \in T_1X$ we can associate a graph G_v by flowing in the direction of v and $-v$ at equal speeds until the first intersection in both directions. A neighborhood of this graph then determines either a geometric pair of pants or a one-holed torus; we call this neighborhood F_v. Let F be a geometric pair of pants or one-holed torus in X, then we can define $U_F = \{v \in T_1X \colon F_v = F\}$. The U_F then give a full measure decomposition of T_1X. By studying the volumes of these sets, we obtain:

Theorem 2.6.5 (Luo-Tan Identity, [LT14])**.** *Let X be a closed hyperbolic surface of genus $g \geq 2$. There exist functions f and g involving dilogarithms of the lengths of simple closed geodesics in a 3-holed sphere or 1-holed torus, such that*

$$\sum_P f(P) + \sum_T g(T) = 8\pi^2(g-1)$$

where the first sum is over all properly embedded geometric 3-holed spheres $P \subset X$ and the second sum is over all properly embedded geometric 1-holed tori $T \subset X$.

Chapter 3

Quasiconformal homogeneity and subgroups of the mapping class group

3.1 Torelli Groups and Congruence Subgroups

For a closed orientable surface S_g with genus $g \geq 2$, the *Torelli group*, $\mathcal{I}(S_g)$, is the kernel of the action of $\mathrm{Mod}(S_g)$ on $H_1(S_g, \mathbb{Z})$, the first homology with $\mathbb{Z}$ coefficients. We similarly define the *level m congruence subgroup*, $\mathrm{Mod}(S_g)[m]$, as the kernel of the action of $\mathrm{Mod}(S_g)$ on $H_1(S_g, \mathbb{Z}/m\mathbb{Z})$. For the rest of this section all the results stated will hold for both classes of subgroups just mentioned with $m \geq 3$ in the latter case; we will set $\Gamma(S) = \mathcal{I}(S), \mathrm{Mod}(S)[m]$.

In [FLM08] the authors prove that for a pseudo-Anosov element $f \in \Gamma(S)$ that $\log \lambda(f) \geq 0.197$. We would like to have a similar result for reducible elements of these subgroups. We can get such a result directly from the authors' original proof with understanding how their pseudo-Anosov assumption is being used.

In their proof, they use a cone metric on S coming from a quadratic differential with stable and unstable foliations corresponding to the stable and unstable foliations for f. They use this metric to compare lengths of curves. The same proof can be given using a hyperbolic metric on S yielding $2\tau(f) = \log(\lambda(f)^2) \geq 0.197$. The authors' proof over a hyperbolic metric views f as a quasiconformal map and uses Wolpert's lemma:

Lemma 3.1.1 (Wolpert's Lemma, Lemma 12.5 in [FM11]). *Let X, Y be hyperbolic surfaces and let $f : X \to Y$ be a K-quasiconformal homeomorphism. For any isotopy class c of simple closed curves in X, the following holds:*

$$\frac{\ell_X(c)}{K} \leq \ell_Y(f(c)) \leq K\ell_X(c),$$

where $\ell_X(c)$ denotes the length of the unique geodesic representative of c in X.

This is also explained in a remark in [FLM08]. By replacing the cone metric coming from the pseudo-Anosov with a hyperbolic metric we remove the first instance of the pseudo-Anosov assumption.

The second way that the pseudo-Anosov assumption is used is to state that f does not fix the homotopy class of a shortest curve. We can remove this assumption by looking at mapping classes that do not fix a shortest curve:

Theorem 3.1.2 (Farb, Leininger, Margalit, [FLM08]). *Let X be a hyperbolic surface and γ the homotopy class of a shortest curve in X. If $f : X \to X$ is a quasiconformal homeomorphism with $[f] \in \mathcal{I}(X)$ or $[f] \in \mathrm{Mod}(X)[m]$ for some $m \geq 3$ such that $f(\gamma) \neq \gamma$, then $\log K(f) \geq 0.197$.*

For studying quasiconformal homogeneity with respect to $\Gamma(S)$, this theorem will allow us to discard any elements not fixing a shortest curve. This will be enough to prove our theorem. We start with a lemma describing the situation for large genus surfaces.

Lemma 3.1.3. *There exists g_0 such that if X is a closed hyperbolic surface of genus $g > g_0$ and X is Γ_K-homogeneous for either $\Gamma = \mathcal{I}(X)$ or $\Gamma = \mathrm{Mod}(X)[m]$ for $m \geq 3$, then $\log K > 0.197$.*

Proof. From Theorem 2.3.2 above, we know that if $f : X \to X$ is K-quasiconformal, then f is a $(K, K\log 4)$-quasi-isometry. In particular, there is some $C(K) \geq 0$ such that if γ is a geodesic in X, then $f(\gamma)$ is contained in a $C(K)$-neighborhood of $\tilde{\gamma}$, call it $N_{C(K)}(\tilde{\gamma})$, for some geodesic $\tilde{\gamma}$ in X. Define $C_0 = C(\exp(0.197))$. Also, if X is a genus g hyperbolic surface, then $\ell(X) \leq A \log g$, where A is a constant independent of genus (this is Gromov's inequality for surfaces, see [Gro83]). Now choose g_0 such that

$$\frac{4\pi(g_0 - 1)}{A \log g_0} > 2 \sinh C_0.$$

Assume that the genus of X is $g > g_0$ and that X is Γ_K-homogeneous. Let γ be a closed geodesic in X of shortest length, then it satisfies $\ell_X(\gamma) \leq A \log g$. For every $y \in X$ and $x \in \gamma$ there exists $f : X \to X$ such that $[f] \in \Gamma_K$ and $f(x) = y$. If $\log K < 0.197$, then $[f(\gamma)] = [\gamma]$ implying every point of X must be in the C_0-neighborhood of γ. Let us identify the universal cover of X with $\mathbb{H}^2$, so that $X = \mathbb{H}^2/G$ for $G < \mathrm{Isom}^+(\mathbb{H}^2)$. In the upper half plane model we can translate a lift of γ to be the imaginary axis so that the geodesic segment $[i, ie^{\ell_X(\gamma)}]$ maps onto γ. If U is a C_0-neighborhood of this segment in $\mathbb{H}^2$, then from above we know there exists a fundamental domain for the action of G on $\mathbb{H}^2$ contained in U. In particular, this implies $\mathrm{Area}(U) \geq \mathrm{Area}(X)$. However,

$$\mathrm{Area}(U) = 2\ell_X(\gamma) \sinh C_0 < 2A \log(g) \sinh(C_0) < 4\pi(g - 1).$$

But, $4\pi(g - 1) = \mathrm{Area(X)}$; hence, we found $\mathrm{Area}(U) < \mathrm{Area}(X)$. This is a contradiction; thus, we must have $\log K > 0.197$. □

Theorem 1.1.2 *There exists a constant $K_T > 1$ such that if X is a closed hyperbolic surface that is Γ_K-homogeneous for $\Gamma = \mathcal{I}(X)$ or $\Gamma = \mathrm{Mod}(X)[r]$ with $r \geq 3$, then $K \geq K_T$.*

Proof. Given a sequence of hyperbolic surfaces $\{X_n\}$, let g_n be the genus of X_n and $\Gamma_n = \mathcal{I}(X_n), \mathrm{Mod}(X_n)[m]$ for $m \geq 3$. We proceed by contradiction: Suppose the statement is false, then there exists a sequence of hyperbolic surfaces $\{X_n\}$ that are $(\Gamma_n)_{K_n}$-homogeneous such that $\lim_{n\to\infty} K_n = 1$. As $K_n \to 1$, Proposition 2.5.2 tells us that $\ell(X_n) \to \infty$ and Gromov's inequality implies that $g_n \to \infty$. Pick N such that $g_N > g_0$, where g_0 is from Lemma 3.1.3. For all $n > N$ we have $\log K_n > 0.197$ contradicting $K_n \to 1$. This completes the proof. □

3.2 A Counting Problem in Teichmüller Space

For the rest of the paper, our main method of studying quasiconformal homogeneity will be to translate the problem of understanding the homogeneity constants to one of counting orbit points in Teichmüller space under the action of the mapping class group. Before stating the lemma that will allow us to accomplish this we recall a proposition in [BTBCT07]:

Proposition 3.2.1 (Proposition 6.2 in [BTBCT07])**.** *Let $f : \mathbb{H}^2 \to \mathbb{H}^2$ be a quasiconformal map which extends to the identity on $\partial_\infty \mathbb{H}^2$ and let $x \in \mathbb{H}^2$. Then $K(f) \geq \psi(d(x, f(x)))$, where $\psi : [0, \infty) \to [1, \infty)$ is the increasing homeomorphism given by the function*

$$\psi(d) = \coth^2\left(\frac{\pi^2}{4\mu(e^{-d})}\right) = \coth^2 \mu\left(\sqrt{1 - e^{-2d}}\right), \tag{3.1}$$

where $\mu(r)$ is the modulus of the Grötsch ring whose complementary components are $\overline{\mathbb{B}^2}$ and $[1/r, \infty]$ for $0 < r < 1$.

The explicit formula for ψ was originally due to Teichmüller [Tei44]. In what follows, we will define $K(\varphi)$ for $\varphi \in \mathrm{Mod}(X)$ by

$$K(\varphi) = \min\{K_f \colon f \in \mathrm{QC}(X) \text{ and } [f] = \varphi\},$$

where $[f]$ denotes the homotopy class of f.

Lemma 3.2.2. *Let X be a genus g closed hyperbolic surface and $\Gamma < \mathrm{Mod}(X)$ such that X is Γ_K-homogeneous. If the set*

$$\{\varphi \in \Gamma \colon K(\varphi) < K\}$$

is finite with cardinality n, then

$$K \geq \sqrt{\psi\left(2\,\mathrm{arccosh}\left(\frac{2}{n}(g-1) + 1\right)\right)},$$

where ψ is defined in (3.1)*.*

Proof. As the action of $\mathrm{Mod}(X)$ on $\mathrm{Teich}(X)$ is properly discontinuous there can only be finitely many mapping classes with dilatation less than K. Let $\varphi_1, \ldots, \varphi_n$ be the n elements in Γ such that $K(\varphi_i) \leq K$. Fix $a \in X$ and let

$$U_i = \{x \in X \colon \exists f \in QC_K(X) \text{ such that } [f] = \varphi_i \text{ and } f(a) = x\}.$$

In particular, $X = \bigcup_{i=1}^n U_i$. Now $\operatorname{Area}(X) = 4\pi(g-1) \leq \sum \operatorname{Area}(U_i)$; hence, there exists $k \in \{1, \ldots, n\}$ such that $U = U_k$ satisfies $\operatorname{Area}(U) \geq \frac{4\pi}{n}(g-1)$. Let d be the diameter of U so that

$$2\pi\left(\cosh\frac{d}{2} - 1\right) \geq \operatorname{Area}(U) \geq \frac{4\pi}{n}(g-1),$$

where the leftmost term is the area of the hyperbolic ball of diameter d. This implies

$$d \geq 2\operatorname{arccosh}\left(\frac{2}{n}(g-1)+1\right).$$

For $\epsilon > 0$, let $x, y \in U$ such that $d_X(x,y) = d - \epsilon$ and pick $f, g \in \mathrm{QC}_K(X)$ with $[f] = [g] = \varphi_i$ such that $f(a) = x$ and $g(a) = y$, then $h = g \circ f^{-1}$ is isotopic to the identity and $h(x) = y$. Let $\tilde{h} : \mathbb{H}^2 \to \mathbb{H}^2$ be a lift of h which extends to the identity on $\partial_\infty \mathbb{H}^2$. The above proposition implies

$$K(\tilde{h}) = K(h) \geq \psi(d(x,y)) = \psi(d-\epsilon).$$

We now have

$$K^2 \geq K(f) \cdot K(g^{-1}) \geq K(f \circ g^{-1}) = K(h) \geq \psi(d_X(x,y)) = \psi(d-\epsilon).$$

The result follows by letting ϵ tend to zero and the fact that ψ is increasing. $\square$

Let us wrap the above lemma in the language of Teichmüller theory. Given $\mathcal{X} = (X, \varphi) \in \operatorname{Teich}(S_g)$ we can identify $f \in \operatorname{Mod}(S_g)$ with $\varphi \circ f \circ \varphi^{-1} \in \operatorname{Mod}(X)$, then

$$|\{g \in \operatorname{Mod}(X) \colon K(g) < K\}| = |\{f \in \operatorname{Mod}(S_g) \colon f \cdot \mathcal{X} \in B_{\log\sqrt{K}}(\mathcal{X})\}|,$$

where $B_R(\mathcal{X})$ is the ball of radius R in the Teicmüller metric centered at $\mathcal{X} \in \operatorname{Teich}(S_g)$. This allows us to think about orbits in $\operatorname{Teich}(S_g)$. Lemma 3.2.2 provides a possible route to proving that there exists an universal constant $K_2 > 1$ such that if X is a K-quasiconformally homogeneous closed hyperbolic surface, then $K \geq K_2$.

Theorem 1.1.3 *Suppose there exist constants $\epsilon, R, C > 0$ such that for any $\mathcal{X} \in \mathrm{Teich}_{(\epsilon,\infty)}(S_g)$ with $g > 1$*

$$|\{f \in \mathrm{Mod}(S_g)\colon f \cdot \mathcal{X} \in B_R(\mathcal{X})\}| \leq Cg.$$

Then, there exists a constant $K_2 > 1$ such that any closed K-qch surface must have $K \geq K_2$.

Proof. We proceed by contradiction: Assume there exists a sequence of closed hyperbolic surfaces $\{X_n\}$ such that X_n is K_n-quasiconformally homogeneous and $K_n \to 1$. This implies $\ell(X_n) \to \infty$ by Proposition 2.5.2 and $g_n \to \infty$ by Gromov's inequality, where g_n is the genus of X_n. By Lemma 3.2.2 and the cardinality assumption we have that

$$K_n \geq \sqrt{\psi\left(2\,\mathrm{arccosh}\left(\frac{2}{Cg_n}(g_n-1)+1\right)\right)}.$$

(Note that we use that both ψ and arccosh are increasing functions.) In particular, we have

$$\lim_{n\to\infty} K_n \geq \sqrt{\psi\left(2\,\mathrm{arccosh}\left(\frac{2}{C}+1\right)\right)} > 1.$$

This contradicts the assumption $K_n \to 1$, which completes the proof. $\square$

3.3 Finite Subgroups

For a closed orientable surface S with negative Euler characteristic, there are well known bounds for the order of finite groups and elements in $\mathrm{Mod}(S)$: it is a theorem of Hurwitz that the the group $\mathrm{Isom}^+(X)$ for a closed hyperbolic surface X of genus $g \geq 2$ has order bounded above by $84(g-1)$. Also, it was proved by Wiman [Wim] that any element in $\mathrm{Isom}^+(X)$ has order bounded above by $4g+2$ (both of these are proved in [FM11]). In addition, the Nielsen realization theorem proved by Kerckhoff [Ker83] tells us that a finite subgroup of $\mathrm{Mod}(S)$ can be realized as a subgroup of $\mathrm{Isom}^+(X)$ for some hyperbolic surface X homeomorphic to S. Combining these results with Lemma 3.2.2, we get the following results:

Theorem 1.1.5 *There exists a constant $K_F > 1$ such that if a closed hyperbolic surface X is Γ_K-homogeneous, where $\Gamma < \mathrm{Mod}(X)$ has finite order, then $K \geq K_F$. Furthermore, we have*

$$K_F \geq \sqrt{\psi\left(2\,\mathrm{arccosh}\left(\frac{1}{42}+1\right)\right)} = 1.11469\ldots,$$

where ψ is defined in equation (3.1)*.*

Proof. From the above discussion, we know that $|\Gamma| \leq 84(g-1)$. The result follows by setting $n = 84(g-1)$ in Lemma 3.2.2. □

Theorem 3.3.1. *There exists a constant $K_P > 1$ such that if a closed hyperbolic surface X is Γ_K-homogeneous, where $\Gamma = \langle f \rangle$ and $f \in \mathrm{Mod}(X)$ is periodic, then $K \geq K_P$. In particular, we have*

$$K_P \geq \sqrt{\psi\left(2\,\mathrm{arccosh}\left(\frac{6}{5}\right)\right)} = 1.35547\ldots.$$

Proof. From the above discussion, we know that $|\varphi| \leq 4g+2$, so we can use Lemma 3.2.2 with $n = 4g+2$. We see the worst case is $n = 4g+2$ when $g = 2$. □

3.4 Pure Cyclic Subgroups

We follow [Iva92] in calling a homeomorphism $f : S \to S$ *pure* if for some closed one-dimensional submanifold C of S the following are true:

(1) the components of C are nontrivial,

(2) $f|_C$ is the identity,

(3) f does not rearrange the components of $S \setminus C$, and

(4) f induces on each component of S cut along C a homeomorphism either homotopic to a pseudo-Anosov or the identity homeomorphism.

An element of $\mathrm{Mod}(S)$ is called *pure* if the homotopy class contains a pure homeomorphism. Note that we allow $C = \emptyset$ so that pseudo-Anosov homeomorphisms are pure. Recall that for a mapping class $f \in \mathrm{Mod}(S)$ we let $\tau(f)$ denote its translation length in $\mathrm{Teich}(S)$. We can then break pure mapping class elements into three categories along the lines of Bers's classification of surface diffeomorphisms: if $f \in \mathrm{Mod}(S)$ is pure, then

(i) $\tau(f) > 0$ and realized, so that f is a (full) pseudo-Anosov,

(ii) $\tau(f) > 0$ and not realized, so that f induces a pseudo-Anosov homeomorphism on some component of S cut along the canonical reduction system for f (we will call these *partial pseudo-Anosov*), or

(iii) $\tau(f) = 0$ and not realized, so that f is a Dehn twist about a multicurve, which we will call a *multi-twist*.

We will consider homogeneity with respect to cyclic subgroups generated by each type of pure mapping class in turn.

3.4.1 Full and Partial Pseudo-Anosov Mapping Classes

Let S be a closed surface and $f \in \mathrm{Mod}(S)$ be a pure partial pseudo-Anosov mapping class. Then there exists a multicurve C and a representative of f, which we will also call f, such that f fixes C pointwise. Let R be a component of the (possibly disconnected) surface resulting from cutting S along C such that $f|_R$ is pseudo-Anosov. We can build a punctured surface F by gluing punctured disks to each of the boundary components of R, so that R is embedded in F. Furthermore, since f restricted to ∂R is the identity, we can extend $f|_R$ to a map $\hat{f} : F \to F$ by defining $\hat{f}|_R = f|_R$ and $\hat{f}|_{F \setminus R} = id$. We have constructed $\hat{f}$ so that $[\hat{f}] \in \mathrm{Mod}(F)$ is a full pseudo-Anosov map on a punctured surface and our first goal will be to relate the the translation length, $\tau(f)$, of f in $\mathrm{Teich}(S)$ to the translation length, $\tau(\hat{f})$, of $\hat{f}$ in $\mathrm{Teich}(F)$.

Lemma 3.4.1. *Let $S, f, F, \hat{f}$ be defined as above, then $\tau(f) \geq \tau(\hat{f})$.*

Proof. Recall that $\tau(f)$ is not realized, so let $\{(X_n, \varphi_n)\}$ be a sequence in $\mathrm{Teich}(S)$ and $f_n : X_n \to X_n$ be the Teichmüller map in the homotopy class of $\varphi_n \circ f \circ \varphi_n^{-1}$ so that $\lim_{n\to\infty} K(f_n) = e^{2\tau(f)}$. Define R_n to be the geometric straightening of $\varphi_n(R)$ in X_n so that ∂R_n is a disjoint union of simple closed geodesics. The collaring lemma provides disjoint neighborhoods around each boundary component of R_n; let N_n be the union of these neighborhoods. We can then pick points $x_n \in R_n \smallsetminus N_n$ such that $f_n(x_n) \in R_n \smallsetminus N_n$. The sequence - possibly a subsequence - of pointed surfaces (X_n, x_n) converges geometrically to (X_∞, x_∞), where X_∞ is homeomorphic to F as the collection of curves permuted by f must be pinched. This convergence is clear as this limit agrees with the visual limit from the viewpoint of x_n. With this setup we will construct a quasiconformal map on X_∞ that has the same translation length in Teich(F) as $\hat{f}$ and smaller dilatation then $\lim_{n\to\infty} K(f_n)$.

We will want to work in the hyperbolic plane; in particular, we will use the disk model $(\mathbb{D}, d_H)$, where $\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$ and d_H is the hyperbolic metric. Let us identify the universal cover of (X_n, x_n) with $(\mathbb{D}, 0)$ and let $\Gamma_n < \mathrm{Isom}(\mathbb{D})$ such that $X_n = \mathbb{D}/\Gamma_n$. We may assume that our marking $\varphi_n : S \to X_n$ induces the representation $\rho_n = (\varphi_n)_* : \pi_1 S \to \Gamma_n$. We note that the Γ_n converge to a group Γ_∞ such that $\mathbb{H}^2/\Gamma_\infty = X_\infty$. Let $\tilde{y}_n$ be a lift of $f(x_n)$ such that $d_H(0, \tilde{y}_n) = d_X(x_n, f(x_n))$, then choose a lift $\tilde{f}_n : \mathbb{D} \to \mathbb{D}$ of f_n with $\tilde{f}_n(0) = \tilde{y}_n$. By compactness, the sequence of points $\{\tilde{y}_n\}$ must have a convergent subsequence, which we also call $\{\tilde{y}_n\}$, in $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$. Set $\tilde{y}_\infty = \lim_{n\to\infty}\{\tilde{y}_n\}$, then as the $x_n \in R_n$ have been chosen to avoid going up the cusp, we see that $\tilde{y}_\infty \in \mathbb{D}$. Let $y_n \in X_n$ be the projection of $\widetilde{y}_\infty$ to X_n. Define $h_n : X_n \to X_n$ such that h_n is isotopic to the identity, $h_n(f(x_n)) = y_n$ and $\lim_{n\to\infty} K(h_n) = 1$. Now $g_n = h_n \circ f_n : X_n \to X_n$ with $g_n(x_n) = y_n$; in particular, we can choose lifts $\tilde{g}_n : \mathbb{D} \to \mathbb{D}$ of the g_n with $\tilde{g}_n(0) = \tilde{y}_\infty$.

The family of K-quasiconformal maps

$$\{g : \mathbb{D} \to \mathbb{D}\colon K(g) \leq K \text{ and } g(0) = \tilde{y}_\infty\}$$

is normal [Hub06]; therefore, the sequence $\{\tilde{g}_n\}$ of quasiconformal maps has a convergent subsequence, which we also call $\{\tilde{g}_n\}$. Define $\tilde{g}_\infty = \lim_{n\to\infty}\{\tilde{g}_n\}$, so that $\tilde{g}_\infty(0) = \tilde{y}_\infty$ and

$$K(\tilde{g}_\infty) = \lim_{n\to\infty} K(\tilde{g}_n) = \lim_{n\to\infty} K(g_n) \leq \lim_{n\to\infty}[K(h_n) \cdot K(f_n)] = e^{2\tau(f)}.$$

It is left to show that $\tilde{g}_\infty$ descends to a map $g_\infty : X_\infty \to X_\infty$ and $\tau(\hat{f}) \leq \frac{1}{2}\log K(g_\infty)$.

In order to finish the proof we will look at a particular definition of the geometric limit (details for geometric limits can found in §E.1 in [BP92]). Let $p_n : \mathbb{H}^2 \to X_n$ be the canonical projections (where we identify $X_n = \mathbb{H}^2/\Gamma_n$). As the sequence (X_n, x_n) converges to (X_∞, x_∞) geometrically, we can find bilipschitz maps $\tilde{\psi}_n : \overline{B(0, r_n)} \to \mathbb{H}^2$, where $B(z, r)$ is the ball of radius r about z, such that $\tilde{\psi}_n(0) = 0$, the $\tilde{\psi}_n$ converge to the identity on $\mathbb{H}^2$, and for all z_1, z_2 in the domain of $\tilde{\psi}_n$

$$p_\infty(z_1) = p_\infty(z_2) \iff p_n(\tilde{\psi}_n(z_1)) = p_n(\tilde{\psi}_n(z_2)). \tag{3.2}$$

In particular, the maps $\tilde{\psi}_n^{-1} \circ \tilde{g}_n \circ \tilde{\psi}_n$ converge to $\tilde{g}_\infty$. Combining (3.2) with the fact that $\tilde{g}_n$ is Γ_n-equivariant we see that $\tilde{\psi}_n^{-1} \circ \tilde{g}_n \circ \tilde{\psi}_n$ is Γ_∞-equivariant on its domain. This implies that $\tilde{g}_\infty$ is Γ_∞-equivariant and descends to $g_\infty : X_\infty \to X_\infty$.

It is left to show $\tau(\hat{f}) \leq \frac{1}{2} \log K(g_\infty)$. Condition (3.2) implies that the maps $\tilde{\psi}_n$ descend to $\psi_n : K_n \hookrightarrow X_n$, where K_n is a compact set in X_∞. From above we know the domain of $\psi_n^{-1} \circ g_n \circ \psi_n$ is converging to X_∞ and $\psi_n^{-1} \circ g_n \circ \psi_n$ is converging to g_∞. Choose N such that for $n > N$ if removing the domain of $\psi_n^{-1} \circ g_n \circ \psi_n$ from X_∞ results in a disjoint union of punctured disks. We can then extend $\psi_n^{-1} \circ g_n \circ \psi_n : X_\infty \to X_\infty$ without affecting convergence. We therefore see that for large n that $\psi_n^{-1} \circ g_n \circ \psi_n$ is homotopic to g_∞, which implies

$$g_\infty \simeq \psi_n^{-1} \circ \varphi_n \circ f \circ \varphi_n^{-1} \circ \psi_n.$$

On the domain of interest, we are really looking at restricting the φ_n and f to R and then extending. In fact, we see that

$$g_\infty \simeq \psi_n^{-1} \circ \varphi_n \circ \hat{f} \circ \varphi_n^{-1} \circ \psi_n.$$

We can think of an extension of $\psi_n^{-1} \circ \varphi_n|_R$ as a marking $F \to X_\infty$, which implies $\tau(\hat{f}) \leq \frac{1}{2} \log K(g_\infty) \leq \tau(f)$ as desired. □

We will consider both full and partial pseudo-Anosov homeomorphisms at the same time. We will rely on a result of Penner [Pen91], which provides a lower bound for the dilatation of a pseudo-Anosov $f \in \mathrm{Mod}(S)$:

$$\log \lambda(f) \geq \frac{\log 2}{|\chi(S)|},$$

where $\chi(S)$ denotes the Euler characteristic of S. This holds for both closed and punctured surfaces.

Theorem 3.4.2. *There exists a constant $K_A > 1$ such that if a closed hyperbolic surface X is Γ_K-homogeneous, where $\Gamma = \langle f \rangle$ with $f \in \mathrm{Mod}(X)$ either pseudo-Anosov or partial pseudo-Anosov, then $K \geq K_A$. In particular, we have $K_A \geq 1.42588$.*

Proof. Let $[f] \in \mathrm{Mod}(X)$ and $R \subseteq X$ a connected subsurface such that $f|_R$ is pseudo-Anosov and $f(R)$ is isotopic to R. Note that in the case f is not reducible, then $R = X$. We will keep with our notation above, so that we can extend $f|_R$ to $\hat{f} : F \to F$, where F is a punctured surface in the reducible case or again $F = X$ and $\hat{f} = f$ in the pseudo-Anosov case. If we let $\tau(\hat{f})$ denote the translation length of $\hat{f}$ in Teich(F), then, as $|\chi(F)| \leq |\chi(X)|$, we have $\tau(\hat{f}) \geq \frac{\log 2}{12(g-1)}$, where g is the genus of X (see [Pen91]). Let $m \in \mathbb{Z}$ such that

$$\frac{m \log 2}{6(g-1)} \geq \log K.$$

As $\widehat{(f^2)} = \hat{f}^2$ and $\tau(\hat{f}^2) = 2\tau(\hat{f})$, we find

$$\tau(f^m) \geq \tau(\hat{f}^m) = m\tau(\hat{f}) \geq \frac{m \log 2}{12(g-1)} \geq \frac{1}{2} \log K.$$

In particular, $K(f^m) \geq K$. We can now appeal to Lemma 3.2.2 with $n \leq 2m+1$ (accounting for negative powers and the identity) to find that

$$K \geq \mu_g(K),$$

where we define

$$\mu_g(K) = \sqrt{\psi\left(2\,\mathrm{arccosh}\left(\frac{2\log 2}{12(g-1)\log K + \log 2}(g-1)+1\right)\right)}.$$

As $\mu_g(K)$ increases with g, we have that $K \geq \mu_2(K)$. For $K \geq 1$, we see that $\mu_2(K)$ is decreasing and so there exists a unique solution to $K - \mu_2(K) = 0$, call it K_A. A computation shows that $K_A = 1.42588...$ and the result follows. □

3.4.2 Multi-twists

We start this section with finding a lower bound for the dilatation of a quasiconformal homeomorphism homotopic to a multi-twist. We do this by understanding the map induced on the boundary of the hyperbolic plane. Let X be a closed hyperbolic surface and $f \in \mathrm{QC}(X)$, then by identifying the universal cover of X with $\mathbb{H}^2$ we can choose $\widetilde{f} : \mathbb{H}^2 \to \mathbb{H}^2$ to be a lift of f. Furthermore, we can extend $\widetilde{f}$ to the boundary of $\mathbb{H}^2$ continuously, which we identify with $\overline{\mathbb{R}}$. Let $\overline{f} : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be the restriction of $\widetilde{f}$ to $\overline{\mathbb{R}} = \partial\mathbb{H}^2$. We can choose $\widetilde{f}$ such that $\overline{f}(\infty) = \infty$. In this setup there exists an M such that $\overline{f}$ is $\mathbb{R}$-quasisymmetric with modulus M, that is

$$\frac{1}{M} \leq \frac{\overline{f}(x+t) - \overline{f}(x)}{\overline{f}(x) - \overline{f}(x-t)} \leq M,$$

for all $x \in \mathbb{R}$ and $t > 0$ (see §4.9 of [Hub06]). Sharp bounds are known for the modulus M above associated to a K-quasiconformal homeomorphism of $\mathbb{H}^2$: define

$$\lambda(K) = \frac{1}{(\mu^{-1}(\pi K/2))^2} - 1,$$

where $\mu(r)$ is the modulus of the Grötsch ring whose complementary components are $\overline{\mathbb{B}^2}$ and $[1/r, \infty]$ for $0 < r < 1$. Then (see [LV73]) we have

$$\frac{1}{\lambda(K(f))} \leq \frac{\overline{f}(x+t) - \overline{f}(x)}{\overline{f}(x) - \overline{f}(x-t)} \leq \lambda(K(f)). \tag{3.3}$$

If f is homotopic to a multi-twist, then this is enough information to produce a lower bound for $K(f)$ in terms of the lengths of the curves f twists about.

Lemma 3.4.3. *Let X be a closed hyperbolic surface and $f \in \mathrm{QC}(X)$ be homotopic to a multi-twist T_C about a multicurve $C = \{\gamma_1, \ldots, \gamma_n\}$, so that $T_C = T_{\gamma_1}^{m_1} \circ \cdots \circ T_{\gamma_n}^{m_n}$. If*

$m = |m_k|, \ell = \ell_X(\gamma_k)$ *such that* $m\ell = \max_i\{|m_i| \cdot \ell_X(\gamma_i)\}$, *then*

$$K(f) \geq \frac{2}{\pi}\,\mu\left(\sqrt{\frac{2}{2+e^{(m-1)\ell}+e^{(m-\frac{1}{2})\ell}}}\right),$$

where $\mu(r)$ *is the modulus of the Grötsch ring whose complementary components are* $\overline{\mathbb{B}^2}$ *and* $[1/r, \infty]$ *for* $0 < r < 1$.

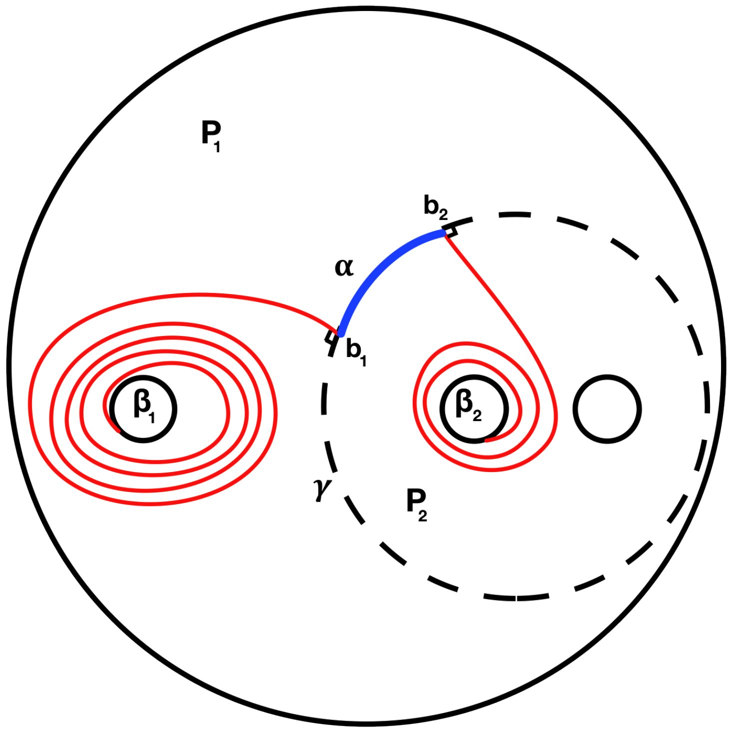

Figure 3.1: A 4-punctured sphere in X with γ bounding two embedded pairs of pants. The curve α intersects γ once and spirals towards both β_1 and β_2 so that it is disjoint from all boundary components.

Proof. Let $\gamma = \gamma_k$ so that $\ell = \ell_X(\gamma)$ and extend the collection $C = \{\gamma_1, \ldots, \gamma_n\}$ of disjoint simple closed curves to a maximal collection, call it C', giving a pants decomposition for X. We want to construct an infinite simple complete geodesic in X, which does not intersect any element of C' other than γ. First assume that γ bounds two pairs of pants, P_1 and P_2 as in Figure 3.1. Let β_i be a component of ∂P_i for $i = 1, 2$ such that $\beta_i \neq \gamma$, then there exists a geodesic ray in P_i spiraling towards β_i and meeting γ perpendicularly at b_i. In X, P_1 and P_2 are glued together with a twist along γ, so we can create a geodesic α by

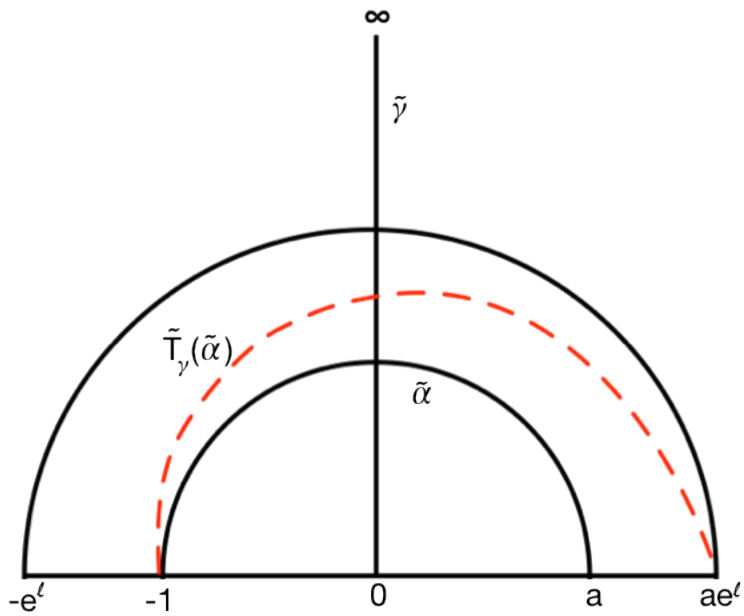

Figure 3.2: Lifts of α and γ in the upper half plane. Also drawn is a copy of $\tilde{\alpha}$ under a translation by the element of $\pi_1 X$ representing γ. The dotted geodesic is the image of $\tilde{\alpha}$ under the lift of a Dehn twist about γ.

connecting the two rays via an arc on γ connecting the images of b_1 and b_2 in X and pulling this curve tight. The other possibility is that γ bounds a single pair of pants P. In P we have two copies of γ and one other boundary component. There exists a ray emanating perpendicularly from each copy of γ spiraling towards this other component such that these two rays are disjoint. We then construct α from these rays as above. We see that α is our desired complete geodesic.

We can identify the universal cover $\widetilde{X}$ of X with the upper half plane $\{z \in \mathbb{C}\colon \operatorname{Im}(z) > 0\}$ so that we have lifts $\widetilde{\gamma}, \widetilde{\alpha}$ of γ, α in the configuration showed in Figure 3.2. Let $T_\gamma : X \to X$ be a left Dehn twist about γ and let $\widetilde{T}_\gamma : \mathbb{H}^2 \to \mathbb{H}^2$ be a lift of T_γ fixing $\widetilde{\gamma}$. Let $[x, y]$ denote the geodesic in $\mathbb{H}^2$ with endpoints $x, y \in \partial\mathbb{H}^2$. In our setup, $\widetilde{\alpha} = [-1, a]$ and we see that $\widetilde{T}_\gamma(\widetilde{\alpha})$ is homotopic to the dotted curve shown in Figure 3.2 and has endpoints $[-1, ae^\ell]$. By iterating this map, we can construct a family of geodesics $\{\alpha_n\}$ in X that are the projection of $\widetilde{T}_\gamma^n(\widetilde{\alpha}) = [-1, ae^{n\ell}]$. Furthermore, every α_n is an infinite simple complete geodesic in X that does not intersect any element of C' other than γ. We can then find an integer k such that $ae^{k\ell} \in [\frac{1}{2}(e^{-\ell} + e^{-\ell/2}), \frac{1}{2}(1 + e^{\ell/2})]$; define $\widetilde{\beta} = [-1, ae^{k\ell}] = [-1, b]$ so that the image of

$\widetilde{\beta}$ is $\beta = \alpha_k$.

We now want to investigate $K = K(f)$ by studying $\overline{f} : \partial\mathbb{H}^2 \to \partial\mathbb{H}^2$, which is the induced boundary map from the lift $\widetilde{f} : \mathbb{H}^2 \to \mathbb{H}^2$ fixing $0, -1, \infty$. As two homotopic maps induce the same boundary map on $\mathbb{H}^2$, we have $\bar{f} = \overline{T}_C$ (it is convenient to think of $\bar{f}$ as the map on $\partial\mathbb{H}^2$ coming from a left earthquake along the complete lift of the multicurve C, see [Ker83] for the definition of an earthquake). Let us assume for now that $\frac{1}{2}(e^{-\ell} + e^{-\ell/2}) \leq b \leq 1$ and that f twists left about γ (if not we can just study f^{-1}). By construction β is infinite in X, β intersects γ exactly once, and $\beta \cap \gamma_i = \emptyset$ for $i \neq k$; this implies that $\widetilde{\gamma}$ is the only geodesic in the full lift of C that $\widetilde{\beta}$ intersects. Therefore, we know that $\overline{f}(b) = be^{m\ell}$ and also that $[-1, b^{m\ell}]$ and $[-1, \overline{f}(1)]$ do not intersect as $[-1, b]$ and $[-1, 1]$ do not. In particular, we must have that $\overline{f}(1) \geq b^{m\ell}$. This yields:

$$\lambda(K) \geq \frac{\overline{f}(1) - \overline{f}(0)}{\overline{f}(0) - \overline{f}(-1)} = \overline{f}(1) \geq be^{m\ell} \geq \frac{1}{2}\left(e^{(m-1)\ell} + e^{(m-\frac{1}{2})\ell}\right).$$

From above we can write

$$K = \frac{2}{\pi}\mu\left(\sqrt{\frac{1}{\lambda(K)+1}}\right),$$

and as μ is a decreasing function (see [LV73]), we have

$$K \geq \frac{2}{\pi}\,\mu\left(\sqrt{\frac{2}{2 + e^{(m-1)\ell} + e^{(m-\frac{1}{2})\ell}}}\right).$$

Now assume that $1 \leq b \leq \frac{1}{2}(1 + e^{\ell})$. Furthermore since $K(f) = K(f^{-1})$ for any quasiconformal map, we may assume that f twists to the right along γ. We have the same exact setup as before, except this time the inequality is as follows:

$$\frac{1}{\lambda(K)} \leq \frac{\overline{f}(1) - \overline{f}(0)}{\overline{f}(0) - \overline{f}(-1)} = \overline{f}(1) \leq \overline{f}(b) = be^{-m\ell} \leq \frac{1}{2}\left(e^{-m\ell} + e^{(\frac{1}{2}-m)\ell}\right),$$

yielding

$$K \geq \frac{2}{\pi}\,\mu\left(\sqrt{\frac{1 + e^{\frac{\ell}{2}}}{1 + e^{\frac{\ell}{2}} + 2e^{m\ell}}}\right).$$

As μ is decreasing, for $\ell \geq 0$ the first inequality for K is always smaller. $\square$

We saw in Theorem 2.5.1 that a hyperbolic surface X with a short curve has a large homogeneity constant. We leverage this with the above lemma to get a universal bound for the homogeneity constant with respect to a subgroup of $\mathrm{Mod}(X)$ generated by a multi-twist.

Theorem 3.4.4. *There exists a constant $K_D > 1$ such that if a closed hyperbolic surface X is Γ_K-homogeneous, where $\Gamma = \langle f \rangle < \mathrm{Mod}(X)$ with f being a muti-twist, then $K \geq K_D$. In particular, we have $K_D \geq 1.09297$.*

Proof. Let $\ell = \ell(X)$ be the systole of X. From the definition of $m(2, K)$ in Theorem 2.5.1 given in [BTCMT05] and the inequality $\ell \geq m(2, K)$, we have

$$K \geq \frac{\log\left(\frac{1}{2}\tanh\frac{\ell}{2}\right) - \log 2e}{\log\left(\frac{1}{2}\tanh\frac{d_2}{2}\right) - \log 2e} \equiv \Phi(\ell), \tag{3.4}$$

where d_2 is defined such that every closed hyperbolic surface contains an embedded hyperbolic disk of diameter d_2. It is shown in [Yam81] that we can take $d_2 = 2\log(1 + \sqrt{2})$. From Lemma 3.4.3 we have

$$K(f) \geq \frac{2}{\pi}\,\mu\left(\sqrt{\frac{2}{3 + e^{\frac{1}{2}\ell}}}\right) \equiv \Psi(\ell). \tag{3.5}$$

Now, Φ is decreasing on $\mathbb{R}^{>0}$ with $\Phi(0) = +\infty$ and Ψ is an increasing function on $\mathbb{R}^{>0}$ with $\Psi(0) = 1$; hence, there exists a unique value L such that $\Phi(L) = \Psi(L)$. We note that $L \approx 1.33994$ and $\Phi(L) \approx 1.09297$. If $\ell \leq L$, then $K \geq \Phi(L)$. Assume $\ell \geq L$ and $K < \Psi(L)$. Then $K(f) \geq \Psi(L)$ and every element in Γ_K is isotopic to the identity: this case is handled in [BTBCT07] and tells us it must be that $K \geq 1.626 > \Psi(L)$. This contradiction proves the theorem. □

Theorem 1.1.6 is now just a corollary of the previous two sections with setting $K_C = \min\{K_D, K_A\}$.

3.5 Torsion-Free Subgroups

In this section we investigate a lower bound for the homogeneity constant of a surface in terms of its genus. The idea is to find a lower bound for the dilatation of a quasiconformal map on a thick surface. Periodic elements create serious difficulties that we do not know how to deal with, so we will restrict ourselves to the torsion-free case.

Theorem 1.1.7 *Let X be a closed hyperbolic surface and suppose $\Gamma < \mathrm{Mod}(X)$ is torsion-free. If X is Γ_K-homogeneous, then*

$$\log K \geq \frac{1}{7000g^2},$$

where g is the genus of X.

Proof. Let $\mathcal{F} = \{f \in \Gamma\colon \log K(f) < 7000^{-1}g^{-2}\}$, then our goal will be to show that $\mathcal{F} = \{id\}$. The first observation is that $\mathcal{F}$ cannot contain any pseudo-Anosov or pure partial pseudo-Anosov elements. This is seen by combining the bounds in [Pen91] already mentioned and Lemma 3.4.1.

We can find ℓ_0 such that $\log \Phi(\ell_0) > 1$, where Φ is defined in (3.4); in particular, we can take $\ell_0 = 1.8$. Furthermore, since we know that if $\ell(X) < \ell_0$ then $K > \Phi(\ell_0) > \exp(g^{-2})$. Therefore, we may assume $\ell(X) > \ell_0$ and so $\mathcal{F}$ cannot contain any multi-twists as any mutli-twist will have dilatation bigger than $\Psi(\ell_0) = 1.12$, where Ψ is defined in (3.5). We are left with mapping classes of the form f where some power of f is either a partial pseudo-Anosov or multi-twist.

Let us first consider the partial pseudo-Anosov case: we can find a subsurface $R \subset X$ and a $k > 0$ such that f^k fixes the isotopy class of R and $f^k|R$ is pseudo-Anosov. There are at most $\chi(X)/\chi(R)$ copies of R permuted by f in X, therefore we may choose $k \leq \chi(X)/\chi(R)$. We then have

$$\log K(f^k) \geq \frac{\log 4}{|\chi(R)|}.$$

It follows that

$$k \cdot |\chi(R)| \cdot \log K(f) \geq |\chi(R)| \cdot \log K(f^k) \geq \log 4,$$

and

$$\log K(f) \geq \frac{\log 4}{k \cdot |\chi(R)|} \geq \frac{\log 4}{|\chi(X)|}.$$

This shows that $f \notin \mathcal{F}$.

We may now suppose that some power of f is a multi-twist. Recall $\ell(X) > \ell_0$. Choose a simple closed curve γ and $k > 0$ such that $f^k([\gamma]) = [\gamma]$. Define R_1 and R_2 to be the subsurfaces on either side of γ (possibly $R_1 = R_2$) such that there exists $n > 0$ with $f^n|_{R_1} = f^n|_{R_2} = id$. Let $R = R_1 \cup R_2$, then we can choose $k < \chi(X)/\chi(R)$; furthermore, f^{2k} fixes the isotopy classes of both R_1, R_2. Now choose m_i such that $f^{2km_i}|_{R_i} = id$ for $i = 1, 2$. By doubling R_i, we see that

$$m_i \leq 4|\chi(R_i)| + 6 \leq 10|\chi(R_i)|$$

(recall that for a periodic element $h \in \mathrm{Mod}(S_g)$ that $|\langle h \rangle| \leq 4g + 2 = 2|\chi(S_g)| + 6$). This implies $2km_1m_2 < 800g^2$. The same line of argument as above tells us that

$$2 \cdot k \cdot m_1 \cdot m_2 \cdot \log K(f) \geq \log \Phi(\ell_0)$$

and

$$\log K(f) \geq \frac{\log \Phi(\ell_0)}{800g^2} > \frac{1}{7000g^2}.$$

Again we see $f \notin \mathcal{F}$.

We have exhausted all the torsion free elements in $\mathrm{Mod}(S_g)$; hence, $\mathcal{F} = \{id\}$ as claimed. If $\log K < 7000^{-1}g^{-2}$, we can proceed by contradiction as we did in the cyclic multi-twist case: we must have that the elements in Γ_K are isotopic to the identity: this case is handled in [BTBCT07] and implies $K \geq 1.626$, which is larger than our assumption. This is a contradiction, so we see $\log K > 7000^{-1}g^{-2}$. □

3.6 Functions on Teichmüller Space and Moduli Space

This section looks at building functions on Teichmüller space out of measuring the homogeneity constant at a given point. The statements and techniques follow the related results in [BTBCT07]. For the entirety of this section, let S be a closed orientable surface with $\chi(S) < 0$. Let $\mathcal{X} = [(X, \varphi)] \in \mathrm{Teich}(S)$, then given $\Gamma < \mathrm{Mod}(S)$ define

$$\Gamma_\varphi = \{[\varphi \circ f \circ \varphi^{-1}] \colon f \in \mathrm{Homeo}^+(S) \text{ and } [f] \in \Gamma\} < \mathrm{Mod}(X).$$

We then define $K_\Gamma : \mathrm{Teich}(S) \to (1,\infty)$ by

$$K_\Gamma([(X,\varphi)]) = \min\{K \colon X \text{ is } (\Gamma_\varphi)_K\text{-homogeneous}\}.$$

Lemma 3.6.1. *Given $\Gamma < \mathrm{Mod}(S)$, the function $K_\Gamma : \mathrm{Teich}(S) \to (1,\infty)$ exists and is well-defined.*

Proof. We first need to prove that K_Γ exists, i.e. that the minimum exists. Let X be a hyperbolic surface and let $\varphi : S \to X$ be a diffeomorphism. Set

$$K = \inf\{Q \colon X \text{ is } (\Gamma_\varphi)_Q\text{-homogeneous}\}.$$

We can then find a sequence $\{K_j\}$ converging to K such that X is $(\Gamma_\varphi)_{K_j}$-homogeneous. We want to show that X is $(\Gamma_\varphi)_K$-homogeneous.

Let $x, y \in X$, then we can find a K_j-quasiconformal homeomorphisms f_j such that $f_j(x) = y$. Pick lifts $\tilde{x}, \tilde{y} \in \mathbb{H}^2$ and $\tilde{f}_j : \mathbb{H}^2 \to \mathbb{H}^2$ of x, y, and f_j, respectively, such that $\tilde{f}_j(\tilde{x}) = \tilde{y}$. We recall that the family of all Q-quasiconformal homeomorphisms of $\mathbb{H}^2$ sending $\tilde{x}$ to $\tilde{y}$ is normal (see corollary 4.4.3 in [Hub06]). Therefore, there exists a subsequence of $\{\tilde{f}_j\}$ that converges to a K-quasiconformal homeomorphism $\tilde{f} : \mathbb{H}^2 \to \mathbb{H}^2$ with $\tilde{f}(\tilde{x}) = \tilde{y}$. Furthermore, $\tilde{f}$ descends to a K-quasiconformal mapping $f : X \to X$. It is left to show that $[f] \in \Gamma_\varphi$. As the connected components of $\mathrm{QC}(X)$ are given by isotopy classes, we must have that for j large $[f_j] = [f]$ and as each $[f_j] \in \Gamma_\varphi$, so is $[f]$. This shows the minimum exists.

As a point in Teichmüller space is an equivalence class we must check that K_Γ is well-defined. Let $(X,\varphi) = (X,\psi) \in \mathrm{Teich}(S)$, so that φ and ψ are isotopic. As $\mathrm{Mod}(X)$ is defined up to isotopy, it is clear that $\Gamma_\varphi = \Gamma_\psi$ and $K_\Gamma((X,\varphi)) = K_\Gamma((X,\psi))$. Now let $(X,\varphi) = (Y,\xi) \in \mathrm{Teich}(S)$, so that $\varphi \circ \xi^{-1} \simeq I$ for some conformal map $I : Y \to X$. As conformal maps preserve quasiconformal dilatations it is clear $K_\Gamma((Y,\xi)) = K_\Gamma((X, I \circ \xi))$. By definition $I \circ \xi \simeq \psi$, so that by the previous argument $K_\Gamma((X,\varphi)) = K_\Gamma((Y,\xi))$. This shows that $K_\Gamma : \mathrm{Teich}(S) \to (1,\infty)$ is well-defined. $\square$

We now associated to each subgroup of the mapping class group a continuous function of Teichmüller space. In the following we closely adhere to the proof of Lemma 7.1 in [BTBCT07].

Proposition 3.6.2. *For* $\Gamma < \mathrm{Mod}(S)$, *the function* $K_\Gamma : \mathrm{Teich}(S) \to (1, \infty)$ *is continuous.*

Proof. We will prove continuity in two steps: we will first prove that K_Γ is lower semicontinuous and then that it is upper semicontinuous. We make the following definitions for the entirety of the proof: Let $\{\mathcal{X}_n\} = \{(X_n, \varphi_n)\}$ be a sequence in $\mathrm{Teich}(S)$ converging to $\mathcal{X} = (X, \varphi) \in \mathrm{Teich}(S)$. Let $f_n = \varphi \circ \varphi_n^{-1} : X_n \to X$ and observe $\lim_{n\to\infty} K(f_n) = 1$.

Pick $x, y \in X$ and set $x_n = f_n^{-1}(x)$ and $y_n = f_n^{-1}(y)$. Then there is a $K_\Gamma(X_n)$-qc mapping $g_n : X_n \to X_n$ such that $g_n(x_n) = y_n$ with $[g_n] \in \Gamma_{\varphi_n}$. Let $\{\mathcal{X}_{n_j}\}$ be a subsequence of $\{\mathcal{X}_n\}$ such that $\lim K_\Gamma(\mathcal{X}_{n_j}) = \liminf K_\Gamma(\mathcal{X}_n)$. As $f_{n_j} \circ g_{n_j} \circ f_{n_j}^{-1} : X \to X$ with $f_{n_j} \circ g_{n_j} \circ f_{n_j}^{-1}(x) = y$ and $\lim K(f_{n_j} \circ g_{n_j} \circ f_{n_j}^{-1}) \leq \liminf K(f_{n_j})^2 K(g_{n_j}) = \liminf K(g_{n_j})$ we can pass to another subsequence, still labelled $\{\mathcal{X}_{n_j}\}$, such that $f_{n_j} \circ g_{n_j} \circ f_{n_j}^{-1}$ converges to a quasiconformal mapping $g : X \to X$ such that $g(x) = y$ (this is again due to normality as in the above lemma). For j large we must have that $f_{n_j} \circ g_{n_j} \circ f_{n_j}^{-1}$ is homotopic to g, again as the connected components of $\mathrm{QC}(X)$ are given by isotopy classes. As $g_{n_j} \in \Gamma_{\varphi_{n_j}}$ we have $[g] \in \Gamma_{f_{n_j} \circ \varphi_{n_j}}$, but $f_{n_j} \circ \varphi_{n_j} = \varphi$, so that $[g] \in \Gamma_\varphi$. By our setup we now have

$$K(g) \leq \liminf K(g_{n_j}) \leq \lim K_\Gamma(\mathcal{X}_n) = \liminf(\mathcal{X}_n).$$

As x, y were arbitrary

$$K_\Gamma(\mathcal{X}) \leq \liminf K_\Gamma(\mathcal{X}_n).$$

Therefore, K_Γ is lower semicontinuous.

It is left to show that K_Γ is upper semicontinuous. Fix n and choose $x_n, y_n \in X_n$ and set $x = f_n(x_n)$ and $y = f_n(y_n)$. Then there exists a $K_\Gamma(\mathcal{X})$-qc mapping $g_n : X \to X$ such that $g_n(x) = y$. We then have that $h_n = f_n^{-1} \circ g_n \circ f_n$ is a qc mapping of X_n such that $h_n(x_n) = y_n$ and $[h_n] \in \Gamma_{\varphi_n}$. Furthermore,

$$K(h_n) \leq K(f_n)^2 K(g_n) \leq K(f_n)^2 K_\Gamma(\mathcal{X}).$$

As x_n, y_n were arbitrary we have that

$$K_\Gamma(\mathcal{X}_n) \leq K(f_n)^2 K_\Gamma(\mathcal{X})$$

and thus

$$\limsup K_\Gamma(\mathcal{X}_n) \leq \lim K(f_n)^2 K_\Gamma(\mathcal{X}) = K_\Gamma(\mathcal{X}).$$

Therefore, K_Γ is upper semicontinuous. $\square$

It is natural to ask when these functions descend to functions on Moduli space. Recall that if $X \in \mathcal{M}(S)$, then two points $\mathcal{X}, \mathcal{Y} \in \operatorname{Teich}(S)$ are in the preimage of X under the projection $\operatorname{Teich}(S) \to \mathcal{M}(S)$ if there exists $[f] \in \operatorname{Mod}(S)$ with $\mathcal{Y} = [f] \cdot \mathcal{X}$. If $\mathcal{X} = [(X, \varphi)]$, then $[f] \cdot \mathcal{X} = [(X, \psi)]$ with $\psi = \varphi \circ f^{-1}$. Given a normal subgroup $\Gamma \triangleleft \operatorname{Mod}(S)$, then by definition we have

$$\begin{aligned}
\Gamma_\psi &= \{[\psi \circ g \circ \psi^{-1}] \colon g \in \operatorname{Homeo}^+(S) \text{ and } [g] \in \Gamma\} \\
&= \{[\varphi \circ f^{-1} \circ g \circ f \circ \psi^{-1}] \colon g \in \operatorname{Homeo}^+(S) \text{ and } [g] \in \Gamma\} \\
&= \{[\varphi \circ g' \circ \varphi^{-1}] \colon g' \in \operatorname{Homeo}^+(S) \text{ and } [g'] \in \Gamma\} \\
&= \Gamma_\varphi
\end{aligned}$$

As $\Gamma_\psi = \Gamma_\varphi$ it is clear that $K_\Gamma(\mathcal{X}) = K_\Gamma(f \cdot \mathcal{X})$. This proves the following:

Proposition 3.6.3. *For a normal subgroup $\Gamma \triangleleft \operatorname{Mod}(S)$, the function $K_\Gamma : \operatorname{Teich}(S) \to (1, \infty)$ descends to a continuous function $K_\Gamma : \mathcal{M}(S) \to (1, \infty)$.*

Remark 3.6.4. The normality of the subgroup in the above lemma is required: Dehn twists about curves with different lengths have different dilatations and all Dehn twists about non-separating simple closed curves are conjugates. If we take $\Gamma = \langle f \rangle$ where $f \in \operatorname{Mod}(S)$ is a Dehn twist about a curve γ, then for $X \in \mathcal{M}(S)$ with $\ell(X)$ very small we can choose $\varphi : S \to X$ and $\psi : S \to X$ and some K such that $|(\Gamma_\varphi)_K| = 1$ (where $\ell(\varphi(\gamma))$ is very large) and $|(\Gamma_\psi)_K| = 1000$ (where $\ell(\psi(\gamma))$ is very small). In the latter case you have more quasiconformal maps at your disposal.

Chapter 4

Moments of a length function on the boundary of a hyperbolic manifold

4.1 Finite Moments

Let $M = M^n$ be a compact n-dimensional hyperbolic manifold with totally geodesic boundary. As above, let L denote the time to the boundary of the unit normal flow on the boundary. We let dV be the induced hyperbolic volume measure on the boundary and define $dm = dV/V(\partial M)$, so that $(\partial M, m)$ is a probability space and $L : \partial M \to \mathbb{R}$ is a random variable on this space. We let $A_k(\partial M) = E[L^k] = \int_{\partial M} L^k \, dm$ be the k^{th} moment of L. In this section we will show that $A_k(M)$ is finite for all nonnegative integers k.

We first need to recall the following two theorems from Patterson-Sullivan theory (see [Nic89]):

Theorem 4.1.1. *Let $\Gamma < \mathrm{Isom}^+(\mathbb{H}^n)$ be a convex cocompact Kleinian group and let $\delta = \delta(\Gamma)$ be the Hausdorff dimension of the limit set of Γ. There exists r_0 such that for $r \geq r_0$,*

$$N_x(r) = |\{\gamma \in \Gamma \colon d(\gamma(x), x) < r\}| \leq a e^{\delta r},$$

for some constant a depending on Γ and x.

Theorem 4.1.2. *Let $\Gamma < \mathrm{Isom}^+(\mathbb{H}^n)$ be a convex cocompact Kleinian group and let $\delta = \delta(\Gamma)$ be the Hausdorff dimension of the limit set of Γ. Then $\delta = n-1$ if and only if $\mathbb{H}^n/\Gamma$ is finite volume.*

Also, a basic lemma from hyperbolic geometry:

Lemma 4.1.3. *Let U be a hyperplane in $\mathbb{H}^n$ and B_R a hyperbolic n-ball of radius R a distance s from U. The orthogonal projection of B_R to U has radius $r < \log\coth(s/2)$.*

Proof. Let $p \in \partial B_R$ be the point closest to U, so that $d(p,U) = s$ and let V be the hyperplane containing p such that $d(U,V) = s$. Then the orthogonal projection of B_R is contained in the orthogonal projection of V. The orthogonal projection of V to U has radius $\log\coth(s/2)$ (see [Bas93]), implying that $r < \log\coth(s/2)$ as desired. □

We can now show that $A_k(M)$ is finite:

Theorem 1.2.1 *Let $M = M^n$ be an n-dimensional compact hyperbolic manifold with nonempty totally geodesic boundary, then $A_k(M)$ is finite for all $k \in \mathbb{Z}^{\geq 0}$.*

Proof. We want to work in hyperbolic space: identify the universal cover $\widetilde{M}$ of M with a convex subset of $\mathbb{H}^n$, so that $\pi_1 M = \Gamma < \mathrm{Isom}^+(\mathbb{H}^n)$ is a convex cocompact Kleinian group. As M has a finite number of disjoint boundary components and we are investigating the integral over the boundary, it is enough to prove finiteness for a single component. Fix $S \subset \partial M$ a component and a lift $\widetilde{S} \subset \widetilde{M}$ of S (note: $\widetilde{S}$ is a copy of $\mathbb{H}^{n-1}$ sitting in $\mathbb{H}^n$). Let U be a convex fundamental domain for the action of Γ on $\widetilde{M}$. Pick $p \in U$ and let $B_R(p)$ be a ball centered at p of radius R such that $U \subset B_R(p)$. Set $W = U \cap \widetilde{S}$ to be a fundamental domain for the action of $\mathrm{Stab}(\widetilde{S}) < \Gamma$ on $\widetilde{S}$. Define $n_t \colon W \to \mathbb{H}^n$ to be the unit geodesic flow normal to $\widetilde{S}$ into $\widetilde{M}$ for a time t and set $X_t = \{x \in W \colon n_t(x) \in \widetilde{M}\}$. Define $\pi \colon \mathbb{H}^n \to \widetilde{S}$ to be orthogonal projection.

We will now bound $V(X_t)$ for $t \geq r_0$, where r_0 is taken from Theorem 4.1.1. If $x \in X_t$, then $n_t(x) \in \gamma U$ for some $\gamma \in \Gamma$. If $n_t(x) \in n_t(X_t) \cap \gamma U$, then $d(p, \gamma(p)) < t + 2R$. Let $\Gamma_t = \{\gamma \in \Gamma \colon n_t(X_t) \cap \gamma U \neq \emptyset\}$, then from the above theorem, we know that $|\Gamma_t| \leq$

$N_p(t+2R) \leq ae^{\delta(t+2R)}$, where δ is the Hausdorff dimension of the limit set of Γ. As $n_t(X_t) \subset \bigcup_{\gamma\in\Gamma_t} \gamma U$ and $\pi(n_t(X_t)) = X_t$, we have

$$V(X_t) \leq \sum_{\gamma\in\Gamma_t} V(\pi(\gamma U)).$$

Now, fix $\gamma \in \Gamma_t$, then $\gamma \cdot U \subset B_R(\gamma \cdot p)$. Suppose that $B_R(\gamma \cdot p)$ is a distance s from W and let r be the radius of its projection, we then have that $t < r + s + 2R$ by the triangle inequality; in particular, $s > t - r - 2R$. Furthermore, as orthogonal projection is always distance decreasing in hyperbolic space, $r < R$, so that $s > t - 3R$. From the above lemma, we can conclude that

$$r \leq \log\coth(s/2) \leq \log\coth\left(\frac{t-3R}{2}\right) \equiv f(t).$$

As the above bound for the radius does not depend on γ, we have

$$V(X_t) \leq |\Gamma_t| V_{n-1}(f(t)) \leq N_p(t+2R) V_{n-1}(f(t))) \leq ae^{\delta(t+2R)}\, V_{n-1}(f(t)),$$

where $V_n(r)$ is the volume of a n-dimensional hyperbolic ball of radius r. We observe two asymptotics: 1) $\lim_{x\to\infty} e^x \log\coth(x/2) = 2$ and 2) $\lim_{r\to 0} V_n(r)/r^n = C_n$ for some constant $C_n > 0$. From these facts and the above inequality, we see that

$$\lim_{t\to\infty} e^{(n-1-\delta)t} \cdot V(X_t) \leq A,$$

for some constant A. From the theorem stated above, we know that $n-1-\delta > 0$.

We now move to the moments. We have setup the following situation:

$$\int_S L^k dV = \sum_{t=0}^{\infty} \int_{L^{-1}(t,t+1)} L^k dV \leq \sum_{t=0}^{\infty} (t+1)^k \int_{L^{-1}(t,t+1)} dV \leq \sum_{t=0}^{\infty} (t+1)^k V(X_t).$$

But, we saw that the asymptotics of $V(X_t)$ are less than a multiple of e^{-bt} with $b > 0$, which implies the above sum converges since $\sum (t+1)^k e^{-bt}$ converges. $\square$

4.2 The Moments as a Sum over the Orthospectrum

4.2.1 Basmajian's Ball Decomposition of the Boundary

In this section we introduce a decomposition of ∂M into a disjoint union of $n-1$ balls (affectionately known as "leopard spots"). We will recall Danny Calegari's method of accomplishing this in [Cal10].

Definition 4.2.1. Let X and Y be totally geodesic copies of $\mathbb{H}^{n-1}$ sitting inside of $\mathbb{H}^n$ with disjoint closure in $\mathbb{H}^n \cup S_\infty^{n-1}$. A *chimney* is the closure of the union of the geodesic arcs from X to Y that are perpendicular to X.

The distance between the hyperplanes X and Y defining the chimney is realized by a unique geodesic perpendicular to both planes, called the *core*, the length of which is the *height* of the chimney. The chimney cuts out a disk in X, which is called the *base*. Let α be the geodesic containing the core and β a geodesic containing a diameter of the base. Then α and β span a copy of $\mathbb{H}^2$ in $\mathbb{H}^n$. Furthermore, the intersection of this plane with the chimney cuts out half an ideal quadrilateral with orthospectrum $\{2\ell, 2r\}$, where ℓ is the length of the core and r the radius of the base. We then have $\sinh(r)\sinh(\ell) = 1$, which yields $r = \log\coth\frac{\ell}{2}$ (see [Bea83]).

Theorem 4.2.2 (Chimney Decomposition, [Cal10]). *Let M be a compact hyperbolic n-manifold with totally geodesic boundary S. Let M_S be the covering space of M associated to S. Then M_S has a canonical decomposition into a piece of zero measure, together with two chimneys of height ℓ_i for each ℓ_i in the (unordered) orthospectrum.*

If we take the bases of the chimneys in the decomposition of the above theorem, then we get a decomposition of ∂M into $(n-1)$-balls. With this decomposition, we can give the quick proof of Basmajian's identity in [Cal10]:

Proof of Basmajian's identity. Recall that we are working with the ordered orthospectrum. S in M_S is decomposed into a set of measure zero together with the union of the bases of the chimneys. Thus

$$V(S) = \sum_i V_{n-1}\left(\log\coth\frac{\ell_i}{2}\right),$$

where $V_n(r)$ is the volume of a hyperbolic n-ball of radius r. □

4.2.2 Deriving the Length Function

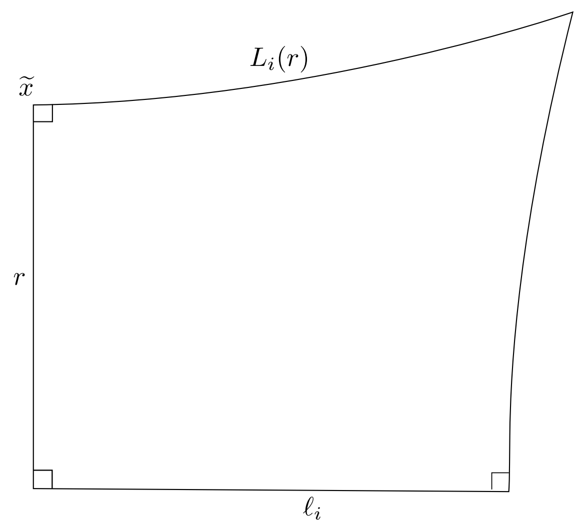

Figure 4.1: A Lambert quadrilateral showing the setup for $L(r)$.

Let U_i be the interior of the ball associated to $\ell_i \in |O_M|$. By above, the union of the U_i's is a full measure set in $S = \partial M$. The measurable function $L : S \to \mathbb{R}$ assigning to each $x \in S$ the length of the arc emanating perpendicularly from S at x can be written as $L = \sum_{\ell_i \in |O_M|} L_i$, where $L_i = L|U_i \colon U_i \to \mathbb{R}$ since the U_i's are pairwise disjoint. As a chimney has rotational symmetry about its core, we see that $L(x)$ for $x \in S$ depends only on the distance between x and the core, ie L_i is a function of the radius; hence, deriving a formula for L_i is a problem in the hyperbolic plane. Associated to each U_i are two components of the boundary, R_i and T_i, and two lifts of these components to hyperplanes in $\mathbb{H}^n$, $\widetilde{R}_i$ and $\widetilde{T}_i$. If $x \in R_i$, then we are interested in the chimney with its base in $\widetilde{R}_i$ and the lift of x sitting in $\widetilde{R}_i$, call it $\widetilde{x}$. There is a unique copy of $\mathbb{H}^2 \subset \mathbb{H}^n$ containing the core of the chimney, $\widetilde{x}$, and the geodesic connecting the two. The geodesic contained in this plane going through $\widetilde{x}$ and intersecting $\widetilde{R}_i$ perpendicularly intersects $\widetilde{T}_i$; furthermore, the length of this arc is $L_i(x)$. The diagram showing this situation in $\mathbb{H}^2$ is shown in Figure 4.1.

We see that $L_i(x)$ is the length of a side in a Lambert quadrilateral (a hyperbolic quadrilateral with three right angles). Let r be the distance from x to the core, then as we noted L_i is solely a function of the radius, we will write $L_i(x) = L_i(r)$. From hyperbolic trigonometry we find $\coth L_i(r) = \operatorname{sech}(r)\coth(\ell_i)$ (see [Bea83]) or

$$L_i(r) = \operatorname{arccoth}(\operatorname{sech}(r)\coth(\ell_i)) = \frac{1}{2}\log\left(\frac{\coth \ell_i + \cosh r}{\coth \ell_i - \cosh r}\right), \tag{4.1}$$

where the second equality holds as $\operatorname{sech}(r)\coth(\ell_i) > 1$ on the domain of interest $r \in [0, \log\coth(\ell_i/2))$.

4.2.3 Proof of Theorem 1.2

For completeness, we restate the result:

Theorem 1.2.2. *Let $M = M^n$ be an n-dimensional compact hyperbolic manifold with nonempty totally geodesic boundary, then for all $k \in \mathbb{Z}^{\geq 0}$*

$$A_k(M) = \frac{1}{V(\partial M)} \sum_{\ell \in |O_M|} F_{n,k}(\ell),$$

where

$$F_{n,k}(x) = \Omega_{n-2} \int_0^{\log\coth(x/2)} \left[\log\left(\frac{\coth x + \cosh r}{\coth x - \cosh r}\right)\right]^k \sinh^{n-2}(r)\ dr$$

and Ω_n is the volume of the standard n-sphere. Furthermore, the identity for $A_0(M)$ is Basmajian's identity.

Proof. From the additivity property of measures we have $\int L^k\ dm = \sum \int_{U_i} L_i^k\, dm$. As $dm = dV/V(\partial M)$ and dV is the $(n-1)$-dimensional hyperbolic volume form we can write it in spherical coordinates as

$$dm = \frac{1}{V(\partial M)} \sinh^{n-2}(r)\, dr\, d\Omega_{n-2},$$

where $d\Omega_{n-2}$ is the volume form on the standard unit sphere. Above we saw that L_i is a function solely of the radius and U_i is a $(n-1)$-dimensional hyperbolic ball with radius

$\log(\coth(\ell_i/2))$, so that

$$\begin{aligned}\int_{U_i} L_i^k\, dm &= \frac{1}{V(\partial M)} \int_{S^{n-2}} \int_0^{\log(\coth(\ell_i/2))} L_i^k(r) \sinh^{n-2}(r)\, dr\, d\Omega_{n-2} \\ &= \frac{\Omega_{n-2}}{V(\partial M)} \int_0^{\log(\coth(\ell_i/2))} L_i^k(r) \sinh^{n-2}(r)\, dr,\end{aligned}$$

where we write Ω_{n-2} for the volume of the standard $(n-2)$-dimensional unit sphere. Define $F_{n,k}(x)$ as stated in the theorem, so that the equality holds for $A_k(M)$ by (4.1).

Observe that $F_{n,0}(x)$ gives the volume of a hyperbolic $(n-1)$-ball of radius $\log\coth(x/2)$. As $A_0(M) = 1$, we see that this identity yields

$$1 = \frac{1}{V(\partial M)} \sum_{\ell \in |O_M|} V_{n-1}(\log\coth(\ell/2)),$$

which is Basmajian's identity. $\square$

4.3 Surface Case

4.3.1 Dilogarithms

For $|z| < 1$ in $\mathbb{C}$ the dilogarithm is defined as

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}.$$

Using the Taylor series for $\log(1-z)$ about $z = 0$, we can write

$$\mathrm{Li}_2(z) = \int_z^0 \frac{\log(1-z)}{z} dz.$$

One can then take a branch of $\log(z)$ in order to analytically continue $\mathrm{Li}_2(z)$ to the complex plane minus a branch cut. The standard definition of the dilogarithm assumes the branch cut for $\log(z)$ is along $(-\infty, 0]$; however, for our purposes we will be interested in a different branch cut. Define the function $\mathcal{D}(z)$ to be the resulting dilogarithm by using the branch cut along $(-i\infty, 0]$ for $\log(z)$ such that $\log(-1) = i\pi$. We note that $\mathrm{Li}_2(z) = \mathcal{D}(z)$ for

$z \in (-\infty, 1)$.

The dilogarithm $\mathrm{Li}_2(z)$ has the following well-known identity (see [Lew91]):

$$\mathrm{Li}_2(z) + \mathrm{Li}_2\left(\frac{1}{z}\right) = -\frac{1}{2}\log^2(-z) - \frac{\pi^2}{6}.$$

This identity is verified by differentiating both sides. As $\mathrm{Li}_2' = \mathcal{D}'$ and $\mathrm{Li}_2(z) = \mathcal{D}(z)$ on the negative real axis, the identity holds for $\mathcal{D}$; hence,

$$\mathcal{D}(z) + \mathcal{D}\left(\frac{1}{z}\right) = -\frac{1}{2}\log^2(-z) - \frac{\pi^2}{6}. \tag{4.2}$$

(The branch of logarithm being used should be clear from context.)

4.3.2 Deriving the formula for $F_{2,1}(x)$

For a fixed positive value of x, we define the map $H_x : [0, \log\coth(x/2)] \to \mathbb{C}$ as follows:

$$H_x(r) = \mathcal{D}(-e^{-r}\coth(x/2)) - \mathcal{D}(e^{-r}\coth(x/2)) + \mathcal{D}(-e^{-r}\tanh(x/2)) - \mathcal{D}(e^{-r}\tanh(x/2)).$$

Lemma 4.3.1.

$$\frac{d(\Re H_x)}{dr} = \log\frac{\coth x + \cosh r}{\coth x - \cosh r}$$

Proof. We first calculate H_x' and then take real parts. Given the definition of the dilogarithm and the fact that $\coth(x/2) + \tanh(x/2) = 2\coth x$, we have that

$$\begin{aligned}
H_x'(r) &= \log(1 + e^{-r}\coth(x/2)) - \log(1 - e^{-r}\coth(x/2)) + \\
&\quad + \log(1 + e^{-r}\tanh(x/2)) - \log(1 - e^{-r}\tanh(x/2)) \\
&= \log[(1 + e^{-r}\coth(x/2))(1 + e^{-r}\tanh(x/2))] - \log[(1 - e^{-r}\coth(x/2))(1 - e^{-r}\tanh(x/2))] \\
&= \log[2e^{-r}(\cosh r + \coth x)] - \log[2e^{-r}(\cosh r - \coth x)] \\
&= \log(\coth x + \cosh r) - \log(\cosh r - \coth x) \\
&= \log\frac{\coth x + \cosh r}{\coth x - \cosh r} - i\pi.
\end{aligned}$$

Given the domain for H_x, the argument of the logarithm above is always a positive real number. □

We therefore see that $F_{2,1}(x) = 2 \cdot \Re[H_x(\log\coth(x/2)) - H_x(0)]$ as $\Omega_0 = 2$. For a surface S with boundary, let $\ell(\partial S)$ be the sum of the lengths of the boundary components. Given the above we can now prove the following:

Corollary 1.2.4. *Let S be a compact hyperbolic surface with nonempty totally geodesic boundary. Then*

$$A_1(S) = \frac{2}{\ell(\partial S)} \sum_{\ell \in |O_S|} \left[\mathrm{Li}_2\left(-\tanh^2\frac{\ell}{2}\right) - \mathrm{Li}_2\left(\tanh^2\frac{\ell}{2}\right) + \frac{\pi^2}{4}\right].$$

Proof. From the above formulation of $F_{2,1}(x)$, we get the following:

$$F_{2,1}(x) = 2 \cdot \Re\left[\mathcal{D}(a) + \mathcal{D}\left(\frac{1}{a}\right) - \mathcal{D}\left(-a\right) - \mathcal{D}\left(-\frac{1}{a}\right) + \mathcal{D}\left(-\frac{1}{a^2}\right) - \mathcal{D}\left(\frac{1}{a^2}\right) - \frac{\pi^2}{4}\right],$$

where $a = \coth\left(\frac{x}{2}\right)$. From applying (4.2) twice we see that:

$$\mathcal{D}(a) + \mathcal{D}\left(\frac{1}{a}\right) - \mathcal{D}\left(-a\right) - \mathcal{D}\left(-\frac{1}{a}\right) = \frac{1}{2}(\log^2(a) - \log^2(-a)).$$

Recalling that $\log(-1) = i\pi$, for $a > 0$ we have $\log(a) - \log(-a) = -\log(-1)$, so that simplifying the above

$$\mathcal{D}(a) + \mathcal{D}\left(\frac{1}{a}\right) - \mathcal{D}\left(-a\right) - \mathcal{D}\left(-\frac{1}{a}\right) = \frac{\pi^2}{2} - i\pi\log a \text{ for } a > 0.$$

In particular, for positive values of a, the real part is always $\pi^2/2$. As ℓ_i is always positive this identity holds in the case of interest. Furthermore, $\mathrm{Li}_2\left(\pm\tanh(\ell_i/2)\right) = \mathcal{D}\left(\pm\tanh(\ell_i/2)\right)$ as $\pm\tanh(\ell_i/2) \in (-1, 1)$; hence, the result follows. □

4.3.3 Asymptotics for $F_{2,1}(x)$

We will use the following notation throughout the rest of the paper: For $f, g : \mathbb{R} \to \mathbb{R}$ we will write $f \sim g$ if $\lim_{x\to\infty}[f(x)/g(x)] = 1$. This is clearly an equivalence relation on real-valued functions. Below we find the asymptotic behavior of $F_{2,1}(x)$ from our above result; we note that we will also come to the same result later in the paper when we study the asymptotics of $F_{n,k}$ from the integral definition.

Proposition 4.3.2. *Let $F_{2,1}(x)$ be defined as above, then $F_{2,1}(x) \sim 8xe^{-x}$.*

Proof. We start with the following observation:

$$\lim_{x\to 1} \frac{\mathrm{Li}_2(-x) - \mathrm{Li}_2(x) + \pi^2/4}{(x-1)\log(1-x)} = 1,$$

which is a direct application of L'Hôpital's rule and the definition of the dilogarithm. From this, we can gather the following:

$$\begin{aligned} F_{2,1}(x) &\sim 2(\tanh^2(x/2) - 1)\log(1 - \tanh^2(x/2)) = 4\,\mathrm{sech}^2(x/2)\log\cosh(x/2) \\ &= 4\left(\frac{2}{e^{\frac{x}{2}} + e^{-\frac{x}{2}}}\right)^2 \log\left[e^{\frac{x}{2}}\left(\frac{1+e^{-x}}{2}\right)\right] \sim 8xe^{-x} \end{aligned}$$

□

4.4 Connecting with Bridgeman's identity in dimension 2

4.4.1 Liouville measure and Bridgeman's identity

We first need to recall Bridgeman's identity. Denote the space of oriented geodesics in $\mathbb{H}^2$ by $G(\mathbb{H}^2)$, then by identifying a geodesic with its endpoints in $\mathbb{S}^1_\infty$ we see

$$G(\mathbb{H}^2) \cong (\mathbb{S}^1_\infty \times \mathbb{S}^1_\infty) - \Delta,$$

where Δ denotes the diagonal and we view the geodesic $[x,y] \in G(\mathbb{H}^2)$ as oriented from x to y. The *Liouville measure* μ is an $\mathrm{Isom}(\mathbb{H}^2)$ invariant measure on $G(\mathbb{H}^2)$. If we identify $\mathbb{S}^1_\infty$ with $\overline{\mathbb{R}}$, then μ is characterized by

$$\mu((a,b)\times(c,d)) = 2\left|\log\left|\frac{(a-c)(b-d)}{(a-d)(b-c)}\right|\right|$$

for disjoint intervals (a, b) and (c, d) in $\overline{\mathbb{R}}$. The Liouville measure on $G(\mathbb{H}^2)$ is derived from the Liouville measure on the unit tangent bundle $T_1(\mathbb{H}^2)$, which is given by $dV d\theta$, where dV is the hyperbolic volume measure and $d\theta$ is the standard measure on $\mathbb{S}^1$ (see [Bon88, Nic89] for details). In fact, the natural fibering $T_1(\mathbb{H}^2) \to G(\mathbb{H}^2)$ is such that the volume measure on $T_1(\mathbb{H}^2)$ is $d\mu\, dl$, where dl is the length along a fiber. Note that the factor of 2 appears above so that $d\mu\, dl$ agrees with $dV\, d\theta$.

There are local coordinates for $G(\mathbb{H}^2)$ where the Liouville measure can be written as a product measure. Let $\gamma \in G(\mathbb{H}^2)$ and $p \in \gamma$ a base point. Let η be a geodesic intersecting γ, then η is determined by the signed hyperbolic distance $s = \pm d(\gamma \cap \eta, p)$ coming from the orientation of γ and the angle θ between γ and η measured from η to γ. In these local coordinates, we have

$$d\mu_{(s,\theta)} = \sin\theta\, ds\, d\theta. \tag{4.3}$$

These coordinates are described in the appendices of [Bon88].

Given a hyperbolic surface S with totally geodesic boundary, identify the universal cover of S with a convex subset $\tilde{S} \subseteq \mathbb{H}^2$. Define $G(\tilde{S}) \subseteq G(\mathbb{H}^2)$ to be the set of all geodesics intersecting $\tilde{S}$. Let $\Gamma \subset \mathrm{Isom}(\mathbb{H}^2)$ such that $S = \tilde{S}/\Gamma$, then we can set $G(S) = G(\tilde{S})/\Gamma$ to be the space of geodesics in S. The Liouville measure descends to a measure on $G(S)$.

Let S be an orientable compact hyperbolic surface with nonempty totally geodesic boundary and given $v \in T_1(S)$ let $\alpha_v \in G(S)$ such that $\alpha_v'(t) = v$ for some $t \in \mathbb{R}$. For each orthogeodesic α_i set $W_i = \{v \in T_1(S) \colon \alpha_v \text{ is properly isotopic to } \alpha_i\}$. We then have $\mathrm{Vol}(T_1(S)) = 4\pi^2|\chi(S)| = \sum \mathrm{Vol}(W_i)$. If we define $L_S : G(S) \to \mathbb{R}$ by $L_S(g) = \mathit{length}(g)$, where length is measured in S, and set $\overline{W}_i = \{\alpha_v \in G(S) \colon v \in W_i\}$, then it was proved in [Bri11] that

$$\mathrm{Vol}(W_i) = \int_{\overline{W}_i} L_S\, d\mu = 8\mathcal{L}\left(\operatorname{sech}^2 \frac{\ell_i}{2}\right),$$

where $\mathcal{L}(x) = \mathrm{Li}_2(x) + \frac{1}{2}\log|x|\log(1-x)$ for $x \leq 1$ is the Rogers dilogarithm. This gives Bridgeman's identity:

$$\sum_i \mathcal{L}\left(\operatorname{sech}^2 \frac{\ell_i}{2}\right) = \frac{\pi^2}{2}|\chi(S)|.$$

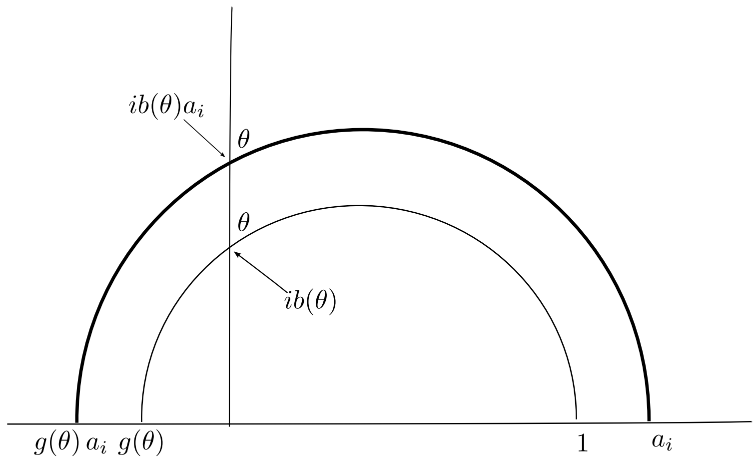

Figure 4.2: The figure shows the θ-projection of the geodesic $[1, a_i]$ to $[0, \infty]$ as in Lemma 4.4.1.

4.4.2 Random Variables

Let S be an orientable compact hyperbolic surface with nonempty totally geodesic boundary. Given an angle $\theta \in (0, \pi)$, we define a new random variable on $(\partial S, dm)$ as follows: let γ be a component of ∂S and $x \in \gamma$. Let $v \in T_x(S)$ such that the unit speed geodesic α_v resulting from the geodesic flow in the direction of v makes an angle θ with γ when measured from α_v to γ (where the orientation of γ is such that the surface is on the right). Define $\alpha_x^\theta = \alpha_v$, then the function $L_\theta : \partial S \to \mathbb{R}$ defined by $L_\theta(x) = length(\alpha_x^\theta)$ is a random variable on $(\partial S, dm)$. We define its moments

$$A_k^\theta(S) = E[L_\theta^k] = \int_{\partial S} L_\theta^k \, dm.$$

As above, we can decompose our boundary into intervals associated to orthogeodesics: for $\alpha_i \in O_S$ let

$$U_i^\theta = \{x \in \partial S \colon \alpha_x^\theta \text{ is properly isotopic to } \alpha_i\}.$$

Lemma 4.4.1. *For every $\theta \in (0, \pi)$, the set U_i^θ is an interval of length* $2 \log \coth(\ell_i/2)$.

Proof. Let γ_1, γ_2 be the components of ∂S such that α_i travels from γ_1 to γ_2 (possibly $\gamma_1 = \gamma_2$). We may then put this picture in the upper half plane with $\tilde{\gamma}_1 = [0, \infty]$ and $\tilde{\gamma}_2 = [1, a_i]$, where $a_i = \coth^2(\ell_i/2)$. For $x \in U_i^\theta$ there exists a unique lift $\tilde{\alpha}_x^\theta$ intersecting $\tilde{\gamma}_1$ in angle θ and also intersecting $\tilde{\gamma}_2$. As in the proof of Basmajian's identity, we see U_i^θ lifts to the θ-projection of γ_2 onto γ_1. Define $g(\theta)$ such that the geodesic $\beta = [g(\theta), 1]$ intersects $\tilde{\gamma}_1$ at angle θ as in Figure 4.2. Define $b(\theta)$ so that β intersects $\tilde{\gamma}_1$ at the point $ib(\theta)$. Observe that the geodesic $[a_i g(\theta), a_i]$ intersects $\tilde{\gamma}_1$ at angle θ at the point $ia_i b(\theta)$ as it is the image of β under the Möbius transformation $z \mapsto a_i z$. In particular, the length of U_i^θ is $\log(a_i)$. $\square$

In the above proof we see that the picture of U_i^θ only depends on ℓ_i, so as in Theorem 1.2.2 we have that there exists functions

$$F_k^\theta(\ell_i) = \int_{U_i^\theta} L_\theta^k \, dm,$$

such that

$$A_k^\theta(S) = \sum_i F_k^\theta(\ell_i).$$

In particular, each A_k^θ gives an orthospectrum identity.

Proposition 4.4.2. *For $\theta \in (0, \pi)$, the identity for $A_0^\theta(S)$ is Basmajian's identity.*

Proof. As $m(\bigcup_i U_i^\theta) = 1$ and $U_i^\theta \cap U_j^\theta = \emptyset$ for $i \neq j$, we have

$$1 = \sum_i m(U_i^\theta) = \frac{1}{\ell(\partial S)} \sum_i 2 \log \coth(\ell_i/2).$$

$\square$

We now have the following connection between $A_1(S)$ and Bridgeman's identity:

Theorem 4.4.3. *Suppose S is a compact hyperbolic surface with nonempty totally geodesic boundary. Let $F_S : [0, \pi] \to \mathbb{R}$ be defined by*

$$F_S(\varphi) = \ell(\partial S) \int_0^\varphi A_1^\theta(S) \sin\theta \, d\theta,$$

then the identity for $F_S(\pi)$ is Bridgeman's identity and $F'(\pi/2) = A_1(S)$.

Proof. Let $\tilde{S}$ be the universal cover of S identified with a convex subset of the upper half plane $\mathbb{H}$. Let α_i be an orthogeodesic traveling from the boundary component γ_1 to the component γ_2. Assume that the geodesic $[0, \infty] \subset \mathbb{H}^2$ is a lift of γ_1 and the geodesic

$[1, a_i] \subset \mathbb{H}^2$ is a lift of γ_2, where $a_i = \log\coth(\ell_i/2)$. As in the proof of Lemma 4.4.1, we lift the set $\overline{W}_i$ to the set $\widetilde{W}_i = (-\infty, 0) \times (1, a_i) \subset G(\tilde{S})$. Then every geodesic $[x, y] \in \widetilde{W}_i$ can be given coordinates (s, θ), where

$$s([x,y]) = \log\left(\frac{[0,\infty] \cap [x,y]}{i}\right)$$

and $\theta([x, y])$ is the the angle from $[x, y]$ to $[0, \infty]$. Using these local coordinates the Liouville measure can be written as in (4.3) and from the notation above it follows that

$$\mathrm{Vol}(W_i) = \int_{\overline{W}_i} L_S\, d\mu = \int_0^{\pi} \int_{\log(b(\theta))}^{\log(b(\theta)a_i)} L_\theta(s) \sin\theta\, ds\, d\theta$$

and thus

$$\begin{aligned} F_S(\pi) &= \ell(\partial S) \int_0^{\pi} A_1^{\theta}(S) \sin\theta\, d\theta \\ &= \sum_i \int_0^{\pi} \int_{\log(b(\theta))}^{\log(b(\theta)a_i)} L_\theta(s) \sin\theta\, ds\, d\theta \\ &= \sum_i \mathrm{Vol}(W_i) \\ &= 8 \sum_i \mathcal{L}\left(\mathrm{sech}^2 \frac{\ell_i}{2}\right). \end{aligned}$$

Using the notation from the beginning of the section, we see that almost every element of $G(S)$ can be realized as α_p^{θ} for $p \in \partial S$ and $\theta \in (0, \pi)$ implying that $\partial S \times (0, \pi)$ is full measure in $G(S)$. In particular,

$$F_S(\pi) = \ell(\partial S) \int_0^{\pi} A_1^{\theta}(S) \sin\theta\, d\theta = \int_0^{\pi} \int_{\partial S} L_\theta \sin\theta\, ds\, d\theta = \int_{G(S)} L_S\, d\mu = 4\pi^2 |\chi(S)|.$$

This shows that the identity for $F_S(\pi)$ is Bridgeman's identity. Further, it is clear from the definition that $F_S'(\pi/2) = A_1(S)$. $\square$

Remark. This also shows that $A_1^{\theta}(S)$ is finite. It can also be seen that $A_k^{\theta}(S)$ is finite for all k using the same approach as in the proof for the finiteness of $A_n(M)$ given earlier.

4.5 Odd Dimensions

In this section we will write an explicit formula for $A_1(M^n)$ with n odd. For n odd, we can simplify the integral in the definition of $F_{n,k}$ by using the substitution $u = \cosh r$ to get:

$$F_{n,1}(x) = \Omega_{n-2} \int_1^{\coth x} (u^2 - 1)^{\frac{n-3}{2}} \log \frac{\coth x + u}{\coth x - u}\, du.$$

An elementary calculation gives the following integrals (up to a constant) when m is even:

$$F_m^+(u, y) = \textstyle\int u^m \log(y+u)\, du = \frac{1}{m+1} \left[\left(u^{m+1} + y^{m+1}\right) \log(y+u) + \sum_{k=1}^{m+1} (-1)^{m-k} \frac{y^{m-k+1} u^k}{k} \right]$$

$$F_m^-(u, y) = \int u^m \log(y-u)\, du = \frac{1}{m+1} \left[\left(u^{m+1} - y^{m+1}\right) \log(y-u) - \sum_{k=1}^{m+1} \frac{y^{m-k+1} u^k}{k} \right].$$

Now set

$$f_m(x) = F_m^+(\coth x, \coth x) - F_m^+(1, \coth x) + F_m^-(1, \coth x) - \lim_{u \to (\coth x)^-} F_m^-(u, \coth x).$$

After some routine manipulation and simplification, we find:

$$f_m(x) = \frac{2\coth^{m+1}(x)}{m+1} \left[\log(2\cosh x) - x \tanh^{m+1}(x) + \sum_{k=1}^{\frac{m}{2}} \frac{1 - \tanh^{2k}(x)}{2k} \right].$$

If we expand out the binomial in $F_{n,1}(x)$, we find

$$F_{n,1}(x) = \Omega_{n-2} \sum_{j=0}^{\frac{n-3}{2}} (-1)^{\frac{n-3}{2} - j} \binom{\frac{n-3}{2}}{j} f_{2j}(x).$$

We then immediately have:

Corollary 1.2.5. *Let M be an n-dimensional compact hyperbolic manifold with nonempty totally geodesic boundary where n is odd. Then*

$$A_1(M) = \frac{2\Omega_{n-2}}{Vol(\partial M)} \sum_{\ell_i \in |O_M|} \sum_{j=0}^{\frac{n-3}{2}} \frac{(-1)^{\frac{n-3}{2} - j} \binom{\frac{n-3}{2}}{j}}{2j+1} \coth^{2j+1}(\ell_i) \left[\log(2\cosh \ell_i) - \ell_i \tanh^{2j+1}(\ell_i) + \sum_{k=1}^{j} \frac{1 - \tanh^{2k}(\ell_i)}{2k} \right].$$

4.6 The Asymptotics of $F_{n,k}$

In this section, we explore the asymptotic behavior of the $F_{n,k}$'s.

Theorem 4.6.1. *For all $n, k \in \mathbb{Z}^+$*

$$\lim_{x\to\infty} \frac{e^{-(n-1)x} F_{n,k}(x)}{x^k} = \frac{2^{n+k-1}\Omega_{n-2}}{n-1},$$

Furthermore, for n odd we have

$$\lim_{x\to 0} x^{n-2} F_{n,1}(x) = \frac{2}{n-2}[\log 2 + \frac{1}{2} H_{(n-1)/2}],$$

where H_n is the n^{th} harmonic number.

Proof. Recall that $F_{n,k}(x) = \Omega_{n-2} \int_0^{\log\coth(x/2)} L_x^k(r) \sinh^{n-2}(r)\, dr$. Using the substitution $u = \cosh r$, we have

$$F_{n,k}(x) = \Omega_{n-2} \int_1^{\coth x} (u^2-1)^{\frac{n-3}{2}} \left[\log\left(\frac{\coth x + u}{\coth x - u}\right)\right]^k du.$$

For the moment, let $n \geq 3$, so that $(n-3)/2 \geq 0$, then

$$F_{n,k}(x) \geq 2^{\frac{n-3}{2}} \Omega_{n-2} \int_1^{\coth x} (u-1)^{\frac{n-3}{2}} \left[\log\left(\frac{\coth x + u}{\coth x - u}\right)\right]^k du$$
$$F_{n,k}(x) \leq (\coth x + 1)^{\frac{n-3}{2}} \Omega_{n-2} \int_1^{\coth x} (u-1)^{\frac{n-3}{2}} \left[\log\left(\frac{\coth x + u}{\coth x - u}\right)\right]^k du.$$

As $\coth x + 1$ approaches 2 in the limit as x goes to infinity, we see from the above two inequalities that

$$F_{n,k}(x) \sim 2^{\frac{n-3}{2}} \Omega_{n-1} \int_1^{\coth x} (u-1)^{\frac{n-3}{2}} \left[\log\left(\frac{\coth x + u}{\coth x - u}\right)\right]^k du.$$

In the case $n = 2$, the inequalities above are reversed, but yield the same result; hence, what follows will hold for all n. We now look at the following two inequalities:

$$\int_1^{\coth x} (u-1)^{\frac{n-3}{2}} \left[\log\left(\frac{\coth x+u}{\coth x-u}\right)\right]^k du \geq \int_1^{\coth x} (u-1)^{\frac{n-3}{2}} [\log(\coth x + 1) - \log(\coth x - u)]^k du$$

$\int_1^{\coth x}(u-1)^{\frac{n-3}{2}}\left[\log\left(\frac{\coth x+u}{\coth x-u}\right)\right]^k du \leq \int_1^{\coth x}(u-1)^{\frac{n-3}{2}}\left[\log(2\coth x)-\log(\coth x-u)\right]^k du$

Note for large x that $\coth x - u < 1$ for all $u \in [1, \coth x]$, so that $\log(\coth x - u) < 0$. As both $\log(2\coth x)$ and $\log(\coth x + 1)$ limit to $\log 2$, we see that both the integrals in the inequalities are asymptotic to $\int_1^{\coth x}(u-1)^{(n-3)/2}[\log(\coth x - u)]^k du$. Let us write $a(x) = \coth x - 1$ and $v = \frac{u-1}{a}$, so that we now have

$$F_{n,k}(x) \sim 2^{\frac{n-3}{2}}\Omega_{n-2}a^{\frac{n-1}{2}} \int_0^1 v^{\frac{n-3}{2}}[\log(a-av)]^k dv = 2^{\frac{n-3}{2}}\Omega_{n-2}a^{\frac{n-1}{2}} \int_0^1 v^{\frac{n-2}{2}}\left[\log a + \log(1-v)\right]^k dv.$$

As $\int_0^1 v^{(n-3)/2}[\log(1-v)]^m dv$ is finite for all m, we find that

$$F_{n,k}(x) \sim (-1)^k 2^{\frac{n-3}{2}}\Omega_{n-2}(\log a)^k a^{\frac{n-1}{2}} \int_0^1 v^{\frac{n-3}{2}} dv = \frac{2^{\frac{n-1}{2}}\Omega_{n-2}}{n-1}\left(\log\frac{1}{a}\right)^k a^{\frac{n-1}{2}}.$$

Since, $a(x) = \coth x - 1 \sim 2e^{-2x}$, we get the stated result.

When n is odd, we have the following when x approaches 0: As x tends to 0, it is easy to see that $\tanh^{m+1}(x)f_m(x)$ is finite. As $\lim_{x\to 0}[x\coth x]$ is finite, we see that $\lim_{x\to 0} x^{m+1}f_m(x) < \infty$. Again, as $F_{n,1}(x)$ is a sum of the f_m's, the largest exponent dominates, which gives the result. $\square$

4.7 The Moment Generating Function in Dimension 3

Let M be a hyperbolic 3-manifold with totally geodesic boundary and let $S = \partial M$. We define the *moment-generating function* $M_L(t) = E[e^{tL}]$, where $E[X]$ denotes the expected value of a random variable X with respect to our probability measure $dm = dV/V(\partial M)$. The moment-generating function encodes all the moments of L in its derivatives: $A_k(M) = E[L^k] = M_L^{(k)}(0)$. In particular, by calculating $M_L(0)$ we will recover Basmajian's identity and $A_1(M)$ by calculating $M'_L(0)$. The goal of this section is to prove that following theorem:

Theorem 4.7.1. *Let M be a compact hyperbolic 3-manifold with totally geodesic boundary S and let δ be the Hausdorff dimension of the limit set of M. For $t \in (\delta - 2, 2 - \delta)$*

$$M_L(t) = \frac{4\pi}{V(S)} \sum_{\ell_i \in |O_M|} \coth(\ell_i) \cdot B\left(\frac{1-\tanh \ell_i}{2}, 1-t, 1+t\right),$$

where B is the incomplete beta function.

4.7.1 Hypergeometric Function and Incomplete Beta Function

The *hypergeometric functions* ${}_2F_1(a,b,c,z)$ for $z \in \mathbb{C}$ with $|z| < 1$ are given by the power series:

$$ {}_2F_1(a,b,c,z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, $$

provided $c \notin \mathbb{Z}^{\leq 0}$ and where

$$ (a)_n = \begin{cases} 1 & \text{for } n = 0 \\ a(a+1)\cdots(a+n-1) & \text{for } n > 0 \end{cases}. $$

We will use the following identity below:

$$ (1-z)^{-a} = {}_2F_1(a,1,1,z). $$

The *incomplete beta functions* $B(x,a,b)$ are defined as

$$ B(x,a,b) = \int_0^x s^{a-1}(1-s)^{b-1} ds. $$

We can also write an incomplete beta function in terms of a hypergeometric function as follows (see [Luk69]):

$$ B(x,a,b) = \frac{x^a}{a} {}_2F_1(a, 1-b, a+1, x). \tag{4.4} $$

We present two calculations as technical lemmas that will allow us to derive the moment generating function.

Lemma 4.7.2.

$$ \frac{\partial}{\partial x} {}_2F_1(1+t,t,2+t,x) = \frac{1+t}{x} \left[(1-x)^{-t} - {}_2F_1(1+t,t,2+t,x) \right] $$

Proof. We calculate:

$$\begin{aligned}
\frac{\partial}{\partial x}{}_2F_1(1+t,t,2+t,x) &= \sum_{n=1}^{\infty} \frac{(t)_n(1+t)_n}{(2+t)_n} \frac{x^{n-1}}{(n-1)!} \\
&= \frac{1+t}{x} \sum_{n=1}^{\infty} (t)_n \frac{n}{t+n+1} \frac{x^n}{n!} \\
&= \frac{1+t}{x} \sum_{n=1}^{\infty} \left[(t)_n - \frac{(t)_n(1+t)}{t+n+1} \right] \frac{x^n}{n!} \\
&= \frac{1+t}{x} \left[\sum_{n=0}^{\infty} (t)_n \frac{x^n}{n!} - \sum_{n=0}^{\infty} \frac{(t)_n(1+t)_n}{(t+2)_n} \frac{x^n}{n!} \right] \\
&= \frac{1+t}{x} [{}_2F_1(t,1,1,x) - {}_2F_1(1+t,t,2+t,x)] \\
&= \frac{1+t}{x} \left[(1-x)^{-t} - {}_2F_1(1+t,t,2+t,x) \right]
\end{aligned}$$

□

Lemma 4.7.3. *Let* $g(u,a,t) = (1+t)^{-1}(a+u)^{t+1}(2a)^{-t}{}_2F_1\left(1+t,t,2+t,\frac{a+u}{2a}\right)$, *then*

$$\frac{\partial g}{\partial u} = \left(\frac{a+u}{a-u} \right)^t .$$

Proof. This is an immediate consequence of the previous lemma. □

4.7.2 Proof of Theorem 4.7.1

We can now find the moment generating function of L.

Proof of Theorem 4.7.1. Let $S = \partial M$ and recall that $\Omega_1 = 2\pi$. By definition,

$$\begin{aligned}
M_L(t) = E[e^{tL}] &= \int_S e^{tL} dm = \sum_i \int_{U_i} e^{tL_i} dm \\
&= \frac{2\pi}{V(S)} \sum_i \int_0^{\log(\coth(\ell_i/2))} \left(\frac{\coth(\ell_i) + \cosh r}{\coth(\ell_i) - \cosh r} \right)^t \sinh r \, dr. \\
&= \frac{2\pi}{V(S)} \sum_i \int_1^{\coth(\ell_i)} \left(\frac{\coth(\ell_i) + u}{\coth(\ell_i) - u} \right)^t du,
\end{aligned}$$

where $u = \cosh r$. By replacing L^k with e^{tL} in the proof of Theorem 1.2.1, we see that $\int_S e^{tL} dm$ converges for $t < 2 - \delta$. Note that if $t \in (2-\delta, \delta-2)$ then $|t| < 1$. From the above lemma, we then have that

$$M_L(t) = (2\pi/V(S)) \sum_{\ell_i \in |O_M|} [g(\coth(\ell_i), \coth(\ell_i), t) - g(0, \coth(\ell_i), t)].$$

After expanding the above terms using the definition of g, some simplifications get us to:

$$M_L(t) = \frac{2\pi}{V(S)} \sum_{\ell_i \in |O_M|} \frac{\coth(\ell_i)}{1+t} \left[2{}_2F_1(1+t, t, 2+t, 1) - \frac{1}{2} \left(\frac{1+\tanh(\ell_i)}{2} \right)^{t+1} {}_2F_1 \left(1+t, t, 2+t, \frac{1+\tanh(\ell_i)}{2} \right) \right].$$

By (4.4) this becomes

$$M_L(t) = \frac{4\pi}{V(S)} \sum_{\ell_i \in |O_M|} \coth(\ell_i) \left[B(1, 1+t, 1-t) - B\left(\frac{1+\tanh(\ell_i)}{2}, 1+t, 1-t \right) \right].$$

It is left to investigate $B(1, 1+t, 1-t) - B(a, 1+t, 1-t)$:

$$B(1, 1+t, 1-t) - B(a, 1+t, 1-t) = \int_a^1 s^t (1-s)^{-t} ds = -\int_{1-a}^0 (1-u)^t u^{-t} du = B(1-a, 1-t, 1+t),$$

where $u = 1 - a$. Therefore, we can conclude

$$M_L(t) = \frac{4\pi}{V(S)} \sum_{\ell_i \in |O_M|} \coth(\ell_i) \cdot B\left(\frac{1-\tanh(\ell_i)}{2}, 1-t, 1+t \right).$$

□

4.7.3 Recovering Basmajian's Identity in Dimension 3

As $M_L(0) = 1$ we have

$$1 = \frac{4\pi}{V(S)} \sum_{\ell_i \in |O_M|} \coth(\ell_i) \cdot B\left(\frac{1-\tanh(\ell_i)}{2}, 1, 1 \right)$$

and as $B(a, 1, 1) = a$, we have

$$V(S) = \sum_{\ell_i \in |O_M|} 2\pi(\coth(\ell_i) - 1) = \sum_{\ell_i \in |O_M|} \frac{2\pi e^{-\ell_i}}{\sinh(\ell_i)} = \sum_{\ell_i \in |O_M|} V_2(\log(\coth(\ell_i/2))),$$

where $V_2(r)$ is the area of a hyperbolic circle of radius r.

Bibliography

[Ahl66] Lars V. Ahlfors, *Lectures on quasiconformal mappings*, Manuscript prepared with the assistance of Clifford J. Earle, Jr. Van Nostrand Mathematical Studies, No. 10, D. Van Nostrand Co., Inc., Toronto, Ont.-New York-London, 1966. MR 0200442 (34 #336)

[Bas93] A. Basmajian, *The orthogonal spectrum of a hyperbolic manifold*, American Journal of Mathematics **115** (1993), no. 5, 1139–1159.

[Bea83] Alan F Beardon, *The geometry of discrete groups*, vol. 91, Springer New York, 1983.

[Ber78] Lipman Bers, *An extremal problem for quasiconformal mappings and a theorem by Thurston*, Acta Mathematica **141** (1978), no. 1, 73–98.

[BK10] Martin Bridgeman and Jeremy Kahn, *Hyperbolic volume of manifolds with geodesic boundary and orthospectra*, Geometric And Functional Analysis **20** (2010), no. 5, 1210–1230.

[Bon88] Francis Bonahon, *The geometry of Teichmüller space via geodesic currents*, Invent. Math. **92** (1988), no. 1, 139–162. MR 931208 (90a:32025)

[BP92] Riccardo Benedetti and Carlo Petronio, *Lectures on hyperbolic geometry*, Springer Verlag, 1992.

[Bri11] Martin Bridgeman, *Orthospectra and dilogarithm identities on moduli space*, Geometry and Topology **15** (2011), no. 2.

[BT13] Martin Bridgeman and Ser Peow Tan, *Moments of the boundary hitting function for the geodesic flow on a hyperbolic manifold*, arxiv.org/abs/1302.0527, February 2013.

[BTBCT07] Petra Bonfert-Taylor, Martin Bridgeman, Richard D Canary, and Edward C Taylor, *Quasiconformal homogeneity of hyperbolic surfaces with fixed-point full automorphisms*, Mathematical Proceedings of the Cambridge Philosophical Society, vol. 143, Cambridge Univ Press, 2007, pp. 71–84.

[BTCMT05] Petra Bonfert-Taylor, Richard D Canary, Gaven Martin, and Edward Taylor, *Quasiconformal homogeneity of hyperbolic manifolds*, Mathematische Annalen **331** (2005), no. 2, 281–295.

[BTMRT11] Petra Bonfert-Taylor, Gaven Martin, Alan W Reid, and Edward C Taylor, *Teichmüller mappings, quasiconformal homogeneity, and non-amenable covers of riemann surfaces*, Pure and Applied Mathematics Quarterly **7** (2011), no. 2.

[Cal10] D. Calegari, *Chimneys, leopard spots, and the identities of Basmajian and Bridgeman*, Algebr. Geom. Topol. **10** (2010), no. 3, 1857–1863.

[FLM08] Benson Farb, Christopher J Leininger, and Dan Margalit, *The lower central series and pseudo-anosov dilatations*, American journal of mathematics **130** (2008), no. 3, 799–827.

[FM11] Benson Farb and Dan Margalit, *A primer on mapping class groups*, vol. 49, Princeton University Press, 2011.

[GP76] Fredrick W Gehring and Bruce P Palka, *Quasiconformally homogeneous domains*, Journal d'analyse Mathematique **30** (1976), no. 1, 172–199.

[Gre13] Mark Greenfield, *A lower bound for Torelli-K-quasiconformal homogeneity*, Geometriae Dedicata (2013), 1–10.

[Gro83] Mikhael Gromov, *Filling riemannian manifolds*, J. Differential Geom **18** (1983), no. 1, 1–147.

[Hub06] John H Hubbard, *Teichmüller theory and applications to geometry, topology, and dynamics*, vol. 1, Matrix Pr, 2006.

[Iva92] Nikolai V Ivanov, *Subgroups of teichmuller modular groups*, vol. 115, American Mathematical Society, 1992.

[Kap01] Michael Kapovich, *Hyperbolic manifolds and discrete groups*, vol. 183, Birkhauser, 2001.

[Ker83] Steven P Kerckhoff, *The nielsen realization problem*, Annals of mathematics **117** (1983), no. 2, 235–265.

[KM11] Ferry Kwakkel and Vladimir Markovic, *Quasiconformal homogeneity of genus zero surfaces*, Journal d'Analyse Mathématique **113** (2011), no. 1, 173–195.

[Lew91] Leonard Lewin, *Structural properties of polylogarithms*, vol. 37, Amer Mathematical Society, 1991.

[LT14] Feng Luo and Ser Peow Tan, *A dilogarithm identity on moduli spaces of curves*, J. Differential Geom. **97** (2014), no. 2, 255–274. MR 3263507

[Luk69] Yudell Leo Luke, *The special functions and their approximations*, vol. 53, Academic Press, 1969.

[LV73] Olli Lehto and Kalle I Virtanen, *Quasiconformal mappings in the plane*, vol. 126, Springer New York, 1973.

[McS91] Greg McShane, *A remarkable identity for lengths of curves*, Ph.D. thesis, University of Warwick, 1991.

[McS98] ———, *Simple geodesics and a series constant over Teichmuller space*, Invent. Math. **132** (1998), no. 3, 607–632. MR 1625712 (99i:32028)

[Mir07] Maryam Mirzakhani, *Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces*, Invent. Math. **167** (2007), no. 1, 179–222. MR 2264808 (2007k:32016)

[Nic89] Peter J Nicholls, *The ergodic theory of discrete groups*, vol. 143, Cambridge University Press, 1989.

[Pen91] Robert C Penner, *Bounds on least dilatations*, Proc. Amer. Math. Soc, vol. 113, 1991, pp. 443–450.

[Roy71] H. L. Royden, *Automorphisms and isometries of Teichmüller space*, Advances in the Theory of Riemann Surfaces (Proc. Conf., Stony Brook, N.Y., 1969), Ann. of Math. Studies, No. 66. Princeton Univ. Press, Princeton, N.J., 1971, pp. 369–383. MR 0288254 (44 #5452)

[Tei44] Oswald Teichmüller, *Ein verschiebungssatz der quasikonformen abbildung*, Deutsche Math **7** (1944), no. 336-343, 8.

[Thu79] William P Thurston, *The geometry and topology of three-manifolds*, Princeton University, 1979.

[Thu88] ———, *On the geometry and dynamics of diffeomorphisms of surfaces*, Bulletin (New Series) of the American Mathematical Society **19** (1988), no. 2, 417–431.

[Vuo88] Matti Vuorinen, *Conformal geometry and quasiregular mappings*, Lecture Notes in Mathematics, no. 1319, Springer-Verlag, Berlin, 1988.

[Wim] A Wiman, *Ueber die hyperelliptischen curven und diejenigen vom geschlechte p= 3*, Welche eindeutigen transformationen in Sich Zulassen and Ueber die algebraischen curven von den Geschlechtern p **4**, no. 5, 1895–96.

[Yam81] Akira Yamada, *On Marden's universal constant of fuchsian groups*, Kodai Mathematical Journal **4** (1981), no. 2, 266–277.