# Essays in Applied Microeconomic Theory

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Boston College

The Graduate School of Arts and Scineces

Department of Economics

#### ESSAYS IN APPLIED MICROECONOMIC THEORY

a dissertation

by

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submitted in partial fulfillment of the requirements

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#### ESSAYS IN APPLIED MICROECONOMIC THEORY

Abstract

by

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This dissertation consists of three essays in microeconomic theory: two focusing on insurance theory and one on matching theory.

The first chapter is concerned with catastrophe insurance. Motivated by the aftermath of hurricane Katrina, it studies a strategic model of catastrophe insurance in which consumers know that they may not get reimbursed if too many other people file claims at the same time. The model predicts that the demand for catastrophe insurance can "bend backwards" to zero, resulting in multiple equilibria and especially in market failure, which is always an equilibrium. This shows that a catastrophe market can fail entirely due to demand-driven reasons, a result new to the literature. The model suggests that pricing is key for the credibility of catastrophe insurers: instead of increasing demand, price cuts may backfire and instead cause a "race to the bottom." However, small amounts of extra liquidity can restore the system to stable equilibrium, highlighting the importance of a functioning reinsurance market for large risks. These results remain robust both for expected utility consumer preferences and for expected utility's most popular alternative, rank-dependent expected utility. The second chapter develops a model of quality differentiation in insurance markets, focusing on two of their specific features: the fact that costs are uncertain, and the fact that firms are averse to risk. Cornerstone models of price competition predict that firms specialize in products of different quality (differentiate their products) as a way of softening price competition. However, real-world insurance markets feature very little differentiation. This chapter offers an explanation to this phenomenon by showing that cost uncertainty fundamentally alters the nature of price competition among risk-averse firms by creating a drive against differentiation. This force becomes particularly pronounced when consumers are picky about quality, and is capable of reversing standard results, leading to minimum differentiation instead. The chapter concludes with a study of how the costs of quality affect differentiation by considering two benchmark cases: when quality is costless and when quality costs are convex (quadratic).

The third chapter focuses on the theory of two-sided matching. Its main topic are inefficiencies that arise when agent preferences permit indifferences. It is well-known that two-sided matching under weak preferences can result in matchings that are stable, but not Pareto efficient, which creates bad incentives for inefficiently matched agents to stay together. In this chapter I show that in one-to-one matching with weak preferences, the fraction of inefficiently matched agents decreases with market size if agents are sufficiently diverse; in particular, the proportion of agents who can Pareto improve in a randomly chosen stable matching approaches zero when the number of agents goes to infinity. This result shows that the relative degree of the inefficiency vanishes in sufficiently large markets, but this does not provide a "cure-all" solution in absolute terms, because inefficient individuals remain even when their fraction is vanishing. Agent diversity is represented by the diversity of each person's preferences, which are assumed randomly drawn, i.i.d. from the set of all possible weak preferences. To demonstrate its main result, the chapter relies on the combinatorial properties of random weak preferences.

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### Chapter 1

# Does Catastrophe Insurance Obey the Law of Demand?

#### 1.1 Introduction

Catastrophe insurance is insurance against large-scale, low-probability accidents with correlated damage, such as hurricanes, earthquakes, and terrorist acts, which affect many individuals in a single blow. Every year, millions of Americans buy hurricane insurance in Florida, earthquake insurance in California, and flood insurance in Mississippi; insurance against financial instrument default can also be considered a form of catastrophe insurance. Major disasters, however, are often problematic even for the best-prepared insurers. Although catastrophe risk may not be uninsurable in principle (Borch, 1990), in reality, major catastrophes have devastating effects not only on the insured but on the insurers as well (Born and Viscusi, 2006).

Motivated by the aftermath of hurricane Katrina, I study a strategic model of catastrophe insurance in which consumers know that they may not get reimbursed if too many other people file insurance claims at the same time. The model predicts that the demand curve for catastrophe insurance can bend backwards to zero, resulting in multiple equilibria and especially in market failure, which is always an equilibrium in this model. This result is important for two reasons: firstly, it shows that a catastrophe insurance market can fail entirely due to demand-driven reasons, a result new to the literature; and secondly, it sheds light on why catastrophe insurance markets fail so often and why California homeowners buy so little earthquake insurance (Cummins, 2006). The model suggests that when consumers use the price as a signal of solvency, then competitive price-cutting may backfire – instead of generating more demand, it leads to a "race to the bottom."

The backward-bending demand curve permits multiple equilibria with different stability properties. I show that even small amounts of extra liquidity can restore the system to a stable equilibrium, highlighting the importance of a working reinsurance market. For expositional purposes I use expected utility, but the results extend readily to alternative risk preferences such as rank-dependent utility (Quiggin 1982; Segal 1987a; Chew, Karni and Safra, 1987).

The paper is structured as follows. Section 1.2 reviews specific features of catastrophe insurance markets that matter for the model. Section 1.3 reviews the related literature, and Sections 1.4 and 1.5 present the model. Section 1.6 interprets the results, while Section 1.7 looks at the effect of aid from reinsurers or the government. Section 1.8 extends the results to rank-dependent utility; Section 1.9 concludes.

#### **1.2** The Market for Catastrophe Insurance

Catastrophe insurance markets operate very differently from the markets for regular insurance. In regular insurance, such as auto, theft, and industrial accident insurance, accidents are statistically independent and relatively frequent so that expected losses can be predicted accurately using laws of large numbers. It is not so in the market for catastrophe insurance, where losses are "lumpy" in that they occur rarely, arise from correlated claims, and require large amounts of liquidity. Due to this, major disasters often have catastrophic effects on the insurace industry (Born and Viscusi, 2006).

One reason for this is that losses associated with a once-in-a-lifetime catastrophe, such as Hurricane Katrina, the Northridge earthquake, or the terrorist attacks of 9/11, are statistically difficult to predict. Since these are very infrequent events, small sample size makes laws of large numbers uninformative in forecasting future losses; as a result, the insurer may not have enough funds to service all simultaneous claims. Although some authors have argued that catastrophic risk is, at least in principle, fully insurable (Borch (1990), p. 315; Jaffee and Russel, 1997; Zeckhauser, 1995), opinions agree that in reality, large disasters remain difficult to handle. For example, Hurricane Katrina, which inflicted over 45 billion dollars of insured losses,<sup>1</sup> forced several major insurers to withdraw from the catastrophe market: Allstate Insurance exited several West coast states, and State Farm, another major insurer, stopped renewing policies after 2005. One of Florida's largest insurers, Poe Financial, went bankrupt.<sup>2</sup> Even major insurers who did not go bankrupt refused to honor thousands of hurricane policies, leading to an unprecedented wave of lawsuits.

Another reason for insurers' vulnerability to catastrophes is that the firm may still face a liquidity problem even if the premium accurately reflects risk. Due to the simultaneous nature of losses, catastrophe risks require insurers to hold large amounts of liquid capital, but institutional factors and capital market imperfections make insurance companies reluctant to do this (Jaffee and Russel, 1997). For example, accounting standards prevent insurers from dedicating a capital surplus to fund a specific future

 $<sup>^{1}</sup>$ SwissRe (2006).

<sup>&</sup>lt;sup>2</sup>Born and Viscusi (2006); "The Price of Sunshine," The Economist, June 8, 2006, p. 76.

loss, even if that loss is likely.<sup>3</sup> Setting aside retained earnings to fund catastrophe risks is also disadvantageous, since retained earnings are taxed as corporate income in the year in which they are set aside. Finally, holding large amounts of free cash can make the company a target for a hostile takeover; Blanchard, Lopez-de-Silanes, and Schleifer (1994) find that "firms that hold the cash [...] are themselves acquired within a few years." As a result, holding liquid reserves is unattractive for insurers. This gives rise to an additional liquidity constraint which prevents the disbursement of numerous large payments at the same time.

The third reason for insurers' vulnerability to catastrophes is the lack of an adequate reinsurance market (Froot, 2001). Why the market for disaster reinsurance is so thin is still an active area of research. Individual companies' vulnerability to large risks, while unsurprising, does not explain why insurers are not able to pool and diversify risks on a larger scale, such as a national market for reinsurance. It is tempting to think that catastrophe risks are too big to handle even for the nationwide market, but this reasoning is disputed by Cummins, Doherty, and Lo (2002), who estimate that the national reinsurance industry has enough capacity to fund up to a 100 billion dollar accident (more than twice the size of hurricane Katrina), albeit with some disruptions. Jaffee and Russell (1997) similarly conclude that "there is nothing in the nature of catastrophe risk as such which prevents the operation of a private market [for reinsurance]."

Theories differ as to why the market for catastrophe reinsurance is so thin. Froot (2001) examines eight different explanations and finds that the two most likely causes are supply restrictions caused by capital market imperfections and market power exerted by traditional reinsurers. By contrast, Ibragimov, Jaffee and Walden (2009) see the weak reinsurance market for catastrophe risks as a consequence of "non-

<sup>&</sup>lt;sup>3</sup>See Financial Accounting Standards Board (FASB) Statement No. 5, "Accounting for Contingencies". The effects of this accounting rule are discussed in Jaffee and Russel (1997).

diversification traps". They argue that insurance companies have no incentives to diversify catastrophe risks, since diversifying risks with heavy-talied distributions can *increase* the riskiness of the resulting portfolio. For the diversification to be successful, the process requires coordination between firms, a feature absent from reality. Thus Ibragimov et al. see the thin reinsurance market as the result of a coordination problem.

Failures of primary insurance markets have often resulted in government intervention.<sup>4</sup> For example, the failure of the private market for flood insurance prompted the U.S. government to charter the National Flood Insurance Program (NFIP) in 1968, which survived financially until the hurricane losses from 2004 and 2005. The California and Florida hurricane markets similarly experienced difficulties after hurricanes Andrew in 1992 and Katrina in 2005; in response, the State of Florida created the FRPCJUA, a "residual market facility" providing insurance to those unable to find coverage on the private market.

The government also stepped in to save the California earthquake insurance market after the 1994 Northridge quake. According to the California Department of Insurance (1995), companies representing 93 % of the market stopped offering homeowners' insurance or imposed strict liability limits after the earthquake. In response, the California legislature chartered the California Earthquake Authority (CEA) – a quasigovernment entity with a mandate to write insurance at actuarially sound prices. The intervention succeeded in bringing companies back, as by 2005, more than 150 private insurers had joined the market.<sup>5</sup> Despite this, California homeowners remain remarkably reluctant to buy earthquake insurance even at actuarially fair prices. Cummins (2006) writes that this remains "somewhat of a puzzle in the California

<sup>&</sup>lt;sup>4</sup>This summary is based on the discussion in Cummins (2006).

<sup>&</sup>lt;sup>5</sup>See California Department of Insurance (2005).

market". My model shows that such consumer behavior is optimal in the presence of liquidity constraints that affect the insurer's ability to pay many simultaneous claims.

Hurricane Katrina's aftermath showed that the risk of claim denial can be substantial even if the insurer does not go bankrupt. In the wake of the hurricane, insurers denied payments to thousands of Mississippi homeowners, using a legal loophole that allowed them to interpret the policies as insurance against wind damage but not against winddriven water damage. As a result, three thousand Mississippi policyholders filed a class-action lawsuit against insurers Allstate, MetLife, State Farm, and USAA, while Mississippi's Attorney General, Jim Hood, independently filed a separate lawsuit.<sup>6</sup> Two especially famous cases ruled by the same judge were Leonard vs. Nationwide<sup>7</sup> and Broussard vs. State Farm.<sup>8</sup> The court ruled against the Leonard family who claimed that the policy terms were misleading, but sided with the Broussard family, which argued that irreparable wind damage had already occurred before their house was flooded.

The broad media coverage of the lawsuits suggests that homeowners likely take such information into account when considering how much insurance to buy. Here I propose a model of catastrophe insurance in which consumers take each other's insurance purchases into account, knowing that the more people file claims at the same time, the lower is the likelihood of obtaining a payment. This leads to a strategic setup in in which consumers strategize against other consumers, rather than against the insurer, as in the traditional literature.

<sup>&</sup>lt;sup>6</sup>Fox News, July 11, 2006.

<sup>&</sup>lt;sup>7</sup> "Katrina damage ruling bad news for homeowners," The Houston Chronicle, Aug. 15, 2006.

<sup>&</sup>lt;sup>8</sup> "Jury Picked to Hear Katrina Insurance Lawsuit," The Washington Post, Jan. 8, 2007.

#### **1.3 Related Literature**

Authors studying insurance contracts with uncertain repayment typically refer to them either as "nonperforming contracts" (Doherty and Schlesinger, 1990) or "probabilistic insurance" (Kahneman and Tversky, 1979). The common feature of most such models is that the probability of insurer default is either exogenous or determined by factors independent of the price of insurance and of consumers' actions.

Doherty and Schleisinger (1990) were among the first to analyze the response of insurance demand to exogenous changes in the probability of default; they consider the long-run case when the insurer always has a non-negative profit. I begin with a similar theoretical setup, but improve on their model by endogenizing the probability of default and by extending the analysis to the short run, where firms can incur losses as well as profits. Next I introduce correlation between insurance claims, a feature typical of catastrophe insurance, and permit insurance customers to strategize against each other, a feature not considered by Doherty and Schleisinger or the remaining literature.

Strategic situations are common in the context of insurance, but they usually take place either among competing oligipolists, or between buyers and sellers. Strategizing is typically made possible by the presence of asymmetric information combined with a signal, such as a price or an interest rate (e.g., Akerlof (1970); Rotschild and Stiglitz (1976); Shavell (1979); Wilson (1977); Hellwig (1987); Green (1973), Grossman and Stiglitz (1976), and others.<sup>9</sup> For example, in Rotschild and Stiglitz (1976), the prices of separating insurance contracts act as signals that sort customers into high-risk and low-risk types.) However, the existing insurance literature has focused mainly on games where consumers play against the insurer, and more rarely, on games where

 $<sup>^{9}</sup>$ The reader is referred to Laffont (1989) for a survey and to Dionne et al (2000) for a detailed summary of this literature.

insurers play against each other.<sup>10</sup> By contrast, I introduce an insurance model in which consumers strategize against other consumers; this feature is meant to model the aftermath of hurricane Katrina, where the large number of simultaneous claims caused serious difficulties for insurers.

This paper is related to Zanjani (2002) and Cummins and Danzon (1997), who also consider savvy consumers mindful of the probability of default. Zanjani (2002) studies a model of catastrophe insurance pricing where demand is assumed decreasing in the price and increasing in the chance of reimbursement, without considering the endogenous relation between the two. Cummins and Danzon (1997) model the relationship between loss shocks, capitalization and insurance prices, but only in markets without strategic consumer interactions. They find empirically that insurers with high default risk price their policies cheaper than insurers with low default risk, an observation consistent with the predictions of my model.

Newer work by Cummins and Mahul (2003) studies the market for catastrophe reinsurance when the buyer and the insurer have divergent beliefs about the probability of default, and Hoy and Robson (1981) along with Briys, Dionne and Eeckhoudt (1989) study whether insurance can be a Giffen good, but only in the context of certain, non-probabilistic insurance. To my knowledge, the present literature features little discussion of any variation of the probability of default due to consumers' actions, an issue that here is key.

By contrast, in my model the probability of default arises endogenously as a result of consumers' response to the current price. Price-taking consumers observe the going price and compute the probability of default, taking into account remaining demand by other consumers; they consider whether at this value of total demand, the

<sup>&</sup>lt;sup>10</sup>For example, Ibragimov, Jaffee and Walden (2009).

going price guarantees enough solvency in order to pay everyone, and buy insurance accordingly.

#### 1.4 Model

I perform a short-run analysis of a stylized insurance market based on the description of the catastrophe insurance market in Section 1.2. I assume a large number of identical consumers, represented mathematically by a continuum of agents of measure 1; there is a large finite number K of identical firms. The evidence from Section 1.2 suggests that despite the presence of large government-created entities in some catastrophe markets, private insurers can be modeled as price-takers, as government entities do not engage in monopoly pricing<sup>11</sup> (Cummins, 2006). The existence of the California Earthquake Authority, for instance, does not seem to have hindered either competition or market entry in the California market, which has more than 150 private insurers operating at present; based on this, I model private firms as price-takers.

Each firm k begins with an initial set of consumers  $S_k$  of measure  $s_k$  ( $0 < s_k < 1$ ,  $\sum_{k=1}^{K} s_k = 1$ ), interpretable as the firm's market share. It will be shown that market shares become equal given a fixed price, therefore the initial distribution of market shares does not matter and can be assumed arbitrary.<sup>12</sup> Aggregated demand x for insurance at firm k is the sum (integral) of individuals' consumption amounts  $x_i$  integrated over the set  $S_k$ , so that  $x \equiv \int_{S_k} x_i di$ . This formulation captures the idea of an individual consumer who is too small to affect total demand. Indeed, if we

<sup>&</sup>lt;sup>11</sup>For example, the CEA's legal mandate obliges it to price policies close to the expected actuarial loss.

<sup>&</sup>lt;sup>12</sup>Formally, one can also look at this as a two-stage game where the customer first chooses from which firm to buy, and then a quantity demanded. However, solving by backward induction reveals that the first stage is trivial: as in second-stage equilibrium, all firms will offer exactly the same terms, the choice of firm is not a strategic variable.

denote the summary consumption of all agents other than i with  $x_{-i}$ , it is trivial to verify that

$$x = \int_{\{S_k\}} x_j dj = \int_{\{S_k \setminus i\}} x_j dj = x_{-i}, \text{ so } \frac{dx}{dx_i} = 0.$$

Consumers are price-takers: each consumer is too small relative to the market in order to be able to move the price, total market demand, or the probability of non-payment by his insurance company. Each consumer owns initial wealth  $w_0$ ; conditional on a catastrophe, there is a chance  $\gamma$  ( $0 < \gamma \leq 1$ ) that his property is destroyed, inflicting a loss of \$ L (the value of his property); then his wealth becomes ( $w_0 - L$ ). A catastrophe occurs with exogenous probability  $\pi$ , while  $\gamma$  is the conditional chance that the particular individual is affected, so the unconditional probability that each agent suffers a loss is  $p = \gamma \pi$ .

Every consumer *i* has to decide how much coverage  $x_i$  ( $0 \le x_i \le L$ ) to buy at the going price of \$ *q* per dollar of coverage, which the consumer takes as given. The consumer first chooses an insurance company and then the quantity of insurance demanded. Each company's product has two characteristics: price and "quality." By "quality" I understand that in the event of accident, there is a (conditional) chance *r* that the chosen insurance company denies the consumer's claim and refuses payment, resulting in final wealth  $w_0 - qx_i - L$ . In Proposition 2, it will be shown that in competitive equilibrium, all firms offer the same terms (price and quality) of insurance, so that the initial distribution of customers across firms does not matter (if one firm's terms are worse than another's, the consumer can switch). Consumers do not distinguish whether they are denied because of a bankruptcy affecting everyone else, or due to a partial default affecting only some people; what matters to the individual is only his own reimbursement. Each consumer is too small to affect *r* by his choice of coverage and takes *r* as given, so when deciding on the optimal choice of coverage, each consumer i faces the lottery

$$X' = (w_0 - qx_i, \ 1 - p \ ; \ w_0 - L + (1 - q)x_i, \ p(1 - r) \ ; \ w_0 - L - x_iq, \ pr).$$

Each consumer has the same, strictly concave vonNeumann-Morgenstern utility function  $u(\cdot)$  over money, and evaluates the lottery X' using expected utility. To determine the optimal choice of insurance, she solves the problem

$$\max_{x_i} EU[X'(x_i)] \text{ s.t. } qx_i \le w_0$$

with associated first-order condition

$$-q(1-p)u'(w_0-qx_i) + p(1-q)[1-r]u'(w_0-L+(1-q)x_i) - pqr\ u'(w_0-L-qx_i) = 0.$$

The probability r of non-payment, although taken as given by each individual agent, is determined endogenously within the model; individual agents are simply too small to move it by varying  $x_i$ . In reality, r varies depending on how much liquid resources i's chosen firm has, compared to how much it needs in order to cover the accident A. Because individual losses are correlated, the firm needs more liquidity than the unconditional expected loss  $\pi \gamma x$ , in contrast with regular insurance. Since each consumer is affected with probability  $\gamma$  conditional on a catastrophe, a fraction  $\gamma$  of insured properties is destroyed when the disaster arrives  $(1 \ge \gamma > 0)$ ; the parameter  $\gamma$  reflects the severity of the disaster and captures the correlation between claims. Then the insurer's expected payout in the event of accident A is

$$E[\text{Claims}|A] = \gamma x.$$

At price q and demand x, the firm collects total revenue t = qx; therefore, when  $q \ge \gamma$ , the insurer is fully solvent even if he starts with zero liquid reserve from previous years. I consider this case first; section 1.7 extends the results for the case where the insurer has a previously accumulated reserve of R.

Even when the firm sells insurance at a price that guarantees solvency, additional factors typically prevent it from holding its entire revenue liquid. Jaffee and Russell (1997) have found that exogenous institutional factors – accounting, taxes, and takeover risk – make insurers unwilling to hold large amounts of liquid capital, thereby exacerbating the liquidity strain caused by the thin reinsurance market. I model these external factors with an exogenous liquidity function  $\alpha(t)$ , a rule that tells the firm how many dollars to keep liquid out of a given total revenue t. Since the influence of institutions is external, I assume  $\alpha$  exogenous and make only minimal assumptions about it; my results depend only on the assumption that liquidity increases with revenue and does so without abrupt jumps ( $\alpha(t)$  is increasing, continuous,  $\alpha(0) = 0$  and  $\alpha(t) \leq t \; (\forall t)$ ). This assumption is plausible and is the only assumption I make about  $\alpha$ . To reflect insurers' unwillingness to keep large liquid funds, one can additionally assume that liquid funds as a fraction of total revenue (the quantity  $\alpha(t)/t$ ) decreases in t, that is, that the liquidity constraint  $\alpha$  is binding, but this assumption is not essential.<sup>13</sup> The dollar amount of expected unpaid claims in the event of accident Ais therefore

$$E$$
[Unpaid Claims $|A] \equiv \gamma x - \alpha(qx)$ 

If an accident occurs and the insurer has less liquidity than needed to pay the claims, he randomly picks whom to pay and whom to refuse until he exhausts all available money. Claims are either paid in full or denied in full; the evidence quoted in Section 1.2 suggests that cases of partial payment are very rare. If a disaster does not occur,

<sup>&</sup>lt;sup>13</sup>However it is more realistic and suggests an explanation for the California earthquake insurance puzzle (see Section 1.6). One can also assume  $\alpha(t) \equiv t$ ; the details are given in the proofs.

the insurer keeps the entire revenue qx, and either invests it in long-term assets, or sets aside a liquid reserve of  $R \leq qx$  for the future; that case is considered in Section 1.7. I define the probability of claim rejection r as the ratio of the expected unpaid claims to total claims:

$$r = \frac{E[\text{ Unpaid claims}|A]}{E[\text{ Claims}|A]} = \frac{\gamma x - \alpha(qx)}{\gamma x} = 1 - \frac{\alpha(qx)}{\gamma x}.$$

Since consumer *i*'s demand also factors in x, technically r depends on the choice of  $x_i$ . However, since  $x = x_{-i}$ , each consumer ignores his own influence on r as negligible and faces a probability of being rejected equal to

$$r = 1 - \frac{\alpha(qx_{-i})}{\gamma x_{-i}}.$$

Since  $r = r(x_{-i})$ , others' consumption  $x_{-i}$  also enters consumer *i*'s first-order condition

$$-q(1-p)u'(w_0 - qx_i) + p(1-q)[1-r(x_{-i})]u'(w_0 - L + (1-q)x_i)$$
$$-pqr(x_{-i})u'(w_0 - L - qx_i) = 0$$

which from here on, I denote as  $G(x_i, x_{-i}) = 0$ . In this setup, *i*'s optimal choice of insurance depends on other agents' aggregate consumption  $x_{-i}$ .

#### 1.5 Strategic Competitive Equilibrium

I approach the problem using the solution concept of competitive equilibrium. Since all agents are the same, I further narrow down the solution concept to symmetric competitive equilibrium. **Definition 1.** A symmetric competitive equilibrium is a triple  $(x^*, q^*, r^*)$  such that:

(a)  $x_i^* = \operatorname{argmax}_{x_i} EU[X']$  for each i, and  $x_i^* = x_j^*$  ( $\forall i, j \text{ at firm } k$ ) (b) At price  $q^*$  the market clears, and (c)  $r_k^* = r_\ell^* = \ldots = r_K^*$  for all firms  $k, \ell, \ldots K$ .

Since individual optimal choices strategically depend on each other, it is more appropriate to refer to this as strategic competitive equilibrium, although incentive compatibility follows from the Definition. (If no profile of strategies  $\{x^*\}$  constitute mutually best responses at the market-clearing price, clearly existence of competitive equilibrium also fails). Definition 1 also implies that in competitive equilibrium, no agent has an incentive to switch firms, as every firm offers the same probability of rejection and the same price.

I will prove existence of strategic competitive equilibrium in two steps: first I will show that given a fixed price q > 0, there exist values of  $x_i$   $(i \in S_k)$  and  $r_1, \ldots, r_K$ that satisfy parts (a) and (c) of Definition 1, and then I will proceed to show that there is a price q that clears the market; at that price  $q^*$ , all three parts of Definition 1 will hold.

I first show that given a positive price, a mutually compatible profile of best consumption choices exists for all agents within a firm. For a given *fixed* price q > 0, this is easiest to represent as a game  $\Gamma$  parametrized by the price. One can think of  $\Gamma$ as a game in which each agent *i* plays against remaining consumers at the same firm (henceforth, *rest of the firm's market*) about how much consumption to choose at a given fixed price. This is consistent with the previous discussion that the resource constraint occurs on the firm level: that is, insurers cannot lend or borrow from each other, neither are they allowed to pool risks by purchasing reinsurance due to the failure of the reinsurance market. In the interest of clarity, I suppress all firm-specific notation; I reintroduce firm indices in the aggregation stage.

Given a price of isnurance q, define the infinite-player game played at firm k as

$$\Gamma = \{ (i \in S_k, -i); \ (x_{i \in S_k}, x_{-i}); \ (v_{i \in S_k}, v_{-i}) \mid q \},\$$

in which:

- Players are agents i ∈ S<sub>k</sub>, plus the rest of the firm's market, denoted as player -i;
- Each  $i \in S_k$  plays against rest of the firm's market -i, but not directly against other individual consumers  $j \in S_k$ ;
- Agents choose actions  $x_i \in [0, L]$ , and rest of the firm's market -i "selects" consumption  $x_{-i} \in [0, s_k L]$ ;
- The behavior of the individuals is governed by payoff functions  $v_i(x_i, x_{-i}) = EU[X'];$
- The behavior of the rest of the firm's market is governed by the response function  $R_{-i} = \int_{\{S_k \setminus i\}} x_i di$ , equal to the integral of individual responses of other consumers.

Using this formalization, it is possible to show that given a postive price q, part (a) of Definition 1 is satisfied:

**Proposition 1.** Given a price q > 0, at each firm k there exists a symmetric consumption profile  $\{x_i^*\}_{i \in S_k}$  such that  $x_i^* = \operatorname{argmax}_{x_i} EU[X']$  for each i, and  $x_i^* = x_j^*$   $(\forall i, j)$ .

*Proof:* In the Appendix.  $\Box$ 

It remains to be seen whether the market can equilibrate in terms of the rejection probabilies  $r_k$  at different firms k. So far I looked at the symmetric consumption levels  $x^{*,k}$  at each particular firm k, but there is no guarantee that consumers at a different firm  $\ell$  choose the same symmetric amounts. The next Proposition shows that it is again possible to approach this question separately for every price. Here I prove three important results:

#### **Proposition 2.** Given a price q > 0:

a) Symmetric equilibrium consumption levels are the same across firms:  $x^{*,k} = x^{*,\ell}$ for every two firms k and  $\ell$ .

b) All firms have equal shares and equal amounts of consumers:  $s_k = s_\ell = \frac{1}{K} \; (\forall k, \ell)$ c) All firms have the same probability of non-payment:  $r_k = r_\ell, \; (\forall k, \ell).$ 

#### *Proof.* In the Appendix. $\Box$

The last proposition implies that, given a price q > 0, part (c) of Definition 1 also holds. Since (a) and (c) hold for any price including the market-clearing price, this permits us to look at demand and supply separately as a function of the price, as in a standard model.

To find symmetric competitive equilibrium, it remains only to prove that a marketclearing price exists, that is, that supply and demand intersect. To do this I first derive the market demand curve, which is of special interest on its own. Note that the last Proposition also simplifies the aggregation of demand curves into market demand. Each consumer has the same demand curve  $x_i^*(q)$ ; each firm also has the same firm-specific demand curve  $x^k(q) = (1/K)x_i^*(q)$ . Therefore, **market** demand is  $D(q) = K \cdot x^k(q) = x_i^*(q)$ . In other words, market demand and individual demand are the same, a shortcut I use to simplify aggregation. From here, it suffices to describe the shape of the individual consumer's demand; the result for the market follows automatically. Next I show that consumer demand in this model is described by an unusual-looking demand curve which converges to zero at low prices; the intuition behind this is given by three steps:

Step 1. When the price  $q \to 0$ , the probability of non-payment  $r \to 1$  regardless of the path of demand x(q). Step 2. At r = 1, optimal consumption is  $x_i^* = 0$  at every positive price. Step 3. By continuity of expected utility preferences, optimal consumption  $x_i^*$  approaches 0 when  $q \to 0$ .

Steps 2 and 3 are very intuitive, so they are proven in the Appendix. The claim in Step 1 is less apparent. If we take  $q \to 0$  and demand x(q) indeed converges to zero as hypothesized, then r's limit is an indeterminacy of the kind [0/0]. It is not obvious where this expression tends to (indeterminacies of the kind [0/0] can theoretically converge to any real number, or go to infinity). The next proposition resolves the indeterminacy by showing that when q goes to zero, the probability of non-payment r approaches unity regardless of the path of demand, a result that is key.

#### **Proposition 3.** The function

$$r(x,q) = 1 - rac{lpha(qx)}{\gamma x}$$
 satisfies  $\lim_{q \to 0} r(x,q) = 1$ 

on the domain  $x \in (0, s_k L]$ , where  $\alpha(t)$  is an increasing, continuous function s.t.  $\alpha(0) = 0$  and  $\alpha(t) \leq t$ .

#### *Proof.* In the Appendix. $\Box$

The last Proposition provides justification for Step 1, and jointly the three steps establish the fact that at near-zero prices, demand tends to zero:  $\lim_{q\to 0} [x_i^*] = 0$ . Note that this statement does not imply that  $x_i^*(0) = 0$ . Indeed, at the zero price, demand is multiple-valued: when insurance is free, one can buy any amount even if that insurance is useless, i.e. never pays back; however, none of the main results depend on this, as shown in Proposition 5.

With the help of the last three propositions, we are now in possession of the following facts:

- Fact 1. As the price of insurance converges to zero, so does individual demand: lim<sub>q→0</sub>[x<sup>\*</sup><sub>i</sub>] = 0.
- Fact 2. From standard theory, at a sufficiently high price  $q^H$ , the quantity demanded becomes zero:  $x_i^*(q^H) = 0$ .
- Fact 3. By Lemma 2 in the proof of Proposition 1 (see the Appendix), there exists a price  $\bar{q}$  at which demand is positive:  $x_i^*(\bar{q}) > 0$ .
- Fact 4. By the proof of Proposition 1 and the implicit function theorem, demand  $x_i^*(q)$  is differentiable and therefore continuous in the price q.

Facts (1)-(3) pin down three points on the price-quantity plane through which the demand curve must pass. Facts (2), (3) and (4) jointly imply that the demand curve has a downward-sloping portion at sufficiently high prices,<sup>14</sup> while facts (1) and (4) imply that the demand curve has an upward-sloping portion near q = 0. Fact (4) rules out the possibility that quantity demanded returns to zero in a discontinuous "jump." Since in this model, market demand and individual demand coincide, this implies that

**Proposition 4.** Facts (1)-(4) imply that the market demand curve has an upwardsloping portion at prices near q = 0 and a downward-sloping portion at prices near  $q^{H}$ .

<sup>&</sup>lt;sup>14</sup>Because demand is continuous by Fact 4.

The last Proposition implies that the demand curve for catastrophe insurance is backward-bending, one of the main results of this paper. To complete the existence proof for strategic competitive equilibrium, one needs to verify that a market-clearing price exists, that is, that supply and demand intersect; this is done in the next section.

#### **1.6** Discussion and Interpretation

The backward-bending demand curve permits anomalies that cannot arise in a regular market. In particular, it can generate multiple equilibria with distinct stability properties, but the exact equilibria that will occur depend on the properties of supply. To permit for the richest set of outcomes, I again make minimal assumptions about supply. Consistent with price-taking,<sup>15</sup> I assume that the competitive firm's supply equals its marginal cost above the shut-down price (i.e. above minimum average variable cost), and that marginal cost is non-decreasing in quantity, but make no further assumptions. For example, marginal cost could be constant or irregular-shaped, but the model still delivers a stark result.

**Proposition 5.** There exists at least one price  $q^*$  that clears the market. In particular, the zero-price, zero-quantity combination ( $q^* = 0$ ,  $x^* = 0$ ) is always a strategic competitive equilibrium.

*Proof:* In the Appendix.  $\Box$ 

Proposition 5 shows that at least one strategic competitive equilibrium exists, but more importantly, highlights that market failure is *always* an equilibrium in this model, in addition to any non-degenerate equilibria. This provides a distinct demandside view on the frequent failures of catastrophe markets. What Proposition 5 implies

 $<sup>^{15}{\</sup>rm Recall}$  price-taking is consistent with the evidence about the private earthquake insurance market in California.

is that consumer distrust in the paying ability of insurers is already a sufficient cause for the market to fail, a result new to the catastrophe literature. In addition, the model suggests that when consumers use the price as a signal of solvency – which in this line of business seems more than likely – then competitive price-cutting techniques may backfre: instead of generating more demand, they lead to a race to the bottom. An illustration of a few such anomalies appears in Fig. 1.1.



Figure 1.1: Demand anomalies in the market for catastrophe insurance

The first anomaly that can arise is the existence of multiple equilibria, and in particular, of market failure, which is *always* an equilibrium in this market, as shown in Proposition 5. A competitive firm will stop producing when the price falls below the minimum of average variable cost, so supply at the zero price is S(0) = 0; since demand at the zero price is a horizontal line (multiple-valued, as discussed before), this ensures that also x(0) = 0, so supply and demand always intersect at the zero-price, zero-quantity combination. Unlike other models, market failure here is generated entirely by the demand side.

However, other equilibria with distinct stability properties may also exist. For example, the market in Fig. 1.1 (left) has three distinct equilibria: a stable equilibrium

 $(q^*, x^*)$ , an unstable equilibrium at  $(q^{**}, x^{**})$ , and a stable market failure point (0, 0). The equilibrium  $(q^{**}, x^{**})$  is unstable: if q is between  $q^{**}$  and  $q^*$ , excess demand puts upward pressure on prices, while at prices  $q < q^{**}$ , excess supply pushes prices down to the market failure point (0,0); therefore the smallest price perturbation results either in going back to the stable equilibrium or in market failure. Fig. 1.1 (right) illustrates the case of flat marginal costs, where marginal cost equals the shutdown price. In this case there are only two equilibria: market failure and one stable equilibrium where demand intersects marginal cost. It is also possible to have a situation where market failure is the only equilibrium, such as when the shutdown pice is zero and marginal cost is sufficiently flat, as reflected by the supply curve  $S_1$ . In all cases market failure remains as a possible equilibrium.

Another interesting anomaly in the market for catastrophe insurance is the so-called California earthquake insurance puzzle, described by Cummins (2006):

"Something of a puzzle in the California market, however, is that only a small proportion of eligible property owners actually purchase the insurance. In the homeowners market, 33 percent of eligible properties purchased earthquake insurance in 1996, [...] but only 13.6 percent had insurance in 2003. The rationale usually given for the low market penetration is that most buyers consider the price of insurance too high for the coverage provided, even though premiums are close to the expected losses."

My model formalizes the intuition suggested by Cummins (2006), but in addition suggests an alternative explanation of the California puzzle. Jaffee (2005) observes that "It seems that many homeowners consider the premiums to be high relative to the coverage provided" (p. 208), but provides no formal analysis. If prices are "too high" yet pricing is close to the expected loss, the surcharge must be coming

from the correlation of claims. To illustrate this, consider a rare disaster which occurs only once per century with a chance  $\pi = 1\%$ , but is quite severe and destroys 80% of insured properties, so  $\gamma = 0.8$ . Each individual is affected with probability  $p = \pi \gamma = 0.8\%$ , but the fully-solvent price in the model is higher than 0.8 cents. To see this, consider the extreme case where the firm starts with no prior reserves;<sup>16</sup> then the fully-solvent price is  $q = \gamma = 0.8$ , since the expected payout in case of accident is  $E[\text{Claims } |A] = \gamma x$ . In this case, the consumer pays 80 cents per dollar of coverage in order to cover a risk smaller than 1%, a clearly unattractive pricing exceeding the "actuarially fair" price (from the standpoint of the consumer) by a factor of 10. Of course, in reality firms accumulate disaster reserves throughout multiple years, which mitigates the price differential between the fully-solvent and actuarially fair price and moderates some of the worst anomalies, as will be shown in Section 1.7. Nonetheless, a surcharge above the actuarially fair price always remains, because individual risks are correlated; thus the fully-solvent price is not actuarially fair, and a risk-averse consumer will not buy full insurance at that price, consistent with what we observe in the California market. Therefore my model agrees with the intuition in Jaffee (2005), but in addition, suggests additional insights on the California puzzle.

Even at the fully-solvent price, the model is capable of generating reduced demand if the insurer keeps less than 100% of the revenue liquid. If the liquidity constraint  $\alpha$  is binding, as suggested by Jaffee and Russell (1997), the institutional factors described in Section 1.2 create strong disincentives for insurers to keep revenue liquid, so they prefer to convert revenues to illiquid, but more advantageous assests (from the standpoint of accounting and tax law). But the lower the fraction of liquid revenue, the higher is the probability of claim refusal, and the lower is demand compared to the textbook model. The tighter is the constraint  $\alpha$  imposed by the institutional factors, the lower is the quantity demanded. This provides additional insight into the

 $<sup>^{16}</sup>$ The effect of previously accumulated reserves is discussed in the next Section 1.7.

California earthquake insurance puzzle, as facts about the solvency and liquidity of the CEA suggest that consumers may simply distrust that they will be paid back.

Cummins (2006) observes that the CEA has claims-paying ability of about \$7 billion, or just above one-third of the \$18 billion loss inflicted by the Northridge earthquake and states that "it is probable that the CEA could withstand damages on the scale of Northridge." However, this statement reflects solvency, not *liquidity*. In the presence of Jaffee-Russell type liquidty constraints, it is unlikely that the CEA's liquidity is sufficient to address a major disaster, even if one is willing to disregard the \$11 billion difference between the Northridge damage and CEA's paying ability. Since FEMA disaster assistance is not intended as a substitute for property insurance<sup>17</sup> and does not guarantee help in the event of claim denial,<sup>18</sup> homeowners' distrust in the paying ability of their insurer may be another demand-reducing factor in addition to the already high price.

Low-demand situations such as the California insurance puzzle provide a different, demand-side perspective on why a catastrophe insurance market can fail. So far, insurance market failures have been viewed as strictly supply-side phenomena driven entirely by firm exit, based on evidence of insurers discontinuing or limiting coverage after major disasters (e.g. in Ibragimov, Jaffee and Walden (2009) or Born and Viscusi (2006)). The California earthquake puzzle, however, suggests that there may be a demand-side component to market failures as well. A market failure can occur not only because nobody is selling, but also because nobody is buying, and the model highlights precisely this possibility.

<sup>&</sup>lt;sup>17</sup>For example, FEMA (2011) defines disaster assistance as follows: "Disaster assistance is money or direct assistance to individuals, families and businesses in an area whose property has been damaged or destroyed and whose losses are not covered by insurance. It is meant to help you with critical expenses that cannot be covered in other ways. This assistance is not intended to restore your damaged property to its condition before the disaster."

<sup>&</sup>lt;sup>18</sup>Owners of insured properties can receive FEMA assistance only if they have already received the maximum settlement from their insurer and can still demonstrate they have an unmet need, but even then aid is not guaranteed. See FEMA (2011).

# 1.7 Policy Implications and the Role of Reinsurance

I conducted the analysis in Section 1.4 based on the axiom that the insurer starts with no liquid reserves from previous years. Here I relax this assumption to allow an initial reserve of R, coming either from previous years or from an external source, such as a reinsurer or the government. I show that the qualitative result about the shape of demand remains largely intact, but highlights the importance of extra liquidity and particularly of reinsurance. The next proposition shows that having a reserve somewhat mitigates the degree of bending of the demand curve, depending on the reserve size. If the firm "inherits" (or receives) an extra R and keeps A(R) liquid in accordance with its liquidity rule, then the liquidity constraint function becomes  $\alpha(R + qx)$ , and the following result holds:

**Proposition 6.** As q converges to 0, individual demand  $x_i^*$  converges to a limit that satisfies

$$\lim_{q \to 0} x_i^*(q) \ge \frac{\alpha(R)}{\gamma} > 0.$$

*Proof.* In the Appendix.  $\Box$ 

Proposition 6 implies that extra liquidity shifts the backward-bending portion of the demand curve to the right. Consider the effect of providing extra liquidity in the case of the equilibrium  $(q^{**}, x^{**})$  in Fig. 1.1 (left). A reinsurer or the government provides additional liquidity  $\alpha(R)$ , thereby shifting demand's horizontal intercept to the right to a positive constant. At  $q^{**}$ , now there is excess demand which puts upward pressure on prices, resulting in recovery to the stable equilibrium  $(q^*, x^*)$ . A similar story holds for Figrue 1 (right): extra liquidity is sufficient to "walk" the system to a stable equilibrium even if it starts from the point (0,0). Moreover, even a very small amount of aid,  $\alpha(R) = \varepsilon > 0$  is sufficient to do the job. I interpret this

as a highlight of the important of the reinsurance market, which acts positively on consumer confidence not so much by providing actual liquidity but by its ability to do so, similar to the role of the FDIC in preventing bank runs.

#### **1.8** Other Risk Preferences

In the context of small-probability, large-destruction events such as catastrophes, one can argue that decision-makers may mentally "amplify" the probabilities associated with particularly bad outcomes; if so, expected utility is no longer the appropriate risk preference to use, as illustrated by a number of classical examples such as the Allais paradox. For instance, a decision-maker who exaggerates the probability of catastrophe may in fact want to purchase more, not less insurance, at low prices. Here I argue that my model's results remain intact for a large class of non-expected utility preferences, known as rank-dependent expected utility (Quiggin, 1982; Chew, Karni and Safra 1987), which accommodate the Allais paradox and other behavioral phenomena (Segal 1987a, 1987b).

The rank-dependent utility of a lottery  $X = (x_1, p_1, \dots, x_n, p_n)$  is defined as:

$$RD(X) = \sum_{i=1}^{n} u(x_i) \left[ \nu \left( \sum_{k=1}^{i} p_k \right) - \nu \left( \sum_{k=1}^{i-1} p_k \right) \right],$$

where  $\nu$  is a concave increasing differentiable function, called *probability transfor*mation function, from [0,1] onto [0,1], which satisfies  $\nu(0) = 0$ ,  $\nu(1) = 1$  (Segal 1987a; Chew, Karni and Safra, 1987). Rank-dependent risk preferences use "decision weights" instead of the underlying probabilities of each outcome. If one regards an outcome's probability as the change in cumulative probability in the lottery's distribution, then the rank-dependent functional assigns to each outcome a cumulative probability change  $\Delta \nu$ , interpreted as a decision weight. Decision weights allow an "amplification" of the importance of some outcomes relative to others in the decisionmaking process; for a survey of insurance problems using rank-dependent utility, see Dionne (2002).

The rank-dependent valuation of the insurance lottery X' faced by the customer is:

$$RD(X') = u(w_0 - qx_i)\nu(1 - p) + u(w_0 - L + (1 - q)x_i)[\nu(1 - pr) - \nu(1 - p)] + u(w_0 - L - qx_i)[1 - \nu(1 - pr)]$$

with associated first-order condition  $\mathcal{G}(x_i, x_{-i}) = 0$  which takes the form

$$-qu'(w_0 - qx_i)\nu(1 - p) + (1 - q)u'(w_0 - L + (1 - q)x_i)[\nu(1 - pr) - \nu(1 - p)]$$
$$-qu'(w_0 - L - qx_i)[1 - \nu(1 - pr)] = 0.$$

To reflect the new risk preference, I modify the definition of competitive equilibrium by requiring individuals to maximize their rank-dependent valuations of the lottery X', thereby replacing EU[X'] in Definition 1(a) with RD[X']. The payoff function of agent i in the game  $\Gamma$  is now

$$v_i = RD(X'(x_i, x_{-i}))$$

and as before, it remains strictly concave in the consumption of insurance  $x_i$  because

$$\begin{split} \frac{\partial^2 v_i}{\partial x_i^2} &= q^2 u''(w_0 - qx_i)\nu(1-p) + (1-q)^2 u''(w_0 - L + (1-q)x_i)[\nu(1-pr) - \nu(1-p)] + q^2 u''(w_0 - L - qx_i)[1 - \nu(1-pr)] < 0 \end{split}$$

as  $u''(\cdot) < 0$  and the increasing, differentiable probability transformation function  $\nu$  satisfies  $\nu(0) = 0, \nu(1) = 1$  so that the resulting weights satisfy  $\nu(1-p) > 0$ ,  $[\nu(1-pr)-\nu(1-p)] \ge 0$ , and  $[1-\nu(1-pr)] > 0$ , which implies a strictly negative sum.

Using these results, it is not difficult to formulate a result analogous to Proposition 1. To this end we only need to modify the part of Proposition 1's proof contained in Lemma 1 in the appendix. I denote its rank-dependent version with Lemma 1<sup>\*</sup>.

**Lemma 1\*.** When each agent *i* uses rank-dependent preferences to evaluate risk, then:

(a) The best response  $BR_i(x_{-i})$  is single-valued.

(b)  $BR_i(x_{-i})$  is differentiable and always weakly decreasing in  $x_{-i}$ ; if in addition, the liquidity constraint  $\alpha(t)$  is binding, then  $BR_i(x_{-i})$  is stictly decreasing in  $x_{-i}$ .

*Proof.* In the Appendix.  $\Box$ 

The remaining propositions carry over without modification. For example, Lemma 2 in the proof of Proposition 1 (existence of Nash equilibrium) uses only the fact that  $BR_i(x_{-i})$  is continuous and decreasing, which is preserved by the rank-dependent risk preference, so the proof carries over without change. The same is true of Propositions 3, 5, and 6, which do not refer to individual risk preferences. It remains only to verify Steps 2 and 3 that lead to the main result in Proposition 4, for the rank-dependent case; I relabel their rank-dependent versions to Step 2<sup>\*</sup> and 3<sup>\*</sup>, respectively.

**Step 2\*.** When r = 1, optimal consumption is  $x_i^* = 0$  at any q > 0.

*Proof.* The optimal choice of  $x_i^*$  using RD preferences and given r = 1 is the solution to

$$\max_{x_i} \left\{ u(w_0 - qx_i)\nu(1 - p) + u(w_0 - L - qx_i)[1 - \nu(1 - p)] \right\}$$
with first-order condition

$$-qu'(w_0 - qx_i)\nu(1 - p) - qu'(w_0 - L - qx_i)[1 - \nu(1 - p)] < 0 \qquad (\forall x \ge 0, \forall q > 0)$$

which imples that the maxium occurs at the left corner solution  $x_i^* = 0$ .

**Step 3\***. Optimal consumption  $x_i^* \to 0$  approaches  $\theta$  when  $q \to 0$ .

*Proof.* Since  $q \to 0$  implies  $r \to 1$  regardless of x, and since at r = 1 optimal consumption is 0, by continuity of RD preferences (proven in Lemma 3 in the Appendix) it follows that  $x_i^* \to 0$ .

The last two steps make sure the main result from Proposition 4 continues to hold and the demand curve retains its backward-bending shape, preserving all qualitative results obtained so far.

## 1.9 Conclusion

I study a strategic model of catastrophe insurance in which consumers take each other's actions into account, knowing that they are competing for the same, limited financial resource. The model shows that the demand for catastrophe insurance can "bend backwards," resulting in low-price, low-demand equilibria and especially in market failure, which is always an equilibrium. This highlights a new fact: that a competitive insurance market can fail entirely due to demand-side reasons, a scenario new to the catastrophe literature. The model also suggests that pricing is key for the credibility of catastrophe insurers: price-cutting techniques originally aimed at increasing demand can backfire and instead pull it down, leading to market failure. This further highlights the importance of a working reinsurance market for large risks. The backward-bending demand curve permits multiple equilibria with distinct stability properties. The typical low-price, low-demand equilibrium is unstable, but even small amounts of extra liquidity restore the system to stable equilibrium, highlighting the importance of a functioning reinsurance market. My model supplements existing supply-side studies by providing a demand-side perspective on the observed instability and frequent failures of catastrophe insurance markets.

# 1.10 Appendix: Proofs

**Proposition 1.** Given a price q > 0, at each firm k there exists a symmetric consumption profile  $\{x_i^*\}_{i \in S_k}$  such that  $x_i^* = \operatorname{argmax}_{x_i} EU[X']$  for each i, and  $x_i^* = x_j^* \quad (\forall i, j).$ 

Proposition 1 is proved as a sequence of two lemmas. I begin by looking at some properties of *i*'s best response to  $x_{-i}$ , which are helpful in establishing the existence of Nash equilibrium of the game  $\Gamma$  for a fixed price q > 0.

#### Lemma 1.

(a) The best response  $BR_i(x_{-i})$  of individual *i* is single-valued.

(b)  $BR_i(x_{-i})$  is differentiable and always weakly decreasing in  $x_{-i}$ ; if in addition the liquidity constraint  $\alpha(t)$  is binding, then  $BR_i(x_{-i})$  is stictly decreasing in  $x_{-i}$ .

*Proof.* (a) Observe that the payoff function is strictly concave in  $x_i$ :

$$\frac{\partial^2 EU(X'(x_i))}{\partial x_i^2} = q^2 (1-p) \underbrace{u''(w_0 - qx_i)}_{<0} + p(1-q)^2 [1-r] \underbrace{u''(w_0 - L + (1-q)x_i)}_{<0} + pq^2 r \underbrace{u''(w_0 - qx_i - L)}_{<0} < 0$$

since u is strictly concave and  $r(x_{-i})$  doesn't depend on  $x_i$ . A strictly concave function over a compact set  $x_i \in [0, L]$  always has a unique maximum. Therefore  $\operatorname{argmax}_{x_i} EU[X'(x_i|x_{-i})]$  is unique.

(b). The first-order condition  $G(x_i, x_{-i}) = 0$  implicitly defines the optimal choice of coverage as a function  $x_i^* = \varphi(x_{-i})$  w.r.t -i's action in a neighborhood of  $(x_i^*, x_{-i}^*)$ . By the implicit function theorem, in this neighborhood there exists the derivative

$$\frac{dx_i^*(x_{-i})}{dx_{-i}} = -\frac{\frac{\partial G}{\partial x_{-i}}(x_i^*, x_{-i}^*)}{\frac{\partial G}{\partial x_i}(x_i^*, x_{-i}^*)}.$$

The denominator of this fraction,  $\frac{\partial G}{\partial x_i}(x_i^*, x_{-i}^*) = \frac{\partial^2 EU(X'(x_i))}{\partial x_i^2} < 0$  by part (a), so the sign of the numerator determines that of  $\frac{dx_i^*(x_{-i})}{dx_{-i}}$ . Since the best response  $BR_i(x_{-i}) = \varphi(x_{-i})$  is single-valued,  $\varphi$  exists not only locally, but for every value of  $x_{-i}$ , so we can apply the implicit function theorem everywhere. When the constraint  $\alpha$  is binding so that  $\alpha(t)/t$  is decreasing, from the definition of r it is easy to derive that

$$\frac{\partial r}{\partial x_{-i}} = -\gamma \frac{qx_{-i}\alpha'(qx_{-i}) - \alpha(qx_{-i})}{(\gamma x_{-i})^2} > 0,$$

because the decreasing ratio  $\alpha(t)/t$  implies that  $t\alpha'(t) - \alpha(t) < 0$ . (Alternatively, for the case where  $\alpha(t) \equiv t$ , observe that then  $r = 1 - q/\gamma$  and so  $\partial r/\partial x_{-i} = 0$ .) Therefore, it is always true that  $\partial r/\partial x_{-i} \geq 0$  and hence

$$\frac{\partial G}{\partial x_{-i}}(x_i^*, x_{-i}^*) = -p(1-q)\frac{\partial r}{\partial x_{-i}}u'(w_2) - pq\frac{\partial r}{\partial x_{-i}}u'(w_3) \le 0.$$

From this it follows that  $dx_i^*/dx_{-i} \leq 0$  always holds; therefore  $BR_i(x_{-i})$  always weakly decreases in  $x_{-i}$ , and strictly decreases in  $x_{-i}$  whenever  $\alpha$  is binding. Continuity follows from the existence of the derivative  $\frac{dx_i^*(x_{-i})}{dx_{-i}}$ .

Next, observe that when we impose symmetry on individual outcomes  $x_i^* = x_j^* = \bar{x}$ at a given price, the response of rest of the firm's market simplifies to

$$R_{-i}(x_i) \equiv \int_{i \in S_k} x_i di = \bar{x} \int_{i \in S_k} i \ di = s_k \bar{x} = s_k x_i.$$

Using this fact we can prove the existence of Nash equilibrium in the game  $\Gamma$  for a given price q.

**Lemma 2.** Given a price q > 0, symmetric pure-strategy Nash equilibrium of the game  $\Gamma$  exists and is unique.

Proof. Observe that  $\varphi : [0, s_k L] \to [0, L]$  is weakly decreasing, continuous, and  $\varphi(0) > 0$  for all prices q at which  $BR_i \neq 0$ . I will show that there exists a unique  $x^* \in [0, s_k L]$  s.t.  $BR_i(x^*) = R_{-i}(x^*)$ , that is,  $\exists x^* : \varphi(x^*) = (1/s_k)x^*$ . Put  $\phi(x) = s_k \varphi(x) - x$  and observe that  $\phi(0) > 0$  while  $\phi(s_k L) \leq 0$ . Indeed  $\phi(0) = s_k \varphi(0) - 0 > 0$  and  $\varphi(s_k L) = s_k [\varphi(s_k L) - L] \leq 0$ , because  $0 \leq \varphi(\cdot) \leq L$ . By the intermediate value theorem, the continuous function  $\phi$  takes all intermediate values between  $\phi(0)$  and  $\phi(s_k L)$ . Since  $\phi(0) > 0$  and  $\phi(s_k L) \leq 0$ , there exists  $x^* \in [0, s_k L]$  for which  $\phi(x^*) = 0$ . It is trivial to verify that  $x^* \neq 0$ , since  $\phi(0) > 0$ implies  $\exists \varepsilon > 0$  such that  $\varphi(\varepsilon) > 0$ ; therefore  $x^* > 0$ .

**Uniqueness.** Suppose  $\exists$  two solutions  $x_1 < x_2$  for which  $\phi = 0$ . Then  $s_k \varphi(x_1) > s_k \varphi(x_2)$  since  $\varphi$  is non-increasing; then  $0 = \phi(x_1) > \phi(x_2) = 0$ , a contradiction.

Lemma 2 establishes the existence of a symmetric consumption profile  $\{x_i^*\}_{i\in S_k}$ among the clients of firm k, such that, for a given price, each agent is maximing expected utility of the lottery X', consumption choices are symmetric  $(x_i^* = x_j^*, \forall i, j)$ , and no agent has an incentive to deviate from  $x^*$ . This completes the proof of Proposition 1.

#### **Proposition 2.** Given a price q > 0:

a) Symmetric equilibrium consumption levels are the same across firms:  $x^{*,k} = x^{*,\ell}$ for every two firms k and  $\ell$ .

b) All firms have equal shares and equal amounts of consumers:  $s_k = s_\ell = \frac{1}{K} (\forall k, \ell)$ c) All firms have the same probability of non-payment:  $r_k = r_\ell$ ,  $(\forall k, \ell)$ .

*Proof.* (a) Suppose that optimal  $x^{*,k}$  at frim k satisfies  $x^{*,k} < x^{*,\ell}$  given the price q. Then k's clients will prefer to go to  $\ell$  by monotonicity, as at the same price they get strictly more coverage; the reverse argument also holds.

*Proof (b).* Suppose  $s_k > s_\ell$  at price q. Since symmetric consumption  $x^*$  is the same at each firm, then  $x^k = s_k x^* > s_\ell x^* = x^\ell$ , and

$$r_k = 1 - \frac{\alpha(s_k x^* q)}{\gamma x^*} < 1 - \frac{\alpha(s_\ell x^* q)}{\gamma x^*} = r_\ell$$

because  $\alpha$  is increasing, so  $\alpha(s_k x^* q) > \alpha(s_\ell x^* q)$ . Clients purchase the same coverage,  $x^*$ , at k and  $\ell$ , but the probability of rejection is higher at  $\ell$ ; therefore by first-order stochastic dominance, clients have an incentive to switch to k until it attracts all of firm  $\ell$ 's customers (as at the same price, k's probability of default is lower). This implies that the firm with the biggest share will capture the entire market, but that contradicts the assumption that there is more than one seller. (The same argument holds for the case where  $\alpha(t) = t$ .)

*Proof* (c). Follows as a corollary of parts a) and b).  $\blacksquare$ 

#### **Proposition 3.** The function

$$r(x,q) = 1 - rac{lpha(qx)}{\gamma x}$$
 satisfies  $\lim_{q \to 0} r(x,q) = 1$ 

on the domain  $x \in (0, s_k L]$ , where  $\alpha(t)$  is an increasing, continuous function s.t.  $\alpha(0) = 0$  and  $\alpha(t) \leq t$ .

Proof. Let  $\{q_n\}$  be a sequence of positive numbers such that  $q_n \to 0$ . Form a corresponding sequence of functions  $\{r_n\}$  with terms defined as  $r_n(x) = r(x, q_n)$ . Each  $r_n$  is now a function only of x. I will prove that the sequence of functions  $\{r_n\}$  converges to 1 uniformly on the domain  $[\delta, s_k L]$ , for an arbitrary small positive number  $\delta$ .

Fix  $\delta > 0$ , a small positive number, and consider r(x) on the domain  $[\delta, s_k L]$ . Since  $q_n \to 0$  and x is bounded, we have  $q_n x \to 0$ . Since  $\alpha$  is continuous,  $\lim_{q_n \to 0} \alpha(q_n x) = \alpha(\lim_{q_n \to 0} q_n x) = \alpha(0) = 0$ . Therefore, given some  $\varepsilon_1 > 0$ , exists N s.t. for  $n \ge N$ ,  $\alpha(q_n x) < \varepsilon_1$ . At the same time  $\gamma x \ge \gamma \delta$ . Putting  $\varepsilon_1 = \varepsilon \gamma \delta$ , for  $n \ge N$  we have

$$\sup_{[\delta, s_k L]} \left| \frac{\alpha(q_n x)}{\gamma x} \right| \le \frac{\varepsilon_1}{\gamma \delta} = \varepsilon \quad \text{for } n \ge N$$

which implies that for  $n \ge N$ 

$$\sup_{[\delta, s_k L]} |r_n(x) - 1| < \varepsilon$$

so the sequence of functions  $\{r_n(x)\}$  uniformly converges to 1. And since  $\delta$  is arbitrary, now let  $\delta \to 0$  to obtain convergence on the entire domain  $(0, s_k L]$ . Therefore  $\lim_{q\to 0} r(x,q) = 1$  everywhere on the domain  $x \in (0, s_k L]$ .

**Proposition 5.** There exists at least one price  $q^*$  that clears the market. In particular, the zero-price, zero-quantity combination ( $q^* = 0$ ,  $x^* = 0$ ) is always a strategic competitive equilibrium.

Proof. Recall that the quantity demanded x(q) is differentiable and therefore continuous w.r.t. the price q; we also know that  $\lim_{q\to 0} x(q) \to 0$ , and  $x(0) = R_{0+}$ , meaning that demand at q = 0 is multiple-valued (can be any number  $x \ge 0$ ). Next recall that the supply curve is a sum of firms' nondecreasing, continuous marginal cost schedules and so is nondecreasing, continuous in q. Moreover the quantity supplied at the zero price is always zero, because in the short run the firm will shut down when the price falls below average variable cost; hence the quantity supplied at the zero price is always zero. Since  $x^s(0) = 0$  and  $x^d(0) = R_{0+}$ , clearly the combination  $x^s = 0$  and  $x^d = 0$  always clears the market, so market failure (q = 0, x = 0) is always an equilibrium. Therefore at least one market-clearing price exists.

**Step 2**. When r = 1, optimal consumption is  $x_i^* = 0$  at any q > 0.

*Proof.* The optimal choice of  $x_i^*$  when given r = 1 is:

$$\max_{x_i} \left\{ (1-p)u(w_0 - qx_i) + pu(w_0 - L - qx_i) \right\}$$

FOC: 
$$-q(1-p)u'(w_0-qx_i) - pq \ u'(w_0-L-qx_i) < 0 \qquad (\forall x \ge 0, \ \forall q > 0)$$

Therefore the maximum occurs at x = 0.

**Step 3**. Optimal consumption  $x_i^* \to 0$  approaches 0 when  $q \to 0$ .

*Proof.* Since  $q \to 0, \Rightarrow r \to 1$  regardless of x, and since at r = 1 optimal consumption is 0, by continuity of EU preferences it follows that  $x_i^* \to 0$ .

**Proposition 6.** As q converges to 0, individual demand  $x_i^*$  converges to a limit that satisfies

$$\lim_{q \to 0} x_i^*(q) \ge \frac{\alpha(R)}{\gamma} > 0.$$

*Proof.* Since  $q \to 0$  and  $0 \le x \le L$ , it is trivial to verify that  $\lim_{q\to 0} [\alpha(R+qx)] = \alpha(R) > 0$ . Therefore in the limit  $q \to 0$ ,

$$r \equiv 1 - \frac{\alpha(R+qx)}{\gamma x} \to 1 - \frac{\alpha(R)}{\gamma \lim_{q \to 0} x(q)}.$$

Since r is bounded  $(0 \le r \le 1)$ , clearly  $\lim_{q\to 0} x(q) \ne 0$ , so the limit of x must be positive. Then

$$0 \le 1 - \frac{\alpha(R)}{\gamma \lim_{q \to 0} x(q)} \ \Rightarrow \ \alpha(R) \le \gamma \lim_{q \to 0} x(q) \ \Rightarrow \ \lim_{q \to 0} x(q) \ge \frac{\alpha(R)}{\gamma} > 0,$$

since  $\alpha(R) > 0$  and  $\gamma > 0$ .

**Lemma 1\*.** When each agent *i* uses rank-dependent preferences to evaluate risk, then:

(a) The best response  $BR_i(x_{-i})$  is single-valued.

(b)  $BR_i(x_{-i})$  is differentiable and always weakly decreasing in  $x_{-i}$ ; if in addition, the liquidity constraint  $\alpha(t)$  is binding, then  $BR_i(x_{-i})$  is stictly decreasing in  $x_{-i}$ .

*Proof.* (a) The strict concavity of the function  $v_i$  in  $x_i$  over the compact set  $x_i \in [0, L]$ implies that given a value for  $x_{-i}$ , then  $\operatorname{argmax}_{x_i} v_i(x_i|x_{-i})$  is unique; therefore, *i*'s best-response is single-valued.

(b). As before, the equation  $\mathcal{G}(x_i, x_{-i}) = 0$  implicitly defines optimal own consumption of insurance as a function of that of all other agents in a neighborhood of the point  $(x_i^*, x_{-i}^*)$ , so that  $x_i^* = \varphi(x_{-i})$ . The implicit function theorem guarantees that in this neighborhood exists the derivative

$$\frac{dx_i^*(x_{-i})}{dx_{-i}} = -\frac{\frac{\partial \mathcal{G}}{\partial x_{-i}}(x_i^*, x_{-i}^*)}{\frac{\partial \mathcal{G}}{\partial x_i}(x_i^*, x_{-i}^*)}.$$

whose denominator  $\partial \mathcal{G}/\partial x_i < 0$  by part (a). Since  $BR_i(x_{-i})$  is single-valued,  $\varphi$  exists not only locally but for every value of  $x_{-i}$  and therefore so does the above derivative. Since  $\frac{\partial r}{\partial x_{-i}} \geq 0$  remains true (from the proof of Proposition 1), we use this fact to establish

$$\frac{\partial \mathcal{G}}{\partial x_{-i}} = -p \frac{\partial r}{\partial x_{-i}} (1-q) u'(w_2) \nu'(1-pr) - pq \frac{\partial r}{\partial x_{-i}} u'(w_3) \nu'(1-pr) \le 0$$

where I also use the fact that u and  $\nu$  are increasing and differentiable. This determines the sign of the numerator of  $dx_i^*/dx_{-i}$  as nonnegative and implies that always  $dx_i^*/dx_{-i} \leq 0$ . In addition, whenever  $\alpha$  is binding,  $\partial r/dx_{-i} > 0$  implies that  $dx_i^*/dx_{-i} < 0$ , so the best response is differentiable, continuous and decreasing in  $x_{-i}$ .

#### Lemma 3. Rank-dependent preferences are continuous.

*Proof.* A preference relation  $\succeq$  over lotteries is continuous if for every money outcome m, there is exists a probability p that makes the lotteries (m, 1) and (0, p; a, 1 - p) exactly indifferent (where a is the maximal element of  $\{m\}$ ). We must therefore show

that for every m, there exists p such that RD(m, 1) = RD(0, p; a, 1-p). Using the definition of RD in section 1.8,

$$RD(m, 1) = u(m)[\nu(1) - \nu(0)] = u(m)$$
, because  $\nu(1) = 1$ , and

$$RD(0,p; \ a, 1-p) = u(0)[\nu(p) - \nu(0)] + u(a)[\nu(1) - \nu(p)] = u(a)[1 - \nu(p)].$$

We want to show that exists p that makes  $u(m) = u(a)[1 - \nu(p)]$ . Since m and a are fixed, put  $\xi \equiv u(m)/u(a)$ ; since  $m \leq a, \Rightarrow \xi \leq 1$ . We must show that  $\exists p$ :  $\nu(p) = 1 - \xi$ . Since  $1 - \xi \in [0, 1]$  and  $\nu$  is a continuous function from [0, 1] onto [0, 1], the intermediate value theorem guarantees that exists  $p \in [0, 1]$  s.t.  $\nu(p) = 1 - \xi$ .

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# Chapter 2

# How Uncertain Costs Affect Quality Differentiation in the Insurance Market

# 2.1 Introduction

Classical oligopoly models predict that firms differentiate product quality in order to soften price competition.<sup>1</sup> The conventional wisdom is that by making their products different, firms attract consumers of different types, and in doing so avoid prices being driven down to marginal cost.

However, in some oligopoly markets, quality differences are muted. One such example is the market for auto insurance, where the median auto policy looks very similar across sellers in different states, regardless of varying state-mandated minimum re-

<sup>&</sup>lt;sup>1</sup>For example, see Shaked and Sutton (1982) or Tirole (1988).

quirements.<sup>2</sup> Insurance markets also differ from conventional markets in two other respects: the fact that insurers are risk-averse and face uncertain costs.

In this paper, I propose a model that accounts for minimum quality differentiation by showing that it is generated by the cost uncertainty of risk-averse firms. Whereas cost uncertainty is a phenomenon not limited to one particular industry, insurance is a particularly relevant application, because, as argued by Wambach (1999) and Polborn (1998), uncertainty about accident probability can result in aggregate cost uncertainty even when accidents are independent and the insured pool is large. In addition, many authors consider insurance companies risk-averse, a view reinforced by modern finance and insurance theory (see Froot, Scharfstein and Stein, 1993, or Hardelin and Lemoyne-Deforges 2012). Thus, insurance companies are a good example of risk-averse firms with uncertain costs.

The paper is organized as follows. Section 2.2 discusses the motivation and Section 2.3 the related literature. Section 2.4 presents the model's basic assumptions. Section 2.5 derives the outcome of price competition, while Section 2.6 considers quality choice. Section 2.7 concludes the essay.

# 2.2 Motivation

Uncertain costs can occur in a variety of settings, but acquire special significance in the context of insurance. Textbook models often assume that when the number of insureds is large and risks are uncorrelated, there is no aggregate cost uncertainty; typically, laws of large numbers are invoked as a justification. However, that conclusion is an artifact of the stylized assumption that there is a single category of risk (e.g. only one type of auto accident, resulting only in vehicle damage). In reality, there is

 $<sup>^{2}</sup>$ See the discussion in Section 2.2 for more details.

more than one risk category associated even with a standard auto policy: for example, bodiliy injury liabilities, property damage liabitlities, uninsured motorist bodily injury, medical payments, collision vehicle damage, comprehensive risk, and personal legal protection. Therefore "risk" is an umbrella term that combines different types of risks having different probabilities, which have to be estimated from samples much smaller than the insured population. For example, to estimate the probability of damage to uninsured motorist, the insurer will look at the subsample of insureds to whom this particular accident type occured, which is much smaller than the insured pool. Therefore, laws of large numbers need not apply, and the estimate is subject to sampling error. Wambach (1999, p. 946) explains how this can lead to aggregate cost uncertainty despite the fact that individual risks are uncorrelated:

"One example of such a market [with uncertain costs] could be the insurance industry where the accident probability is not exactly known to the insurers [...], only a distribution over this probability exists. Thus, even if the number of insured people is large, so that individual uncertainty does not matter, the costs of all contracts N ( probability of loss) (size of the loss) is uncertain."

In reality, things are even more complicated. Real-world actuaries use regression tools to estimate the so-called *exceedance probability* (EP) curves – graphs depicting the probability that a certain level of loss will be exceeded on an annual basis. According to actuaries, "By its nature, the EP curve incorporates uncertainty associated with the probability of an event occurring and the magnitude of dollar losses. This uncertainty is reflected in the 5% and 95% confidence interval curves." (OECD 2005, p. 115; italics added). Actuaries will typically be interested in the 95% confidence bands around the regression line, so the fewer data points are available for a particular loss size, the wider is the estimated confidence interval. This means that if losses of

certain size occur more rarely than others, their probability will be estimated with a wider margin of error. In a typical insurance market, the largest-size losses (or "tail events") will occur most rarely, but that means their exceedance probabilities will be estimated least accurately. Therefore, uncertainty about accident probabilities is an issue, and concerns that it leads to aggregate cost uncertainty are legitimate.

Another reason to choose the insurance market as an application is that risk-aversion is an appropriate assumption for this industry. Whenever losses exceed estimates, insurers routinely borrow from capital markets or reinsurers, but this comes at a cost because verifying the true financial state of the borrower is costly. According to Froot, Scharfstein and Stein (1993), costly external finance is one of the leading factors causing firms to exhibit risk-aversion, although other factors such as corporate taxes and managers holding company stock also exist (Smith and Stulz, 1985). Because of this, many insurance authors prefer to model insurance firms as risk averters (in addition to Polborn (1998) and Wambach (1999), see Raviv (1979), Eliashberg and Winkler (1981), Blazenko (1985), among others).

Finally, insurance is an appropriate focus for a differentiated goods model because some insurance markets, such as the market for auto insurance, seem to exhibit surprisingly little quality differentiation. A database maintained by StateFarm<sup>3</sup> shows that the median auto policy sold by insurers looks strikingly similar across different states, despite varying state minima. For example, the required minimum coverage for bodily injury is \$ 50,000 in Maine, \$ 25,000 in Vermont, and \$ 20,000 in Massachusetts, but the median auto policy sold in each of these states covers 100,000 with a cap of 300,000 for this category, and this similarity persists both across remaining categories of coverage and across most other states. This sits well with the anecdotal observation that insurers do not like to portray themselves as cheap, affordable, and barely covering the state minimum, but on the contrary, try to project an image of

 $<sup>{}^3\</sup>mathrm{Available\ at\ }http://www.allstate.com/auto-insurance/state-coverages.aspx}$ 

solidity and reliability, which suggests a concentration of quality near the high end. As a main application of the model, I show that cost uncertainty can result exactly in such a drive to minimum differentiation when insurers are risk-averse.

## 2.3 Related Literature

There is a large literature on oligopolies facing uncertainty. One of the earliest analyses belongs to Sandmo (1971), who first formally showed that a risk-averse Bertrand competitor will charge a positive markup in any market with potential losses. In subsequent years, the effects of uncertainty have been studied in a variety of oligopoly contexts. The first generation of such papers focused mostly on risk-neutral oligopolists who feature product differentiation.

For example, Harrington (1992) considers risk-neutral firms who are uncertain of the degree of product homogeneity. He shows that uncertainty about product homogeneity is enough to cause equilibrium pricing above marginal cost. Bester (1998) introduces a model with both horizontal and vertical product differentiation and shows that uncertainty about vertical characteristics may result in a drive to horizontal agglomeration. Klemperer and Meyer (1986), by contrast, consider a model where the nature of competition (price or quantity) is determined endogenously, so firms can choose either a price to charge or a quantity to produce. Vives (1983) is concerned with information exchange between Cournot or Bertrand duopolists with differentiated goods who face uncertain demand, while Spulber (1995) focuses on Bertrand competition when the rival's costs are uncertain. In this literature, firms are riskneutral, so large variances of risk are inconsequential; implicit is the assumption that a monetary loss does not cause more disutility than the utility from an equivalent monetary gain. However, when losses trigger the need to borrow, which comes at an additional cost, firms may no longer act as risk-neutral. As shown by the influential work of Froot, Scharfstein and Stein (1993), costly external finance in and of itself is sufficient to cause risk-averse behavior on part of the firm, although other potential causes also exist. For example, Stulz (1984) argues that firm risk-aversion may stem from the risk aversion of managers who hold a relatively large portion of their wealth in the firm's stock. Smith and Stulz (1985) argue that corporate taxes, which are often convex in earnings, can also lead to risk-averse firm behavior, as a more volatile earnings stream leads to higher average taxes. Other convincing rationales for firm risk aversion also exist, such as the costs of bankruptcy (Smith and Stulz, 1985) and capital market imperfections. Because of this, the second generation of models, beginning roughly in the 90's, focus on risk-averse oligopolies.

To this generation of models belong papers by Wambach (1999), Polborn (1998), Hviid (1989), Kao and Hughes (1993), and Asplund (1995), most of which assume a homogeneous good. Hviid (1989) and Kao and Hughes (1993) are interested mainly in the information exchange between Cournot competitors, while Asplund (1995) studies whether risk aversion leads to softer or fiercer competition.

A key paper in this literature is Polborn (1998), who considers an insurance oligopoly facing uncertain costs. In Polborn's setup, competitors are risk-averse insurers competing in price, who sell a homogeneous good and whose marginal costs are uncertain because the probability of accident is not exactly known to them. This setup is motivated by the fact that in reality insurers always deal with estimates from a distribution, not with hard numbers; the idea is that even very small variances, when incorporated into the model, alter the nature of Bertrand competition. Polborn shows that under the classical assumptions (lowest-price firm captures entire market), Bertrand competition results in a continuum of equilibria, and considers criteria for equilibrium selection. Wambach (1999) employs a similar setup, but with a different equilibrium selection criterion.

This paper adopts the uncertain cost approach of Polborn (1998) and applies it to a setting with differentiated goods in order to study how cost uncertainty affects quality differentiation. I introduce Polborn's uncertain cost framework into a suitably modified framework of vertical differentiation (Tirole, 1988, p. 296), in which firms first choose product quality, and then compete on price. The solution concept I use for this two-stage model is subgame-perfect Nash equilibirum in pure strategies. I modify Tirole's model by introducing firms with risk-averse utility functions and uncertain marginal costs the way it is done in Polborn (1998), which significantly complicates the analysis. I consider two cases: when higher quality is costless, and when quality costs are convex.

In a risk-averse setting, firms care not only about expected profits, but also about their variances, both of which vary nontrivially with the degree of product differentiation. When we additionally make quality costly, tracking the effect on differentiation becomes mathematically challenging, which motivates the choice of a relatively simple differentiated goods framework such as Tirole (1988). Nonetheless, the model delivers several sharp results.

The conventional wisdom is that in order to soften price competition, firms differentiate their products in order to avoid prices being driven down all the way to marginal cost (Shaked and Sutton, 1982; Tirole, 1988). Thus, the classical literature explains the simultaneous existence of expensive, high-quality goods together with cheap, lowquality substitutes. By contrast, I find that uncertain costs fundamentally alter the nature of price competition by introducing a drive *against* quality differentiation. This economic force becomes particularly pronounced when consumers are somewhat picky about quality, and is capable of reversing conventional results by making it optimal for firms to differentiate *minimally* instead. In the realistic case when quality is costly, this setup is capable of producing multiple minimum-differentiation equilibria at both high and low quality levels.

### 2.4 Model

There are two firms, i and j, who can produce a good of quality  $s_i$  and  $s_j$  respectively, charging prices  $p_i$  and  $p_j$ . Chronologically, firms first choose product quality and then, given quality levels, compete on price. Quality can be either costless or costly (see sections 2.6.1 and 2.6.2); when costly, the firms pay for it in the first stage, before entering price competition. Quality costs are certain and known in advance; however, production costs (including insurer claim payments), are not.

Since production costs are uncertain, both the quality choice stage and the price competition stage take place before the realization of costs; firms use information only from the cost distribution, an approach standard in this literature (see Hardelin and Lemoyne-DeForges, 2012 and Polborn, 1998).

Quality is a positive real number  $s \in [\underline{s}, \overline{s}]$ . Since firm indices are arbitrary, we can assume that  $s_j > s_i$  so that the difference in quality always satisfies  $\Delta s \equiv s_j - s_i > 0$ . Due to the mathematics involved, the minimum differentiation case ( $\Delta s = 0$ ) will be defined as the limiting case of the model when  $\Delta s \to 0$ ; the limit will be shown to be well-behaved (see Appendix B).<sup>4</sup> To economize on notation, I will denote the frequently occurring expression  $\overline{\theta}\Delta s$  with  $\overline{\Delta}$  and the expression  $\underline{\theta}\Delta s$  with  $\underline{\Delta}$ .

On the demand side, there is a continuum of consumers with total mass one, who choose from which firm to buy and can purchase at most one unit of the good (this is

<sup>&</sup>lt;sup>4</sup>Alternatively, one can assume min  $\Delta s = \varepsilon > 0$ , which can be motivated as in Harrington (1992), who argues that quality cannot be reproduced perfectly. Results are very similar.

particularly relevant to insurance, where customers typically buy at most one policy per property).

Consumers care both about the price p and the quality s of the good consumed. Each consumer's preferences are described with

$$U(s,p) = \begin{cases} \theta s - p & \text{if buys 1 unit} \\ 0 & \text{if no purchase} \end{cases}$$

where  $\theta$  is a taste parameter regulating how sensitive the given consumer is to the quality *s*. Utility is separable in price and quality and should be interpreted as the surplus from the consumption of the good (Tirole 1988, p.96). For a given fixed price, each consumer prefers a higher-quality good to a lower-quality one, but those consumers who have higher values of  $\theta$  are more willing to pay for high quality (or equivalently, derive higher surplus). The positive number  $\theta$  is distributed uniformly over the interval  $[\underline{\theta}, \overline{\theta}]$ , where  $\overline{\theta} = \underline{\theta} + 1$  to ensure unit mass.

Following Tirole (1988), I impose two standard assumptions to guarantee that the model outcome is non-degenerate:

#### Assumption 1 (Heterogeneity): $\overline{\theta} > 2\underline{\theta}$ .

This assumption makes sure that, roughly speaking, there is enough consumer heterogeneity to rule out degenerate model behavior (negative equilibrium profits and markups). Notice that since  $\overline{\theta} = \underline{\theta} + 1$ , and  $\underline{\theta}$  is positive, this assumption restricts the range of  $\underline{\theta}$  between  $0 < \underline{\theta} < 1$ .

Assumption 2 (Market is covered): Minimum quality  $\underline{s}$  is large enough that no firm faces zero demand in equilibrium.

Assumption 2 guarantees that the market is a duopoly as opposed to a monopoly. If minimum quality  $\underline{s}$  is so low that at the equilibrium price  $p_i^*$  and quality  $\underline{s}$  Firm *i*'s clients prefer to buy nothing (which will happen if  $p_i^* > \underline{\theta} \underline{s}$ ), then Firm *i* faces the possibility of zero demand, and the model is no more a duopoly. For this we require minimum quality  $\underline{s}$  to be high enough so that  $p_i^* < \underline{\theta} \underline{s}$ . (As will be seen from the expression for equilibrium prices, it is always possible to pick such an  $\underline{s}$ .)

When the market is covered, consumers can choose between a higher- and a lowerquality good. A consumer of type  $\theta$  is exactly indifferent between the two goods when

$$\theta s_i - p_i = \theta s_j - p_j \quad \Leftrightarrow \quad \theta(s_j - s_i) = p_j - p_i.$$

Therefore, given prices and qualities, those consumers with  $\theta > \frac{p_j - p_i}{\Delta s}$  will buy the higher-quality good, and those with  $\theta < \frac{p_j - p_i}{\Delta s}$ , the lower-quality good, effectively splitting the interval  $[\underline{\theta}, \overline{\theta}]$  in two parts: those types  $\theta$  who go to Firm *i*, and those who go to Firm *j*. Since  $\theta$  is uniformly distributed over the interval  $[\underline{\theta}, \overline{\theta}]$  of length 1, and since quantity demanded is 1 unit for each person who buys, demand at each firm simply equals the measure (length) of consumers positioned on  $[\underline{\theta}, \overline{\theta}]$  buying from each respective firm:

$$D_i(p_i, p_j) = \frac{p_j - p_i}{\Delta s} - \underline{\theta}; \qquad D_j(p_j, p_i) = \overline{\theta} - \frac{p_j - p_i}{\Delta s}.$$

Each firm's profit equals its quantity demanded times the markup charged above marginal cost c:

$$\Pi_i = (p_i - c) \left[ \frac{p_j - p_i}{\Delta s} - \underline{\theta} \right]; \qquad \Pi_j = (p_j - c) \left[ \overline{\theta} - \frac{p_j - p_i}{\Delta s} \right].$$

So far the setup is the same as in Tirole (1988), where all costs are certian. However, we are interested in price competition with uncertain costs. Following Polborn (1998), I introduce an uncertain marginal cost c, which is normally distributed with mean  $\mu_c$ and variance  $\sigma^2$ , which results in profit uncertainty:

$$c \sim N(\mu_c, \sigma^2) \implies \Pi \sim N(\mu_\Pi, \sigma_\Pi^2)$$

In his paper "A model of an Oligopoly in an Insurance Market," Polborn (1998) argues that this setup is particularly applicable to insurance companies when the accident probability is uncertain: when the number of insureds is large, losses will be approximately normally distributed, carrying normality over to marginal costs. The realization of costs at the two firms is assumed independent, which will be true for most lines of non-catastrophe insurance (there are no common shocks).

For the resons explained in Section 2.2 and discussed in detail by Froot, Scharfstein, and Stein (1993) and Smith and Stulz (1985), firms are assumed risk averse and maximize the expected utility of uncertain profits. The two firms have identical von Neumann-Morgenstern utility functions with constant absolute risk aversion (CARA)

$$u(\Pi_k) = -e^{-r\Pi_k} \quad (k = i, j),$$

where r is the coefficient of absolute risk aversion.

The choice of CARA utility with a normal distribution of costs is motivated mainly by the fact that its expectation has a closed-form solution in terms of the distribution's mean and variance. As shown in Freund (1956), if x is a normally distributed random variable  $x \sim N(\mu_x, \sigma_x^2)$ , then

$$\mathbb{E}[-e^{-rx}] = -\exp\left(-r\mu_x + \frac{1}{2}r^2\sigma_x^2\right).$$

Thus, the expectation of a CARA utility function can be expressed in terms of the mean and variance of the random variable. In the firm's maximization problem, the uncertain variable is profits, which have means and variances as follows:

$$\mu_{\Pi_i} = (p_i - \mu_c) \frac{p_j - p_i - \underline{\Delta}}{\Delta s} \qquad \qquad \mu_{\Pi_j} = (p_j - \mu_c) \frac{\overline{\Delta} - p_j + p_i}{\Delta s}$$
$$\operatorname{Var}(\Pi_i) = \left[\frac{p_j - p_i - \underline{\Delta}}{\Delta s}\right]^2 \sigma^2 \qquad \qquad \operatorname{Var}(\Pi_j) = \left[\frac{\overline{\Delta} - p_j + p_i}{\Delta s}\right]^2 \sigma^2$$

Substituting mean profits and their variances into the firms' utility functions yields expected utilities

$$\mathbb{E}u(\Pi_i) = -\exp\left\{-r(p_i - \mu_c)\frac{p_j - p_i - \underline{\Delta}}{\Delta s} + \frac{1}{2}r^2\sigma^2\left[\frac{p_j - p_i - \underline{\Delta}}{\Delta s}\right]^2\right\}$$
$$\mathbb{E}u(\Pi_j) = -\exp\left\{-r(p_j - \mu_c)\frac{\overline{\Delta} - p_j + p_i}{\Delta s} + \frac{1}{2}r^2\sigma^2\left[\frac{\overline{\Delta} - p_j + p_i}{\Delta s}\right]^2\right\}$$

where I will often denote the expression within the exponent as  $\phi_i$  and  $\phi_j$ , respectively. In the price competition stage, qualities have already been selected so they are treated as given. Firms compete on price by solving:

$$\max_{p_i} \mathbb{E}u(\Pi_i) \quad \text{and} \quad \max_{p_j} \mathbb{E}u(\Pi_j).$$

# 2.5 Price Competition

In order to find maxima using first-order conditions, the expected utility functions must be concave, which is ensured by the next proposition.

**Proposition 1.** Both firms' utility functions are concave with respect to the own price. In particular,

- a)  $\mathbb{E}u(\Pi_i)$  is strictly concave in  $p_i$
- b)  $\mathbb{E}u(\Pi_j)$  is strictly concave in  $p_j$ .

*Proof.* In the Appendix.

If we denote the expected utility functions' exponents above as  $\phi_i$  and  $\phi_j$  respectively, the first-order condition for Firm *i* can be reduced to

$$\frac{d\mathbb{E}u(\Pi_i)}{dp_i} = -e^{\phi_i}\frac{d\phi_i}{dp_i} = 0 \quad \Longleftrightarrow \quad \frac{d\phi_i}{dp_i} = 0$$

which, written fully, takes the form

$$-r\left[\frac{p_j - 2p_i - \underline{\Delta} + \mu_c}{\Delta s}\right] + r^2 \sigma^2 \left[\frac{-p_j + p_i + \underline{\Delta}}{(\Delta s)^2}\right] = 0.$$

This yields Firm *i*'s reaction function in terms of  $p_j$ :

$$p_i(p_j) = \frac{\mu_c \Delta s + (r\sigma^2 + \Delta s)(p_j - \underline{\Delta})}{r\sigma^2 + 2\Delta s}.$$

Similarly Firm j's first-order condition reduces to

$$-r\left[\frac{\overline{\Delta}-2p_j+p_i+\mu_c}{\Delta s}\right]+r^2\sigma^2\left[\frac{p_j-p_i-\overline{\Delta}}{(\Delta s)^2}\right]=0,$$

resulting in the reaction function

$$p_j(p_i) = \frac{\mu_c \Delta s + (r\sigma^2 + \Delta s)(p_i + \overline{\Delta})}{r\sigma^2 + 2\Delta s}.$$

Solving the system of reaction functions results in the following equilibrium prices:

$$p_i^* = \mu_c + \frac{(r\sigma^2 + \Delta s)[r\sigma^2 + (\overline{\theta} - 2\underline{\theta})\Delta s]}{2r\sigma^2 + 3\Delta s}$$
$$p_j^* = \mu_c + \frac{(r\sigma^2 + \Delta s)[r\sigma^2 + (2\overline{\theta} - \underline{\theta})\Delta s]}{2r\sigma^2 + 3\Delta s}$$

where I will often denote the equilibrium markup as  $m_k^* \equiv (p_k^* - \mu_c)$ .

First observe that, unlike the risk-neural case, risk averse firms charge a positive markup above marginal cost, because risk-averse agents require compensation in order to hold risk (Sandmo, 1971). This softens price competition a bit, but as will be seen, not enough to reverse standard results by itself: absent additional incentives, firms will still find it optimal to differentiate maximally even when they are risk averse. As in Tirole (1988), the higher-quality firm earns higher profits and charges a higher markup, as  $(2\overline{\theta} - \underline{\theta}) > (\overline{\theta} - 2\underline{\theta})$ :

$$\Pi_i^* = (r\sigma^2 + \Delta s) \left[ \frac{r\sigma^2 + (\overline{\theta} - 2\underline{\theta})\Delta s}{2r\sigma^2 + 3\Delta s} \right]^2, \qquad \Pi_j^* = (r\sigma^2 + \Delta s) \left[ \frac{r\sigma^2 + (2\overline{\theta} - \underline{\theta})\Delta s}{2r\sigma^2 + 3\Delta s} \right]^2$$

and

$$\operatorname{Var}(\Pi_i^*) = \left[\frac{r\sigma^2 + (\overline{\theta} - 2\underline{\theta})\Delta s}{2r\sigma^2 + 3\Delta s}\right]^2 \sigma^2, \qquad \operatorname{Var}(\Pi_j^*) = \left[\frac{r\sigma^2 + (2\overline{\theta} - \underline{\theta})\Delta s}{2r\sigma^2 + 3\Delta s}\right]^2 \sigma^2.$$

It is also intuitive that the higher the firm's risk aversion (or the variance of costs), the higher the equilibrium markup. Product quality, however, does not always have unambiguous effects on equilibrium profits. The expressions above reveal an interesting tradeoff which is essential in the analysis that follows. Whereas Firm j's profit always increases in own quality, the effect of own quality on Firm i's profits is generally ambiguous. Nonetheless, above a certain quality level,  $s_i$  begins to increase Firm i's profits, so after some point both firms get better profits from higher quality. Higher profits, however, come at the cost of increased variance, as shown in the next proposition.

#### Proposition 2. All else equal,

- a)  $\Pi_i^*$  is increasing in  $s_j$ ,  $(\forall s_i \in [\underline{s}, \overline{s}])$
- b) The overall effect of  $s_i$  on  $\Pi_i^*$  is ambiguous. However, above some quality  $s_i^0$ , Firm *i*'s quality starts to increase profits  $(\partial \Pi_i^* / \partial s_i > 0)$ .
- c) Both firms' equilibrium variances increase in own qualities.

#### *Proof.* In the Appendix.

Proposition 2 suggests an interesting tradeoff. Above some quality level, higher quality leads to higher expected profits for both firms, but extra profits come at the cost of increased variance. This creates a tradeoff between high, variable profits, versus lower, but more predictable ones. If Firm i offers lower quality, it also earns lower profit than Firm j, but it is very predictable. If Firm i offers a higher quality closer to Firm j, its profits may start growing, but so will their variance. Without a quantitative analysis, it is not obvious which combination is better. Therefore, forces acting both for and against quality differentiation exist, and the model's task is to find out which force prevails and when. To find out, we need to look at the effect of quality on equilibrium utility.

Substituting the profit and variance equations into the firms' utility functions, the utilities attained in Bertrand equilibrium are:

$$V_{i} = \mathbb{E}u(\Pi_{i}^{*}) = -\exp\left\{-\left(r\Delta s + \frac{1}{2}r^{2}\sigma^{2}\right)\left[\frac{r\sigma^{2} + (\overline{\theta} - 2\underline{\theta})\Delta s}{2r\sigma^{2} + 3\Delta s}\right]^{2}\right\}$$
$$V_{j} = \mathbb{E}u(\Pi_{j}^{*}) = -\exp\left\{-\left(r\Delta s + \frac{1}{2}r^{2}\sigma^{2}\right)\left[\frac{r\sigma^{2} + (2\overline{\theta} - \underline{\theta})\Delta s}{2r\sigma^{2} + 3\Delta s}\right]^{2}\right\}$$

Having computed what prices, profits and utilities will prevail in the Bertrand equilibrium, firms proceed to find optimal qualities by backward induction.

# 2.6 Quality Choice

#### 2.6.1 Costless Quality

I consider two cases: when quality is costless (higher quality doesn't cost more), and when quality costs are convex (quadratic). It is assumed that the firm pays the costs of quality at the same time when it selects the quality level and before entering the price competition stage. The costless quality case isolates the forces acting for and against differentiation more clearly, so it is considered first.

In subgame-perfect Nash equilibrium, each firm knows it has no incentive to deviate from the Bertrand equilibrium prices and quantities, which are parametrized by quality levels. Therefore when deciding on optimal quality, the firms solve

$$\max_{s_i} \mathbb{E}u(\Pi_i^*|s_j), \qquad \max_{s_j} \mathbb{E}u(\Pi_j^*|s_i).$$

Just as in the standard model (Tirole, 1988), when quality is costless, it will turn out that the higher-quality firm (Firm j) wants to pick the maximum quality regardless of what Firm i does; the explanation for this, however, is different. Recall that by Proposition 2, equilibrium variances increase with own quality; therefore, by increasing quality, the firm increases not only its expected profits, but also their variance. The model tells us that in this particular case, the profit effect dominates over that of variance. Interestingly, this remains true regardless of the degree of risk-aversion r and the variance of risk  $\sigma^2$ . **Proposition 3.** When quality is costless, Firm j will always choose the highest quality  $\overline{s}$ .

*Proof.*  $dV_j/ds_j > 0$  regardless of  $s_i$  (see Appendix).

However, model results begin to diverge from the classical paradigm for the *lower-or-equal-quality* Firm i. The next proposition shows that, by contrast, the outcome for Firm i begins to depend on how much consumers care for quality.

**Proposition 4.** With costless quality,

- a) When consumers are picky enough ( $\underline{\theta}$  exceeds some  $\theta_*$ ), Firm i will choose minimum differentiation regardless of  $r, \sigma^2$ .
- b) Moreover, such a critical value  $\theta_*$  always exists.
- c) When  $\underline{\theta}$  falls below some  $\theta_{**}$ , Firm *i* chooses **maximum differentiation**. Moreover, such a  $\theta_{**} > 0$  always exists as long as  $r\sigma^2 > 3$ .

*Proof.* In the Appendix.

Proposition 3 tells us that we can still get the maximum-differentiation result back, but it will hold only if consumers are not too picky (their quality parameter  $\theta$  starts relatively low). When  $\theta$ 's range starts relatively high, however, the classical result is reversed and the equilibrium outcome is *minimim* differentiation at the point  $\overline{s}$  where both firms offer maximum quality.

To understand this result, first note that Firm *i*'s equilibrium demand  $D_i^*$  and equilibrium markup  $m_i^*$  are both decreasing in  $\underline{\theta}$ :

$$\frac{\partial D_i^*}{\partial \underline{\theta}} = -\frac{\Delta s}{2r\sigma^2 + 3\Delta s} \le 0; \quad \frac{\partial m_i^*}{\partial \underline{\theta}} = -\frac{(r\sigma^2 + \Delta s)[2r\sigma^2 + 3\Delta s]\Delta s}{(2r\sigma^2 + 3\Delta s)^2} \le 0.$$

Therefore, for a fixed quality, higher  $\underline{\theta}$  unambiguously lowers equilibrium profits  $(\partial \Pi_i^* / \partial \underline{\theta} \leq 0$  by the chain rule). Then, if faced by picky consumers with high  $\underline{\theta}$ , Firm *i* may try to improve profits by instead offering higher quality in order to capture more consumers to boost demand, because

$$\frac{\partial D_i^*}{\partial s_i} = -\frac{[2(1-\underline{\theta})-3]r\sigma^2}{(2r\sigma^2+3\Delta s)^2} > 0.$$

Higher demand alone, however, may not be enough to increase profits (relative to the maximum differentiation case) because profit also depends on the equilibirum markup.

The behavior of the equilibrium markup is the decisive factor determining whether the traditional result will reverse. At low levels of  $\underline{\theta}$ , equilibrium markup falls in own quality, stimulating the firm to stay at the lowest-quality point  $\underline{s}$ ; this is how we get the classical, maximum-differentiation equilibrium in part (c). At high enough levels of  $\underline{\theta}$ , however, equilibrium markup begins to increase in  $s_i$ , so now both demand and the markup pull profits in the upper direction. This can be seen from the behavior of the derivative

$$\frac{\partial m_i^*}{\partial s_i} = -\frac{(1-2\underline{\theta})r^2\sigma^4 + (1-\underline{\theta})[3(\Delta s)^2 + 4r\sigma^2\Delta s)}{(2r\sigma^2 + 3\Delta s)^2}.$$

When  $\underline{\theta} < 1/2$ , the expression  $1 - 2\underline{\theta}$  is positive and  $\frac{\partial m_i^*}{\partial s_i} < 0$ , so higher quality lowers the markup, although it increases demand. If we keep lowering  $\underline{\theta}$ , eventually there comes a point  $\theta_{**}$  where the markup effect prevails over that of demand: at low levels of  $\underline{\theta}$  profits are falling in own quality, which stimulates the firm to stay at the left corner  $\underline{s}$ . The fact that the variance of equilibrium profits increases in own quality only reinforces this result, because Firm *i*'s profit variance is lowest at  $\underline{s}$ ; this is the intuition behind the traditional result in part (c). However, when  $\underline{\theta} > 1/2$ , the numerator is positive and markup begins to *increase* in own quality:  $\frac{\partial m_i^*}{\partial s_i} > 0$ . Therefore at high values of  $\underline{\theta}$ , offering higher quality increases both demand and the equilibrium markup, pushing up profits as well. This is the main force behind the drive to minimum differentiation in part (a). However, profits alone cannot tell us whether high quality is optimal, because quality-driven profits come at the cost of increased variance (see Proposition 2(c)). Based on intuition alone, one cannot predict which profit-variance combination is better, or whether the benefits of non-differentiation exceed those of differentiation. The model resolves this tradeoff by showing that when  $\underline{\theta} > \theta_*$ , Firm *i* will unambiguously shoot for the high-quality, minimum differentiation option. This tells us that the utility from extra profits dominates disutility from increased variance, something that is not obvious. It is especially striking that this result is true regardless of the cost variance  $\sigma^2$  and the risk aversion *r*.

#### 2.6.2 Costly Quality

The previous discussion permits both firms to shoot for the highest quality, because quality is assumed to be costless. In more realistic settings, however, achieving high quality requires costly investments. For example, if we measure the quality of an insurance policy by coverage levels in each risk category above the required minimum, offering higher coverage means maintaining higher reserves, which involves higher opportunity costs (every liquid dollar could instead be invested to earn interest). Therefore high quality costs more in the insurance business just as in other industries. The question naturally arises whether with costly quality, the previous results are preserved.

To study this case, I introduce certain quality costs which are paid at the qualitychoice stage (before price competition). Quality costs are commonly assumed convex (higher quality costs increasingly more); I focus on the quadratic quality case as mathematically more tractable.

Assume that both firms can invest in product quality s at cost  $C(s) = bs^2$ , where b > 0. Since quality costs are certain and are paid before entering price competition, they reduce equilibrium profits one-for-one without changing their variance. The equilibrium utilities attained in Bertrand equilibrium therefore are now

$$V_{i} = -\exp\left\{-\left(r\Delta s + \frac{1}{2}r^{2}\sigma^{2}\right)\left[\frac{r\sigma^{2} + (\overline{\theta} - 2\underline{\theta})\Delta s}{2r\sigma^{2} + 3\Delta s}\right]^{2} + rbs_{i}^{2}\right\}$$
$$V_{j} = -\exp\left\{-\left(r\Delta s + \frac{1}{2}r^{2}\sigma^{2}\right)\left[\frac{r\sigma^{2} + (2\overline{\theta} - \underline{\theta})\Delta s}{2r\sigma^{2} + 3\Delta s}\right]^{2} + rbs_{j}^{2}\right\}$$

Each firm's solution to the problem  $\max_{s_k} V_k(s_k|s_{-k})$  (k = i, j) provides its reaction function with respect to opponent's quality. Unfortunately, even with quadratic quality, these reaction functions are highly non-linear, so the Nash equilibrium solutions are analytically untractable. However, it is possible to make at least partial conclusions about minimum-differentiation equilibria at the lowest and highest-quality point.

**Proposition 5.** When quality costs for both firms are given by  $C(s) = bs^2$ ,

- a) If  $\underline{\theta} < 1/2$  and  $b > \frac{5}{16s}$ , both firms choose minimum quality with minimal differentiation.
- b) If  $\underline{\theta} > \frac{1}{2} + 8b\overline{s}$  and  $b < \frac{1}{16\overline{s}}$ , both firms choose maximum quality with minimal differentiation.

#### *Proof.* In the Appendix.

Proposition 5 confirms that the relationship between consumer selectivity  $\underline{\theta}$  and minimum differentiation is not an artifact of the costless quality assumed in section 2.6.1. Again, high values of  $\underline{\theta}$  are conducive to a concentration of quality at the high point  $\overline{s}$ . In addition, however, now a minimum-differentiation equilibrium also becomes possible at the low quality point  $\underline{s}$  when consumers are not very quality-sensitive. Interior quality equilibria with minimum differentiation likely also exist, but their analysis is complicated by the reaction functions' non-linearity, coupled with the fact that the functions  $V_k$  can switch from concave to convex (in quality) depending on parameter combinations. Nonetheless, Proposition 5 carries a very important message: that the same force against product differentiation not only survives, but gets amplified in the costly quality setting, leading to an even richer set of non-differentiation outcomes at both high and low quality levels. This reinforces the notion that cost uncertainty is a fundamental driver behind low quality differentiation in risk-averse industries such as insurance. The model's predictions are especially consistent with the low quality differentiation in the market for auto insurance polcies, suggested by some market surveys and casual observation.

# 2.7 Conclusion

This chapter develops a model of quality differentiation in insurance markets, focusing on two of their specific features: the fact that costs are uncertain, and the fact that firms are averse to risk. Cornerstone models of price competition predict that firms specialize in products of different quality as a way of softening price competition. However, some real-world insurance markets feature very little quality differentiation. This chapter offers an explanation to this phenomenon by showing that cost uncertainty fundamentally alters the nature of price competition among risk-averse firms by creating a drive against differentiation. This force becomes particularly pronounced when consumers are picky about quality, and is capable of reversing standard results, leading to minimum differentiation instead. This result is preserved regardless of whether quality is costly or costless, but in the costly quality case the set of minimum-differentiation outcomes is richer.
# 2.8 Appendix A: Proofs

**Proposition 1.** Both firms' utility functions are concave with respect to the own price. In particular,

- a)  $\mathbb{E}u(\Pi_i)$  is strictly concave in  $p_i$
- b)  $\mathbb{E}u(\Pi_j)$  is strictly concave in  $p_j$ .

*Proof.* Recall that

$$\mathbb{E}u(\Pi_i) = -\exp\left\{-r(p_i - \mu_c)\frac{p_j - p_i - \underline{\Delta}}{\Delta s} + \frac{1}{2}r^2\sigma^2\left[\frac{p_j - p_i - \underline{\Delta}}{\Delta s}\right]^2\right\} = -e^{\phi_i}$$
$$\mathbb{E}u(\Pi_j) = -\exp\left\{-r(p_j - \mu_c)\frac{\overline{\Delta} - p_j + p_i}{\Delta s} + \frac{1}{2}r^2\sigma^2\left[\frac{\overline{\Delta} - p_j + p_i}{\Delta s}\right]^2\right\} = -e^{\phi_j}$$

labeling the expressions inside the exponents as  $\phi_i$  and  $\phi_j$  respectively. I will first show that  $\phi_i$  is convex in  $p_i$  and therefore  $d^2 \mathbb{E} u(\Pi_i)/dp_i^2 < 0$ .

An inspection of the derivatives of  $\phi_i$  shows that

$$\frac{d\phi_i}{dp_i} = -r \left[ \frac{p_j - 2p_i - \underline{\Delta} + \mu_c}{\Delta s} \right] + \sigma^2 r^2 \frac{p_i - p_j + \underline{\Delta}}{\Delta s}$$
$$\frac{d^2 \phi_i}{dp_i^2} = \frac{2r}{\Delta s} + \sigma^2 \left( \frac{r}{\Delta s} \right)^2 > 0,$$

therefore  $\phi_i$  is strictly convex in  $p_i$ . Taking this into account, we can now look at the sign of

$$\frac{d^2}{dp_i}\left[-e^{\phi_i^2}\right] = -e^{\phi_i}\left[\left(\frac{d\phi_i}{dp_i}\right)^2 + \frac{d^2\phi_i}{dp_i^2}\right] < 0$$

which is negative because  $-e^{\phi_i} < 0$  while the convexity of  $\phi_i$  ensures that  $\frac{d^2\phi_i}{dp_i^2} > 0$ .

The same procedure applied to firm j reveals that  $\phi_j$  is likewise convex in  $p_j$  and therefore

$$\frac{d^2}{dp_j^2}[-e^{\phi_j}] = -e^{\phi_j} \left[ \left(\frac{d\phi_j}{dp_j}\right)^2 + \frac{d^2\phi_j}{dp_j^2} \right] < 0. \quad \blacksquare$$

**Proposition 2.** All else equal,

- a)  $\Pi_j^*$  is increasing in  $s_j$ ,  $(\forall s_i \in [\underline{s}, \overline{s}])$
- b) The overall effect of  $s_i$  on  $\Pi_i^*$  is ambiguous. However, above some quality  $s_i^0$ , Firm *i*'s quality starts to increase profits  $(\partial \Pi_i^* / \partial s_i > 0)$ .
- c) Both firms' equilibrium variances increase in own qualities.

*Proof.* a) The partial derivative

$$\begin{aligned} \frac{\partial \Pi_j^*}{\partial s_j} &= \\ &= \frac{4(1+\underline{\theta})r^3\sigma^6 + 3(2+\underline{\theta})^2(\Delta s)^3 + 6r\sigma^2(2+\underline{\theta})^2(\Delta s)^2 + r^2\sigma^2(4\underline{\theta}^2 + 18\underline{\theta} + 17)\Delta s}{(3\Delta s + 2r\sigma^2)^3} > 0 \end{aligned}$$

is strictly positive because all terms are  $\geq 0$ , while  $4(1 + \underline{\theta})r^3\sigma^6 > 0$ .  $\Box$ 

b) The partial of  $\Pi_i^*$  with respect to  $s_i$  is:

$$\frac{\partial \Pi_i^*}{\partial s_i} = -\frac{3[1+2r\sigma^2](1-\underline{\theta})^2(\Delta s)^2 + r^2\sigma^2(4\underline{\theta}^2 - 10\underline{\theta} + 3)\Delta s - 4\underline{\theta}(r\sigma^2)^3}{(3\Delta s + 2r\sigma^2)^3}.$$

In the general case, the sign of  $\frac{\partial \Pi_i^*}{\partial s_i}$  is ambiguous because it depends on the values of  $\Delta s, \underline{\theta}, r$  and  $\sigma^2$ . However observe that as  $\Delta s \to 0$ ,

$$\frac{\partial \Pi_i^*}{\partial s_i} \to \frac{4\underline{\theta}(r\sigma^2)^3}{3\Delta s + 2r\sigma^2} > 0.$$

Recalling that  $\Delta s \equiv s_j - s_i$ , next observe that  $\partial \Pi_i^* / \partial s_i$  is continuous in  $s_i$ , which imples that for any given level of  $s_j$ , there exists a neighborhood where the function's sign is preserved. Therefore there is some  $s_i^0 < s_j$  above which the sign of  $\partial \Pi_i^* / \partial s_i$ remains positive:

Given 
$$s_j$$
,  $\exists s_i^0 : \partial \Pi_i^* / \partial s_i > 0$  for  $s_i > s_i^0$ .  $\Box$ 

c) If we put

$$\left[\frac{r\sigma^2 + (\overline{\theta} - 2\underline{\theta})\Delta s}{2r\sigma^2 + 3\Delta s}\right]^2 \equiv g_i(s_i) \qquad \left[\frac{r\sigma^2 + (2\overline{\theta} - \underline{\theta})\Delta s}{2r\sigma^2 + 3\Delta s}\right]^2 \equiv g_j(s_j),$$

then we can write  $\operatorname{Var}(\Pi_i^*) = g_i(s_i)\sigma^2$  and  $\operatorname{Var}(\Pi_j^*) = g_j(s_j)\sigma^2$ . To find the effect of qualities on the variances, we need the derivatives  $g'_i$  and  $g'_j$ :

$$g_i'(s_i) = \frac{2r\sigma^2[r\sigma^2 + (1-\underline{\theta})\Delta s][1+2\underline{\theta}]}{[2r\sigma^2 + 3\Delta s]^3} > 0.$$
$$g_j'(s_j) = \frac{2r\sigma^2[r\sigma^2 + (\underline{\theta}+2)\Delta s][1+2\underline{\theta}]}{[2r\sigma^2 + 3\Delta s]^3} > 0.$$

where I used the fact that  $\overline{\theta} = \underline{\theta} + 1$ . Both derivatives are positive since  $\underline{\theta} > 0$ ,  $\Delta s \ge 0$ , r > 0,  $\sigma^2 > 0$ . Therefore equilibrium variances increase in own quality.

**Proposition 3.** When quality is costless, Firm j will always choose the highest quality  $\overline{s}$ .

*Proof.* I will show that  $dV_j/ds_j > 0$  regardless of  $s_i$ . To simplify the expression for equilibrium utility, write the negative exponent as a product of two functions,  $f_j$  and  $g_j$ :

$$V_{j} = -\exp\left\{-\underbrace{\left(r\Delta s + \frac{1}{2}r^{2}\sigma^{2}\right)}_{f_{j}}\underbrace{\left[\frac{r\sigma^{2} + (2\overline{\theta} - \underline{\theta})\Delta s}{2r\sigma^{2} + 3\Delta s}\right]^{2}}_{g_{j}}\right\} = -e^{-f_{j}g_{j}}$$

We already found that  $g'_j > 0$ , and  $f'_j = r$ . Therefore

$$\frac{dV_j}{ds_j} = -e^{-f_j g_j} [-(f'_j g_j + f_j g'_j)] = e^{-f_j g_j} [f'_j g_j + f_j g'_j] = e^{-f_j g_j} [rg_j + f_j g'_j].$$

Since both  $f_j > 0$ ,  $g_j > 0$  and on the other hand  $g'_j > 0$ , it follows that always  $\frac{dV_j}{ds_j} > 0$ so the solution for  $s_j$  must be at the right corner  $s_j = \overline{s}$ .

**Proposition 4.** With costless quality,

- a) When consumers are picky enough ( $\underline{\theta}$  exceeds some  $\theta_*$ ), Firm i will choose minimum differentiation regardless of  $r, \sigma^2$ .
- b) Moreover, such a critical value  $\theta_*$  always exists.
- c) When  $\underline{\theta}$  falls below some  $\theta_{**}$ , Firm *i* chooses **maximum differentiation**. Moreover, such a  $\theta_{**} > 0$  always exists as long as  $r\sigma^2 > 3$ .

*Proof.* a) As before, write Firm i's equilibrium utility in simpler form as

$$V_i = -\exp\left\{-\left(r\Delta s + \frac{1}{2}r^2\sigma^2\right)\left[\frac{r\sigma^2 + (\overline{\theta} - 2\underline{\theta})\Delta s}{2r\sigma^2 + 3\Delta s}\right]^2\right\} = -e^{-f_ig_i}$$

Then

$$\frac{dV_i}{ds_i} = e^{-f_i g_i} [f'_i g_i + f_i g'_i], = e^{-f_i g_i} [-rg_i + f_i g'_i],$$

since  $f'_i = -r$ . Notice that the derivative's sign depends on the sign of  $[-rg_i + f_ig'_i]$ , where the first term is negative while the second is positive (both  $f_i > 0$  and  $g'_i > 0$  as shown in the proof to Proposition 2). Which term dominates will turn out to depend on the value of  $\underline{\theta}$ . To avoid dealing with the (strictly positive) denominators of  $g_i$  and its derivatives, which do not change the sign, I bring the expression  $[-rg_i + f_ig'_i]$  to a common denominator and then focus only the numerator, so

$$\operatorname{sgn}[-rg_i + f_i g'_i] = \operatorname{sgn}\left\{-r[r\sigma^2 + (\overline{\theta} - 2\underline{\theta})\Delta s]^2(2r\sigma^2 + 3\Delta s) + 2r\sigma^2\left(r\Delta s + r\sigma^2/2\right)\left[r\sigma^2 + (\overline{\theta} - 2\underline{\theta})\Delta s\right]\left[3 - 2(\overline{\theta} - 2\underline{\theta})\right]\right\}.$$

Keeping in mind that  $[r\sigma^2 + (\overline{\theta} - 2\underline{\theta})\Delta s] > 0$  and  $[3 - 2(\overline{\theta} - 2\underline{\theta})] = 1 + 2\underline{\theta}$ , simplifies to

$$\operatorname{sgn}[-rg_i + f_i g'_i] = \operatorname{sgn}\left\{-(r\sigma^2 + (1 - \underline{\theta})\Delta s)(2r\sigma^2 + 3\Delta s) + 2\sigma^2[r\Delta s + r^2\sigma^2/2](1 + 2\underline{\theta})\right\}.$$

First I will prove that there exists a critical value  $\theta_*$  above which  $[-rg_i + f_ig'_i] < 0$ . For this to happen, it must be the case that

$$2\sigma^{2}[r\Delta s + r^{2}\sigma^{2}/2](1 + 2\underline{\theta}) > (r\sigma^{2} + (1 - \underline{\theta})\Delta s)(2r\sigma^{2} + 3\Delta s)$$

or after reduction,

$$r\sigma^{2}[3\Delta s + r\sigma^{2}] > [6r\sigma^{2}\Delta s + 2r2\sigma^{4} + 3(\Delta s)^{2}](1 - \underline{\theta}).$$

To find a  $\theta_*$  high enough to work independent of  $\Delta s$ , I substitute "the worst possible values"  $\Delta s = 0$  in the LHS and  $\Delta s = 1$  in the RHS, resulting in

$$r^{2}\sigma^{4} > (6r\sigma^{2} + 2r^{2}\sigma^{4} + 3)(1 - \underline{\theta}).$$

Therefore, when

$$\underline{\theta} > \underbrace{1 - \frac{r^2 \sigma^4}{6r \sigma^2 + 2r^2 \sigma^4 + 3}}_{\theta_*} \quad \Rightarrow \frac{dV_i}{ds_i} > 0.$$

(b) Notice that since  $0 < \frac{r^2 \sigma^4}{6r \sigma^2 + 2r^2 \sigma^4 + 3} < \frac{1}{2}$ , it follows that  $1 > \theta_* > 1/2$ , which means  $\theta_*$  is always within the allowed range (0,1) for  $\underline{\theta}$ .

(c). To reverse the sign of  $dV_i/ds_i$ , we need the opposite inequality to hold:

$$(1-\underline{\theta})[\ 6r\sigma^2\Delta s + 2r^2\sigma^4 + 3(\Delta s)^2\ ] > r\sigma^2[3\Delta s + r\sigma^2]$$

which reduces to

$$\underline{\theta} < 1 - \frac{r\sigma^2[3\Delta s + r\sigma^2]}{6r\sigma^2\Delta s + 2r^2\sigma^4 + 3(\Delta s)^2}$$

To find a  $\theta_{**}$  that works for all  $\Delta s$ , I substitute "the worst possible cases"  $\Delta s = 1$  in the numerator and  $\Delta s = 0$  in the denominator, yielding

$$\theta_{**} = 1 - \frac{r\sigma^2[3 + r\sigma^2]}{2r^2\sigma^4} = \frac{r\sigma^2 - 3}{2r\sigma^2}$$

which is smaller than 1 and positive as long as  $r\sigma^2 > 3$  (for example, r = 3 and  $\sigma^2 > 1$ , which are economically reasonable values according to commonplace estimates of risk aversion).

**Proposition 5.** When quality costs for both firms are given by  $C(s) = bs^2$ ,

- a) If  $\underline{\theta} < 1/2$  and  $b > \frac{5}{16\underline{s}}$ , both firms choose minimum quality with minimal differentiation.
- b) If  $\underline{\theta} > 8b\overline{s} + \frac{1}{2}$  and  $b < \frac{1}{16\overline{s}}$ , both firms choose maximum quality with minimal differentiation.

*Proof.* a). Since we look for zero-differentiation equilibria at the corners, evaluate the quality derivatives of  $V_i$  and  $V_j$  at  $\Delta s = 0$  and look for a relationship between them

and costs that will make them positive or negative:

$$\frac{dV_i}{ds_i}\Big|_{\Delta s=0} = re^{-f_i g_i + rbs_i} \left[ -\frac{1}{8} + \frac{1}{4}\underline{\theta} - 2bs_i \right]$$
$$\frac{dV_j}{ds_j}\Big|_{\Delta s=0} = re^{-f_j g_j + rbs_j} \left[ \frac{3}{8} + \frac{1}{4}\underline{\theta} - 2bs_j \right].$$

First notice that if  $\underline{\theta} < 1/2$ , then  $\frac{dV_i}{ds_i}|_{\Delta s=0} < 0$ , so Firm *i* will have an incentive to go for the minimum quality. For this to be an equilibrium, however, Firm *j* will also need to have incentives to stay there.  $V_j$ 's derivative shows that

$$\left. \frac{dV_j}{ds_j} \right|_{\Delta s=0} < 0 \quad \text{whenever} \quad \underline{\theta} < 8b\underline{s} - 3/2.$$

The inequality on the RHS will always be satisfied when  $8b\underline{s} - 3/2 > 1$  (because  $\underline{\theta} < 1$  by assumption). So a sufficient condition for  $\frac{dV_j}{ds_j}|_{\Delta s=0} < 0$  is

$$b > \frac{5}{16\underline{s}}.$$

When  $\underline{\theta} < \frac{1}{2}$  and  $b > \frac{5}{16\underline{s}}$ , both firms's quality choice ( $\underline{s}$ ) is a best response to the opponent's quality, so the quality pair  $(s_i, s_j) = (\underline{s}, \underline{s})$  is a Nash equilibrium.

b) Let us now look if minimum differentiation is possible at the maximum quality. Begin with Firm *i*. If  $\underline{\theta} > 1/2 + 8b\overline{s}$ , then  $dV_i/ds_i|_{\Delta s=0} > 0$ , but  $\underline{\theta}$  also needs to be less than 1. A neccessary condition for this is  $b < \frac{1}{16\overline{s}}$ .

For Firm j, the derivative evaluated at  $\Delta s = 0$  will be positive as long as

$$\underline{\theta} > 8b\overline{s} - \frac{3}{2}$$

which, given that  $\underline{\theta} < 1$ , is possible only when

$$8b\overline{s} - \frac{3}{2} < 1 \Rightarrow b < \frac{5}{16\overline{s}}$$

Collecting the conditions for both firms to go to  $\overline{s}$ , we get:

$$\begin{split} \underline{\theta} &> \frac{1}{2} + 8b\overline{s} \\ \underline{\theta} &> 8b\overline{s} - \frac{3}{2} \end{split} \qquad \qquad b < \frac{1}{16\overline{s}} \\ b &< \frac{5}{16\overline{s}} \end{split}$$

Taking their intersection provides the sufficient conditions

$$\underline{\theta} > \frac{1}{2} + 8b\overline{s} \qquad \qquad b < \frac{5}{16\overline{s}}$$

which guarantee that both firms choose  $\overline{s}$ . Since each firm is maximizing utility given its opponent's action, again this is a Nash equilibrium.

# **2.9** Appendix B: Asymptotics at $\Delta s \rightarrow 0$

**Lemma.** As  $\Delta s \rightarrow 0$ , equilibrium demand, profits and their variances converge to the following qualtities:

$$\lim_{\Delta s \to 0} D_i^* = \lim_{\Delta s \to 0} D_j^* = \frac{1}{2}$$
$$\lim_{\Delta s \to 0} \operatorname{Var}(\Pi_i^*) = \lim_{\Delta s \to 0} \operatorname{Var}(\Pi_j^*) = \frac{1}{4}\sigma^2$$
$$\lim_{\Delta s \to 0} \Pi_i^* = \lim_{\Delta s \to 0} \Pi_j^* = \frac{r\sigma^2}{4}$$

*Proof.* It is sufficient to find the limit of the expression  $\frac{p_j^* - p_i^*}{\Delta s}$ , which enters  $D^*$ ,  $\Pi^*$  and  $Var(\Pi^*)$ . Recall that equilibrium prices  $p_i^*$  and  $p_j^*$  satisfy the system of reaction

functions

$$p_i(p_j) = \frac{\mu_c \Delta s + (r\sigma^2 + \Delta s)(p_j - \underline{\Delta})}{r\sigma^2 + 2\Delta s},$$
$$p_j(p_i) = \frac{\mu_c \Delta s + (r\sigma^2 + \Delta s)(p_i + \overline{\Delta})}{r\sigma^2 + 2\Delta s}.$$

By subtracting the first equation from the second, obtain

$$\frac{p_j^* - p_i^*}{\Delta s} = \frac{r\sigma^2 + \Delta s}{2r\sigma^2 + 3\Delta s} (\overline{\theta} + \underline{\theta}).$$

In the limit  $\Delta s \to 0$ , it becomes

$$\lim_{\Delta s \to 0} \left[ \frac{r\sigma^2 + \Delta s}{2r\sigma^2 + 3\Delta s} (\overline{\theta} + \underline{\theta}) \right] = \frac{(\overline{\theta} + \underline{\theta})}{2} = \underline{\theta} + \frac{1}{2}.$$

Since 
$$D_i^* = \frac{p_j^* - p_i^*}{\Delta s} - \underline{\theta}, \quad \Rightarrow \lim_{\Delta s \to 0} D_i^* = 1/2.$$
  
Analogously  $D_j^* = \overline{\theta} - \frac{p_j^* - p_i^*}{\Delta s} \Rightarrow \lim_{\Delta s \to 0} D_j^* = \overline{\underline{\theta}} - \underline{\underline{\theta}} - \frac{1}{2} = \frac{1}{2}.$ 

From here it easily follows that  $\lim_{\Delta s \to 0} \operatorname{Var}(\Pi_i^*) = \lim_{\Delta s \to 0} \operatorname{Var}(\Pi_j^*) = \frac{1}{4}\sigma^2$ . To prove the result for profits, observe that the equilibrium markups converge to

$$\lim_{\Delta s \to 0} m_i^* = \lim_{\Delta s \to 0} \left[ \frac{(r\sigma^2 + \Delta s)[r\sigma^2 + (\overline{\theta} - 2\underline{\theta})\Delta s]}{2r\sigma^2 + 3\Delta s} \right] = \frac{r\sigma^2}{2},$$
$$\lim_{\Delta s \to 0} m_j^* = \lim_{\Delta s \to 0} \left[ \frac{(r\sigma^2 + \Delta s)[r\sigma^2 + (2\overline{\theta} - \underline{\theta})\Delta s]}{2r\sigma^2 + 3\Delta s} \right] = \frac{r\sigma^2}{2}.$$

so it is enough to multiply the limit of the markup times that of demand.  $\blacksquare$ 

In particular, observe when  $\Delta s \to 0$ , the firms split the market equally in the limit  $(D_i^* = D_j^* = 1/2)$ , thereby recovering the classical Bertrand competition result for homogeneous goods. Therefore, both economically and mathematically it makes sense

to define model variables at the  $\Delta s=0$  point as

$$D_i^*(\Delta s = 0) = D_j^*(\Delta s = 0) = 1/2$$
$$Var(\Pi_i^*(\Delta s = 0)) = Var(\Pi_j^*(\Delta s = 0)) = \frac{1}{4}\sigma^2$$
$$\Pi_i^*(\Delta s = 0) = \Pi_j^*(\Delta s = 0) = \frac{r\sigma^2}{4}.$$

Since a function f is continuous at a point  $x_0$  if  $f(x_0) = \lim_{x \to x_0} [f(x)]$ , this definition also ensures that model variables are continuous in  $\Delta s$ .

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# Chapter 3

# On Pareto Efficiency in the Marriage Problem with Weak Preferences and Many Agents

## 3.1 Introduction

In recent years, results from matching theory have been used to analyze a variety of real-life situations: student-school matching, medical resident-to-hospital matching, centralized university admissions, and pairwise kidney exchange, to name just a few. Many of these applications involve a large number of agents who face different incentives and different outcomes depending on market size.

This paper studies the large-market behavior of the marriage problem with weak preferences. When preferences are weak, the set of stable matchings and that of Pareto efficient matchings diverge, so that some stable matchings are no longer efficient. This creates incentives for inefficiently matched agents to stay together, which is undesirable from both a theoretical and a practical viewpoint. However, several recent studies have shown that the finite-market shortcomings of a mechanism can disappear when the market is large.

For example, Kojima and Pathak (2009) have recently found that the student-optimal mechanism becomes approximately strategy-proof in large markets. Kojima and Manea (2008) have obtained a similar result for the Probabilistic Serial mechanism, and Che and Kojima (2010) demonstrate that even seemingly unrelated mechanisms can converge to the same outcome in a large market. These results show that even well-studied mechanisms can behave surprisingly in large allocation problems, often-times allowing the mechanism designer to overcome or improve on existing shortcomings.

Here I show that the inefficiency associated with weak preferences in the marriage problem vanishes in large markets where agents' preferences are random and sufficiently diverse. In particular, I demonstrate that the proportion of agents who can Pareto improve in a randomly chosen stable matching approaches zero when the number of agents goes to infinity. This result provides a partial alleviation to the inefficiency of stable matchings under weak preferences, but it should be emphasized that this alleviation is only in *relative* terms, as it refers to the expected proportion of inefficient agents. Nothing in this result suggests that the *absolute* number of agents who can Pareto improve goes to zero for large n. Therefore, the result itself is not simply a way to get rid of indifferences with a resulting "cure-all" for the inefficiency problem; on the contrary, even though the frequency of pairwise indifferences vanishes in the limit, this has no implications for the *absolute* number of inefficient agents.

The rest of the paper is organized as follows. Section 3.2 provides a brief literature review. Section 3.3 presents a non-technical summary of the problem and the main results, and Section 3.4 presents the model and technical terms. Section 3.5 proves an important impossibility result that I use throughout, section 3.6 discusses the largemarket aspect and the derivation of main results, while section 3.7 extends the existing results to more general settings. Section 3.8 concludes. For better readability, longer proofs are relegated to the Appendix. Stand-alone lemmas are numbered sequentially; lemmas belonging to the proof of a given theorem are included and named after it (e.g. Lemma 1.2 is used in Theorem 1).

#### 3.2 Related Literature

The main question of this paper belongs to the intersection of two distinct literatures: studies on large markets and studies on matching with weak preferences. The large-market approach dominates throughout my results, but its implementation relies on several crucial tools from the weak-preference literature. The main motivating fact for this study – the inefficiency of stable matchings in the presence of indifferences – also comes from the literature on weak preferences.

It is well-known that weak preferences can generate inefficient stable matchings, but several recent papers underscore the practical importance of this inefficiency. For example, Abdulkadiroğlu, Pathak and Roth (2008) show that the mechanism used by New York City to allocate students to high-schools results in more than 6,800 inefficiently matched students each year because indifferences among students are resolved at random. A similar inefficiency is pervasive in the marriage problem, so improving the efficiency of stable mechanisms with indifferences remains in the focus of current work (for example, Erdil and Ergin (2008)).

I base my approach on tools adopted from the weak-preference literature, which I subsequently introduce into a new asymptotic setting. In particular, I extensively use the concept of Pareto-improvement cycles and chains (Erdil and Ergin (2006)). Pareto-Improvement cycles and chains are coalitions of agents who are willing to trade

partners so that at least one person in the coalition benefits strictly, while nobody is made worse off; the existence of such cycles or chains uniquely identifies inefficient matchings. Pareto-improvement cycles and chains provide a convenient way to identify both inefficient matchings and the particular agents who can Pareto improve, because checking for Pareto-dominant matchings is computationally difficult in large markets. I introduce this identification tool into a *random market environment*, in which preferences are drawn stochastically, and look at the expected proportion of agents who can Pareto improve as the number of agents goes to infinity.

Roth and Peranson (1999) and Immorlica and Mahdian (2005) first introduced twosided matching in the context of a random environment where agents' preferences are drawn from a probability distribution. This technique allows the mechanism designer to look at the mechanism outcome in expectation, averaged across different markets. This method has subsequently been extended and refined for many-to-one matching by Kojima and Pathak (2009), who reintroduced this probabilistic setting under the name *random market*. The random market device is the second main building block I adopt in order to frame my problem. I consider a sequence of random markets with increasing number of agents  $n \to \infty$  in order to show that the expected proportion of inefficiently matched agents converges to zero.

My study differs from Immorlica and Mahdian (2005) and Kojima and Pathak (2009) significantly. Since these authors study the tradeoff between stability and strategy-proofness, they consider neither weak preferences nor the inefficiency associated with them. By contrast, I study the convergence of the set of stable matchings to that of Pareto efficient stable matchings, a subject that does not involve game-theoretic considerations. The paper builds on the combinatorial properties of random weak preferences to demonstrate that the inefficiency vanishes in large markets. The full details of the method are presented in Section 3.6, while the next section presents a brief non-technical summary.

### 3.3 Non-technical Summary

Gale and Shapley's (1962) marriage problem is one of the simplest bilateral matching problems. In it, each of n agents, called men, must be matched to a partner from another set of n agents, called women, based on their mutual preferences. A systematic allocation procedure to assign partners to each other is called a *mechanism*, and the outcome of a mechanism is called a *matching*. A matching  $\mu$  is stable if two conditions hold:

- (1) No agent is matched to a spouse (s)he finds unacceptable, and
- (2) There are no blocking pairs that is, no man and no woman prefer to

be matched to each other rather than to their current spouse given by  $\mu$ .

In this paper, I focus extensively on one particular way of selecting a stable matching, called the random stable mechanism (RSM).

**Example 1a.** The Random Stable Mechanism.

Consider the stable matchings generated by the preferences below (denoted with  $R_i$  when they are weak and  $P_i$  when strict):

$$P_{m_1}: w_1, w_2, m_1; \quad R_{w_1}: \{m_1, m_2, \}, w_1$$
$$P_{m_2}: w_1, w_2, m_2; \quad R_{w_2}: \{m_1, m_2, \}, w_2$$

There are two stable matchings,  $\mu_1$  and  $\mu_2$ , because neither of them exhibits blocking pairs or blocking individuals:

$$\mu_1 = \begin{pmatrix} m_1 & m_2 \\ w_1 & w_2 \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} m_1 & m_2 \\ w_2 & w_1 \end{pmatrix}.$$

(By contrast, matchings where one or both men remain single are not stable, because each agent strictly prefers having a spouse to remaining single and in doing so forms a blocking pair with a member of the opposite gender.)

The RSM mechanism operates by generating a list of all stable matchings and by selecting one at random. For example, given the preferences above, the probability that either  $\mu_1$  or  $\mu_2$  is selected is  $\Pr(\mu_1) = \Pr(\mu_2) = 1/2$ .  $\Box$ 

A matching  $\mu$  is Pareto efficient if there is no other matching  $\nu$  that can reassign at least one person to a strictly better spouse (than under  $\mu$ ) without making anyone else worse off. When agents' preferences are strict, it is well known that all stable matchings are Pareto efficient (for example, see Roth and Sotomayor 1990). When preferences permit indifferences, however, a stable matching need not be efficient, as shown in Example 1b.

**Example 1b.** Consider the preferences and the matching  $\mu$  given below:

$$P_{m_1}: w_1, w_2, w_3, m_1; \quad R_{w_1}: \{m_1, m_2, m_3\}, w_1$$

$$P_{m_2}: w_1, w_3, w_2, m_2; \quad R_{w_2}: \{m_1, m_2, m_3\}, w_2$$

$$P_{m_3}: w_2, w_1, w_3, m_3; \quad R_{w_3}: \{m_1, m_2, m_3\}, w_3$$

$$\mu = \begin{pmatrix} m_1 & m_2 & m_3 \\ w_2 & w_3 & w_1 \end{pmatrix}$$

It is straightforward to verify that the allocation  $\mu$  is stable, but inefficient. Under  $\mu$ , each man is matched to his second choice. However, if men  $m_1$  and  $m_3$  switched partners, each of them would receive his first choice without hurting anyone else or upsetting stability.  $\Box$ 

For expositional clarity, I first permit weak preferences only on the women's side, and then extend the results for both men and women. I characterize inefficiently matched agents using Pareto Improvement cycles and chains (Erdil and Ergin (2006)); my purpose is to show that the proportion of agents involved in cycles or chains converges to zero in expectation. To do this I first obtain a necessary condition for cycle formation in terms of agents' randomly drawn preferences, and show that this condition occurs less and less frequently as the number of agents grows. In this context, the preference-generating process acquires special importance. I show that when agents are *stochastically diverse*, efficiency can be restored in large markets. Specifically, I assume that each agent with weak preferences draws her preferences independently from an urn containing all possible weak preferences. Repetition of agent types is possible, but becomes increasingly unlikely as the market size grows. This technical axiom means that in the limit, agents need to be sufficiently different, or that agent diversity increases with market size. With this assumption, one can obtain an upper bound for the expected proportion of inefficiently matched agents and look at how this bound changes with n.

To show that the proportion of inefficiently matched agents engaged in cycles or chains converges to zero, I recast the economic problem in combinatorial terms. I use combinatorial enumeration methods to obtain a recurrence relation for the maximum number of agents admitting cycles (chains), and show that as a fraction of the total, such agents decrease to zero in the limit. This allows me to conclude that a randomly chosen stable matching is *asymptotically efficient*.

#### 3.4 Model

The triple  $(M, W, R)_n$  is called a marriage problem with n men and n women to be matched, where M denotes the set of men, W the set of women, and R the collection of weak preferences  $R = \{R_m \cup R_w \ m \in M, \ w \in W\}$ . Preferences are ordinal and can be thought of as "rank lists," where each agent lists agents of the opposite sex in order of preference. It is standard to denote the weak preference relation "at least as good as" with the letter R indexed by the "name" of the agent: for example,  $aR_ib$ denotes "agent i weakly prefers a to b". The corresponding strict preference relation is denoted with  $P_i$ , and the indifference relation with  $\sim_i$ . I will often refer to n as simply the size of the market.

In the classical marriage problem (Gale and Shapley (1962)), preferences are treated as given. However, they can also be chosen stochastically: for example, each agent may be drawing his or her preferences from some distribution over all possible preference lists. Such an environment allows one to look at the allocation in a given marriage problem on average, where the averaging takes place across different preference realizations. A marriage problem with stochastically determined preferences is is called a random market (Kojima and Pathak (2009); Immorlica and Mahdian (2005)). In our context, a random market is a quadruple  $(M, W, R, \mathcal{D})_n$  where  $\mathcal{D}$  is a distribution over the set of all possible weak preferences  $\{R\}$ . (Clearly, the set of all strict preferences is also included in  $\{R\}$ ). Each random market can have different realizations, depending on what preferences are jointly drawn. Given a fixed preference realization, one can use a systematic procedure to allocate partners. An allocation that specifies who is matched to whom (and who remains single) is called a **matching** and is often denoted as  $\mu$ . The notation  $\mu(i)$  denotes the partner assinged to agent i by the matching  $\mu$ . Formally, a matching is a function  $\mu : M \cup W \to M \cup W$  that satisfies three conditions:

(1) 
$$\mu(m) \notin W \Rightarrow \mu(m) = m, \forall m \in M;$$
  
(2)  $\mu(w) \notin M \Rightarrow \mu(w) = w, \forall w \in W;$  and  
(3)  $\mu(m) = w \Leftrightarrow \mu(w) = m, \forall m \in M, \forall w \in W$ 

Different preferences can result in different matchings; the set of all matchings is denoted as  $\mathcal{M}$ .

A matching  $\mu$  is **Pareto efficient**<sup>1</sup> when there exists no other matching  $\nu$  such that: (1)  $\nu(i)R_i\mu(i)$  for all  $i \in M \cup W$ , and

(2)  $\nu(i)P_i\mu(i)$  for some  $i \in M \cup W$ .

A matching  $\mu$  is **individually rational** if for every agent *i*, it is true that  $\mu(i)R_i$  *i*. A matching  $\mu$  is **blocked by an individual** *i* if  $iP_i\mu(i)$ . A matching  $\mu$  is **blocked by a pair** (m, w) if both  $wP_m\mu(m)$  and  $mP_w\mu(w)$  hold. A matching is **stable** if it is not blocked by any individual or pair.<sup>2</sup>

A mechanism is a function  $f : R \to \Delta(\mathcal{M})$  from preferences R to the set of distributions  $\Delta$  over the set of possible matchings  $\mathcal{M}$ . Given a set of preferences, a mechanism outputs a (unique) distribution over matchings. A mechanism f is **Pareto-efficient** if the matching f(R) is Pareto efficient for every preference profile R.

To gauge the efficiency of a matching, I first identify agents who can Pareto improve, using the notion of Pareto-Improving cycles and chains (Erdil and Ergin (2006)).

**Definition 1.** Given a matching  $\mu$ , a **Pareto-Improvement Cycle** is a set of agents of the same sex  $i_1, i_2, \ldots, i_K$   $(K \ge 2)$  such that:

<sup>&</sup>lt;sup>1</sup>The terms in this section are explained in greater detail in Roth and Sotomayor (1990).

<sup>&</sup>lt;sup>2</sup>One can also define a stricter notion of stability. A matching is **strictly stable** if it is individually rational and there is no pair (m, w) such that either  $wR_m\mu(m)$  and  $mP_w\mu(w)$  hold together, or  $wP_m\mu(m)$  and  $mR_w\mu(w)$  hold together. Throughout the paper, I use the regular notion of stability, sometimes also called "weak stability" (as in Gusfield and Irving 1989); it is trivial to see that all strictly stable matchings are Pareto efficient.

- 1. Each agent  $i_t$  is matched to a spouse of the opposite sex.
- 2.  $\mu(i_{t+1})R_{i_t}\mu(i_t)$  and  $i_tR_{\mu(i_{t+1})}i_{t+1}$  for  $t \in \{0, 1, \dots, K-1\}$ .
- 3. At least one of the preferences in (2) is strict for some  $t \in \{0, 1, \dots, K-1\}$ .

Pareto-Improvement cycles and chains are sets of agents who are willing to exchange partners among themselves so that at least one agent in the cycle (chain) is better off and nobody else is worse off. If we call the relation  $\mu(i_{t+1})R_{i_t}\mu(i_t)$  an **envy relation** and and denote it with an arrow, Pareto-Improvement cycles and chains can be represented more intuitively as graphs.

**Definition 1'.** Given a matching  $\mu$ , a **Pareto improvement cycle** is a sequence of men  $m_1, m_2, \ldots, m_K$  and their respective spouses  $\mu(m_1), \mu(m_2), \ldots, \mu(m_K)$ , such that

$$\mu(m_1) \to \mu(m_2) \to \ldots \to \mu(m_K) \to \mu(m_1)$$
 and (3.1)

$$m_K \leftarrow m_1 \leftarrow m_2 \leftarrow \ldots \leftarrow m_{K-1} \leftarrow m_K.$$
 (3.2)

where at least one of the envy relations  $\rightarrow$  is strict.

If a matching contains single agents, then such a coalition of agents can begin with and end with an unmatched agent, in which case the relevant concept is the Pareto-Improvement chain. A graphical example of a Pareto-Improvement cycle and a Pareto-improvement chain is shown in Fig. 3.1; the formal definition of chains is deferred to Section 7.1.

Pareto-Improvement cycles and chains (henceforth, PI-cycles and PI-chains) provide a means to construct a matching that Pareto dominates a given matching  $\mu$ , so if a cycle or chain exists, it is immediate that the matching  $\mu$  is inefficient. The converse



Figure 3.1: A Pareto-Improvement Cycle and a Pareto-Improvement Chain.

is also true, as shown in Proposition 1 in the Appendix, so a key feature of Pareto-Improvement cycles and chains is that they identify inefficiently matched agents in the marriage problem.<sup>3</sup>

From here on, my main strategy will be to obtain a necessary condition for cycle (chain) formation, and to show that when certain assumptions are met, this necessary condition occurs less and less frequently as  $n \to \infty$ . I begin with a set of six assumptions, some of which I subsequently relax:

A1. The mechanism designer is facing a sequence of random markets of increasing size  $\{(M, W, R, D)_n\}_{n=3}^{\infty}$ .

**A2.** Men and women are matched using the random stable mechanism (RSM).

**A3.** Everybody is *acceptable*: remaining single is strictly the last choice of every agent.

A4. Given a market size n, each man independently draws strict preferences from the uniform distribution over all possible strict preferences

 $<sup>^{3}</sup>$ Erdil and Ergin (2006, Theorem 1) prove a similar result for many-to-one matchings, but under different assumptions. For example, they rule out indifferences between a mate and remaining unmatched.

over the n women. The draw is repeated for each market size n; successive draws are independent.

A5. Given a market size n, each woman independently draws weak preferences from a distribution  $\mathcal{D}_n$  over all possible weak preferences over the n men. The draw is repeated for each market size n; successive draws are independent.

A6.  $\mathcal{D}_n$  is the uniform distribution.

Assumptions A1 and A2 are self-explanatory. Assumption A3 (acceptability) is introduced for convenience; its immediate corollary is that when all agents are acceptable, any stable mechanism leaves no agent single, so it is enough to consider only Pareto-Improvement cycles. (I relax this assumption in Section 3.7.) For expositional clarity, assumption A5 specifies that weak preferences occur only on one side of the market (for concreteness, women); it is relaxed in Section 3.7.

Assumptions A4 and A5 (stochastic diversity) merit longer discussion. They specify the frequency of occurence of preference types, and imply that recurrence of the same agent (preference) type gets less and less frequent as  $n \to \infty$ : as market size grows, agent diversity increases. When women are stochastically diverse, agents likely to end up in a PI-Cycle occur less and less frequently until their share converges to zero. For technical aspects of this result, the reader is referred to Section 3.6.

Whether assumptions A4 and A5 are realistic or not depends on the market under consideration. They will work well for markets with many agents with heterogeneous, uncorrelated preferences. By contrast, in settings where preferences are correlated (for example, a dating market in which women agree on the ranking of men), A4 an A5 are unsuitable. Thus my results apply to large heterogeneous markets but not to markets with many similar agents. For simplicity, in assumption A6 I also assume that  $\mathcal{D}_n$  is the uniform distribution, but Section 7.2 provides an example that this can be relaxed as well.

#### 3.5 An impossibility result

The next Lemma relates Pareto-improvement cycles to preferences; this impossibility result separates those agents who can enter a cycle from those who cannot.

**Lemma 1.** (Necessary condition for cycle formation). Suppose that  $\mu$  is a stable, 1:1 matching. Then two fixed men  $m_1$  and  $m_2$  cannot be in the same Pareto-Improvement cycle unless at least one of their spouses is indifferent between  $m_1$  and  $m_2$ , so that

$$[m_1 \sim_{\mu(m_2)} m_2]$$
 or  $[m_1 \sim_{\mu(m_1)} m_2]$ , or both.<sup>4</sup>

*Proof.* First consider the (strict) preferences of  $m_1$ . Given any matching  $\mu$ , either  $\mu(m_2)P_{m_1}\mu(m_1)$ , or  $\mu(m_1)P_{m_1}\mu(m_2)$ .

Initially suppose that  $\mu(m_1)P_{m_1}\mu(m_2)$ ; then  $m_1$  will not be willing to trade with  $m_2$ , and  $m_1$  and  $m_2$  cannot participate in the same PI-Cycle.

Now suppose that  $\mu(m_2)P_{m_1}\mu(m_1)$  and consider the weak preferences of  $\mu(m_2)$ . If  $m_1P_{\mu(m_2)}m_2$ , then  $(m_1,\mu(m_2))$  form a blocking pair and so  $\mu$  is not stable, a contradiction. If  $m_2P_{\mu(m_2)}m_1$ , then  $\mu(m_2)$  cannot point to  $\mu(m_1)$  and by Definition 1, these two women and their spouses  $m_1$  and  $m_2$  cannot be part of the same PI-Cycle. The only preference of  $\mu(m_2)$  consistent with a PI-Cycle is  $m_1 \sim_{\mu(m_2)} m_2$ . Reversing the places of  $m_1$  and  $m_2$  yields the second part of the necessary condition,  $[m_1 \sim_{\mu(m_1)} m_2]$ .

<sup>&</sup>lt;sup>4</sup>This "non-exclusive OR" and the notation  $\cup$  will be used interchangeably where needed to save space.

The next step is to determine how often preferences leading to cycle formation occur in the preference-generating setup in A1-A6. I begin by demonstrating that the probability of any two given men (and hence, their corresponding matches) entering a cycle will go to zero when the number of agents  $n \to \infty$ . I do this by showing that this probability is bounded from above and that the upper bound goes to zero.

**Theorem 1.** Let  $\mu$  be a one-to-one matching produced by the Random Stable Mechanism under assumptions A1-A6. Then in the limit  $n \to \infty$ , the probability that two arbitrary fixed men  $m_1$  and  $m_2$  admit a Pareto-Improvement cycle C satisfies the upper bound

$$\Pr(m_1, m_2 \in \mathcal{C}) \le 2\Pr(m_1 \sim_w m_2), \tag{3.3}$$

where w is an arbitrary woman.

#### *Proof:* In the Appendix. $\blacksquare$

The statement in Theorem 1 is less obvious than it appears. Firstly, the theorem uses the probability of pairwise indifference by a *fixed* arbitrary woman w as an upper bound for the probability of indifference by a *variable* woman  $\mu(m_2)$  selected by the mechanism, therefore the upper bound also depends on the mechanism's properties. In the Appendix, I show that the Random Stable Mechanism satisfies the upper bound (3); however, this need not be true for stable mechanisms in general. For example, stable mechanisms that under assumptions A1-A6 match a fixed pair  $(m^*, w^*)$  with probability other than 1/n, may violate the upper bound from the theorem (see the proof in the Appendix).

Theorem 1 implies that the frequency of cycle formation is related to the frequency of indifferences in the underlying preference-generating process. Next I will show that such pairwise indifferences vanish in the limit  $n \to \infty$ .

#### 3.6 Asymptotic Method

This section shows that under assumptions A1-A6, the upper bound  $Pr(m_1 \sim_w m_2)$  converges to zero in large markets. Assumption A5 posits that each woman selects her (weak) preference at random from the list  $\{R\}$  of all possible weak preferences;<sup>5</sup> it specifies a stochastic *preference-generating process* (PGP), which I translate in combinatorial terms. If we denote:

 $\tilde{T}_n$  = The number of weak preferences over n men in which  $m_1 \sim_w m_2$ 

 $T_n$  = The total number of weak preferences over  $n {\rm ~men},$ 

then a woman w selecting a weak preference at random will have a probability of pairwise indifference between two fixed men  $m_1$  and  $m_2$  equal to

$$\Pr(m_1 \sim_w m_2) = \tilde{T}_n / T_n. \tag{3.4}$$

To find the terms  $T_n$  and  $\tilde{T}_n$ , I generate weak preferences as ordered partitions of a set with *n* elements. For example, the ordered partition of a set of 8 men

$${m_7 \mid m_2 \; m_3 \; m_5 \mid m_4 \mid m_6 \; m_1 \mid m_8}$$

generates the weak preference

$$R_w: m_7 \succ \{m_2 \ m_3 \ m_5\} \succ m_4 \succ \{m_6 \ m_1\} \succ m_8.$$

One can think of the woman's preference-generation task as the task of distributing the n men into ordered blocks of various sizes, where each block of size greater than one represents an indifference class (a group of men among whom the woman is indifferent). So I posit a random weak preference to be **a randomly chosen ordered** 

<sup>&</sup>lt;sup>5</sup>Again it is understood that  $\{R\}$  also includes all possible strict orderings of men.

**partition** of the set of men M. As another example, consider the task of generating a random weak preference over a set of n = 3 fixed men.

**Example 2.** The set of men  $M = \{m_1 \ m_2 \ m_3\}$  has  $T_3 = 13$  ordered partitions, corresponding to the 3!=6 permutations corresponding to strict preferences, plus the following 7 preferences containing at least one indifference:

$$\begin{array}{ll} m_1, \{m_2, m_3\}; & \{m_2, m_3\}, m_1; & m_2, \{m_1, m_3\}; & \{m_1, m_3\}, m_2; \\ \\ m_3, \{m_1, m_2\}; & \{m_1, m_2\}, m_3; & \{m_1, m_2, m_3\}. \end{array}$$

Each possible preference is therefore selected with probability 1/13.  $\Box$ .

The next Theorem shows how to count the large numbers  $\tilde{T}_n$  and  $T_n$  using recurrence relations, as they have no exact closed formulas in n.

**Theorem 2.** (a) The total number  $T_n$  of weak preferences over n partners satisfies the recurrence relation

$$T_n = \sum_{i=0}^{n-1} \binom{n}{i} T_i.$$
 (3.5)

(b) The total number  $\tilde{T}_n$  of weak preferences over n partners in which two fixed agents are indifferent, satisfies the recurrence relation

$$\tilde{T}_n = T_{n-1} \tag{3.6}$$

*Proof.* (a). The first partition block of size k  $(1 \le k \le n)$  can be selected from the set of n partners in exactly  $\binom{n}{k}$  ways, because within blocks, order does not matter. To each selection of this 1st k-block, there corresponds a subset of (n - k) remaining elements to be partitioned into ordered blocks, which can be done in exactly  $T_{n-k}$ 

ways. Summing over all possible sizes k = 1, 2, ..., n results in the sum

$$T_{n} = \sum_{k=1}^{n} \binom{n}{k} T_{n-k} = \sum_{k=1}^{n} \binom{n}{n-k} T_{n-k} = \sum_{i=0}^{n-1} \binom{n}{i} T_{i},$$
(3.7)

where i = n - k.

(b). Let the set M be of cardinality n. If two fixed elements  $m_1, m_2 \in M$  always appear in the same block of an ordered partition of M, one can treat them as a single element  $\bar{m}$  for partitioning purposes. The resulting set  $\{\bar{m}, m_3, m_4, \ldots, m_n\}$  consists of n - 1 elements and can therefore be partitioned into ordered blocks in exactly  $T_{n-1}$  ways. Therefore the number of ordered partitions  $\tilde{T}_n$  in which the two fixed elements  $m_1$  and  $m_2$  always occur in the same block, satisfies the recurrence relation  $\tilde{T}_n = T_{n-1}$ .

Using the recurrence relations from Theorem 2, one can demonstrate that the probability of a pairwise indifference (a necessary condition for cycle formation), goes to zero in a large market. To emphasize that this probability depends on market size, here it will be specifically denoted as  $\Pr(m_1 \sim_w m_2; n)$ .

**Theorem 3.** The probability of pairwise indifference between two fixed agents  $m_1$  and  $m_2$  as  $n \to \infty$  satisfies the limit

$$\lim_{n \to \infty} \Pr(m_1 \sim_w m_2 \ ; \ n) = \lim_{n \to \infty} \frac{\tilde{T}_n}{T_n} = 0.$$
(3.8)

*Proof:* In the Appendix.  $\blacksquare$ 

The last theorem shows that a 2-ple of fixed men is less and less likely to enter a cycle as the number of agents increases. As n grows, however, the number of such

2-ples also increases; the next step shows that with the random stable mechanism, the expected *proportion* of 2-ples admitting a cycle (and therefore also the expected proportion of agents admitting a cycle) converges to zero.

**Theorem 4.** Let assumptions A1-A6 be satisfied. Then the expected proportion of agents that can participate in a Pareto-Improvement cycle converges to zero as  $n \rightarrow \infty$ .

*Proof.* In the Appendix.

Theorem 4 establishes the main result of the paper: that the expected proportion of individuals admitting a PI-Cycle is vanishingly small at infinity. Sections 7.1 - 7.3 extend this result to setups allowing Pareto-improvement chains, two-sided indifferences, and non-uniform distributions over preferences, respectively.

# 3.7 Extensions

#### 3.7.1 Pareto Improvement Chains

Here I drop the assumption that all agents are acceptable, allowing an agent to prefer remaining single to being matched. Partners who appear as choises worse than iare *unacceptable* for i, and if  $\mu(i) = i$  we say that agent i remains single under the matching  $\mu$ .

If an agent remains single, he or she cannot be part of a Pareto-Improvement cycle. However, a *chain* of agents willing to exchange spouses to their mutual benefit may still exist. A Pareto-Improvment chain (Ergin and Erdil (2006)) is a sequence of men and women, beginning with an unmatched man and ending with an unmatched woman, who are willing to exchange spouses so that nobody is worse off and at least one person is better off. This idea is made precise in Definition 2. Let the *envy* relation between two agents  $x \to y$  denote the fact that  $\mu(y)R_x\mu(x)$ .

**Definition 2.** Given a matching  $\mu$ , a **Pareto-Improvement Chain** is a set of men  $m_1, \ldots, m_K$  and women  $\mu(m_2), \mu(m_3), \ldots, \mu(m_K), w_K, K \ge 2$  such that:

- 1.  $m_1$  and  $w_K$  are single;
- 2.  $m_1 \rightarrow m_2, \ldots, m_{K-1} \rightarrow m_K$  and  $m_K \rightarrow w_K$ .
- 3.  $w_K \to w_{K-1}, \ldots, w_2 \to w_1$  and  $w_1 \to m_1$ .
- 4. At least one of the envy relations " $\rightarrow$ " is strict.

In an inefficient matching where some agents are unacceptable to others, either a PI-Cycle or a PI-Chain must occur. I am again interested in finding a limiting condition for the frequency of Pareto improvement cycles and chains. In a setting with single agents, a man can envy (i.e. point to) either a matched man or a single woman, so instead of envy 2-ples of men of the type  $m_i \to m_j$ , now I will consider generalized envy 2-ples  $m \to i$  where the envied agent i can be either a matched man or a single woman. Given a matching  $\mu$ , denote the set of matched men as  $\overline{M}$  and the set of single women as  $\underline{W}$ ; so, I will consider 2-ples of the type (m, i) s.t.  $i \in \overline{M} \cup \underline{W}$ .

#### Lemma 2. (Necessary condition for PI-cycle/chain formation).

Suppose that  $\mu$  is a stable, one-to-one matching. Then the pair of agents m and i cannot be in the same PI-Cycle or PI-Chain unless

$$m \sim_{\mu(i)} i \ \cup \ m \sim_{\mu(m)} i \tag{3.9}$$

*Proof.* First suppose that  $\mu(i)P_m\mu(m)$ , where  $m \in M$  and  $i \in \overline{M} \cup \underline{W}$ . (If instead  $\mu(m)P_m\mu(i)$ , agents m and i will never voluntarily trade partners).

1. For cycles: Suppose not, so  $m \not\sim_{\mu(i)} i$ . Then either  $\mu(m_2)$  will be unwilling to trade, or else  $(m_1, \mu(m_2))$  form a blocking pair, so  $\mu$  is not stable.  $\Box$ 

2. For chains. If both m and i are matched men, the proof is the same as for cycles. If  $m \in \overline{M}$  (so m has a wife) and  $i \in \underline{W}$  (i.e. i is a single woman), then obviously  $\mu(i) = i$ . If i is not indifferent between m and remaining single, either she won't agree to trade, or else (m, i) are a blocking pair. Alternatively, if m is a single man, then i is a matched man, and if  $m \not\sim_{\mu(i)} i$ , again  $\mu(i)$  will either refuse to trade, or form a blocking pair with m.  $\Box$ 

It is also possible that  $\mu(m)P_i\mu(i)$ ; (if instead  $\mu(i)P_i\mu(m)$ , agents m and i will not want to trade). In this case, simply reverse the roles of m and i in the proofs above to obtain the second part of the necessary condition  $m \sim_{\mu(m)} i$ .

The probability of the event  $m \sim_{\mu(i)} i$ ,  $(i \in \overline{M} \cup \underline{W})$  will again be an upper bound for the chance that these two agents can be in a cycle (chain). Since in the case of a single woman w,  $\mu(w) = w$ , again this frequency depends only on the preferences of women; the only difference is that now we include into consideration any indifferences between the "stay single" position and being matched to a man m. Each woman's preference is now defined over n + 1 agents: the n men plus herself (to indicate her individual rationality point). As before, we can decompose

$$\Pr(m \sim_{\mu(i)} i) = \Pr\left(\bigcup_{i=1}^{n} [\mu(i) = w_i \cap m \sim_{w_i} i]\right)$$
(3.10)

and by replacing  $m_2$  with *i* in the proof of Theorem 1, directly obtain

$$\Pr(m, i \in \mathcal{C}) \le 2\Pr(i \sim_w j) \tag{3.11}$$

where C is a PI-chain or PI-Cycle, w is an arbitrary woman and i and j are any two fixed agents in the ordinal preference of w.

From this it is evident that the combinatorial problem remains unchanged. The only difference is that now we have to partition a set of n + 1 elements, as the individual rationality point may now occur anywhere in the rank list, including an indifference class. A statement analogous to Theorem 4 immediately follows, because

$$\lim_{n \to \infty} \Pr(i \sim_w j \ ; \ n) = \lim_{n \to \infty} \frac{\tilde{T}_{n+1}}{T_{n+1}} = \lim_{n \to \infty} \frac{\tilde{T}_n}{T_n} = 0.$$
(3.12)

Repeating without change the reasoning in Theorem 4, it follows that the expected proportion of men pointing to either other men or to single women, i.e. the fraction of men that admits a PI-cycle or a PI-chain, converges to zero. Since the matching is one-to-one, so does the expected proportion of inefficient women.

#### 3.7.2 Two-sided Weak Preferences

Similar results hold when both men and women have random weak preferences, except that in this case the inefficiency vanishes slower. Instead of assumption A4, now assume that:

A4'. Given a market size n, each man independently draws *weak* preferences from the uniform distribution over all possible weak preferences over the n women. The draw is repeated for each market size n; successive draws are independent.

In this setting, cycles can occur more frequently, because there are more indifferent agents who can agree to switch partners. The necessary condition for a cycle involving two fixed men  $m_1$  and  $m_2$  is now the following.

**Lemma 3.** Suppose that  $\mu$  is a stable, 1:1 matching. Then two fixed men  $m_1$  and  $m_2$  cannot be in the same Pareto-Improvement cycle unless

 $[m_1 \sim_{\mu(m_1)} m_2] \cup [m_1 \sim_{\mu(m_2)} m_2] \cup [\mu(m_1) \sim_{m_1} \mu(m_2)] \cup [\mu(m_1) \sim_{m_2} \mu(m_2)].$ (3.13)

Proof. The first two indifferences  $[m_1 \sim_{\mu(m_1)} m_2] \cup [m_1 \sim_{\mu(m_2)} m_2]$  follow without change from the proof of Lemma 1. In addition, however, now a cycle can also occur when one of the women  $\mu(m_1), \mu(m_2)$  has a *strict* preference over the men  $m_1, m_2$ , but one of these men is indifferent between  $\mu(m_1), \mu(m_2)$ . Again, if  $m_1 P_{\mu(m_1)} m_2$ , then  $\mu(m_1)$  and  $\mu(m_2)$  will not trade, and when  $m_2 P_{\mu(m_1)} m_1$ , the two fixed men cannot be in the same cycle unless  $\mu(m_1) \sim_{m_2} \mu(m_2)$ ; the same logic applies to  $\mu(m_1)$ , resulting in the indifference  $[\mu(m_1) \sim_{m_1} \mu(m_2)]$ .

To obtain a similar efficiency result, I use a modified version of the upper bound theorem.

**Theorem 5.** Let  $\mu$  be a matching produced by the Random Stable Mechanism and let assumptions A1-A3, A4', A5-A6 hold. Then the probability that two arbitrary fixed men  $m_1$  and  $m_2$  admit a Pareto-Improvement cycle C satisfies the upper bound

$$\Pr(m_1, m_2 \in \mathcal{C}) \le 2\Pr(m_1 \sim_w m_2) + 2\Pr(w' \sim_m w''), \tag{3.14}$$

where w, w' and w'' are any fixed women and m is any fixed man.

*Proof.* In the Appendix.

When wen and women draw weak preferences using the same i.i.d. uniform generation process, the upper bound is  $4 \Pr(m_1 \sim_w m_2)$ , twice larger than before. Applying without change the reasoning from Theorem 4, it follows that **Theorem 6.** Let both men and women have weak preferences drawn from the uniform distribution so that assuptions A1-A3,A4', and A5-A6 hold. Then the expected proportion of agents that can participate in a Pareto-Improvement cycle converges to zero as  $n \to \infty$ .

This result completes the extension for two-sided weak preferences. As expected, when both sides of the market can have indifferences, the associated inefficiency is larger, but it goes to zero as well.

#### 3.7.3 A Non-Uniform Distribution

So far I discussed weak preferences generated as random ordered partitions of the set of agents of the opposite sex. Now suppose we eliminate all strict orders from that list, and allow the agent to choose only from preferences containing at least one indifference. It can be shown that this non-uniform distribution over preferences preserves the results. In this case, given two fixed men  $m_1$  and  $m_2$ ,

$$\Pr(m_1 \sim_w m_2) = \frac{\tilde{T}_n}{T_n - n!} = \frac{T_{n-1}}{T_n - n!}$$
(3.15)

because there are  $T_{n-1}$  preferences where  $m_1 \sim_w m_2$ , and from the total  $T_n$  we must subtract the *n*! strict preferences. One can represent

$$\Pr(m_1 \sim_w m_2) = \frac{T_{n-1}/T_n}{(T_n - n!)/T_n} = \frac{T_{n-1}/T_n}{1 - (n!/T_n)}.$$
(3.16)

I will show that  $(n!/T_n) \to 0$  which implies that,

$$\lim_{n \to \infty} \Pr(m_1 \sim_w m_2) = \lim_{n \to \infty} \frac{T_{n-1}}{T_n} = 0.$$
 (3.17)

**Lemma 4.** The quantity  $(n!/T_n)$  satisfies  $\lim_{n\to\infty} (n!/T_n) = 0$ .
*Proof.* Equation 45 in the proof of Theorem 3 directly implies that  $\Delta T_n > nT_{n-1}$ . Using this inequality recursively yields

$$\frac{n!}{T_n} < \frac{n}{n+1} \frac{(n-1)!}{T_{n-1}} < \frac{n}{(n+1)} \frac{(n-1)}{n} \frac{(n-2)!}{(n-1)!} \cdots \frac{2}{3} \frac{1!}{T_1} = \frac{2}{n+1}$$
(3.18)

since  $T_1 = 1$  and cross-terms cancel. Since  $(n!/T_n) \ge 0$  and  $2/(n+1) \to 0$  it follows immediately that

$$\lim_{n \to \infty} \frac{n!}{T_n} = 0. \qquad \blacksquare \tag{3.19}$$

This statement implies that the above non-uniform preference generating process (PGP) also results in  $Pr(m_1 \sim_w m_2) \rightarrow 0$ . Moreover, notice that if we denote the standard uniform PGP as  $\mathcal{P}$  and the PGP requiring at least one indifference as  $\mathcal{P}^*$ , then the following asymptotic approximation holds:

$$\Pr(m_1 \sim_w m_2 \mid \mathcal{P}^*) \to \Pr(m_1 \sim_w m_2 \mid \mathcal{P}) \equiv T_{n-1}/T_n.$$
(3.20)

This property allows us to preserve the proofs of all Theorems proved so far in their entirety because the limiting distribution of preferences is the same. All results obtained in Theorems 1-6 carry over immediately without any modification.

Clearly, other non-uniform PGPs may also preserve the limit  $\Pr(m_1 \sim_w m_2) \rightarrow 0$  without satisfying the last property. Extending Theorems 1-6 for such PGPs is considerably more difficult and is therefore relegated to a separate paper.

## 3.8 Conclusion

This paper establishes a simple-to-state result for the marriage problem with weak preferences: that with heterogeneous agents, a randomly selected stable matching will be approximately Pareto efficient when the number of agents approaches infinity. Specifically, the expected proportion of agents who can Pareto improve by exchanging partners is vanishingly small in the limit. This result provides a partial alleviation to the inefficiency of stable matchings under weak preferences, but it should be emphasized that this alleviation is only in *relative* terms, as it refers to the expected proportion of inefficient agents. Nothing in this result suggests that the *absolute* number of agents who can Pareto improve goes to zero for large n. Therefore, the result itself is not simply a way to get rid of indifferences with a resulting "cure-all" for the inefficiency problem; on the contrary, even though the frequency of pairwise indifferences vanishes in the limit, this has no implications for the absolute number of inefficient agents.

This result joins an interesting class of other asymptotic results recently obtained by Kojima and Pathak (2009), Kojima and Manea (2008), and Che and Kojima (2010). These authors have shown that certain finite-market shortcomings of stable mechanisms, such as the lack of strategyproofness, can vanish in large markets; my paper demonstrates an analogous result for Pareto efficiency. Taken together, these results suggest that in large markets, mechanism designers can improve not only on agens' incentives, but also on the efficiency of the stable match.

## 3.9 Appendix: Theorem Proofs

**Proposition 1.** Any one-to-one matching is Pareto Efficient if and only if it admits neither PI-Cycles nor PI-Chains.

*Proof.* Sufficiency. If the matching  $\mu$  admits a PI-Chanin or a PI-Cycle C, we can construct a matching  $\nu$  that Pareto-dominatss  $\mu$  by simply executing the trades suggested by the arrows in C. By Definitions 1 and 2,  $\nu$  Pareto-dominates  $\mu$ ; therefore  $\mu$  is not Pareto efficient.

Necessity. Assume that no PI-Cycles and no PI-Chains exist, but that the one-toone matching  $\mu$  is inefficient. Then there exists a matching  $\nu$  that Pareto dominates  $\mu$ . This implies that  $\nu(i)R_i\mu(i)$  for every i, and that there exists at least one agent  $i^*$  s.t.  $\nu(i^*)P_{i^*}\mu(i^*)$ ; therefore,  $\nu(i^*) \neq \mu(i^*)$  and therefore  $i^*$  has a different partner under  $\nu$  compared to  $\mu$ . Hence there is a non-empty set S of agents, including  $i^*$ , who switch partners from  $\mu$  to  $\nu$ .

Since  $\nu$  Pareto-dominates  $\mu$ , in particular,  $\nu(i)R_i\mu(i)$  and  $\nu(i^*)P_{i^*}\mu(i^*)$  for  $\forall i, i^* \in S$ . But this implies that each  $i \in S$  weakly points to  $\mu(\nu(i))$ , that is,  $i \stackrel{R}{\to} \mu(\nu(i))$ , while  $i^* \stackrel{P}{\to} \mu(\nu(i))$ . Moreover, notice that

$$i \in S \Rightarrow \mu(\nu(i)) \in S$$
 (3.21)

because if i is reassigned to  $\nu(i)$ , then  $\mu(\nu(i))$  must also switch partners. Since  $\mu(\nu(i)) \in S$  and S is finite, we can construct the sequence

$$i \stackrel{R}{\to} \mu(\nu(i)) \stackrel{R}{\to} \mu(\nu(\mu(v(i)))) \stackrel{R}{\to} \dots \stackrel{R}{\to} i, \quad i \in S$$
 (3.22)

where for  $i = i^*$ , the envy relation is strict. At the same time, since  $\mu(\nu(i)) \in S$ , and S is finite,

$$\mu(\nu(i)) \xrightarrow{R} \mu(\nu(\mu(\nu(i)))) \xrightarrow{R} \dots \xrightarrow{R} \mu(\nu(i)) \quad i \in S$$
(3.23)

where again for  $i^*$ , the envy relation is strict. By Definitions 1 and 2, equations 22 and 23 above imply that:

- If i is single under  $\mu$ , then S is a Pareto-improvement chain.
- If i is matched under  $\mu$ , then S is a Pareto-improvement cycle.

But this contradicts the starting assumption that no chains or cycles exist. Therefore the absence of cycles or chains implies that  $\mu$  is efficient. This completes the proof.

**Theorem 1.** Let  $\mu$  be a stable mechanism for one-to-one matching and let assumptions A1-A6 hold. Then the probability that two arbitrary fixed men  $m_1$  and  $m_2$  admit a Pareto-Improvement cycle C satisfies the upper bound

$$\Pr(m_1, m_2 \in \mathcal{C}) \le 2 \Pr(m_1 \sim_w m_2), \tag{3.24}$$

where w is an arbitrary woman.

*Proof.* My strategy will be to bound the complementary probability  $Pr(m_1 \not\sim_{\mu(m_2)} m_2)$  from **below** and show that, in the limit, it satisfies the **lower** bound

$$\Pr(m_1 \not\sim_{\mu(m_2)} m_2) \ge 1 - \Pr(m_1 \sim_w m_2), \tag{3.25}$$

which is not an identity because the woman  $\mu(m_2)$  in the LHS is variable, while the woman w in the RHS is generic (fixed). Then by reversing the last inequality I will obtain the upper bound

$$\Pr(m_1 \sim_{\mu(m_2)} m_2) \le \Pr(m_1 \sim_w m_2), \tag{3.26}$$

which is what I need; the rest follows trivially.

First, I will decompose the probability  $\Pr(m_1 \not\sim_{\mu(m_2)} m_2)$  as a union of disjoint events:

$$\Pr(m_1 \not\sim_{\mu(m_2)} m_2) = \Pr\left(\bigcup_{i=1}^n [\mu(m_2) = w_i \cap m_1 \not\sim_{w_i} m_2]\right) = (3.27)$$

$$=\sum_{i=1}^{n} \Pr[\mu(m_2) = w_i \cap m_1 \not\sim_{w_i} m_2]$$
(3.28)

To bound this sum from below, I want to separate the intersected events inside the sum using the following fact:

**Lemma 1.1.** In the limit  $n \to \infty$ ,

$$\lim_{n \to \infty} \Pr[\mu(m_2) = w_i \cap m_1 \not\sim_{w_i} m_2] \ge \lim_{n \to \infty} \Pr(\mu(m_2) = w_i) \cdot \Pr(m_1 \not\sim_{w_i} m_2) \quad (3.29)$$

*Proof of Lemma 1.1:* I first transform the inequality I want to prove to a more convenient conditional form:

$$\Pr(\mu(m_2) = w_i \cap m_1 \not\sim_{w_i} m_2) \ge \Pr(\mu(m_2) = w_i) \cdot \Pr(m_1 \not\sim_{w_i} m_2)$$
(3.30)

$$\Pr(\mu(m_2) = w_i \mid m_1 \not\sim_{w_i} m_2) \ge \Pr(\mu(m_2) = w_i).$$
(3.31)

I am interested whether this inequality holds in the limit:

$$\lim_{n \to \infty} \Pr(\mu(m_2) = w_i \mid m_1 \not\sim_{w_i} m_2) \ge \lim_{n \to \infty} \Pr(\mu(m_2) = w_i)$$
(3.32)

When women's preferences are weak, uniformly drawn, and men's preferences are strict, also from the uniform distribution, the random stable mechanism gives each woman an equal chance of being matched to a fixed man:  $\Pr(\mu(m_2) = w_i) = 1/n$ , as will be proved in the next Lemma. Therefore  $\lim_{n\to\infty} \Pr(\mu(m_2) = w_i) = 0$  and all we need is to verify whether

$$\lim_{n \to \infty} \Pr(\mu(m_2) = w_i \mid m_1 \not\sim_{w_i} m_2) \ge 0$$
(3.33)

which is always true because  $Pr(\cdot) \ge 0, (\forall n)$ .

Lemma 1.2. When preferences  $R_W$  and  $P_M$  are drawn from the uniform distribution, each woman  $w_i$  has an equal chance of being matched to a fixed man  $(m_2)$  by the Random Stable Mechanism  $\mu$ . Specifically,  $\Pr(\mu(m_2) = w_i) = 1/n$ .

Proof of Lemma 1.2: There are n! possible matchings and n matchings of each type  $\mu(m_2) = w_1, \mu(m_2) = w_2, \ldots, \mu(m_2) = w_n$ . Indeed, to see this fix  $\mu(m_2) = w_k$  and observe that the remaining n-1 men can be matched to the remaining (n-1) women in (n-1)! ways. The fraction of matchings where  $\mu(m_2) = w_k$  is therefore (n-1)!/n! = 1/n. Next observe that with uniform preferences, each given matching out of the n! possible matchings has an equal chance of being stable, so the proportion of stable matching types does not change. Indeed, fix a matching  $\mu : \mu(m_2) = w_k$ ; then each man and woman (m, w) who are not spouses blocks  $\mu$  with equal probability, because men and women draw preferences independently from the respective uniform distribution. Since stable matchings are selected at random, the RSM selects a matching s.t.  $\mu(m_2) = w_1, \mu(m_2) = w_2, \ldots, \mu(m_2) = w_n$  with equal probability 1/n.  $\Box$ 

Using Lemma 1.1, I can be sure that in the limit,

$$\lim_{n \to \infty} \Pr(m_1 \not\sim_{\mu(m_2)} m_2) = \lim_{n \to \infty} \sum_{i=1}^n \Pr[\mu(m_2) = w_i \cap m_1 \not\sim_{w_i} m_2] \ge (3.34)$$

$$\geq \lim_{n \to \infty} \sum_{i=1}^{n} \underbrace{\Pr(\mu(m_2) = w_i)}_{1/n} \cdot \underbrace{\Pr(m_1 \not\sim_{w_i} m_2)}_{1 - \Pr(m_1 \sim_{w_i} m_2)}$$
(3.35)

Since all women draw the same iid preferences,  $\Pr(m_1 \sim_{w_i} m_2)$  does not depend on the index *i*, so the summation becomes

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{n} \cdot [1 - \Pr(m_1 \sim_w m_2)] = \lim_{n \to \infty} n \left[ \frac{1}{n} \left( 1 - \Pr(m_1 \sim_w m_2) \right) \right] = \lim_{n \to \infty} [1 - \Pr(m_1 \sim_w m_2)]$$

where w is now a generic, fixed woman because the probability of pairwise indifference is the same for each woman i. (Notice that the fact  $Pr(\mu(m_2) = w_i) = 1/n$  is key.) Therefore I obtained

$$\lim_{n \to \infty} \Pr(m_1 \not\sim_{\mu(m_2)} m_2) \ge \lim_{n \to \infty} [1 - \Pr(m_1 \sim_w m_2)]$$
(3.36)

which, in turn, has implications for the complementary probability  $\Pr(m_1 \sim_{\mu(m_2)} m_2)$ :

$$\lim_{n \to \infty} \Pr(m_1 \sim_{\mu(m_2)} m_2) \equiv 1 - \lim_{n \to \infty} \Pr(m_1 \not\sim_{\mu(m_2)} m_2)$$
(3.37)

$$\leq 1 - \lim_{n \to \infty} [1 - \Pr(m_1 \sim_w m_2)]$$
(3.38)

therefore

$$\lim_{n \to \infty} \Pr(m_1 \sim_{\mu(m_2)} m_2) \le \lim_{n \to \infty} \Pr(m_1 \sim_w m_2)$$
(3.39)

which is exactly the needed upper bound.  $\blacksquare$ 

**Theorem 3.** The probability of pairwise indifference between two fixed agents  $m_1$  and  $m_2$  as  $n \to \infty$  satisfies the limit

$$\lim_{n \to \infty} \Pr(m_1 \sim_w m_2 ; n) = \lim_{n \to \infty} \frac{\tilde{T}_n}{T_n} = 0.$$
(3.40)

Proof.

Observe that

$$\frac{\tilde{T}_n}{T_n} = \frac{T_{n-1}}{T_n} = \frac{T_{n-1}}{T_{n-1} + \Delta T_n} = \frac{1}{1 + \frac{\Delta T_n}{T_{n-1}}}.$$
(3.41)

I will prove that  $\frac{\Delta T_n}{T_{n-1}} \to \infty$ . From the recurrence relation

$$T_n = \sum_{i=0}^{n-1} \binom{n}{i} T_i,$$
(3.42)

and taking into account that  $T_0 = 1$  and  $\binom{n}{0} = \binom{n-1}{0} = 1$ , it follows that

$$\Delta T_n \equiv T_n - T_{n-1} = \tag{3.43}$$

$$= \left[ \binom{n}{1} - \binom{n-1}{1} \right] T_1 + \ldots + \left[ \binom{n}{n-2} - \binom{n-1}{n-2} \right] T_{n-2} + \binom{n}{n-1} T_{n-1}.$$
(3.44)

We can simplify this using Pascal's formula  $\binom{n}{k} - \binom{n-1}{k} = \binom{n-1}{k-1}$  and the fact that  $\binom{n}{n-1} = n$ , from which we get

$$\Delta T_n = nT_{n-1} + \sum_{i=0}^{n-3} \binom{n-1}{i} T_{i+1}.$$
(3.45)

From this recurrence it is evident that:

$$\frac{\Delta T_n}{T_{n-1}} = n + \underbrace{\frac{\sum_{i=0}^{n-3} {\binom{n-1}{i}} T_{i+1}}{T_{n-1}}}_{\equiv Z(n)>0}$$
(3.46)

Therefore

$$\lim_{n \to \infty} \frac{\Delta T_n}{T_{n-1}} = \lim_{n \to \infty} [n + Z(n)] = \infty.$$
(3.47)

and

$$\lim_{n \to \infty} P(n) = \lim_{n \to \infty} \frac{1}{1 + \frac{\Delta T_n}{T_{n-1}}} = \frac{1}{1 + \infty} = 0. \quad \blacksquare$$
(3.48)

**Theorem 4.** Let assumptions A1-A6 be satisfied. Then the expected proportion of agents that can participate in a Pareto-Improvement cycle converges to zero as  $n \to \infty$ .

*Proof.* I will consider all  $\binom{n}{2}$  2-ples that can be formed from the set of n men and argue that (1) the expected proportion of 2-ples admitting a cycle converges to 0, and (2) the expected proportion of *men* involved in 2-ples admitting a cycle tends to 0.

1) Since indifferences of the type  $m_i \sim_{\mu(w_j)} m_j$  occur independently and the assignment is random, whether an arbitrary 2-ple  $(m_i, m_j)$  admits a cycle is independent of whether any other 2-ple  $(m_k, m_z)$  admits a cycle, so the expected proportion of 2-ples admitting cycles is the same as the probability of a generic 2-ple admitting a cycle. In Theorem 3 it was shown that for arbitrary two men  $m_i$  and  $m_j$ ,  $\Pr(m_i, m_j \in \mathcal{C}) \to 0$ , therefore so does the expected proportion of 2-ples admitting cycles.

2) Now it remains to translate this result from the expected proportion of 2-ples to the expected proportion of *agents*. Let  $a_n$  be the expected *number* of men admitting a cycle, so that  $\frac{a_n}{n}$  is the expected *proportion* of men admitting a cycle. Since the expected proportion of 2-ples admitting a cycle  $\binom{a_n}{2} / \binom{n}{2}$  goes to 0, this implies

$$\frac{\binom{a_n}{2}}{\binom{n}{2}} \to 0 \quad \Rightarrow \quad \frac{a_n(a_n-1)}{n(n-1)} \to 0 \tag{3.49}$$

Further notice that  $\lim_{n\to\infty} \frac{a_n}{n} = \lim_{n\to\infty} \frac{a_n-1}{n-1}$ , so if one limit exists, so does the other and

$$\left(\lim_{n \to \infty} \frac{a_n}{n}\right)^2 = \lim_{n \to \infty} \frac{a_n}{n} \cdot \lim_{n \to \infty} \frac{a_n - 1}{n - 1} = \lim_{n \to \infty} \frac{a_n(a_n - 1)}{n(n - 1)} = 0, \quad (3.50)$$

from which it follows that  $\lim_{n\to\infty} \frac{a_n}{n} = 0$ . Therefore, the expected proportion of men admitting a cycle converges to zero. Since the matching is one-to-one, the same is true for the expected proportion of women engaged in cycles.

(Alternatively, if the limit  $\lim_{n\to\infty} \frac{a_n}{n}$  doesn't exist, then neither does  $\left(\lim_{n\to\infty} \frac{a_n}{n}\right)^2 = \lim_{n\to\infty} \left[\binom{a_n}{2} / \binom{n}{2}\right]$ , in contradiction to  $\lim_{n\to\infty} \left[\binom{a_n}{2} / \binom{n}{2}\right] = 0$ , so this case is impossible.)

**Theorem 5.** Let  $\mu$  be a matching produced by the Random Stable Mechanism and let assumptions A1-A3,A4',A5-A6 hold. Then the probability that two arbitrary fixed men  $m_1$  and  $m_2$  admit a Pareto-Improvement cycle C satisfies the upper bound

$$\Pr(m_1, m_2 \in \mathcal{C}) \le 2\Pr(m_1 \sim_w m_2) + 2\Pr(w' \sim_m w''), \tag{3.51}$$

where w, w' and w'' are any fixed women and m is any fixed man.

*Proof.* The logic is very similar to that in Theorem 1. Decompose

$$\Pr(\mu(m_1) \not\sim_{m_1} \mu(m_1)) = \Pr\left(\bigcup_{i \neq j} [\mu(m_1) = w_i] \cap [\mu(m_2) = w_j] \cap [w_i \not\sim_{m_1} w_j]\right) = (3.52)$$

$$= \sum_{i \neq j} \Pr([\mu(m_1) = w_i] \cap [\mu(m_2) = w_j] \cap [w_i \not\sim_{m_1} w_j]) \ge \text{ (by Lemma 5.1)} \quad (3.53)$$

$$\geq \sum_{i \neq j} \Pr([\mu(m_1) = w_i] \cap [\mu(m_2) = w_j]) \cdot \Pr(w_i \not\sim_{m_1} w_j) =$$
(3.54)

$$= \sum_{i \neq j} \frac{1}{\binom{n}{2}} (1 - \Pr(w_i \sim_{m_1} w_j)) = \binom{n}{2} \frac{1}{\binom{n}{2}} (1 - \Pr(w_i \sim_{m_1} w_j)) = (3.55)$$

 $= 1 - \Pr(w_i \sim_{m_1} w_j). \tag{3.56}$ 

Therefore

$$\Pr(\mu(m_1) \sim_{m_1} \mu(m_1)) \le \Pr(w_i \sim_m w_j) \tag{3.57}$$

and because men and women's preferences are independent,

$$\Pr[m_1 \sim_{\mu(m_1)} m_2 \cup m_1 \sim_{\mu(m_2)} m_2 \cup \mu(m_1) \sim_{m_1} \mu(m_2) \cup \mu(m_1) \sim_{m_2} \mu(m_2)] \le \le 2\Pr(m_1 \sim_w m_2) + 2\Pr(w' \sim_m w'').$$

**Lemma 5.1.** In the limit  $n \to \infty$ ,

$$\Pr([\mu(m_1) = w_i] \cap [\mu(m_2) = w_j] \cap [w_i \not\sim_{m_1} w_j]) \ge$$
$$\ge \Pr([\mu(m_1) = w_i] \cap [\mu(m_2) = w_j]) \cdot \Pr([w_i \not\sim_{m_1} w_j])$$

Proof. Rewriting the inequality in conditional form, we need to verify that

$$\lim_{n \to \infty} \Pr(\mu(m_1) = w_i \cap \mu(m_2) = w_j \mid w_i \not\sim_{m_1} w_j) \ge \underbrace{\lim_{n \to \infty} \Pr(\mu(m_1) = w_i \cap \mu(m_2) = w_j)}_{0}$$
(3.58)

which is verified using the fact that  $Pr(\mu(m_1) = w_i \cap \mu(m_2) = w_j) \leq Pr(\mu(m_1) = w_i) = 1/n \to 0$  as in Lemma 1.1.

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