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TESTING THE NULL HYPOTHESIS OF STATIONARITY AGAINST AN AUTOREGRESSIVE UNIT ROOT ALTERNATIVE

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1 Introduction

There is a large literature in time series econometrics on the debate about whether economic time series are best characterized as trend stationary processes or difference stationary processes. Since the influential article by Nelson and Plosser (1982), hundreds of economic time series have been examined by unit root tests and empirical evidence has accumulated that many economic and financial time series contain a unit root (Meese and Singleton, 1982; Perron, 1988; Christano, 1992; Banerjee, et al., 1990; Gil-Alana and Robinson 1997; among others). However, as argued elsewhere (see for example Kwiatkowski et al., 1992), most standard testing procedures consider the null hypothesis of a unit root which ensures that the null hypothesis is accepted unless there is strong evidence against it. Monte Carlo evidence (Schwert, 1989; Diebold and Rudebusch, 1991; Dejong, et al., 1992; Ng and Perron, 1995; Stock, 1995) show that the discriminatory power of unit root tests is often low, indicating standard unit root tests are not very powerful against trend stationary alternatives. Indeed, different results have been obtained from other approaches. By allowing for structural breaks in the deterministic trend, Perron (1989) rejected the unit root hypothesis at the 5% level of significance for eleven out of fourteen of the Nelson-Plosser series. Using a flat prior Bayesian technique, DeJong and Whiteman (1989b) challenged the classical unit root tests results in many cases. Phillips (1991) provided an alternative Bayesian approach using a Jeffrey's prior and found support for a unit root in five of the series (also see, inter alia, Zivot and Andrews, 1992; Schotman and van Dijk, 1990; Zivot and Phillips, 1994; Phillips and Ploberger, 1994; Stock 1994).

Given these empirical results and Monte Carlo evidence, to decide whether a time series is trend stationary or difference stationary, it would be useful to perform tests for the null hypothesis of stationarity as well as tests for a unit root. However, although the literature on testing the null hypothesis of a unit root is huge (see, inter alia, Dickey and Fuller 1979; Phillips 1987; Phillips and Perron 1988; Robinson 1994), there have been only several attempts on testing stationarity (Park and Choi, 1988; Rudebusch, 1988; Kwiatkowski et al., 1992; Leybourne and McCabe, 1994; Fukushige, Hatanaka, and Koto, 1994). In particular, Kwiatkowski et al. (1992) (hereafter KPSS, see also King, 1980; King and Hiller, 1985; Nyblom and Makelainen, 1983; Nyblom, 1986; Saikkonen and Luukkonen, 1993; and Tanaka, 1990) considered a time series model that can be decomposed as the sum of a deterministic trend, a random walk, and a stationary error, and proposed an LM test for the null hypothesis of stationarity. Leybourne and McCabe (1994) suggested a similar test which differs from the KPSS test in its treatment of autocorrelation and applies when the null hypothesis is an AR(k) process.

We believe that fluctuation tests for structural stability can provide another way to distinguish between stationary and unit root processes. Testing for structural stability has long been an important topic in statistics and econometrics (Hawkins, 1977; Andrews, 1993; Andrews and Ploberger, 1994; Chu, et al., 1995; Perron, 1991; Kuan and Hornik, 1995; Bai, 1996; Kuan 1998). This paper provides a straightforward test for the null hypothesis of stationarity (or trend stationarity) by an application of fluctuation tests. The driving force behind the proposed test is as follows: If y_t is a stationary time series, it has a fixed mean, finite variance and cannot grow indefinitely. However, a unit root process has unbounded variance and grows in a secular way over long period of time. As a result, the fluctuation of a unit root process is much larger than that of a stationary process. This suggests that we can test whether or not y_t is stationary by looking at the fluctuation in the time series. If a time series displays too much fluctuation, we should reject the null hypothesis of stationarity. More generally, if a time series y_t can be represented as the summation of a deterministic trend $\gamma' x_t$ and a stochastic component y_t^s, we can test the hypothesis of trend stationarity against difference stationarity by looking at the fluctuation in the detrended time series.

Notice that the KPSS test can be derived as a special case of the test by Nabeya and Tanaka (1988) for random coefficients, and thus it can also be treated as an application of structural stability test. Our test provides an alternative to the KPSS test in this sense. However, in our test, instead of decomposing the stochastic process into a random walk and a stationary component and deriving an LM multiplier test under the Gaussian assumption, we simply look at the fluctuation in the time series and provide a straightforward test for stationarity. Our test is an asymptotic test. It is shown that the suggested test is consistent and can be applied to general time series models. Limiting distribution of the test is derived under both the null and the unit root alternative, and critical values for the leading cases are provided based on simulation experiments. Size and power properties of these tests in a finite sample are also examined.

The paper is organized as follows: in Section 2 below, we propose the test statistic for stationarity and derive its limiting distribution. Tables of critical values are also provided. Section 3 discusses the asymptotic properties of the test under the unit root alternative. Section 4 reports its finite sample size and power based on

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a Monte Carlo experiment. A small empirical application of the test to some U.S. macroeconomic time series is given in Section 5, and Section 6 concludes. Proofs are provided in the Appendix. For notation, we use "$\Rightarrow$" to signify weak convergence, and $[nr]$ to signify the integer part of nr.

2 Testing the Null Hypothesis of Stationarity

Suppose that the observed time series z_t can be written as the sum of a deterministic trend d_t and a stochastic component y_t :

$$z_t = d_t + y_t, \; t = 1, ..., n, \tag{1}$$

$$y_t = \alpha y_{t-1} + u_t. \tag{2}$$

The deterministic trend d_t depends on unknown parameters and is specified as $d_t = \gamma' x_t$, where $\gamma = (\gamma_0, ..., \gamma_p)'$ is a vector of trend coefficient and x_t is a deterministic trend of known form, say, $x_t = (1, t, ..., t^p)'$. The leading cases of the deterministic component are (i) a constant term $x_t = 1$; and (ii) a linear time trend $x_t = (1, t)'$. y_t is the stochastic component in time series z_t. For convenience in deriving asymptotic theory, we assume that the disturbances u_t follow a general linear process whose coefficients satisfy the summability conditions given in the following Assumptions.

Assumption L$_1$ (Linear Process): $u_t = C(L)\varepsilon_t$, where ε_t is an i.i.d. process with zero mean and finite variance σ_ε^2, and $C(L) = \sum_{j=0}^{\infty} c_j L^j$, where L is the lag operator defined as $L\varepsilon_t = \varepsilon_{t-1}$, $C(1) \neq 0$, $\sum_{j=1}^{\infty} j^2 c_j^2 < \infty$.

The linear process assumption facilitates a straightforward asymptotic analysis by applications of the methods of Phillips and Solo (1992). Similar results could be obtained under strong mixing conditions (e.g. Phillips and Perron, 1988) which also ensure invariance principles for the partial sums of u_t. Notice that the asymptotic analysis of linear processes holds under a variety of conditions, and the limiting results of our test can also be generalized to different classes of time series innovations. For example, with a strengthening of the moment and the summability condition, our results can be generalized to time series with stationary martingale difference sequence innovations.

Assumption L$_2$: $u_t = C(L)\varepsilon_t$, where ε_t is a stationary martingale difference sequence with respect to the natural filtration, $E(\varepsilon_t^{2+\eta}) < \infty$ for some $\eta > 0$, and $C(L) = \sum_{j=0}^{\infty} c_j L^j$, $C(1) \neq 0$, $\sum_{j=1}^{\infty} j\,|c_j| < \infty$.

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The linear process includes quite general classes of time series models like the ARMA process. Assumptions L_1 and L_2 ensure that u_t is covariance stationary and has positive spectral density at the origin, thereby ensuring that the unit root in y_t does not cancel (as it would if u_t had a moving average unit root, in which case the spectral density would be zero at the origin). The summability conditions are useful in validating the following expansion of the operator $C(L)$

$$C(L) = C(1) + \widetilde{C}(L)(L-1), \tag{3}$$

where $\widetilde{C}(L) = \sum_{j=0}^{\infty} \widetilde{c}_j L^j$ and $\widetilde{c}_j = \sum_{j+1}^{\infty} c_s$. This expansion gives rise to an explicit martingale difference decomposition of u_t

$$u_t = C(1)\varepsilon_t + \widetilde{\varepsilon}_{t-1} - \widetilde{\varepsilon}_t, \quad \text{with } \widetilde{\varepsilon}_t = \widetilde{C}(L)\varepsilon_t, \tag{4}$$

This decomposition is sometimes called the martingale decomposition in the probability literature (see Hall and Heyde, 1980) because the first term of (4) is a martingale difference and the partial sums $\sum_{s=1}^{t} u_s$ correspondingly have the leading martingale term $C(1)\sum_{s=1}^{t} \varepsilon_s$. The decomposition (4) was justified by Phillips and Solo (1992), and can be used to prove that the partial sums of the time series u_t satisfy a functional central limit theorem (see Phillips and Solo, 1992, Theorem 3.4 and Theorem 3.15, for a demonstration), i.e., $n^{-1/2}\sum_{t=1}^{[nr]} u_t \Rightarrow B_u(r), 0 \leq r \leq 1$, where $B_u(r)$ is a Brownian motion with variance $C(1)^2\sigma_\varepsilon^2$, $[nr]$ signifies the integer part of nr and $r \in [0,1]$ represents some fraction of the sample data. If we denote the corresponding standardized Brownian motion as $W(r)$, then $B_u(r) = \omega W(r)$, where $\omega^2 = C(1)^2\sigma_\varepsilon^2$ is called the long-run variance of the process u_t, and equals $2\pi f_{uu}(0)$, where $f_{uu}(\cdot)$ is the spectral density of the process u_t. If $|\alpha| < 1$, y_t is a stationary process, and when $\alpha = 1$, y_t has an autoregressive unit root.

We want to test the hypothesis that y_t is stationary, or in another word, z_t is trend stationary, corresponding to $H_0 : |\alpha| < 1$, against the unit root alternative $H_1 : \alpha = 1$. Under Assumption L and H_0, y_t is stationary and $n^{-1/2}\sum_{t=1}^{[nr]} y_t \Rightarrow B_y(r) = (1-\alpha)^{-1}B_u(r)$, where the limiting process $B_y(r)$ is a Brownian motion of variance $\omega_y^2 = \omega^2/(1-\alpha)^2$. Under H_1, y_t is an integrated process such that $y_t = \sum_{j=1}^{t} u_j + O_p(1)$, and $n^{-1/2}y_{[nr]} \Rightarrow B_u(r)$.

As mentioned in the previous section, an important difference between a unit root process and a stationary time series is that a unit root process has unbounded variance and wanders around in a random way with no fixed mean. Thus, to test whether or not y_t is stationary, we look at the fluctuation in the data and reject

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the null hypothesis of stationarity whenever there is excessive fluctuation. If y_t were observable and ω_y were known, consider the following quantity as a measurement of fluctuation in time series y_t (Sen, 1980; Ploberger, Kramer, and Kortrus, 1989),

$$\max_{k=1,\ldots,n} \frac{k}{\omega_y \sqrt{n}} \left| \frac{1}{k} \sum_{t=1}^{k} y_t - \frac{1}{n} \sum_{t=1}^{n} y_t \right| . \tag{5}$$

This is the recursive-estimates test statistic for fluctuation. Ploberger, Kramer, and Kortrus (1989) use a similar statistic to test the structural stability in linear regression models. Under H_0, it can be shown that the statistic (5) converges weakly to $\sup_{0\leq r\leq 1} \left|\widetilde{W}(r)\right|$, where $\widetilde{W}(r) = W(r) - rW(1)$ is a standard Brownian bridge which is tied down to the origin at the end of the $[0, 1]$ interval. Under the alternative hypothesis, y_t is a unit root process and it is easy to verify that the statistic (5) has much larger order of magnitude, diverging to ∞ at rate n.

Notice that in practical analysis the long-run variance parameter ω_y is unknown and thus (5) can not be used directly for testing stationarity. However, ω_y^2 can be consistently estimated (Phillips, 1987; Andrews, 1991). In this paper, we consider the following nonparametric kernel estimator for ω_y^2 given by $\widehat{\omega}_y^2 = 2\pi \widehat{f}_{yy}(0)$, where

$$\widehat{f}_{yy}(0) = \frac{1}{2\pi} \sum_{h=-M}^{M} k(\frac{h}{M}) C_{yy}(h) \tag{6}$$

is the conventional spectral density estimator. In formula (6), $C_{yy}(h)$ is the sample variance defined as $n^{-1} \sum' \widehat{y}_t \widehat{y}_{t+h}$, where $\sum'$ signifies summation over $1 \leq t, t+h \leq n$, $k(\cdot)$ is the lag window defined on $[-1, 1]$ with $k(0) = 1$, and M is the bandwidth parameter satisfying the property that $M \to \infty$ and $M/n \to 0$ (say $M = O(n^{1/3})$ as in Andrews, 1991) as the sample size $n \to \infty$. Then, $\widehat{\omega}_y^2$ is a consistent estimator of ω_y^2 under H_0. Candidate kernel functions can be found in standard texts (e.g. Hannan, 1970; Brillinger, 1980; and Priestley, 1981). For example, when we use $k(x) = 1 - |x|$, we get the Bartlett estimator.

However, y_t is generally unobservable since the deterministic component $\gamma' x_t$ is unknown. In order to test H_0, we need to estimate y_t (detrend z_t) first and then test stationarity by looking at the fluctuation in the detrended data. Assume that there is a standardizing matrix D such that $D^{-1} x_{[nr]} \to X(r)$ as $n \to \infty$, uniformly in $r \in [0, 1]$. For the case of a linear trend, $D = diag[1, n]$ and $X(r) = (1, r)'$. More generally, if x_t is a polynomial trend $D = diag[1, n, ..., n^p]$, then $X(r) = (1, r, ..., r^p)$. We detrend z_t by least squares regression

$$z_t = \widehat{\gamma}' x_t + \widehat{y}_t, \tag{7}$$

and denote the detrended time series as $\widehat{y}_t = z_t - \widehat{\gamma}' x_t$, where $\widehat{\gamma} = \left[\sum_t x_t x_t'\right]^{-1} \left[\sum_t x_t z_t\right]$. The following statistic can then be used in testing (trend) stationarity in time series (1)

$$\underline{S}_n = \max_{k=1,\dots,n} \frac{k}{\widehat{\omega}_y \sqrt{n}} \left| \frac{1}{k} \sum_{t=1}^{k} \widehat{y}_t - \frac{1}{n} \sum_{t=1}^{n} \widehat{y}_t \right|. \tag{8}$$

Under H_0, the partial sum of the detrended time series converges to the following limiting process:

$$\begin{aligned} n^{-1/2} \sum_{t=1}^{[nr]} \widehat{y}_t \;&\Rightarrow\; \widetilde{B}_{y,X}(r) = B_y(r) - \left[\int_0^1 dB_y(s) X(s)'\right] \left[\int_0^1 X(s)X(s)'ds\right]^{-1} \int_0^r X(s)ds \\ &= \omega_y \left\{ W(r) - \left[\int_0^1 dW(s)X(s)'\right] \left[\int_0^1 X(s)X(s)'ds\right]^{-1} \int_0^r X(s)ds \right\} \\ &= \omega_y \widetilde{W}_X(r). \end{aligned}$$

where $\widetilde{W}_X(r) = W(r) - \left[\int_0^1 dW(s)X(s)'\right] \left[\int_0^1 X(s)X(s)'ds\right]^{-1} \int_0^r X(s)ds.$

The limiting process, $\widetilde{B}_{y,X}(r) = \omega_y \widetilde{W}_X(r)$, is a generalized Brownian bridge process. When x_t has a constant element, the process $\widetilde{W}_X(r)$ (or $\widetilde{B}_{y,X}(r)$) is tied down to the origin at the ends of the [0,1] interval just like a Brownian bridge. In the case that x_t is a constant, $\widetilde{W}_X(r) = W(r) - rW(1)$ is a standard Brownian bridge. If x_t is a linear trend, i.e. $x_t = (1, t)'$,

$$\widetilde{W}_X(r) = [W(r) - rW(1)] + 6r(1-r)\left[\frac{1}{2}W(1) - \int_0^1 W(s)ds\right],$$

which is the sum of a standard Brownian bridge plus another factor

$$6r(1-r)\left[\frac{1}{2}W(1) - \int_0^1 W(s)ds\right],$$

brought by the addition of a time trend t. This process is usually called a second-level Brownian bridge (MacNeill, 1978).

We summarize the asymptotic results in the following Theorem.

THEOREM 1: *Under H_0 and Assumption L_1 (or L_2), when x_t has a constant element, as $n \to \infty$,*

$$\underline{S}_n \Rightarrow \sup_{0 \le r \le 1} \left| \widetilde{W}_X(r) \right|,$$

where $\widetilde{W}_X(r) = W(r) - \left[\int_0^1 dW(s)X(s)'\right] \left[\int_0^1 X(s)X(s)'ds\right]^{-1} \int_0^r X(s)ds.$

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Similar to many other testing procedures in the unit root context, the asymptotic distribution of $\underline{S}_n$ depends on the limiting function of the deterministic trend. For the leading cases where x_t equals a constant and a linear trend, we denote the test statistics as S_n^μ and S_n^τ respectively, with the superscripts μ and τ indicating that S_n^μ uses the demeaned data and S_n^τ uses the detrended data.

REMARK 1: *In the case that x_t equals a constant, $\widetilde{W}_X(r) = W(r) - rW(1)$ is a standard Brownian bridge. As shown in Billingsley (1968), the corresponding distribution function of this limiting variate $\sup_{0\le r\le 1}\left|\widetilde{W}_X(r)\right|$ has the classical Kolmogoroff-Smirnoff form that $F(x) = \Pr(\sup_{0\le r\le 1}\left|\widetilde{W}(r)\right| \le x) = 1 + 2\sum_{j=1}^{\infty}(-1)^j \exp(-2j^2x^2)$, for $x \ge 0$, and 0, for $x < 0$. As a comparison, notice that by using a different measure (Cramer-von Mises) of the fluctuation in time series y_t, we can immediately derive the (demeaned) KPSS test, which has the Cramer-von Mises limiting distribution and can be represented as an infinite weighted sum of independent central chi-squared random variables. In this sense, both the $\underline{S}_n$ test and the KPSS test can be obtained by testing the fluctuations in the detrended time series. Our procedure corresponds to the Kolmogoroff-Smirnoff test and the KPSS approach is of Cramer-von Mises type.*

REMARK 2: *If there is no deterministic trend in z_t, y_t is observable and the test statistic can be constructed by simply using y_t in the formula of (8). It is easy to show that under the conditions in Theorem 1, the test statistic converges to a standard Brownian bridge.*

Table 1 gives critical values for the test statistics S_n^μ and S_n^τ. The critical values of S_n^τ are calculated by a direct simulation using a sample size of 3,000 and 50,000 replications.

TABLE 1. Upper tail critical values for S_n^μ and S_n^τ

	S_n^μ			S_n^τ		
Critical level	0.1	0.05	0.01	0.1	0.05	0.01
Critical value	1.22	1.36	1.63	0.827	0.901	1.041

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3 Consistency

It is critical that a statistical test be able to discriminate between the null and the alternative in large sample. Under the alternative hypothesis and Assumption L, $n^{-1/2}y_{[nr]} \Rightarrow B_u(r)$, and

$$n^{-1/2}\widehat{y}_{[nr]} \Rightarrow B_{u,X}(r) = B_u(r) - \left[\int_0^1 B_u(s)X(s)'ds\right]\left[\int_0^1 X(s)X(s)'ds\right]^{-1} X(r).$$

Thus, $n^{-1}\sum_{t=1}^{[nr]}(\widehat{y}_t/\sqrt{n}) \Rightarrow \int_0^r B_{u,X}(s)ds$, and we have from the fact that $[nr]/n \to r$ and the continuous mapping theorem that, as $n \to \infty$,

$$\begin{aligned}
&\frac{1}{n}\left\{\max_{k=1,\ldots,n}\frac{k}{\sqrt{n}}\left|\frac{1}{k}\sum_{t=1}^{k}\widehat{y}_t - \frac{1}{n}\sum_{t=1}^{n}\widehat{y}_t\right|\right\} \\
= &\sup_{1\le r\le 1}\left|\frac{1}{n}\sum_{t=1}^{[nr]}\left(\frac{\widehat{y}_t}{\sqrt{n}}\right) - \frac{[nr]}{n}\frac{1}{n}\sum_{t=1}^{n}\left(\frac{\widehat{y}_t}{\sqrt{n}}\right)\right| \\
\Rightarrow &\sup_{0\le r\le 1}\left|\int_0^r B_{u,X}(s)ds - r\int_0^1 B_{u,X}(s)ds\right|.
\end{aligned}$$

Thus $\max_{k=1,\ldots,n}\frac{k}{\sqrt{n}}\left|\frac{1}{k}\sum_{t=1}^{k}\widehat{y}_t - \frac{1}{n}\sum_{t=1}^{n}\widehat{y}_t\right|$ diverges at rate n under H_1.

However, under H_1, the nonparametric spectral density estimate $\widehat{f}_{yy}(0)$ diverges as well. In order to show the consistency of the test, we need to prove that $\widehat{\omega}_y$ diverges at a slower rate. This is confirmed by the following Lemma.

LEMMA 1: *Under H_1 and Assumption L_1 (or L_2), as $n \to \infty$,*

$$\frac{1}{nM}\widehat{\omega}_y^2 \Rightarrow 2\pi K(0)\int_0^1 B_{u,X}(r)^2 dr,$$

where $B_{u,X}(r) = B_u(r) - \left[\int_0^1 B_u(s)X(s)'ds\right]\left[\int_0^1 X(s)X(s)'ds\right]^{-1} X(r)$ is a detrended Brownian motion, and $K(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} k(x)e^{-i\lambda x}dx$ is the spectral window.

In consequence, we obtain the following Theorem on the consistency of the test.

THEOREM 2: *Under H_1 and Assumption L, as $n \to \infty$, $\Pr[\underline{S}_n > B_n] \to 1$, for any nonstochastic sequence $B_n = o(n^{1/2}M^{-1/2})$.*

REMARK: *Here we get a similar result as in Kwiatkowski et al. (1992) that the divergence rate of $\underline{S}_n$ under H_1 is dependent on the bandwidth parameter.*

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4 Finite Sample Performance

A Monte Carlo experiment was conducted to examine the finite sample performance of these tests. We considered the leading cases where the deterministic components are a constant term and a linear time trend, i.e., the S_n^μ and S_n^τ tests. From the construction of these test statistics, the finite sample performance of S_n^μ and S_n^τ depends on the sample size n and the bandwidth parameter M that is used to calculate $\widehat{\omega}_y^2$. Thus, special attention was paid to the effects of the bandwidth and sample sizes on the performance of these tests. We considered the following sample sizes in our experiment: $n = 50, 80, 100, 120, 150, 200, 300, 500$. These sample sizes were chosen because they represent the most relevant range of sample sizes in many empirical analyses. Four bandwidth choices were considered, the first two bandwidth values, $M1 = 1$, $M2 = 2$, are small and fixed, while the third and fourth bandwidth, $M3 = [4(n/100)^{1/4}]$, $M4 = [12(n/100)^{1/4}]$, are functions of the sample sizes and are increasing with n. These bandwidth values were used because similar choices had been used in Schwert (1989), Kwiatkowski et al. (1992), and other simulations. In the presence of serial correlation, we need the bandwidth increase with n in estimating the long-run variance. Thus, we expect that small fixed bandwidth will have relatively better effect for the iid case and cases of small serial correlation, and $M3$ and $M4$ will work better for cases with high serial correlation. All experiments used 10,000 replications. For the kernel function, following Kwiatkowski et al. (1992), we used the Bartlett window $k(x) = 1 - |x|$ so that the nonnegativity of $\widehat{\omega}_y^2$ was guaranteed.

We examined the size and power of the S_n^μ and S_n^τ tests. For the purpose of comparison, we also calculated the empirical size and power of demeaned and detrended KPSS tests. In each iteration, $\underline{S}_n$ and the KPSS test were calculated based on the same data. First, we consider the size of these tests when the process y_t is a sequence of *i.i.d.* random variables. Table 2 reports the size of S_n^μ and the demeaned KPSS test corresponding to different n and M values at the 5% level, and Table 3 gives the results for S_n^τ and the detrended KPSS test. We can see that these tests have reasonable size except for the cases with a small sample and large M. Since y_t is an *iid* sequence, we expect that a small M will produce better performance, and this is confirmed in the simulation. We can also see that as sample size increases, size distortion reduces even for large M, corroborating the asymptotic theory.

TABLE 2: Size of Demeaned Tests, 5% level, iid case

	S_n^μ			KPSS		
n	M1	M2	M3	M1	M2	M3
50	0.030	0.026	0.016	0.044	0.042	0.037
80	0.036	0.030	0.024	0.046	0.043	0.039
100	0.038	0.033	0.027	0.046	0.045	0.043
120	0.041	0.035	0.033	0.045	0.044	0.040
150	0.042	0.039	0.034	0.048	0.049	0.047
200	0.044	0.042	0.036	0.045	0.045	0.043
300	0.046	0.044	0.041	0.052	0.050	0.048
500	0.048	0.048	0.046	0.049	0.048	0.048

TABLE 3: Size of Detrended Tests, 5% level, iid case

	S_n^τ			KPSS		
n	M1	M2	M3	M1	M2	M3
50	0.024	0.019	0.013	0.047	0.042	0.036
80	0.026	0.021	0.017	0.049	0.048	0.045
100	0.029	0.025	0.021	0.055	0.052	0.047
120	0.032	0.029	0.024	0.047	0.045	0.044
150	0.033	0.031	0.027	0.054	0.049	0.046
200	0.036	0.034	0.032	0.048	0.047	0.046
300	0.040	0.037	0.037	0.049	0.049	0.048
500	0.044	0.043	0.043	0.054	0.053	0.053

We next examined the size properties of S_n^μ and S_n^τ in the presence of serial correlation. The data were generated from $y_t = \alpha y_{t-1} + u_t$, where $u_t \equiv iidN(0, 1)$. In this model, the AR coefficient α is a convenient nuisance parameter to investigate. It measures the distance of the null from the alternative. As α approaches unity, y_t behaves more and more like a random walk. In consequence, it is anticipated that the tests will overreject the null hypothesis for positive α, and that as α increases, the empirical rejection rate of these tests will also increase, depending on how close α is to unity (see Tables 4a&b and Tables 5a&b for the cases of large α values). We examined the empirical rejection rates for cases with $\alpha = 0.1, 0.5, 0.8$, 0.85, 0.9, 0.95. Our choices of α put a particular emphasis on those values close to unity because many macroeconomic time series contain a large autoregressive root. The values 0.95, 0.9, 0.85, and 0.8 are typical values used in the "unit root" Monte Carlo experiments in literature.

Notice that the bandwidth parameter M corresponds to the number of lags used to calculate $\widehat{\omega}_y^2$. Intuitively, for $\alpha > 0$, the larger α is, the longer lags we need. In the case that $\alpha = 0$, y_t is an independent sequence and the long-run variance of y_t equals the variance of y_t. Thus, for small α, we expect that a small bandwidth value can provide reasonably good finite sample performance. As α increases, we need a larger M to estimate ω_y^2. These are confirmed in the simulation. In cases of large α values the problem of overrejection is severe for both the KPSS test and the $\underline{S}_n$ test when M is small ($M = M1 = 1$, and $M = M2 = 2$) because, according to the asymptotic theory, the validity of the tests requires M to increase with n in this circumstance. However, as will become clear in Tables 6 and 7, a large value of M reduces the power of these tests and a trade off has to be made. Tables 4a&b report the empirical size of the S_n^μ and the demeaned KPSS test for the cases with AR(1) errors, corresponding to different choices of AR coefficient, at the 5% level. Results of the detrended tests are provided in Tables 5a&b.

Tables 6 and 7 report the empirical rejection rates for the case of $\alpha = 1$, giving the power of these testing procedures. In particular, Table 6 gives the result of demeaned tests and Table 7 gives those of the detrended tests. Again, we consider the effects of the bandwidth and the sample size on the power of the tests. The tests have reasonable power in most cases (except for those with small sample and large bandwidth). As we can anticipate from the consistency, for each bandwidth choice, power usually increases as n increases. Also, according to the asymptotic analysis, the distribution of our test under the alternative hypothesis depends on n/M; a large M will generally reduce the power. This is also confirmed by the results in Tables 6 and 7.

A word on the comparison between $\underline{S}_n$ and the KPSS test. Although results differ across α values and sample sizes, it can be seen that the bandwidth choice and the value of α have similar effects on these tests. It is clear from the Monte Carlo evidence that in general these two tests have very similar finite sample behavior, corroborating Remark 1 that both the $\underline{S}_n$ and the KPSS tests can be derived from fluctuation tests, but using different matrics.

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TABLE 4a : Size of Demeaned Tests, 5% level, AR(1) Errors

		S_n^μ				KPSS			
α	n	M1	M2	M3	M4	M1	M2	M3	M4
0.8	50	0.458	0.270	0.052	0.015	0.487	0.350	0.128	0.069
	80	0.503	0.349	0.072	0.023	0.537	0.380	0.162	0.065
	100	0.528	0.391	0.109	0.028	0.556	0.409	0.181	0.084
	120	0.584	0.408	0.121	0.033	0.563	0.414	0.177	0.091
	150	0.563	0.425	0.140	0.038	0.579	0.427	0.189	0.095
	200	0.582	0.428	0.162	0.042	0.599	0.435	0.190	0.097
	300	0.597	0.438	0.198	0.048	0.602	0.447	0.210	0.107
	500	0.606	0.452	0.210	0.049	0.638	0.475	0.212	0.103
0.5	50	0.271	0.135	0.042	0.037	0.319	0.165	0.074	0.065
	80	0.310	0.166	0.044	0.012	0.368	0.186	0.066	0.045
	100	0.362	0.180	0.050	0.019	0.383	0.190	0.072	0.047
	120	0.398	0.192	0.053	0.028	0.414	0.197	0.070	0.049
	150	0.411	0.197	0.062	0.029	0.428	0.202	0.078	0.051
	200	0.412	0.221	0.067	0.032	0.431	0.224	0.077	0.052
	300	0.432	0.227	0.072	0.039	0.449	0.251	0.082	0.055
	500	0.436	0.242	0.080	0.044	0.461	0.264	0.088	0.062
0.1	50	0.058	0.036	0.020	0.011	0.071	0.052	0.040	0.065
	80	0.062	0.041	0.026	0.029	0.072	0.054	0.044	0.035
	100	0.065	0.045	0.030	0.021	0.074	0.057	0.049	0.038
	120	0.068	0.049	0.036	0.025	0.072	0.055	0.045	0.039
	150	0.070	0.051	0.038	0.025	0.075	0.059	0.054	0.038
	200	0.072	0.055	0.042	0.032	0.071	0.055	0.049	0.039
	300	0.078	0.055	0.046	0.038	0.077	0.061	0.053	0.043
	500	0.081	0.058	0.049	0.041	0.082	0.060	0.052	0.051

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TABLE 4b: Size of Demeaned Tests, 5% level, AR(1) Errors

		S_n^τ				KPSS			
α	n	M1	M2	M3	M4	M1	M2	M3	M4
0.95	50	0.707	0.537	0.232	0.012	0.759	0.616	0.445	0.076
	80	0.734	0.588	0.361	0.022	0.752	0.629	0.537	0.196
	100	0.787	0.659	0.456	0.051	0.818	0.713	0.525	0.231
	120	0.819	0.701	0.506	0.052	0.811	0.715	0.554	0.259
	150	0.836	0.742	0.564	0.106	0.826	0.779	0.592	0.277
	200	0.863	0.790	0.605	0.174	0.851	0.807	0.612	0.286
	300	0.889	0.827	0.626	0.242	0.887	0.841	0.622	0.302
	500	0.912	0.859	0.649	0.273	0.918	0.866	0.664	0.297
0.90	50	0.623	0.401	0.153	0.013	0.665	0.516	0.340	0.045
	80	0.639	0.517	0.231	0.021	0.668	0.529	0.404	0.118
	100	0.687	0.626	0.299	0.034	0.706	0.633	0.378	0.137
	120	0.722	0.657	0.337	0.046	0.779	0.655	0.382	0.150
	150	0.745	0.686	0.385	0.053	0.792	0.671	0.405	0.161
	200	0.784	0.674	0.407	0.082	0.809	0.689	0.425	0.162
	300	0.810	0.728	0.404	0.114	0.816	0.721	0.391	0.165
	500	0.838	0.732	0.411	0.132	0.835	0.736	0.410	0.157
0.85	50	0.517	0.351	0.084	0.018	0.574	0.426	0.264	0.052
	80	0.538	0.359	0.133	0.014	0.635	0.479	0.305	0.084
	100	0.585	0.405	0.205	0.025	0.663	0.506	0.277	0.095
	120	0.611	0.462	0.229	0.036	0.678	0.513	0.275	0.102
	150	0.638	0.515	0.263	0.045	0.689	0.532	0.288	0.111
	200	0.671	0.557	0.291	0.052	0.701	0.538	0.301	0.111
	300	0.711	0.581	0.254	0.072	0.724	0.559	0.266	0.114
	500	0.745	0.616	0.284	0.087	0.731	0.575	0.280	0.109

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TABLE 5a: Size of Detrended Tests, 5% level, AR(1) Errors

		S_n^{τ}				KPSS			
α	n	M1	M2	M3	M4	M1	M2	M3	M4
0.8	50	0.406	0.141	0.021	0.020	0.642	0.456	0.242	0.037
	80	0.597	0.346	0.079	0.024	0.732	0.546	0.319	0.065
	100	0.652	0.407	0.074	0.025	0.757	0.569	0.271	0.075
	120	0.692	0.462	0.112	0.032	0.781	0.596	0.292	0.079
	150	0.739	0.513	0.160	0.036	0.802	0.624	0.311	0.087
	200	0.781	0.570	0.211	0.048	0.822	0.638	0.328	0.088
	300	0.835	0.631	0.202	0.054	0.845	0.661	0.283	0.088
	500	0.861	0.693	0.259	0.056	0.863	0.698	0.303	0.088
0.5	50	0.310	0.103	0.024	0.013	0.454	0.259	0.086	0.026
	80	0.358	0.158	0.042	0.013	0.483	0.281	0.104	0.026
	100	0.447	0.180	0.042	0.016	0.492	0.285	0.096	0.039
	120	0.468	0.204	0.046	0.029	0.504	0.290	0.094	0.041
	150	0.498	0.229	0.053	0.033	0.512	0.300	0.101	0.051
	200	0.521	0.245	0.067	0.036	0.535	0.299	0.102	0.050
	300	0.556	0.278	0.066	0.045	0.582	0.302	0.095	0.053
	500	0.583	0.314	0.079	0.051	0.637	0.316	0.103	0.056
0.1	50	0.038	0.023	0.014	0.004	0.063	0.054	0.041	0.022
	80	0.051	0.030	0.016	0.006	0.070	0.061	0.051	0.015
	100	0.059	0.035	0.021	0.018	0.073	0.65	0.054	0.029
	120	0.064	0.037	0.025	0.022	0.072	0.058	0.051	0.031
	150	0.068	0.039	0.029	0.028	0.080	0.063	0.053	0.039
	200	0.071	0.046	0.034	0.033	0.081	0.062	0.053	0.039
	300	0.082	0.052	0.038	0.042	0.084	0.064	0.053	0.045
	500	0.086	0.054	0.042	0.046	0.092	0.070	0.059	0.052

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TABLE 5b: Size of Demeaned Tests, 5% level, AR(1) Errors

		S_n^{μ}				KPSS			
α	n	M1	M2	M3	M4	M1	M2	M3	M4
0.95	50	0.650	0.3034	0.038	0.047	0.823	0.700	0.469	0.112
	80	0.759	0.676	0.257	0.035	0.847	0.805	0.660	0.221
	100	0.812	0.774	0.269	0.058	0.885	0.836	0.638	0.249
	120	0.846	0.842	0.451	0.071	0.906	0.882	0.665	0.296
	150	0.867	0.886	0.546	0.083	0.915	0.908	0.756	0.316
	200	0.907	0.899	0.667	0.094	0.914	0.917	0.770	0.350
	300	0.926	0.914	0.707	0.144	0.938	0.926	0.782	0.396
	500	0.948	0.932	0.722	0.254	0.949	0.935	0.804	0.424
0.90	50	0.571	0.239	0.028	0.027	0.788	0.629	0.378	0.081
	80	0.682	0.563	0.168	0.025	0.789	0.756	0.426	0.139
	100	0.746	0.652	0.165	0.037	0.823	0.807	0.461	0.156
	120	0.785	0.717	0.252	0.040	0.841	0.820	0.537	0.174
	150	0.816	0.767	0.286	0.054	0.852	0.833	0.551	0.181
	200	0.849	0.809	0.345	0.063	0.870	0.851	0.573	0.193
	300	0.905	0.862	0.355	0.077	0.914	0.873	0.582	0.208
	500	0.928	0.888	0.458	0.105	0.929	0.892	0.580	0.213
0.85	50	0.483	0.184	0.022	0.011	0.720	0.485	0.199	0.067
	80	0.690	0.333	0.081	0.018	0.806	0.516	0.309	0.098
	100	0.754	0.444	0.106	0.028	0.805	0.538	0.315	0.108
	120	0.775	0.521	0.124	0.033	0.814	0.618	0.328	0.118
	150	0.804	0.580	0.203	0.032	0.828	0.651	0.359	0.125
	200	0.836	0.637	0.257	0.042	0.870	0.686	0.381	0.129
	300	0.877	0.697	0.286	0.056	0.891	0.717	0.403	0.139
	500	0.903	0.766	0.350	0.065	0.911	0.752	0.405	0.138

TABLE 6: Power of Demeaned Tests, 5% level

	S_n^μ				KPSS			
n	M1	M2	M3	M4	M1	M2	M3	M4
50	0.934	0.820	0.586	0.201	0.944	0.866	0.655	0.341
80	0.964	0.913	0.754	0.272	0.969	0.919	0.765	0.534
100	0.977	0.936	0.775	0.375	0.979	0.949	0.782	0.591
120	0.989	0.964	0.818	0.439	0.989	0.969	0.826	0.636
150	0.996	0.981	0.887	0.551	0.995	0.983	0.905	0.669
200	1.000	0.997	0.921	0.682	1.000	0.996	0.923	0.725
300	1.000	1.000	0.948	0.792	1.000	1.000	0.957	0.813
500	1.000	1.000	0.997	0.886	1.000	1.000	0.997	0.897

TABLE 7: Power of Detrended Tests, 5% level

	S_n^μ				KPSS			
n	M1	M2	M3	M4	M1	M2	M3	M4
50	0.925	0.682	0.440	0.180	0.960	0.756	0.523	0.289
80	0.946	0.886	0.541	0.212	0.979	0.895	0.710	0.353
100	0.973	0.935	0.580	0.305	0.989	0.946	0.739	0.416
120	0.989	0.965	0.716	0.341	0.997	0.971	0.785	0.495
150	0.996	0.980	0.813	0.415	1.000	0.982	0.868	0.563
200	1.000	0.992	0.894	0.546	1.000	0.994	0.938	0.663
300	1.000	1.000	0.951	0.712	1.000	1.000	0.973	0.807
500	1.000	1.000	0.986	0.882	1.000	1.000	0.995	0.909

5 Application to the U.S. Economy

The test was also applied to several post-war quarterly U.S. macroeconomic time series. The data set in our empirical analysis consists of Real GDP, Real Investment, Real Consumption, and Employment. All these variables are from Citibase, over the period 1947:1 - 1993:4. The number of observations for these time series is 188. Because of the obvious tendency of growth presented in these series, we tested the null hypothesis of stationarity around a linear trend. Thus, S_n^τ is the appropriate statistic. (Testing for the null of level stationarity using S_n^μ has also been performed. Despite of the fact that the values of the test statistics are sensitive to the bandwidth choice, we can reject the null hypothesis of level stationarity for all these series, as expected.)

For comparison, we also calculated the Augmented Dickey-Fuller (ADF) tests for a unit root on these series (we used the BIC criterion of Schwarz, 1978 and Rissanen, 1978 in selecting the appropriate lag length of the autoregression). Table 8 presents the ADF coefficient (ADF_α) and t-ratio (ADF_t) statistics for the unit root hypothesis. The critical values at 5% level of significance for the ADF tests are -21.20 and -3.44 respectively. We can not reject the null hypothesis of a unit root in the series of Real GDP, Real Consumption, and Employment in both the coefficient and t-ratio tests. For the series of Real Investment, the unit root hypothesis is rejected in both tests.

TABLE 8. ADF tests applied to U.S. Macro data

Series	ADF_α	ADF_t
Real GDP	-8.5	-1.94
Real Investment	-37.28	-3.84
Real Consumption	-14.77	-3.07
Employment	-18.58	-3.11

TABLE 9. S_n^τ test applied to U.S. macro data

Series	M=2	M=4	M=6	M=8	M=10
Real GDP	1.688	1.335	1.152	1.038	0.960
Real Investment	1.254	0.998	0.869	0.791	0.738
Real Consumption	1.458	1.174	1.036	0.956	0.904
Employment	1.696	1.345	1.166	1.058	0.983

Table 9 gives the S_n^τ test statistic for the null hypothesis of stationarity around a linear trend. We consider bandwidth choices from two to ten since the values of the test statistics are sensitive to bandwidth. The empirical results seem to be in accord with those in Kwiatkowski et al. (1992). It is clear from Table 9 that the test statistics decline monotonically as M increases. If we choose a small bandwidth, say $M = 2$, we would reject the null hypothesis of trend stationarity for all these series. However, these series are obviously temporally dependent and such a serial dependence should be taken into account when we estimate the long-run variance. For bandwidth choices $M = 6, 8, 10$, we find that we cannot reject the null hypothesis of trend stationarity at the 5% level for the series of real investment. But we can reject trend stationarity in the other three series. Combining the results from Table 8 and Table 9, we reach the following conclusion: the series Real GDP, Real

Consumption, and Employment appear to have unit roots and, the Real Investment series is likely to be trend stationary.

6 Conclusion

We have proposed statistical tests for the null hypothesis of stationarity (or trend stationarity) by looking at the fluctuation in a (detrended) time series. The results apply to a wide class of time series models. Asymptotic distributions of these tests are derived under both the null hypothesis and the unit root alternative. These limiting distributions are nonstandard and are functions of Brownian motions, involving higher order Brownian bridges. The principle of the approach is general and can be applied to other types of alternatives. Table of critical values is provided based on the asymptotic null distributions. The consistency of the tests are proved in this paper. The asymptotic behavior of the proposed test is similar to that of the KPSS test and the divergence rate of the statistics under H_1 depends on the bandwidth parameter. A Monte Carlo experiment was conducted to examine the finite sample performance of these tests. In particular, finite sample size and power were studied. As do other tests for stationarity, these tests provide a useful complement to the conventional unit root tests.

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7 Appendix: Proofs

7.1 Proof of Theorem 1

By definition

$$\begin{aligned}
\underline{S}_n &= \max_{1\le k\le n} \frac{k}{\widehat{\omega}_y\sqrt{n}}\left|\frac{1}{k}\sum_{t=1}^{k}\widehat{y}_t - \frac{1}{n}\sum_{t=1}^{n}\widehat{y}_t\right| \\
&= \max_{0\le r\le 1} \frac{1}{\widehat{\omega}_y}\left|\frac{1}{\sqrt{n}}\sum_{t=1}^{[nr]}\widehat{y}_t - \frac{[nr]}{n}\left(\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\widehat{y}_t\right)\right|,
\end{aligned}$$

Notice that

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{[nr]}\widehat{y}_t = \frac{1}{\sqrt{n}}\sum_{t=1}^{[nr]} y_t - \frac{1}{\sqrt{n}}(\widehat{\gamma}-\gamma)'\sum_{t=1}^{[nr]} x_t$$

where $Y_n(r) = n^{-1/2}\sum_{t=1}^{[nr]} y_t$ is a stochastic process in $D[0,1]$, the space of functions on $r \in [0,1]$ that are right continuous with left-hand limits. We endow the space $D[0,1]$ with the Skorohod topology (Billingsley, 1968). Under the null of stationarity and Assumption L, the partial sum process $n^{-1/2}\sum_{t=1}^{[nr]} y_t$ satisfies the following invariance principle (Phillips and Solo, 1992, Theorem 3.4)

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{[nr]} y_t \Rightarrow B_y(r).$$

Thus, by the continuous mapping theorem,

$$\begin{aligned}\sqrt{n}D(\widehat{\gamma}-\gamma) &= \left[n^{-1}\sum D^{-1}x_t x_t' D^{-1}\right]^{-1}\left[\sum D^{-1}x_t y_t/\sqrt{n}\right] \\ &\Rightarrow \left[\int_0^1 X(s)X(s)'ds\right]^{-1}\left[\int_0^1 X(s)dB_y(s)\right],\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\sqrt{n}}\sum_{t=1}^{[nr]}\widehat{y}_t &= \frac{1}{\sqrt{n}}\sum_{t=1}^{[nr]} y_t - \frac{1}{\sqrt{n}}(\widehat{\gamma}-\gamma)'\sum_{t=1}^{[nr]} x_t \\ &= \frac{1}{\sqrt{n}}\sum_{t=1}^{[nr]} y_t - \left[\sqrt{n}(\widehat{\gamma}-\gamma)'D\right]\left[\frac{1}{n}\sum_{t=1}^{[nr]} D^{-1}x_t\right] \\ &\Rightarrow B_y(r) - \left[\int_0^1 dB_y(s)X(s)'\right]\left[\int_0^1 X(s)X(s)'ds\right]^{-1}\int_0^r X(s)ds \\ &= \omega_y\left\{W(r) - \left[\int_0^1 dW(s)X(s)'\right]\left[\int_0^1 X(s)X(s)'ds\right]^{-1}\int_0^r X(s)ds\right\} \\ &= \omega_y\widetilde{W}_X(r).\end{aligned}$$

Thus, by the fact that $[nr]/n \to r$ and the continuous mapping theorem,

$$\max_{0\le r\le 1}\left|\frac{1}{\sqrt{n}}\sum_{t=1}^{[nr]}\widehat{y}_t - \frac{[nr]}{n}\left(\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\widehat{y}_t\right)\right| \Rightarrow \sup_{0\le r\le 1}\left|\omega_y\left(\widetilde{W}_X(r) - r\widetilde{W}_X(1)\right)\right|.$$

and

$$\underline{S}_n = \max_{1\le k\le n}\frac{k}{\widehat{\omega}_y\sqrt{n}}\left|\frac{1}{k}\sum_{t=1}^{k}\widehat{y}_t - \frac{1}{n}\sum_{t=1}^{n}\widehat{y}_t\right| \Rightarrow \sup_{0\le r\le 1}\left|\widetilde{W}_X(r) - r\widetilde{W}_X(1)\right|.$$

Notice that $\widetilde{W}_X(r)$ is a generalized Brownian bridge, when x_t has a constant element it is tied down to the origin at the ends of the [0,1] interval just like a Brownian bridge, thus $\widetilde{W}_X(1) = 0$, and

$$\underline{S}_n \Rightarrow \sup_{0 \le r \le 1} \left| \widetilde{W}_X(r) \right|.$$

7.2 Proof of Lemma 1

Notice that

$$\widehat{f}_{yy}(0) = \frac{1}{2\pi} \sum_{h=-M}^{M} k(\frac{h}{M}) C_{yy}(h),$$

where

$$C_{yy}(h) = \frac{1}{n} \sum_{1}^{n} \widehat{y}_t \widehat{y}_{t+h}, \;\; 1 \le t+h \le n.$$

Under the alternative hypothesis and Assumption L, $n^{-1/2} y_{[nr]} \approx n^{-1/2} \sum_{t=1}^{[nr]} u_t \Rightarrow B_u(r)$, and thus

$$n^{-1/2} \widehat{y}_{[nr]} \Rightarrow B_{u,X}(r) = B_u(r) - \left[\int_0^1 B_u(s) X(s)' ds \right] \left[\int_0^1 X(s) X(s)' ds \right]^{-1} X(r).$$

Following the same argument as in Phillips (1991), notice that $M = O(n^{1/3})$ as $n \to \infty$, we have

$$\begin{aligned} \frac{1}{nM} \widehat{\omega}_y^2 &= \frac{1}{M} \sum_{h=-M}^{M} k(\frac{h}{M}) \left[\frac{1}{n} C_{yy}(h) \right] \\ &\Rightarrow 2\pi \left(\frac{1}{2\pi} \int_{-1}^{1} k(s) ds \right) \int_0^1 B_{u,X}(r)^2 dr \\ &= 2\pi K(0) \int_0^1 B_{u,X}(r)^2 dr. \end{aligned}$$

7.3 Proof of Theorem 2

By the result of Lemma 1,

$$\begin{aligned}
\sqrt{\frac{M}{n}}S_n &= \max_{k=1,\ldots,n}\sqrt{\frac{M}{n}}\frac{k}{\widehat{\omega}_y\sqrt{n}}\left|\frac{1}{k}\sum_{t=1}^{k}\widehat{y}_t-\frac{1}{n}\sum_{t=1}^{n}\widehat{y}_t\right| \\
&= \max_{k=1,\ldots,n}\frac{1}{\sqrt{n^{-1}M^{-1}\widehat{\omega}_y^2}}\left|\frac{1}{n}\sum_{t=1}^{k}\frac{\widehat{y}_t}{\sqrt{n}}-\frac{k}{n^2}\sum_{t=1}^{n}\frac{\widehat{y}_t}{\sqrt{n}}\right| \\
&= \sup_{0\le r\le 1}\frac{1}{\sqrt{n^{-1}M^{-1}\widehat{\omega}_y^2}}\left|\frac{1}{n}\sum_{t=1}^{[nr]}\frac{\widehat{y}_t}{\sqrt{n}}-\frac{[nr]}{n^2}\sum_{t=1}^{n}\frac{\widehat{y}_t}{\sqrt{n}}\right| \\
&\Rightarrow \left[2\pi K(0)\int_0^1 B_{u,X}(r)^2dr\right]^{-1/2}\sup_{0\le r\le 1}\left|\int_0^r B_{u,X}(s)ds-r\int_0^1 B_{u,X}(s)ds\right|.
\end{aligned}$$

Thus the result of Theorem 2 follows immediately.

8 References

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